Due Thursday, April 12 before 5:00 pm

1. Let $x \in \mathbb{R}$. Consider the following statement: *"$x^2 - 5x + 6 = 0$ if and only if $x = 3$ or $x = 2$"*.

   (a) Express the statement in propositional notation by identifying the parts of the statement (labeling them as p, q, etc.) and giving the form of the proposition.

   (b) Express the negation of the statement in propositional notation, and in written English. Simplify the expression as much as possible.

2. Suppose $f(x)$ is a twice differentiable function (i.e. $f(x)$, $f'(x)$ and $f''(x)$ are defined for all x), and let $a \in \mathbb{R}$. Consider the following statement: *"If $f'(a) = 0$ and $f''(a) < 0$, then a is a strict local maximum of f."*

   (a) Express the statement in propositional notation by identifying the parts of the statement (labeling them as p, q, etc.) and giving the form of the proposition.

   (b) Express the converse and contrapositive of the statement, using propositional logic, and in written English. Simplify the expressions as much as possible.

   (c) Express the negation of the statement in propositional notation, and in written English. Simplify the expression as much as possible.

   (d) Suppose that $f'(a) = 0$ and $f''(a) = 0$. What does the statement tell us about whether a is a strict local maximum? Explain.

3. Use truth tables to show that the following statements are tautologies:

   (a) $\{\sim p \to [q \wedge (\sim q)]\} \to p$

   (b) $[p \vee q \vee r] \leftrightarrow [(\sim p \;\wedge\; \sim q) \to r]$

4. Express each of the following statements in propositional notation, using quantifiers. Then, express the negation of the statement, both in propositional notation, and in written English.

   (a) For all real numbers x, there exists a real number y such that $x = y^2$, or $y^2 + x = 0$.

   (b) For all real numbers x and y, $f(x) < f(y)$ whenever $x < y$. (This defines what it means for f to be a monotonically increasing function).

5. Use proof by contradiction to prove the following results:

   (a) Let x and y be positive real numbers. Then, $\sqrt{x+y} < \sqrt{x} + \sqrt{y}$.

   (b) Let x be a rational number and let y be an irrational number. Then, $z = x + y$ is irrational. (Recall that a rational number can be written as $q = \dfrac{a}{b}$ where a and b are integers; an irrational number cannot be written in this way).

   (c) **[Bonus: up to +10 points]** It is well known that there are consecutive integers such that $a^2 + b^2 = c^2$; namely, $a = 3$, $b = 4$, $c = 5$. Prove using contradiction that there are no consecutive integers a, b and c such that $a^3 + b^3 = c^3$. Citing Fermat's Last Theorem is not allowed! :-) You should be able to arrive at a contradiction using basic ideas about divisibility and/or modular arithmetic.

Note: While solutions will be provided to all of these problems, not all problems may be fully graded.