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Assignment 1

Introduction

Legend has it that, in 1971, two college students at Yale University used a large slingshot (made with 8, 2-metre lengths of surgical rubber) to launch a loaf of frozen bread over their college master's house. The physical environment and approximate trajectory is shown below.

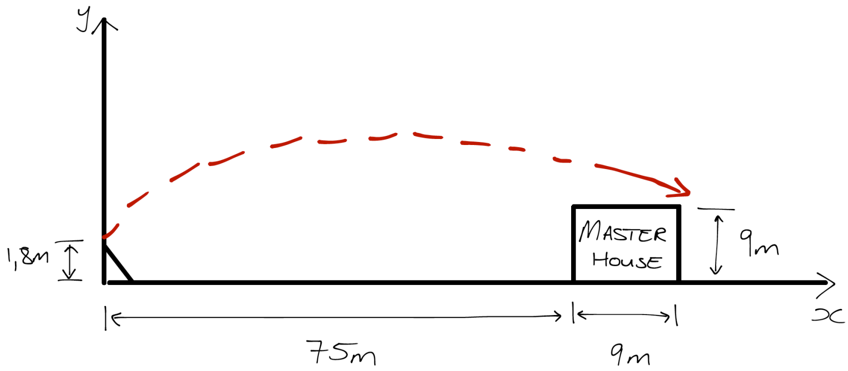

Figure 1: Scenario

In order to verify the legend a numerical approach is required. The numerical approach with discrete time steps allows for a better and more practical approximation of the problem than a continuous approach with a closed-form solution. Python was used to build various functions with discrete time steps eliminating the need for tedious hand calculations also allowing fast recalculations.

A function was also created to calculate the optimal firing velocity and angle. Performing this optimisation by hand would be time-consuming, not as accurate and a continuous function would be required. One value would have to be optimised first and then the second would have to be calculated. The computer is able to make this exercise much more computationally efficient with the computer being able to perform multiple calculations per second.

Method

Drag Coefficient

The loaf of bread can be modelled as sphere. An equivalent object that has undergone extensive testing and research is a tennis ball. A tennis player instinctively knows whether to put backspin or topspin in order to make the ball move in various configurations, fascinating researchers for years. The drag coefficients of sports balls have been measured by a number of people and published in various journal papers.

Alam et. al. (2007) published an article with extensive research detailing the drag coefficients over various speeds. To simplify the problem a single value was chosen and used regardless of speed however, at a later stage the program could be updated to use various coefficients as the speed changed. Figure 2 details the coefficients of various brands of tennis ball over a range of speeds.

The initial choice of drag coefficient was 0.55 as it was believed the initial launch velocity would be between 80 to 100 km.h^{-1} (22.2 to 27.7 m.s^{-1}).

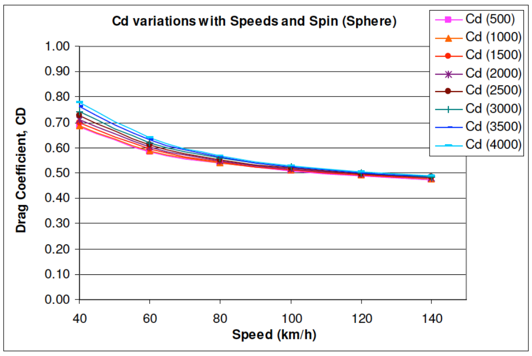

Figure 2: Tennis Ball Drag Coefficients (Alam et. al. 2007)

Assumptions

As with any legend and theory assumptions are required in order to make the problem workable. Values of the initial / object conditions are specified in Table 1 while simplifying assumptions are listed below.

- *The drag coefficient remains constant over the flight path. In reality the drag coefficient increases as velocity decreases. This could be incorporated into the model providing a more realistic outcome at a later stage if required but would increase the program complexity. The drag also does not include any kind of spin.*
- *Drag in the y direction is assumed to be neglected as the forces cancel each other out on the upward and downward travel and only acts on the last 1.8 meters. As above, this could be incorporated into the model but would increase the complexity.*

Table 1: Initial Conditions

Condition	Amount	Details
Starting coordinates	trajXY = (0, 1.8)	Initial x and y coordinates in a tuple (immutable)
Drag	0.55	Dimensionless drag coefficient
Diameter	0.15	Diameter of the loaf of bread in meters
Radius	Diameter / 2	Radius of the bread in meters
Mass	0.7	Mass of the loaf of bread in kg
Rho	1.2	Density of the ambient air in $kg.m^{-3}$
Gravity	9.81	Gravitational acceleration in $m.s^{-2}$

Task

Excerpt from the task sheet:

"For our trajectory simulation, let's assume that the frozen loaf of bread has a mass m = 0.7 kg and has an effective diameter d = 15 cm (to be used for determining the frontal-area of the loaf in flight).

At t = 0, the loaf leaves the slingshot located at (Xo, Yo) with a speed of Vo m/s and a flight-path angle, θo, with respect to the horizontal. These are the initial conditions.

While in flight, the only forces acting on the load are self-weight because of the Earth's gravity (g = 9.81 m/s^2) and aerodynamic drag. The magnitude of the aerodynamic drag is given by the expression

$$F_{drag} = \frac{1}{2}\rho v^2 A_{front} C_{drag}$$

where ρ = 1.2kg/m^3 is the ambient air density, v is the speed of the projectile, A_{front} is the frontal area of the loaf (i.e. its cross-sectional area) and C_{drag} is a dimensionless coefficient that has been determined by experiment. If we specify our quantities in metres, kilograms and seconds, F_{drag} will be in Newtons. The direction of the aerodynamic drag force is opposite to that of the projectile's velocity."

Numerical Proof

In order to determine the numerical analysis we have to approximate the state of motion:

$$F_{drag} = \frac{1}{2}\rho v^2 A_{front} C_{drag} = m a_x$$

Re-arranging for acceleration:

$$a_x = \frac{\rho v^2 A_{front} C_{drag}}{2m}$$

$$a_y = g$$

And then discreetly integrating:

$$v_x = v_o cos\theta - a_x \Delta t$$
$$v_y = v_o sin\theta - mg\Delta t$$

Once again integrating discretely:

$$x(t) = x_0 + v_x \Delta t$$
$$y(t) = y_0 + v_y \Delta t$$

Free-Body Diagrams

The next step is to determine the vector components of the accelerations, velocities and directions by means of Free-Body Diagrams:

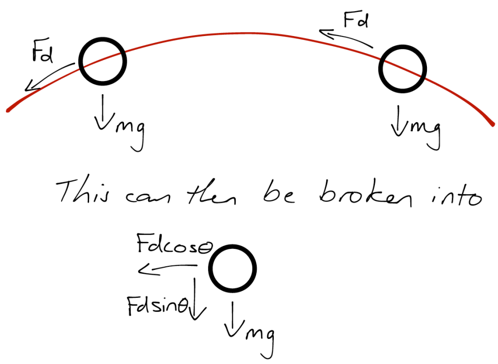

Figure 3: Free-Body Diagrams

No Drag Solution

If we assume there is no drag then the formulas can be assumed to be continuous and upon integrating the following is calculated:

$$x(t) = x_0 + v_0 cos\theta(t)$$
$$y(t) = y_0 + v_0 sin\theta(t) - \frac{1}{2}g(t)^2$$

From the No Drag graph in Figure 4 it can be seen that the graph is parabolic in nature and as such agrees with the formulas above. The only acceleration acting upon the bread is gravity.

Time-Step

The time-step that gave a sufficiently accurate estimate of the trajectory for all the functions was 0.1. Any greater than this and there was a noticeable discretisation in the plot. The smaller the time-step the closer the function is to becoming *continuous. However, there is a trade off, the smaller the time-step the more computational power required. For this problem a time-step of 0.0001 was tested without issues but as the complexity of the program increases, so will the run-time.*

Optimal Solution

Finally, once all the mathematics has been proven and tested with various parameters an optimal solution can be calculated. The idea behind this calculation is that due to the physical limitations of the students and their system, the optimal solution will require the smallest launch velocity possible and fine-tuning the launch angle. In order to achieve this 2 nested "for" loops were implemented. As velocity increased by 1 $m.s^{-1}$, angles between 1 and 89 degrees were tested.

This continued until the loaf cleared the furthest roof corner of the master's house (84, 9). As soon as this condition was met the function finalised the trajectory, plotted the results and returned the optimal launch velocity and angle along with the bread's landing location.

Results

The functions for the Bread Slingshot were run with initial velocities and angles of 30m.s^{-1} and 45 degrees respectively. The graphs can be seen below in Figure 4. The initial guess (includes drag) does not even reach the master's house while the same launch conditions made it over the master's house when no drag was included. When drag is included the graph is less parabolic and "skewed" more to one end. This is because the as the velocity decreases the drag and acceleration have a greater effect on the velocity.

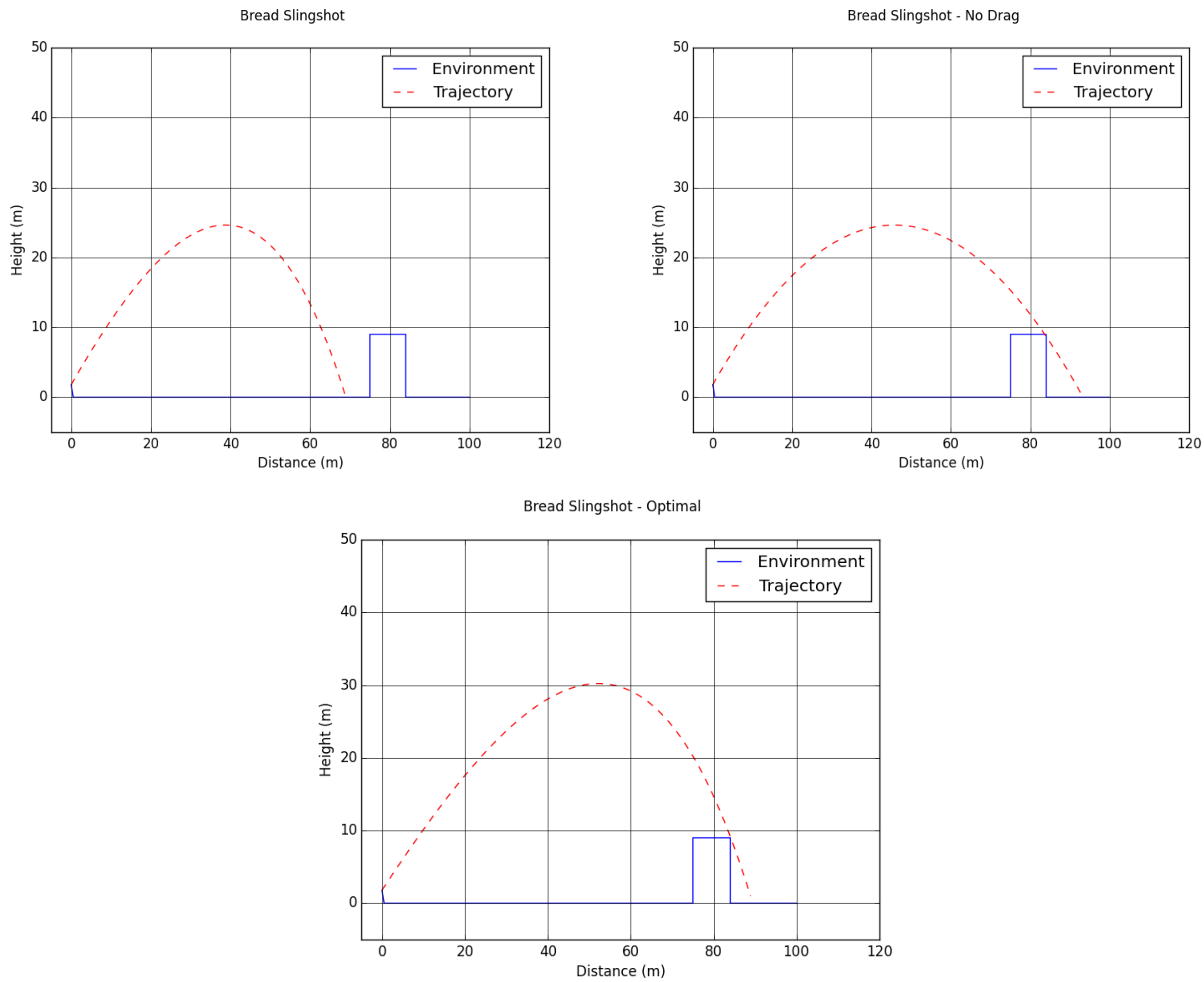

Figure 4: Graphed Solutions

When the optimal solution function is run the following optimal conditions are found:

- *Velocity: 36 m.s^{-1} (129.6 km.h^{-1})*
- *Angle: 42 degrees*
- *Landing location: (89.001, 0.856)*

Conclusion

The legend could be verified as being true provided a launch velocity of 36 $m.s^{-1}$ could be achieved. It would be interesting to incorporate a more accurate drag approximation and re-run the functions to see if there is a much greater discrepancy in the results.

References

Alam, F., Tio, W., Watkins, S., Subic, A. & Naser, J., 2007, ***Effects of Spin on Tennis Ball Aerodynamics: An Experimental and Computational Study****, School of Aerospace, Mechanical and Manufacturing Engineering, RMIT University, Melbourne, VIC 3083, AUSTRALIA, 16th Australasian Fluid Mechanics Conference, Crown Plaza, Gold Coast, Australia, 2-7 December 2007.*