Lecture 1

Tuesday, March 27, 2018 8:44 AM

§1.1 Linear Systems

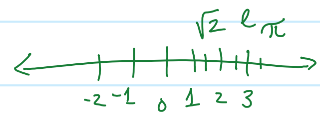
Def'n: A linear equation (in the variables $x_1, \dots, x_n$) is any equation of the form

$$a_1 x_1 + \dots + a_n x_n = \sum_{i=1}^{n} a_i x_i = b,$$

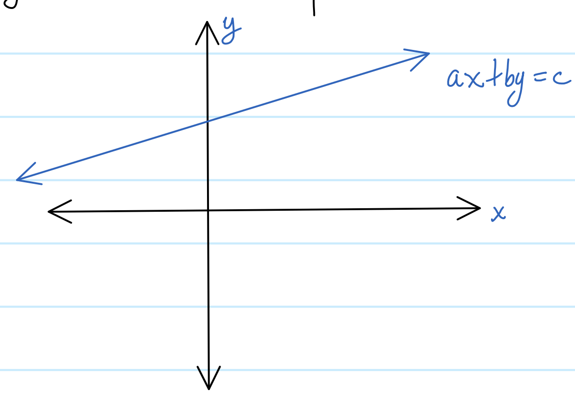
in which $a_1, \dots, a_n, b$ (called the coefficients) are real numbers and n is a natural number.

$\downarrow$ $\mathbb{N} = \{1, 2, 3, \dots\}$

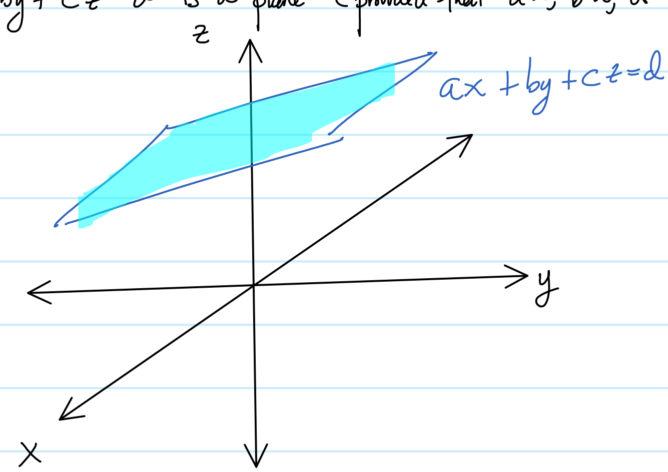
$\downarrow$ $\mathbb{R}$



Remark: The graph of $ax + by = c$ is a line (provided that $a \neq 0$ or $b \neq 0$).



The graph of $ax + by + cz = d$ is a plane (provided that $a \neq 0$, $b \neq 0$, or $c \neq 0$)



Linear System

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Def'n: (1) A linear system is a collection of one or more linear equations in the same variables.

E.g. $\begin{cases} 2x + y = 3 \\ x - y = -1 \end{cases}$, $\begin{cases} 2x_1 - x_2 + 1.5x_3 = 8 \\ 1x_1 + 0x_2 - 4x_3 = -7 \end{cases}$

2x2 system 2x3 system

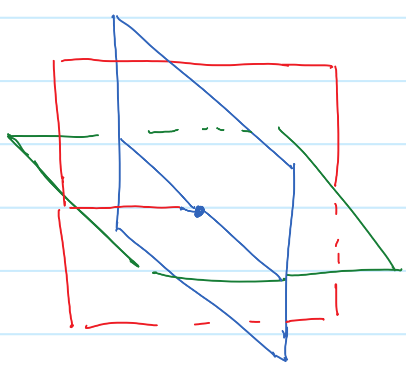
(2) A solution of a linear system is any ordered n-tuple, say $(s_1, \dots, s_n)$, that satisfies each equation (i.e., makes each equation a true statement). For instance, $(5, 6.5, 3)$ satisfies the second linear system.

(3) The set of all possible solutions to a LS is called the solution-set (of the LS). Two or more LSs are called equivalent if they have the same solution set.

* Possible Solution Sets

2x2 LSs

Case (i): unique solution Case (ii): no solution Case (iii): infinitely-many solutions



Fact: Any LS will have a unique solution, no solution, or infinitely-many solutions.

LS

- has a solution (feasible)
  - Unique "1"
  - Infinitely-many "∞"
- does not have a solution (infeasible) "0"

Matrix Notation

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Given the LS

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \quad\quad 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases} \qquad (*)$$

with coefficients of each variable aligned in columns, the matrix (rectangular array of numbers)

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

is called the coefficient matrix of the LS (*) and

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

Caution: The coefficient matrix of

$$\cancel{\begin{cases} 2x_1 - x_2 = 4 \\ 3x_2 - 4x_1 = 1 \end{cases}} \qquad \begin{cases} 2x_1 - x_2 = 4 \\ -4x_1 + 3x_2 = 1 \end{cases}$$

is

$$\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$$

Consider the LS

$$\begin{cases} x_1 \quad\quad\quad = a \\ \quad x_2 \quad\quad = b \\ \quad\quad x_3 = c. \end{cases}$$

The solution is (a, b, c).

Row-reduction

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* Row operations

Elementary Row-operations

1. $R_j \leftarrow cR_i + R_j$: replace row j with the sum of c times row i and row j. $c \neq 0$

   $\downarrow$ "replace with" "row-replacement"

2. $R_i \leftrightarrow R_j$: Swap row i with row j. "row-interchange"
3. $R_i \leftarrow cR_i$: multiply row i by c, $c \neq 0$. "row-Scaling"

E.g. Solve

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 & R_1 \\ 2x_2 - 8x_3 = 8 & R_2 \\ -4x_1 + 5x_2 + 9x_3 = -9 & R_3 \end{cases}$$

Sol'n:

$R_3 \leftarrow 4R_1 + R_3$
$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -3x_2 + 13x_3 = -9 \end{cases}$$

$R_1 \leftarrow R_1 - R_3$
$$\begin{cases} x_1 - 2x_2 = -3 \\ x_2 = 16 \\ x_3 = 3 \end{cases}$$

$R_2 \leftarrow \frac{1}{2}R_2$
$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ -3x_2 + 13x_3 = -9 \end{cases}$$

$R_1 \leftarrow 2R_2 + R_1$
$$\begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3 \end{cases}$$

$R_3 \leftarrow 3R_2 + R_3$
$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

The solution-set is $\{(29, 16, 3)\}$.

$R_2 \leftarrow 4R_3 + R_2$
$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 = 16 \\ x_3 = 3 \end{cases}$$

Augmented Form

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The augmented matrix of the LS in the previous example is

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

Exercise: Apply the row-ops in the previous example to arrive at

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$