Problem (Putnam 2011 - B1)**.** *Let h and k be positive integers. Prove that for every $\epsilon > 0$, there are positive integers m and n such that*

$$\epsilon < |h\sqrt{m} - k\sqrt{n}| < 2\epsilon.$$

Throughout the solution, we will use the following identity (valid for all $n \in \mathbb{N}$):

$$\sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}},$$

We will need the following lemma.

Lemma 1. *For all $b \in \mathbb{N}$, we have*

$$\sqrt{b+2} - \sqrt{b+1} > \frac{1}{2}(\sqrt{b+1} - \sqrt{b}).$$

Proof. We have:

$$\begin{aligned}
2\sqrt{b} > 1 > \frac{1}{\sqrt{b+2} + \sqrt{b+1}} &= \sqrt{b+2} - \sqrt{b+1} \\
2\sqrt{b+1} + 2\sqrt{b} &> \sqrt{b+2} + \sqrt{b+1} \\
\frac{2}{\sqrt{b+2} + \sqrt{b+1}} &> \frac{1}{\sqrt{b+1} + \sqrt{b}} \\
2(\sqrt{b+2} - \sqrt{b+1}) &> \sqrt{b+1} - \sqrt{b} \\
\sqrt{b+2} - \sqrt{b+1} &> \frac{1}{2}(\sqrt{b+1} - \sqrt{b}).
\end{aligned}$$

□

We are now ready to solve the problem. Let $\epsilon > 0$ be given. Choose $s \in \mathbb{N}$ to be large enough that

$$\frac{\epsilon}{hks} < \sqrt{2} - 1.$$

This ensures there exists some $b \in \mathbb{N}$ with

$$\frac{\epsilon}{hks} < \sqrt{b+1} - \sqrt{b}.$$

Namely, $b = 1$ achieves this. But let us choose $b \in \mathbb{N}$ to be as large as possible so that this inequality holds. (There *is* a largest such b, since $\sqrt{b+1} - \sqrt{b} = 1/(\sqrt{b+1} + \sqrt{b})$, which converges to 0 as $b \to \infty$.)

Now, suppose for contradiction that

$$\sqrt{b+1} - \sqrt{b} \geq \frac{2\epsilon}{hks}.$$

Then, by Lemma 1,

$$\sqrt{b+2} - \sqrt{b+1} > \frac{1}{2}(\sqrt{b+1} - \sqrt{b}) \geq \frac{\epsilon}{hks},$$

contradicting that b was as large as possible. This establishes that

$$\frac{\epsilon}{hks} < \sqrt{b+1} - \sqrt{b} < \frac{2\epsilon}{hks}.$$

Manipulating the inequality gives

$$\epsilon < hks(\sqrt{b+1} - \sqrt{b}) < 2\epsilon$$

$$\epsilon < \left| hks\sqrt{b+1} - hks\sqrt{b} \right| < 2\epsilon$$

$$\epsilon < \left| h\sqrt{k^2 s^2 (b+1)} - k\sqrt{h^2 s^2 b} \right| < 2\epsilon.$$

So we have found satisfactory values for m and n.