Problem 1 (Putnam 2011 - B1)**.** Let h and k be positive integers. Prove that for every $\epsilon > 0$, there are positive integers m and n such that

$$\epsilon < |h\sqrt{m} - k\sqrt{n}| < 2\epsilon.$$

Throughout the solution, we will use the following identity (valid for all $n \in \mathbb{N}$):

$$\sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}},$$

We will need the following lemma.

Lemma 1. *For all $b \in \mathbb{N}$, we have*

$$\sqrt{b+2} - \sqrt{b+1} > \frac{1}{2}(\sqrt{b+1} - \sqrt{b}).$$

Proof. We have:

$$\begin{aligned}
2\sqrt{b} > 1 > \frac{1}{\sqrt{b+2} + \sqrt{b+1}} &= \sqrt{b+2} - \sqrt{b+1} \\
2\sqrt{b+1} + 2\sqrt{b} &> \sqrt{b+2} + \sqrt{b+1} \\
\frac{2}{\sqrt{b+2} + \sqrt{b+1}} &> \frac{1}{\sqrt{b+1} + \sqrt{b}} \\
2(\sqrt{b+2} - \sqrt{b+1}) &> \sqrt{b+1} - \sqrt{b} \\
\sqrt{b+2} - \sqrt{b+1} &> \frac{1}{2}(\sqrt{b+1} - \sqrt{b}).
\end{aligned}$$

□

We are now ready to solve the problem. Let $\epsilon > 0$ be given. Choose $s \in \mathbb{N}$ to be large enough that

$$\frac{\epsilon}{hks} < \sqrt{2} - 1.$$

This ensures there exists some $b \in \mathbb{N}$ with

$$\frac{\epsilon}{hks} < \sqrt{b+1} - \sqrt{b}.$$

Namely, $b = 1$ achieves this. But let us choose $b \in \mathbb{N}$ to be as large as possible so that this inequality holds. (There *is* a largest such b, since $\sqrt{b+1} - \sqrt{b} = 1/(\sqrt{b+1} + \sqrt{b})$, which converges to 0 as $b \to \infty$.)

Now, suppose for contradiction that

$$\sqrt{b+1} - \sqrt{b} \geq \frac{2\epsilon}{hks}.$$

Then, by Lemma 1,

$$\sqrt{b+2} - \sqrt{b+1} > \frac{1}{2}(\sqrt{b+1} - \sqrt{b}) \geq \frac{\epsilon}{hks},$$

contradicting that b was as large as possible. This establishes that

$$\frac{\epsilon}{hks} < \sqrt{b+1} - \sqrt{b} < \frac{2\epsilon}{hks}.$$

Manipulating the inequality gives

$$\epsilon < hks(\sqrt{b+1} - \sqrt{b}) < 2\epsilon$$

$$\epsilon < \left| hks\sqrt{b+1} - hks\sqrt{b} \right| < 2\epsilon$$

$$\epsilon < \left| h\sqrt{k^2s^2(b+1)} - k\sqrt{h^2s^2b} \right| < 2\epsilon.$$

So we have found satisfactory values for m and n.

Problem 2 (Putnam 2009 - A1)**.** Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A)+f(B)+f(C)+f(D)=0$. Does it follow that $f(P)=0$ for all points in the plane?

The answer is yes. To see this, let (x,y) be an arbitrary point in the plane. Then, we have

$$\begin{aligned}
f(x,y)+f(x+1,y)+f(x,y+1)+f(x+1,y+1)&=0\\
f(x,y)+f(x-1,y)+f(x-1,y+1)+f(x,y+1)&=0\\
f(x,y)+f(x-1,y)+f(x-1,y-1)+f(x,y-1)&=0\\
f(x,y)+f(x+1,y)+f(x+1,y-1)+f(x,y-1)&=0\\
-\left[f(x-1,y-1)+f(x+1,y+1)+f(x+1,y-1)+f(x-1,y-1)\right]&=-(0)=0\\
-2\left[f(x,y+1)+f(x+1,y)+f(x,y-1)+f(x-1,y)\right]&=-2(0)=0.
\end{aligned}$$

Adding the equations and simplifying, we get

$$4f(x,y)=0,$$

which means

$$f(x,y)=0.$$

Problem 3 (Engel Problem-Solving Strategies 1.6)**.** There are a white, b black, and c red chips on a table. In one step, you may choose two chips of different colors and replace them by a chip of the third color. If just one chip will remain at the end, its color will not depend on the evolution of the game. When can this final state be reached?

The "final state" can be reached exactly when two conditions are met:

1. a, b, and c don't all have the same parity.
2. If two of a, b, and c are 0, then the other is 1.

The last remaining chip will be the color whose parity differs from the other two.

Let us first show that if either of these conditions fails, then the final state cannot be reached. If the second condition fails, then two of a, b, and c are 0 and the other is not 1. In this situation, we are out of legal moves, and we are not in the final state, so the final state is unreachable. On the other hand, if the first condition fails, then a, b, and c all have the same parity. In this case, a chip exchange involves changing each of a, b, and c by one, so the parities all get reversed, and they are still the same as each other after the exchange. It follows that the colors will always have the same parity, so a configuration with just one chip is unreachable.

It remains to show that if both conditions are met, then the final state can be reached, and the last remaining chip will be the color whose parity differs from the other two. We prove this statement by induction on the number of chips on the table.

Base cases: If there are 0 chips, then the conditions cannot be met, so the statement is vacuously true. For 1 chip, we are already in the final state, and the last remaining chip has parity different from the other two, so the statement is again true. For two chips, there are 6 cases to check, and the statement can easily be verified for each of those cases.

Induction step: Now let $k \geq 2$ and suppose the statement holds for k chips. Consider a table with $k+1$ chips, and assume both conditions are met. Since there is more than one chip, it cannot be the case that some two of a, b, and c are 0 (or else condition 2 would be violated). Thus, we know that the two most numerous colors are both greater than 0. We also know that they are not both 1, or else all three colors would be 1, contradicting condition 1. Thus, of the two most numerous colors, one is greater than 0 and the other is greater than 1. We may then perform a chip exchange, removing from those two colors and adding to the third. Clearly, this leaves at least two nonzero colors, so condition 2 is vacuously true for the k remaining chips. But condition 1 is also true for the k remaining chips, since it was assumed true prior to the chip exchange, and the chip exchange just reversed the three parities. Moreover, the color whose parity is the "odd man out" didn't change. Applying the induction hypothesis to the k remaining chips, we get that the final state can be reached, and the last remaining chip will be the color whose parity differs from the other two.

Problem 4 (Engel Problem-Solving Strategies 1.8)**.** There is a positive integer in each square of a rectangular table. In each move, you may double each number in a row or subtract 1 from each number of a column. Prove that you can reach a table of zeros by a sequence of these permitted moves.

Choose one of the columns of positive integers. If it is not only 1's, then double all its 1's (by doubling their rows), and finally subtract 1 from the column. This causes any 1's in the column to remain 1's, and the greater integers to decrease by 1. We get a column of positive integers whose maximum is one less than before the operation. Apply this operation repeatedly until the maximum reaches 1, at which point the column is only 1's. Now, subtract 1 from the column to make it 0's.

We have now made one column into 0's. The other columns are still positive integers, since we never subtracted from them. We may therefore apply the same procedure one-by-one to the remaining columns, until all of them are zeroed. Since doubling rows has no effect on a column already turned to 0's, we will never damage any of our previously zeroed columns.

Problem 5 (Engel Problem-Solving Strategies 1.10). The vertices of an n-gon are labeled by real numbers $x_1, \ldots, x_n$. Let a, b, c, d be four successive labels. If $(a-d)(b-c) < 0$, then we may switch b with c. Decide if this switching operation can be performed infinitely often.

It cannot.

Consider the sum

$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1.$$

Performing the switching operation only affects one part of this sum. Specifically,

$$ab + bc + cd$$

becomes

$$ac + cb + bd.$$

But we have

$$\begin{aligned}(a-d)(b-c) &< 0 \\ ab - ac - bd + cd &< 0 \\ ab + cd &< ac + bd,\end{aligned}$$

so our sum has increased. Since there are only finitely many orderings of labels, these increases cannot go on forever. Therefore, we must eventually eventually run out of legal switches.

Problem 6 (Engel Problem-Solving Strategies 1.12)**.** There is a row of 1000 integers. There is a second row below, which is constructed as follows. Under each number a of the first row, there is a positive integer $f(a)$ such that $f(a)$ equals the number of occurrences of a in the first row. In the same way, we get the 3rd row from the 2nd row, and so on. Prove that, finally, one of the rows is identical to the next row.

Let $N(r)$ be the number of distinct integers that appear in the rth row, and let $f_r(a)$ be the number of occurrences of the integer a in the rth row.

As we move from the rth row to the $(r+1)$th row, all occurrences of a number a get replaced by the same number $f_r(a)$, so $N(r) \geq N(r+1)$. Therefore, $N(r)$ is nonincreasing as r grows. But $N(r)$ is at least 1, so eventually it must be unchanging.

Let s be the row where that happens, so that $N(s) = N(s+1) = N(s+2) = \ldots$. In row s, call the distinct integers $x_1, x_2, \ldots, x_{N(s)}$. Since $N(s+1) = N(s)$, we must have $f_s(x_1), f_s(x_2), \ldots, f_s(x_{N(s)})$ all distinct. Thus, in row $s+1$, $f_s(x_1)$ appears only below x_1, $f_s(x_2)$ appears only below x_2, and so on. It follows that for each $1 \leq i \leq N(s)$, the integer $f_s(x_i)$ appears in row $s+1$ exactly as many times as x_i appears in row s. That is, $f_s(x_i)$ appears $f_s(x_i)$ times in row $s+1$. But that means row $s+2$ must be identical to row $s+1$.

Problem 7 (Engel Problem-Solving Strategies 1.14)**.** We strike the first digit of the number 7^{1996}, and then add it to the remaining number. This is repeated until a number with 10 digits remains. Prove that this number has two equal digits.

Given a positive integer, call its first digit d. Striking the first digit means subtracting $d \cdot 10^p$ for some $p \geq 0$. Adding it back means adding d. Thus, the whole operation adds

$$\begin{aligned} -d \cdot 10^p + d &\equiv -d \cdot (1)^p + d \pmod 9 \\ &\equiv -d + d \pmod 9 \\ &\equiv 0 \pmod 9. \end{aligned}$$

Therefore, the value mod 9 of our number is invariant under this operation.

Now suppose for contradiction that, when we arrive at a 10-digit number, it has no two equal digits. Then all the digits from 0 to 9 must appear exactly once. A positive integer is congruent mod 9 to the sum of its digits, so our number is congruent mod 9 to

$$\begin{aligned} 0 + 1 + \cdots + 9 &= \frac{9(10)}{2} \\ &\equiv 0 \pmod 9. \end{aligned}$$

Since the striking and adding operations did not change the value mod 9, we must also have our original number 7^{1996} divisible by 9. But its only prime factors are 7, so the Fundamental Theorem of Arithmetic forbids this.

Problem 8 (Engel Problem-Solving Strategies 1.15)**.** There is a checker at point $(1,1)$ of the lattice (x, y) with x, y positive integers. It moves as follows. At any move it may double one coordinate, or it may subtract the smaller coordinate from the larger. Which points of the lattice can the checker reach?

It can reach exactly the points (a, b) where $\gcd(a, b)$ is a power of 2.

First, we argue that no other points can be reached. Note that the checker starts at the point $(1, 1)$, whose gcd is a power of 2. Now suppose it is at an arbitrary point (a, b) having gcd a power of 2. Say, $\gcd(a, b) = 2^k$ for some $k \geq 0$. If we perform the first kind of move, the checker goes to $(2a, b)$ or $(a, 2b)$. The gcd either remains 2^k or becomes 2^{k+1}. If we perform the second kind of move, the checker goes to $(a - b, b)$ or $(a, b - a)$, and the gcd remains 2^k. We see that no matter how we move, the gcd remains a power of 2. Thus, this property is invariant, so no other points can be reached.

It remains to show that for all points (a, b), if $\gcd(a, b)$ is a power of 2, then (a, b) can indeed be reached. We prove this by strong induction on the sum $a + b$.

Base Case: Let $a + b = 2$, and assume $\gcd(a, b)$ is a power of 2. In this case, we must have $(a, b) = (1, 1)$, which can clearly be reached because it our starting position.

Induction Step: Let $k \geq 3$, and assume that for all $2 \leq m < k$, if $a + b = m$ and $\gcd(a, b)$ is a power of 2, then (a, b) can be reached. Now suppose that $a + b = k$ and $\gcd(a, b)$ is a power of 2. We will prove that (a, b) can be reached by considering two cases.

Case 1. a is even or b is even. Without loss of generality, say a is even. Observe that $(\frac{a}{2}, b)$ has gcd a power of 2, and $2 \leq \frac{a}{2} + b < k$. Thus, by hypothesis, $(\frac{a}{2}, b)$ can be reached. But then we can double $\frac{a}{2}$, reaching (a, b).

Case 2. a and b are both odd. Then a and b must be distinct, or else we would have $\gcd(a, b) = a$, which is not a power of 2. Without loss of generality, say $a > b$. Then, $\gcd(a + b, b)$ is a power of 2, so $\gcd(\frac{a+b}{2}, b)$ is a power of 2. Moreover, $2 \leq \frac{a+b}{2} + b < a + b = k$. Thus, by hypothesis, $(\frac{a+b}{2}, b)$ can be reached. Doubling the first coordinate and subtracting the second from it, we reach (a, b).