

Problem (Engel Problem-Solving Strategies 1.10). *The vertices of an n -gon are labeled by real numbers x_1, \dots, x_n . Let a, b, c, d be four successive labels. If $(a - d)(b - c) < 0$, then we may switch b with c . Decide if this switching operation can be performed infinitely often.*

Answer. It cannot.

Proof. Consider the sum

$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1.$$

Performing the switching operation only affects one part of this sum. Specifically,

$$ab + bc + cd$$

becomes

$$ac + cb + bd.$$

But we have

$$\begin{aligned}(a - d)(b - c) &< 0 \\ ab - ac - bd + cd &< 0 \\ ab + cd &< ac + bd,\end{aligned}$$

so our sum has increased. Since there are only finitely many orderings of labels, these increases cannot go on forever. Therefore, we must eventually eventually run out of legal switches. \square