Lecture 1-B

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§1.1 Linear Systems (Systems of Linear Equations)

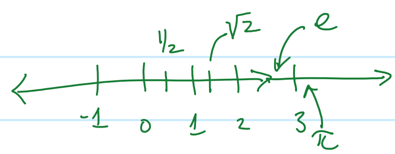
Defn. A linear equation (in the variables $x_1, \dots, x_n$) is any equation of the form

$$a_1 x_1 + \dots + a_n x_n = b$$

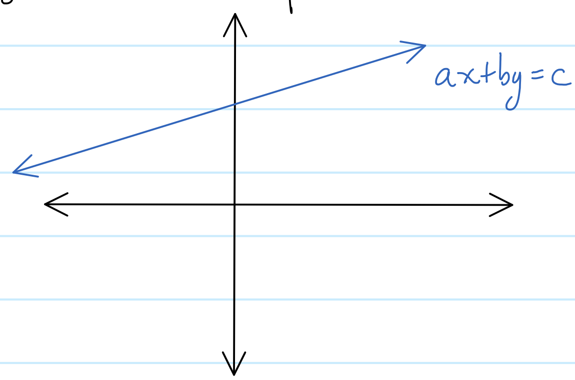
in which $a_1, \dots, a_n, b$, called the coefficients, are $\underbrace{\text{real numbers}}_{\text{reals}}$ and n is a natural number.

$\downarrow$ $\mathbb{N} = \{1, 2, 3, \dots\}$

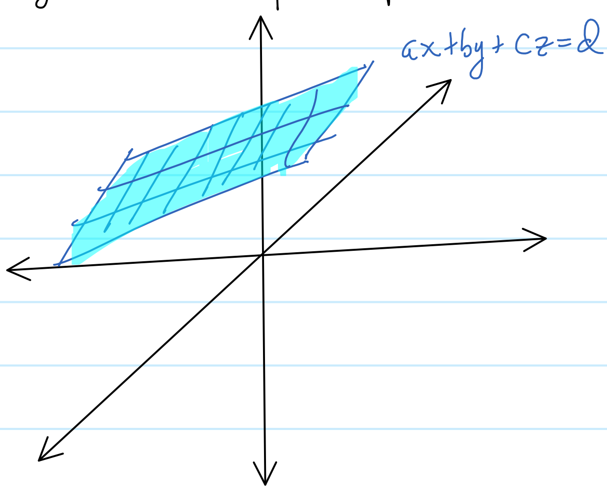
$\downarrow$ $\mathbb{R}$



Remark. The graph of $ax + by = c$ is a line (provided that $a \neq 0$ or $b \neq 0$).



The graph of $ax + by + cz = d$ is a plane (provided that $a \neq 0$, $b \neq 0$, or $c \neq 0$)



Linear System

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Def'n. (1) A linear system (LS) is a collection of one or more linear equations in the same variables.

E.g. (a) $\begin{cases} 2x + y = 3 \\ x - y = -1 \end{cases}$ 2x2 LS (b) $\begin{cases} 2x_1 - x_2 + 1.5x_3 = 8 \\ 1x_1 + 0x_2 - 4x_3 = -7 \end{cases}$ 2x3 LS

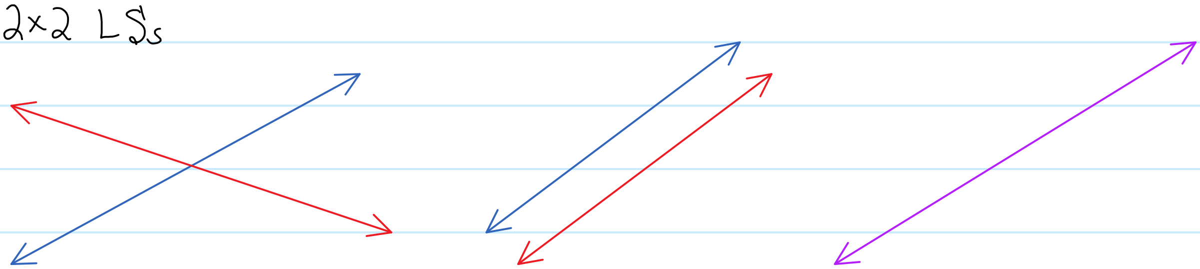
(2) A solution of a linear system is any ordered n-tuple, say $(s_1, \ldots, s_n)$, that satisfies each equation (ie., makes each equation a true statement). For instance, the triple $(5, 6.5, 3)$ satisfies the LS in (b).

- Object of the course: we want to solve any linear system.

(3) The set of all solutions to a LS is called the solution-set (of the LS). Two or more LSs are called equivalent if they have the same solution set.

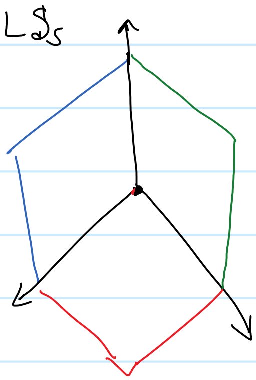
* Possible Solution Sets

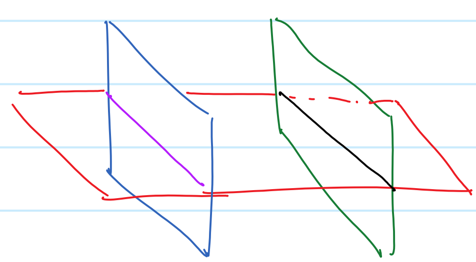
- 2x2 LSs



Unique solution No solution Infinitely-many solutions

- 3x3 LSs





Unique solution

Solution-sets

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Fact: Any LS will have a unique solution, no solution, or infinitely-many solutions.

LS

- has a solution (feasible)
  - Unique — 1
  - infinitely-many — ∞
- does not have a solution (infeasible) — 0

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* Matrix Notation

Given the LS

Solution column

$$\begin{array}{c} \begin{matrix} C_1 & & C_2 & & C_3 & & \end{matrix} \\ \left\{ \begin{array}{rcl} x_1 - 2x_2 + x_3 &=& 0 \\ 2x_2 - 8x_3 &=& 8 \\ -4x_1 + 5x_2 + 9x_3 &=& -9 \end{array} \right. \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \end{array} \qquad (*)$$

the matrix ← "rectangular array of numbers"

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

is called the coefficient matrix of (*) and the matrix

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right]$$

is called the augmented matrix of (*).

Caution: The coefficient matrix of

$$\left\{ \begin{array}{l} 2x_1 - x_2 = 4 \\ 3x_2 - 4x_1 = 1 \end{array} \right.$$

is

$$\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$$

Row-operations

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Consider the LS

$$\begin{cases} x_1 & & & = a \\ & x_2 & & = b \\ & & x_3 & = c. \end{cases}$$

The solution is $\{(a,b,c)\}$.

* Elementary row-operations ← "replace with"

1. $R_j \leftarrow cR_i + R_j$: replace row-j with the sum of c times row-i and row-j.
   "row-replacement"
2. $R_i \leftrightarrow R_j$: swap row-i with row-j. "row-interchange"
3. $R_i \leftarrow cR_i$: replace row-i with c times row-i "row-scaling" $(c \neq 0)$

Eg. Solve

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

Sol'n:

$R_3 \leftarrow 4R_1 + R_3$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -3x_2 + 13x_3 = -9 \end{cases} \quad R_2 \leftarrow 4R_3 + R_2 \quad \begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 = 16 \\ x_3 = 3 \end{cases}$$

$R_2 \leftarrow \frac{1}{2} R_2$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ -3x_2 + 13x_3 = -9 \end{cases} \quad R_1 \leftarrow R_1 - R_3 \;\; (-R_3 + R_1) \quad \begin{cases} x_1 - 2x_2 = -3 \\ x_2 = 16 \\ x_3 = 3 \end{cases}$$

$R_3 \leftarrow 3R_2 + R_3$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases} \quad R_1 \leftarrow 2R_2 + R_1 \quad \begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3 \end{cases}$$

solution

Row-ops to Matrices

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Exercise: Apply the previous operations to the matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right].$$