Lecture 8-A

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§1.5 QIA

$$\vec{x} = \sum_{i=1}^{n} c_i \vec{v}_i \quad \text{\} parametric vector form}$$

c_i's are parameters

$A\vec{x} = \vec{0}$

9. Quiz

$$\begin{bmatrix} 3 & -9 & 6 \\ -1 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 \\ -1 & 3 & -2 \end{bmatrix} \sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & -3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \rightarrow x_1 = 3x_2 - 2x_3$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{x_2 \overset{\vec{u}}{\overset{\shortparallel}{\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}}} + x_3 \overset{\vec{v}}{\overset{\shortparallel}{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}}}_{\text{parametric vector form}}, \quad x_2, x_3 \in \mathbb{R}$$

The solution-set is Span$(\vec{u}, \vec{v})$.

row sums equal to zero

35.

$$\underbrace{\underset{3\times3}{\begin{bmatrix} 1 & 1 & -2 \\ 0 & 3 & -3 \\ 1 & 0 & -1 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

1.5 Q/A

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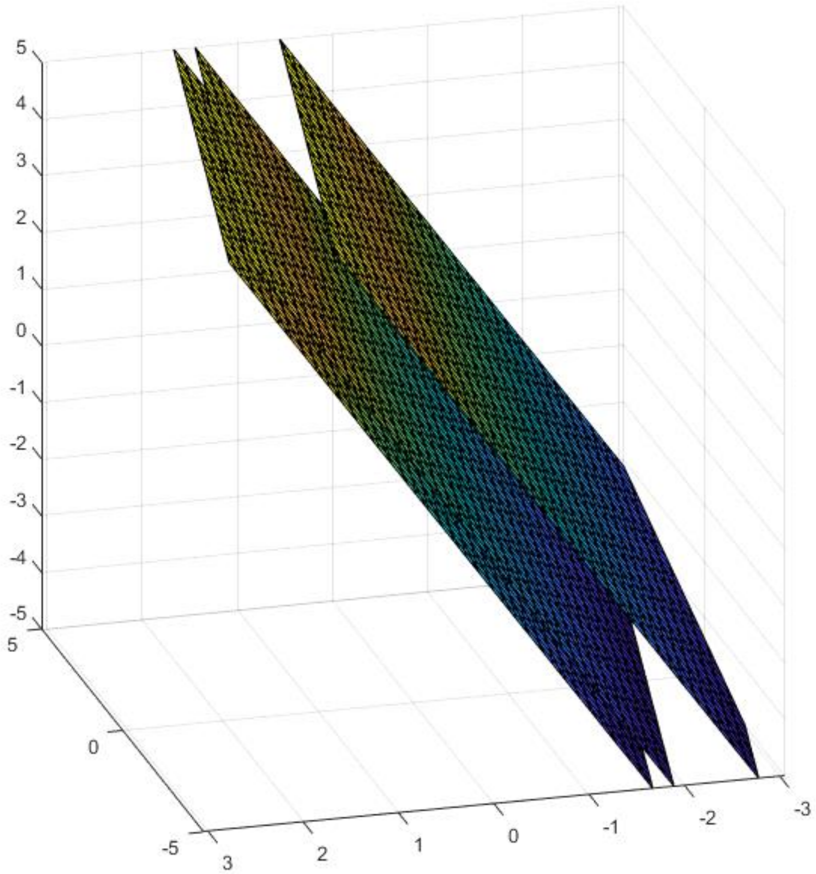
#17.



§1.7 Linear Independence

Def'n: If $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$, $\vec{v}_i \in \mathbb{R}^n$, then S (or $\vec{v}_1, \ldots, \vec{v}_p$) is (are) linearly independent (LI) if the equation

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + \ldots + x_p \vec{v}_p = \vec{0}$$

has only the trivial solution (i.e., $x_1 = x_2 = \cdots = x_p = 0$). Otherwise, the set S (or the vectors $\vec{v}_1, \ldots, \vec{v}_p$) is (are) called linearly dependent (LD), i.e., there are weights (i.e., scalars) $c_1, \ldots, c_p$, not all zero, s.t.

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_p \vec{v}_p = \vec{0}. \quad \Big\} \text{ linear dependence relation (LDR)}$$

LI & LD

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E.g. If $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, then $\vec{v}_1 - \vec{v}_2 - \vec{v}_3 = \vec{0}$;

i.e., $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are LD.

Remark: Since $x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{0}$ is equivalent to $A\vec{x} = \vec{0}$, in which $A = [\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n]$, it follows that $\{\vec{a}_1, \ldots, \vec{a}_n\}$ is LI (LD) iff the HLS $A\vec{x} = \vec{0}$ does not have a nontrivial solution (resp., the HLS $A\vec{x} = \vec{0}$ has a nontrivial solution).

E.g. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

(a) Determine whether the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is LI.

(b) If it is LD, determine a LDR.

Sol'n: (a)

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & -6 & -6 \end{bmatrix} \sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\Rightarrow$ no; the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is LD.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad x_3 \in \mathbb{R}.$$

(b) $2\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = \vec{0}$ LDRs of vectors correspond to nontrivial solutions to a HLS.

E.g. Determine if the columns of $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$ (viewed as vectors in $\mathbb{R}^3$) are LI

Sol'n:

$$\begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 5 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & -2 & 5 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{array}\right]$$

$13x_3 = 0 \Rightarrow x_3 = 0 \Rightarrow x_2 = 0 \Rightarrow x_1 = 0$. Thus, the columns of A are LI.

Observations

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* A set consisting of a single vector

If $\vec{v} \in \mathbb{R}^n$, $\vec{v} \neq \vec{0}$, then $c\vec{v} \neq \vec{0}$ if $c \neq 0$. Moreover, if $c\vec{v} = \vec{0}$, then $c = 0$.

Observation: If $S = \{\vec{v}\}$, then S is LI iff $\vec{v} \neq \vec{0}$. If $S = \{\vec{v}\}$, then S is LD iff $\vec{v} = \vec{0}$.

* Sets w/ two vectors

Suppose $S = \{\vec{u}, \vec{v}\}$, $\vec{u}, \vec{v} \in \mathbb{R}^n$. If S is LD, then there are scalars $c, d \in \mathbb{R}$ (not both zero) s.t.

$$c\vec{u} + d\vec{v} = \vec{0}.$$

Without loss of generality (WLOG), assume that $c \neq 0$. Then

$$\vec{u} = -\left(\frac{d}{c}\right)\vec{v},$$

i.e., $\vec{u}$ is a scalar multiple of $\vec{v}$.

Conversely, suppose that $\vec{u}$ is a scalar multiple of $\vec{v}$. Thus, there is a scalar $\alpha \in \mathbb{R}$ s.t

$$\vec{u} = \alpha\vec{v},$$

i.e.,

$$1\,\vec{u} - \alpha\vec{v} = \vec{0}.$$

Thus, $\{\vec{u}, \vec{v}\}$ is LD.

Obs: If $S = \{\vec{u}, \vec{v}\}$, $\vec{u}, \vec{v} \in \mathbb{R}^n$, then S is LD iff $\vec{u}$ is a scalar multiple of $\vec{v}$. Or, S is LI iff $\vec{u}$ is not a scalar multiple of $\vec{v}$.

E.g. Determine whether the following vectors are LI or LD.

(a) $\begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ 2 \\ 4 \end{bmatrix}$ LI

(b) $\begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ 2 \\ 0 \end{bmatrix}$ LD; since $\begin{bmatrix} -10 \\ 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}$.

Sets with more than two vectors

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In a previous example, the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \text{ and } \vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

were shown to be LD and that $2\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = \vec{0}$. Notice that

$$\vec{v}_2 = 2\vec{v}_1 + \vec{v}_3,$$

i.e., $\vec{v}_2 \in \text{span}(\vec{v}_1, \vec{v}_3)$. LD $\Rightarrow$ one of the vectors is in the span of the remaining vectors.

Notation: If $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$, then $S_k = \{\vec{v}_1, \ldots, \vec{v}_{k-1}, \vec{v}_{k+1}, \ldots, \vec{v}_p\}$.
For instance, if $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$, then $S_2 = \{\vec{v}_1, \vec{v}_3, \vec{v}_4\}$, $S_3 = \{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$.

Thm. If $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$, $p \geq 2$, then S is LD if and only if $\vec{v}_k \in \text{span}(S_k)$, where $1 \leq k \leq p$.

Pf: If S is LD, then there are scalars $c_1, \ldots, c_p$, not all zero, s.t

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0}.$$

Let k be the smallest pos. int. s.t. $c_k \neq 0$. Then

$$\vec{v}_k = -\left(\frac{c_1}{c_k}\right)\vec{v}_1 - \left(\frac{c_2}{c_k}\right)\vec{v}_2 - \cdots - \left(\frac{c_{k-1}}{c_k}\right)\vec{v}_{k-1} - \left(\frac{c_{k+1}}{c_k}\right)\vec{v}_{k+1} - \cdots - \left(\frac{c_p}{c_k}\right)\vec{v}_p$$

i.e., $\vec{v}_k \in \text{Span}(S_k)$.

If $\vec{v}_k \in \text{Span}(S_k)$, $1 \leq k \leq p$, then there are scalars $c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_p$ s.t

$$\vec{v}_k = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_{k-1}\vec{v}_{k-1} + c_{k+1}\vec{v}_{k+1} + \cdots + c_p\vec{v}_p.$$

i.e.,

$$\vec{0} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + (-1)\vec{v}_k + \cdots + c_p\vec{v}_p.$$

Thus, S is LD.

An example

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E.g. If $S = \left\{ \overset{\vec{v}_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \overset{\vec{v}_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}, \overset{\vec{v}_3}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} \right\}$, then S is LD since

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Notice that span$(S) = \mathbb{R}^2$ and span$(S_3) = \mathbb{R}^2$. Notice that S_3 is LI and if we remove either vector, the span is no longer $\mathbb{R}^2$.

Thm: If $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$, $\vec{v}_i \in \mathbb{R}^n$ $(1 \le i \le p)$, and $p > n$, then S is LD.

Pf: The HLS $[\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_p \mid \vec{0}]$ is undetermined (more variables than rows). Thus, it must have free variables, ie, it must possess non trivial solutions.

Quiz: Focus on §1.7