1 The Magnetic Laplacian and its Spectrum

Definition 1. A *simple graph* is a set of vertices V and a set of oriented edges $E \subseteq V \times V$, where for all $(v, w) \in E$, we have $v \neq w$ and $(w, v) \in E$.

Definition 2. A *magnetic graph* is a simple graph (V, E) together with a *signature* function

$$\sigma : E \to \mathbb{C}_{|z|=1}$$

which has the property $\sigma(v, w) = \overline{\sigma(w, v)}$ for all $(v, w) \in E$. We may abbreviate $\sigma(v, w)$ as σ_{vw}.

We may sometimes treat a simple graph as a magnetic graph, in which case all signatures are assumed to be 1. Unless we explicitly state the vertex set of a graph, it is assumed to be $\{1, \ldots, n\}$ for some n.

Definition 3. Suppose $G = (V, E)$ is a magnetic graph with vertices $\{1, \ldots, n\}$. The *Laplacian* of G is the n x n matrix (l_{ij}) given by

$$l_{ij} = \begin{cases} d_i & \text{if } i = j \\ -\sigma_{ij} & \text{otherwise} \end{cases}$$

for all $i, j \in \{1 \ldots n\}$.

Note that this "magnetic" definition for a Laplacian matches the familiar definition of a Laplacian when the graph is simple. The Laplacian always has a nice spectrum to analyze, as the following theorem shows.

Theorem 4. *If G is a magnetic graph with Laplacian L, then there is an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of L, and the eigenvalues of L are real.*

Proof. L is Hermitian because $L = L^*$. Therefore, the Complex Spectral Theorem gives this statement exactly. □

We also want to establish that the eigenvalues of the Laplacian are non-negative. We will follow Jiang's paper (Theorem 3.5 and the preceding discussion), but modified slightly for magnetic graphs.

Definition 5. Let $G = (E, V)$ be a magnetic graph on n vertices with Laplacian L. For an edge (e, v) of G, the *contribution* of (e, v) to L is the n x n matrix (l_{ij}) given by

$$l_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } i \in \{e, v\} \\ -\sigma_{ij} & \text{if } \{i, j\} = \{e, v\} \\ 0 & \text{otherwise} \end{cases} .$$

We denote this matrix $L_{(e,v)}$.

As the name "contribution" suggests, the Laplacian of a graph is the sum of the contributions of its edges (only counting one edge between each pair of vertices):

Proposition 6. *If G is a magnetic graph with Laplacian L, then*

$$L = \sum_{\substack{(i,j) \in E \\ i<j}} L_{(i,j)}.$$

For the next few propositions, remember that if $z \in \mathbb{C}$, then $z \geq 0$ means z is real and nonnegative. Also, $\cdot$ is the complex dot product.

Proposition 7. *Let $G = (E, V)$ be a magnetic graph on n vertices with Laplacian L, and let $(i, j) \in E$. Then for any vector $v \in \mathbb{C}^n$, we have*

$$v \cdot L_{(i,j)} v \geq 0.$$

Proof. Let $v = (v_1, \ldots, v_n)$. By direct computation, the vector $L_{(i,j)}v$ consists of all 0s, except for $v_i - \sigma_{ij} v_j$ in the ith slot and $v_j - \sigma_{ji} v_i$ in the jth slot. Therefore,

$$\begin{aligned}
v \cdot L_{(i,j)} v &= v_i \overline{v_i - \sigma_{ij} v_j} + v_j \overline{v_j - \sigma_{ji} v_i} \\
&= v_i \left(\overline{v_i} - \overline{\sigma_{ij} v_j}\right) + v_j \left(\overline{v_j} - \overline{\sigma_{ji} v_i}\right) \\
&= |v_i|^2 - v_i \overline{\sigma_{ij} v_j} - v_j \overline{\sigma_{ji} v_i} + |v_j|^2 \\
&= |v_i|^2 - \sigma_{ji} v_i \overline{v_j} - \overline{\sigma_{ji} v_i \overline{v_j}} + |v_j|^2 \\
&= |v_i|^2 - \left(\sigma_{ji} v_i \overline{v_j} + \overline{\sigma_{ji} v_i \overline{v_j}}\right) + |v_j|^2 \\
&= |v_i|^2 - 2\mathrm{Re}\left(\sigma_{ji} v_i \overline{v_j}\right) + |v_j|^2 \\
&\geq |v_i|^2 - 2|\mathrm{Re}\left(\sigma_{ji} v_i \overline{v_j}\right)| + |v_j|^2 \\
&\geq |v_i|^2 - 2|\sigma_{ji} v_i \overline{v_j}| + |v_j|^2 \\
&= |v_i|^2 - 2|\sigma_{ji}||v_i||\overline{v_j}| + |v_j|^2 \\
&= |v_i|^2 - 2|\sigma_{ji}||v_i||\overline{v_j}| + |v_j|^2 \\
&= |v_i|^2 - 2|v_i||v_j| + |v_j|^2 \\
&= (|v_i| - |v_j|)^2 \\
&\geq 0.
\end{aligned}$$

□

Proposition 8. *Let $G = (E, V)$ be a magnetic graph on n vertices with Laplacian L. Then for any vector $v \in \mathbb{C}^n$, we have*

$$v \cdot Lv \geq 0.$$

Proof. We have:

$$v \cdot Lv = v \cdot \left(\sum_{\substack{(i,j) \in E \\ i<j}} L_{(i,j)} \right) v = v \cdot \sum_{\substack{(i,j) \in E \\ i<j}} L_{(i,j)} v = \sum_{\substack{(i,j) \in E \\ i<j}} \left(v \cdot L_{(i,j)} v\right),$$

where the last equality holds because inner products are additive in the second slot. By Proposition 7, this is a sum of nonnegative real numbers, so $v \cdot Lv \geq 0$. □

Theorem 9. *If G is a magnetic graph with Laplacian L, then the eigenvalues of L are nonnegative.*

Proof. Let λ be an eigenvalue of L, and let $v \in \mathbb{C}^n$ be a nonzero eigenvector corresponding to λ. Then,

$$\begin{aligned}
\lambda(v \cdot v) &= v \cdot \overline{\lambda} v \\
&= v \cdot \lambda v && \text{(since } \lambda \text{ is real by Theorem 4)} \\
&= v \cdot Lv \\
&\geq 0 && \text{(by Proposition 8).}
\end{aligned}$$

But v is nonzero, so $v \cdot v \geq 0$ by the definition of inner product. Thus we may conclude that $\lambda \geq 0$. □

2 Renumbering Vertices

The Laplacian of a magnetic graph is only defined when the graph's vertices are numbered $1, \dots, n$, and changing the numbering changes the Laplacian. However, we will show in this section that changing the vertex numbering has no effect on the spectrum of the Laplacian. This will allow us to unambiguously refer to the "spectrum of the Laplacian" of any magnetic graph, even if its vertices are not numbered or we don't know how they're numbered. With that goal in mind, we will first need a few facts about similar matrices.

Proposition 10. *Suppose A and B are similar $n \times n$ matrices over $\mathbb{C}$. Then A and B have the same nullity.*

Proof. Since A and B are similar, there exists an invertible matrix $P \in M_{n\times n}(\mathbb{C})$ such that

$$B = P^{-1}AP.$$

Let k be the nullity of A. Then there exists a basis $x_1, \dots, x_k$ of $\ker A$, where each $x_i \in \mathbb{C}^n$. To show that k is also the nullity of B, we will show that

$$P^{-1}x_1, \dots, P^{-1}x_k$$

is a basis for $\ker B$. First note that those vectors are linearly independent, since $x_1, \dots, x_k$ are linearly independent, and an invertible linear map preserves linear independence. To see that $\operatorname{span}(P^{-1}x_1, \dots, P^{-1}x_k) \subseteq \ker B$, observe that for any $i \in \{1, \dots, k\}$, we have

$$\begin{aligned} B(P^{-1}x_i) &= (P^{-1}AP)P^{-1}x_i \\ &= P^{-1}Ax_i \\ &= P^{-1}(0) \\ &= 0, \end{aligned}$$

so $P^{-1}x_i \in \ker B$. It remains to show that $\ker B \subseteq \operatorname{span}(P^{-1}x_1, \dots, P^{-1}x_k)$. To that end, let $v \in \ker B$. Then,

$$\begin{aligned} &P^{-1}APv = Bv = 0 \\ &\qquad \to APv = 0 \\ &\qquad \to Pv \in \ker A \\ &\qquad \to Pv = c_1x_1 + \cdots + c_kx_k && \text{for some } c_1, \dots, c_k \in \mathbb{C} \\ &\qquad \to v = c_1P^{-1}x_1 + \cdots + c_kP^{-1}x_k && \text{for some } c_1, \dots, c_k \in \mathbb{C} \\ &\qquad \to v \in \operatorname{span}(P^{-1}x_1, \dots, P^{-1}x_k). \end{aligned}$$

□

Proposition 11. *Suppose A and B are similar $n \times n$ matrices over $\mathbb{C}$. Then A and B have the same eigenvalues, counting multiplicity.*

Proof. Given $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$, we must show that λ is an eigenvalue of A with multiplicity k if and only if λ is an eigenvalue of B with multiplicity k. The directions are analogous, so we will only show the forward direction. Suppose λ is an eigenvalue of A with multiplicity k. Since A and B are similar, there exists an invertible matrix $P \in M_{n\times n}(\mathbb{C})$ such that

$$B = P^{-1}AP.$$

Then, we have

$$\begin{aligned}
\dim\ker(B-\lambda I) &= \dim\ker(P^{-1}AP-\lambda I)\\
&= \dim\ker(P^{-1}(A-\lambda I)P)\\
&= \dim\ker(A-\lambda I) && \text{(by Proposition 10)}\\
&= k.
\end{aligned}$$

Therefore, λ is an eigenvalue of B with multiplicity k. □

Proposition 12. *Let ϕ be a permutation of $\{1,\ldots,n\}$, and let $A, B \in M_{n\times n}(\mathbb{C})$ be related by*

$$A_{ij} = B_{\phi(i)\phi(j)}$$

for all $i, j \in \{1,\ldots,n\}$. Then

$$A = P^{-1}BP$$

for some permutation matrix $P \in M_{n\times n}(\mathbb{C})$.

Proof. Define P by

$$P_{ij} = \begin{cases} 1 & \text{if } i = \phi(j) \\ 0 & \text{otherwise} \end{cases}.$$

Since P is a permutation matrix, its inverse is its transpose, so the elements of P^{-1} are given by

$$P^{-1}_{ij} = P_{ji} = \begin{cases} 1 & \text{if } j = \phi(i) \\ 0 & \text{otherwise} \end{cases}.$$

It can now be verified by computation that for all $i, j \in \{1,\ldots,n\}$, we have

$$\left(P^{-1}BP\right)_{ij} = B_{\phi(i)\phi(j)} = A_{ij}.$$

□

Finally we can prove the main theorem of this section:

Theorem 13. *Let G be a magnetic graph with vertices $\{1,\ldots,n\}$. Then renumbering the vertices of G does not change the spectrum of its Laplacian.*

Proof. Let L be the Laplacian of G. Renumbering the vertices of G means selecting a permutation ϕ of $\{1,\ldots,n\}$ and replacing each vertex v with $\phi(v)$. This results in a new Laplacian L', where clearly $L'_{ij} = L_{\phi^{-1}(i)\phi^{-1}(j)}$ for all i, j. Applying Proposition 12, we get that

$$L'_{ij} = P^{-1}LP$$

for some permutation matrix P. Finally, Proposition 11 gives that L' has the same spectrum as L. □

As an example of the importance of Theorem 13, the following theorem would be confusing without it. In this theorem, we can assume that the vertices of G are the numbers $1, \ldots, n$, so we know exactly what matrix is meant by the Laplacian L of G. However, if the vertices of G are indeed $1, \ldots, n$, then it isn't possible for all the connected components to also have vertices in that format (only one of the components could get the vertex 1, for example). Therefore, the Laplacians $L_1, \ldots, L_k$ are ambiguous. The key is to remember that every numbering for the vertices of a component gives a Laplacian with the same spectrum, so it doesn't matter which one we choose. $L_1, \ldots, L_k$ are not really *the* Laplacians of $G_1, \ldots, G_k$, but rather some Laplacians that result from an arbitrary choice of vertex numberings. And the theorem holds regardless of the choice.

It might simplify the numbering considerations if we define the *Laplacian spectrum* of a graph, which would be the unique spectrum resulting from every numbering of the vertices. The Laplacian spectrum would then be independent of the choice of numbering, meaning we could often ignore numberings entirely, or just choose them as we need them. But I'm not sure that's worth creating a new definition.

Theorem 14. *Let G be a magnetic graph with Laplacian L, and let $G_1, \ldots G_k$ be the connected components of G, with Laplacians $L_1, \ldots, L_k$. Then the spectrum of L is equal to the joined spectra of $L_1, \ldots, L_k$.*

Proof. Renumber the vertices of G so that all the vertices of G_1 precede all the vertices of G_2, and so on. By Theorem 13, this doesn't affect the spectrum of L. But it makes L into a block diagonal matrix whose blocks $B_1, \ldots, B_k$ are some Laplacians of $G_1, \ldots, G_k$, respectively. Since $B_1, \ldots, B_k$ are Laplacians of $G_1, \ldots, G_k$, Theorem 13 says they have the same spectra as $L_1, \ldots, L_k$. Therefore, it suffices to prove the spectrum of L is equal to the joined spectra of $B_1, \ldots, B_k$.

Let $\lambda \in \mathbb{C}$, and let $m_1, \ldots, m_k$ be the multiplicities of λ as an eigenvalue of $B_1, \ldots, B_k$, respectively. We will be done if we can show that the multiplicity of λ as an eigenvalue of L is equal to $m_1 + \cdots + m_k$. Indeed, the multiplicity of λ as an eigenvalue of L is

$$\begin{aligned}
\dim\ker(L - \lambda I) &= \dim\ker\left(\begin{pmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_k \end{pmatrix} - \lambda I\right) \\
&= \dim\ker\begin{pmatrix} B_1 - \lambda I & & & 0 \\ & B_2 - \lambda I & & \\ & & \ddots & \\ 0 & & & B_k - \lambda I \end{pmatrix} \\
&= \dim\ker(B_1 - \lambda I) + \cdots + \dim\ker(B_k - \lambda I) \\
&= m_1 + \cdots + m_k.
\end{aligned}$$

Here, the second to last equality holds because the nullity of a block diagonal matrix is equal to the sum of the nullities of the blocks (to see this, prove the analogous statement for rank, apply the rank nullity theorem, and simplify). $\square$

3 Balanced Magnetic Graphs

Definition 15. A magnetic graph is called *balanced* if the signatures along any closed walk multiply to 1.

Proposition 16. *Let $G = (E, V, \sigma)$ be a magnetic graph. Then the following are equivalent.*

1. *G is balanced.*
2. *The signatures along any cycle multiply to 1 (in either direction).*
3. *For every pair of connected vertices $u, v \in V$, there exists $z \in \mathbb{C}_{|z|=1}$ such that along every walk from u to v, the signatures multiply to z.*

Because the number z in condition 3 is clearly unique for each pair of connected vertices, we can make the following definition.

Definition 17. Let $G = (E, V, \sigma)$ be a balanced magnetic graph, and let $u, v \in V$ be connected. Then the *signature distance* from u to v, written $\operatorname{sigdist}(u, v)$, is the product of the signatures along every walk from u to v.

4 Switching Equivalence

Definition 18. If $G = (V, E, \sigma)$ is a magnetic graph, then a function $c : V \to \mathbb{C}_{|z|=1}$ is called a *switching function* for G. We may write c_v as shorthand for $c(v)$.

Definition 19. Let $G = (V, E, \sigma)$ be a magnetic graph, and let c be a switching function for G. Then *c applied to G* is the magnetic graph with the same edges and vertices, whose signature function τ is given by

$$\tau_{vw} = \overline{c_v}\sigma_{vw}c_w$$

for all $v \sim w$.

Definition 20. A magnetic graph G is *switching equivalent* to a magnetic graph H if there exists a switching function c for G such that c applied to G gives H.

Proposition 21. *Switching equivalence is an equivalence relation.*

Proposition 22. *Let $G^\sigma = (V, E, \sigma)$ and $G^\tau = (V, E, \tau)$ be two magnetic graphs with the same vertices and edges. Then G^σ and G^τ are switching equivalent if and only if each connected component of G^σ is switching equivalent to the corresponding connected component of G^τ.*

Proof. ($\to$) Suppose G^σ and G^τ are switching equivalent. Then there is a switching function c for G^σ that, when applied to G^σ, gives G^τ. Now let $C^\sigma = (U, D, \sigma|_D)$ be an arbitrary connected component of G^σ. Then $C^\tau = (U, D, \tau|_D)$ is the corresponding connected component of G^τ. Choose b to be the restriction of c to U, and note that b is a switching function for C^σ. It remains to show that b applied to C^σ gives C^τ. Indeed, for any adjacent vertices $v, w \in U$, the signature of (v, w) given by b applied to C^σ is

$$\overline{b_v}(\sigma|_D)_{vw}b_w = \overline{c_v}\sigma_{vw}c_w = \tau_{vw} = (\tau|_D)_{vw},$$

which is the signature of (v, w) in C^τ.

($\leftarrow$) Suppose each connected component of G^σ is switching equivalent to the corresponding connected component of G^τ. Call the connected components $C_1, \ldots, C_k$. Then, for each $i \in \{1, \ldots, k\}$, let b^i be the switching function that takes the ith connected component of G^σ to the ith connected component of G^τ. Define a switching function c for G^σ by $c_v = (b^i)_v$, where i is such that v belongs to C_i. It is easy to verify that c applied to G^σ gives G^τ. $\square$

We now work towards another characterization of switching equivalence. We want to show that two magnetic graphs with the same vertices and edges are switching equivalent if and only if the product of the signatures along any closed walk is the same for the two graphs. To make this easier, we will start with a lemma.

Lemma 23. *If $G = (V, E, \sigma)$ is a balanced magnetic graph, then G is switching equivalent to the simple graph on (V, E).*

Proof. Because of Proposition 22 and the fact that connected components of a balanced graph are also balanced, it suffices to prove the lemma in the case where V is connected. So assume V is connected. We must construct a switching function that takes every signature in G to 1. To do this, fix a vertex $u_0 \in V$ and define the switching function $c : V \to \mathbb{C}_{|z|=1}$ by

$$c_u = \mathrm{sigdist}(u, u_0)$$

for all $u \in V$. Then for every pair of adjacent vertices $v, w \in V$, we have

$$\begin{aligned}\overline{c_v}\sigma_{vw}c_w &= \overline{\mathrm{sigdist}(v, u_0)}\sigma_{vw}\mathrm{sigdist}(w, u_0)\\ &= \overline{\mathrm{sigdist}(v, u_0)}\mathrm{sigdist}(v, u_0)\\ &= 1,\end{aligned}$$

as desired. $\square$

Theorem 24. *Let $G^\sigma = (V, E, \sigma)$ and $G^\tau = (V, E, \tau)$ be two magnetic graphs with the same vertices and edges. Then G^σ and G^τ are switching equivalent if and only if along every closed walk, the product of the signatures of G^σ equals the product of the signatures of G^τ.*

Proof. ($\to$) Suppose G^σ and G^τ are switching equivalent. Then there is a switching function $c : V \to \mathbb{C}_{|z|=1}$ that takes G^σ to G^τ. Let $v_1, \ldots, v_k$ be the vertices along a closed walk, so that $v_1 = v_k$. Then,

$$\begin{aligned}\sigma_{v_1v_2}\sigma_{v_2v_3}\ldots\sigma_{v_{k-1}v_k} &= \overline{c_{v_1}}\sigma_{v_1v_2}\sigma_{v_2v_3}\ldots\sigma_{v_{k-1}v_k}c_{v_1}\\ &= \overline{c_{v_1}}\sigma_{v_1v_2}\sigma_{v_2v_3}\ldots\sigma_{v_{k-1}v_k}c_{v_k}\\ &= (\overline{c_{v_1}}\sigma_{v_1v_2}c_{v_2})(\overline{c_{v_2}}\sigma_{v_2v_3}c_{v_3})\ldots(\overline{c_{v_{k-1}}}\sigma_{v_{k-1}v_k}c_{v_k})\\ &= \tau_{v_1v_2}\tau_{v_2v_3}\ldots\tau_{v_{k-1}v_k}.\end{aligned}$$

Therefore, along every closed walk, the product of the signatures of G^σ equals the product of the signatures of G^τ.

($\leftarrow$) Suppose that along every closed walk, the product of the signatures of G^σ equals the product of the signatures of G^τ. Define a new graph $G^s = (V, E, s)$, where the signatures s are given by

$$s_{vw} = \overline{\tau_{vw}}\sigma_{vw}$$

for all vertices $v \sim w$. It is easy to see that along any closed walk, the signatures of G^s multiply to 1. Therefore, G^s is balanced. By Lemma 23, it follows that G^s is switching equivalent to the simple graph on (V, E), whose signatures are all 1. Call the switching function that takes G^s to the simple graph c. Then for any vertices $v \sim w$, we have

$$\begin{aligned}\overline{c_v}s_{vw}c_w = 1 &\to \overline{c_v}\left(\overline{\tau_{vw}}\sigma_{vw}\right)c_w = 1\\ &\to \overline{c_v}\sigma_{vw}c_w = \tau_{vw}.\end{aligned}$$

Therefore, applying the switching function c to G^σ gives G^τ, so G^σ and G^τ are switching equivalent. $\square$

Many nice consequences flow from this characterization of switching equivalence.

Corollary 25. *A magnetic graph is balanced if and only if it is switching equivalent to the simple graph with the same vertices and edges.*

Corollary 26. *If a magnetic graph is switching equivalent to a balanced magnetic graph, then it is also balanced.*

Corollary 27. *Any two balanced magnetic graphs with the same vertices and edges are switching equivalent.*

Corollary 28. *Every acyclic magnetic graph is switching equivalent to the simple graph.*

Corollary 29. *Every magnetic cycle graph is switching equivalent to a graph which differs from the simple graph by at most one edge.*

The next theorem is why we care about switching equivalence in the context of spectral graph theory.

Theorem 30. *Let $G^\sigma = (V, E, \sigma)$ and $G^\tau = (V, E, \tau)$ be magnetic graphs with the same vertices and edges. Then if G^σ and G^τ are switching equivalent, it follows that their Laplacians have the same spectrum.*

Proof. Let n be the number of vertices in V. Let L^σ and L^τ be the Laplacians for G^σ and G^τ. Let c be a switching function that takes G^σ to G^τ. Then,

$$\begin{aligned} L^\tau &= \operatorname{diag}(\overline{c_1}, \ldots, \overline{c_n})\,(L^\sigma)\operatorname{diag}(c_1, \ldots, c_n) \\ &= \operatorname{diag}(c_1, \ldots, c_n)^{-1}\,(L^\sigma)\operatorname{diag}(c_1, \ldots, c_n). \end{aligned}$$

We see that L^σ and L^τ are similar matrices. Therefore, by Proposition 11, they have the same spectrum. □

The converse of Theorem 30 doesn't hold. For example, the 3-cycle graph with signatures 1, 1, and i (going in order around the cycle) has the same spectrum as the 3-cycle graph with signatures 1, 1, and $-i$. However, the product along the cycles is different, so by Theorem 24, the graphs cannot be switching equivalent.

5 The Eigenvalue 0 of the Laplacian

If G is a magnetic graph with vertices $\{1, \ldots, n\}$ and x is a vector in $\mathbb{C}^n$, then it makes sense to imagine x as consisting of values for the vertices of G. Specifically, the entry x_i is like a value for the vertex i. With this interpretation, multiplying x by the Laplacian of G creates a new list of vertex values by taking linear combinations of the current vertex values. The weights of the combinations are determined by the structure of the Laplacian. In fact, the next proposition shows that a vertex's new value after the multiplication depends only on its current value and the values of its neighbors.

Proposition 31. *Let G be a magnetic graph on n vertices with Laplacian L. Let $x \in \mathbb{C}^n$ with $x = (x_1, \ldots, x_n)$. Then*

$$(Lx)_i = d_i x_i - \sum_{j \sim i} \sigma_{ij} x_j$$

for each $i \in \{1, \ldots, n\}$.

Proof. Let l_{ij} denote the element of L at row i and column j. Let $i \in \{1, \dots, n\}$ be given. Then,

$$\begin{aligned}(Lx)_i &= \sum_{j=1}^{n} l_{ij} x_j \\ &= l_{ii} x_i + \sum_{j \neq i} l_{ij} x_j \\ &= d_i x_i + \sum_{j \sim i} (-\sigma_{ij}) x_j \\ &= d_i x_i - \sum_{j \sim i} \sigma_{ij} x_j.\end{aligned}$$

□

We are particularly interested in when the "vertex values" all go to 0 under multiplication by the Laplacian. It turns out that this happens exactly when each value is a signature-weighted average of the neighboring values, as the next lemma demonstrates.

Lemma 32. *Let G be a magnetic graph on n vertices with Laplacian L. Suppose $x \in \mathbb{C}^n$ with $x = (x_1, \dots, x_n)$. Then $Lx = 0$ if and only if*

$$x_i = \frac{1}{d_i} \sum_{j \sim i} \sigma_{ij} x_j$$

for each $i \in \{1, \dots n\}$ with $d_i \neq 0$.

Proof. This follows easily from Proposition 31. □

We now show that this condition is equivalent to a much simpler one.

Theorem 33. *Let G be a magnetic graph on n vertices with Laplacian L. Suppose $x \in \mathbb{C}^n$ with $x = (x_1, \dots, x_n)$. Then $Lx = 0$ if and only if*

$$x_i = \sigma_{ij} x_j$$

whenever $i \sim j$.

Proof. ($\leftarrow$) Suppose $x_i = \sigma_{ij} x_j$ whenever $i \sim j$. We will show that $Lx = 0$ using the condition from Lemma 32. Fix $i \in \{1, \dots, n\}$ with $d_i \neq 0$. Then,

$$x_i = \frac{1}{d_i} d_i x_i = \frac{1}{d_i} \sum_{j \sim i} x_i = \frac{1}{d_i} \sum_{j \sim i} \sigma_{ij} x_j.$$

($\rightarrow$) Suppose $Lx = 0$. We first argue that whenever a and b are vertices in the same connected component of G, we have $|x_a| = |x_b|$. To see this, take an arbitrary connected component of G, and choose a vertex m in that component which maximizes $|x_m|$. If $d_m = 0$, then m is the only vertex in the component, so our conclusion holds trivially. On the other hand, if $d_m \neq 0$, then we can apply Lemma 32 to get

$$\begin{aligned}
|x_m| &= \left|\frac{1}{d_m}\sum_{l\sim m}\sigma_{ml}x_l\right| \\
&= \frac{1}{d_m}\left|\sum_{l\sim m}\sigma_{ml}x_l\right| \\
&\le \frac{1}{d_m}\sum_{l\sim m}|\sigma_{ml}x_l| && \text{(by the Triangle Inequality)} \\
&= \frac{1}{d_m}\sum_{l\sim m}|x_l|.
\end{aligned}$$

That is, $|x_m|$ is the mean of the values $|x_l|$ where $l \sim m$. But our choice of m ensures that $|x_m| \geq |x_l|$ for each $l \sim m$, so in fact $|x_m| = |x_l|$ for each $l \sim m$. It follows that any vertex l which is adjacent to m also maximizes the quantity $|x_l|$, so we can apply the same argument to each l, repeating the process until we conclude that $|x_a| = |x_b|$ for every pair of vertices a, b in the connected component.

Now let i and j be vertices of G with $i \sim j$. We want to prove that $x_i = \sigma_{ij}x_j$. From the previous paragraph, we already have that

$$|x_i| = |x_j| = |\sigma_{ij}x_j|,$$

so it suffices to show x_i differs from $\sigma_{ij}x_j$ by a positive real factor. Observe that

$$\begin{aligned}
\left|\sum_{l\sim i}\sigma_{il}x_l\right| &= |d_i x_i| && \text{(by Lemma 32)} \\
&= d_i|x_i| \\
&= \sum_{l\sim i}|x_i| \\
&= \sum_{l\sim i}|x_l| && \text{(by the previous paragraph)} \\
&= \sum_{l\sim i}|\sigma_{il}x_l|.
\end{aligned}$$

Therefore, assuming $\sigma_{ij}x_j \neq 0$ (the other case is trivial), we may conclude that

$$\sum_{l\sim i}\sigma_{il}x_l = r(\sigma_{ij}x_j)$$

for some positive real r. Then,

$$\begin{aligned}
x_i &= \frac{1}{d_i}\sum_{l\sim i}\sigma_{il}x_l && \text{(by Lemma 32)} \\
&= \frac{1}{d_i}r(\sigma_{ij}x_j),
\end{aligned}$$

so x_i differs from $\sigma_{ij}x_j$ by a positive real factor, as desired. □

Corollary 34. *Let G be a simple graph on n vertices with Laplacian L. Suppose $x \in \mathbb{C}^n$ with $x = (x_1, \ldots, x_n)$. Then $Lx = 0$ if and only if*

$$x_i = x_j$$

whenever $i \sim j$.

Proof. This follows from Theorem 33 and the fact that every signature in a simple graph is 1. $\square$

In a way, Theorem 33 completely answers the question of when 0 is an eigenvalue of the Laplacian. 0 is an eigenvalue exactly when there is a nonzero vector x satisfying the condition in Theorem 33. But that condition is phrased in terms of local properties of the graph. We want to find global properties that determine whether 0 is an eigenvalue, and if so, reveal its multiplicity. The final few results of the section achieve that.

Proposition 35. *If G is a connected and balanced magnetic graph, then 0 is an eigenvalue of its Laplacian with multiplicity 1.*

Proof. Since G is balanced, Corollary 25 gives that it is switching equivalent to the simple graph on the same edges and vertices. By Theorem 30, the spectrum of that simple graph is the same as the spectrum of G. So we only have to show that 0 is an eigenvalue with multiplicity 1 for the Laplacian of a connected, simple graph. Indeed, it is clear from Corollary 34 that the kernel of such a Laplacian is exactly the one-dimensional space of vectors whose entries are all the same. $\square$

Proposition 36. *If G is a connected and unbalanced magnetic graph, then 0 is not an eigenvalue of its Laplacian.*

Proof. Let n be the number of vertices in G, and let $x = (x_1, \ldots, x_n) \in \ker L$. We will be done if we can show that x must be 0. Since G is unbalanced, it has a closed walk along which the signatures multiply to some $c \neq 1$. Let k be the starting and ending vertex of the walk. Repeated application of Theorem 33 along that walk gives that $x_k = cx_k$, so $x_k = 0$. Then applying Theorem 33 outward from k, we get $x_j = 0$ for every $j \in \{1, \ldots, n\}$. This means $x = 0$, as desired. $\square$

Theorem 37. *Let G be a magnetic graph with Laplacian L. Then the multiplicity of 0 as an eigenvalue of L is equal to the number of balanced connected components of G.*

Proof. By Theorem 14, it suffices to sum the multiplicities of 0 as an eigenvalue for G's connected components. By Proposition 35, the balanced components each contribute 1 multiplicity, and by Proposition 36, the unbalanced components each contribute 0 multiplicity. Therefore, the total multiplicity is the number of balanced components. $\square$

Corollary 38. *The Laplacian of a simple graph always has 0 as an eigenvalue, and its multiplicity is equal to the number of connected components in the graph.*

Proof. A simple graph is balanced because all its signatures are 1. Thus all its connected components are balanced. Since the multiplicity of 0 is equal to the number of balanced components, in this case it is just equal to the total number of components. $\square$

6 The Frustration Index

The frustration index of a magnetic graph is a way to measure how close the graph is to being balanced.

Definition 39. Let $G = (V, E, \sigma)$ be a magnetic graph. Then the frustration index of G is

$$\min_{c:V\to\mathbb{C}_{|z|=1}} \sum_{i\sim j} |\overline{c_i}\sigma_{ij}c_j - 1|.$$

Note that a balanced graph has frustration index 0 (since it is switching equivalent to a graph whose signatures are all 1). Cycle graphs also have a nice frustration index, as we will show next. For this we need a lemma.

Lemma 40. *If $z_1, \dots, z_n \in \mathbb{C}_{|z|=1}$, then*

$$|z_1 z_2 \dots z_n - 1| \le |z_1 - 1| + \dots + |z_n - 1|.$$

Proof. We use induction on n. The claim is clearly true in the base case $n = 1$. Now let $k \in \mathbb{N}$, and assume the claim holds for $n = k$. Then,

$$\begin{aligned}
|z_1 z_2 \dots z_{k+1} - 1| &= |z_1 z_2 \dots z_{k+1} - z_{k+1} + z_{k+1} - 1| \\
&\le |z_1 z_2 \dots z_{k+1} - z_{k+1}| + |z_{k+1} - 1| \\
&= |z_{k+1}(z_1 z_2 \dots z_k - 1)| + |z_{k+1} - 1| \\
&= |z_{k+1}||z_1 z_2 \dots z_k - 1| + |z_{k+1} - 1| \\
&= |z_1 z_2 \dots z_k - 1| + |z_{k+1} - 1| \\
&\le |z_1 - 1| + \dots + |z_k - 1| + |z_{k+1} - 1|.
\end{aligned}$$

So the claim also holds for $n = k + 1$, and we are done. ∎

Proposition 41. *Let $G = (V, E, \sigma)$ be a cycle graph on the vertices $\{1, \dots, n\}$. Let p be the product of the signatures along the cycle in one direction. Then the frustration index of G is $2|p - 1|$.*

Proof. Without loss of generality, let

$$p = \sigma_{12}\sigma_{23} \dots \sigma_{n1}.$$

By Theorem 24, G is switching equivalent to the graph with signatures τ, where

$$\begin{aligned}
&\tau_{12} = \tau_{23} = \dots = \tau_{(n-1)n} = 1, \text{ and} \\
&\tau_{n1} = p.
\end{aligned}$$

Let b be a switching function which takes G's signatures to τ. Then,

$$\begin{aligned}
\sum_{i\sim j} \left|\overline{b_i}\sigma_{ij}b_j - 1\right| &= \sum_{i\sim j} |\tau_{ij} - 1| \\
&= |\tau_{n1} - 1| + |\tau_{1n} - 1| && \text{(eliminating 0 terms)} \\
&= |p - 1| + |\overline{p} - 1| \\
&= |p - 1| + |p - 1| \\
&= 2|p - 1|.
\end{aligned}$$

So we have shown that there exists a switching function b for G with

$$\sum_{i\sim j} \left|\overline{b_i}\sigma_{ij}b_j - 1\right| = 2|p-1|.$$

To finish the proof, we must show that whenever c is a switching function for G, we have

$$\sum_{i\sim j} |\overline{c_i}\sigma_{ij}c_j - 1| \geq 2|p-1|.$$

To see this, let c be an arbitrary switching function for G, and let s be the new signatures obtained by applying c to G. Then,

$$\begin{aligned}
\sum_{i\sim j} |\overline{c_i}\sigma_{ij}c_j - 1| &= \sum_{i\sim j} |s_{ij} - 1| \\
&= 2\left(|s_{12} - 1| + |s_{23} - 1| + \cdots + |s_{n1} - 1|\right) \\
&\geq 2|s_{12}s_{23}\ldots s_{n1} - 1| && \text{(Lemma 40)} \\
&= 2|\sigma_{12}\sigma_{23}\ldots\sigma_{n1} - 1| && \text{(switching preserves products along cycles)} \\
&= 2|p-1|.
\end{aligned}$$

$\square$

7 Magnetic Graphs as Group Elements

Definition 42. Let G be a magnetic graph whose signatures are all roots of unity. The *multiplicative order* of G is the smallest $k \in \mathbb{N}$ such that raising all signatures to the power of k makes G balanced.

There are a few different ways to characterize this idea:

Proposition 43. *Let G be a magnetic graph whose signatures are all roots of unity, and let $k \in \mathbb{N}$. Then the following are equivalent.*

1. *G has multiplicative order k.*
2. *S'_k is the subgroup of S' generated by the cycle products of G.*
3. *k is the smallest natural number such that the cycle products of G all belong to S'_k.*

From 3, it is clear that we have:

Proposition 44. *Multiplicative order is invariant under switching.*

Definition 45. A connected magnetic graph G is called *quasi-simple* if there is a spanning tree S such that the signatures along S are all equal to 1. We may also be more specific and say that G is quasi-simple along S.

Quasi-simple graphs are useful to us because of the following properties:

Proposition 46. *Let G be a connected magnetic graph, and let S be a spanning tree for G. Then there is a unique magnetic graph which is switching equivalent to G and quasi-simple along S.*

8 Lifts

We want a way to convert a magnetic graph into a simple graph while preserving most of its important information. Unfortunately, there is no way to do this in general, since there are too many possible signatures that may appear in a magnetic graph. However, if the signatures are roots of unity (in other words, if they all belong to some S'_k), then we can use a lift.

Definition 47. Let $k \in \mathbb{N}$, and let $G = (V, E, \sigma)$ be a magnetic graph whose signatures all belong to S'_k. Then the *k-lift* of G is the simple graph on the vertices $V \times S'_k$, where $(u, z) \sim (v, \omega)$ exactly when $u \sim v$ and $z\sigma_{uv} = \omega$.

When looking at a lift, we can consider just the vertices of the form (v, ω_0) for some fixed ω_0, or we can consider just the vertices of the form (v_0, ω) for some fixed v_0. These are called levels and fibers, respectively.

Definition 48. Let H be the k-lift of a magnetic graph G, and let $\omega \in S'_k$. Then *level* ω of H is the subgraph of H whose vertices have the form (v, ω) for some $v \in V(G)$.

Definition 49. Let H be the k-lift of a magnetic graph G, and let $v \in S'_k$. Then *fiber* v of H is the subgraph of H whose vertices have the form (v, ω) for some $\omega \in S'_k$.

Lifts interact nicely with connected components and switching equivalence, as the next two facts show.

Proposition 50. *Let G be a magnetic graph whose signatures all belong to S'_k. Then the k-lift of G is isomorphic to the union of the k-lifts of its connected components.*

Proof. This follows easily from the definition of k-lift. □

Theorem 51. *Let $G^\sigma = (V, E, \sigma)$ and $G^\tau = (V, E, \tau)$ be magnetic graphs on the same vertices and edges, where all signatures belong to S'_k. If G^σ and G^τ are switching equivalent, then their k-lifts are isomorphic.*

Proof. Because of Proposition 50 and Proposition 22, we may assume G^σ and G^τ are connected. Now suppose G^σ and G^τ are switching equivalent. Let H^σ and H^τ be the k-lifts of G^σ and G^τ. We seek an isomorphism $\phi : V(H^\sigma) \to V(H^\tau)$. To construct this isomorphism, first define a new magnetic graph $G^s = (V, E, s)$, where the signatures s are given by

$$s_{ij} = \sigma_{ij}\overline{\tau_{ij}}$$

for all vertices $i \sim j$. Then G^s is balanced (since G^σ and G^τ are switching equivalent), so we may refer to $\text{sigdist}^s(i, j)$ for any vertices $i, j \in V$. Now fix a vertex $w \in V$, and define our isomorphism $\phi : V(H^\sigma) \to V(H^\tau)$ by

$$\phi(v, z) := (v, z\ \text{sigdist}^s(v, w))$$

for all $v \in V$ and $z \in S'_k$.

To show that ϕ is indeed an isomorphism, we must first show that it is bijective. But its domain and its codomain are the same finite set, so it suffices to show that it's injective. Let $(v_1, z_1), (v_2, z_2) \in V(H^\sigma)$, and assume

$$\phi(v_1, z_1) = \phi(v_2, z_2).$$

From the definition of ϕ, it follows that $v_1 = v_2$ and

$$z_1 \text{ sigdist}^s(v_1, w) = z_2 \text{ sigdist}^s(v_2, w).$$

Substituting v_2 for v_1 and dividing away the sigdist, the previous equation becomes $z_1 = z_2$. So we have $(v_1, z_1) = (v_2, z_2)$, and we have proven that ϕ is injective.

It remains to show that two vertices are adjacent in H^σ if and only if their images under ϕ are adjacent in H^τ. To that end, let (v_1, z_1) and (v_2, z_2) be vertices of H^σ. If $v_1 \not\sim v_2$, then neither $(v_1, z_1) \sim (v_2, z_2)$ in H^σ nor $\phi(v_1, z_1) \sim \phi(v_2, z_2)$ in H^τ, so the desired equivalence holds. On the other hand, assume $v_1 \sim v_2$. Then we have

$$\begin{aligned}
& (v_1, z_1) \sim (v_2, z_2) \text{ in } H^\sigma \\
\iff & z_1 \sigma_{v_1 v_2} = z_2 \\
\iff & z_1 (\sigma_{v_1 v_2} \overline{\tau_{v_1 v_2}}) \tau_{v_1 v_2} = z_2 \\
\iff & z_1 s_{v_1 v_2} \tau_{v_1 v_2} = z_2 \\
\iff & z_1 s_{v_1 v_2} \left(\text{sigdist}^s(v_2, w)\right) \tau_{v_1 v_2} = z_2 \left(\text{sigdist}^s(v_2, w)\right) \\
\iff & z_1 \text{ sigdist}^s(v_1, w) \tau_{v_1 v_2} = z_2 \text{ sigdist}^s(v_2, w) \\
\iff & (v_1, z_1 \text{ sigdist}^s(v_1, w)) \sim (v_2, z_2 \text{ sigdist}^s(v_2, w)) \text{ in } H^\tau \\
\iff & \phi(v_1, z_1) \sim \phi(v_2, z_2) \text{ in } H^\tau.
\end{aligned}$$

□

What happens if we apply an unnecessarily large lift to a graph? For example, what if we take the 6-lift of a graph that has a 2-lift? The answer is that we get multiple copies of the smaller lift.

Theorem 52. *Let G be a magnetic graph whose signatures all belong to S'_k, and let $m \in \mathbb{N}$. Then the mk-lift of G is isomorphic to m copies of the k-lift of G.*

Proof. Let $\omega_0, \ldots, \omega_{mk-1}$ be the mkth roots of unity, starting with 1 and going counterclockwise. Let J be the k-lift of G and let H be the mk-lift of G. For $i \in \{0, \ldots, m-1\}$, let J_i be the subgraph of H consisting of all the levels ω_l with $l \equiv i \pmod{m}$. In fact, each J_i is made up of k different levels of H.

It is easy to see that the J_i's are disjoint and cover all the vertices of H. Also, we claim that the J_i's are mutually disconnected. It suffices to show that every edge of H begins and ends in the same J_i. Take an arbitrary edge of H; say that it goes from J_a to J_b. That means the edge connects a level ω_α to a level ω_β, where $\alpha \equiv a \pmod{m}$ and $\beta \equiv b \pmod{m}$. So the edge corresponds to an edge in G whose signature is $\omega_{\beta-\alpha}$ (assuming without loss of generality that $\beta \geq \alpha$). All the signatures of G belong to S'_k, so in particular $\omega_{\beta-\alpha} \in S'_k$. But S'_k is exactly the set $\left\{\omega_0, \omega_m, \omega_{2m}, \ldots, \omega_{(k-1)m}\right\}$, so we must have $m \mid \beta - \alpha$. It follows that $\alpha \equiv \beta \pmod{m}$, so $a \equiv b \pmod{m}$, so $a = b$, as desired.

It remains to show that each J_i is isomorphic to J. Fix $i \in \{0, \ldots, m-1\}$. Note that every vertex in J_i is of the form (v, ω_{am+i}) for some $v \in V(G)$ and unique $a \in \{0, \ldots, k-1\}$. So we may define our isomorphism $\phi : V(J_i) \to V(J)$ by

$$\phi(v, \omega_{am+i}) := (v, \omega_{am}).$$

Since $\omega_{am} \in S'_k$, ϕ is indeed a function from $V(J_i)$ into $V(J)$. In fact, ϕ is clearly surjective, and thus bijective.

We must finally show that two vertices are adjacent in J_i if and only if their images under ϕ are adjacent in J. To that end, let (u, ω_{pm+i}) and (v, ω_{qm+i}) be arbitrary vertices of J_i. If $u \not\sim v$, then neither $(u, \omega_{pm+i}) \sim (v, \omega_{qm+i})$ in J_i, nor $\phi(u, \omega_{pm+i}) \sim \phi(v, \omega_{qm+i})$ in J, so the desired equivalence holds. On the other hand, suppose $u \sim v$. Then, letting σ be the signature function of G, we have

$$\begin{aligned}
& (u, \omega_{pm+i}) \sim (v, \omega_{qm+i}) \text{ in } J_i \\
\iff \quad & \omega_{pm+i}\sigma_{uv} = \omega_{qm+i} \\
\iff \quad & \omega_{pm}\omega_i\sigma_{uv} = \omega_{qm}\omega_i \\
\iff \quad & \omega_{pm}\sigma_{uv} = \omega_{qm} \\
\iff \quad & (u, \omega_{pm}) \sim (v, \omega_{qm}) \text{ in } J \\
\iff \quad & \phi(u, \omega_{pm+i}) \sim \phi(v, \omega_{qm+i}) \text{ in } J.
\end{aligned}$$

□

We can now describe the lift of a balanced graph.

Proposition 53. *Let $G = (V, E, \sigma)$ be a balanced graph with all signatures in S'_k. Then the k-lift of G is isomorphic to k copies of the simple graph (V, E).*

Proof. By Corollary 25, we can apply a switching function to G to make it simple. Theorem 51 says that this does not affect G's k-lift, up to isomorphism. But now G has a 1-lift, which is just the simple graph (V, E). By Theorem 52, the k-lift of G is isomorphic to k copies of the 1-lift of G, so we are done. □

Given a magnetic graph G, it would be great if we could find a $k \in \mathbb{N}$ such that

1. G (or some switching equivalent graph) can be k-lifted; and

2. Whenever G has an l-lift, $k \mid l$.

If this were the case, then every lift of G would be an mk-lift of G for some m, so it could be described as m copies of the k-lift. In fact, choosing k to be the multiplicative order of G gives these properties, as we will now show.

Proposition 54. *Let G be a magnetic graph with multiplicative order k. Then there is a switching function for G which takes all its signatures into S'_k.*

Proof. We may assume G is connected. Pick a spanning tree S for G, and apply the switching function which reduces G along S. Then all the edges in S get signature 1, which belongs to S'_k, as desired. It remains to consider an arbitrary edge $d \notin S$. Since S is a spanning tree, d makes a cycle with the edges of the S. And since k is the multiplicative order of G (this did not change when we applied our switching function), raising each signature in that cycle to the power of k makes the product along the cycle 1. But all the edges in the cycle have signature 1 except for (possibly) d, so raising d's signature to the power of k must give 1. In other words, $d \in S'_k$. □

This establishes that if k is the multiplicative order of G, then there is a graph which is switching equivalent to G that can be k-lifted. The next proposition shows that (thanks to Theorem 52) every lift is made up of one or more copies of this "atomic" k-lift.

Proposition 55. *Let G be a magnetic graph with multiplicative order k. Then whenever G has an l-lift, $k \mid l$.*

Proof. Since G has an l-lift, all the signatures in G belong to S'_l. Thus, viewing G as a group element, $G^l = e$. But any power of a group element that takes it to the identity must be divisible by the order of that element, so $k \mid l$. $\square$

Proposition 55 shows that, when k is the multiplicative order of G, a k-lift has no redundancy in the sense that it is not composed of multiple copies of a smaller lift (and in fact, there is no smaller lift). But we can go further and say that if G is connected, then the k-lift is connected too.

Theorem 56. *Let G be a connected graph with multiplicative order k, with all signatures in S'_k. Then the k-lift of G is connected.*

Proof. Reduce G along a spanning tree S. This does not affect the multiplicative order, the fact that all the signatures are in S'_k, or the k-lift (up to isomorphism). Call the k-lift of the reduced graph H. Since G has all signatures 1 along S, each level of H is spanned by a copy of S, so the levels are internally connected. It remains to show that every level can be reached from every other level. We will show that every level can be reached from level 1. Let $\omega \in S'_k$, and consider level ω. Finding a path from level 1 to level ω in H means finding a path in G whose signatures multiply to ω. Since k is the multiplicative order of G, part 2 of Proposition 43 says there exists a series of cycle products $p_1, \ldots, p_m$ in G (not necessarily distinct) which multiply to ω. Walking each of these cycles once and otherwise traveling along S gives a path whose signatures multiply to ω. $\square$

We finish this section by completely describing the spectrum of a graph's k-lift.

Theorem 57. *Let G be a magnetic graph with signatures in S'_k, and let H be the k-lift of G. Then the spectrum of H is the joined spectra of the graphs $G, G^2, \ldots, G^k$.*

Proof. Let n be the number of vertices of G, and let $\omega_0, \ldots, \omega_{k-1}$ be the kth roots of unity, starting at 1 and going counterclockwise. For each $i \in \{1, \ldots, k\}$, let $\lambda_1^i, \ldots, \lambda_n^i$ be the eigenvalues for the graph G^i, and let $v_1^i, \ldots, v_n^i$ be corresponding nonzero eigenvectors. We have thus defined nk eigenvectors. We will be done if we can convert them to eigenvectors for H (corresponding to the same eigenvalues) and show that the converted eigenvectors are linearly independent.

We now describe the procedure to convert an eigenvector v_j^i for G to an eigenvector x_j^i for H. Begin by magnetizing the edges of H so that every edge from level ω_a to ω_b (with $a \leq b$) has signature ω_{b-a}^i in that direction. What we are really doing is applying the switching function whose value at each vertex in level ω is ω^i. Now define y_j^i to be k concatenated copies of the eigenvector v_j^i (one for each level of H). If we treat y_j^i as a list of vertex values on the magnetized version of H, then each level of H exactly resembles G^i with the vertex values v_j^i. More precisely, any vertex (u, ω) in the magnetized H has exactly the same value, neighboring signatures, and neighboring vertex values as the vertex u in G^i. It follows that the Laplacian of the magnetized H maps the value of (u, ω) to the same place that the Laplacian of G^i maps the value of u (see Proposition 31). But we know that the Laplacian of G^i scales each entry of v_j^i by λ_j^i, so the Laplacian of the magnetized H does the same to each entry of y_j^i. In other words, y_j^i is an eigenvector for the magnetized H corresponding to λ_j^i. Finally, using Proposition 11, we can convert y_j^i to an eigenvector x_j^i of the unmagnetized H by multiplying by the matrix

$$\operatorname{diag}(\omega_0^i, \ldots, \omega_0^i, \omega_1^i, \ldots, \omega_1^i, \ldots, \omega_{k-1}^i, \ldots, \omega_{k-1}^i)^{-1},$$

which is equal to

$$\operatorname{diag}(1, \ldots, 1, \omega_{k-1}^i, \ldots, \omega_{k-1}^i, \ldots, \omega_1^i, \ldots, \omega_1^i).$$

Carrying out that multiplication gives us our eigenvector

$$x_j^i := \begin{bmatrix} v_j^i \\ \omega_{k-1}^i v_j^i \\ \vdots \\ \omega_1^i v_j^i \end{bmatrix}.$$

It remains to show that the x_j^i's are linearly independent. The key is that the vectors

$$\begin{bmatrix} 1 \\ \omega_{k-1} \\ \vdots \\ \omega_1 \end{bmatrix} ...TODO$$

$\square$