Lecture 10-A

Thursday, April 26, 2018 8:43 AM

* Announcements

(i) Midterm Tuesday (1 hr; 8.5x11 notesheet)

(ii) Quizzes will be 15 mins in length.

$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$$
$$T(c\vec{u}) = c\,T(\vec{u})$$

§1.8 Q/A

#19. $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{y}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $\vec{y}_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$

$T: \mathbb{R}^2 \to \mathbb{R}^2$ $T(\vec{e}_1) = \vec{y}_1$, $T(\vec{e}_2) = \vec{y}_2$. Find $T\left(\begin{bmatrix} 5 \\ -3 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$.

Sol'n: Since

$$\begin{bmatrix} 5 \\ -3 \end{bmatrix} = 5\vec{e}_1 + (-3)\vec{e}_2$$

and since T is linear, it follows that

$$\begin{aligned} T\left(\begin{bmatrix} 5 \\ -3 \end{bmatrix}\right) &= T(5\vec{e}_1 + (-3)\vec{e}_2) \\ &= T(5\vec{e}_1) + T((-3)\vec{e}_2) \\ &= 5T(\vec{e}_1) + (-3)T(\vec{e}_2) \\ &= 5\begin{bmatrix} 2 \\ 5 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 13 \\ 7 \end{bmatrix}. \end{aligned}$$

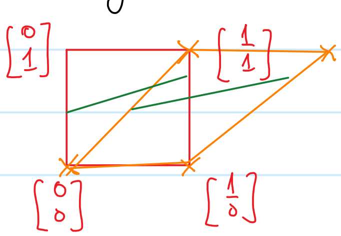
Moreover,

$$\begin{aligned} T\Big(\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{=\vec{x}}\Big) &= T(x_1\vec{e}_1 + x_2\vec{e}_2) \\ &= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) \qquad \text{(linearity conditions)} \\ &= x_1\begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \end{aligned}$$

$\longleftarrow$ implies T is a matrix transformation.

-1

E.g. Consider $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}$.



Midterm Example

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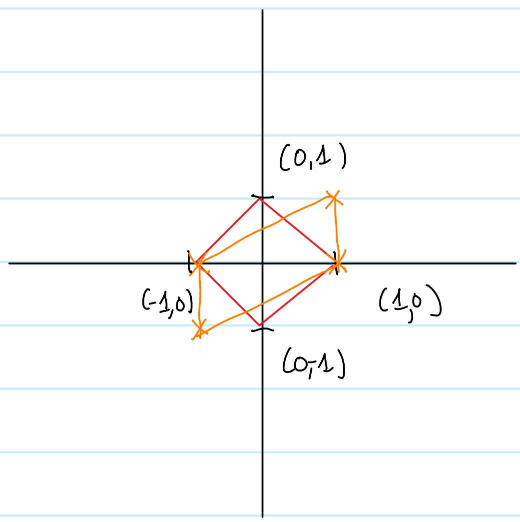
E.g. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and let $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\vec{x}) = A\vec{x}$.

What is the image of the unit diamond?

$$A\vec{x} = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}$$



Prop: If $T: \mathbb{R}^n \to \mathbb{R}^m$ and T is linear, then

(i) $T(\vec{0}_n) = \vec{0}_m$; and

(ii) $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$. $\Longrightarrow$ T is linear. exercise

Pf: (i) If $\vec{u} \in \mathbb{R}^n$, then $T(\vec{0}_n) = T(0 \cdot \vec{u}) = 0T(\vec{u}) = \vec{0}_m$.

(ii) Notice that

$$T(c\vec{u} + d\vec{v}) = T(c\vec{u}) + T(d\vec{v}) = cT(\vec{u}) + dT(\vec{v}).$$

Remark: If $T: \mathbb{R}^n \to \mathbb{R}^m$ and T is linear, then

$$T\left(\sum_{i=1}^{p} x_i \vec{a}_i\right) = \sum_{i=1}^{p} x_i T(\vec{a}_i).$$

The Standard Matrix

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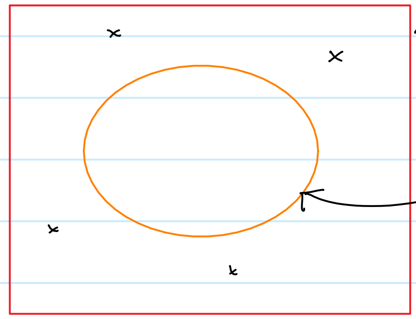
§1.9 The Matrix of a Linear Transformation



All linear transformations with domain $\mathbb{R}^n$ and co-domain $\mathbb{R}^m$.

Matrix transformations; $T(\vec{x}) = A\vec{x}$

$\uparrow$ $m \times n$ matrix

Q: Are there LTs that are not matrix transformations?

A: Surprisingly, the answer is 'no'. Recall that the (n-by-n) identity matrix is the matrix defined by

$$I = I_n = \begin{array}{c} \begin{matrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{matrix} \\ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \end{array}.$$

Let $\vec{e}_j$ denote the j^{th}-column of I. Notice that if $\vec{x} \in \mathbb{R}^n$, then

$$\vec{x} = I\vec{x} = \sum_{i=1}^{n} x_i \vec{e}_i .$$

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ and assume that T is linear and that $T(\vec{e}_j) = \vec{y}_j \in \mathbb{R}^m$, $j = 1, \dots, n$. Since T is linear, it follows that

$$T(\vec{x}) = T\left(\sum_{i=1}^{n} x_i \vec{e}_i\right) = \sum_{i=1}^{n} x_i T(\vec{e}_i) = \sum_{i=1}^{n} x_i \vec{y}_i = [\vec{y}_1 \ \vec{y}_2 \cdots \vec{y}_n]\vec{x}.$$

Def'n: If $T: \mathbb{R}^n \to \mathbb{R}^m$ and T is linear, then the matrix A defined by

$$A = [T(\vec{e}_1) \ T(\vec{e}_2) \cdots T(\vec{e}_n)],$$

is called the standard matrix for T.

Thm: If $T: \mathbb{R}^n \to \mathbb{R}^m$ and T is linear, then $T(\vec{x}) = A\vec{x}$, where A is the standard matrix for T. Moreover, this matrix is unique.

Rotations

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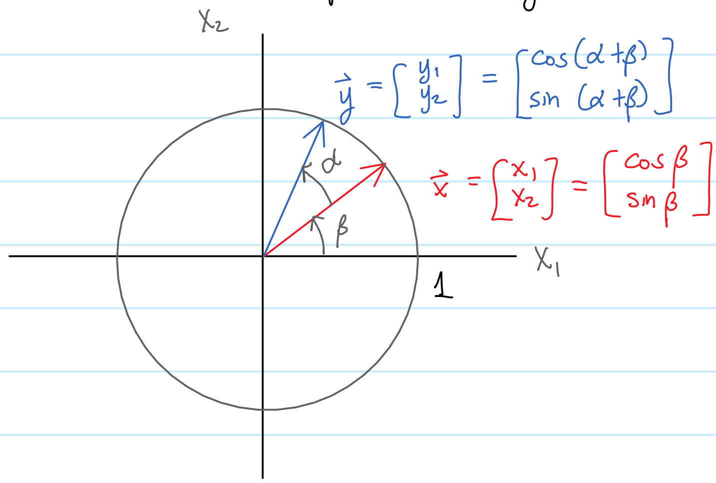
Consider the transformation that rotates every point (vector) in $\mathbb{R}^2$ by a fixed positive angle α counterclockwise with respect to the origin.



Following the trig-identities

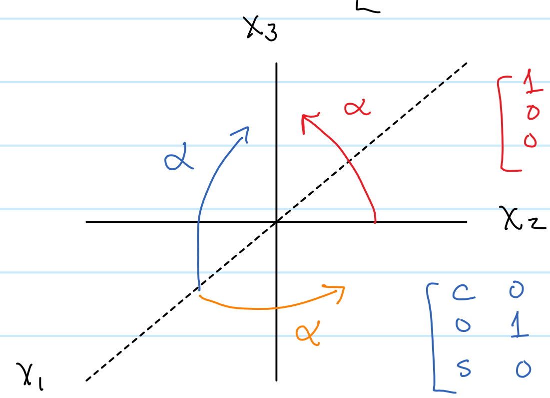
$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos(\alpha+\beta) \\ \sin(\alpha+\beta) \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{bmatrix} = \begin{bmatrix} \cos\alpha\, x_1 - \sin\alpha\, x_2 \\ \sin\alpha\, x_1 + \cos\alpha\, x_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Q: What effect does the matrix

$$\left[\begin{array}{cc|c} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

have on points in $\mathbb{R}^3$?



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}$$

$$\begin{bmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{bmatrix}$$

Matrix Algebra

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§2.1 Matrix Operations

Recall: If A is an m-by-n matrix, then a_{ij} denotes the entry in the (i,j)-position of A; in particular,

$$A = \begin{array}{cc} & \begin{array}{ccccc} 1 & \cdots & j & \cdots & n \end{array} \\ & \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ \vdots \\ i \\ \vdots \\ m \end{array}$$

Entries of the form a_{ii} are called the diagonal entries of A.

E.g. $A = \begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}$ or $A = \begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}$

If $m=n$, then A is called a square matrix. A diagonal matrix is a square matrix whose off-diagonal entries are all zero.

E.g. $\begin{bmatrix} * & & 0 \\ & * & \\ & & \ddots & \\ 0 & & & * \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$: "off-diagonal"

Suppose that A and B are m-by-n matrices and consider the matrix transformations T and S defined by $T(\vec{x}) = A\vec{x}$ and $S(\vec{x}) = B\vec{x}$. Notice that we may define a new function by adding the vectors $T(\vec{x})$ and $S(\vec{x})$; in particular,

$$R(\vec{x}) = T(\vec{x}) + S(\vec{x}) = A\vec{x} + B\vec{x} = \sum_{i=1}^{n} x_i \vec{a}_i + \sum_{i=1}^{n} x_i \vec{b}_i$$

$$= \sum_{i=1}^{n} x_i(\vec{a}_i + \vec{b}_i) = C\vec{x},$$

where $C = \overbrace{[(\vec{a}_1+\vec{b}_1) | \cdots | (\vec{a}_n+\vec{b}_n)]}^{m\times n}$. It can be shown that $c_{ij} = a_{ij} + b_{ij}$.

Defn: If $A = [a_{ij}]$ and $B = [b_{ij}]$ are m-by-n matrices, then the sum of A and B, denoted by $A+B$, is the m-by-n matrix whose (i,j)-entry is $a_{ij}+b_{ij}$, i.e.,

$$A+B = [a_{ij}+b_{ij}].$$

Matrix Addition

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$T: \mathbb{R}^3 \to \mathbb{R}^2$ $S: \mathbb{R}^3 \to \mathbb{R}^2$ $T+S: \mathbb{R}^3 \to \mathbb{R}^2$

Eg. If $A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}$, then $A+B = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}$.

With A defined as above, what can be meant by $3A$? It is reasonable to define

$$\begin{aligned} 3A &= A + A + A \\ &= \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 8 & 0 & 10 \\ -2 & 6 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 0 & 15 \\ -3 & 9 & 6 \end{bmatrix} = \begin{bmatrix} 3(4) & 3(0) & 3(5) \\ 3(-1) & 3(3) & 3(2) \end{bmatrix}. \end{aligned}$$

Def'n: If A is an m-by-n matrix and $c \in \mathbb{R}$, then cA is the m-by-n matrix whose (i,j)-entry is $c\,a_{ij}$. In particular, $-A := (-1)A$.