Lecture 8-B

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§1.5 Q/A

#7. $A \sim \begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix}$

$$\Rightarrow [A \mid \vec{0}] \sim \left[\begin{array}{cccc|c} 1 & 3 & -3 & 7 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 9 & -8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right] \end{array}$$

All solutions to $A\vec{x} = \vec{0}$ are of the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 + 8x_4 \\ 4x_3 - 5x_4 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\underset{s}{x_3} \overset{\vec{u}}{\overset{\parallel}{\begin{bmatrix} -9 \\ 4 \\ 1 \\ 0 \end{bmatrix}}} + \underset{t}{x_4} \overset{\vec{v}}{\overset{\parallel}{\begin{bmatrix} 8 \\ -5 \\ 0 \\ 1 \end{bmatrix}}}}_{\text{parametric vector form}}, \quad x_3, x_4 \in \mathbb{R} \quad s, t \in \mathbb{R}$$

Solution-set to $A\vec{x} = \vec{0}$ is Span$(\vec{u}, \vec{v})$.

#33. $A = \begin{array}{c} \begin{array}{cc} \vec{a}_1 & \vec{a}_2 \end{array} \\ \begin{bmatrix} -2 & -6 \\ 7 & 21 \\ -3 & -9 \end{bmatrix} \end{array}$

$A\vec{x} = \vec{0}$ Since $\vec{a}_2 = 3\vec{a}_1$, it follows that $3\vec{a}_1 - \vec{a}_2 = \vec{0}$. Thus, $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is a nontrivial solution.

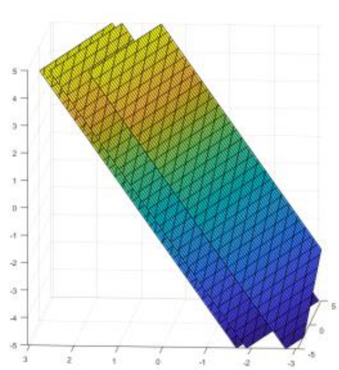
#35.

$$\underset{3\times 3}{\begin{bmatrix} 2 & -1 & -1 \\ 5 & 10 & -15 \\ 2 & 3 & -5 \end{bmatrix}} \overset{3\times 1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = \underset{3\times 1}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} \begin{matrix} 1 \\ 1 \\ 1 \end{matrix}$$

row sums equal 1.

all-ones vector

#17.



Linear (In)dependence

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§1.7. Linear Independence

Def'n: If $S=\{\vec{v}_1,..,\vec{v}_p\}$, $\vec{v}_i \in \mathbb{R}^n$, $1 \le i \le p$, then S (or $\vec{v}_1,..,\vec{v}_p$) is (are) linearly independent (LI) if the equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + .. + x_p\vec{v}_p = \vec{0}$$

has only the trivial solution (i.e., $x_1 = x_2 = \dots = x_p = 0$). Otherwise, the set S (or the vectors $\vec{v}_1,..,\vec{v}_p$) is (are) called linearly dependent (LD), i.e., there are weights (scalars) $c_1,..,c_p$, not all zero, s.t.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \quad + c_p\vec{v}_p = \vec{0}. \qquad \text{(linear dependence relation [LDR])}$$

E.g. If $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, then $-2\vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{0}$.

Thus, the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are LD.

Remark: Since $x_1\vec{a}_1 + x_2\vec{a}_2 + \quad + x_n\vec{a}_n = \vec{0}$ is equivalent to $A\vec{x} = \vec{0}$, in which $A = [\vec{a}_1\,\vec{a}_2 \cdots \vec{a}_n]$, it follows that $\{\vec{a}_1,..,\vec{a}_n\}$ is LI iff the HLS $A\vec{x} = \vec{0}$ has only the trivial solution (every column is a pivot column). Similarly, the set $\{\vec{a}_1,..,\vec{a}_n\}$ is LD iff $A\vec{x} = \vec{0}$ has a nontrivial solution (i.e., free-vars are present).

E.g. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

(a) Determine whether the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is LI or LD.

(b) If it is LD, determine an LDR.

Sol'n: (a)

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & -6 & -6 \end{bmatrix} \sim \begin{array}{ccc} x_1 & x_2 & x_3 \\ \hline \multicolumn{3}{c}{\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}} \end{array} \sim \begin{array}{ccc} x_1 & x_2 & x_3 \\ \hline \multicolumn{3}{c}{\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}} \end{array}. \qquad \text{LD}$$

(b) $$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \; x_3 \in \mathbb{R}.$$

LDRs of vectors correspond to nontrivial solutions of a HLS and vice-versa.

LDR: $2\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = \vec{0}$

LI & LD

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Eg. Determine if the columns of the matrix $A = \begin{array}{ccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ \end{array} \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$ (viewed as vectors in $\mathbb{R}^3$) are LI.

Caution: it is not appropriate to say that a matrix is LI (LD).

Sol'n: Notice that

$$\begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 5 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & -2 & 5 \end{bmatrix} \sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 13 \end{bmatrix} \end{array} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, $x_3 = 0 \Rightarrow x_2 = 0 \Rightarrow x_1 = 0$. Thus, $A\vec{x} = \vec{0}$ has only the trivial solution, ie, $\vec{a}_1, \vec{a}_2, \vec{a}_3$ are LI.

* A set containing a single vector

If $\vec{v} \in \mathbb{R}^n$, $\vec{v} \neq \vec{0}$, then $c\vec{v} \neq \vec{0}$ if $c \neq 0$. Moreover, if $c\vec{v} = \vec{0}$, $c \neq 0$, then $\vec{v} = \vec{0}$.

Observation: If $S = \{\vec{v}\}$, $\vec{v} \in \mathbb{R}^n$, then S is LI iff $\vec{v} \neq \vec{0}$.
——— " ——— LD iff $\vec{v} = \vec{0}$.

* Sets w/ two vectors

Suppose $S = \{\vec{u}, \vec{v}\}$, $\vec{u}, \vec{v} \in \mathbb{R}^n$. If S is LD, then there are scalars $c, d \in \mathbb{R}$ (not both zero) s.t

$$c\vec{u} + d\vec{v} = \vec{0}.$$

If $c \neq 0$, then $\vec{u} = -\left(\frac{d}{c}\right)\vec{v}$, ie, $\vec{v}$ is a scalar mult. of $\vec{u}$; if $d \neq 0$, then $\vec{v} = -\left(\frac{c}{d}\right)\vec{u}$, ie, $\vec{u}$ is a scalar mult. of $\vec{v}$.

Conversely, suppose that $\vec{u}$ is a scalar mult. of $\vec{v}$. Then there is a scalar $\alpha \in \mathbb{R}$ s.t. $\vec{u} = \alpha\vec{v}$, i.e., $1\vec{u} - \alpha\vec{v} = \vec{0}$. Thus, $\{\vec{u}, \vec{v}\}$ is LD.

Obs: If $S = \{\vec{u}, \vec{v}\}$, $\vec{u}, \vec{v} \in \mathbb{R}^n$, then S is LD iff one of the vectors is a scalar multiple of the other.

Observations

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E.g. Without solving a LS, determine whether the following vectors are LI or LD.

(a) $\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\4\\-2\end{bmatrix}$ LI

(b) $\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}-3\\0\\3\end{bmatrix}$ LD $\begin{bmatrix}-3\\0\\3\end{bmatrix} = -3\begin{bmatrix}1\\0\\-1\end{bmatrix}$

(c) $\begin{bmatrix}2\\-4\\1\\3\end{bmatrix}, \begin{bmatrix}4\\-5\\9\\8\end{bmatrix}$ LI

not scalar multiples

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* Sets w/ two or more vectors

In a previous example, the vectors

$$\vec{v}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix},\ \vec{v}_2 = \begin{bmatrix}4\\5\\6\end{bmatrix},\ \text{and } \vec{v}_3 = \begin{bmatrix}2\\1\\0\end{bmatrix}$$

were shown to be LD and a LDR is given by $2\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = \vec{0}$. Notice that $\vec{v}_2 = 2\vec{v}_1 + \vec{v}_3$, i.e., $\vec{v}_2 \in \text{Span}(\vec{v}_1, \vec{v}_3)$.

Notation:

(i) If $S = \{\vec{v}_1, \dots, \vec{v}_p\}$, then $S_k = \{\vec{v}_1, \dots, \vec{v}_{k-1}, \vec{v}_{k+1}, \dots, \vec{v}_p\}$.

E.g. If $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, then $S_1 = \{\vec{v}_2, \vec{v}_3\}$, $S_2 = \{\vec{v}_1, \vec{v}_3\}$, and $S_3 = \{\vec{v}_1, \vec{v}_2\}$.

(ii) $\sum_{\substack{i=1\\i\neq k}}^{n} a_i = a_1 + a_2 + \dots + a_{k-1} + a_{k+1} + \dots + a_n$

E.g. $\sum_{\substack{i=1\\i\neq 4}}^{5} a_i = a_1 + a_2 + a_3 + a_5$

Characterizing LD

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Thm: If $S = \{\vec{v}_1, .., \vec{v}_p\}$, $p \geq 2$, then S is LD if and only if $\vec{v}_k \in \text{span}(S_k)$, for some k.

Pf: ~~If~~ LD, then there are scalars $c_1, .., c_p$, not all zero, s.t.

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_p \vec{v}_p = \vec{0}.$$

Let k be the smallest pos. int s.t $c_k \neq 0$. Then

$$\vec{v}_k = \sum_{\substack{i=1 \\ i \neq k}}^{p} -\left(\frac{c_i}{c_k}\right) \vec{v}_i \in \text{span}(S_k), \quad 1 \leq k \leq p,$$

Conversely, if $\vec{v}_k \in \text{span}(S_k)$, then there are scalars $c_1, \dots, c_{k-1}, c_{k+1}, .., c_p$ s.t

$$\vec{v}_k = \sum_{\substack{i=1 \\ i \neq k}}^{p} c_i \vec{v}_i$$

i.e.,

$$\vec{0} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + (-1)\vec{v}_k + \dots + c_p \vec{v}_p.$$

As this is an LDR, it follows that S is LD.

Thm: If $S = \{\vec{v}_1, .., \vec{v}_p\}$, $\vec{v}_i \in \mathbb{R}^n$, and $p > n$, then S is LD.

Pf: The HLS $[\vec{v}_1 \ \vec{v}_2 \cdots \vec{v}_p \mid \vec{0}]$ is underdetermined and therefore must possess free variables, i.e., must possess nontrivial solutions.

E.g. $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -5 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are LD (more vecs than entries).