Lecture 29
March 23rd, 2015

$x^2 - Dy^2 = 1$ D not perfect square

(x_1, y_1) smallest solution

$x_k + y_k\sqrt{D} = (x_1 + y_1\sqrt{D})^k$

Suppose D perfect square $D = A^2$

$$x^2 - A^2y^2 = 1$$

$$(x+Ay)(x-Ay) = 1$$

$$\begin{cases} x+Ay=1 \\ x-Ay=1 \end{cases} \text{ or } \begin{cases} x+Ay=-1 \\ x-Ay=-1 \end{cases}$$

$$\downarrow \qquad\qquad \downarrow$$

$$x=1, y=0 \qquad x=-1, y=0$$

we get only trivial solutions $(\pm 1, 0)$

Finding a pattern for the smallest solution to $x^2 - Dy^2 = 1$ is an often problem

$x^2 - 61y^2 = 1$

Best bound so far is due to Siegal: (x_1, y_1) smallest solution.

$$h \cdot \log(x_1 + y_1\sqrt{D}) \sim \sqrt{D}$$

h class number of $\mathbb{Q}(\sqrt{D})$ if $h=1$, $x_1 + y_1\sqrt{D} \sim e^{\sqrt{D}}$

e.g. $D = 2$ $(3,2)$

$$x_k = \frac{1}{2}((3+2\sqrt{2})^k + (3-2\sqrt{2})^k)$$

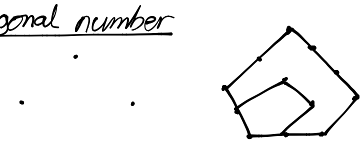
$$x_k^2 - 2y_k^2 = 1 \quad x_k \text{ odd since } x_k^2 = 1 + 2y_k^2$$

$$x_{10} \sim \frac{1}{2}(3+2\sqrt{2})^{10}$$

$$\sim 22619536.9999\,9998895$$

$$x_{10} = 22619537$$

<u>Pentagonal number</u>



5   5+7   5+7+10=22

nth pentagonal number

$$= 5 + 7 + \cdots + (3n+1) \qquad n \geq 2$$

$$= 5 + \sum_{k=2}^{n} (3k+1) = \frac{n(3n+5)}{2} + 1$$

Pentagonal Number which is also a triangular number.

$$\frac{n(3n+5)}{2} + 1 = \frac{m(m+1)}{2}$$

$$n(3n+5) + 2 = m(m+1)$$

$$12n^2 + 20n + 8 = 4m^2 + 4m$$

$$12(n^2 + \frac{5}{3}n) + 9 = (2m+1)^2$$

$$-\frac{2}{3} + 12 \cdot \frac{1}{36}(6n+5)^2 = 12(n+\frac{5}{6})^2 + \frac{2}{3} = (2m+1)^2$$

$$\frac{3(2m+1)^2}{y} = \frac{(6n+5)^2}{x}+2$$

$$\boxed{x^2-3y^2=-2}$$

$x^2-3y^2=1$ $(2,1)$ is the smallest soln.

All solns are $(2+\sqrt{3})^k = x_k + y_k\sqrt{3}$

Identity: $(x_1^2-Dy_1^2)(x_2^2-Dy_2^2)=(x_1x_2+Dy_1y_2)^2-D(x_1y_2+x_2y_1)^2$

The solutions of $x^2-Dy^2=1-D$ are given by

$(x_0x_k+Dy_0y_k,\ x_0y_k+x_ky_0)$

where (x_0,y_0) is a solution to $x^2-Dy^2=1-D$

(x_k,y_k) is a solution to $x^2-Dy^2=1$.

true since $(x_0^2-Dy_0^2)(x_k^2-Dy_k^2)=(1-D)\cdot 1=(x_0y_k+Dy_0y_k)^2-D(x_0y_k+x_ky_0)^2$

Conversely, (u,v) is a solution to $x^2-Dy^2=1-D$.

use method of descent.

$$\begin{cases} u = x_1x_2+Dy_1y_2 \\ v = x_1y_2+x_1y_2 \end{cases}$$

$\boxed{(x_1,y_1) \text{ is the smallest solution to } x^2-Dy^2=1.}$

solve for (x_2,y_2)
$$\begin{cases} x_2 = x_1u-Dy_1v \\ y_2 = x_1v-y_1u \end{cases}$$

Claim: $x_2, y_2>0$

(x_2,y_2) is a solution to $x^2-Dy^2=1-D$.

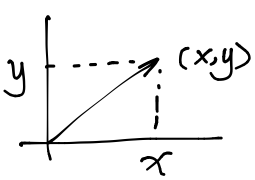
Finally, you will get (x_0,y_0)

Chap 35–36

Number Theory and Imaginary numbers (complex numbers)

$z=x+iy$

$x^2+1=0$ has no soln in real numbers $\quad i=\sqrt{-1}$ is a solution.



norm of $x+iy=z$

norm $(x+iy)=|z|=\sqrt{x^2+y^2}$