1 The Magnetic Laplacian and its Spectrum

Definition 1. A *simple graph* is a set of vertices V and a set of oriented edges $E \subseteq V \times V$, where for all $(v, w) \in E$, we have $v \neq w$ and $(w, v) \in E$.

Definition 2. A *magnetic graph* is a simple graph (V, E) together with a *signature* function

$$\sigma : E \to \mathbb{C}_{|z|=1}$$

which has the property $\sigma(v, w) = \overline{\sigma(w, v)}$ for all $(v, w) \in E$. We may abbreviate $\sigma(v, w)$ as σ_{vw}.

We may sometimes treat a simple graph as a magnetic graph, in which case all signatures are assumed to be 1. Unless we explicitly state the vertex set of a graph, it is assumed to be $\{1, \dots, n\}$ for some n.

Definition 3. Suppose $G = (V, E)$ is a magnetic graph with vertices $\{1, \dots, n\}$. The *Laplacian* of G is the n x n matrix (l_{ij}) given by

$$l_{ij} = \begin{cases} d_i & \text{if } i = j \\ -\sigma_{ij} & \text{otherwise} \end{cases}$$

for all $i, j \in \{1 \dots n\}$.

Note that this "magnetic" definition for a Laplacian matches the familiar definition of a Laplacian when the graph is simple. The Laplacian always has a nice spectrum to analyze, as the following theorem shows.

Theorem 4. *If G is a magnetic graph with Laplacian L, then there is an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of L, and the eigenvalues of L are real.*

Proof. L is Hermitian because $L = L^*$. Therefore, the Complex Spectral Theorem gives this statement exactly. $\square$

We also want to establish that the eigenvalues of the Laplacian are non-negative. We will follow Jiang's paper (Theorem 3.5 and the preceding discussion), but modified slightly for magnetic graphs.

Definition 5. Let $G = (E, V)$ be a magnetic graph on n vertices with Laplacian L. For an edge (e, v) of G, the *contribution* of (e, v) to L is the n x n matrix (l_{ij}) given by

$$l_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } i \in \{e, v\} \\ -\sigma_{ij} & \text{if } \{i, j\} = \{e, v\} \\ 0 & \text{otherwise} \end{cases} .$$

We denote this matrix $L_{(e,v)}$.

As the name "contribution" suggests, the Laplacian of a graph is the sum of the contributions of its edges (only counting one edge between each pair of vertices):

Proposition 6. *If G is a magnetic graph with Laplacian L, then*

$$L = \sum_{\substack{(i,j) \in E \\ i<j}} L_{(i,j)}.$$

For the next few propositions, remember that if $z \in \mathbb{C}$, then $z \geq 0$ means z is real and nonnegative. Also, $\cdot$ is the complex dot product.

Proposition 7. *Let $G = (E, V)$ be a magnetic graph on n vertices with Laplacian L, and let $(i, j) \in E$. Then for any vector $v \in \mathbb{C}^n$, we have*

$$v \cdot L_{(i,j)} v \geq 0.$$

Proof. Let $v = (v_1, \ldots, v_n)$. By direct computation, the vector $L_{(i,j)}v$ consists of all 0s, except for $v_i - \sigma_{ij}v_j$ in the ith slot and $v_j - \sigma_{ji}v_i$ in the jth slot. Therefore,

$$\begin{aligned}
v \cdot L_{(i,j)}v &= v_i\overline{v_i - \sigma_{ij}v_j} + v_j\overline{v_j - \sigma_{ji}v_i} \\
&= v_i\left(\overline{v_i} - \overline{\sigma_{ij}v_j}\right) + v_j\left(\overline{v_j} - \overline{\sigma_{ji}v_i}\right) \\
&= |v_i|^2 - v_i\overline{\sigma_{ij}v_j} - v_j\overline{\sigma_{ji}v_i} + |v_j|^2 \\
&= |v_i|^2 - \sigma_{ji}v_i\overline{v_j} - \overline{\sigma_{ji}v_i\overline{v_j}} + |v_j|^2 \\
&= |v_i|^2 - \left(\sigma_{ji}v_i\overline{v_j} + \overline{\sigma_{ji}v_i\overline{v_j}}\right) + |v_j|^2 \\
&= |v_i|^2 - 2\mathrm{Re}\left(\sigma_{ji}v_i\overline{v_j}\right) + |v_j|^2 \\
&\geq |v_i|^2 - 2|\mathrm{Re}\left(\sigma_{ji}v_i\overline{v_j}\right)| + |v_j|^2 \\
&\geq |v_i|^2 - 2|\sigma_{ji}v_i\overline{v_j}| + |v_j|^2 \\
&= |v_i|^2 - 2|\sigma_{ji}||v_i||\overline{v_j}| + |v_j|^2 \\
&= |v_i|^2 - 2|\sigma_{ji}||v_i||\overline{v_j}| + |v_j|^2 \\
&= |v_i|^2 - 2|v_i||v_j| + |v_j|^2 \\
&= (|v_i| - |v_j|)^2 \\
&\geq 0.
\end{aligned}$$

□

Proposition 8. *Let $G = (E, V)$ be a magnetic graph on n vertices with Laplacian L. Then for any vector $v \in \mathbb{C}^n$, we have*

$$v \cdot Lv \geq 0.$$

Proof. We have:

$$v \cdot Lv = v \cdot \left(\sum_{\substack{(i,j)\in E \\ i<j}} L_{(i,j)}\right) v = v \cdot \sum_{\substack{(i,j)\in E \\ i<j}} L_{(i,j)}v = \sum_{\substack{(i,j)\in E \\ i<j}} \left(v \cdot L_{(i,j)}v\right),$$

where the last equality holds because inner products are additive in the second slot. By Proposition 7, this is a sum of nonnegative real numbers, so $v \cdot Lv \geq 0$. □

Theorem 9. *If G is a magnetic graph with Laplacian L, then the eigenvalues of L are nonnegative.*

Proof. Let λ be an eigenvalue of L, and let $v \in \mathbb{C}^n$ be a nonzero eigenvector corresponding to λ. Then,

$$\begin{aligned}
\lambda(v \cdot v) &= v \cdot \overline{\lambda}v \\
&= v \cdot \lambda v && \text{(since } \lambda \text{ is real by Theorem 4)} \\
&= v \cdot Lv \\
&\geq 0 && \text{(by Proposition 8).}
\end{aligned}$$

But v is nonzero, so $v \cdot v \geq 0$ by the definition of inner product. Thus we may conclude that $\lambda \geq 0$. $\square$

2 Renumbering Vertices

The Laplacian of a magnetic graph is only defined when the graph's vertices are numbered $1, \ldots, n$, and changing the numbering changes the Laplacian. However, we will show in this section that changing the vertex numbering has no effect on the spectrum of the Laplacian. This will allow us to unambiguously refer to the "spectrum of the Laplacian" of any magnetic graph, even if its vertices are not numbered or we don't know how they're numbered. With that goal in mind, we will first need a few facts about similar matrices.

Proposition 10. *Suppose A and B are similar $n \times n$ matrices over $\mathbb{C}$. Then A and B have the same nullity.*

Proof. Since A and B are similar, there exists an invertible matrix $P \in M_{n\times n}(\mathbb{C})$ such that

$$B = P^{-1}AP.$$

Let k be the nullity of A. Then there exists a basis $x_1, \ldots, x_k$ of $\ker A$, where each $x_i \in \mathbb{C}^n$. To show that k is also the nullity of B, we will show that

$$P^{-1}x_1, \ldots, P^{-1}x_k$$

is a basis for $\ker B$. First note that those vectors are linearly independent, since $x_1, \ldots, x_k$ are linearly independent, and an invertible linear map preserves linear independence. To see that $\operatorname{span}(P^{-1}x_1, \ldots, P^{-1}x_k) \subseteq \ker B$, observe that for any $i \in \{1, \ldots, k\}$, we have

$$\begin{aligned} B(P^{-1}x_i) &= (P^{-1}AP)P^{-1}x_i \\ &= P^{-1}Ax_i \\ &= P^{-1}(0) \\ &= 0, \end{aligned}$$

so $P^{-1}x_i \in \ker B$. It remains to show that $\ker B \subseteq \operatorname{span}(P^{-1}x_1, \ldots, P^{-1}x_k)$. To that end, let $v \in \ker B$. Then,

$$\begin{aligned} P^{-1}APv = Bv = 0 & \\ &\to APv = 0 \\ &\to Pv \in \ker A \\ &\to Pv = c_1x_1 + \cdots + c_kx_k && \text{for some } c_1, \ldots, c_k \in \mathbb{C} \\ &\to v = c_1P^{-1}x_1 + \cdots + c_kP^{-1}x_k && \text{for some } c_1, \ldots, c_k \in \mathbb{C} \\ &\to v \in \operatorname{span}(P^{-1}x_1, \ldots, P^{-1}x_k). \end{aligned}$$

$\square$

Proposition 11. *Suppose A and B are similar $n \times n$ matrices over $\mathbb{C}$. Then A and B have the same eigenvalues, counting multiplicity.*

Proof. Given $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$, we must show that λ is an eigenvalue of A with multiplicity k if and only if λ is an eigenvalue of B with multiplicity k. The directions are analogous, so we will only show the forward direction. Suppose λ is an eigenvalue of A with multiplicity k. Since A and B are similar, there exists an invertible matrix $P \in M_{n\times n}(\mathbb{C})$ such that

$$B = P^{-1}AP.$$

Then, we have

$$\begin{aligned}
\dim\ker(B - \lambda I) &= \dim\ker(P^{-1}AP - \lambda I) \\
&= \dim\ker(P^{-1}(A - \lambda I)P) \\
&= \dim\ker(A - \lambda I) && \text{(by Proposition 10)} \\
&= k.
\end{aligned}$$

Therefore, λ is an eigenvalue of B with multiplicity k. $\square$

Proposition 12. *Let ϕ be a permutation of $\{1, \dots, n\}$, and let $A, B \in M_{n\times n}(\mathbb{C})$ be related by*

$$A_{ij} = B_{\phi(i)\phi(j)}$$

for all $i, j \in \{1, \dots, n\}$. Then

$$A = P^{-1}BP$$

for some permutation matrix $P \in M_{n\times n}(\mathbb{C})$.

Proof. Define P by

$$P_{ij} = \begin{cases} 1 & \text{if } i = \phi(j) \\ 0 & \text{otherwise} \end{cases}.$$

Since P is a permutation matrix, its inverse is its transpose, so the elements of P^{-1} are given by

$$P^{-1}_{ij} = P_{ji} = \begin{cases} 1 & \text{if } j = \phi(i) \\ 0 & \text{otherwise} \end{cases}.$$

It can now be verified by computation that for all $i, j \in \{1, \dots, n\}$, we have

$$\left(P^{-1}BP\right)_{ij} = B_{\phi(i)\phi(j)} = A_{ij}.$$

$\square$

Finally we can prove the main theorem of this section:

Theorem 13. *Let G be a magnetic graph with vertices $\{1, \dots, n\}$. Then renumbering the vertices of G does not change the spectrum of its Laplacian.*

Proof. Let L be the Laplacian of G. Renumbering the vertices of G means selecting a permutation ϕ of $\{1, \dots, n\}$ and replacing each vertex v with $\phi(v)$. This results in a new Laplacian L', where clearly $L'_{ij} = L_{\phi^{-1}(i)\phi^{-1}(j)}$ for all i, j. Applying Proposition 12, we get that

$$L'_{ij} = P^{-1}LP$$

for some permutation matrix P. Finally, Proposition 11 gives that L' has the same spectrum as L. $\square$

As an example of the importance of Theorem 13, the following theorem would be confusing without it. In this theorem, we can assume that the vertices of G are the numbers $1, \dots, n$, so we know exactly what matrix is meant by the Laplacian L of G. However, if the vertices of G are indeed $1, \dots, n$, then it isn't possible for all the connected components to also have vertices in that format (only one of the components could get the vertex 1, for example). Therefore, the Laplacians $L_1, \dots, L_k$ are ambiguous. The key is to remember that every numbering for the vertices of a component gives a Laplacian with the same spectrum, so it doesn't matter which one we choose. $L_1, \dots, L_k$ are not really *the* Laplacians of $G_1, \dots, G_k$, but rather some Laplacians that result from an arbitrary choice of vertex numberings. And the theorem holds regardless of the choice.

It might simplify the numbering considerations if we define the *Laplacian spectrum* of a graph, which would be the unique spectrum resulting from every numbering of the vertices. The Laplacian spectrum would then be independent of the choice of numbering, meaning we could often ignore numberings entirely, or just choose them as we need them. But I'm not sure that's worth creating a new definition.

Theorem 14. *Let G be a magnetic graph with Laplacian L, and let $G_1, \dots G_k$ be the connected components of G, with Laplacians $L_1, \dots, L_k$. Then the spectrum of L is equal to the joined spectra of $L_1, \dots, L_k$.*

Proof. Renumber the vertices of G so that all the vertices of G_1 precede all the vertices of G_2, and so on. By Theorem 13, this doesn't affect the spectrum of L. But it makes L into a block diagonal matrix whose blocks $B_1, \dots, B_k$ are some Laplacians of $G_1, \dots, G_k$, respectively. Since $B_1, \dots, B_k$ are Laplacians of $G_1, \dots, G_k$, Theorem 13 says they have the same spectra as $L_1, \dots, L_k$. Therefore, it suffices to prove the spectrum of L is equal to the joined spectra of $B_1, \dots, B_k$.

Let $\lambda \in \mathbb{C}$, and let $m_1, \dots, m_k$ be the multiplicities of λ as an eigenvalue of $B_1, \dots, B_k$, respectively. We will be done if we can show that the multiplicity of λ as an eigenvalue of L is equal to $m_1 + \dots + m_k$. Indeed, the multiplicity of λ as an eigenvalue of L is

$$\begin{aligned}
\dim\ker(L - \lambda I) &= \dim\ker\left(\begin{pmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_k \end{pmatrix} - \lambda I\right) \\
&= \dim\ker\begin{pmatrix} B_1 - \lambda I & & & 0 \\ & B_2 - \lambda I & & \\ & & \ddots & \\ 0 & & & B_k - \lambda I \end{pmatrix} \\
&= \dim\ker(B_1 - \lambda I) + \dots + \dim\ker(B_k - \lambda I) \\
&= m_1 + \dots + m_k.
\end{aligned}$$

Here, the second to last equality holds because the nullity of a block diagonal matrix is equal to the sum of the nullities of the blocks (to see this, prove the analogous statement for rank, apply the rank nullity theorem, and simplify). □

3 Balanced Magnetic Graphs

Definition 15. A magnetic graph is called *balanced* if the signatures along any closed walk multiply to 1.

Proposition 16. *Let $G = (E, V, \sigma)$ be a magnetic graph. Then the following are equivalent.*

1. *G is balanced.*
2. *The signatures along any cycle multiply to 1 (in either direction).*
3. *For every pair of connected vertices $u, v \in V$, there exists $z \in \mathbb{C}_{|z|=1}$ such that along every walk from u to v, the signatures multiply to z.*

Because the number z in condition 3 is clearly unique for each pair of connected vertices, we can make the following definition.

Definition 17. Let $G = (E, V, \sigma)$ be a balanced magnetic graph, and let $u, v \in V$ be connected. Then the *signature distance* from u to v, written $\operatorname{sigdist}(u, v)$, is the product of the signatures along every walk from u to v.

4 Switching Equivalence

Definition 18. If $G = (V, E, \sigma)$ is a magnetic graph, then a function $c : V \to \mathbb{C}_{|z|=1}$ is called a *switching function* for G. We may write c_v as shorthand for $c(v)$.

Definition 19. Let $G = (V, E, \sigma)$ be a magnetic graph, and let c be a switching function for G. Then c *applied to* G is the magnetic graph with the same edges and vertices, whose signature function τ is given by

$$\tau_{vw} = \overline{c_v} \sigma_{vw} c_w$$

for all $v \sim w$.

Definition 20. A magnetic graph G is *switching equivalent* to a magnetic graph H if there exists a switching function c for G such that c applied to G gives H.

Proposition 21. *Switching equivalence is an equivalence relation.*

Proposition 22. *Let $G^\sigma = (V, E, \sigma)$ and $G^\tau = (V, E, \tau)$ be two magnetic graphs with the same vertices and edges. Then G^σ and G^τ are switching equivalent if and only if each connected component of G^σ is switching equivalent to the corresponding connected component of G^τ.*

Proof. ($\rightarrow$) Suppose G^σ and G^τ are switching equivalent. Then there is a switching function c for G^σ that, when applied to G^σ, gives G^τ. Now let $C^\sigma = (U, D, \sigma|_D)$ be an arbitrary connected component of G^σ. Then $C^\tau = (U, D, \tau|_D)$ is the corresponding connected component of G^τ. Choose b to be the restriction of c to U, and note that b is a switching function for C^σ. It remains to show that b applied to C^σ gives C^τ. Indeed, for any adjacent vertices $v, w \in U$, the signature of (v, w) given by b applied to C^σ is

$$\overline{b_v} (\sigma|_D)_{vw} b_w = \overline{c_v} \sigma_{vw} c_w = \tau_{vw} = (\tau|_D)_{vw},$$

which is the signature of (v, w) in C^τ.

($\leftarrow$) Suppose each connected component of G^σ is switching equivalent to the corresponding connected component of G^τ. Call the connected components $C_1, \ldots, C_k$. Then, for each $i \in \{1, \ldots, k\}$, let b^i be the switching function that takes the ith connected component of G^σ to the ith connected component of G^τ. Define a switching function c for G^σ by $c_v = (b^i)_v$, where i is such that v belongs to C_i. It is easy to verify that c applied to G^σ gives G^τ. ∎

We now work towards another characterization of switching equivalence. We want to show that two magnetic graphs with the same vertices and edges are switching equivalent if and only if the product of the signatures along any closed walk is the same for the two graphs. To make this easier, we will start with a lemma.

Lemma 23. *If $G = (V, E, \sigma)$ is a balanced magnetic graph, then G is switching equivalent to the simple graph on (V, E).*

Proof. Because of Proposition 22 and the fact that connected components of a balanced graph are also balanced, it suffices to prove the lemma in the case where V is connected. So assume V is connected. We must construct a switching function that takes every signature in G to 1. To do this, fix a vertex $u_0 \in V$ and define the switching function $c : V \to \mathbb{C}_{|z|=1}$ by

$$c_u = \overline{\operatorname{sigdist}(u_0, u)}$$

for all $u \in V$. Then for every pair of adjacent vertices $v, w \in V$, we have

$$\begin{aligned}\overline{c_v}\sigma_{vw}c_w &= \operatorname{sigdist}(u_0, v)\sigma_{vw}\overline{\operatorname{sigdist}(u_0, w)}\\ &= \operatorname{sigdist}(u_0, w)\overline{\operatorname{sigdist}(u_0, w)}\\ &= 1,\end{aligned}$$

as desired. □

Theorem 24. *Let $G^\sigma = (V, E, \sigma)$ and $G^\tau = (V, E, \tau)$ be two magnetic graphs with the same vertices and edges. Then G^σ and G^τ are switching equivalent if and only if along every closed walk, the product of the signatures of G^σ equals the product of the signatures of G^τ.*

Proof. ($\to$) Suppose G^σ and G^τ are switching equivalent. Then there is a switching function $c : V \to \mathbb{C}_{|z|=1}$ that takes G^σ to G^τ. Let $v_1, \ldots, v_k$ be the vertices along a closed walk, so that $v_1 = v_k$. Then,

$$\begin{aligned}\sigma_{v_1v_2}\sigma_{v_2v_3}\ldots\sigma_{v_{k-1}v_k} &= \overline{c_{v_1}}\sigma_{v_1v_2}\sigma_{v_2v_3}\ldots\sigma_{v_{k-1}v_k}c_{v_1}\\ &= \overline{c_{v_1}}\sigma_{v_1v_2}\sigma_{v_2v_3}\ldots\sigma_{v_{k-1}v_k}c_{v_k}\\ &= (\overline{c_{v_1}}\sigma_{v_1v_2}c_{v_2})(\overline{c_{v_2}}\sigma_{v_2v_3}c_{v_3})\ldots(\overline{c_{v_{k-1}}}\sigma_{v_{k-1}v_k}c_{v_k})\\ &= \tau_{v_1v_2}\tau_{v_2v_3}\ldots\tau_{v_{k-1}v_k}.\end{aligned}$$

Therefore, along every closed walk, the product of the signatures of G^σ equals the product of the signatures of G^τ.

($\leftarrow$) Suppose that along every closed walk, the product of the signatures of G^σ equals the product of the signatures of G^τ. Define a new graph $G^s = (V, E, s)$, where the signatures s are given by

$$s_{vw} = \overline{\tau_{vw}}\sigma_{vw}$$

for all vertices $v \sim w$. It is easy to see that along any closed walk, the signatures of G^s multiply to 1. Therefore, G^s is balanced. By Lemma 23, it follows that G^s is switching equivalent to the simple graph on (V, E), whose signatures are all 1. Call the switching function that takes G^s to the simple graph c. Then for any vertices $v \sim w$, we have

$$\begin{aligned}\overline{c_v}s_{vw}c_w = 1 &\to \overline{c_v}\left(\overline{\tau_{vw}}\sigma_{vw}\right)c_w = 1\\ &\to \overline{c_v}\sigma_{vw}c_w = \tau_{vw}.\end{aligned}$$

Therefore, applying the switching function c to G^σ gives G^τ, so G^σ and G^τ are switching equivalent. □

Many nice consequences flow from this characterization of switching equivalence.

Corollary 25. *A magnetic graph is balanced if and only if it is switching equivalent to the simple graph with the same vertices and edges.*

Corollary 26. *If a magnetic graph is switching equivalent to a balanced magnetic graph, then it is also balanced.*

Corollary 27. *Any two balanced magnetic graphs with the same vertices and edges are switching equivalent.*

Corollary 28. *Every acyclic magnetic graph is switching equivalent to the simple graph.*

Corollary 29. *Every magnetic cycle graph is switching equivalent to a graph which differs from the simple graph by at most one edge.*

The next theorem is why we care about switching equivalence in the context of spectral graph theory.

Theorem 30. *Let $G^\sigma = (V, E, \sigma)$ and $G^\tau = (V, E, \tau)$ be magnetic graphs with the same vertices and edges. Then if G^σ and G^τ are switching equivalent, it follows that their Laplacians have the same spectrum.*

Proof. Let n be the number of vertices in V. Let L^σ and L^τ be the Laplacians for G^σ and G^τ. Let c be a switching function that takes G^σ to G^τ. Then,

$$\begin{aligned} L^\tau &= \operatorname{diag}(\overline{c_1}, \ldots, \overline{c_n})\,(L^\sigma)\,\operatorname{diag}(c_1, \ldots, c_n) \\ &= \operatorname{diag}(c_1, \ldots, c_n)^{-1}\,(L^\sigma)\,\operatorname{diag}(c_1, \ldots, c_n). \end{aligned}$$

We see that L^σ and L^τ are similar matrices. Therefore, by Proposition 11, they have the same spectrum. □

The converse of Theorem 30 doesn't hold. For example, the 3-cycle graph with signatures 1, 1, and i (going in order around the cycle) has the same spectrum as the 3-cycle graph with signatures 1, 1, and $-i$. However, the product along the cycles is different, so by Theorem 24, the graphs cannot be switching equivalent.

5 The Eigenvalue 0 of the Laplacian

If G is a magnetic graph with vertices $\{1, \ldots, n\}$ and x is a vector in $\mathbb{C}^n$, then it makes sense to imagine x as consisting of values for the vertices of G. Specifically, the entry x_i is like a value for the vertex i. With this interpretation, multiplying x by the Laplacian of G creates a new list of vertex values by taking linear combinations of the current vertex values. The weights of the combinations are determined by the structure of the Laplacian. In fact, the next proposition shows that a vertex's new value after the multiplication depends only on its current value and the values of its neighbors.

Proposition 31. *Let G be a magnetic graph on n vertices with Laplacian L. Let $x \in \mathbb{C}^n$ with $x = (x_1, \ldots, x_n)$. Then*

$$(Lx)_i = d_i x_i - \sum_{j \sim i} \sigma_{ij} x_j$$

for each $i \in \{1, \ldots, n\}$.

Proof. Let l_{ij} denote the element of L at row i and column j. Let $i \in \{1, \ldots, n\}$ be given. Then,

$$\begin{aligned}(Lx)_i &= \sum_{j=1}^{n} l_{ij} x_j \\ &= l_{ii} x_i + \sum_{j \neq i} l_{ij} x_j \\ &= d_i x_i + \sum_{j \neq i} (-\sigma_{ij}) x_j \\ &= d_i x_i - \sum_{j \neq i} \sigma_{ij} x_j.\end{aligned}$$

□

We are particularly interested in when the "vertex values" all go to 0 under multiplication by the Laplacian. It turns out that this happens exactly when each value is a signature-weighted average of the neighboring values, as the next lemma demonstrates.

Lemma 32. *Let G be a magnetic graph on n vertices with Laplacian L. Suppose $x \in \mathbb{C}^n$ with $x = (x_1, \ldots, x_n)$. Then $Lx = 0$ if and only if*

$$x_i = \frac{1}{d_i} \sum_{j \sim i} \sigma_{ij} x_j$$

for each $i \in \{1, \ldots n\}$ with $d_i \neq 0$.

Proof. This follows easily from Proposition 31. □

We now show that this condition is equivalent to a much simpler one.

Theorem 33. *Let G be a magnetic graph on n vertices with Laplacian L. Suppose $x \in \mathbb{C}^n$ with $x = (x_1, \ldots, x_n)$. Then $Lx = 0$ if and only if*

$$x_i = \sigma_{ij} x_j$$

whenever $i \sim j$.

Proof. ($\leftarrow$) Suppose $x_i = \sigma_{ij} x_j$ whenever $i \sim j$. We will show that $Lx = 0$ using the condition from Lemma 32. Fix $i \in \{1, \ldots, n\}$ with $d_i \neq 0$. Then,

$$x_i = \frac{1}{d_i} d_i x_i = \frac{1}{d_i} \sum_{j \sim i} x_i = \frac{1}{d_i} \sum_{j \sim i} \sigma_{ij} x_j.$$

($\rightarrow$) Suppose $Lx = 0$. We first argue that whenever a and b are vertices in the same connected component of G, we have $|x_a| = |x_b|$. To see this, take an arbitrary connected component of G, and choose a vertex m in that component which maximizes $|x_m|$. If $d_m = 0$, then m is the only vertex in the component, so our conclusion holds trivially. On the other hand, if $d_m \neq 0$, then we can apply Lemma 32 to get

$$\begin{aligned}
|x_m| &= \left| \frac{1}{d_m} \sum_{l \sim m} \sigma_{ml} x_l \right| \\
&= \frac{1}{d_m} \left| \sum_{l \sim m} \sigma_{ml} x_l \right| \\
&\leq \frac{1}{d_m} \sum_{l \sim m} |\sigma_{ml} x_l| && \text{(by the Triangle Inequality)} \\
&= \frac{1}{d_m} \sum_{l \sim m} |x_l|.
\end{aligned}$$

That is, $|x_m|$ is the mean of the values $|x_l|$ where $l \sim m$. But our choice of m ensures that $|x_m| \geq |x_l|$ for each $l \sim m$, so in fact $|x_m| = |x_l|$ for each $l \sim m$. It follows that any vertex l which is adjacent to m also maximizes the quantity $|x_l|$, so we can apply the same argument to each l, repeating the process until we conclude that $|x_a| = |x_b|$ for every pair of vertices a, b in the connected component.

Now let i and j be vertices of G with $i \sim j$. We want to prove that $x_i = \sigma_{ij} x_j$. From the previous paragraph, we already have that

$$|x_i| = |x_j| = |\sigma_{ij} x_j|,$$

so it suffices to show x_i differs from $\sigma_{ij} x_j$ by a positive real factor. Observe that

$$\begin{aligned}
\left| \sum_{l \sim i} \sigma_{il} x_l \right| &= |d_i x_i| && \text{(by Lemma 32)} \\
&= d_i |x_i| \\
&= \sum_{l \sim i} |x_i| \\
&= \sum_{l \sim i} |x_l| && \text{(by the previous paragraph)} \\
&= \sum_{l \sim i} |\sigma_{il} x_l|.
\end{aligned}$$

Therefore, assuming $\sigma_{ij} x_j \neq 0$ (the other case is trivial), we may conclude that

$$\sum_{l \sim i} \sigma_{il} x_l = r(\sigma_{ij} x_j)$$

for some positive real r. Then,

$$\begin{aligned}
x_i &= \frac{1}{d_i} \sum_{l \sim i} \sigma_{il} x_l && \text{(by Lemma 32)} \\
&= \frac{1}{d_i} r(\sigma_{ij} x_j),
\end{aligned}$$

so x_i differs from $\sigma_{ij} x_j$ by a positive real factor, as desired. □

Corollary 34. *Let G be a simple graph on n vertices with Laplacian L. Suppose $x \in \mathbb{C}^n$ with $x = (x_1, \ldots, x_n)$. Then $Lx = 0$ if and only if*

$$x_i = x_j$$

whenever $i \sim j$.

Proof. This follows from Theorem 33 and the fact that every signature in a simple graph is 1. □

In a way, Theorem 33 completely answers the question of when 0 is an eigenvalue of the Laplacian. 0 is an eigenvalue exactly when there is a nonzero vector x satisfying the condition in Theorem 33. But that condition is phrased in terms of local properties of the graph. We want to find global properties that determine whether 0 is an eigenvalue, and if so, reveal its multiplicity. The final few results of the section achieve that.

Proposition 35. *If G is a connected and balanced magnetic graph, then 0 is an eigenvalue of its Laplacian with multiplicity 1.*

Proof. Since G is balanced, Corollary 25 gives that it is switching equivalent to the simple graph on the same edges and vertices. By Theorem 30, the spectrum of that simple graph is the same as the spectrum of G. So we only have to show that 0 is an eigenvalue with multiplicity 1 for the Laplacian of a connected, simple graph. Indeed, it is clear from Corollary 34 that the kernel of such a Laplacian is exactly the one-dimensional space of vectors whose entries are all the same. □

Proposition 36. *If G is a connected and unbalanced magnetic graph, then 0 is not an eigenvalue of its Laplacian.*

Proof. Let n be the number of vertices in G, and let $x = (x_1, \ldots, x_n) \in \ker L$. We will be done if we can show that x must be 0. Since G is unbalanced, it has a closed walk along which the signatures multiply to some $c \neq 1$. Let k be the starting and ending vertex of the walk. Repeated application of Theorem 33 along that walk gives that $x_k = cx_k$, so $x_k = 0$. Then applying Theorem 33 outward from k, we get $x_j = 0$ for every $j \in \{1, \ldots, n\}$. This means $x = 0$, as desired. □

Theorem 37. *Let G be a magnetic graph with Laplacian L. Then the multiplicity of 0 as an eigenvalue of L is equal to the number of balanced connected components of G.*

Proof. By Theorem 14, it suffices to sum the multiplicities of 0 as an eigenvalue for G's connected components. By Proposition 35, the balanced components each contribute 1 multiplicity, and by Proposition 36, the unbalanced components each contribute 0 multiplicity. Therefore, the total multiplicity is the number of balanced components. □

Corollary 38. *The Laplacian of a simple graph always has 0 as an eigenvalue, and its multiplicity is equal to the number of connected components in the graph.*

Proof. A simple graph is balanced because all its signatures are 1. Thus all its connected components are balanced. Since the multiplicity of 0 is equal to the number of balanced components, in this case it is just equal to the total number of components. □

6 The Frustration Index

The frustration index of a magnetic graph is a way to measure how close the graph is to being balanced.

Definition 39. Let $G = (V, E, \sigma)$ be a magnetic graph. Then the frustration index of G is

$$\min_{c: V \to \mathbb{C}_{|z|=1}} \sum_{i \sim j} |\overline{c_i} \sigma_{ij} c_j - 1|.$$

Note that a balanced graph has frustration index 0 (since it is switching equivalent to a graph whose signatures are all 1). Cycle graphs also have a nice frustration index, as we will show next. For this we need a lemma.

Lemma 40. *If $z_1, \dots, z_n \in \mathbb{C}_{|z|=1}$, then*

$$|z_1 z_2 \dots z_n - 1| \leq |z_1 - 1| + \dots + |z_n - 1|.$$

Proof. We use induction on n. The claim is clearly true in the base case $n = 1$. Now let $k \in \mathbb{N}$, and assume the claim holds for $n = k$. Then,

$$\begin{aligned} |z_1 z_2 \dots z_{k+1} - 1| &= |z_1 z_2 \dots z_{k+1} - z_{k+1} + z_{k+1} - 1| \\ &\leq |z_1 z_2 \dots z_{k+1} - z_{k+1}| + |z_{k+1} - 1| \\ &= |z_{k+1}(z_1 z_2 \dots z_k - 1)| + |z_{k+1} - 1| \\ &= |z_{k+1}||z_1 z_2 \dots z_k - 1| + |z_{k+1} - 1| \\ &= |z_1 z_2 \dots z_k - 1| + |z_{k+1} - 1| \\ &\leq |z_1 - 1| + \dots + |z_k - 1| + |z_{k+1} - 1|. \end{aligned}$$

So the claim also holds for $n = k + 1$, and we are done. □

Proposition 41. *Let $G = (V, E, \sigma)$ be a cycle graph on the vertices $\{1, \dots, n\}$. Let p be the product of the signatures along the cycle in one direction. Then the frustration index of G is $2|p - 1|$.*

Proof. Without loss of generality, let

$$p = \sigma_{12} \sigma_{23} \dots \sigma_{n1}.$$

By Theorem 24, G is switching equivalent to the graph with signatures τ, where

$$\begin{aligned} &\tau_{12} = \tau_{23} = \dots = \tau_{(n-1)n} = 1, \text{ and} \\ &\tau_{n1} = p. \end{aligned}$$

Let b be a switching function which takes G's signatures to τ. Then,

$$\begin{aligned} \sum_{i \sim j} \left|\overline{b_i} \sigma_{ij} b_j - 1\right| &= \sum_{i \sim j} |\tau_{ij} - 1| \\ &= |\tau_{n1} - 1| + |\tau_{1n} - 1| \qquad \text{(eliminating 0 terms)} \\ &= |p - 1| + |\overline{p} - 1| \\ &= |p - 1| + |p - 1| \\ &= 2|p - 1|. \end{aligned}$$

So we have shown that there exists a switching function b for G with

$$\sum_{i\sim j}\left|\overline{b_i}\sigma_{ij}b_j - 1\right| = 2|p-1|.$$

To finish the proof, we must show that whenever c is a switching function for G, we have

$$\sum_{i\sim j}|\overline{c_i}\sigma_{ij}c_j - 1| \geq 2|p-1|.$$

To see this, let c be an arbitrary switching function for G, and let s be the new signatures obtained by applying c to G. Then,

$$\begin{aligned}
\sum_{i\sim j}|\overline{c_i}\sigma_{ij}c_j - 1| &= \sum_{i\sim j}|s_{ij} - 1| \\
&= 2\left(|s_{12}-1| + |s_{23}-1| + \cdots + |s_{n1}-1|\right) \\
&\geq 2|s_{12}s_{23}\dots s_{n1} - 1| && \text{(Lemma 40)} \\
&= 2|\sigma_{12}\sigma_{23}\dots\sigma_{n1} - 1| && \text{(switching preserves products along cycles)} \\
&= 2|p-1|.
\end{aligned}$$

□