# Question 6:

If $a$ and $b$ are positive integers, and $ab + 1$ divides $a^2 + b^2$, then

$$\frac{a^2 + b^2}{ab + 1}$$

is a perfect square.

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If $ab + 1$ divides $a^2 + b^2$, then:

$$a^2 + b^2 = m_1(ab + 1)$$

$$b^2 - m_1 ab = m_1 - a^2$$

$$b(b - m_1 a) = m_1 - a^2 \tag{1}$$

Thus $b$ divides $m_1 - a^2$, so we can denote the ratio by

$$m_2 = \frac{m_1 - a^2}{b}$$

$$m_1 = m_2 b + a^2$$

Applying to eq:1:

$$b(b - (m_2 b + a^2)a) = (m_2 b + a^2) - a^2$$

$$b(b - m_2 ab - a^3 - m_2) = 0$$

Since $b$ is non-zero:

$$b - m_2 ab - a^3 - m_2 = 0$$

$$b(1 - m_2 a) = a^3 + m_2 \tag{2}$$

Thus $b$ divides $a^3 + m_2$, so we can denote the ratio by

$$m_3 = \frac{a^3 + m_2}{b}$$

$$m_2 = m_3 b - a^3$$

Applying to eq:2:

$$b(1 - (m_3 b - a^3)a) = a^3 + (m_3 b - a^3)$$

$$b(1 - m_3 ab + a^4 - m_3) = 0$$

$$1 - m_3 ab + a^4 - m_3 = 0$$

$$\frac{a^4+1}{ab+1} = m_3 \tag{3}$$

This is where we take a step back and appreciate what we've arrived at. If $a^2 + b^2$ is a multiple of $ab+1$, then $a^4+1$ is also a multiple of $ab+1$. We can then say that a pair of integers $(x, y) = (a, b)$ "solves eq:3" if $\frac{x^4+1}{xy+1}$ is an integer. It can easily be shown using the steps above in reverse that if $a$ and $b$ are both non-zero:
*$ab+1$ divides $a^4+1$ if and only if $ab+1$ divides $a^2+b^2$.*
So if $x \neq 0$, $y \neq 0$, and $(x, y)$ solves eq:3, then $\frac{x^2+y^2}{xy+1}$ is an integer, which implies $(y, x)$ also solves eq:3.

Eq:3 is equivalent to:

$$a^4 + 1 = m_3 + m_3 ab$$

$$a(a^3 - m_3 b) = m_3 - 1$$

which shows that $a$ divides $m_3 - 1$. Thus there exists an integer $c$ such that $m_3 = ac + 1$. So eq:3 becomes:

$$\frac{a^4+1}{ab+1} = ac + 1 \tag{4}$$

$$\frac{a^4+1}{ac+1} = ab + 1 \tag{5}$$

By eq:4 and eq:5 it can easily be shown that:

$$a^3 = abc + b + c \tag{6}$$

It is important to note from eq:6 that one of $b$ or $c$ is less than $a$.

If $c = 0$, then $b = a^3$ is the solution to eq:6, so $(x, x^3)$ solves eq:3 for all integers $x$. By arguments already established, this implies $(x^3, x)$ also solves eq:3. By letting $a = x^3$ and $b = x$ we find that $c = x^5 - x$ by eq:6. This means that $(x^3, x^5 - x)$ and $(x^5 - x, x^3)$ solve eq:3. Again, by letting $(a, b) = (x^5 - x, x^3)$ and applying eq:6, we find that $c = x^7 - 2x^3$. Adjacent pairs of these polynomials:

$$x,\ x^3,\ x^5 - x,\ x^7 - 2x^3,\ x^9 - 3x^5 + x,\ ...$$

solve eq:3.

It is easy to show that all pairs that solve eq:3 are adjacent pairs of these polynomials. I will do so now:

Consider the smallest $\alpha > 0$ such that the pair of integers $(\alpha, \beta)$, solves eq:3, and $\alpha$ and $\beta$ are not adjacent polynomials above evaluated at any integer. It follows that $\beta > \alpha$. Therefore, there exists a $\gamma < \alpha$ such that $(\alpha, \gamma)$ solves eq:3, and so does $(\gamma, \alpha)$, contradicting the minimality of $\alpha$.

The only thing left to prove is that the integer ratio of $a^2 + b^2$ and $ab + 1$ is a square when $a$ and $b$ are adjacent polynomials described above. I will do so by using induction. The base case is proved as follows:

$$b = x, a = x^3$$

$$\frac{a^2 + b^2}{ab + 1} = \frac{x^6 + x^2}{x^4 + 1} = x^2$$

Now for the induction step, I will assume that any adjacent polynomials when applied to $\frac{a^2+b^2}{ab+1}$ yields $x^2$, and I will show that the next two polynomials also share that property:

Two polynomials $p_{n-1}$ and $p_n$, $p_{n-1} < p_n$ can produce the next polynomial in the sequence by eq:6:

$$p_n^3 = p_{n-1}p_np_{n+1} + p_{n-1} + p_{n+1}$$

$$\frac{p_n^3 - p_{n-1}}{p_{n-1}p_n + 1} = p_{n+1}$$

And likewise,

$$\frac{p_n^3 - p_{n+1}}{p_np_{n+1} + 1} = p_{n-1}$$

I now apply the induction assumption, then substitute the above expression for $p_{n-1}$:

$$\frac{p_n^2 + p_{n-1}^2}{p_{n-1}p_n + 1} = x^2$$

$$\frac{p_n^2 + (\frac{p_n^3 - p_{n+1}}{p_n p_{n+1} + 1})^2}{(\frac{p_n^3 - p_{n+1}}{p_n p_{n+1} + 1})p_n + 1} = x^2$$

Multiplying both the numerator and denominator by $(p_n p_{n+1} + 1)^2$ yields:

$$\frac{p_n^6 + p_n^4 p_{n+1}^2 + 2p_n^3 p_{n+1} - 2p_n^3 p_{n+1} + p_n^2 + p_{n+1}^2}{p_n^5 p_{n+1} + p_n^4 + p_n^2 p_{n+1}^2 - p_n^2 p_{n+1}^2 + 2p_n p_{n+1} - p_n p_{n+1} + 1} = x^2$$

$$\frac{p_n^6 + p_n^4 p_{n+1}^2 + p_n^2 + p_{n+1}^2}{p_n^5 p_{n+1} + p_n^4 + p_n p_{n+1} + 1} = x^2$$

$$\frac{(p_n^4 + 1)(p_n^2 + p_{n+1}^2)}{(p_n^4 + 1)(p_n p_{n+1} + 1)} = x^2$$

$$\frac{p_n^2 + p_{n+1}^2}{p_n p_{n+1} + 1} = x^2$$

Q.E.D.