- Let A, B, C and D be 4 points on one line. Prove that $(A, B, C, D) = (B, A, D, C) = (D, C, B, A) = (C, D, A, B)$
- Let L_1, L_2, L_3 and L_4 be 4 planes in the space passing through one line μ. Take any line l which does not belong to the planes and which is not parallel to any of them. Consider 4 points of intersections $A_i = l \cup L_i, i = 1, 2, 3, 4$. Prove that the cross ratio (A_1, A_2, A_3, A_4) does not depend on the choice of the line l.
- Let l_1, l_2, l_3 be 3 lines in general position in the space and let a, b, c, d be 4 lines in the space such that each of them intersects all lines l_1, l_2 and l_3. Prove that the three cross ratios of the points of such intersections on each line l_1, l_2 and l_3 are the same (hint: use the previous problem).
- Take an angle between 2 rays l_1 and l_2 with vertex O. Take a point A inside the angle. Consider all triangles with vertex O two sides of which belong to l_1 and l_2 and third side of which passes through the point A. Prove that the area of the triangle will be the smallest if the point A is the middle of the third side.
- Take 3 lines l, l_1 and l_2 in general position in the space. Consider the following map F from line l_1 to line l_2. By definition for each point $A \in l_1$ the point $F(A)$ is the point of intersection of the line l_2 with the plane passing through the line l and the point A. Prove that the map F is projective, i.e. F preserves the cross ratio of each 4 points.
- Take a projective map $f : l \to l$ from a line l to itself. Assume that there is a point $a \in l$ such that $f(a) \neq a$ but $f(f(a)) = a$. Prove that $f(f(x)) = x$ for every $x \in l$.
- Assume that for four lines a, b, c, d passing though a point P the cross-ratio (a, b, c, d) equals -1. Prove: if the ray c bisects the angle between a and b, then d is perpendicular to c.
- Let P_1, P_2, P_3, P_4 be four points on an ellipse E. Show that the cross ratio of lines QP_1, QP_2, QP_3, QP_4 does not depend on the choice of the point $Q \in E$.
- If P_1, P_2, P_3, P_4 are vertices of a square inscribed in a circle E, what is the cross ratio of lines QP_1, QP_2, QP_3, QP_4 for $Q \in E$?
- Let A_1, A_2, A_3 and A_4 be 4 points on one line. Assume that the cross ratio (A_1, A_2, A_3, A_4) is equal to t. Prove that among 24 numbers $(A_{i_1}, A_{i_2}, A_{i_3}, A_{i_4})$ with different indices i_1, i_2, i_3, i_4 there are at

most 6 different numbers. Prove that these numbers are equal to $t, 1-t, 1/t, (t-1)/t, t/(t-1), 1/(1-t)$.

- Take a sheet of graph paper, draw a horizontal segment of length 11 (which occupies roughly the middle third of the sheet), draw a line through the left endpoint A of the segment with the slope of $-45°$, draw another line through the right endpoint B of the segment with the slope of $45°$. Draw a series of lines through the following pairs of points: pick an integer x between -9 and 9, move the point A along the first line by x squares and get a new point $A(x)$, move the point B along the second line by x squares and get a new point $B(x)$; (for positive x $A(x)$ and $B(x)$ lie to the right of A and B respectively); and then draw a line through $A(x)$ and $B(x)$. The resulting picture will clearly show a conic section (parabola) that is tangent to all lines that you have drawn. Explain why all these lines are tangent to a conic section and why this conic section is a parabola.