Bedford & Fowler Engineering Mechanics Dynamics Principles

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MECH2700 Minor Assignment 1

Notes for tutors

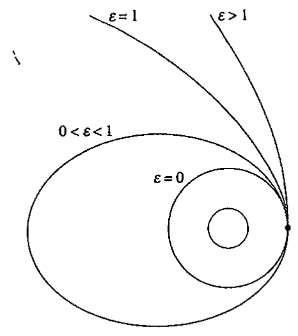

Figure 14.19
Orbits for different of eccentricity.

Circular Orbit If the initial velocity v_0 is chosen so that $\varepsilon = 0$, Eq. (14.22) reduces to $r = r_0$ and the orbit is circular (Fig. 14.19). Setting $\varepsilon = 0$ in Eq. (14.23) and solving for v_0, we obtain

$$v_0 = \sqrt{\frac{gR_E^2}{r_0}}, \tag{14.24}$$

which agrees with the velocity for a circular orbit we obtained by a different method in Example 13.10.

Elliptic Orbit If $0 < \varepsilon < 1$, orbit is an ellipse (Fig. 14.19). The maximum radius of the ellipse occurs when $\theta = 180°$. Setting θ equal to 180° in Eq. (14.22), we obtain an expression for the maximum radius of the ellipse in terms of the initial radius and ε:

$$r_{\max} = r_0\left(\frac{1+\varepsilon}{1-\varepsilon}\right). \tag{14.25}$$

Parabolic Orbit Notice from Eq. (14.25) that the maximum radius of the elliptic orbit increases without limit as $\varepsilon \to 1$. When $\varepsilon = 1$, the orbit is a parabola (Fig. 14.19). The corresponding velocity v_0 is the minimum initial velocity for which the radius r increases without limit, which is the escape velocity. Setting $\varepsilon = 1$ in Eq. (14.23) and solving for v_0, we obtain

$$v_0 = \sqrt{\frac{2gR_E^2}{r_0}}.$$

This is the same value for the escape velocity we obtained in Example 13.5 for the case of an object moving in a straight path directly away from the center of the earth.

Hyperbolic Orbit If $\varepsilon > 1$, the orbit is a hyperbola (Fig. 14.19).

The solution we have presented, based on the assumption that the earth is a homogeneous sphere, approximates the orbit of an earth satellite. Determining the orbit accurately requires taking into account the variations in the earth's gravitational field due to its actual mass distribution. Similarly, depending on the accuracy required, determining the orbit of a planet around the sun may require accounting for perturbations due to the gravitational attractions of the other planets.

Example 14.8

Orbit of an Earth Satellite

An earth satellite is in an elliptic orbit with a minimum radius of 6600 km and a maximum radius of 16,000 km. The earth's radius is 6370 km.
(a) Determine the satellite's velocity when it is at perigee (its minimum radius) and when it is at apogee (its maximum radius).
(b) Draw a graph of the orbit.

Strategy

We can regard the radius and velocity of the satellite at perigee as the initial conditions r_0 and v_0 used in obtaining Eq. (14.22). Since the maximum radius of the orbit is given, we can solve Eq. (14.25) for the eccentricity of the orbit and then use Eq. (14.23) to determine v_0. From Eq. (14.14), the product of r and the transverse component of the velocity is constant. From this condition, we can determine the velocity at apogee.

Solution

(a) The ratio of the radius at apogee to the radius at perigee is

$$\frac{r_{\max}}{r_0} = \frac{1.60 \times 10^7\ \text{m}}{6.60 \times 10^6\ \text{m}} = 2.42.$$

Solving Eq. (14.25) for ε, we find that the eccentricity is

$$\varepsilon = \frac{\dfrac{r_{\max}}{r_0} - 1}{\dfrac{r_{\max}}{r_0} + 1} = \frac{2.42 - 1}{2.42 + 1} = 0.416.$$

From Eq. (14.23), the velocity at perigee is

$$v_0 = \sqrt{\frac{(\varepsilon + 1)gR_E^2}{r_0}} = \sqrt{\frac{(0.416 + 1)(9.81)(6.37 \times 10^6)^2}{6.60 \times 10^6}}$$
$$= 9240\ \text{m/s}.$$

At both perigee and apogee, the velocity has only a transverse component. From Eq. (14.14), the velocity at apogee, v_a, is related to the velocity v_0 by

$$r_0 v_0 = r_{\max} v_a.$$

Therefore, the velocity at apogee is

$$v_a = \frac{v_0}{r_{\max}/r_0} = \frac{9240}{2.42} = 3810\ \text{m/s}.$$

(b) By plotting Eq. (14.22) with $\varepsilon = 0.416$, we obtain the graph of the orbit (Fig. 14.20).

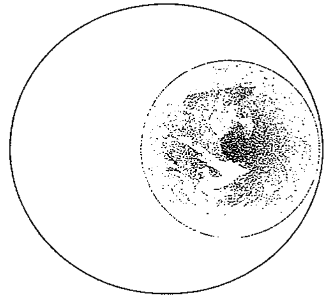

Figure 14.20
Orbit of an earth satellite with a perigee of 6600 km and an apogee of 16,000 km.

14.6 Numerical Solutions

So far in this chapter, we have described many situations in which we were able to determine the motion of an object by a simple procedure: After using Newton's second law to determine the acceleration, we integrated to obtain analytical, or *closed-form*, expressions for the object's velocity and position. These examples are very valuable—they demonstrate how to use free-body diagrams and express problems in different coordinate systems, and they develop intuitive understanding of forces and motions. But it would be misleading if we presented examples of this kind only, because most problems that must be dealt with in engineering cannot be solved in that way. The functions

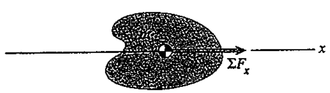

Figure 14.21
An object moving along the x axis.

describing the forces, and therefore the acceleration, are often too complicated to integrate and obtain closed-form solutions. In other situations, the forces are not known in terms of functions, but instead are specified in terms of data, either as a continuous recording of force as a function of time (analog data) or as values of force measured at discrete times (digital data).

We can obtain approximate solutions to such problems by using numerical integration. Consider an object of mass m in straight-line motion along the x-axis (Fig. 14.21) and assume that the x component of the total force may depend on the time and on the position and velocity of the object:

$$\Sigma F_x = \Sigma F_x(t, x, v_x). \tag{14.26}$$

Suppose that at a particular time t_0 we know the position $x(t_0)$ and velocity $v_x(t_0)$ of the object. The acceleration of the object at t_0 is

$$\frac{dv_x}{dt}(t_0) = \frac{1}{m}\Sigma F_x(t_0, x(t_0), v_x(t_0)). \tag{14.27}$$

To determine the velocity at time $t_0 + \Delta t$, we express it as a Taylor series:

$$v_x(t_0 + \Delta t) = v_x(t_0) + \frac{dv_x}{dt}(t_0)\Delta t + \frac{1}{2}\frac{d^2v_x}{dt^2}(t_0)(\Delta t)^2 + \cdots.$$

By choosing a sufficiently small value of Δt, we can neglect terms in this equation that are of second and higher order in Δt and substitute Eq. (14.27) to obtain an approximation for the velocity at $t_0 + \Delta t$:

$$v_x(t_0 + \Delta t) = v_x(t_0) + \frac{1}{m}\Sigma F_x(t_0, x(t_0), v_x(t_0))\Delta t. \tag{14.28}$$

We approximate the position at $t_0 + \Delta t$ in the same way. Expressing it as a Taylor series, we have

$$x(t_0 + \Delta t) = x(t_0) + \frac{dx}{dt}(t_0)\Delta t + \frac{1}{2}\frac{d^2x}{dt^2}(t_0)(\Delta t)^2 + \cdots,$$

and neglecting higher-order terms in Δt, we obtain

$$x(t_0 + \Delta t) = x(t_0) + v_x(t_0)\Delta t. \tag{14.29}$$

Thus, if we know the position and velocity at a time t_0, we can approximate their values at $t_0 + \Delta t$ by using Eqs. (14.28) and (14.29). We can then repeat the procedure, using $x(t_0 + \Delta t)$ and $v_x(t_0 + \Delta t)$ as initial conditions to determine the approximate position and velocity at $t_0 + 2\Delta t$. By continuing in this way, we obtain approximate solutions for the position and velocity in terms of time. This procedure is easy to carry out using a calculator or a computer. It is called a *finite-difference method*, because it determines changes in the dependent variables over finite intervals of time. The particular finite-difference method we describe, due to Leonhard Euler (1707–1783), is called *forward differencing*: The value of the derivative of a function at t_0 is approximated by using its value at t_0 and its value forward in time at $t_0 + \Delta t$.

More elaborate finite-difference methods, based on retaining more terms in the Taylor series, produce smaller errors in each time step. For example, in the fourth-order Runge–Kutta method, terms through the fourth order in Δt are retained. Euler's method is adequate to introduce you to numerical solutions of problems in dynamics.

Notice that Eq. (14.26) does not need to be a functional expression to carry out the process we have described. The values of the total force must be known at times $t_0, t_0 + \Delta t, \dots$, and can be determined either from a function or from analog or digital data.

We can determine the velocity and position of an object in curvilinear motion by the same approach. Suppose that the object moves in the xy-plane and that the components of force may depend on the time and on the position and velocity of the object; that is,

$$\Sigma F_x = \Sigma F_x(t, x, y, v_x, v_y) \quad \text{and} \quad \Sigma F_y = \Sigma F_y(t, x, y, v_x, v_y).$$

If the position and velocity are known at a time t_0, we can use the same steps leading to Eqs. (14.28) and (14.29) to obtain approximate expressions for the components of position and velocity at $t_0 + \Delta t$, yielding

$$x(t_0 + \Delta t) = x(t_0) + v_x(t_0)\Delta t,$$

$$y(t_0 + \Delta t) = y(t_0) + v_y(t_0)\Delta t,$$

$$v_x(t_0 + \Delta t) = v_x(t_0) + \frac{1}{m}\Sigma F_x(t_0, x(t_0), y(t_0), v_x(t_0), v_y(t_0))\Delta t, \tag{14.30}$$

and

$$v_y(t_0 + \Delta t) = v_y(t_0) + \frac{1}{m}\Sigma F_y(t_0, x(t_0), y(t_0), v_x(t_0), v_y(t_0))\Delta t.$$

Computational Mechanics

The following example and problems are designed to be worked with the use of a programmable calculator or computer.

Computational Example 14.9

A 50-kg projectile is launched from $x = 0$, $y = 0$ with initial velocity $v_x = 100$ m/s, $v_y = 100$ m/s. (The y axis is positive upward.) The aerodynamic drag force on the projectile is of magnitude $C|\mathbf{v}|^2$, where C is a constant. Determine the trajectory of the projectile for $C = 0.005$, 0.01, and 0.02.

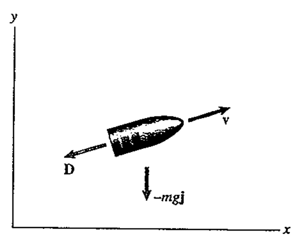

Figure 14.22
The forces on the projectile are its weight and the drag force **D**.

Solution

To apply Eqs. (14.30), we must determine the x and y components of the total force on the projectile. Let **D** be the drag force (Fig. 14.22). Because $\mathbf{v}/|\mathbf{v}|$ is a unit vector in the direction of **v**, we can write

$$\mathbf{D} = -C|\mathbf{v}|^2\frac{\mathbf{v}}{|\mathbf{v}|} = -C|\mathbf{v}|\mathbf{v}.$$

The external forces on the projectile are its weight and the drag, so we have

$$\Sigma \mathbf{F} = -mg\mathbf{j} - C|\mathbf{v}|\mathbf{v},$$

and the components of the total force are

$$\Sigma F_x = -C\sqrt{v_x^2 + v_y^2}\,v_x, \qquad \Sigma F_y = -mg - C\sqrt{v_x^2 + v_y^2}\,v_y. \quad (14.31)$$

Consider the case in which $C = 0.005$, and let $\Delta t = 0.2$ s. At the initial time $t_0 = 0$, the position coordinates and the components of the velocity are $x(t_0) = 0$, $y(t_0) = 0$, $v_x(t_0) = 100$ m/s, and $v_y(t_0) = 100$ m/s. The x coordinate after the first time step is

$$x(t_0 + \Delta t) = x(t_0) + v_x(t_0)\Delta t,$$

or

$$\begin{aligned} x(0.2) &= x(0) + v_x(0)\Delta t \\ &= 0 + (100)(0.2) \\ &= 20 \text{ m.} \end{aligned}$$

The y coordinate after the first time step is

$$y(t_0 + \Delta t) = y(t_0) + v_y(t_0)\Delta t,$$

or

$$\begin{aligned} y(0.2) &= y(0) + v_y(0)\Delta t \\ &= 0 + (100)(0.2) \\ &= 20 \text{ m.} \end{aligned}$$

The x component of the velocity after the first time step is

$$v_x(t_0 + \Delta t) = v_x(t_0) + \frac{1}{m}\Sigma F_x(t_0, x(t_0), y(t_0), v_x(t_0), v_y(t_0))\Delta t,$$

or

$$\begin{aligned} v_x(0.2) &= v_x(0) + \frac{1}{m}\left\{-C\sqrt{[v_x(0)]^2 + [v_y(0)]^2}\,v_x(0)\right\}\Delta t \\ &= 100 + \frac{1}{50}\left[-0.005\sqrt{(100)^2 + (100)^2}\,(100)\right](0.2) \\ &= 99.72 \text{ m/s,} \end{aligned}$$

and the y component of the velocity after the first time step is

$$v_y(t_0 + \Delta t) = v_y(t_0) + \frac{1}{m}\Sigma F_y(t_0, x(t_0), y(t_0), v_x(t_0), v_y(t_0))\Delta t,$$

or

$$\begin{aligned} v_y(0.2) &= v_y(0) + \frac{1}{m}\left\{-mg - C\sqrt{[v_x(0)]^2 + [v_y(0)]^2}\,v_y(0)\right\}\Delta t \\ &= 100 + \frac{1}{50}\left[-(50)(9.81) - 0.005\sqrt{(100)^2 + (100)^2}\,(100)\right](0.2) \\ &= 97.76 \text{ m/s.} \end{aligned}$$

Continuing in this way, we obtain the following results for the first five time steps:

Time, s	x, m	y, m	v_x, m/s	v_y, m/s
0.0	0.00	0.00	100.00	100.00
0.2	20.00	20.00	99.72	97.76
0.4	39.94	39.55	99.44	95.52
0.6	59.83	58.66	99.16	93.29
0.8	79.66	77.31	98.89	91.08
1.0	99.44	95.53	98.63	88.87

When there is no drag ($C = 0$), we can obtain the closed-form solution for the trajectory. Figure 14.23 compares the closed-form solution with numerical solutions obtained using $\Delta t = 2.0$ s, 1.0 s, and 0.2 s. The numerical solutions approach the exact solution as Δt decreases.

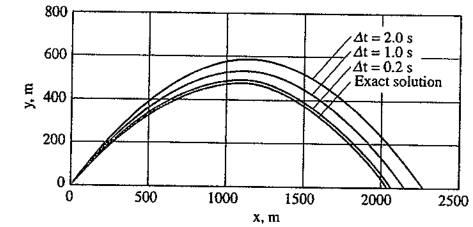

Figure 14.23
The closed-form solution for the trajectory when $C = 0$, compared with numerical solutions.

Figure 14.24 shows numerical solutions (obtained using $\Delta t = 0.01$ s) for the various values of C. As expected, the range of the projectile decreases as C increases. Also, notice that drag changes the shape of the trajectory: The angle at which the projectile descends is steeper than the angle at which it was launched.

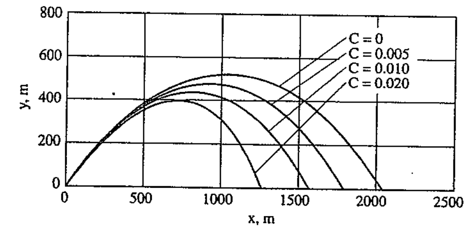

Figure 14.24
Trajectories for various values of C.

Discussion

The development of the first completely electronic digital computer, the ENIAC (Electronic Numerical Integrator and Computer), at the University of Pennsylvania between 1943 and 1945 was motivated in part by the need to calculate trajectories of projectiles. A room-sized machine with 18,000 vacuum tubes, the ENIAC had 20 bytes of RAM and 450 bytes of ROM.