Due Thursday, May 31 by 5:00 pm

1. Let $f : A \to B$ and $g : B \to C$. Prove that if $g \circ f$ is onto and g is one-to-one, then f must be onto.
2. Let $f : A \to A$ be a bijection. Prove that $f \circ f^{-1} = I$, where I is the identity function.
3. Let $f : A \to A$ be a bijection, and let $S \subseteq A$. Prove that $\overline{f(S)} = f(\overline{S})$.
4. Let $f : A \to B$ be one-to-one, and let $S \subseteq A$. Prove that $S = (f^{-1} \circ f)(S)$.

   Hint: The direction $S \subseteq (f^{-1} \circ f)(S)$ is true even when f is not one-to-one (i.e. if one takes f^{-1} to be the inverse relation, which will not be a function). However the other direction only holds if f is one-to-one.
5. Define $a * b = \max(a, b) = \begin{cases} a & \text{if } a \geq b \\ b & \text{if } b > a \end{cases}$.
   (a) Prove that $*$ is a binary operation on $\mathbb{N}$.
   (b) Is $*$ associative? Explain.
   (c) Does $\mathbb{N}$ have an identity under $*$? Explain.
   (d) (If answer to (c) is yes): Does every element in $\mathbb{N}$ have an inverse under $*$?
6. Define $a * b = \sqrt{ab}$, and let $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$.
   (a) Prove that $*$ is a binary operation on $\mathbb{R}^+$.
   (b) Is $*$ associative? Explain.
   (c) Does $\mathbb{N}$ have an identity under $*$? Explain.
   (d) (If answer to (c) is yes): Does every element in $\mathbb{R}^+$ have an inverse under $*$?
7. We saw that $\cap$ is a binary operation on $\mathcal{P}(U)$ for any nonempty set U. Do $\cap$ and $\mathcal{P}(U)$ form a group? Explain why or why not.