University of Toronto
FACULTY OF ARTS AND SCIENCE
Final Examinations, April 25, 2005

MAT325H Classical Plane Geometries and their Transformations
Instructor: Prof. A. Khovanskii
Duration – 3 hours

NO AIDS ALLOWED
Total marks = 100. All questions have equal value

Problem 1. Prove Ceva's theorem: *Let the sides of a triangle ABC be divided at L, M, N in the respective ratios $\lambda\colon 1$, $\mu\colon 1$, $\nu\colon 1$. Then the three lines AL, BM, CN are passing through one point if and only if $\lambda\mu\nu = 1$.*

Problem 2. Take an angle between 2 rays l_1 and l_2 with vertex O and a point A inside the angle. Consider all triangles with vertex O such that two sides of them belong to l_1 and l_2 and the third side l passes through A. Find the location of line l for which the area of the triangle is minimal. Hint: consider the parallelogram with two sides in l_1 and l_2 and with center A and look how line l cuts this parallelogram.

Problem 3. Consider a square $ABCD$ inscribed in a circle. Let P be an arbitrary point on the circle. Explain why the cross-ratio of the lines AP, BP, CP, and DP is independent of the choice of point P. Find this cross-ratio.

Problem 4. Consider two non-concentric circles S_1 and S_2, one inside another. Assume that there exists a chain of circles $S_3, ..., S_{2005}$, such that each circle in the chain is tangent to the circles S_1 and S_2, and also to the next circle (i.e. S_3 is tangent to S_4, S_4 is tangent to S_5 and so on), and S_{2005} is tangent to S_3. Prove that for any other chain of circles $S'_3, \ldots, S'_{2005}$, such that each circle in the chain is tangent to the circles S_1 and S_2, and also to the next circle the last one S'_{2005} will be also tangent to S'_3. Hint: Using an inversion reduce the problem to a simpler form.

Problem 5. Consider a sphere S of radius R. Cover it by equal triangles assuming that each angle of each triangle equals to $2\pi/5$ and assuming that two different triangles or have a common vertex or have a common side or have no points of intersections. How many triangles are there in such covering? (Hint: use areas)