(1) Solve the following linear systems:

(a) $\left\{\begin{array}{rcrcr} x_1 & - & 4x_2 & = & 6 \\ 2x_1 & + & 5x_2 & = & -1 \end{array}\right.$

Solution. Because

$$\left[\begin{array}{rr|r} 1 & -4 & 6 \\ 2 & 5 & -1 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & -4 & 6 \\ 0 & 13 & -13 \end{array}\right] \qquad (R_2 \longleftarrow (-2)R_1 + R_2)$$

$$\sim \left[\begin{array}{rr|r} 1 & -4 & 6 \\ 0 & 1 & -1 \end{array}\right] \qquad (R_2 \longleftarrow (1/13)R_2)$$

$$\sim \left[\begin{array}{rr|r} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right], \qquad (R_1 \longleftarrow 4R_2 + R_1)$$

it follows that the unique solution of the linear system is

$$\vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

□

(b) $\left\{\begin{array}{rcrcrcr} 2x_1 & + & x_2 & - & x_3 & = & 0 \\ -4x_1 & - & x_2 & - & 5x_3 & = & 0 \end{array}\right.$

Solution. Because

$$\begin{bmatrix} 2 & 1 & -1 \\ -4 & -1 & -5 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & -7 \end{bmatrix} \qquad (R_2 \longleftarrow 2R_1 + R_2)$$

$$\sim \begin{bmatrix} 2 & 0 & 6 \\ 0 & 1 & -7 \end{bmatrix} \qquad (R_1 \longleftarrow -R_2 + R_1)$$

$$\sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -7 \end{bmatrix}, \qquad (R_1 \longleftarrow (1/2)R_1)$$

it follows that any general solution to linear system is of the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ 7x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 7 \\ 1 \end{bmatrix}, x_3 \in \mathbb{R}.$$

□

(2) Determine, by solving a linear system, if the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent or linearly dependent.

Solution. Because

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} \qquad (R_3 \longleftarrow R_1 + R_3)$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad (R_3 \longleftarrow R_2 + R_3)$$

it follows that the homogeneous linear system $[A \mid 0]$ has only the trivial solution.

□

(3) Without solving a linear system, determine whether the following sets are linearly independent or dependent (please provide adequate justification):

(a) $\left\{ \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}, \begin{bmatrix} -4 \\ 4 \end{bmatrix} \right\}$

Solution. Since

$$\begin{bmatrix} -4 \\ 4 \end{bmatrix} = -8 \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix},$$

it follows that the vectors are linearly dependent. □

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

Solution. Since there are more vectors than entries in the vectors, it follows that the vectors are linearly dependent. □

(c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Solution. Since the set contains the zero-vector, it follows that the vectors are linearly dependent. □

(4) Let $\vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$, $\vec{a}_3 = \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} -20 \\ 25 \\ 24 \end{bmatrix}$.

(a) Is $\vec{b}$ a linear combination of $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$? If so, find scalars x_1, x_2, and x_3 such that

$$x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{b}.$$

Solution. Since

$$\left[\begin{array}{ccc|c} 1 & -2 & -6 & -20 \\ 0 & 3 & 7 & 25 \\ 1 & -2 & 5 & 24 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -6 & -20 \\ 0 & 3 & 7 & 25 \\ 0 & 0 & 11 & 44 \end{array}\right] \qquad (R_3 \leftarrow -R_1 + R_3)$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -6 & -20 \\ 0 & 3 & 7 & 25 \\ 0 & 0 & 1 & 4 \end{array}\right] \qquad (R_3 \leftarrow 1/11R_3)$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -6 & -20 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right] \qquad (R_2 \leftarrow (-7)R_3 + R_2)$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -6 & -20 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{array}\right] \qquad (R_2 \leftarrow 1/3R_2)$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{array}\right] \qquad (R_1 \leftarrow 6R_3 + R_1)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{array}\right], \qquad (R_1 \leftarrow 2R_2 + R_1)$$

it follows that $2\vec{a}_1 - \vec{a}_2 + 4\vec{a}_3 = \vec{b}$. □

(b) Let $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}$. If $\vec{b} \in \mathbb{R}^3$, is $A\vec{x} = \vec{b}$ consistent?

Solution. From part (a) of problem four, it follows that A is row-equivalent to the identity and that A has a pivot in every row; thus, $A\vec{x} = \vec{b}$ is consistent for every $\vec{b} \in \mathbb{R}^3$. $\square$