Magnetic Versions of PageRank, Hitting Time, and Effective Resistance

Ryan Chimienti

April 2020

Abstract

Classical spectral graph theory studies the relationship between the properties of a simple graph and the eigenvalues of its graph Laplacian. It also deals with other graph quantities like PageRank, hitting time, and effective resistance. In this paper, we explore how these ideas extend to magnetic graphs (graphs where each edge is assigned a complex number of modulus 1). We begin by surveying some generally useful ideas for understanding magnetic graphs: switching equivalence and balance. Then we give some known results about the magnetic version of the graph Laplacian. Finally, equipped with these tools, we are able to introduce and analyze magnetic versions of PageRank, hitting time, and effective resistance.

1 Introduction

A *simple graph* is a finite set of vertices $V \neq \varnothing$ and a set of undirected edges E, where each edge has the form $\{u, v\}$ for some distinct vertices u and v. We will use d_v to mean the degree of a vertex v and $u \sim v$ to mean the vertices u and v are connected by an edge. It will sometimes be convenient to think of an edge as going in a particular direction, so we implicitly associate each graph with an oriented edge set $E^{or} := \{(u, v), (v, u) : \{u, v\} \in E\}$. A *magnetic graph* is a simple graph (V, E) together with a function $\sigma : E^{or} \to S^1 := \{z \in \mathbb{C} : |z| = 1\}$ which has the property $\sigma(u, v) = \overline{\sigma(v, u)}$ for all $u \sim v$. We will usually abbreviate $\sigma(u, v)$ as σ_{uv}. The whole function σ is called the *signature* of the graph, or we can refer to the signature of an individual oriented edge (u, v), by which we mean σ_{uv}.

Magnetic graphs arise naturally in the study of quantum mechanics on a graph, as seen in Lieb and Loss [10]. In that paper, the signature of a magnetic graph is used to capture the properties of a magnetic field, which explains the name "magnetic." With an eye to similar physical implications, Dodziuk and Mathai [6] , Shubin [12], and Bédos [1] study (variants of) a linear operator called the discrete magnetic Laplacian. That operator is analogous to (and in Dodziuk and Mathai exactly the same as) the magnetic version of the graph Laplacian that we will study here.

Magnetic graphs have also been studied at different levels of generality. Lange et al. [9] consider a more general version of our magnetic graphs, which have positive real edge weights (in addition to a signature) and positive real vertex values. When the edge weights and vertex values are all taken to be 1, their graphs are equivalent to ours. Elsewhere in the literature, signatures take values on an arbitrary group instead of S^1, with the requirement that the signature of an edge in one direction be the inverse of the signature in the other direction. Graphs with such signatures are called voltage graphs, or gain graphs. As voltage graphs generalize magnetic graphs, so do magnetic graphs generalize signed graphs, whose signatures take values on $\{-1, 1\}$. Many of the results on balance and switching we will list for magnetic graphs were given by Harary [7] or Zaslavsky [13] in the context of signed graphs, with Zaslavsky observing that they would generalize to voltage graphs.

The specific idea of extending important graph quantities like PageRank to the magnetic context has also been considered. Chung et al. [5] arrive at the same definition as us for the PageRank of magnetic graphs, but a different definition for effective resistance. In that paper, graph edges are weighted with rotations of arbitrary dimension, which of course match our magnetic signatures when the dimension is 2. The authors were motivated partially by applications in image processing and electron cryomicroscopy.

2 Balance and Switching

One of the most convenient properties a magnetic graph can have is called balance.

Definition 1. A magnetic graph is *balanced* if the signatures along every (directed) cycle multiply to 1.

We can characterize this idea a few different ways.

Proposition 2. *Let $G = (V, E, \sigma)$ be a magnetic graph. Then the following are equivalent.*

1. *G is balanced.*
2. *The signatures along every closed walk multiply to 1.*
3. *For every pair of connected vertices $u, v \in V$, there exists $z \in S^1$ such that along every walk from u to v, the signatures multiply to z.*

Proof. $(1 \to 2)$ Suppose G is balanced, so that the product of the signatures along any cycle is 1. We will show that the signatures along every closed walk multiply to 1 using induction on the length of the walk. In the base case, a walk of length 0 vacuously has product of signatures 1. Now let $k > 0$, and assume all closed walks of length less than k have signature product 1. Consider an arbitrary closed walk of length k. Let w be the first vertex that appears twice in the walk. Then the portion of the walk from the first occurrence of w to the second occurrence of w can't have any duplicate vertices other than w. In fact, this makes that portion a cycle. By assumption the cycle has signature product 1, and deleting the cycle from the walk leaves a shorter closed walk, which by the induction hypothesis also has signature product 1. Thus, the total product of the signatures along the walk is $1 \cdot 1 = 1$, which completes the induction.

$(2 \to 3)$ Suppose the signatures along every closed walk multiply to 1. Let u, v be an arbitrary pair of connected vertices. Then there is a walk p from u to v. Let $z \in S^1$ be the product of the signatures along p. Now consider an arbitrary walk q from u to v, with signature product $y \in S^1$. Then we may form a closed walk by joining p with the reverse of q. The signature product along that closed walk is $z\overline{y}$, and since the walk is closed we have $z\overline{y} = 1$. Thus $y = z$, so the product of the signatures along q is z.

$(3 \to 1)$ This implication follows from a similar argument to the previous one. □

Because the number z in condition 3 is clearly unique for each pair of connected vertices, we can make the following definition.

Definition 3. Let $G = (V, E, \sigma)$ be a balanced magnetic graph, and let $u, v \in V$ be connected. Then the *signature distance* from u to v, written $\operatorname{sigdist}(u, v)$, is the product of the signatures along every walk from u to v.

As far as I know, the term "signature distance" is new to this paper, but certainly the idea is well-known. We call it a distance because, intuitively, it is what one must travel through to get from one vertex to another. Now we turn our attention to switching, which is a natural transformation that can be applied to magnetic graphs.

Definition 4. If $G = (V, E, \sigma)$ is a magnetic graph, then a function $c : V \to S^1$ is called a *switching function* for G. We may write c_v as shorthand for $c(v)$.

Definition 5. Let $G = (V, E, \sigma)$ be a magnetic graph, and let c be a switching function for G. Then *G switched by c* is the magnetic graph with the same edges and vertices, whose signature τ is given by

$$\tau_{vw} = \overline{c_v}\sigma_{vw}c_w$$

for all $v \sim w$.

Definition 6. A magnetic graph G is *switching equivalent* to a magnetic graph H if there exists a switching function c for G such that G switched by c is H.

Proposition 7. *Switching equivalence is an equivalence relation.*

The following proposition means that we can often work with connected magnetic graphs, and the results will automatically extend to disconnected magnetic graphs.

Proposition 8. *Let $G^\sigma = (V, E, \sigma)$ and $G^\tau = (V, E, \tau)$ be two magnetic graphs with the same vertices and edges. Then G^σ and G^τ are switching equivalent if and only if each connected component of G^σ is switching equivalent to the corresponding connected component of G^τ.*

Now, we give a key fact about switching equivalence. This proposition was proved by Zaslavsky [13] in the context of signed graphs, essentially the same way we prove it here. It was also noted to extend to magnetic graphs in [11]. We nevertheless include a proof because the approach is enlightening, specifically in the $(\leftarrow)$ direction.

Proposition 9. *Let $G^\sigma = (V, E, \sigma)$ and $G^\tau = (V, E, \tau)$ be two magnetic graphs with the same vertices and edges. Then G^σ and G^τ are switching equivalent if and only if along every cycle, the product of the signatures of G^σ equals the product of the signatures of G^τ.*

Proof. ($\to$) Suppose G^σ and G^τ are switching equivalent. Then there is a switching function $c : V \to S^1$ that switches G^σ to G^τ. Let $v_1, \ldots, v_k, v_1$ be the consecutive vertices along a cycle. Then,

$$\begin{aligned}\sigma_{v_1v_2}\sigma_{v_2v_3}\ldots\sigma_{v_kv_1} &= (\overline{c_{v_1}}\sigma_{v_1v_2}c_{v_2})(\overline{c_{v_2}}\sigma_{v_2v_3}c_{v_3})\ldots(\overline{c_{v_k}}\sigma_{v_kv_1}c_{v_1})\\ &= \tau_{v_1v_2}\tau_{v_2v_3}\ldots\tau_{v_kv_1}.\end{aligned}$$

Therefore, along every cycle, the product of the signatures of G^σ equals the product of the signatures of G^τ.

($\leftarrow$) Suppose that along every cycle, the product of the signatures of G^σ equals the product of the signatures of G^τ. Because of Proposition 8, we may assume the graph (V, E) is connected. Select a rooted spanning tree S. For G^σ, begin at the root of S and work outwards to choose a switching function that sends all of the signatures in S to 1. Now obtain a switching function that does the same for G^τ. We argue that the switched versions of G^σ and G^τ are identical, which will complete the proof that the original graphs were switching equivalent. Clearly the edges in S are identical, since they all have signature 1. It remains to consider an edge $e \notin S$. That edge forms a cycle with the edges of S. The cycle had the same signature product p in G^σ and G^τ prior to switching, and by the ($\to$) direction, switching preserves products along cycles. Therefore, the cycle still has product p in the switched versions of G^σ and G^τ. It follows that the signature of e equals p in both the switched version of G^σ and the switched version of G^τ, so we are done. ☐

We note a few nice consequences of this characterization of switching equivalence:

Corollary 10. *A magnetic graph is balanced if and only if it is switching equivalent to the graph with all signatures 1.*

Corollary 11. *If a magnetic graph is switching equivalent to a balanced magnetic graph, then it is also balanced.*

Corollary 12. *Any two balanced magnetic graphs with the same vertices and edges are switching equivalent.*

3 The Magnetic Laplacian and its Spectrum

Given a magnetic graph (V, E, σ), its *discrete magnetic Laplacian* is a linear operator L on the space of functions $f : V \to \mathbb{C}$, where $(Lf)(u) := \sum_{v\sim u}(f(u) - \sigma_{uv}f(v))$. For simplicity, we may call L simply the Laplacian. We can also view L as a matrix. Working in terms of the standard basis and indexing by the vertices, we get the matrix

$$L(u, v) := \begin{cases} d_u & \text{if } u = v \\ -\sigma_{uv} & \text{if } u \sim v \\ 0 & \text{otherwise} \end{cases}.$$

In the case where all the signatures are 1, this matches the classical definition of the graph Laplacian for the underlying simple graph. Thus, the results we give about magnetic Laplacians will reduce to results about classical Laplacians when the graph has all signatures 1. We will start by showing that the magnetic Laplacian is positive-semidefinite. Recall the definition:

Definition 13. Let B be a linear operator on a complex inner product space W. Then B is *positive-semidefinite* if it is Hermitian and

$$\langle Bf, f\rangle \geq 0$$

for every $f \in W$. (Note that $\langle Bf, f\rangle$ is automatically real since B is Hermitian.)

In this paper, the inner product space is always the set of functions from a vertex set V into $\mathbb{C}$, and the inner product is always defined by $\langle f, g\rangle := \sum_{u\in V} f(u)\overline{g(u)}$. We want the magnetic Laplacian to be positive-semidefinite because then it will have nice eigenfunctions and eigenvalues, as the following proposition shows.

Proposition 14. *If B is a positive-semidefinite operator on a complex inner product space W, then there exists an orthonormal basis for W made up of eigenfunctions of B, and the eigenvalues are real and nonnegative.*

The next proof is modeled after Jiang's argument for the non-magnetic case (see pages 3 and 4 in [8]).

Proposition 15. *Let L be the magnetic Laplacian of a magnetic graph (V, E, σ). Then L is positive-semidefinite.*

Proof. From the matrix representation of L, we can see that $L = L^*$, so L is Hermitian. Now for each edge $\{u, v\} \in E$, we define an operator $L_{\{u,v\}}$ as follows. For every function function $f : V \to \mathbb{C}$, we let $L_{\{u,v\}}f$ take the value 0 at every vertex except u and v. But at u and v we let

$$\begin{aligned}(L_{\{u,v\}}f)(u) &:= f(u) - \sigma_{uv} f(v)\\ (L_{\{u,v\}}f)(v) &:= f(v) - \sigma_{vu} f(u).\end{aligned}$$

With these operators defined, let $f : V \to \mathbb{C}$ be arbitrary. Then we have

$$\langle Lf, f\rangle = \left\langle \sum_{\{u,v\}\in E} (L_{\{u,v\}}f), f \right\rangle = \sum_{\{u,v\}\in E} \left\langle L_{\{u,v\}}f, f\right\rangle.$$

So to show L is positive-semidefinite, it suffices to show $\left\langle L_{\{u,v\}}f, f\right\rangle$ is real and nonnegative for each $\{u, v\} \in E$. Indeed, we have

$$\begin{aligned}\left\langle L_{\{u,v\}}f, f\right\rangle &= (L_{\{u,v\}}f)(u)\overline{f(u)} + (L_{\{u,v\}}f)(v)\overline{f(v)}\\ &= (f(u) - \sigma_{uv}f(v))\,\overline{f(u)} + (f(v) - \sigma_{vu}f(u))\,\overline{f(v)}\\ &= (f(u) - \sigma_{uv}f(v))\overline{(f(u) - \sigma_{uv}f(v))}\\ &= |f(u) - \sigma_{uv}f(v)|^2.\end{aligned}$$

□

We have now established that the magnetic Laplacian has a nice spectrum of positive real eigenvalues. In fact, the spectrum doesn't change when switching functions are applied to the graph.

Proposition 16. *Let $G^\sigma = (V, E, \sigma)$ and $G^\tau = (V, E, \tau)$ be magnetic graphs with the same vertices and edges, and assume they are switching equivalent. Then their magnetic Laplacians have the same spectrum.*

Proof. Let L^σ and L^τ be the magnetic Laplacian matrices for G^σ and G^τ. Let c be a switching function that takes G^σ to G^τ. Define C to be the diagonal matrix (indexed by the vertices) with $C(u, u) = c_u$ for each $u \in V$. Then C^{-1} is the diagonal matrix with $C^{-1}(u, u) = \overline{c_u}$ for each $u \in V$. It follows that $L^\tau = C^{-1}L^\sigma C$, so L^σ and L^τ are similar matrices, which means they have the same spectrum. □

4 The Eigenvalue 0 of the Magnetic Laplacian

In this section we identify the magnetic graphs for which the magnetic Laplacian has 0 as an eigenvalue, and we describe the corresponding eigenfunctions.

Lemma 17. *Let $G := (V, E, \sigma)$ be a magnetic graph with magnetic Laplacian L. Let $f : V \to \mathbb{C}$. Then Lf is the zero function if and only if*

$$f(u) = \frac{1}{d_u}\sum_{v\sim u}\sigma_{uv}f(v)$$

for every $u \in V$ with $d_u \neq 0$.

Proof. We have:

$$\begin{aligned}& (Lf)(u) = 0 && \text{for every } u \in V\\ \iff\ & \sum_{v\sim u}(f(u) - \sigma_{uv}f(v)) = 0 && \text{for every } u \in V\\ \iff\ & d_u f(u) - \sum_{v\sim u}\sigma_{uv}f(v) = 0 && \text{for every } u \in V\\ \iff\ & f(u) = \frac{1}{d_u}\sum_{v\sim u}\sigma_{uv}f(v) && \text{for every } u \in V \text{ with } d_u \neq 0.\end{aligned}$$

□

The previous lemma says that the magnetic Laplacian sends a function to 0 iff that function assigns each vertex a value which is the signature-weighted average of the neighboring values. The next theorem strengthens that condition. Not only are the vertex values averages of the neighboring values; they are exactly equal to the neighboring values (in a signature-weighted sense).

Theorem 18. *Let $G := (V, E, \sigma)$ be a magnetic graph with magnetic Laplacian L. Let $f : V \to \mathbb{C}$. Then Lf is the zero function if and only if $f(u) = \sigma_{uv} f(v)$ whenever $u \sim v$.*

Proof. ($\leftarrow$) Suppose $f(u) = \sigma_{uv} f(v)$ whenever $u \sim v$. Then, for every $u \in V$ we have

$$(Lf)(u) = \sum_{v \sim u} (f(u) - \sigma_{uv} f(v)) = \sum_{v \sim u} (f(u) - f(u)) = 0.$$

($\rightarrow$) Suppose Lf is the zero function. We first argue that whenever u and v are vertices in the same connected component of G, we have $|f(u)| = |f(v)|$. To see this, take an arbitrary connected component of G, and choose a vertex m in that component which maximizes $|f(m)|$. If $d_m = 0$, then m is the only vertex in the component, so our conclusion holds trivially. On the other hand, if $d_m \neq 0$, then we can apply Lemma 17 to get

$$|f(m)| = \left| \frac{1}{d_m} \sum_{l \sim m} \sigma_{ml} f(l) \right| \leq \frac{1}{d_m} \sum_{l \sim m} |\sigma_{ml} f(l)| = \frac{1}{d_m} \sum_{l \sim m} |f(l)|.$$

That is, $|f(m)|$ is no greater than the mean of the values $|f(l)|$ where $l \sim m$. But our choice of m ensures that $|f(m)| \geq |f(l)|$ for each $l \sim m$, so in fact $|f(m)| = |f(l)|$ for each $l \sim m$. It follows that any vertex l which is adjacent to m also maximizes the quantity $|f(l)|$, so we can apply the same argument to each l, repeating the process until we conclude that $|f(u)| = |f(v)|$ for every pair of vertices u, v in the connected component.

Now fix vertices $u \sim v$. We want to prove that $f(u) = \sigma_{uv} f(v)$. From the previous paragraph, we already have that

$$|f(u)| = |f(v)| = |\sigma_{uv} f(v)|,$$

so it suffices to show $f(u)$ differs from $\sigma_{uv} f(v)$ by a positive real factor. Observe that

$$\begin{aligned}
\left| \sum_{l \sim u} \sigma_{ul} f(l) \right| &= |d_u f(u)| && \text{(by Lemma 17)} \\
&= d_u |f(u)| \\
&= \sum_{l \sim u} |f(u)| \\
&= \sum_{l \sim u} |f(l)| && \text{(by the previous paragraph)} \\
&= \sum_{l \sim u} |\sigma_{ul} f(l)|.
\end{aligned}$$

Therefore, assuming $\sigma_{uv} f(v) \neq 0$ (the other case is trivial), we may conclude that

$$\sum_{l \sim u} \sigma_{ul} f(l) = r(\sigma_{uv} f(v))$$

for some positive real r. Then, applying Lemma 17 once more, we have

$$f(u) = \frac{1}{d_u} \sum_{l \sim u} \sigma_{ul} f(l) = \frac{1}{d_u} r(\sigma_{uv} f(v)),$$

so $f(u)$ differs from $\sigma_{uv} f(v)$ by a positive real factor, as desired. $\square$

Proposition 19. *If G is a connected and balanced magnetic graph, then 0 is an eigenvalue of its magnetic Laplacian with multiplicity 1.*

Proof. Since G is balanced, Corollary 10 gives that it is switching equivalent to the trivial graph with all signatures 1. Then by Proposition 16, the spectrum of the trivial graph's magnetic Laplacian matches the spectrum of G's magnetic Laplacian. So we only have to show that 0 is an eigenvalue with multiplicity 1 for the magnetic Laplacian of the trivial graph. Indeed, it is clear from Theorem 18 that the kernel of that Laplacian is exactly the one-dimensional space of constant functions. $\square$

Conversely to the previous proposition, if a connected graph has 0 as an eigenvalue of its magnetic Laplacian, then it must be balanced. It follows that sigdist is defined on any pair of vertices (see definition 3), and in fact sigdist is related to the eigenfunctions with eigenvalue 0.

Proposition 20. *Let $G = (V, E, \sigma)$ be a connected graph. If 0 is an eigenvalue of the magnetic Laplacian, then G is balanced. Moreover, for any nonzero $f \in \ker L$ and pair of vertices $u, v \in V$, we have*

$$f(u)\overline{f(v)} = \mathrm{sigdist}(u, v).$$

Proof. We will prove balance by the third characterization in Proposition 2. Namely, we will show that for every pair of vertices $u, v \in V$, there exists a $z \in S^1$ such that the signatures along every walk from u to v multiply to z. To that end, let $u, v \in V$. Choose z to be $f(u)\overline{f(v)}$. Let $u \to w_1 \to w_2 \to \cdots \to w_k \to v$ be an arbitrary walk from u to v. Applying Theorem 18 repeatedly, we have

$$\begin{aligned} f(u) &= \sigma_{uw_1} f(w_1) \\ &= \sigma_{uw_1}(\sigma_{w_1 w_2} f(w_2)) \\ &\vdots \\ &= \sigma_{uw_1}\sigma_{w_1 w_2} \ldots (\sigma_{w_k v} f(v)). \end{aligned}$$

Multiplying both sides by $\overline{f(v)}$, we find that $z = f(u)\overline{f(v)}$ is the product of the signatures along the walk, as desired. Furthermore, this confirms that $z \in S^1$. Now recall that $\mathrm{sigdist}(u, v)$ is defined to be the product of the signatures along every walk from u to v, so immediately we have $\mathrm{sigdist}(u, v) = f(u)\overline{f(v)}$. ☐

5 Magnetic PageRank

Let $G = (V, E, \sigma)$ be a magnetic graph. Then G's *magnetic PageRank matrix* is the matrix W given by

$$W(u, v) := \begin{cases} \frac{\sigma_{uv}}{d_v} & \text{if } u \sim v \\ 0 & \text{if } u \not\sim v \end{cases}$$

for all $u, v \in V$.

In the case where G has all signatures 1, this is the non-magnetic PageRank matrix for the underlying simple graph. Viewed as a linear operator on the space of functions $f : V \to \mathbb{R}$, the non-magnetic PageRank matrix W' has a nice interpretation. Suppose a person is standing on a vertex of G, but we don't know which vertex. Let $g : V \to \mathbb{R}$, with $g(u)$ being the probability that our person is standing on vertex u. Then $(W'g)(u)$ is the probability that he is standing on vertex u, after he has stepped away from his original vertex along a random edge. If we apply W' many times in a row, the probabilities converge to some "PageRank vector" $x : V \to \mathbb{R}$ satisfying $W'x = x$, where $x(u)$ is intuitively the probability that a walker starting anywhere will be at vertex u after a large number of random steps. If the vertices of the graph represent webpages and the edges represent links between the pages, then the probabilities in x can serve as rankings of the importance of each webpage. This is the idea behind the PageRank algorithm originally given by Brin and Page in [2].

We will show that our magnetic PageRank matrix shares some of the properties of the non-magnetic PageRank matrix, starting with the following proposition:

Proposition 21. *Let $G = (V, E, \sigma)$ be a magnetic graph with magnetic PageRank matrix W. Then $\|W\|_1 \leq 1$, where $\|\cdot\|_1$ is the operator norm induced by the vector 1-norm (which is also denoted $\|\cdot\|_1$).*

Proof. It suffices to show that for any $x : V \to \mathbb{C}$ with $\|x\|_1 = 1$, we have $\|Wx\|_1 \leq 1$. Indeed, letting $W_{\bullet v}$ stand for the vth column vector of W, we have

$$\begin{aligned}
\|Wx\|_1 &= \left\|\sum_{v\in V} x(v)W_{\bullet v}\right\|_1 \\
&\le \sum_{v\in V}\|x(v)W_{\bullet v}\|_1 \\
&= \sum_{v\in V}\left(|x(v)|\|W_{\bullet v}\|_1\right) \\
&= \sum_{v\in V}\left(|x(v)|\sum_{u\in V}|W_{uv}|\right) \\
&= \sum_{v\in V}\left(|x(v)|\sum_{u\sim v}\left|\frac{\sigma_{uv}}{d_v}\right|\right) \\
&= \sum_{v\in V}\left(|x(v)|\frac{1}{d_v}\sum_{u\sim v}|\sigma_{uv}|\right) = \sum_{v\in V}\left(|x(v)|\frac{1}{d_v}d_v\right) = \sum_{v\in V}|x(v)| = \|x\|_1 = 1.
\end{aligned}$$

□

We have said that the non-magnetic PageRank matrix can be used to define a PageRank vector made up of probabilities that a random walker on the graph will occupy a particular vertex after a large number of steps, but the non-magnetic PageRank matrix can also be used to define a "personalized PageRank vector." This too can be thought of as a vector of probabilities that a random walker ends up at a particular vertex, but instead of always stepping along a random edge, the walker will sometimes (with probability α) jump to a random vertex drawn from a probability distribution s. We can give a magnetic definition of this personalized PageRank vector, following Chung's non-magnetic definition in [3] (page 7).

Definition 22. Let G be a magnetic graph with magnetic PageRank matrix W. Then the graph's *personalized PageRank vector* with *jumping constant* $\alpha \in (0,1)$ and *seed vector* $s : V \to \mathbb{C}$, written $\mathrm{pr}(\alpha, s)$, is defined to be the unique vector $x : V \to \mathbb{C}$ satisfying

$$\alpha s + (1-\alpha)Wx = x.$$

To show that our definition is valid, we need to show that there is indeed a unique $x : V \to \mathbb{C}$ satisfying that equation. The equation can be rewritten as

$$\alpha s = (I - (1-\alpha)W)x, \tag{1}$$

so it suffices to show $(I-(1-\alpha)W)$ is invertible. Since $\|W\|_1 \le 1$, we have $\|(1-\alpha)W\|_1 < 1$. Therefore, $(I-(1-\alpha)W)$ is invertible as desired, and moreover

$$(I-(1-\alpha)W)^{-1} = \sum_{k=0}^{\infty}((1-\alpha)W)^k = \sum_{k=0}^{\infty}(1-\alpha)^k W^k.$$

Multiplying both sides of (1) by $(I-(1-\alpha)W)^{-1}$ gives the following fact.

Proposition 23. *Let $G = (V, E, \sigma)$ be a magnetic graph with magnetic PageRank matrix W. Let $\alpha \in (0,1)$ and $s : V \to \mathbb{C}$. Then,*

$$\mathrm{pr}(\alpha, s) = \alpha\left(\sum_{k=0}^{\infty}(1-\alpha)^k W^k\right)s.$$

We can also define a "lazy" variant of the magnetic PageRank matrix, which is sometimes easier to work with.

Definition 24. Let G be a magnetic graph with magnetic PageRank matrix W. Then the graph's (magnetic) *lazy PageRank matrix* is $Z := (I+W)/2$.

If we can show that $\|Z\|_1 \le 1$, then Z will be associated with a personalized PageRank vector just like W. Indeed, we have

$$\|Z\|_1 = \|(I+W)/2\|_1 \le \frac{1}{2}\|I\|_1 + \frac{1}{2}\|W\|_1 \le \frac{1}{2}(1) + \frac{1}{2}(1) = 1.$$

So we can make the following definition:

Definition 25. Let G be a magnetic graph with (magnetic) lazy PageRank matrix Z. Then the graph's *lazy personalized PageRank vector* with *jumping constant* $\alpha \in (0, 1)$ and *seed vector* $s : V \to \mathbb{C}$, written $\mathrm{lazypr}(\alpha, s)$, is defined to be the unique vector $x : V \to \mathbb{C}$ satisfying

$$\alpha s + (1 - \alpha)Zx = x.$$

6 Magnetic PageRank as a Green's Function

From the previous section, we have that for fixed α, the value of $\mathrm{lazypr}(\alpha, s)$ is linear with respect to s. In this section, we will show that the matrix for this linear transformation is in fact a variant on the inverse of the magnetic Laplacian, which is called a Green's function. Our argument is modeled after section 4 of [3].

Throughout this section, let $G = (V, E, \sigma)$ be a magnetic graph on n vertices, with magnetic PageRank matrix W, magnetic lazy PageRank matrix Z, and magnetic Laplacian L. We note that $W = AD^{-1}$ and $L = D - A$, where D is the diagonal matrix with $D(u, u) = d_u$ and A is the magnetic adjacency matrix. That is, $A(u, v) = \sigma_{uv}$ if $u \sim v$ and 0 otherwise. Assume G is connected and has at least 2 vertices, so that D is invertible. Fix a jumping constant $\alpha \in (0, 1)$, and let

$$\beta := \frac{2\alpha}{1 - \alpha}.$$

It will be more convenient going forward to work with normalized versions of some of our matrices, which we now define.

Definition 26. The *normalized magnetic Laplacian* of G is

$$\mathcal{L} := D^{-1/2} L D^{-1/2}.$$

Definition 27. The *normalized magnetic adjacency matrix* of G is

$$\mathcal{A} := D^{-1/2} A D^{-1/2}.$$

Normalizing the Laplacian and adjacency matrix in this way is nice, because instead of the unnormalized relationship $L = D - A$, we have the cleaner relationship

$$\mathcal{L} = I - \mathcal{A}.$$

Moreover, we have a nice relationship with the PageRank matrix:

$$W = D^{1/2} \mathcal{A} D^{-1/2},$$

so W can be viewed as an "asymmetric" (or more precisely non-Hermitian) version of $\mathcal{A}$. This idea of working with asymmetric versions of Hermitian matrices is important because it bridges the gap from the asymmetric PageRank world of W and Z to the symmetric Laplacian world of $\mathcal{A}$ and $\mathcal{L}$.

Since L is positive-semidefinite and $D^{-1/2}$ is Hermitian, $\mathcal{L}$ is positive semidefinite too. Therefore $\mathcal{L}$ has n real, nonnegative eigenvalues. Going forward we will refer to these as

$$0 \leq \lambda_1 \leq \cdots \leq \lambda_n$$

with the corresponding orthonormal eigenfunctions

$$\phi_1, \ldots, \phi_n.$$

Definition 28. The *β-normalized magnetic Laplacian* is

$$\mathcal{L}_\beta := \beta I + \mathcal{L}.$$

We can see that $\mathcal{L}_\beta$ shares the eigenfunctions $\phi_1, \ldots, \phi_n$ with $\mathcal{L}$, except the corresponding eigenvalues of $\mathcal{L}_\beta$ are $\beta + \lambda_1, \ldots, \beta + \lambda_n$. Since $\beta > 0$, we can conclude that $\mathcal{L}_\beta$ does not have 0 as an eigenvalue, and is therefore invertible. So we can make the following definition.

Definition 29. The *β-normalized magnetic Green's function* is

$$\mathcal{G}_\beta := \mathcal{L}_\beta^{-1}.$$

Next we want to define an asymmetric version of the Green's Function. Asymmetrizing the Green's function will help us relate it to PageRank, as we discussed earlier. We asymmetrize the Green's function the same way we asymmetrized $\mathcal{A}$ to get W.

Definition 30. The *asymmetric β-normalized magnetic Green's function* is

$$\mathbf{G}_\beta := D^{1/2}\mathcal{G}_\beta D^{-1/2}.$$

There is another way to arrive at this definition. We could follow the same procedure we used to define $\mathcal{G}_\beta$, where we start with $\mathcal{L}$, add βI, and then take the inverse; except instead we start with an asymmetric version of $\mathcal{L}$. The asymmetric version of $\mathcal{L}$ we need is the following.

Definition 31. The *magnetic Laplace operator* is the matrix

$$\Delta := D^{1/2}\mathcal{L}D^{-1/2}.$$

If we were to define $\mathbf{G}_\beta$ using Δ, taking the same approach as for $\mathcal{G}_\beta$, we would want to say

$$\mathbf{G}_\beta = (\beta I + \Delta)^{-1}.$$

Indeed, this definition is equivalent to ours, as the next proposition shows.

Proposition 32. *$\beta I + \Delta$ is invertible, and $\mathbf{G}_\beta = (\beta I + \Delta)^{-1}$.*

Proof. Note that Δ has n nonnegative eigenvalues because it is similar to $\mathcal{L}$. Since $\beta > 0$, it follows that $\beta I + \Delta$ has n nonzero eigenvalues, meaning $\beta I + \Delta$ is invertible. Finally, we have

$$\begin{aligned}
(\beta I + \Delta)^{-1} &= \left(\beta I + D^{1/2}\mathcal{L}D^{-1/2}\right)^{-1} \\
&= \left(D^{1/2}\left(\beta I + \mathcal{L}\right)D^{-1/2}\right)^{-1} \\
&= \left(D^{1/2}\mathcal{L}_\beta D^{-1/2}\right)^{-1} \\
&= D^{1/2}\mathcal{G}_\beta D^{-1/2} \\
&= \mathbf{G}_\beta.
\end{aligned}$$

□

Since Δ is an asymmetric version of $\mathcal{L}$ and W is an asymmetric version of $\mathcal{A}$, we naturally have the following relationship.

Proposition 33. *We have $\Delta = I - W$.*

Proof.

$$\Delta = D^{1/2}\mathcal{L}D^{-1/2} = D^{1/2}(I - \mathcal{A})D^{-1/2} = I - D^{1/2}\mathcal{A}D^{-1/2} = I - W.$$

□

Finally, we can describe the matrix for the linear transformation $\mathrm{lazypr}(\alpha, s)$.

Theorem 34. *For any seed vector $s \in \mathbb{C}^n$, we have*

$$\mathrm{lazypr}(\alpha, s) = \beta\mathbf{G}_\beta s.$$

Proof.

$$\begin{aligned}
\frac{\mathrm{lazypr}(\alpha, s)}{\beta} &= \frac{\alpha}{\beta}\left[I - (1-\alpha)Z\right]^{-1} s \\
&= \left(\frac{\beta}{\alpha}I - 2Z\right)^{-1} s \\
&= \left[(\beta + 2)I - 2Z\right]^{-1} s \\
&= \left[(\beta + 2)I - (I + W)\right]^{-1} s \\
&= \left[\beta I + (I - W)\right]^{-1} s \\
&= (\beta I + \Delta)^{-1} s && \text{(by Proposition 33)} \\
&= \mathbf{G}_\beta s && \text{(by Proposition 32).}
\end{aligned}$$

□

7 Magnetic Hitting Time and Effective Resistance

In this section we will define and relate magnetic versions of the hitting time and effective resistance, which are important quantities in spectral graph theory. We will base our definitions and results off of sections 3 and 7 of Chung and Zhao [4], which considers non-magnetic versions of these quantities.

We will continue using the variables and assumptions from the previous section. We will add the notation $\mathrm{Vol}(G)$ to mean the sum of all the degrees of the vertices in G. We will also frequently refer to the *characteristic function* of a vertex u, which is the function $\chi_u : V \to \mathbb{C}$ that takes the value 1 at u and the value 0 everywhere else. Moreover, for this section we will need a more explicit understanding of the vector ϕ_1 in the case that G is balanced. For that purpose, we define a vector $l_1 : V \to \mathbb{C}$ by

$$l_1 := \frac{\sqrt{n}}{\left\|D^{-1/2}\phi_1\right\|} D^{-1/2}\phi_1.$$

Then if G is balanced, l_1 allows us to nicely describe ϕ_1 per the following lemma.

Lemma 35. *Suppose G is balanced. Then,*

1. $l_1 \in \ker L$
2. *Every entry of l_1 belongs to S^1*
3. *For any vertex $v \in V$, we have*

$$\phi_1(v) = \frac{\sqrt{d_v}}{\sqrt{\mathrm{Vol}(G)}} l_1(v).$$

Proof. Since G is balanced, $\mathcal{L}$ has 0 as an eigenvalue, so $\mathcal{L}\phi_1 = 0$. Therefore,

$$D^{-1/2}Ll_1 = \frac{\sqrt{n}}{\left\|D^{-1/2}\phi_1\right\|}\mathcal{L}\phi_1 = 0,$$

so $l_1 \in \ker L$ as desired. By Proposition 20 all the entries of l_1 have the same modulus. But $\|l_1\| = \sqrt{n}$, so in fact all the entries of l_1 have modulus 1, as desired. Finally, observe that for any vertex v of G, we have

$$\phi_1(v) = \chi_v^*\phi_1 = \chi_v^*\frac{D^{1/2}l_1}{\|D^{1/2}l_1\|} = \chi_v^*\frac{D^{1/2}l_1}{\sqrt{\mathrm{Vol}(G)}} = \frac{\sqrt{d_v}}{\sqrt{\mathrm{Vol}(G)}}\chi_v^* l_1 = \frac{\sqrt{d_v}}{\sqrt{\mathrm{Vol}(G)}} l_1(v).$$

□

To define the magnetic effective resistance, we first need to define the unnormalized magnetic Green's function $\mathcal{G}$.

Definition 36. The *magnetic Green's function* is

$$\mathcal{G} := \sum_{\substack{1\le i\le n\\ \lambda_i\neq 0}} \frac{1}{\lambda_i}\phi_i\phi_i^*.$$

We can view $\mathcal{G}$ as a "pseudoinverse" of $\mathcal{L}$ (to be technical, the Moore-Penrose pseudoinverse). If our graph G is unbalanced, meaning $\mathcal{L}$ is invertible, then $\mathcal{G} = \mathcal{L}^{-1}$. On the other hand, if G is balanced, meaning $\mathcal{L}$ has nullity 1, then $\mathcal{G}$ acts like an inverse of $\mathcal{L}$ in the orthogonal complement of $\ker \mathcal{L}$, but still takes the vectors in $\ker \mathcal{L}$ to 0. Thinking along these lines, we can see that $\mathcal{G}$ is independent of the choice of ϕ_i's, so the magnetic Green's function is well-defined for any graph. $\mathcal{G}$ has the same orthonormal eigenfunctions $\phi_1, \ldots, \phi_n$ as $\mathcal{L}$, and all its eigenvalues are real and nonnegative. Thus, $\mathcal{G}$ is positive-semidefinite.

We are now ready to define a magnetic version of the effective resistance. The classical effective resistance is a way to measure how robustly two vertices are connected in a non-magnetic graph, with relevance to the theory of electrical circuits. To obtain a definition for the magnetic effective resistance, we borrow an expression for the classical effective resistance from [4] and replace the classical Green's function with our magnetic Green's function.

Definition 37. Let u and v be vertices of G. Then the *magnetic effective resistance* between u and v is

$$R(u,v) := (\chi_v - \chi_u)^* D^{-1/2}\mathcal{G}D^{-1/2}(\chi_v - \chi_u).$$

Since $\mathcal{G}$ is positive-semidefinite, $D^{-1/2}\mathcal{G}D^{-1/2}$ is positive-semidefinite also. Thus, like the classical effective resistance, $R(u,v)$ is real and nonnegative for any vertices u and v. However, the magnetic effective resistance differs from the classical effective resistance in that it doesn't satisfy the triangle inequality. For example, consider a graph with vertices 1, 2, and 3 and signatures $\tau_{12} = -1$, $\tau_{23} = -1$, $\tau_{31} = 1$. For that graph, it is easy to check that

$$R(1,2) + R(2,3) = \frac{2}{9} + \frac{2}{9} < \frac{2}{3} = R(1,3).$$

We can also define a generalized version of the effective resistance, which depends on the parameter α.

Definition 38. Let u and v be vertices of G. Then the *generalized magnetic effective resistance* between u and v is

$$R_\alpha(u,v) := \beta(\chi_v - \chi_u)^* D^{-1/2}\mathcal{G}_\beta D^{-1/2}(\chi_v - \chi_u).$$

Similarly to the effective resistance, this generalized effective resistance is always real and nonnegative. An advantage of the generalized effective resistance is that it can be written in terms of the magnetic PageRank:

Proposition 39. *For any vertices u and v, we have*

$$R_\alpha(u,v) = \frac{\mathrm{lazypr}(\alpha,\chi_v)(v)}{d_v} - \frac{\mathrm{lazypr}(\alpha,\chi_v)(u)}{d_u} + \frac{\mathrm{lazypr}(\alpha,\chi_u)(u)}{d_u} - \frac{\mathrm{lazypr}(\alpha,\chi_u)(v)}{d_v}.$$

Proof.

$$\begin{aligned}
R_\alpha(u,v) &= \beta(\chi_v - \chi_u)^* D^{-1/2}\mathcal{G}_\beta D^{-1/2}(\chi_v - \chi_u)\\
&= \beta(\chi_v - \chi_u)^* D^{-1}\mathbf{G}_\beta(\chi_v - \chi_u)\\
&= \left(\chi_v^* D^{-1} - \chi_u^* D^{-1}\right)\left[\beta\mathbf{G}_\beta\chi_v - \beta\mathbf{G}_\beta\chi_u\right]\\
&= \left(\frac{1}{d_v}\chi_v^* - \frac{1}{d_u}\chi_u^*\right)\left[\mathrm{lazypr}(\alpha,\chi_v) - \mathrm{lazypr}(\alpha,\chi_u)\right]\\
&= \frac{\mathrm{lazypr}(\alpha,\chi_v)(v)}{d_v} - \frac{\mathrm{lazypr}(\alpha,\chi_v)(u)}{d_u} + \frac{\mathrm{lazypr}(\alpha,\chi_u)(u)}{d_u} - \frac{\mathrm{lazypr}(\alpha,\chi_u)(v)}{d_v}.
\end{aligned}$$

□

The next theorem (a magnetic version of Theorem 9 of [4]) bounds the magnetic effective resistance using the generalized magnetic effective resistance. Since the generalized effective resistance can be expressed in terms of PageRank, this allows us to estimate the effective resistance using PageRank.

Theorem 40. *Let u and v be vertices of G. If G is balanced, then we have*

$$|\beta R(u,v) - R_\alpha(u,v)| \leq \left(1 - \frac{\beta^2}{\lambda_2^2}\right)\frac{|1 - \mathrm{sigdist}(u,v)|^2}{\mathrm{Vol}(G)} + \frac{\beta^2}{\lambda_2^2}\left(\frac{1}{d_u} + \frac{1}{d_v}\right),$$

And if G is unbalanced, then

$$|\beta R(u,v) - R_\alpha(u,v)| \leq \frac{\beta^2}{\lambda_1^2}\left(\frac{1}{d_u} + \frac{1}{d_v}\right).$$

Proof. First assume G is balanced. Then,

$$\begin{aligned}
\beta R(u,v) &= \beta(\chi_v - \chi_u)^* D^{-1/2}\mathcal{G}D^{-1/2}(\chi_v - \chi_u)\\
&= \beta\left(\frac{\chi_v}{\sqrt{d_v}} - \frac{\chi_u}{\sqrt{d_u}}\right)^*\left(\sum_{i=2}^{n}\frac{1}{\lambda_i}\phi_i\phi_i^*\right)\left(\frac{\chi_v}{\sqrt{d_v}} - \frac{\chi_u}{\sqrt{d_u}}\right)\\
&= \sum_{i=2}^{n}\frac{\beta}{\lambda_i}\left(\frac{\chi_v}{\sqrt{d_v}} - \frac{\chi_u}{\sqrt{d_u}}\right)^*\phi_i\left[\left(\frac{\chi_v}{\sqrt{d_v}} - \frac{\chi_u}{\sqrt{d_u}}\right)^*\phi_i\right]^*\\
&= \sum_{i=2}^{n}\frac{\beta}{\lambda_i}\left|\frac{\phi_i(u)}{\sqrt{d_u}} - \frac{\phi_i(v)}{\sqrt{d_v}}\right|^2.
\end{aligned}$$

Similarly,

$$R_\alpha(u,v) = \sum_{i=1}^{n}\frac{\beta}{\lambda_i + \beta}\left|\frac{\phi_i(u)}{\sqrt{d_u}} - \frac{\phi_i(v)}{\sqrt{d_v}}\right|^2.$$

It follows that

$$
\begin{aligned}
&|\beta R(u,v) - R_\alpha(u,v)| \\
&= \left| \left[\sum_{i=2}^{n} \left(\frac{\beta}{\lambda_i} - \frac{\beta}{\lambda_i + \beta} \right) \left| \frac{\phi_i(u)}{\sqrt{d_u}} - \frac{\phi_i(v)}{\sqrt{d_v}} \right|^2 \right] - \left| \frac{\phi_i(u)}{\sqrt{d_u}} - \frac{\phi_i(v)}{\sqrt{d_v}} \right|^2 \right| \\
&\leq \left| \frac{\phi_1(u)}{\sqrt{d_u}} - \frac{\phi_1(v)}{\sqrt{d_v}} \right|^2 + \sum_{i=2}^{n} \frac{\beta^2}{\lambda_i(\lambda_i + \beta)} \left| \frac{\phi_i(u)}{\sqrt{d_u}} - \frac{\phi_i(v)}{\sqrt{d_v}} \right|^2 \\
&\leq \left| \frac{\phi_1(u)}{\sqrt{d_u}} - \frac{\phi_1(v)}{\sqrt{d_v}} \right|^2 + \frac{\beta^2}{\lambda_2(\lambda_2 + \beta)} \sum_{i=2}^{n} \left| \frac{\phi_i(u)}{\sqrt{d_u}} - \frac{\phi_i(v)}{\sqrt{d_v}} \right|^2 \\
&\leq \left| \frac{\phi_1(u)}{\sqrt{d_u}} - \frac{\phi_1(v)}{\sqrt{d_v}} \right|^2 + \frac{\beta^2}{\lambda_2^2} \left(- \left| \frac{\phi_1(u)}{\sqrt{d_u}} - \frac{\phi_1(v)}{\sqrt{d_v}} \right|^2 + \sum_{i=1}^{n} \left| \frac{\phi_i(u)}{\sqrt{d_u}} - \frac{\phi_i(v)}{\sqrt{d_v}} \right|^2 \right) \\
&= \left(1 - \frac{\beta^2}{\lambda_2^2} \right) \left| \frac{\phi_1(u)}{\sqrt{d_u}} - \frac{\phi_1(v)}{\sqrt{d_v}} \right|^2 + \frac{\beta^2}{\lambda_2^2} \sum_{i=1}^{n} \left| \frac{\phi_i(u)}{\sqrt{d_u}} - \frac{\phi_i(v)}{\sqrt{d_v}} \right|^2 \\
&= \left(1 - \frac{\beta^2}{\lambda_2^2} \right) \left| \frac{\phi_1(u)}{\sqrt{d_u}} - \frac{\phi_1(v)}{\sqrt{d_v}} \right|^2 + \frac{\beta^2}{\lambda_2^2} \sum_{i=1}^{n} \left[\frac{|\phi_i(u)|^2}{d_u} + \frac{|\phi_i(v)|^2}{d_v} - 2\Re \left(\frac{\phi_i(u)\overline{\phi_i(v)}}{\sqrt{d_u d_v}} \right) \right] \\
&= \left(1 - \frac{\beta^2}{\lambda_2^2} \right) \left| \frac{\phi_1(u)}{\sqrt{d_u}} - \frac{\phi_1(v)}{\sqrt{d_v}} \right|^2 + \frac{\beta^2}{\lambda_2^2} \sum_{i=1}^{n} \left(\frac{1}{d_u} + \frac{1}{d_v} \right),
\end{aligned}
$$

where the last equality holds because the columns (and therefore the rows) of the matrix $[\phi_1, \dots \phi_n]$ are orthonormal. We finish the balanced case by observing that

$$
\begin{aligned}
\left| \frac{\phi_1(u)}{\sqrt{d_u}} - \frac{\phi_1(v)}{\sqrt{d_v}} \right|^2 &= \left| \frac{l_1(u)}{\sqrt{\mathrm{Vol}(G)}} - \frac{l_1(v)}{\sqrt{\mathrm{Vol}(G)}} \right|^2 \\
&= \frac{\left| l_1(u)\overline{l_1(v)} - l_1(v)\overline{l_1(v)} \right|^2}{\mathrm{Vol}(G)} \\
&= \frac{|1 - \mathrm{sigdist}(u,v)|^2}{\mathrm{Vol}(G)} && \text{(by Proposition 20)}.
\end{aligned}
$$

Finally, assume G is unbalanced. Then we have

$$
\begin{aligned}
&|\beta R(u,v) - R_\alpha(u,v)| \\
&= \sum_{i=1}^{n} \left(\frac{\beta}{\lambda_i} - \frac{\beta}{\lambda_i + \beta} \right) \left| \frac{\phi_i(u)}{\sqrt{d_u}} - \frac{\phi_i(v)}{\sqrt{d_v}} \right|^2 \\
&\leq \frac{\beta^2}{\lambda_1(\lambda_1 + \beta)} \sum_{i=1}^{n} \left| \frac{\phi_i(u)}{\sqrt{d_u}} - \frac{\phi_i(v)}{\sqrt{d_v}} \right|^2 \\
&\leq \frac{\beta^2}{\lambda_1^2} \sum_{i=1}^{n} \left[\frac{|\phi_i(u)|^2}{d_u} + \frac{|\phi_i(v)|^2}{d_v} - 2\Re \left(\frac{\phi_i(u)\overline{\phi_i(v)}}{\sqrt{d_u d_v}} \right) \right] \\
&= \frac{\beta^2}{\lambda_1^2} \sum_{i=1}^{n} \left(\frac{1}{d_u} + \frac{1}{d_v} \right).
\end{aligned}
$$

□

We remark that whether or not G is balanced, Theorem 40 implies that $R(u,v)$ can be made arbitrary close to $R_\alpha(u,v)/\beta$ with a sufficiently small choice of α. If G is unbalanced this follows immediately from the second inequality in the theorem. If G is balanced, unbalance the graph by a very small amount and then once again use the second inequality.

The magnetic effective resistance is related to another quantity called the magnetic hitting time. The classical hitting time for a non-magnetic graph is the expected length of a random walk that begins at a vertex u and ends at a vertex v. To define a magnetic version of the hitting time, we borrow an expression for the hitting time from [4], but we replace the classical Green's function with the magnetic Green's function:

Definition 41. Given two vertices u and v, the *magnetic hitting time* from u to v is

$$H(u,v) := \mathrm{Vol}(G)(\chi_v - \chi_u)^* D^{-1/2} \mathcal{G} D^{-1/2} \chi_v.$$

From the definitions, we get the relationship

$$R(u,v) = \frac{H(u,v) + H(v,u)}{\mathrm{Vol}(G)}.$$

8 Switching with Magnetic Hitting Time and Effective Resistance

Switching functions interact nicely with the magnetic effective resistance and magnetic hitting time. This is ultimately because switching functions interact nicely with Green's function, as the next lemma shows.

Lemma 42. *Suppose that a switching function $c : V \to S^1$ is applied to G to obtain a new magnetic graph G'. Let $\mathcal{G}'$ be the magnetic Green's function for G'. Let C be the diagonal matrix indexed by the vertices whose entries are given by $C(u,u) = c_u$. Then,*

$$\mathcal{G}' = C^* \mathcal{G} C.$$

Proof. Recall that $\phi_1, \ldots, \phi_n$ are orthonormal eigenfunctions for $\mathcal{L}$, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. The Laplacian for G' is $\mathcal{L}' := C^* \mathcal{L} C$, so the list $C^*\phi_1, \ldots, C^*\phi_n$ is made up of eigenfunctions of $\mathcal{L}'$, and the eigenvalues are still $\lambda_1, \ldots, \lambda_n$. It is easy to check that these eigenfunctions have norm 1. Moreover, for every $1 \le i < j \le n$ we have

$$\langle C^*\phi_i, C^*\phi_j \rangle = \langle \phi_i, CC^*\phi_j \rangle = \langle \phi_i, I\phi_j \rangle = \langle \phi_i, \phi_j \rangle = 0.$$

So the eigenfunctions are actually orthnormal. Thus, by the definition of Green's function we have

$$\mathcal{G}' = \sum_{\substack{1 \le i \le n \\ \lambda_i \neq 0}} \frac{1}{\lambda_i} (C^*\phi_i)(C^*\phi_i)^* = \sum_{\substack{1 \le i \le n \\ \lambda_i \neq 0}} \frac{1}{\lambda_i} C^* \phi_i \phi_i^* C = C^* \left(\sum_{\substack{1 \le i \le n \\ \lambda_i \neq 0}} \frac{1}{\lambda_i} \phi_i \phi_i^* \right) C = C^* \mathcal{G} C.$$

□

With this lemma, it is easy to check the following series of statements.

Proposition 43. *Suppose that a switching function $c : V \to S^1$ is applied to G to obtain a new graph G'. Let R^c be the magnetic effective resistance function for G'. Then for any vertices u and v, we have*

$$R^c(u,v) = (c_v\chi_v - c_u\chi_u)^* D^{-1/2} \mathcal{G} D^{-1/2} (c_v\chi_v - c_u\chi_u).$$

Proposition 44. *Suppose that a switching function $c : V \to S^1$ is applied to G to obtain a new graph G'. Let H^c be the hitting time function for G'. Then for any vertices u and v, we have*

$$H^c(u,v) = (c_v\chi_v - c_u\chi_u)^* D^{-1/2} \mathcal{G} D^{-1/2} (c_v\chi_v).$$

Proposition 45. *If u, v are vertices of G and b, c are switching functions for G with $\overline{b_u} b_v = \overline{c_u} c_v$, then $R^b(u,v) = R^c(u,v)$ and $H^b(u,v) = H^c(u,v)$.*

Corollary 46. *If u, v are vertices of G and c is a switching function for G with $c_u = c_v$, then $R^c(u,v) = R(u,v)$ and $H^c(u,v) = H(u,v)$.*

Corollary 47. *If u, v are vertices of G and G is balanced with* $\mathrm{sigdist}(u,v) = 1$*, then $R(u,v)$ equals the classical effective resistance between u and v for the underlying simple graph, and $H(u,v)$ equals the classical hitting time between u and v for the underlying simple graph.*

To state the next theorem nicely, we will make a new definition.

Definition 48. For a connected magnetic graph G, the *average signature distance* between two vertices u and v is

$$\operatorname{asd}(u,v) := \begin{cases} \operatorname{sigdist}(u,v) & \text{if } G \text{ is balanced} \\ 0 & \text{otherwise} \end{cases}.$$

Why is it called the average signature distance? When we walk along a path from u to v, we accumulate a product of signatures. Choosing a different path may result in a different product. Informally, the average signature distance is the (unweighted) average of all the unique products we can get by choosing different paths.

The next (and final) theorem is a magnetic version of the non-magnetic hitting time recurrence given in [4] (equation 9). Our proof of the recurrence uses ideas from the proof of Lemma 1 in that same paper.

Theorem 49. *Let u and v be distinct vertices in G. Then,*

$$H(u,v) = \operatorname{asd}(u,v) + \frac{1}{d_u}\sum_{w\sim u} H^w(w,v)$$

where $H^w(w,v)$ denotes the hitting time from w to v, computed on a switched version of G where the switching function c satisfies $\overline{c_w}c_v = \sigma_{uw}$. Or, if $v \sim u$ and $\sigma_{uv} \neq 1$, then $H^v(v,v)$ denotes

$$(1-\sigma_{uv})\operatorname{Vol}(G)\chi_v^* D^{-1/2}\mathcal{G}D^{-1/2}\chi_v.$$

Proof. We have (justifications for numbered steps are below):

$$\begin{aligned}
\frac{H(u,v)}{\operatorname{Vol}(G)} &= (\chi_v - \chi_u)^* D^{-1/2}\mathcal{G}D^{-1/2}\chi_v \\
&= \chi_v^* D^{-1/2}\mathcal{G}D^{-1/2}\chi_v - \chi_u^* D^{-1/2}\mathcal{G}D^{-1/2}\chi_v \\
&= \chi_v^* D^{-1/2}\mathcal{G}D^{-1/2}\chi_v + \chi_u^* D^{-1/2}ID^{-1/2}\chi_v - \chi_u^* D^{-1/2}\mathcal{G}D^{-1/2}\chi_v \\
&= \chi_v^* D^{-1/2}\mathcal{G}D^{-1/2}\chi_v + \chi_u^* D^{-1/2}(I-\mathcal{L}\mathcal{G})D^{-1/2}\chi_v - \chi_u^* D^{-1/2}(I-\mathcal{L})\mathcal{G}D^{-1/2}\chi_v \\
&= \frac{\operatorname{asd}(u,v)}{\operatorname{Vol}(G)} + \chi_v^* D^{-1/2}\mathcal{G}D^{-1/2}\chi_v - \chi_u^* D^{-1/2}(I-\mathcal{L})\mathcal{G}D^{-1/2}\chi_v & (2)\\
&= \frac{\operatorname{asd}(u,v)}{\operatorname{Vol}(G)} + \chi_v^* D^{-1/2}\mathcal{G}D^{-1/2}\chi_v - \chi_u^* D^{-1/2}\mathcal{A}\mathcal{G}D^{-1/2}\chi_v \\
&= \frac{\operatorname{asd}(u,v)}{\operatorname{Vol}(G)} + \chi_v^* D^{-1/2}\mathcal{G}D^{-1/2}\chi_v - \chi_u^* D^{-1}AD^{-1/2}\mathcal{G}D^{-1/2}\chi_v \\
&= \frac{\operatorname{asd}(u,v)}{\operatorname{Vol}(G)} + \chi_v^* D^{-1/2}\mathcal{G}D^{-1/2}\chi_v - \frac{1}{d_u}\chi_u^* AD^{-1/2}\mathcal{G}D^{-1/2}\chi_v \\
&= \frac{\operatorname{asd}(u,v)}{\operatorname{Vol}(G)} + \chi_v^* D^{-1/2}\mathcal{G}D^{-1/2}\chi_v - \frac{1}{d_u}\sum_{w\sim u}\sigma_{uw}\chi_w^* D^{-1/2}\mathcal{G}D^{-1/2}\chi_v & (3)\\
&= \frac{\operatorname{asd}(u,v)}{\operatorname{Vol}(G)} + \frac{1}{d_u}\left(\sum_{w\sim u}\chi_v^* D^{-1/2}\mathcal{G}D^{-1/2}\chi_v\right) - \frac{1}{d_u}\sum_{w\sim u}\sigma_{uw}\chi_w^* D^{-1/2}\mathcal{G}D^{-1/2}\chi_v \\
&= \frac{\operatorname{asd}(u,v)}{\operatorname{Vol}(G)} + \frac{1}{d_u}\sum_{w\sim u}\left(\chi_v^* - \sigma_{uw}\chi_w^*\right) D^{-1/2}\mathcal{G}D^{-1/2}\chi_v \\
&= \frac{\operatorname{asd}(u,v)}{\operatorname{Vol}(G)} + \frac{1}{d_u}\sum_{w\sim u}\left(\chi_v - \sigma_{wu}\chi_w\right)^* D^{-1/2}\mathcal{G}D^{-1/2}\chi_v \\
&= \frac{\operatorname{asd}(u,v)}{\operatorname{Vol}(G)} + \frac{1}{d_u}\sum_{w\sim u}\frac{H^w(w,v)}{\operatorname{Vol}(G)}.
\end{aligned}$$

For (2), observe that if G is unbalanced, then $\mathcal{L}\mathcal{G} = I$ and the result follows. Now assume that G is balanced.

Then $I - \mathcal{L}\mathcal{G} = \phi_1\phi_1^*$, since those matrices act the same way on the basis $\phi_1, \dots, \phi_n$. It follows that

$$\begin{aligned}
\chi_u^* D^{-1/2}(I - \mathcal{L}\mathcal{G})D^{-1/2}\chi_v &= \chi_u^* D^{-1/2}\phi_1\phi_1^* D^{-1/2}\chi_v \\
&= \frac{1}{\sqrt{d_u d_v}}\chi_u^*\phi_1\phi_1^*\chi_v \\
&= \frac{1}{\sqrt{d_u d_v}}\phi_1(u)\overline{\phi_1(v)} \\
&= \frac{1}{\sqrt{d_u d_v}}\left(\frac{\sqrt{d_u}}{\sqrt{\mathrm{Vol}(G)}}l_1(u)\right)\left(\frac{\sqrt{d_v}}{\sqrt{\mathrm{Vol}(G)}}\overline{l_1(v)}\right) && \text{(by Lemma 35)} \\
&= \frac{l_1(u)\overline{l_1(v)}}{\mathrm{Vol}(G)} \\
&= \frac{\mathrm{sigdist}(u,v)}{\mathrm{Vol}(G)} && \text{(by Proposition 20).}
\end{aligned}$$

For (3), note that for any function $y : V \to \mathbb{C}$, we have

$$\chi_u^* A y = (Ay)(u) = \sum_{j=1}^{n} A_{uj}y(j) = \sum_{w\sim u} A_{uw}y(w) = \sum_{w\sim u}\sigma_{uw}y(w) = \sum_{w\sim u}\sigma_{uw}\chi_w^* y.$$

□

References

[1] Erik Bédos. An introduction to 3D discrete magnetic Laplacians and noncommutative 3-tori. *J. Geom. Phys.*, 30(3):204–232, 1999.

[2] Sergey Brin and Lawrence Page. The anatomy of a large-scale hypertextual web search engine. In *Proceedings of the Seventh International Conference on World Wide Web 7*, WWW7, page 107–117, NLD, 1998. Elsevier Science Publishers B. V.

[3] Fan Chung. PageRank as a discrete Green's function. In *Geometry and analysis. No. 1*, volume 17 of *Adv. Lect. Math. (ALM)*, pages 285–302. Int. Press, Somerville, MA, 2011.

[4] Fan Chung and Wenbo Zhao. PageRank and random walks on graphs. In *Fete of combinatorics and computer science*, volume 20 of *Bolyai Soc. Math. Stud.*, pages 43–62. János Bolyai Math. Soc., Budapest, 2010.

[5] Fan Chung, Wenbo Zhao, and Mark Kempton. Ranking and sparsifying a connection graph. *Internet Math.*, 10(1-2):87–115, 2014.

[6] Józef Dodziuk and Varghese Mathai. Kato's inequality and asymptotic spectral properties for discrete magnetic Laplacians. In *The ubiquitous heat kernel*, volume 398 of *Contemp. Math.*, pages 69–81. Amer. Math. Soc., Providence, RI, 2006.

[7] Frank Harary. On the notion of balance of a signed graph. *Michigan Math. J.*, 2:143–146 (1955), 1953/54.

[8] Jiaqi Jiang. Introduction to spectral graph theory. 2012.

[9] Carsten Lange, Shiping Liu, Norbert Peyerimhoff, and Olaf Post. Frustration index and Cheeger inequalities for discrete and continuous magnetic Laplacians. *Calc. Var. Partial Differential Equations*, 54(4):4165–4196, 2015.

[10] Elliott H. Lieb and Michael Loss. Fluxes, Laplacians, and Kasteleyn's theorem. *Duke Math. J.*, 71(2):337–363, 1993.

[11] Shiping Liu, Norbert Peyerimhoff, and Alina Vdovina. Signatures, lifts, and eigenvalues of graphs. 12 2014.

[12] M. A. Shubin. Discrete magnetic Laplacian. *Comm. Math. Phys.*, 164(2):259–275, 1994.

[13] Thomas Zaslavsky. Signed graphs. *Discrete Appl. Math.*, 4(1):47–74, 1982.