Lecture 2-B

Thursday, March 29, 2018 1:14 PM

* §1.1 Q/A

Consistent = feasible, inconsistent = infeasible

No-solution case:
Augmented matrix contains a row of the form
$[0\ 0 \cdots 0 \mid b]$, $b \neq 0$.
The LS requires numbers $x_1, \ldots, x_n$ s.t.
$0 = 0x_1 + 0x_2 + \cdots + 0x_n = b \neq 0$,
which is clearly impossible.

#19. $\left[\begin{array}{cc|c} 1 & h & 4 \\ 3 & 6 & 8 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & h & 4 \\ 0 & 6-3h & -4 \end{array}\right]$ $(R_2 \leftarrow -3R_1 + R_2)$

"row-equivalent"

always nonzero regardless of the value of h.

The LS does not have a solution if $6-3h=0$, i.e., if $h=2$. The LS is consistent for every value of h other than two.

#11. $\left[\begin{array}{ccc|c} 0 & 1 & 4 & -5 \\ 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 3 & 7 & 7 & 6 \end{array}\right]$ $(R_1 \Leftrightarrow R_2)$

$\sim \left[\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & -2 & -8 & 12 \end{array}\right]$ $(R_3 \leftarrow -3R_1 + R_3)$

$\sim \left[\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 2 \end{array}\right]$ $(R_3 \leftarrow 2R_2 + R_3)$

§1.1 (continued)

Recall the elementary row-ops:

1. $R_j \leftarrow cR_i + R_j$, $i \neq j$, $c \neq 0$.
2. $R_i \Leftrightarrow R_j$, $i \neq j$.
3. $R_i \leftarrow cR_i$, $c \neq 0$.

Concluding Remarks

Thursday, March 29, 2018 1:14 PM

rectangular array of numbers

Remarks: (i) Row-operations can be applied to any matrix.

(ii) Two matrices are called row-equivalent if there is a finite sequence of row-ops that transforms one matrix into the other.

E.g. $$\begin{array}{c} R_1 \\ R_2 \end{array}\begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \underset{\text{row-equivalent}}{\sim} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{array}{c} \hat{R}_1 \\ \hat{R}_2 \end{array}$$

(iii) Row-ops are reversible:

(a) $R_j \leftarrow cR_i + R_j$ is undone by $R_j \leftarrow (-c)R_i + R_j$.

(b) $R_i \longleftrightarrow R_j$ " " " $R_i \longleftrightarrow R_j$

(c) $R_i \leftarrow cR_i$ " " " $R_i \leftarrow \frac{1}{c} R_i$.

Thm: If the augmented matrices of two LSs are row-equivalent, then the LSs have the same solution set.

§1.2 Row-reduction Algorithm (RRA) and Echelon Forms.

Defn: (1) A nonzero row (or nonzero column) in a matrix is any row (respectively, column) that contains at least one nonzero entry.

E.g. $$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ \underbrace{1 & 2 & 3}_{\text{totally nonzero row.}} \end{bmatrix}$$

zero-row

nonzero rows

(2) A leading entry of a nonzero row (in a matrix) refers to the leftmost nonzero entry in that row.

E.g. $$\begin{array}{c|c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 1 & -1 & 0 & 3 & 2 & 4 \\ 0 & 0 & 2 & 4 & 0 & 1 \end{bmatrix} \end{array}$$

~~The leading entries are located~~ in ~~the (1,1) and (2,3) entries~~

The leading entries are 1 (from first-row) and 2 (from second-row).

Echelon Forms

Thursday, March 29, 2018 1:14 PM

Def'n: A matrix is said to be in echelon form (EF) if it satisfies the following properties:

(i) all nonzero rows are above any zero-rows;

(ii) each leading entry of a row is in a column to the right of the leading entry of the row above it;

E.g. $$\begin{bmatrix} \boxed{1} & * & * & * & * \\ 0 & 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

(iii) all entries in a column below a leading entry are zero.

Note: this property follows from (ii).

$$\begin{bmatrix} \boxed{1} & * & * & * & * \\ 0 & 0 & 0 & \boxed{1} & * \\ \boxed{1} & * & * & * & * \end{bmatrix}$$

If a matrix in EF satisfies the following additional conditions, then it is said to be in reduced echelon form (REF).

(iv) the leading entry in every nonzero row is a one; and

(v) each leading entry is the only nonzero entry in its column.

E.g. $$\underset{\text{EF}}{\begin{bmatrix} \boxed{1} & * & * & * & * \\ 0 & 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}} \sim \underset{\text{REF}}{\begin{bmatrix} \boxed{1} & * & * & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}}$$

Thm: Each matrix is row-equivalent to a unique matrix in reduced echelon form.

E.g. $$\underbrace{\begin{bmatrix} 2 & -5 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & -5 \\ 0 & 2 \end{bmatrix}}_{\text{EFs}} \sim \underset{\text{REF}}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$\not\sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Echelon Forms (continued)

Thursday, March 29, 2018 1:14 PM

Def'n: If a matrix A is row-equivalent to a matrix U, and U is in echelon form, then U is called an echelon form of A; if U is in reduced echelon form, then U is called the reduced echelon form of A.

Notation: REF(A): the reduced echelon form of A.

* Pivot positions

Remark: When row-ops on a matrix produce an EF, further row-ops do not change the location of the leading entries. Since the REF is unique, the leading entries are always in the same positions in any echelon form obtained via row-ops.

Def'n: A pivot position in a matrix A is any position in A that contains a leading entry in the REF of A.

E.g.

$$\underset{A}{\underset{\parallel}{\begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}}} \sim \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & \boxed{1} & * & 0 \\ 2 & 0 & 0 & \boxed{1} \end{array}$$

pivot positions: $(1,1)$ & $(2,3)$

pivot rows

pivot columns

Eg. Row-reduce A to REF and locate the pivot positions/columns/rows, where

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}.$$

Sol'n:

$$A \sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \quad \begin{matrix} R_2 \leftarrow R_1 + R_2 \\ R_3 \leftarrow 2R_1 + R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \quad R_4 \leftarrow 3R_2 + R_4$$

An illustrative example

Thursday, March 29, 2018 1:14 PM

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \sim \begin{array}{c c} & \begin{array}{ccccc} * & * & & * & \end{array} \\ \begin{array}{c} * \\ * \\ * \\ \\ \end{array} & \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \sim \begin{bmatrix} 1 & 4 & 5 & 0 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_1 \leftarrow -4R_2 + R_1$$

Suppose that A is the augmented matrix of a LS, where A is the matrix in the previous example. Thus,

$$A \sim \begin{array}{c} \begin{array}{ccccc} 1 & 2 & & 4 & \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \end{array} \qquad x_4 = 0.$$

By convention, all variables that do not correspond to pivot columns will be called "free variables".

$$x_1 = 3x_3 + 5$$

$$x_2 = -2x_3 - 3$$

$$x_3 \text{ is free}$$

$$x_4 = 0$$