

The Magnetic Laplacian and its Spectrum

Definition 1. A *simple graph* is a set of vertices V and a set of oriented edges $E \subseteq V \times V$, where for all $(v, w) \in E$, we have $v \neq w$ and $(w, v) \in E$.

Definition 2. A *magnetic graph* is a simple graph (V, E) together with a *signature* function

$$\sigma : E \rightarrow \mathbb{C}_{|z|=1}$$

which has the property $\sigma(v, w) = \overline{\sigma(w, v)}$ for all $(v, w) \in E$. We may abbreviate $\sigma(v, w)$ as σ_{vw} .

We may sometimes treat a simple graph as a magnetic graph, in which case all signatures are assumed to be 1. Unless we explicitly state the vertex set of a graph, it is assumed to be $\{1, \dots, n\}$ for some n .

Definition 3. Suppose $G = (V, E)$ is a magnetic graph with vertices $\{1, \dots, n\}$. The *Laplacian* of G is the $n \times n$ matrix (l_{ij}) given by

$$l_{ij} = \begin{cases} d_i & \text{if } i = j \\ -\sigma_{ij} & \text{otherwise} \end{cases}$$

for all $i, j \in \{1 \dots n\}$.

Note that this “magnetic” definition for a Laplacian matches the familiar definition of a Laplacian when the graph is simple. The Laplacian always has a nice spectrum to analyze, as the following theorem shows.

Theorem 4. *If G is a magnetic graph with Laplacian L , then there is an orthonormal basis of \mathbb{C}^n consisting of eigenvectors of L , and the eigenvalues of L are real.*

Proof. L is Hermitian because $L = L^*$. Therefore, the Complex Spectral Theorem gives this statement exactly. \square

We also want to establish that the eigenvalues of the Laplacian are non-negative. We will follow Jiang’s paper (Theorem 3.5 and the preceding discussion), but modified slightly for magnetic graphs.

Definition 5. Let $G = (E, V)$ be a magnetic graph on n vertices with Laplacian L . For an edge (e, v) of G , the *contribution* of (e, v) to L is the $n \times n$ matrix (l_{ij}) given by

$$l_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } i \in \{e, v\} \\ -\sigma_{ij} & \text{if } \{i, j\} = \{e, v\} \\ 0 & \text{otherwise} \end{cases}.$$

We denote this matrix $L_{(e,v)}$.

As the name “contribution” suggests, the Laplacian of a graph is the sum of the contributions of its edges (only counting one edge between each pair of vertices):

Proposition 6. *If G is a magnetic graph with Laplacian L , then*

$$L = \sum_{\substack{(i,j) \in E \\ i < j}} L_{(i,j)}.$$

For the next few propositions, remember that if $z \in \mathbb{C}$, then $z \geq 0$ means z is real and nonnegative. Also, \cdot is the complex dot product.

Proposition 7. *Let $G = (E, V)$ be a magnetic graph on n vertices with Laplacian L , and let $(i, j) \in E$. Then for any vector $v \in \mathbb{C}^n$, we have*

$$v \cdot L_{(i,j)} v \geq 0.$$

Proof. Let $v = (v_1, \dots, v_n)$. By direct computation, the vector $L_{(i,j)} v$ consists of all 0s, except for $v_i - \sigma_{ij} v_j$ in the i th slot and $v_j - \sigma_{ji} v_i$ in the j th slot. Therefore,

$$\begin{aligned} v \cdot L_{(i,j)} v &= v_i \overline{v_i - \sigma_{ij} v_j} + v_j \overline{v_j - \sigma_{ji} v_i} \\ &= v_i (\overline{v_i} - \overline{\sigma_{ij} v_j}) + v_j (\overline{v_j} - \overline{\sigma_{ji} v_i}) \\ &= |v_i|^2 - v_i \overline{\sigma_{ij} v_j} - v_j \overline{\sigma_{ji} v_i} + |v_j|^2 \\ &= |v_i|^2 - \sigma_{ji} v_i \overline{v_j} - \overline{\sigma_{ji} v_i} \overline{v_j} + |v_j|^2 \\ &= |v_i|^2 - (\sigma_{ji} v_i \overline{v_j} + \overline{\sigma_{ji} v_i} \overline{v_j}) + |v_j|^2 \\ &= |v_i|^2 - 2\operatorname{Re}(\sigma_{ji} v_i \overline{v_j}) + |v_j|^2 \\ &\geq |v_i|^2 - 2|\operatorname{Re}(\sigma_{ji} v_i \overline{v_j})| + |v_j|^2 \\ &\geq |v_i|^2 - 2|\sigma_{ji} v_i \overline{v_j}| + |v_j|^2 \\ &= |v_i|^2 - 2|\sigma_{ji}| |v_i| |\overline{v_j}| + |v_j|^2 \\ &= |v_i|^2 - 2|\sigma_{ji}| |v_i| |\overline{v_j}| + |v_j|^2 \\ &= |v_i|^2 - 2|v_i| |v_j| + |v_j|^2 \\ &= (|v_i| - |v_j|)^2 \\ &\geq 0. \end{aligned}$$

□

Proposition 8. *Let $G = (E, V)$ be a magnetic graph on n vertices with Laplacian L . Then for any vector $v \in \mathbb{C}^n$, we have*

$$v \cdot Lv \geq 0.$$

Proof. We have:

$$v \cdot Lv = v \cdot \left(\sum_{\substack{(i,j) \in E \\ i < j}} L_{(i,j)} \right) v = v \cdot \sum_{\substack{(i,j) \in E \\ i < j}} L_{(i,j)} v = \sum_{\substack{(i,j) \in E \\ i < j}} (v \cdot L_{(i,j)} v),$$

where the last equality holds because inner products are additive in the second slot. By Proposition 7, this is a sum of nonnegative real numbers, so $v \cdot Lv \geq 0$. □

Theorem 9. *If G is a magnetic graph with Laplacian L , then the eigenvalues of L are nonnegative.*

Proof. Let λ be an eigenvalue of L , and let $v \in \mathbb{C}^n$ be a nonzero eigenvector corresponding to λ . Then,

$$\begin{aligned} \lambda(v \cdot v) &= v \cdot \bar{\lambda} v \\ &= v \cdot \lambda v && \text{(since } \lambda \text{ is real by Theorem 4)} \\ &= v \cdot Lv \\ &\geq 0 && \text{(by Proposition 8).} \end{aligned}$$

But v is nonzero, so $v \cdot v \geq 0$ by the definition of inner product. Thus we may conclude that $\lambda \geq 0$. \square

The Eigenvalue 0

If G is a magnetic graph with vertices $\{1, \dots, n\}$ and x is a vector in \mathbb{C}^n , then it makes sense to imagine x as consisting of values for the vertices of G . Specifically, the entry x_i is like a value for the vertex i . With this interpretation, multiplying x by the Laplacian of G creates a new list of vertex values by taking linear combinations of the current vertex values. The weights of the combinations are determined by the structure of the Laplacian. In fact, the next proposition shows that a vertex's new value after the multiplication depends only on its current value and the values of its neighbors.

Proposition 10. *Let G be a magnetic graph on n vertices with Laplacian L . Let $x \in \mathbb{C}^n$ with $x = (x_1, \dots, x_n)$. Then*

$$(Lx)_i = d_i x_i - \sum_{j \sim i} \sigma_{ij} x_j$$

for each $i \in \{1, \dots, n\}$.

Proof. Let l_{ij} denote the element of L at row i and column j . Let $i \in \{1, \dots, n\}$ be given. Then,

$$\begin{aligned} (Lx)_i &= \sum_{j=1}^n l_{ij} x_j \\ &= l_{ii} x_i + \sum_{j \neq i} l_{ij} x_j \\ &= d_i x_i + \sum_{j \neq i} (-\sigma_{ij}) x_j \\ &= d_i x_i - \sum_{j \neq i} \sigma_{ij} x_j. \end{aligned}$$

\square

We are particularly interested in when the “vertex values” all go to 0 under multiplication by the Laplacian. It turns out that this happens exactly when each value is a signature-weighted average of the neighboring values, as the next lemma demonstrates.

Lemma 11. *Let G be a magnetic graph on n vertices with Laplacian L . Suppose $x \in \mathbb{C}^n$ with $x = (x_1, \dots, x_n)$. Then $Lx = 0$ if and only if*

$$x_i = \frac{1}{d_i} \sum_{j \sim i} \sigma_{ij} x_j$$

for each $i \in \{1, \dots, n\}$ with $d_i \neq 0$.

Proof. This follows easily from Proposition 10. \square

We now show that this condition is equivalent to a much simpler one.

Theorem 12. *Let G be a magnetic graph on n vertices with Laplacian L . Suppose $x \in \mathbb{C}^n$ with $x = (x_1, \dots, x_n)$. Then $Lx = 0$ if and only if*

$$x_i = \sigma_{ij}x_j$$

whenever $i \sim j$.

Proof. (\leftarrow) Suppose $x_i = \sigma_{ij}x_j$ whenever $i \sim j$. We will show that $Lx = 0$ using the condition from Lemma 11. Fix $i \in \{1, \dots, n\}$ with $d_i \neq 0$. Then,

$$x_i = \frac{1}{d_i}d_i x_i = \frac{1}{d_i} \sum_{j \sim i} x_i = \frac{1}{d_i} \sum_{j \sim i} \sigma_{ij}x_j.$$

(\rightarrow) Suppose $Lx = 0$. We first argue that whenever a and b are vertices in the same connected component of G , we have $|x_a| = |x_b|$. To see this, take an arbitrary connected component of G , and choose a vertex m in that component which maximizes $|x_m|$. If $d_m = 0$, then m is the only vertex in the component, so our conclusion holds trivially. On the other hand, if $d_m \neq 0$, then we can apply Lemma 11 to get

$$\begin{aligned} |x_m| &= \left| \frac{1}{d_m} \sum_{l \sim m} \sigma_{ml}x_l \right| \\ &= \frac{1}{d_m} \left| \sum_{l \sim m} \sigma_{ml}x_l \right| \\ &\leq \frac{1}{d_m} \sum_{l \sim m} |\sigma_{ml}x_l| && \text{(by the Triangle Inequality)} \\ &= \frac{1}{d_m} \sum_{l \sim m} |x_l|. \end{aligned}$$

That is, $|x_m|$ is the mean of the values $|x_l|$ where $l \sim m$. But our choice of m ensures that $|x_m| \geq |x_l|$ for each $l \sim m$, so in fact $|x_m| = |x_l|$ for each $l \sim m$. It follows that any vertex l which is adjacent to m also maximizes the quantity $|x_l|$, so we can apply the same argument to each l , repeating the process until we conclude that $|x_a| = |x_b|$ for every pair of vertices a, b in the connected component.

Now let i and j be vertices of G with $i \sim j$. We want to prove that $x_i = \sigma_{ij}x_j$. From the previous paragraph, we already have that

$$|x_i| = |x_j| = |\sigma_{ij}x_j|,$$

so it suffices to show x_i differs from $\sigma_{ij}x_j$ by a positive real factor. Observe that

$$\begin{aligned} \left| \sum_{l \sim i} \sigma_{il}x_l \right| &= |d_i x_i| && \text{(by Lemma 11)} \\ &= d_i |x_i| \\ &= \sum_{l \sim i} |x_i| \\ &= \sum_{l \sim i} |x_l| && \text{(by the previous paragraph)} \\ &= \sum_{l \sim i} |\sigma_{il}x_l|. \end{aligned}$$

Therefore, assuming $\sigma_{ij}x_j \neq 0$ (the other case is trivial), we may conclude that

$$\sum_{l \sim i} \sigma_{il}x_l = r(\sigma_{ij}x_j)$$

for some positive real r . Then,

$$\begin{aligned} x_i &= \frac{1}{d_i} \sum_{l \sim i} \sigma_{il}x_l && \text{(by Lemma 11)} \\ &= \frac{1}{d_i} r(\sigma_{ij}x_j), \end{aligned}$$

so x_i differs from $\sigma_{ij}x_j$ by a positive real factor, as desired. \square

Corollary 13. *Let G be a simple graph on n vertices with Laplacian L . Suppose $x \in \mathbb{C}^n$ with $x = (x_1, \dots, x_n)$. Then $Lx = 0$ if and only if*

$$x_i = x_j$$

whenever $i \sim j$.

Proof. This follows from Theorem 12 and the fact that every signature in a simple graph is 1. \square

For what magnetic graphs does the Laplacian have 0 as an eigenvalue? And when 0 is an eigenvalue, what is its multiplicity? In other words, we want to know how a magnetic graph is related to the nullity of its Laplacian. The following definition will help to answer this question.

Definition 14. A magnetic graph is called *consistent* if the signatures along any closed walk multiply to 1.

Proposition 15. *Let $G = (E, V)$ be a magnetic graph. Then the following are equivalent.*

1. *G is consistent.*
2. *The signatures along any cycle multiply to 1 (in either direction).*
3. *For every pair of connected vertices $u, v \in V$, there exists $z \in \mathbb{C}_{|z|=1}$ such that along every walk from u to v , the signatures multiply to z .*

Condition 3 was the reason for the name “consistent.” Because the number z in that condition is clearly unique for each pair of connected vertices, we can make the following definition.

Definition 16. Let $G = (E, V)$ be a consistent magnetic graph, and let $u, v \in V$. Then the *signature distance* from u to v , written $\text{sigdist}(u, v)$, is the product of the signatures along every walk from u to v (or ∞ if u and v are disconnected).

Now we can achieve our goal of relating a magnetic graph to the nullity of its Laplacian.

Theorem 17. *Let G be a magnetic graph with Laplacian L . Then the nullity of L is equal to the number of consistent connected components of G .*

Proof. Let n be the number of vertices in G , and let

$$C_1 = (E_1, V_1), \dots, C_k = (E_k, V_k)$$

be the consistent connected components of V , where k may be 0. Our goal is to construct a basis for the null space of L consisting of k vectors in \mathbb{C}^n . We begin by choosing vertices

$$u_1 \in V_1, \dots, u_k \in V_k.$$

Then, we construct our basis vectors $w_1, \dots, w_k \in \mathbb{C}^n$ as follows. For each $i \in \{1, \dots, k\}$, define the j th entry of w_i to be

$$(w_i)_j = \begin{cases} \text{sigdist}(j, u_i) & \text{if } j \in V_i \\ 0 & \text{otherwise} \end{cases}.$$

It remains to argue that w_1, \dots, w_k is indeed a basis for null L . Start by observing that for each $i \in \{1, \dots, k\}$, w_i is the only basis vector which is nonzero at the indices in V_i . This proves linear independence.

To show that $\text{span}(w_1, \dots, w_k) \subseteq \text{null } L$, it suffices to show each $w_i \in \text{null } L$. We will use the equivalent condition given in Theorem 12. That is, we will show that for any vertices $a \sim b$,

$$(w_i)_a = \sigma_{ab}(w_i)_b.$$

If the connected component containing a and b is not C_i , then $(w_i)_a = (w_i)_b = 0$, so the desired equality holds. On the other hand, suppose the connected component containing a and b is C_i . Then

$$\begin{aligned} (w_i)_a &= \text{sigdist}(a, u_i) \\ &= \sigma_{ab} \text{sigdist}(b, u_i) \\ &= \sigma_{ab}(w_i)_b. \end{aligned}$$

Finally we will prove that $\text{null } L \subseteq \text{span}(w_1, \dots, w_k)$. Let $x \in \text{null } L$. We claim

$$x = x_{u_1} w_1 + \dots + x_{u_k} w_k, \tag{*}$$

which we will prove for each index in the vector. To that end, let j be an index in $\{1, \dots, n\}$.

Case 1. As a vertex, j belongs to an inconsistent connected component. Call it I . Then the RHS of $(*)$ is 0 at index j , so we must show $x_j = 0$ too. Since I is inconsistent, it has a closed walk along which the signatures multiply to some $c \neq 1$. Let z be the starting and ending vertex of the walk. Repeated application of Theorem 12 along that walk gives that $x_z = cx_z$, so $x_z = 0$. Then applying Theorem 12 along a path from z to j , we get $x_j = 0$, as desired. So $(*)$ holds at index j .

Case 2. As a vertex, j belongs to a consistent connected component. Say, C_α where $\alpha \in \{1, \dots, k\}$. Then

$$\begin{aligned} x_j &= x_{u_\alpha} \text{sigdist}(j, u_\alpha) && \text{(repeatedly applying Theorem 12)} \\ &= x_{u_\alpha} (w_\alpha)_j \\ &= x_{u_1} (w_1)_j + \dots + x_{u_k} (w_k)_j && \text{(introducing 0 terms)} \\ &= (x_{u_1} w_1 + \dots + x_{u_k} w_k)_j. \end{aligned}$$

So in this case as well, $(*)$ holds at index j .

□

Corollary 18. *The Laplacian of a simple graph always has 0 as an eigenvalue.*

Proof. A simple graph is consistent because all its signatures are 1. Thus it has at least one consistent connected component, so the nullity of the Laplacian is at least 1. □

Corollary 19. *The Laplacian of an acyclic magnetic graph always has 0 as an eigenvalue.*

Proof. An acyclic graph is consistent because it vacuously satisfies condition 2 of Proposition 15. Thus it has at least one consistent connected component, so the nullity of the Laplacian is at least 1. □