Lecture 3-A

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§1.1 Q/A

#28. $\left[\begin{array}{cc|c} a & b & f \\ c & d & g \end{array}\right]$, $a \neq 0$

$\sim \left[\begin{array}{cc|c} 1 & b/a & f/a \\ c & d & g \end{array}\right]$ $(R_1 \leftarrow 1/a\, R_1)$

$\sim \left[\begin{array}{cc|c} 1 & b/a & f/a \\ 0 & \frac{ad-bc}{a} & \frac{ag-cf}{a} \end{array}\right]$ $(R_2 \leftarrow (-c)R_1 + R_2)$

Notice that $\frac{ad-bc}{a} \neq 0$, which holds iff $ad-bc \neq 0$.

$\underline{\text{Ch. 3}}$ The quantity $ad-bc$ is called the determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

#12. $\left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 3 & -7 & 7 & -8 \\ -4 & 6 & -1 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & -6 & 15 & -9 \end{array}\right]$ $R_2 \leftarrow (-3)R_1 + R_2$, $R_3 \leftarrow 4R_1 + R_3$

$\sim \left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & 0 & 0 & 3 \end{array}\right]$ $R_3 \leftarrow 3R_2 + R_3$

The original LS has no solution. $\emptyset$ $\cancel{\{\emptyset\}}$

§1.2 (Continued)

E.g. Recall that

consider as an augmented matrix

$$A \sim \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow x_4 = 0$$

Declare x_3 as a free-variable (i.e., it is unrestricted), which mean that we may select any value for x_3.

Example (Continued)

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$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & \end{matrix} \\ \left[\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_4 = 0$

$x_3 = \text{free}$

$x_1 = 3x_3 + 5$

$x_2 = -2x_3 - 3$

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$0x_1 + 0x_2 + 0x_3 + 1x_4 = 0$

Parametric description:

All solutions are given by

$$\begin{cases} x_1 = 3x_3 + 5 \\ x_2 = -2x_3 - 3 \\ x_3 \text{ is free} \\ x_4 = 0 \end{cases}$$

For instance, $(8, -5, 1, 0)$ is a solution.

E.g. Find the general solution of the LS whose augmented matrix has been reduced to

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & \end{matrix} \\ \left[\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right] \end{array}$$

← underdetermined LS; can not possibly have a unique solution.

x_1, x_3, x_5 : basic variables

x_2, x_4 : free variables

verify

$$\sim \left[\begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right]$$

$$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 4x_4 + 5 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases}$$

By convention, the free variables are used as parameters.

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \sim \left[\begin{array}{ccc|c} 1 & 5 & 0 & 21 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

← determines x_1 in terms of x_2

← determines x_3 " " " "

A final theorem

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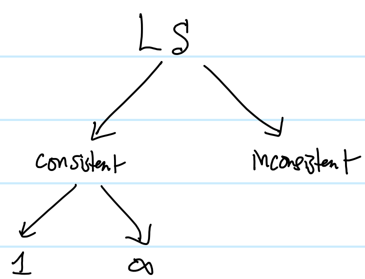
Thm: A LS is consistent if and only if the rightmost column of the augmented matrix is not a pivot column, ie, if and only if an echelon form of the augmented matrix has no row of the form

$$[0 \cdots 0 \mid b], \quad b \neq 0.$$

If LS is consistent, then the solution set contains
(i) a unique solution (no free variables); or
(ii) infinitely-many solutions (free variables present).



Remark: Row-reduction algorithm (RRA): apply row-ops to arrive at the REF of a matrix.

§1.3 Matrices, Vectors, & Vector Equations

Def'n: A matrix is a rectangular array of numbers consisting of rows and columns. Typically, matrices are denoted with capital letters and the entry in the (i,j)-position (ie, row-i, column-j) is denoted by a_{ij}, if the matrix is A.

E.g. If

$$A = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & \pi & -\sqrt{2} \\ e^2 & 0 & 5 \end{bmatrix} \end{array},$$

square brackets

the size of A is 2-by-3.

size of a matrix: # of rows and # of columns.

Then $a_{12} = \pi$, $a_{21} = e^2$, etc.

Def'n: A (column) vector (or row vector) is a matrix with only one column (resp., row).

E.g. Column vectors: $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, $\vec{w} = [\pi] = \pi$

3 x 1 vector

by convention

Remark: Identify all n-tuples of the form $(x_1, x_2, \ldots, x_n)$ with the column vector

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Vectors

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Def'n: The set of all n-by-one column vectors is called the n-dimensional Euclidean space and is denoted by $\mathbb{R}^n$ (read "r-n").

$\in$: "belongs to" or "element of"

Def'n: If $\vec{u}, \vec{v} \in \mathbb{R}^n$, then $\vec{u}$ and $\vec{v}$ are called equal if $u_i = v_i$ for all $i = 1, .., n$.

E.g. $\begin{bmatrix} 4 \\ 7 \end{bmatrix} \neq \begin{bmatrix} 7 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 8/2 \\ 7 \end{bmatrix}$

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* Algebra on $\mathbb{R}^n$

Def'n: (1) Given $\vec{u}, \vec{v} \in \mathbb{R}^2$, the sum of $\vec{u}$ and $\vec{v}$, denoted by $\vec{u}+\vec{v}$, is defined by

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix} \in \mathbb{R}^2.$$

(2) Given $\vec{u} \in \mathbb{R}^2$ and $c \in \mathbb{R}$, the scalar product of $\vec{u}$ and c, denoted by $c\vec{u}$, is defined by

$$c\vec{u} = c\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix} \in \mathbb{R}^2.$$

E.g. If $\vec{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$, then

(i) $\vec{u} + \vec{v} = \begin{bmatrix} 1+(-5) \\ -2+2 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$;

(ii) $2\vec{u} = \begin{bmatrix} 2(1) \\ 2(-2) \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$; and

(iii) $\vec{u} - 3\vec{v} = \vec{u} + (-3)\vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 15 \\ -6 \end{bmatrix} = \begin{bmatrix} 16 \\ -8 \end{bmatrix}$.