Lecture 6-A

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§1.4 (continued)

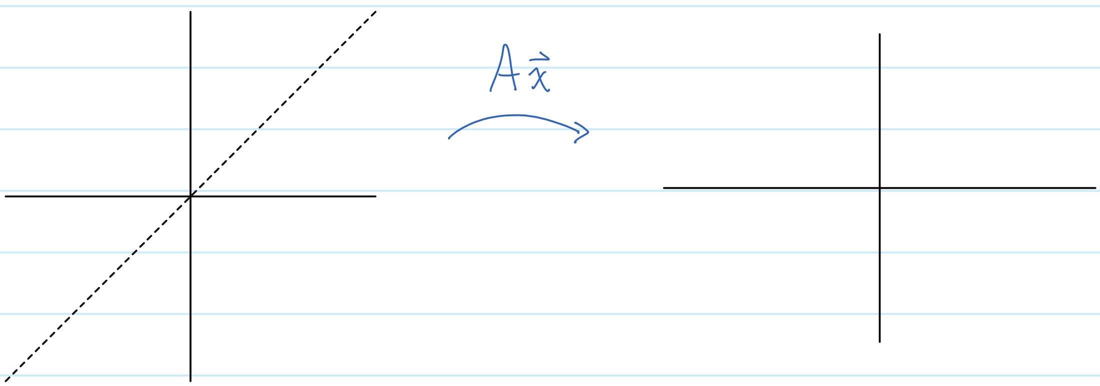
Recall: If $A = [\overbrace{\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_n}^{\in \mathbb{R}^m}]$ is an $m\times n$ matrix and $\vec{x} \in \mathbb{R}^n$, then

"$A\vec{x}$ is defined as"

$$A\vec{x} \underset{\text{"defined as"}}{:=} \sum_{i=1}^{n} x_i \vec{a}_i = x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n \in \mathbb{R}^m.$$

E.g. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$, then

$$A\vec{x} = (-1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$



Notation: If f is a function with domain X and co-domain Y, then we write this as $f: X \to Y$.

E.g. Let $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Notice that that the range of f is $[0,\infty)$ which differs from the co-domain.

$f(x) = ax$

Remark: We will be interested in the function $T: \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\vec{x}) = A\vec{x}$.

Convention: Unless otherwise stated, A is an $m\times n$ matrix, $\vec{x} \in \mathbb{R}^n$, and $\vec{b} \in \mathbb{R}^m$.

Thm: The matrix equation $A\vec{x} = \vec{b}$ is equivalent to the vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n = \vec{b},$$

which, in turn, is equivalent to the LS whose augmented matrix is $[A|b]$.

Section 1.4 (Continued)

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Remark: Solving $A\vec{x} = \vec{b}$ is also the same as determining whether $\vec{b} \in \text{span}(\vec{a}_1, \dots, \vec{a}_n)$.

E.g. Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is $A\vec{x} = \vec{b}$ consistent for every choice of $b_1, b_2,$ and b_3?

Sol'n: Solve

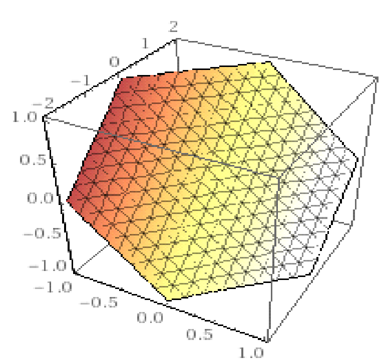
$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 7 & 5 & 3b_1 + b_3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 0 & 0 & 3b_1 + b_3 - \frac{1}{2}(4b_1 + b_2) \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 0 & 0 & b_1 - \frac{1}{2}b_2 + b_3 \end{array}\right] \quad (*)$$

No; we can select $b_1, b_2,$ and b_3 s.t. $b_1 - \frac{1}{2}b_2 + b_3 \neq 0$, in which case $(*)$ would be inconsistent. Notice that $(*)$ is consistent provided that $b_1 - \frac{1}{2}b_2 + b_3 = 0$.

Plot:



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A very important theorem

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Remark: In the previous example, the equation $A\vec{x}=\vec{b}$ failed to be consistent for every $\vec{b}\in\mathbb{R}^3$ because an echelon form of A contained a row of zeros. If A had a pivot in each row, then the expressions in the solution column would be irrelevant

Def'n: If A is an $m\times n$ matrix, then the columns of A are said to span $\mathbb{R}^m$ if every $\vec{b}\in\mathbb{R}^m$ is a linear combination of the columns of A.

Thm: If A is an $m\times n$ matrix, then the following statements are equivalent (they are all true or all false):

different ways of saying the same thing

(a) the equation $A\vec{x}=\vec{b}$ is consistent for every $\vec{b}\in\mathbb{R}^m$;

(b) every vector in $\mathbb{R}^m$ is a linear combination of the columns of A;

(c) the columns of A span $\mathbb{R}^m$;

(d) A has a pivot position in every row.

$T:\mathbb{R}^n\to\mathbb{R}^m$

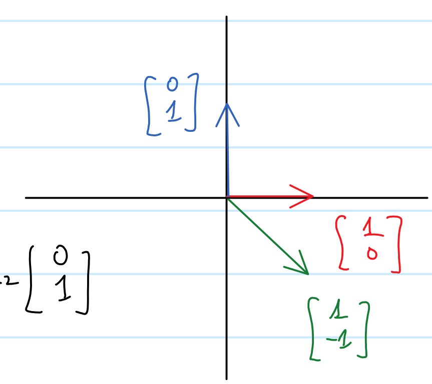
$T(\vec{x})=A\vec{x}$

Eg. Let $A=\begin{bmatrix}1&0&1\\0&1&-1\end{bmatrix}$. Is $A\vec{x}=\vec{b}$ consistent for any $\vec{b}\in\mathbb{R}^2$?

redundant

Sol'n: Yes, A has a pivot position in every row.

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}=x_1\begin{bmatrix}1\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\end{bmatrix}$$



Row-vector Rule

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Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Notice that

$$A\vec{x} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ a_{31}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \\ a_{32}x_2 \end{bmatrix} + \begin{bmatrix} a_{13}x_3 \\ a_{23}x_3 \\ a_{33}x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix}$$

Thm. If $\vec{y} = A\vec{x}$, then

$$y_i = \sum_{j=1}^{n} a_{ij} x_j .$$

(Row-vector rule for $A\vec{x}$)

E.g. Compute

(a)
$$\underset{2\times 3}{\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}} = \begin{bmatrix} 1(4) + 2(3) + (-1)7 \\ 0(4) + (-5)(3) + 3(7) \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

Called the "identity matrix of order three"

(b)
$$\underset{3\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Thm. If A is an $m\times n$ matrix, $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, then

(a) $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$; and

(b) $A(c\vec{u}) = c(A\vec{u})$.

Corollary: $A(c\vec{u} + d\vec{y}) = cA\vec{u} + dA\vec{y}$, for every $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$.

Pf: (a) If $A = [\vec{a}_1\ \vec{a}_2 \dots \vec{a}_n]$, then

$$A(\vec{u}+\vec{v}) = \sum_{i=1}^{n} (u_i + v_i)\vec{a}_i = \sum_{i=1}^{n} u_i\vec{a}_i + \sum_{i=1}^{n} v_i\vec{a}_i = A\vec{u} + A\vec{v}.$$

(b) Notice that

$$A(c\vec{u}) = \sum_{i=1}^{n} (cu_i)\vec{a}_i = \sum_{i=1}^{n} c(u_i\vec{a}_i) = c\sum_{i=1}^{n} u_i\vec{a}_i = cA\vec{u}.$$

Section 1.5

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* Solution Sets of LSs

Def'n: A LS of the form $\underbrace{A\vec{x}}_{}=\vec{0}$ is called a homogeneous LS or homogeneous.

interested in vectors that get mapped to the zero-vector

$\vec{0} \hookrightarrow \mathbb{R}^m$

Remarks: (i) Notice that $A(\underset{\mathbb{R}^n}{\vec{0}}) = \underset{\mathbb{R}^m}{\vec{0}}$ is always a solution and is called the trivial solution.

E.g. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(ii) Thus, it is important to determine whether $A\vec{x}=\vec{0}$ has nontrivial solutions, i.e., whether $A\vec{x}=\vec{0}$ has nonzero solutions.

E.g. If $A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$, determine whether $A\vec{x}=\vec{0}$ has nontrivial solutions and find all solutions.

Sol'n:

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{array}\right]$$

$$\sim \begin{bmatrix} 3 & 5 & -4 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 5 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Homogeneous LSs

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$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

All solutions are of the form

$$\vec{x} = \begin{bmatrix} 4/3\, x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}}_{= \vec{v}}, \quad x_3 \in \mathbb{R}.$$

The solution-set is $\text{Span}(\vec{v})$, with $\vec{v} = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$. It is also

$\text{Span}(\vec{u})$, with $\vec{u} = 3\vec{v} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$.

E.g. Solve $A\vec{x} = \vec{0}$, with $A = \begin{bmatrix} 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

Sol'n:

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

All solutions are of the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_2 - 2x_4 \\ x_2 \\ x_4 \\ x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2x_4 \\ 0 \\ x_4 \\ x_4 \\ 0 \end{bmatrix}$$

$$= x_2 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{= \vec{u}} + x_4 \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}}_{= \vec{v}}, \quad x_2, x_4 \in \mathbb{R}.$$

The solution-set is $\text{Span}(\vec{u}, \vec{v})$.

Q_3: §1.4