Lecture 3-B

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§1.1 Q/A

#27. $\left[\begin{array}{cc|c} 1 & 3 & f \\ c & d & g \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & f \\ 0 & d-3c & g-cf \end{array}\right]$

The LS is consistent provided that $d-3c \neq 0$.

#28. $\left[\begin{array}{cc|c} a & b & f \\ c & d & g \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & b/a & f/a \\ c & d & g \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & b/a & f/a \\ 0 & \frac{ad-bc}{a} & \frac{ag-cf}{a} \end{array}\right]$ $R_2 \leftarrow (-c)R_1 + R_2$

$a \neq 0$

The LS is consistent provided that $ad-bc \neq 0$.

Ch 3: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the quantity $ad-bc$ is called the determinant of A.

§1.2 Q/A

#7. $\left[\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{array}\right]$ $R_2 \leftarrow (-3)R_1 + R_2$

$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$

$\sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right]$ $R_1 \leftarrow (-4)R_2 + R_1$

Variables in pivot columns are called basic variables; variables that are not in pivot columns are called free variables.

$$\begin{cases} x_1 = -3x_2 - 5 \\ x_2 \text{ is free} \\ x_3 = 3 \end{cases}$$

For instance, $(-5, 0, 3)$ and $(-8, 1, 3)$ are solutions.

Basic / free variables

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§1.2 (continued)

E.g. Find the general solution of the LS whose augmented matrix has been row-reduced to

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right] \end{array}$$

fewer rows than variables is called "underdetermined".

basic: x_1, x_3, x_5

free: x_2, x_4

verify

$$\sim \left[\begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right]$$

parametric description

$$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 4x_4 + 5 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases}$$

Remark: The free-variables are not unique:

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \sim \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 5 & 0 & 21 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \qquad R_1 \leftarrow 5R_2 + R_3$$

$$\begin{cases} x_1 = 5x_3 + 1 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases} \qquad \begin{cases} x_1 = -5x_2 + 21 \\ x_2 \text{ is free} \\ x_3 = 4 - x_2 \end{cases}$$

Thm: A LS is consistent if and only if the rightmost column of the augmented matrix is not a pivot column, i.e., iff an echelon form of the augmented matrix has no row of the form

$$[0 \cdots 0 \,|\, b], \; b \neq 0.$$

If LS is consistent, then the solution-set has either

(i) a unique solution (no free variables = every column is a pivot column); or

(ii) infinitely-many solutions (free variables present = not every column is a pivot column).

Section 1.3

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§1.3 Matrices, Vectors, & Vector Equations.

Def'n: A matrix is a rectangular array of real numbers consisting of rows and columns. Typically, matrices are denoted with capital letters and the (i,j)-entry (i.e., the number located in row-i, column-j) is denoted by a_{ij}, if the matrix is A.

E.g. If

square brackets

$$A = \begin{array}{c} 1 \\ 2 \\ \\ \end{array}\begin{array}{c} \begin{bmatrix} 1 & \pi & -\sqrt{2} \\ e^2 & 0 & 5 \end{bmatrix} \\ \begin{matrix} 1 & \;2 & \;\;3 \end{matrix} \end{array},$$

then $a_{12} = \pi$, $a_{23} = 5$, etc.

If A is a matrix consisting of m-rows and n-columns, then the size of A is said to be m-by-n.

Caution:

$$\left[\begin{array}{cc|c} * & * & * \\ * & * & * \end{array}\right]$$

size of matrix: 2×3

size of LS: 2×2

Def'n: A (column) vector (or row vector) is a matrix with only one column (resp., row).

E.g. Column vectors: $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, $\vec{w} = [\,1\,] = 1$.

3×1 vector $\quad$ 4×1 vector $\quad$ convention

Remark: Solutions to linear systems of the form $(x_1, x_2, \dots, x_n)$ will be written as

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Def'n: The set of all n-by-one column vectors is called the n-dimensional Euclidean space and is denoted by $\mathbb{R}^n$ (read "r-n").

Notation: '$\in$': "belongs to" or "element of"

Def'n: If $\vec{u}, \vec{v} \in \mathbb{R}^n$, then $\vec{u}$ and $\vec{v}$ are called equal if $u_i = v_i$ for all $i = 1, \dots, n$.

E.g.

$$\begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix}$$

Algebra on $\mathbb{R}^n$

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Def'n: (i) Given $\vec{u}, \vec{v} \in \mathbb{R}^2$, the sum of $\vec{u}$ and $\vec{v}$, denoted by $\vec{u}+\vec{v}$, is defined by

$$\vec{u}+\vec{v} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix} \in \mathbb{R}^2.$$

(ii) Given $\vec{u} \in \mathbb{R}^2$ and $\underset{\uparrow \text{scalar}}{c} \in \mathbb{R}$, the scalar product of $\vec{u}$ and c, denoted by $c\vec{u}$, is defined by

$$c\vec{u} = c\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix} \in \mathbb{R}^2.$$

Eg. If $\vec{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$, then

(a) $\vec{u} + 2\vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} -10 \\ 4 \end{bmatrix} = \begin{bmatrix} -9 \\ 2 \end{bmatrix}$;

(b) $\vec{u} - 3\vec{v} \underset{\text{definition}}{=} \vec{u} + (-3)\vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 15 \\ -6 \end{bmatrix} = \begin{bmatrix} 16 \\ -8 \end{bmatrix}$; and

(c) $-\vec{u} \underset{\text{definition}}{=} (-1)\vec{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.