The University of Queensland
School of Mechanical and Mining Engineering

MECH2700 Engineering Analysis I (2015)

Assignment 2: Analysis of steady-state flow in networks

Introduction

The mathematical analysis of flow in a network of interacting components usually results in a system of equations with several unknown quantities. For a steady-state situation, the description is in the form of a set of constraints on the unknown flow quantities. The solution, a vector of values, must satisfy all the equations simultaneously. In this assignment, you will analyse two network systems:

- a high-order filter for electrical signals, built with passive components.
- a water-supply pipe network.

For the filter, the system of equations will be linear but the coefficients and the unknown quantities will be complex numbers. The description of the flow within the pipe network requires only real numbers but the constraint equations are nonlinear. In both cases, you will eventually apply a Gauss-Jordan elimination solver for linear systems of equations to get your solutions.

Task 0

Write a Python module with your own implementation of a Gauss-Jordan elimination solver that can work with floating-point or complex numbers. Provide suitable documentation and test cases. Set up the module so that it can be imported into another application conveniently and the solver reused.

Part 1: A Chebyshev filter for electrical signals

Theory

In your introductory electrical engineering course (ENGG1300), you learned to analyse the response of circuits to sinusoidal signals with the aid of phasors. If the circuit is built from ideal linear components, and a sinusoidal forcing function is applied, the steady-state behaviour of the flow quantities (voltage and current) can be described with harmonic functions. For convenience these are written as:

$$V(t) = v\,\mathrm{e}^{j\omega t} \quad , \quad I(t) = i\,\mathrm{e}^{j\omega t}$$

where the v and i amplitudes are complex quantities and $j = \sqrt{-1}$. Note that ω is in radians/s.

Considering the behaviour of individual resistors, capacitors and inductors in terms of these phasor quantities, the simple relation between voltage and current

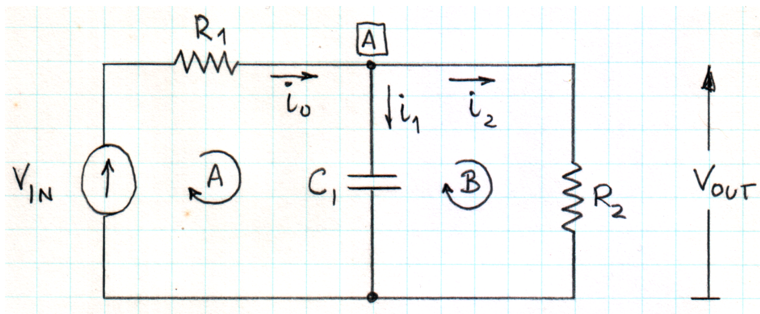
$$v = i\,Z$$

holds if we use the impedances

$$Z_R = R \quad , \quad Z_C = \frac{-j}{\omega C} = \frac{1}{j\omega C} \quad , \quad Z_L = j\omega L \quad .$$

Note that the impedances for capacitors and inductors are complex and change with frequency of the forcing function.

Now, consider a small filter circuit as shown below. This circuit has two loops with currents labelled i_0, i_1 and i_2. The circuit is driven by the alternating-current (AC) voltage V_{IN} and the output of interest is the voltage V_{OUT} across the resistor R_2.



The analysis of this circuit proceeds by using Kirchoff's laws to get the constraint equations:

1. The sum of voltages across components arranged in a closed loop is zero.
2. The sum of all currents into a node (labelled $\boxed{\text{A}}$ above) is zero.

Considering voltages around loop A we have:

$$V_{IN} - Z_{R1}\, i_0 - Z_{C1}\, i_1 \; = \; 0$$

for voltages around loop B:

$$Z_{C1}\, i_1 - Z_{R2}\, i_2 \; = \; 0$$

and for the currents at node $\boxed{\text{A}}$:

$$i_0 - i_1 - i_2 \; = \; 0$$

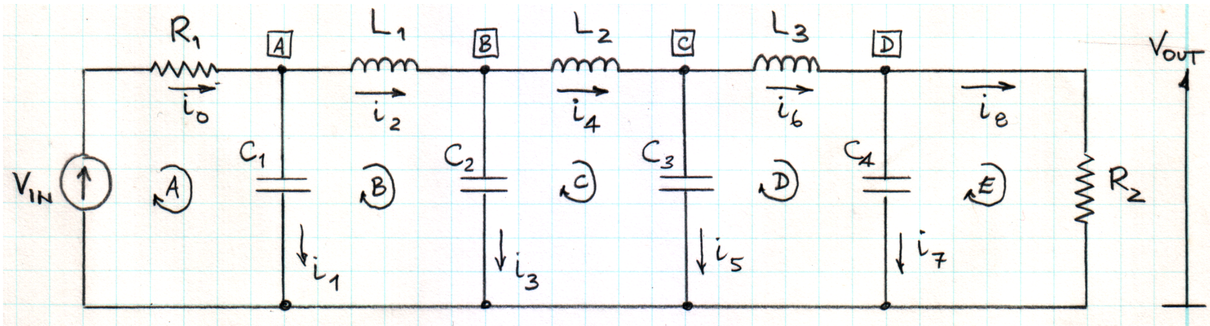
This system of constraint equations is linear in the unknown currents and may be written as

$$\begin{bmatrix} -R_1 & \frac{j}{\omega C_1} & 0 \\ 0 & \frac{-j}{\omega C_1} & -R_2 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} i_0 \\ i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -V_{IN} \\ 0 \\ 0 \end{bmatrix}$$

and, for particular values of $R1$, $R2$, $C1$, V_{IN} and frequency ω, may be solved with one of the standard elimination methods. Once you have the values for the currents, the output voltage across R_2 is easy to compute.

Task 1

Write the constraint equations for the low-pass filter shown below. Use the notation specified in the figure.



You should end up with 5 equations for the voltages around the loops A through E and 4 equations for the conservation of current at nodes $\boxed{\text{A}}$ through $\boxed{\text{D}}$. The unknowns are the 9 currents i_0 through i_8.

Task 2

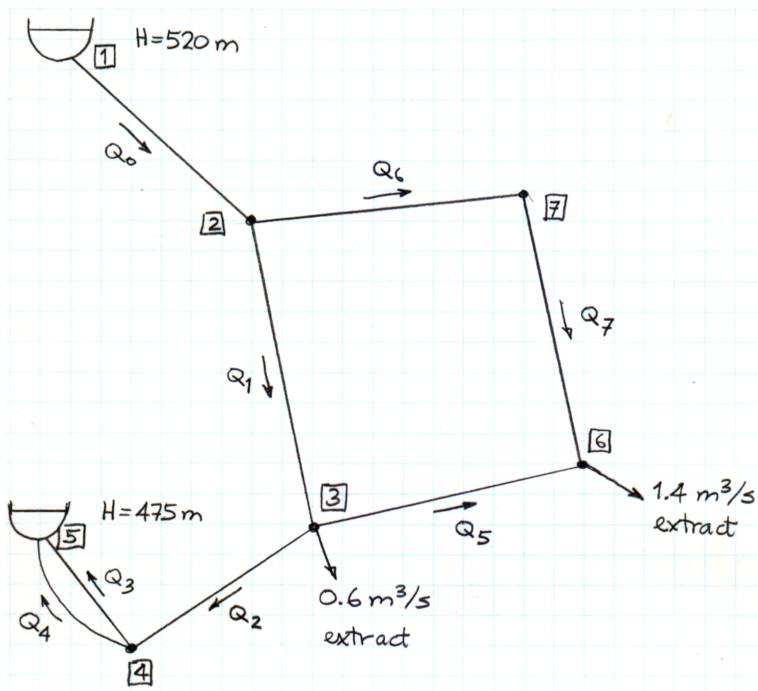
Write a Python program to compute the response of the filter (i.e. V_{OUT}/V_{IN}) as a function of frequency for the following parameter values: $R_1 = R_2 = 378\,\Omega$, $C_1 = C_4 = 532\,\text{pF}$, $C_2 = C_3 = 944\,\text{pF}$, $L_1 = L_3 = 91.3\,\mu\text{H}$, and $L_2 = 101\,\mu\text{H}$. According to Horowitz and Hill (in the third edition of *The Art of Electronics*), this is a low-pass filter with stunning performance.

Task 3

Plot the amplitude response of the filter for frequencies over the range 10 Hz to 10 MHz. Do a plot with linear scales to show the low-frequency response clearly and another plot to show the high-frequency roll-off. Comment of the nature of the filter and its performance.

Part 2: A water-supply network

A water-supply system has 2 reservoirs ($\boxed{1}$ and $\boxed{5}$) and a network of pipes with unknown flowrates, Q_0 through Q_7, as shown below. If the total head of at each reservoir is as shown on the figure and water is drawn from junctions $\boxed{3}$ and $\boxed{6}$ at volume flow rates of $0.6\,\mathrm{m^3/s}$ and $1.4\,\mathrm{m^3/s}$, respectively, the goal of our analysis is to determine the flow rates in each of the pipes.



The remaining parameters that are needed in the analysis are given in the following table.

pipe number	L (metres)	D (metres)	$Q_{initial-guess}$ ($\mathrm{m^3/s}$)
0	3000	0.3	3.0
1	3000	0.3	1.6
2	3000	0.3	1.0
3	3000	0.3	0.7
4	3000	0.1	0.3
5	4000	0.3	0.0
6	4000	0.25	1.4
7	4000	0.25	1.4

Theory

When analysing steady flows in pipe networks, it is often convenient to work in terms of total head (or potential)

$$H = \frac{p}{\rho\,\mathrm{g}} + \frac{V^2}{2\,\mathrm{g}} + z$$

where

- p is static pressure in Pascals
- ρ is the fluid density in kg/m^3
- V is the bulk velocity of the water in the pipe (in m/s)
- g is gravity 9.81 m/s^2
- z is elevation of the pipe centreline in metres

Because of frictional effects, there will be a loss of total head as the water flows along an individual pipe according to

$$\Delta h = f \frac{L}{D} \frac{V^2}{2\mathrm{g}}$$

which, for a circular pipe may be written in terms of the volume flow rate Q as

$$\Delta h = \left(f \frac{16}{\pi^2} \frac{L}{D^5} \right) Q^2 \;=\; R\, Q^2$$

where R is now the pipe resistance. For convenience in this exercise, we will assume the friction factor is constant with value $f = 0.02$. Because of the squared Q, we have to take care with the direction of flow and the sign of the head loss. If we choose a nominal direction for flow from pipe-end a to pipe-end b, assuming a positive Q, we need to compute the change in total head as

$$H_a - H_b \;=\; \Delta h \times \mathrm{sign}(Q)$$

to compute the correct sign of head loss for reverse flow.

Task 4

Formulate the constraint equations for the pipe network by considering

- the head loss along the path $\boxed{1}\!\rightarrow\!\boxed{2}\!\rightarrow\!\boxed{3}\!\rightarrow\!\boxed{4}\!\rightarrow\!\boxed{5}$ which has a reservoir at each end.
- the head loss around the closed loop $\boxed{4}\!\rightarrow\!\boxed{5}\!\rightarrow\!\boxed{4}$ which has to be zero
- the head loss around other closed loop $\boxed{2}\!\rightarrow\!\boxed{7}\!\rightarrow\!\boxed{6}\!\rightarrow\!\boxed{3}\!\rightarrow\!\boxed{2}$
- and the flow continuity at nodes $\boxed{2}$, $\boxed{3}$, $\boxed{4}$, $\boxed{6}$ and $\boxed{7}$

Note that some of the equations are nonlinear.

Task 5

Because some of the equations from Task 4 are nonlinear, let's consider rewriting the flow rates as

$$Q_{solution} = Q_{guess} + \Delta Q$$

Assuming that ΔQ is small, show that we can expand the Q^2 terms so that each head-loss along each pipe can be approximately written as

$$H_a - H_b \;=\; \left(R\, Q_{guess}^2 \right) \times \mathrm{sign}(Q_{guess}) + (2\, R\, Q_{guess}\, \Delta Q) \times \mathrm{sign}(Q_{guess})$$

Now, use this to rewrite the constraint equations for a new set of unknowns ΔQ_i, presuming that suitable set of $Q_{guess,i}$ values are available.

Task 6

Implement your set of constraint equations as a Python program that assembles the linear system of equations for the unknown ΔQ_i and use it to solve for the flow rates in each of the pipes. Use your Gauss-Jordan elimination code as the linear equation solver. You may use the values in the earlier table as an initial guess. Document your results.

Task 7

Rework Task 6 using one of the iterative schemes to solve the set of linear equations for the ΔQ_i values. Comment on the code and performance trade-offs made between the use of direct-elimination solver and the use of the iterative solver.

Submission Notes and Assessment Criteria

This assignment may be done in pairs or individually, however, remember that discussing your formulation and code development with your peers is a useful activity. If working as a pair, do *not* partition the tasks and do them separately, only coming back together to assemble the report. Be sure to work on all tasks together.

Submit one report (per pair of students), detailing your analysis, code and results for the Tasks. This report should be brief but well structured, so that the tutor can easily interpret your results. Include your Python code as and produce a single PDF file for your report. It is fine to scan hand-written elements (such as equations and schematic figures) and include them in your report. There is no good reason to add errors to your report through sloppy typesetting. The plots in Task 3, however, should be computer-generated and appropriately labelled. As well as the report PDF, you will need to submit a zip-file containing all of your Python code.

For assessment of your work, the tutors will be looking for answers to the following questions.

- Have you done what the tasks require? Note that your report should be well structured and tidy so that the tutor can easily find your answers. Your plots having correct titles and axes labels is quite helpful.
- Are your answers accurate?
- Is your code well-structured and clearly written, with suitable documentation comments?
- Does your code correspond to the results shown in your report?