Lecture 2-A

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* §1.1 Q/A

"tilde": similar to

#11. $\left[\begin{array}{ccc|c} 0 & 1 & 4 & -5 \\ 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 3 & 7 & 7 & 6 \end{array}\right]$ $(R_1 \leftrightarrow R_2)$

$\sim \left[\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & -2 & -8 & 12 \end{array}\right]$ $(R_3 \leftarrow -3R_1 + R_3)$

$\sim \left[\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & -1 & -4 & 6 \end{array}\right]$ $(R_3 \leftarrow \frac{1}{2} R_3)$

$\sim \left[\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 1 \end{array}\right]$ $(R_3 \leftarrow R_2 + R_3)$

The third-row of the last matrix tells us that we want to find numbers x_1, x_2, and x_3 s.t

$$\overbrace{0 = 0x_1 + 0x_2 + 0x_3 = 1}$$

"inconsistent" or "infeasible"

#15. $\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & -5 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 0 & 0 & -9 & 7 & -11 \end{array}\right]$ $(R_4 \leftarrow -3R_1 + R_4)$

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#15 (Cont'd)

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 0 & 0 & -9 & 7 & -11 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & -9 & 7 & -11 \end{array}\right] \quad (R_3 \leftarrow 2R_2 + R_3)$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & 0 & -5 & 10 \end{array}\right] \quad (R_4 \leftarrow 3R_3 + R_4)$$

"consistent"

#17.

$$\left[\begin{array}{cc|c} 1 & -4 & 1 \\ 2 & -1 & -3 \\ -1 & -3 & 4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -4 & 1 \\ 0 & 7 & -5 \\ 0 & -7 & 5 \end{array}\right] \quad \left(\begin{array}{l} R_2 \leftarrow -2R_1 + R_2 \\ R_3 \leftarrow R_1 + R_3 \end{array}\right)$$

$$\sim \left[\begin{array}{cc|c} 1 & -4 & 1 \\ 0 & 7 & -5 \\ 0 & 0 & 0 \end{array}\right] \quad R_3 \leftarrow R_2 + R_3$$

$$x_2 = -\frac{5}{7} \qquad x_1 + \frac{20}{7} = \cancel{1}\ \frac{7}{7}$$

$$x_1 = \frac{-13}{7}$$

$$\{(-13/7, -5/7)\}$$

§1.1 (Continued)

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Recall the elementary row-ops:

1. $R_j \leftarrow cR_i + R_j$, $i \neq j$, $c \neq 0$
2. $R_i \leftrightarrow R_j$, $i \neq j$.
3. $R_i \leftarrow cR_i$, $c \neq 0$.

Remarks:

(i) Row-operations can be applied to any matrix.

rectangular array of numbers

$$\begin{bmatrix} \pi & \sqrt{2} & 2+i \\ -1 & i & 0 \end{bmatrix}$$

(ii) Two matrices are called row-equivalent if there is a finite sequence of row-operations that transforms one matrix into the other

E.g.

$$\begin{array}{c} R_1 \\ R_2 \end{array}\begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{array}{c} \hat{R}_1 \\ \hat{R}_2 \end{array}$$

row-equivalent

(iii) Row-operations are reversible:

(a) $R_j \leftarrow cR_i + R_j$ is undone by $R_j \leftarrow (-c)R_i + R_j$.

(b) $R_i \leftrightarrow R_j$ " " " $R_i \leftrightarrow R_j$.

(c) $R_i \leftarrow cR_i$ " " " $R_i \leftarrow \frac{1}{c}R_i$, $c \neq 0$.

Theorem: If the augmented matrices of two LSs are row-equivalent, then the LSs have the same solution set.

§1.2 Row-reduction Algorithm (RRA) and Echelon Forms

Def'n: (1) A nonzero row (or column) in a matrix is a row (respectively, column) that contains at least one nonzero entry.

E.g.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \leftarrow \text{nonzero row} \\ \leftarrow \text{zero row} \end{array}$$

nonzero column

zero columns

(2) A leading entry of a row (in a matrix) refers to the leftmost nonzero entry (in a nonzero row).

E.g.
$$\begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 1 & -1 & 0 & 3 & 2 & 4 \\ 0 & 0 & 2 & 4 & 0 & 1 \end{bmatrix} \end{array}$$
The $(1,1)$ and $(2,3)$ entries are leading entries.

Def'n: A matrix is said to be in echelon form (or row echelon form) if it satisfies the following properties:

(i) all nonzero rows are above any zero-rows;

(ii) each leading entry of a row is in a column to the right of the leading entry of the row above it; and

E.g.
$$\begin{bmatrix} \boxed{1} & * & * & * & * \\ 0 & 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

(iii) all entries in a column below a leading entry are zero.

If a matrix in echelon form satisfies the following additional conditions, then it is said to be in reduced (row) echelon form (REF):

(iv) the leading entry in each nonzero row is a one; and

(v) each leading entry is the only nonzero entry in its column.

E.g.
$$\underbrace{\begin{bmatrix} \boxed{1} & * & * & * & * \\ 0 & 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}}_{\text{echelon form (EF)}} \sim \underbrace{\begin{bmatrix} \boxed{1} & * & * & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}}_{\text{reduced echelon form (REF)}}$$

Def'n: An echelon matrix (reduced echelon matrix) is any matrix that is in echelon form (resp., reduced echelon form).

E.g. The matrix $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ is an echelon matrix and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a reduced echelon matrix.

Echelon Forms

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Thm: Each matrix is row-equivalent to one and only one matrix in reduced echelon form.

E.g. $\begin{bmatrix} 2 & -5 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & -5 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\not\sim \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}$

Def'n: If a matrix A is row-equivalent to a matrix U, and U is in echelon form, then U is called an echelon form of A; if U is in reduced echelon form, then U is called the reduced echelon form of A.

* Pivot positions

Remark: When row-ops on a matrix produce an EF, further row-ops do not change the positions of the leading entries. Since the REF is unique, the leading entries are always in the same positions in any echelon form obtained via row ops.

Def'n: A pivot position in a matrix A is any entry in A that corresponds to a leading entry in the REF of A. A pivot column is any column of A that contains a pivot position.

E.g. $\begin{bmatrix} \boxed{*} & * & * \\ * & * & \boxed{*} \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & * & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$

E.g. Row reduce A to echelon form and locate the pivot positions, pivot columns, and pivot rows, where

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Example of EF

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$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \qquad (R_1 \leftrightarrow R_4)$$

$$\sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \qquad \begin{pmatrix} R_2 \leftarrow R_1 + R_2 \\ R_3 \leftarrow 2R_1 + R_3 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \qquad (R_4 \leftarrow 3R_2 + R_4)$$

$$\sim \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\sim \begin{bmatrix} 1 & 4 & 5 & 0 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (R_1 \longleftarrow -4R_2 + R_1)$$