For problems #1 – 4, prove or disprove the statement.

1. **[8 points]** Let R be a relation on A with $\mathrm{Dom}(R) = A$. If R is symmetric and transitive, then R is also reflexive.

   The statement is **true**. Proof: Let $a \in A$. Since $a \in \mathrm{Dom}(R)$, there exists $b \in A$ such that $(a, b) \in R$. Since R is symmetric, it must also be true that $(b, a) \in R$, and since R is transitive, $(a, b) \in R$ and $(b, a) \in R$ implies that (a, a) in R. So, R is reflexive.

2. **[4 points]** Let R be a relation that is antisymmetric. Then R is not symmetric.

   The statement is **false**. Consider $R = \{(1, 1), (2, 2), (3, 3)\}$. Then R is both symmetric and antisymmetric.

3. **[not graded]** Let R be a relation that is asymmetric. Then R is antisymmetric.

   The statement is **true**. Suppose R is asymmetric. Then there do not exist any ordered pairs (a, b) in R such that (b, a) is also in R. The definition of antisymmetry is then satisfied vacuously – since there aren't any two pairs (a, b), (b, a) in R, the definition $\forall a, b \in A, (a, b) \in R$ and $(b, a) \in R \to a = b$ is true.

   Alternatively, we can prove the contrapositive. Suppose that R is not antisymmetric. Then, there exist $a, b \in A$ such that aRb and bRa and $a \neq b$. But since this requires aRb and bRa, R cannot be asymmetric. So by contraposition, if R is asymmetric, R must be antisymmetric.

4. **[8 points]** Let $(A, \prec)$ be a poset where $\prec$ is a **linear (total)** order. Then any maximal element of $(A, \prec)$ must be a maximum.

   The statement is **true**. Let a be a maximal element of $(A, \prec)$. By definition, $\nexists b \in A, a \prec b$, or in other words, $\forall b \in A, a \not\prec b$. But since $\prec$ is a linear order, if $a \not\prec b$ for all b, it must be true that $b \prec a$ for all b. Therefore a is a maximum.

5. **[14 points]** Let U be a nonempty universal set. As discussed in class, $(\mathcal{P}(U), \subseteq)$ is a partially ordered set. Suppose as well that $\mathcal{B} \subseteq \mathcal{P}(U)$, that is, $\mathcal{B}$ is a set containing some (but not necessarily all) subsets of U.

   (a) Give an example demonstrating why $\subseteq$ is not a total order on $\mathcal{P}(U)$.

   Suppose U is the set of all integers. Then, $\{1\}$, $\{2\}$ are both elements of $\mathcal{P}(U)$, but $\{1\} \not\subseteq \{2\}$ and $\{2\} \not\subseteq \{1\}$. Therefore $\subseteq$ is not a total order.

   (b) Give an example demonstrating why $\mathcal{B}$ might not have a maximum element.

   Similarly to above, suppose $\mathcal{B} = \{\{1\}, \{2\}\}$. Then both $\{1\}$ and $\{2\}$ are maximal, but neither is a maximum, since $\{1\} \not\subseteq \{2\}$ and $\{2\} \not\subseteq \{1\}$.

   (c) Prove that $\displaystyle\bigcup_{B \in \mathcal{B}} B$ is an upper bound on $\mathcal{B}$.

   We need to show that for all $S \in \mathcal{B}$, $S \subseteq \displaystyle\bigcup_{B \in \mathcal{B}} B$. So, let $S \in \mathcal{B}$, and let $x \in S$. By definition, $x \in \displaystyle\bigcup_{B \in \mathcal{B}} B$ if there exists $B \in \mathcal{B}$ such that $x \in B$. But this is clearly the case, since $x \in S$ and $S \in \mathcal{B}$. Therefore $x \in \displaystyle\bigcup_{B \in \mathcal{B}} B$, and so it is an upper bound.

   (d) Prove that $\displaystyle\bigcap_{B \in \mathcal{B}} B$ is a lower bound on $\mathcal{B}$.

We need to show that for all $S \in \mathcal{B}$, $\bigcap_{B\in\mathcal{B}} B \subseteq S$. So, let $x \in \bigcap_{B\in\mathcal{B}} B$. By definition, it follows that $x \in S$ for all $S \in \mathcal{B}$. Therefore $\bigcap_{B\in\mathcal{B}} B \subseteq S$ for all $S \in \mathcal{B}$, and so it is a lower bound.

(e) Prove that $\bigcup_{B\in\mathcal{B}} B$ and $\bigcap_{B\in\mathcal{B}} B$ are in fact the least upper bound and greatest lower bound, respectively, on $\mathcal{B}$.

Let T be any upper bound on $\mathcal{B}$. We must prove that $\bigcup_{B\in\mathcal{B}} B$ is a subset of T. So, let $x \in \bigcup_{B\in\mathcal{B}} B$. By definition, $x \in S$ for at least one $S \in \mathcal{B}$. But since T is an upper bound on $\mathcal{B}$, it must be true that $S \subseteq T$, so $x \in T$. So, $\bigcup_{B\in\mathcal{B}} B \subseteq T$ for any upper bound T, and therefore it is a least upper bound.

Now, Let T be any lower bound on $\mathcal{B}$. We must prove that $T \subseteq \bigcap_{B\in\mathcal{B}} B$. So, let $x \in T$. Since $T \subseteq B$ for all $B \in \mathcal{B}$, it follows that $x \in B$ for all $B \in \mathcal{B}$. Therefore $x \in \bigcap_{B\in\mathcal{B}} B$. So, $T \subseteq \bigcap_{B\in\mathcal{B}} B$ for any lower bound T, and therefore it is a greatest lower bound.

Hint: Parts (c)-(e) require using element chasing arguments, but the proofs should not be too long!

For problems # 6 – 8, prove whether the specified function is

i) one-to-one

ii) onto the specified codomain.

6. **[8 points]** $g : \mathbb{R}\backslash\{1\} \to \mathbb{R}\backslash\{1\}$, $g(x) = \dfrac{x}{x-1}$ (Graphing is not an acceptable proof).

g **is one-to-one** since if $g(x_1) = g(x_2)$, it follows that

$$\begin{aligned} \frac{x_1}{x_1-1} &= \frac{x_2}{x_2-1} \\ x_1(x_2-1) &= x_2(x_1-1) \\ x_1x_2 - x_1 &= x_2x_1 - x_2 \\ x_1 &= x_2 \end{aligned}$$

g **is onto** since, if we take $y \in \mathbb{R}$ and try to find a corresponding x value, we would need

$$\begin{aligned} y &= \frac{x}{x-1} \\ y(x-1) &= x \\ yx - x &= y \\ x &= \frac{y}{y-1}, \end{aligned}$$

which exists provided that $y \neq 1$. So for every $y \in \mathbb{R}\backslash\{1\}$, we can take $x = \frac{y}{y-1}$ to find an x such that $f(x) = y$.

7. **[8 points]** $\mathcal{I} : F \to \mathbb{R}$, $\mathcal{I}(f) = \displaystyle\int_{-1}^{1} f(x)\,dx$, where F is the set of all functions with a finite integral on $[-1, 1]$.

$\mathcal{I}$ is **not one-to-one** since many different functions could have the same definite integral on $[-1,1]$. For example, $\int_{-1}^{1} x\,dx = 0$ and $\int_{-1}^{1} -x\,dx = 0$, but $x \neq -x$.

$\mathcal{I}$ **is onto** $\mathbb{R}$ since we can always find a function f such that $\int_{-1}^{1} f(x)\,dx = y$ for any $y \in \mathbb{R}$. The simplest choice is just to take the constant function, $f(x) = \frac{y}{2}$.

8. **[not graded]** $\varphi : \mathbb{N} \to \mathbb{N}$, where $\varphi(n)$ is the sum of the divisors of n.

   So for example $\varphi(6) = 1+2+3+6 = 12$, $\varphi(12) = 1+2+3+4+6+12 = 28$, etc.

   This function is **not one-to-one** since, for example, $\varphi(11) = 1+11 = 12$, which is the same as $\varphi(6)$.

   It is also **not onto** $\mathbb{N}$, since for example, there are no numbers whose divisors sum to 2. $\varphi(1) = 1$, $\varphi(2) = 3$, and $\varphi(n) \geq n+1$ for any other number n, since 1 and n always divide into n. (Other numbers that are not in the range include 5, 9, 10, 11, etc.).

9. **Bonus: [up to 10 additional points]**. Let $(A, \prec_A)$ and $(B, \prec_B)$ be posets.

   (a) Prove that the lexicographic ordering $\prec_L$ (defined on p. 99 of the text) is a partial ordering of $A \times B$.

   Recall that the lexicographic ordering is defined as: $(a,b) \prec_L (c,d)$ if and only if:

   i. $a \prec_A c$ and $a \neq c$, or
   ii. $a = c$ and $b \prec_B d$.

   We prove that this relation is reflexive, antisymmetric, and transitive.

   First, we note that $(a,b) \prec_L (a,b)$ since in this case, $a = a$, and $b \prec_B b$ since $\prec_B$ is a partial order (and therefore reflexive). So we can conclude that $\prec_L$ is reflexive.

   Second, suppose that $(a,b) \prec_L (c,d)$ and $(c,d) \prec_L (a,b)$. We consider two cases. If $a \neq c$, then this implies that $a \prec_A c$ and $c \prec_A a$. But since $\prec_A$ is a partial ordering, this would imply $a = c$ since $\prec_A$ is antisymmetric. So we conclude that $a \neq c$ is not possible. Since $a = c$, the definition then implies that $b \prec_B d$ and $d \prec_B b$. Since $\prec_B$ is a partial ordering, this implies that $b = d$. Since $a = c$ and $b = d$, we conclude that $(a,b) = (c,d)$, and therefore $\prec_L$ is antisymmetric.

   Finally, suppose that $(a,b) \prec_L (c,d)$ and $(c,d) \prec_L (e,f)$. We consider four cases:

   i. $a \neq c$ and $c \neq e$: in this case, $a \prec_A c$ and $c \prec_A e$, so since $\prec_A$ is transitive, we conclude that $a \prec_A e$. Since $a \neq e$[1], the first rule of lexicographic ordering implies that $(a,b) \prec_L (e,f)$.
   ii. $a = c$ and $c \neq e$: In this case we can conclude that $a \neq e$ and also that $a \prec_A e$, since $a = c$. Thus $(a,b) \prec_L (e,f)$ by the first rule.
   iii. $a \neq c$ and $c = e$: In this case we can also conclude that $a \neq e$ and also that $a \prec_A e$, since $a \prec_A c$. Thus $(a,b) \prec_L (e,f)$ by the first rule.
   iv. $a = c$ and $c = e$: in this case, $a = e$, $b \prec_B d$, and $d \prec_B f$. Since $\prec_B$ is transitive, we can conclude that $b \prec_B f$. Therefore $(a,b) \prec_L (e,f)$ by the second rule.

   We conclude that $\prec_L$ is transitive, and therefore that it is a partial ordering.

   (b) Prove that if $\prec_A$ and $\prec_B$ are total orderings, then $\prec_L$ is also a total ordering.

   Let $(a,b),(c,d) \in A \times B$. Since $a, c \in A$, and since $\prec_A$ is a total ordering, it must be true that $a \prec_A c$ or $c \prec_A a$. If $a \prec_A c$ and $a \neq c$, then the first rule of lexicographic ordering implies that $(a,b) \prec_L (c,d)$, and if $c \prec_A a$ and $c \neq a$, the same rule implies that $(c,d) \prec_L (a,b)$. The only other possibility is that $a = c$. In this case, since $\prec_B$ is a total ordering, it must be true that

[1] There is a subtle point here – note that $a \neq c$ and $c \neq e$ does not imply $a \neq e$, in general. But in this case, we can say that $a \neq e$ by the following reasoning. Suppose that $a = e$. By reflexivity, it must then be true that $e \prec_A a$. But since we also had $a \prec_A c$, this implies by transitivity that $e \prec_A c$, which implies that $e = c$ since we also had $c \prec_A e$. Since we assumed $c \neq e$, by contradiction we can conclude that $a \neq e$.

$b \prec_B d$, in which case $(a, b) \prec_L (c, d)$ by the second rule, or $d \prec_B b$, in which case $(c, d) \prec_L (a, b)$ by the second rule. We see that in every case, either $(a, b) \prec_L (c, d)$, or $(c, d) \prec_L (a, b)$, and so $\prec_L$ is a total ordering on $A \times B$.

Note: While solutions will be provided to all of these problems, not all problems may be fully graded.