

**Problem 1** (Putnam 2011 - B1). *Let  $h$  and  $k$  be positive integers. Prove that for every  $\epsilon > 0$ , there are positive integers  $m$  and  $n$  such that*

$$\epsilon < |h\sqrt{m} - k\sqrt{n}| < 2\epsilon.$$

Throughout the solution, we will use the following identity (valid for all  $n \in \mathbb{N}$ ):

$$\sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}},$$

We will need the following lemma.

**Lemma 1.** *For all  $b \in \mathbb{N}$ , we have*

$$\sqrt{b+2} - \sqrt{b+1} > \frac{1}{2}(\sqrt{b+1} - \sqrt{b}).$$

*Proof.* We have:

$$\begin{aligned} 2\sqrt{b} > 1 &> \frac{1}{\sqrt{b+2} + \sqrt{b+1}} = \sqrt{b+2} - \sqrt{b+1} \\ 2\sqrt{b+1} + 2\sqrt{b} &> \sqrt{b+2} + \sqrt{b+1} \\ \frac{2}{\sqrt{b+2} + \sqrt{b+1}} &> \frac{1}{\sqrt{b+1} + \sqrt{b}} \\ 2(\sqrt{b+2} - \sqrt{b+1}) &> \sqrt{b+1} - \sqrt{b} \\ \sqrt{b+2} - \sqrt{b+1} &> \frac{1}{2}(\sqrt{b+1} - \sqrt{b}). \end{aligned}$$

□

We are now ready to solve the problem. Let  $\epsilon > 0$  be given. Choose  $s \in \mathbb{N}$  to be large enough that

$$\frac{\epsilon}{hks} < \sqrt{2} - 1.$$

This ensures there exists some  $b \in \mathbb{N}$  with

$$\frac{\epsilon}{hks} < \sqrt{b+1} - \sqrt{b}.$$

Namely,  $b = 1$  achieves this. But let us choose  $b \in \mathbb{N}$  to be as large as possible so that this inequality holds. (There *is* a largest such  $b$ , since  $\sqrt{b+1} - \sqrt{b} = 1/(\sqrt{b+1} + \sqrt{b})$ , which converges to 0 as  $b \rightarrow \infty$ .)

Now, suppose for contradiction that

$$\sqrt{b+1} - \sqrt{b} \geq \frac{2\epsilon}{hks}.$$

Then, by Lemma 1,

$$\sqrt{b+2} - \sqrt{b+1} > \frac{1}{2}(\sqrt{b+1} - \sqrt{b}) \geq \frac{\epsilon}{hks},$$

contradicting that  $b$  was as large as possible. This establishes that

$$\frac{\epsilon}{hks} < \sqrt{b+1} - \sqrt{b} < \frac{2\epsilon}{hks}.$$

Manipulating the inequality gives

$$\epsilon < hks(\sqrt{b+1} - \sqrt{b}) < 2\epsilon$$

$$\epsilon < \left| hks\sqrt{b+1} - hks\sqrt{b} \right| < 2\epsilon$$

$$\epsilon < \left| h\sqrt{k^2s^2(b+1)} - k\sqrt{h^2s^2b} \right| < 2\epsilon.$$

So we have found satisfactory values for  $m$  and  $n$ .