The Technique of Center of Mass

0.1 Intuitive meaning of center of mass

The notion of center of mass of a body or a system of particles, which is very useful in kinematics, proves to be very handy in geometry as well. Let's see what the center of mass is for a system of two point masses m_1 and m_2 located at points A and B. One way to define it is to say it is the unique point C on the line AB such that $m_1 \cdot AC = m_2 \cdot CB$ (the notations AC, CB can be interpreted as just lengths of the segments for now, but later, when we introduce negative masses, it will be better to think of them as oriented lenths, or, even better, vectors).
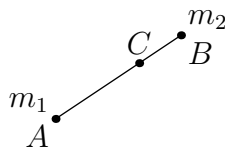

Figure 1: If the mass at B is twice as big as the mass at A, then the center of mass C will be twice as close to B as it is to A

One possible way to understand what center of mass means is to imagine a big flat weightless table top with two small objects of masses m_1 and m_2 pulling down (with forces proportional to their masses). If we want to support the table top on just one leg, we will have to place this leg at the center of mass of these two objects.

This is also the intuitive meaning of the center of mass of any collection of objects - it is the point at which a weightless tabletop should be supported if it has to stay in equilibrium once we put our system of objects on it.
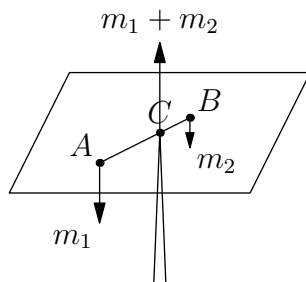
Figure 2: Force proportional to $m_1 + m_2$ must be placed at C to keep the table in equilibrium

For the sake of equilibrium considerations we can think of the effect of putting point masses m_1 and m_2 at points A and B as equal to the effect of placing mass $m_1 + m_2$ at their center of mass (see figure 0.1).

We will use this property to define the center of mass of three point masses m_1, m_2 and m_3 placed at points A_1, A_2 and A_3 respectively. We can first replace the two masses m_1 and m_2 by one mass $m_1 + m_2$ and place this new mass at the center of masses C_{12} of the masses m_1 and m_2. As we mentioned above, this shouldn't change the equilibrium of the tabletop, thus the center of mass of the system shouldn't change as well. So to compute the center of mass of all three bodies it remains to find the center of mass of mass m_3 at A_3 and the combined mass $m_1 + m_2$ at C_{12}.

Note, however, that the definition we proposed for the center of mass has a problem. The tabletop equilibrium considerations depend on our intuitive understanding of the words "mass", "flat weightless tabletop", "forces" etc, and thus can't serve as a basis for a rigorous mathematical definition.

One might argue that we can *define* the center of mass as the answer that the process of finding it described above produces. While this is what we shall do presently, such definition is also problematic. Indeed, why did we first combine m_1 with m_2 and then combined the result with m_3? Why not combine m_2 with m_3 first and then combine the result with m_1? Maybe we will get a different answer and then our "definition" won't be applicable? We will see in next couple of sections that this is not the case. In fact the property that all the computations of the center of mass by using different processes (i.e. different sequences of combining masses into groups) give the same answer will be very useful in geometric problems.

0.2 Axiomatic definition of center of mass

To summarize the discussion above, we give now a list of axioms center of mass should satisfy. The fact that there is a construction that associates to a system of masses its center of mass that satisfies these axioms will be proved later, in section ...

Axiom 1: the center of mass of a system of two point masses m_1 and m_2 located at points A, B is the unique point C on the line AB such that $m_1 \cdot AC = m_2 \cdot CB$ (the masses m_1, m_2 are allowed to be any real numbers, positive or negative).

Axiom 2: Suppose the system of masses S is subdivided to disjoint subsystems $S_1, \ldots, S_k$. Suppose that the total mass of subsystem S_i is M_i and that its center of masses is located at point C_i. Then the center of mass of system S coincides with the center of mass of system of masses M_i located at points C_i.

Using the two axioms above, we can compute the center of mass of any system of point masses: indeed, axiom 1 allows one to compute the center of mass of any system of two masses, and axiom 2 allows recursive calculation of center of mass of any larger system of masses by subdividing it to several subsystems.

The axioms above do not, however, justify why they are consistent, i.e. why there is a way to assign center of mass to any system of masses in a way that both axioms will be satisfied. We will deal with this later. For now we will assume that the center of masses is well-defined and look at some problems from geometry to which this notion can be applied.

0.3 Applications of Notion of Center of Mass

Suppose we start with a triangle ABC and three unit masses at its vertices. How can we compute the center of mass of the resulting system? One way is to combine masses at A and B first. This will give us a mass of two units placed at the midpoint of AB. Now we combine the resulting mass with the unit mass at C. By doing so we get that the center of mass lies on the segment connecting the vertex C to the midpoint of AB and divides it in ratio $2 : 1$ (the part of it adjacent to the vertex being the longer one). But we could proceed in a different way: first combine the masses at A and C and then combine the result with the mass at B. In this way we get that the

center of mass of the system lies on the median from vertex B and divides it in ratio $2 : 1$. By combining the masses in the third possible order (B and C first and adding A afterwards) we get that it also lies on the median from the vertex A and divides it in the ratio $2 : 1$. Thus we get that the center of mass lies on all three medians of triangle ABC and divides them in ratios $2 : 1$. We've seen this theorem in previous lecture, but didn't interpret the intersection point of medians as the center of mass.
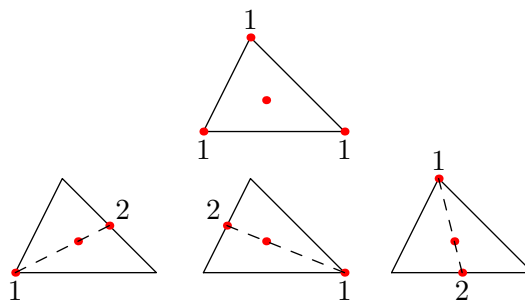
The argument above is very simple, but it is also very typical of how the notion of center of mass is used in geometry. We will see in a moment more complicated applications, but they all use the same simple ideas. For slightly more sophisticated example of use of notion of center of mass, we can reprove Ceva's theorem. Let's remember what it says:

Theorem 1. *Let ABC be a triangle and A', B', C' be points on the sides BC, CA and AB respectively. Then the three lines AA', BB', CC' are concurrent if and only if $\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1$.*

Proof. Place masses m_A, m_B and m_C at points A, B, C so that $m_A \cdot AC' = m_B \cdot C'B$ and $m_B \cdot BA' = m_C A'C$ (note that we assume the points A', B', C' are given to us and we are choosing the masses). For instance we can choose $m_A = 1$, $m_B = \frac{AC'}{C'B}$ and $m_C = \frac{AC'}{C'B} \cdot \frac{BA'}{A'C}$. Then the center of mass of m_A and m_B is at point C' and the center of mass of masses m_B, m_C is at point A'. Thus the center of masses of all three must lie both on the line CC' (if we combine the masses at A, B first and then combine the result with the mass at C) and on the line AA' (if instead we combine the masses at B and C first). So the center of masses lies on the intersection of the lines AA' and CC'. We will call it O. Our goal now is to show that O lies on the line BB' if and only if $\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1$.

Now O, being the center of mass, lies on the segment joining B with the center of mass of m_A and m_C. The latter divides the segment AC in

ratio $\frac{m_C}{m_A} = \frac{AC'}{C'B} \cdot \frac{BA'}{A'C}$. Hence B' coincides with this center of mass if and only if $\frac{m_C}{m_A}$ is equal to $\frac{AB'}{B'C}$, or equivallently $\frac{AC'}{C'B} \cdot \frac{BA'}{A'C} = \frac{B'A}{CB'}$, or, equivalently $\frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} = 1$. Hence O lies on BB' if and only if $\frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} = 1$.

$\square$

Note how a clever choice of masses in the last proof allowed us to prove both directions of Ceva's theorem in a rather simple fashion.

A similar technique can be adopted to prove Menelaus' theorem as well.

Theorem 2. *Let ABC be a triangle and let points A', B' and C' belong to the lines BC, CA and AB respectively. The points A', B' and C' belong to one line if and only if $\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = -1$*

Proof. Place masses m_A, m_B and $m_{B'}$ at points A, B and B' respectively in such a way that C' is the center of mass of m_A and m_B (i.e. $\frac{m_A}{m_B} = \frac{C'B}{AC'}$) while the center of mass of m_A and $m_{B'}$ is at C (i.e. $\frac{m_{B'}}{m_A} = \frac{AC}{CB'}$).
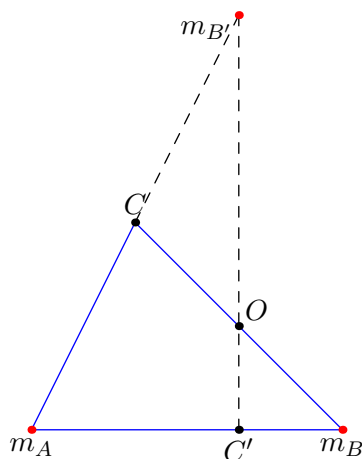
The center of mass of all three then lies both on the segment $B'C'$ and on the segment BC. Call this point O.

We want to show that O coincides with A' if and only if $\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = -1$.

Point O divides the segment BC in ratio $\frac{BO}{OC} = \frac{m_A + m_{B'}}{m_B}$, hence it coincides with point A' if and only if $\frac{BA'}{A'C} = \frac{m_A + m_{B'}}{m_B}$. But the ratio $\frac{m_A + m_{B'}}{m_B}$ can be computed as follows:

$$\frac{m_A + m_{B'}}{m_B} = \frac{m_A + m_{B'}}{m_A} \cdot \frac{m_A}{m_B}$$

The ratio $\frac{m_A+m_{B'}}{m_A} = 1 + \frac{m_{B'}}{m_A}$ is equal to $1 + \frac{AC}{CB'} = \frac{AC+CB'}{CB'} = \frac{AB'}{CB'}$, while the ratio $\frac{m_A}{m_B}$ is equal to $\frac{C'B}{AC'}$. Thus we can conclude that the point O coincides with A' if and only if $\frac{BA'}{A'C} = \frac{AB'}{CB'} \cdot \frac{C'B}{AC'}$ or $\frac{AB'}{CB'} \cdot \frac{C'B}{AC'} \cdot \frac{A'C}{BA'} = 1$. The last equality can be rewritten in the form $\frac{AB'}{B'C} \cdot \frac{BC'}{C'A} \cdot \frac{CA'}{A'B} = -1$, as required.

$\square$

0.4   Analogues of theorems of Ceva and Menelaus in space

After having seen how the notion of center of mass helps to prove Ceva's and Menelaus' theorems, we can try to find and prove three-dimensional analogues of these theorems. In fact instead of having two separate theorems, like in plane, we have only one Ceva-Menelaus theorem, but it is simultaneously the analogue of both planar theorems. Here is what it says:

Theorem 3. *Let $ABCD$ be a non-planar quadrilateral in space. Let points X, Y, Z, W lie on lines AB, BC, CD and DA respectively. Then the following three conditions are equivallent:*

1. The four points X, Y, Z, W are coplanar.

2. The four planes ABZ, BCW, CDX, DAY are concurrent.

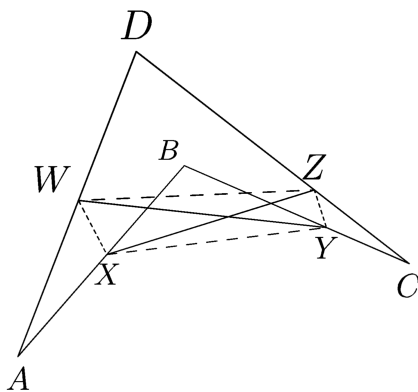*3. $\frac{AX}{XB} \cdot \frac{BY}{YC} \cdot \frac{CZ}{ZD} \cdot \frac{DW}{WA} = 1$*
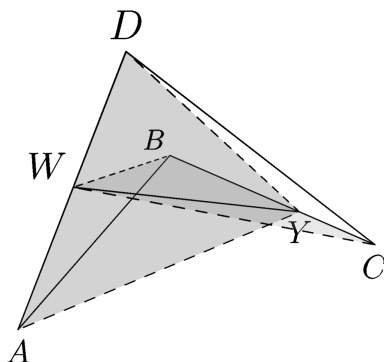

Figure 3:

Figure 4:

Notice that it is reasonable that the same condition $\frac{AX}{XB} \cdot \frac{BY}{YC} \cdot \frac{CZ}{ZD} \cdot \frac{DW}{WA} = 1$ appear in both Ceva's and Menelaus' theorem in space, because the product $\frac{AX}{XB} \cdot \frac{BY}{YC} \cdot \frac{CZ}{ZD} \cdot \frac{DW}{WA}$ is equal to $(-1)^4$ times the product $\frac{XA}{BX} \cdot \frac{YB}{CY} \cdot \frac{ZC}{DZ} \cdot \frac{WD}{AW}$, so that they are in fact equal.

Proof. Condition 1 is equivalent to the condition that lines XZ and WY intersect at a common point (indeed, two lines in space intersect if and only if there is a plane containing both of them). Now the line XZ is the intersection line of planes ABZ and DCX while the line WY is the intersection line of planes BCW and ABZ. Thus the lines XZ and WY intersect if and only if the four planes are concurrent.

Now we will present two proofs that condition 1 is equivalent to condition 3. The first proof relies on the notion of center of mass.

Place masses m_A, m_B, m_C and m_D at points A, B, C, D so that the center of mass of m_A and m_B is at X, the center of mass of m_B and m_C is at Y and finally the center of mass of m_C and m_D is at Z. These conditions are equivalent to $\frac{m_B}{m_A} = \frac{AX}{XB}, \frac{m_C}{m_B} = \frac{BY}{YC}$ and $\frac{m_D}{m_C} = \frac{CZ}{ZD}$ and they can be easily satisfied. Let $\tilde{W}$ be the center of mass of m_A and m_D.

If we combine the masses at A and B and also combine the masses at C and D, we get that the total center of mass lies on the line XZ. If we combine the masses at A and D and also combine the masses at B and C, we get that the same point lies on $Y\tilde{W}$.

Thus the lines XZ and $Y\tilde{W}$ intersect at one point, hence the four points $X, Y, Z, \tilde{W}$ are coplanar.

It follows that the points X, Y, Z, W can be coplanar if and only if $\tilde{W}$, the

center of mass of m_A and m_D, coincides with W, or, equivalently, $\frac{m_D}{m_A} = \frac{AW}{WD}$. But $\frac{m_D}{m_A} = \frac{m_D}{m_C} \cdot \frac{m_C}{m_B} \cdot \frac{m_B}{m_A} = \frac{AX}{XB} \cdot \frac{BY}{YC} \cdot \frac{CZ}{ZD}$. So X, Y, Z, W are coplanar if and only if $\frac{AX}{XB} \cdot \frac{BY}{YC} \cdot \frac{CZ}{ZD} = \frac{AW}{WD}$ or $\frac{AX}{XB} \cdot \frac{BY}{YC} \cdot \frac{CZ}{ZD} \cdot \frac{DW}{WA} = 1$, as required.

Another way to prove the same result is along the lines of proof of Menelaus' theorem in lecture 1. We first prove a lemma about parallel projection on a line.

Figure 5:

Lemma 4. *Let l, l' be two lines. Let π_1, π_2, π_3 be three planes that are parallel to each other and intersect both l and l'. Let A_1, A_2, A_3 be the intersection points of π_1, π_2 and π_3 with the line l. Let also A_1', A_2', A_3' be the intersection points of π_1, π_2 and π_3 with the line l'. Then $\frac{A_1 A_2}{A_2 A_3} = \frac{A_1' A_2'}{A_2' A_3'}$.*

This lemma states that parallel projection from a line to another line in space preserves ratios of oriented lengths.

Proof. In case line l and l' lie in a common plane, the claim follows from the two-dimensional analogue (the parallel projection in the space restricts to a parallel projection in two dimensions).

Figure 6:

In the case when the lines l, l' do not lie in a common plane, we can find another line l'' that intersects both of these lines and is not parallel to the planes π_1, π_2, π_3. Let A_1'', A_2'', A_3'' denote the points of intersection of line l'' with planes π_1, π_2, π_3. Since the lines l, l'' are coplanar, we can conclude from the first case we considered that $\frac{A_1 A_2}{A_2 A_3} = \frac{A_1'' A_2''}{A_2'' A_3''}$. Similarly the lines l'' and l' are coplanar, hence $\frac{A_1'' A_2''}{A_2'' A_3''} = \frac{A_1' A_2'}{A_2' A_3'}$. These two equalities together imply that $\frac{A_1 A_2}{A_2 A_3} = \frac{A_1' A_2'}{A_2' A_3'}$.

Figure 7:

$\square$

With the help of this lemma we can now finish the proof of Ceva-Menelaus theorem. Suppose that the points X, Y, Z, W lie in one plane. Choose a line l that is not parallel to this plane and consider parallel projection onto this line along the direction of the plane $XYZW$. Let A', B', C', D' be the images of points A, B, C, D under this projection. Let O be the image of points X, Y, Z, W. Then clearly $\frac{A'O}{OB'} \cdot \frac{B'O}{OC'} \cdot \frac{C'O}{OD'} \cdot \frac{D'O}{OA'} = 1$ (because of cancellations) and the lemma implies that $\frac{A'O}{OB'} = \frac{AX}{XB}, \frac{B'O}{OC'} = \frac{BY}{YC}, \frac{C'O}{OD'} = \frac{CZ}{ZD}$ and $\frac{D'O}{OA'} = \frac{DW}{WA}$. Thus $\frac{AX}{XB} \cdot \frac{BY}{YC} \cdot \frac{CZ}{ZD} \cdot \frac{DW}{WA} = 1$.

Conversely, suppose that $\frac{AX}{XB} \cdot \frac{BY}{YC} \cdot \frac{CZ}{ZD} \cdot \frac{DW}{WA} = 1$. Let $\tilde{W}$ be the point of intersection of line AD with the plane XYZ. From what we already proved, $\frac{AX}{XB} \cdot \frac{BY}{YC} \cdot \frac{CZ}{ZD} \cdot \frac{D\tilde{W}}{\tilde{W}A} = 1$. Hence $\frac{D\tilde{W}}{\tilde{W}A} = \frac{DW}{WA}$, showing that $\tilde{W} = W$.

$\square$

0.5 Existence of Center of Mass

After having seen some of the applications of the notion of center of mass, time has come to make sure one exists, i.e. that there is a rule that associates a point to a system of masses, which satisfies axioms 1 and 2 from section 0.2.

We will find the center of mass of a system of particles by introducing linear structure to our space, i.e. choosing an origin and allowing ourselves to add and scale points.

Recall that if we choose an origin O in the plane, then we can identify any point A in the plane with the vector $\overline{OA}$ from the origin O to A. Once we do so, we get the ability to speak about sum of points and scalar multiples of points. Note however that the results of scaling and addition depend on the particular choice of origin. Indeed, if we choose another point O' as the origin, then the point A will correspond to the vector $\overline{O'A}$. This vector can be expressed as $\overline{O'O} + \overline{OA}$. Similarly point B corresponds now to vector $\overline{O'B} = \overline{O'O} + \overline{OB}$. The point, expressing the sum of $\overline{OA} + \overline{OB}$ in previous coordinate system corresponds now to the vector $\overline{O'O} + (\overline{OA} + \overline{OB})$. However the sum of $\overline{O'A}$ and $\overline{O'B}$ is equal to $(\overline{O'O} + \overline{OA}) + (\overline{O'O} + \overline{OB}) = 2\overline{O'O} + (\overline{OA} + \overline{OB})$, so it doesn't correspond to the same point.

The considerations above show that it doesn't make sense to talk about "sum of two points in the plane". This sum can be given a meaning only after the origin has been chosen and the meaning depends on the choice. We may ask ourselves whether at least some linear combinations make sense.
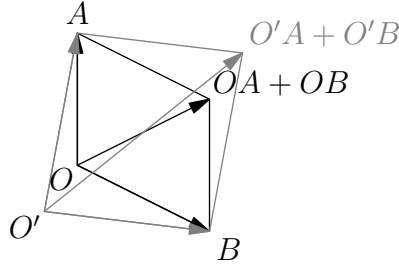
For example we know that the sum $\frac{1}{2}(\overline{OA}+\overline{OB})$ expresses the midpoint of the segment AB, so it has geometrical meaning, which is independent of the choice of origin.

Indeed, if O' is another origin, then the new sum $\frac{1}{2}(\overline{OA}+\overline{OB})$ is equal to $\frac{1}{2}\left((\overline{O'O}+\overline{OA})+(\overline{O'O}+\overline{OB})\right) = \overline{O'O}+\frac{1}{2}(\overline{OA}+\overline{OB})$. Now for any vector v, the vector v with origin O corresponds to the same point as the vector $\overline{O'O}+v$ with origin O'. Thus the vector $\frac{1}{2}(\overline{OA}+\overline{OB})$ with origin O corresponds to the same point as the vector $\frac{1}{2}(\overline{O'A}+\overline{O'B}) = \overline{O'O}+\frac{1}{2}(\overline{OA}+\overline{OB})$ with origin O'.

We can generalize the previous observation:

Claim 1. *Let A_i, i from 1 to n, be n points (in plane or in higher dimensional space). Let λ_i be real numbers such that thir sum is equal to 1: $\sum_{i=1}^{n}\lambda_i = 1$. Then the point that corresponds to the vector sum $\sum_{1}^{n}\lambda_i\overline{OA_i}$ does not depend on the choice of origin O.*

Proof. Let O' be another origin. The the points A_i correspond in the new vector space to vectors $\overline{O'A_i} = \overline{O'O}+\overline{OA_i}$. The vector sum we consider then becomes $\sum_{1}^{n}\lambda_i\overline{O'A_i} = \sum\lambda_i(\overline{O'O}+\overline{OA_i}) = \sum\lambda_i\overline{O'O}+\sum\lambda_i\overline{OA_i} = \overline{O'O}+\sum\lambda_i\overline{OA_i}$. The point of our space that corresponds to the vector $\overline{O'O}+\sum\lambda_i\overline{OA_i}$ in the new coordinate system is the same as the point to which $\sum\lambda_i\overline{OA_i}$ corresponds in the old one.

$\square$

Since the point that corresponds to the sum $\sum_{1}^{n}\lambda_i\overline{OA_i}$ doesn't depend on the choice of origin (provided that $\sum_{i=1}^{n}\lambda_i = 1$), we will omit the point O from the notation and will write the point that corresponds to the sum $\sum_{1}^{n}\lambda_i\overline{OA_i}$ simply as $\sum_{1}^{n}\lambda_i A_i$.

We define the center of mass of a system of point masses m_i located at points A_i as the vector sum $\frac{\sum_i m_i A_i}{M}$ where $M = \sum_i m_i$ is the total mass of the system. We are allowed to do this, because $\sum_i \frac{m_i}{M} = 1$.

It remains to verify that the axioms 1 and 2 from section 0.2 are satisfied by this defintion.

For axiom 1, let m_1, m_2 be two masses located at points A and B. Let $C = \frac{m_1 A + m_2 B}{m_1 + m_2}$ be their center of mass. We should verify that $m_1 \overline{AC} = m_2 \overline{CB}$. Indeed, since the point C corresponds to vector $\overline{OC} = \frac{m_1 \overline{OA} + m_2 \overline{OB}}{m_1 + m_2}$, we have

$$m_1 \overline{AC} = m_1(\overline{OC} - \overline{OA}) = m_1 \left(\frac{m_1 \overline{OA} + m_2 \overline{OB}}{m_1 + m_2} - \overline{OA} \right) = \frac{m_1 m_2 (\overline{OB} - \overline{OA})}{m_1 + m_2} = \frac{m_1 m_2 \overline{AB}}{m_1 + m_2}$$

Similarly $m_2 \overline{CB} = m_2(\overline{OB} - \overline{OC}) = m_2 \left(\overline{OB} - \frac{m_1 \overline{OA} + m_2 \overline{OB}}{m_1 + m_2} \right) = \frac{m_1 m_2 (\overline{OB} - \overline{OA})}{m_1 + m_2} = \frac{m_1 m_2 \overline{AB}}{m_1 + m_2}$, so indeed $m_1 \overline{AC} = m_2 \overline{CB}$.

For verification of axiom 2 we will need more or less the same amount of algebra, but slightly more sophisticated notation. Let S be a system of point masses subdivided to subsystems $S_1, \dots, S_n$. For a point mass x we will denote by m_x its mass and by A_x - the point at which it is located.

The center of mass of the system S corresponds (after a choice of origin) to the vector

$$\frac{\sum_{x \in S} m_x \overline{OA_x}}{\sum_{x \in S} m_x}$$

Similarly the center of mass of system S_i is located at point C_i that corresponds to vector

$$\overline{OC_i} = \frac{\sum_{x \in S_i} m_x \overline{OA_x}}{M_i}$$

where $M_i = \sum_{x \in S_i} m_x$ is the total mass of subsystem S_i.

We want to show that

$$\frac{\sum_{x \in S} m_x \overline{OA_x}}{\sum_{x \in S} m_x} = \frac{\sum_i M_i \overline{OC_i}}{\sum_i M_i}$$

Notice that $\sum_i M_i = \sum_i \sum_{x \in S_i} m_x = \sum_{x \in S} m_x$ (the total mass of the system is the sum of the masses of all the subsystems), so it remains to show that $\sum_{x \in S} m_x \overline{OA_x} = \sum_i M_i \overline{OC_i}$. Remember that

$$\overline{OC_i} = \frac{\sum_{x \in S_i} m_x \overline{OA_x}}{M_i}$$

so we have to prove that $\sum_{x \in S} m_x \overline{OA_x} = \sum_i \sum_{x \in S_i} m_x \overline{OA_x}$, which is obvious.

Now we've finally established the existence of center of mass and some of its its interesting properties.

Add exercise on moment of inertia.