Lecture 10-A

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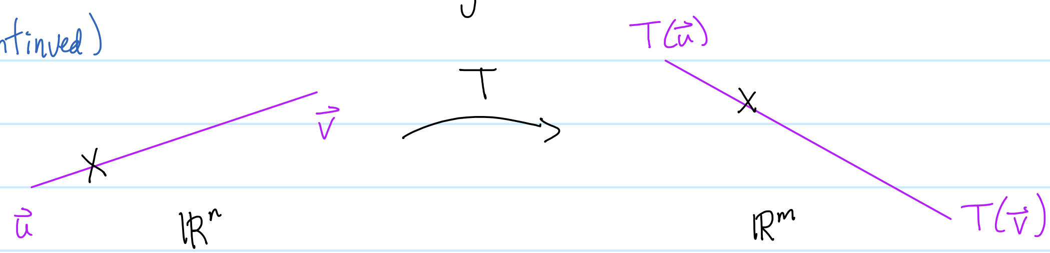
$*$ **Announcements**

(i) Midterm Tuesday (1 hr; 8.5 × 11 notesheet)

(ii) Quizzes will be 15 mins in length.

§1.8 (Continued)

Recall:

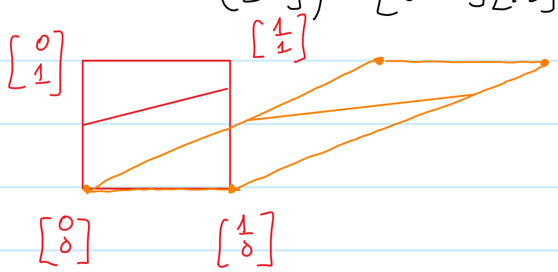


Every point in the line segment from $\vec{u}$ to $\vec{v}$ is of the form $\alpha\vec{u}+(1-\alpha)\vec{v}$, with $\alpha \in [0,1]$. If T is a LT, then

$$T(\alpha\vec{u}+(1-\alpha)\vec{v}) = T(\alpha\vec{u}) + T((1-\alpha)\vec{v}) = \alpha T(\vec{u}) + (1-\alpha)T(\vec{v}).$$

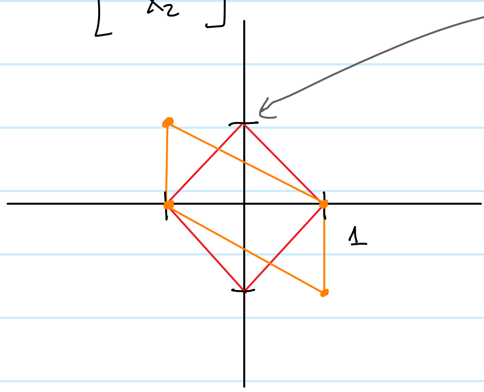
E.g. Consider $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1+2x_2 \\ x_2 \end{bmatrix}$$



E.g. Let $D = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and let $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\vec{x}) = D\vec{x}$.

Notice that $T(\vec{x}) = \begin{bmatrix} x_1 - x_2 \\ x_2 \end{bmatrix}$. What is the image of the unit diamond?



$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Properties of LTs

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Proposition: If $T: \mathbb{R}^n \to \mathbb{R}^m$ and T is linear, then

(i) $T(\vec{0}_n) = \vec{0}_m$; and

(ii) $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$.

$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

$T(c\vec{u}) = cT(\vec{u})$

Pf: (i) If $\vec{u} \in \mathbb{R}^n$, then $0\vec{u} = \vec{0}_n$ and

$$T(\vec{0}_n) = T(0\vec{u}) = 0\underbrace{T(\vec{u})}_{\in \mathbb{R}^m} = \vec{0}_m.$$

(ii) Notice that

$$T(c\vec{u} + d\vec{v}) = T(c\vec{u}) + T(d\vec{v}) = cT(\vec{u}) + dT(\vec{v}).$$

Remark: If $T: \mathbb{R}^n \to \mathbb{R}^m$ and T is linear, then

$$T(x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p) = x_1T(\vec{v}_1) + x_2T(\vec{v}_2) + \cdots + x_pT(\vec{v}_p).$$

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§1.9 The Matrix of a Linear Transformation

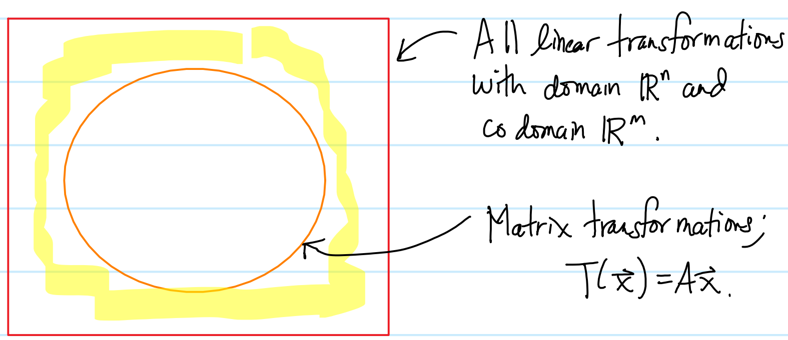
Recap: We know that if A is an m-by-n matrix, then

(i) $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$; and

(ii) $A(c\vec{u}) = cA\vec{u}$.

If $T: \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\vec{x}) = A\vec{x}$ $\left(\vec{x} \overset{T}{\mapsto} A\vec{x}\right)$, then, from above,

"linear" $\begin{cases} \text{(i) } T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}); \text{ and} \\ \text{(ii) } T(c\vec{u}) = cT(\vec{u}). \end{cases}$



Q: Are there LTs that are not matrix transformations?

Problem 19 of Section 1.8

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#19 §1.8 $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{y}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $\vec{y}_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$

LT

$T: \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $T(\vec{e}_1) = \vec{y}_1$, $T(\vec{e}_2) = \vec{y}_2$. Find $T\left(\begin{bmatrix} 5 \\ -3 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$.

Sol'n: Since

$$\begin{bmatrix} 5 \\ -3 \end{bmatrix} = 5\vec{e}_1 + (-3)\vec{e}_2$$

and since T is linear, it follows that

$$\begin{aligned} T\left(\begin{bmatrix} 5 \\ -3 \end{bmatrix}\right) &= T\left(5\vec{e}_1 + (-3)\vec{e}_2\right) \\ &= 5\,T(\vec{e}_1) + (-3)\,T(\vec{e}_2) \\ &= 5\vec{y}_1 + (-3)\vec{y}_2 \\ &= 5\begin{bmatrix} 2 \\ 5 \end{bmatrix} + (-3)\begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 13 \\ 7 \end{bmatrix}. \end{aligned}$$

In general, notice that

$$\begin{aligned} T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) &= T\left(x_1\vec{e}_1 + x_2\vec{e}_2\right) \\ &= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) \\ &= x_1\vec{y}_1 + x_2\vec{y}_2 \\ &= x_1\begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \end{aligned}$$

A: Surprisingly, the answer is 'no'. Recall that the (n-by-n) identity matrix is the matrix defined by

$$I = I_n = \begin{array}{c} \begin{matrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{matrix} \\ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \end{array} = \begin{bmatrix} 1 & & & \\ & 1 & & 0 \\ & 0 & \ddots & \\ & & & 1 \end{bmatrix}$$

For $i \in \{1, \ldots, n\}$, let $\vec{e}_i$ denote the i^{th}-column of I. If $\vec{x} \in \mathbb{R}^n$, then $\vec{x} = \sum_{i=1}^{n} x_i\vec{e}_i$; moreover, if $T: \mathbb{R}^n \to \mathbb{R}^m$ and T is linear, then

$$T(\vec{x}) = T\left(\sum_{i=1}^{n} x_i\vec{e}_i\right) = \sum_{i=1}^{n} x_i T(e_i) = \sum_{i=1}^{n} x_i\vec{y}_i = \left[\vec{y}_1\ \vec{y}_2 \cdots \vec{y}_n\right]\vec{x}.$$

Standard Matrix

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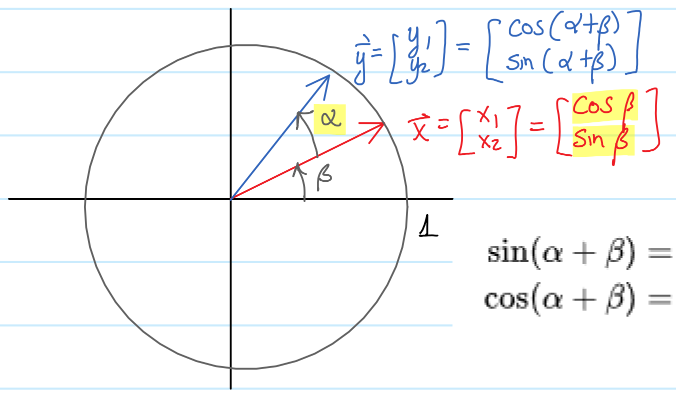
Def'n: If $T : \mathbb{R}^n \to \mathbb{R}^m$ and T is linear, then the matrix A defined by
$$A = [T(\vec{e}_1)\ T(\vec{e}_2)\ \cdots\ T(\vec{e}_n)],$$
is called the standard matrix for T.

Thm: If $T : \mathbb{R}^n \to \mathbb{R}^m$ and T is linear, then $T(\vec{x}) = A\vec{x}$, where A is the std. matrix for T, i.e, T is a matrix transformation. Moreover, this matrix is unique.

Consider the transformation that rotates every point (vector) in $\mathbb{R}^2$ counterclockwise by a fixed positive angle α (radians) with respect to the origin.



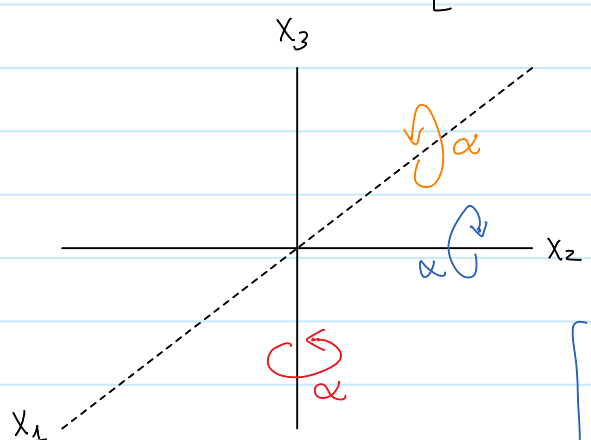
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Notice that

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos(\alpha+\beta) \\ \sin(\alpha+\beta) \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{bmatrix} = \begin{bmatrix} \cos\alpha\, x_1 - \sin\alpha\, x_2 \\ \sin\alpha\, x_1 + \cos\alpha\, x_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Q: What effect does the matrix
$$\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
have on points in $\mathbb{R}^3$?



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}$$

$$\begin{bmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{bmatrix}$$

Matrix Algebra

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§2.1 Matrix Operations

Recall: If A is an m-by-n matrix, then the (i,j)-entry is denoted by a_{ij}; in particular,

$$A = \begin{array}{c c} \begin{array}{ccc} 1 & \cdots\ j \cdots & n \end{array} & \\ \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} & \begin{array}{c} 1 \\ \vdots \\ i \\ \vdots \\ m \end{array} \end{array}.$$

Entries of the form a_{ii} are called the diagonal entries of A.

E.g. $A = \begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}$, $B = \begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}$

If $m=n$, then A is called a square matrix. A diagonal matrix is a square matrix all of whose off-diagonal entries are zero.

E.g. The matrix

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

is a diagonal matrix.

$T, S: \mathbb{R}^n \to \mathbb{R}^m$

Suppose that A and B are m-by-n matrices and consider the matrix transformations T and S defined by $T(\vec{x}) = A\vec{x}$ and $S(\vec{x}) = B\vec{x}$. Consider the function

$$\begin{aligned}(T+S)(\vec{x}) = T(\vec{x}) + S(\vec{x}) &= A\vec{x} + B\vec{x} \\ &= \sum_{i=1}^{n} x_i \vec{a}_i + \sum_{i=1}^{n} x_i \vec{b}_i \\ &= \sum_{i=1}^{n} x_i (\vec{a}_i + \vec{b}_i) \\ &= \underbrace{\left[(\vec{a}_1 + \vec{b}_1) \,|\, \cdots \,|\, (\vec{a}_n + \vec{b}_n) \right]}_{m \times n \quad C} \vec{x}\end{aligned}$$

Since

$$\vec{a}_j + \vec{b}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} + \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{bmatrix} = \begin{bmatrix} a_{1j} + b_{1j} \\ a_{2j} + b_{2j} \\ \vdots \\ a_{mj} + b_{mj} \end{bmatrix},$$

it follows that the (i,j)-entry of C is $a_{ij} + b_{ij}$.

Matrix Addition

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Def'n: If $A=[a_{ij}]$ and $B=[b_{ij}]$ are m-by-n matrices, then the sum of A and B, denoted by $A+B$, is the m-by-n matrix whose (i,j)-entry is $a_{ij}+b_{ij}$, i.e.,

$$A+B=[a_{ij}+b_{ij}].$$