Lecture 5-B

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§1.4 (Continued)

Recall: If $A = [\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n]$ is an m-by-n matrix and $\vec{x} \in \mathbb{R}^n$, then

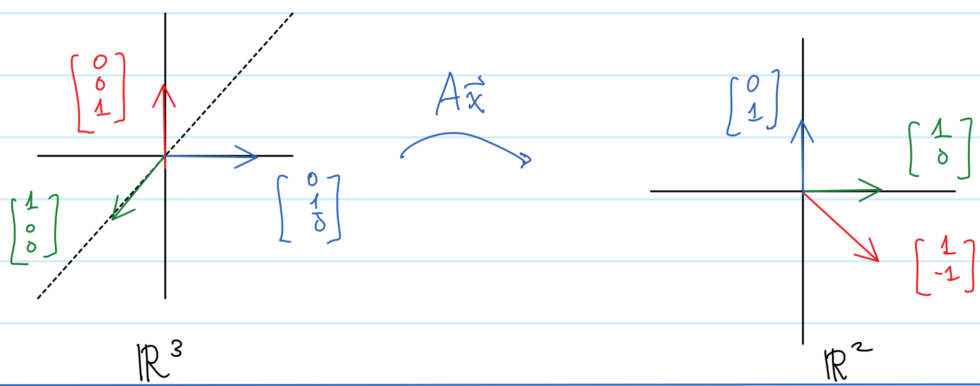
"$A\vec{x}$ is defined as.."

$$A\vec{x} := \sum_{i=1}^{n} x_i \vec{a}_i = x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n \in \mathbb{R}^m.$$

"defined as"

E.g. If $A = \underset{2\times 3}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}}$ and $\vec{x} = \underset{3\times 1}{\begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}}$, then

$$A\vec{x} = (-1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \in \mathbb{R}^2$$



Notation: If f is a function with domain X and co-domain Y, then we write this as $f: X \to Y$.

E.g. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Notice that the range of f is $[0, \infty)$, which differs from the co-domain.

$f(x) = ax$

Def'n: If A is an m-by-n matrix, then the function $T: \mathbb{R}^n \to \mathbb{R}^m$, defined by $T(\vec{x}) = A\vec{x}$, is called a matrix transformation.

Thm: The matrix equation $A\vec{x} = \vec{b}$ is equivalent to the vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n = \vec{b},$$

which, in turn, is equivalent to the LS whose augmented matrix is given by $[A \mid \vec{b}]$. Convention: We will refer to $A\vec{x} = \vec{b}$ as a LS.

Section 1.4 (continued)

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Remark: Solving the LS $A\vec{x}=\vec{b}$ is equivalent to determining whether $\vec{b}\in \text{Span}(\vec{a}_1,\ldots,\vec{a}_n)$.

"Determine whether $\vec{b}$ is a LC of $\vec{a}_1,\ldots,\vec{a}_n$"

or

"Determine whether $\vec{b}\in\text{Span}(\vec{a}_1,\ldots,\vec{a}_n)$"

or

"Solve $x_1\vec{a}_1+\cdots+x_n\vec{a}_n=\vec{b}$"

Solve $[A \mid \vec{b}]$

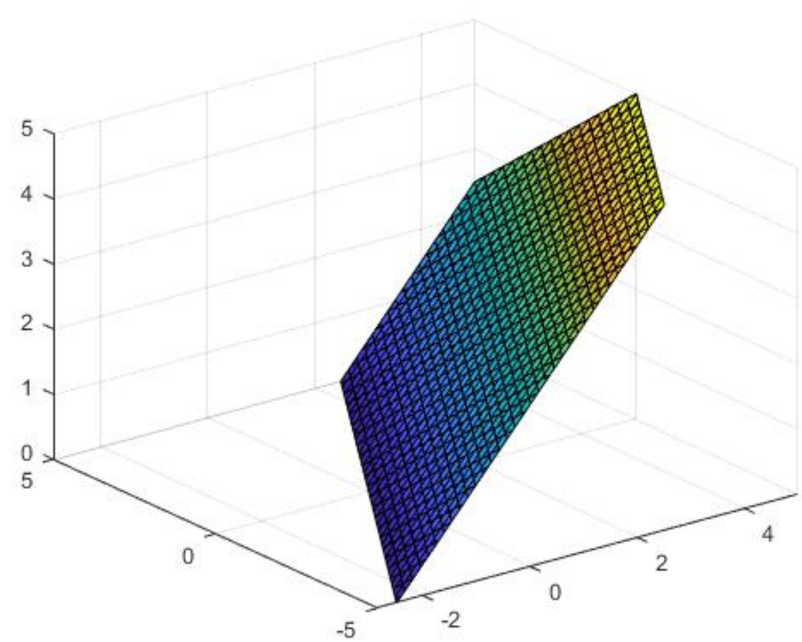
Eg. Solve

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1+b_2 \\ 0 & 7 & 5 & 3b_1+b_3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1+b_2 \\ 0 & 0 & 0 & 3b_1+b_3-\frac{1}{2}(4b_1+b_2) \end{array}\right]$$

Q: Does the original LS have a solution for every $b_1,b_2,b_3\in\mathbb{R}$?

A: Notice that $3b_1+b_3-\frac{1}{2}(4b_1+b_2)=b_1-\frac{1}{2}b_2+b_3$ and we can easily find values of b_1, b_2, and b_3 s.t. $b_1-\frac{1}{2}b_2+b_3\neq 0$.



A very important theorem

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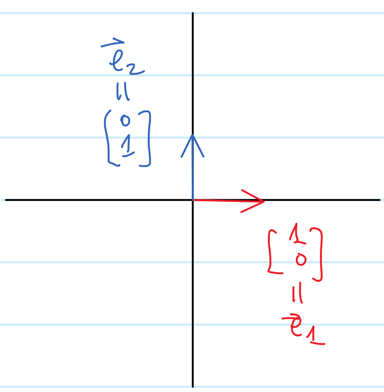
$$\left[\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right]$$

Remark: In the previous example, the LS $A\vec{x} = \vec{b}$ failed to be consistent for every $\vec{b} \in \mathbb{R}^3$ because an echelon form of A contained a row of zeros. Put differently, if every row of a matrix A is a pivot row, then $A\vec{x} = \vec{b}$ is always consistent.

Def'n: If A is an m-by-n matrix, then the columns of A are said to span $\mathbb{R}^m$ or are called a spanning-set for $\mathbb{R}^m$ if $\vec{b} \in \text{Span}(\vec{a}_1, .., \vec{a}_n)$ for every $\vec{b} \in \mathbb{R}^m$.



The set $\{\vec{e}_1, \vec{e}_2\}$ forms a spanning-set for $\mathbb{R}^2$.

Thm: If A is an m-by-n matrix, then the following statements are equivalent (they are all true or all false):

different ways of saying the same thing.

(a) the equation $A\vec{x} = \vec{b}$ is consistent for every $\vec{b} \in \mathbb{R}^m$;

(b) every vector in $\mathbb{R}^m$ is a linear combination of the columns of A;

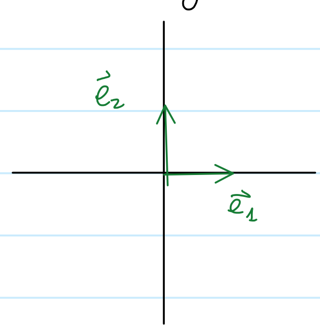
(c) the columns of A span $\mathbb{R}^m$; and

(d) A has a pivot position in every row (every row of A is a pivot row).

E.g. Let $A = \begin{array}{c} \begin{array}{ccc} \vec{e}_1 & \vec{e}_2 & \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \end{array}$. Is $A\vec{x} = \vec{b}$ consistent for every $\vec{b} \in \mathbb{R}^2$?

Sol'n: Yes; A has a pivot position in every row.

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{e}_1 - \vec{e}_2$$



Row-vector Rule

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Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Notice that

$$A\vec{x} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ a_{31}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \\ a_{32}x_2 \end{bmatrix} + \begin{bmatrix} a_{13}x_3 \\ a_{23}x_3 \\ a_{33}x_3 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix} \end{array} \begin{array}{c} \\ 2 \\ 3 \end{array}$$

Thm: If $\vec{y} = A\vec{x}$, A is an m-by-n matrix and $\vec{x} \in \mathbb{R}^n$, then

$$y_i = \sum_{j=1}^{n} a_{ij} x_j$$

(Row-vector rule for $A\vec{x}$)

E.g. Compute

(a) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 1(4) + 2(3) + (-1)(7) \\ 0(4) + (-5)(3) + 3(7) \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ → This matrix is called the identity matrix (of order 3).

Thm: If A is an m-by-n matrix, $\vec{u} \in \mathbb{R}^n$, $\vec{v} \in \mathbb{R}^n$, and $c \in \mathbb{R}$, then

(a) $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$; and $\quad T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

(b) $A(c\vec{u}) = cA\vec{u}$. $\quad T(c\vec{u}) = cT(\vec{u})$.

Pf: (a) If $A = [\vec{a}_1 \vec{a}_2 \dots \vec{a}_n]$, then

$$A(\vec{u}+\vec{v}) = \sum_{i=1}^{n} (u_i + v_i)\vec{a}_i = \sum_{i=1}^{n} (u_i\vec{a}_i + v_i\vec{a}_i) = \sum_{i=1}^{n} u_i\vec{a}_i + \sum_{i=1}^{n} v_i\vec{a}_i = A\vec{u} + A\vec{v}.$$

(b) Notice that

$$A(c\vec{u}) = \sum_{i=1}^{n} (cu_i)\vec{a}_i = \sum_{i=1}^{n} c(u_i\vec{a}_i) = c\sum_{i=1}^{n} u_i\vec{a}_i = cA\vec{u}.$$

Section 1.5

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* Solution-sets of LS's

Def'n: A LS of the form $\underset{\mathbb{R}^m}{\underset{\downarrow}{\underset{m\text{-by-}n}{\underset{\downarrow}{A}}\underset{\mathbb{R}^n}{\underset{\downarrow}{\vec{x}}} = \vec{0}}}$ is called a homogeneous LS or homogeneous.

Remarks: (i) Since $A(\underset{\mathbb{R}^n}{\underset{\downarrow}{\vec{0}}}) = \underset{\mathbb{R}^m}{\underset{\downarrow}{\vec{0}}}$, it follows that every homogeneous LS is consistent.

The solution $\vec{x} = \vec{0}$ is called the trivial solution.

E.g. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(ii) Of interest is to determine whether $A\vec{x} = \vec{0}$ has nontrivial solutions, i.e., solutions that are nonzero.

E.g. Let $A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$. Determine whether $A\vec{x} = \vec{0}$ has nontrivial solutions and, if so, find a general form for all solutions.

Sol'n: $\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right] \sim \begin{bmatrix} 3 & 5 & -4 \\ 0 & 3 & 0 \\ 0 & -9 & 0 \end{bmatrix}$

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \sim \begin{bmatrix} 3 & 5 & -4 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

x_3 ← free variable; nontrivial solutions present

$$\sim \begin{bmatrix} 3 & 5 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

All solutions are of the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/3\, x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}}_{= \vec{v}}, \quad x_3 \in \mathbb{R}.$$

The solution-set is span $(\vec{v})$, with $\vec{v} = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$. It is also

Span $(\vec{u})$, with $\vec{u} = 3\vec{v} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$.