Lecture 7-A

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§1.4 Q|A

#7.
$$x_1\begin{bmatrix}4\\-1\\7\\-4\end{bmatrix}+x_2\begin{bmatrix}-5\\3\\-5\\1\end{bmatrix}+x_3\begin{bmatrix}7\\-8\\0\\2\end{bmatrix}=\begin{bmatrix}6\\-8\\0\\-7\end{bmatrix}$$

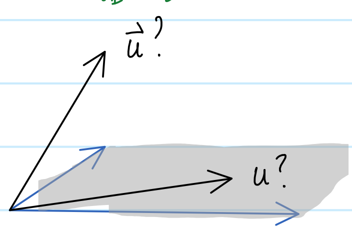
$$\Leftrightarrow \begin{bmatrix}4&-5&7\\-1&3&-8\\7&-5&0\\-4&1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}6\\-8\\0\\7\end{bmatrix}.$$

#13. Let $\vec{u}=\begin{bmatrix}0\\4\\4\end{bmatrix}$ and $A=\begin{bmatrix}3&-5\\-2&6\\1&1\end{bmatrix}$. Is $\vec{u}$ in the plane in $\mathbb{R}^3$ spanned by the columns of A?

$$\begin{array}{cc} \begin{bmatrix}3&-5\\-2&6\\1&1\end{bmatrix} \\ \vec{a}_1 \quad \vec{a}_2 \end{array}$$

- $\vec{u}\in \mathrm{Span}(\vec{a}_1,\vec{a}_2)$
- $A\vec{x}=\vec{u}$ has a solution
- $\vec{u}$ is a LC of $\vec{a}_1$ and $\vec{a}_2$

equivalent



Sol'n: Solve

$$\left[\begin{array}{cc|c}3&-5&0\\-2&6&4\\1&1&4\end{array}\right]\sim\left[\begin{array}{cc|c}1&1&4\\-1&3&2\\3&-5&0\end{array}\right]\sim\left[\begin{array}{cc|c}1&1&4\\0&4&6\\0&-8&-12\end{array}\right]\sim\left[\begin{array}{cc|c}1&1&4\\0&2&3\\0&-2&-3\end{array}\right]$$

$$\sim\left[\begin{array}{cc|c}1&1&4\\0&2&3\\0&0&0\end{array}\right]\Rightarrow \text{LS is feasible} \Rightarrow \vec{u}\in\mathrm{Span}(\vec{a}_1,\vec{a}_2).$$

#12 §1.3 (p.32)

$$\begin{array}{cccc}\vec{a}_1&\vec{a}_2&\vec{a}_3&\vec{b}\end{array}$$
$$\left[\begin{array}{ccc|c}1&0&2&-5\\-2&5&0&11\\2&5&8&-7\end{array}\right]\sim\left[\begin{array}{ccc|c}1&0&2&-5\\0&5&4&1\\0&5&4&3\end{array}\right]\sim\left[\begin{array}{ccc|c}1&0&2&-5\\0&5&4&1\\0&0&0&2\end{array}\right]$$

No, $\vec{b}$ is not a LC of $\vec{a}_1,\vec{a}_2$, and $\vec{a}_3$.

Q1A

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17. $\begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.5 & 0 \\ 0 & 1 & -.5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$A\vec{x} = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^4$?
No, A does not have a pivot in every row.

#14. $\begin{array}{c} \vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3 \\ \left[\begin{array}{ccc|c} 5 & 8 & 7 & 2 \\ 0 & 1 & -1 & -3 \\ 1 & 3 & 0 & 2 \end{array}\right] \end{array} \sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 5 & 8 & 7 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & -7 & 7 & -8 \end{array}\right]$

$\quad\quad\quad\quad \underset{A}{\shortparallel} \quad\quad\quad \vec{u}$

$\sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & -29 \end{array}\right]$

No, $\vec{u}$ is not in span $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$

§1.5 (continued)

Recall: A homogeneous LS (HLS) is any LS of the form

$$A\vec{x} = \vec{0}.$$

Thm: If x satisfies $\underset{m\times n}{A}\vec{x} = \vec{0}$ (ie, $\vec{x}$ is a solution of $A\vec{x} = \vec{0}$), then there are vectors $\vec{v}_1, .., \vec{v}_p$ $(1 \le p \le n)$ s.t. $\vec{x} \in$ span $(\vec{v}_1, .., \vec{v}_p)$.

Eg. Solve $A\vec{x} = \vec{b}$, with

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}.$$

Sol'n: $\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{array}\right] \overset{\text{verify}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

All solutions are of the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/3 x_3 - 1 \\ 2 \\ x_3 \end{bmatrix}$$

Affine spaces

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$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 - 1 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}}_{=\vec{u}} + \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{=\vec{v}}$$

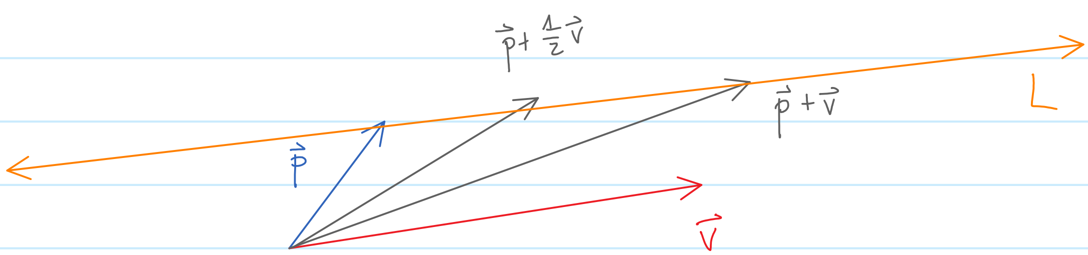
Span$(\vec{u})$ is the solution set to $A\vec{x} = \vec{0}$.

All solutions are of the form

$$\vec{x} = t\vec{u} + \vec{v}, \quad t \in \mathbb{R}.$$

$\mathbb{R}^2$. Suppose that $\vec{p}$ and $\vec{v}$ are nonzero vectors in $\mathbb{R}^2$. Consider the set

$$L = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = \vec{p} + t\vec{v},\ t \in \mathbb{R}\}$$



The set L is an example of an affine space.

E.g. Let $A = \begin{bmatrix} 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. If $\vec{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, then we know that

$A\vec{x} = \vec{b}$ is consistent because A has a pivot in every row.

$$[A \mid \vec{b}] = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \end{array} \Rightarrow \text{every solution is of the form}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_2 - 2x_4 + 2 \\ x_2 \\ x_4 - 1 \\ x_4 \\ 1 \end{bmatrix} = \overbrace{x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}}^{\text{solution to } A\vec{x} = \vec{0}} + \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

General result

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Thm: Suppose $A\vec{x} = \vec{b}$ is consistent for some $\vec{b} \in \mathbb{R}^m$ and let $\vec{p} \in \mathbb{R}^n$ be a solution. Then all solutions of $A\vec{x} = \vec{b}$ are of the form

$$\vec{w} = \vec{p} + \vec{v},$$

where $\vec{v}$ is any solution of the HLS $A\vec{x} = \vec{0}$.

Pf. If $\vec{w} = \vec{p} + \vec{v}$, where $\vec{v}$ is a solution of $A\vec{x} = \vec{0}$, then

$$A\vec{w} = A(\vec{p} + \vec{v}) = A\vec{p} + A\vec{v} = \vec{b} + \vec{0} = \vec{b},$$

i.e., $\vec{w}$ is a solution of $A\vec{x} = \vec{b}$.

If $\vec{w}$ is a solution to $A\vec{x} = \vec{b}$, then

$$\vec{w} = \vec{w} + \vec{0},$$

in which $\vec{0}$ is a solution to $A\vec{x} = \vec{0}$.