Lesson 9

Formula for the Area of a Triangle

Goals

- Compare, contrast, and critique (orally) different strategies for determining the area of a triangle.
- Generalize a process for finding the area of a triangle, and justify (orally and in writing) why this can be abstracted as $\frac{1}{2} \cdot b \cdot h$.
- Recognize that any side of a triangle can be considered its base, choose a side to use as the base when calculating the area of a triangle, and identify the corresponding height.

Learning Targets

- I can use the area formula to find the area of any triangle.
- I can write and explain the formula for the area of a triangle.
- I know what the terms “base” and “height” refer to in a triangle.

Access for Students with Diverse Abilities

- Representation (Activity 1)

Access for Multilingual Learners

- MLR8: Discussion Supports (Activity 1)

Required Materials

Materials to Gather

- Geometry toolkits: Activity 1

Lesson Narrative

In this lesson, students begin to reason about areas of triangles more methodically: by generalizing their observations up to this point and expressing the area of a triangle in terms of its base and height.

Students first learn about bases and heights in a triangle by studying examples and counterexamples. They see that any side of a triangle can be its base, as is the case for parallelograms. They also learn that the height that corresponds to a chosen base is the length of a perpendicular segment that connects the base to the **opposite vertex**.

Next, they identify base-height measurements of triangles and use them to determine area. Then, students look for a pattern in their reasoning to help them write a formula for finding the area of any triangle. They also have a chance to build an informal argument about why the formula works for any triangle.

Student Learning Goal

Let’s write and use a formula to find the area of a triangle.

Lesson Timeline

Warm-up	Activity 1	Activity 2	Lesson Synthesis	Assessment: Cool-down
10 min	15 min	10 min	10 min	5 min

Warm-up
Bases and Heights of a Triangle

10 min

Activity Narrative

In this activity, students think about the meaning of base and height in a triangle by studying examples and non-examples. Then, they examine some statements about bases and heights and determine if the statements are true. The goal is for students to see that in a triangle:

- Any side can be a base.
- A segment that represents a height must be drawn at a right angle to the base, but can be drawn in more than one place. The length of this perpendicular segment is the distance between the base and the vertex opposite it.
- A triangle can have three possible bases, each with a corresponding height.

Monitor for students who notice similarities between the bases and heights in a triangle and those in a parallelogram. Ask them to share their observations later.

As students justify how they know whether the given statements are true and consider others' justifications, they practice constructing logical reasoning and critiquing the reasoning of others.

Students will have many opportunities to make sense of bases and heights in this lesson and an upcoming one, so they do not need to know how to draw a height correctly at this point.

Launch

Remind students that recently they looked at bases and heights of parallelograms. Tell students that they will now examine bases and heights of triangles.

Display the examples and non-examples of bases and heights for all to see. Read aloud the first paragraph of the activity and the description of each set of images. Give students a minute to observe the images. Then, tell students to use the examples and non-examples to determine what is true about bases and heights in a triangle.

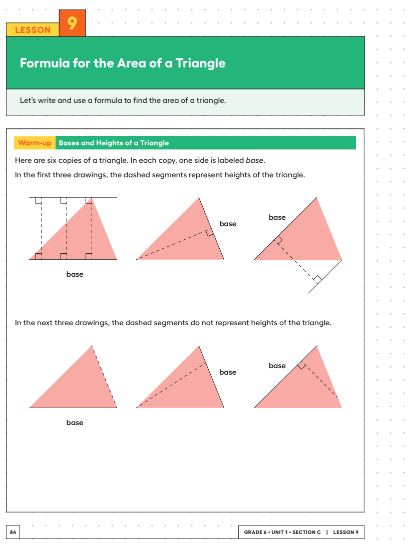
Arrange students in groups of 2. Give them 2–3 minutes of quiet think time and then a minute to discuss their responses with a partner.

Building on Student Thinking

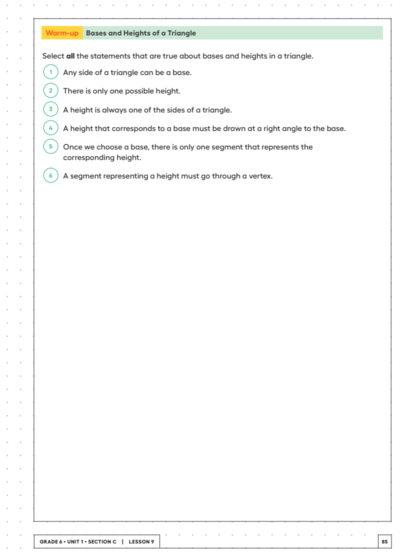
If students are unsure how to interpret the diagrams, ask them to point out parts of the diagrams that might be unclear. Clarify as needed.

Students may not remember from their experience with parallelograms that a height needs to be perpendicular to a base. Consider posting a diagram of a parallelogram—with its base and height labeled—in a visible place in the room so that it can serve as a reference.

Student Workbook



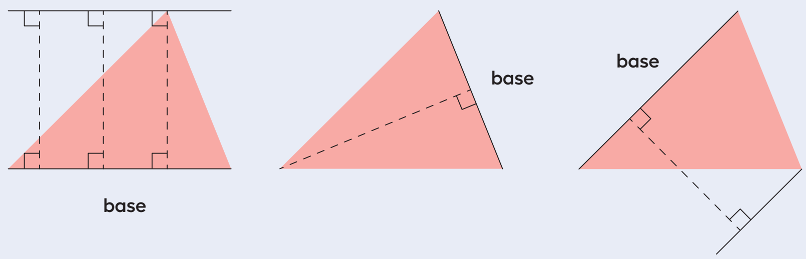
Student Workbook



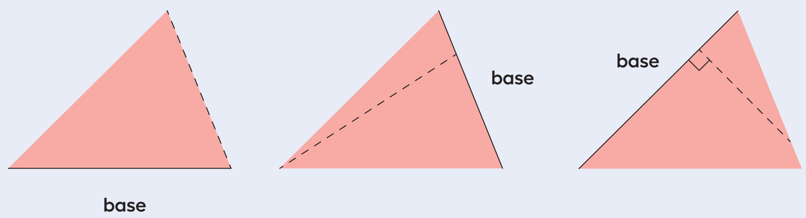
Student Task Statement

Here are six copies of a triangle. In each copy, one side is labeled *base*.

In the first three drawings, the dashed segments represent heights of the triangle.



In the next three drawings, the dashed segments do not represent heights of the triangle.



Select **all** the statements that are true about bases and heights in a triangle.

1. Any side of a triangle can be a base.
2. There is only one possible height.
3. A height is always one of the sides of a triangle.
4. A height that corresponds to a base must be drawn at a right angle to the base.
5. Once we choose a base, there is only one segment that represents the corresponding height.
6. A segment representing a height must go through a vertex.

Activity Synthesis

For each statement, ask students to indicate whether they think it is true. For each statement, ask one or two students to explain how they know. Encourage students to use the examples and counterexamples to support their argument (for instance, "The last statement is not true because the examples show dashed segments or heights that do not go through a vertex"). Make sure that students agree about each statement before moving on. Display the true statements for all to see.

Students should see that only Statements 1 and 4 are true—that any side of a triangle can be a base, and a segment for the corresponding height must be drawn at a right angle to the base. What is missing—an important gap to fill during discussion—is the length of any segment representing a height.

Ask students,

"How long should a segment that shows a height be? If we draw a perpendicular line from the base, where do we stop?"

Solicit some ideas from students. Then, highlight the following:

- The length of each perpendicular segment is the shortest distance between the base and its opposite vertex. The **opposite vertex** is the vertex that is not an endpoint of the base. (Point out the opposite vertex for each base.)
- The segment representing height does not have to be drawn through the vertex (although that would be a natural place to draw it). It does need to maintain that distance between the base and the opposite vertex.

If any students noticed connections between the bases and heights in a triangle with those in a related parallelogram, invite them to share their observations. Otherwise, draw students' attention to it. Consider duplicating a triangle that shows a base and a height (by tracing on patty paper or creating a paper cutout). Use the original and the copy to compose a parallelogram. Ask students:

"Suppose we choose the same side as the base of both the parallelogram and the triangle. What do you notice about the height of each shape?"

The two shapes have the same height.

Activity 1

Finding a Formula for the Area of a Triangle

15 min

Activity Narrative

In this activity, students generalize a formula for the area of triangles. They build on their observations about the relationship between the area of a triangle and the area of a parallelogram with the same base and height.

Students first find the areas of several triangles given base and height measurements. They notice regularity in repeated reasoning and arrive at an expression for finding the area of any triangle. Students might write $b \cdot h \div 2$, $b \cdot h \cdot \frac{1}{2}$, or another equivalent expression.

At the end of the activity, consider giving students a chance to reason more abstractly and think about why the expression $b \cdot h \div 2$ would hold true for all triangles. See the activity synthesis for prompts and diagrams that support such reasoning.

Launch

Arrange students in groups of 2–3. Explain that they will now find the area of some triangles using what they know about base-height pairs in triangles and the relationships between triangles and parallelograms.

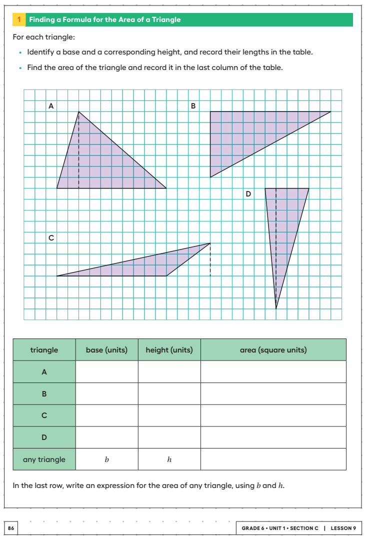
Give students 5 minutes to complete the activity. Provide access to geometry toolkits, especially tracing paper. Ask them to find the area of at least two triangles independently before discussing with their partner(s).

If needed, remind students how they reasoned about the area of triangles in the previous lesson (such as by composing a parallelogram, enclosing a triangle with one or more rectangles, and so on). Encourage them to refer to their previous work and to use tracing paper as needed.

Building on Student Thinking

Students may not be inclined to write an expression using the variables b and h and instead replace the variables with numbers of their choice. Ask them to reflect on what they did with the numbers for the first four triangles. Then, encourage them to write the same operations but using the letters b and h rather than numbers.

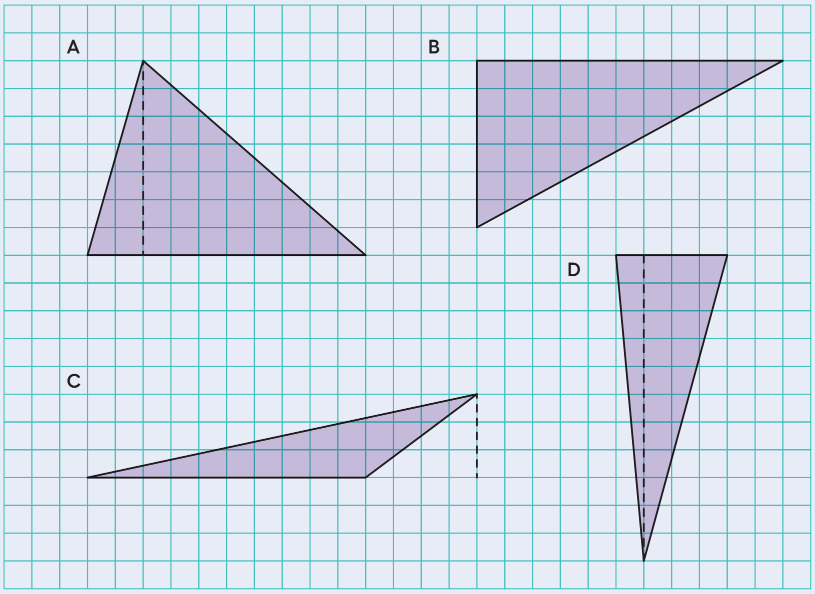
Student Workbook



Student Task Statement

For each triangle:

- Identify a base and a corresponding height, and record their lengths in the table.
- Find the area of the triangle and record it in the last column of the table.



In the last row, write an expression for the area of any triangle, using b and h.

triangle	base (units)	height (units)	area (square units)
A	10	7	35
B	11 (or 6)	6 (or 11)	33
C	10	3	15
D	4	11	22
any triangle	b	h	$b \cdot h \div 2$ (or equivalent)

Sample responses:

- We can make a parallelogram from any triangle using the same base and height. The triangle will be half of the parallelogram. The area of a parallelogram is the length of the base times the length of the height, so the area of the triangle will be $b \cdot h \div 2$.
- I can cut off the top half of a triangle and rotate it to make a parallelogram. That parallelogram has a base of b and a height that is half of the original triangle, which is $\frac{1}{2} \cdot h$, so its area is $b \cdot \frac{1}{2} \cdot h$. Since the parallelogram is just the triangle rearranged, the area of the triangle is also $\frac{1}{2} b \cdot h$.

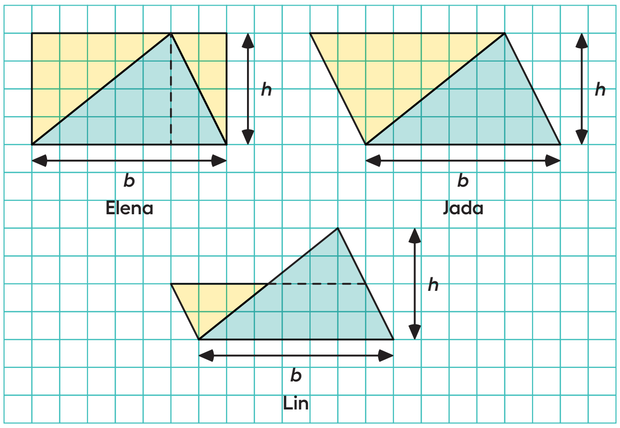
Activity Synthesis

Select a few students to share their expression for finding the area of any triangle. Record each expression for all to see.

To give students a chance to reason logically and deductively about their expression, ask,

"Can you explain why this expression is true for any *triangle?"*

Display the following diagrams for all to see. Give students a minute to observe the diagrams. Ask them to choose one that makes sense to them and use that diagram to explain or show in writing that the expression $b \cdot h \div 2$ works for finding the area of any triangle. (Consider giving each student an index card or a sheet of paper on which to write their reasoning so that their responses could be collected, if desired.)



When dealing only with the variables b and h and no numbers, students are likely to find Jada's and Lin's diagrams more intuitive to explain. Those choosing to use Elena's diagram are likely to suggest moving one of the extra triangles and joining it with the other to form a non-rectangular parallelogram with an area of $b \cdot h$. Expect students to be less comfortable reasoning in abstract terms than in concrete terms. Prepare to support them in piecing together a logical argument using only variables.

If time permits, select students who used different diagrams to share their explanation, starting with the most commonly used diagram (most likely Jada's). Ask other students to support, refine, or disagree with their arguments. If time is limited, consider collecting students' written responses now and discussing them in an upcoming lesson.

Access for Multilingual Learners (Activity 1, Synthesis)

MLR8: Discussion Supports. Provide students with the opportunity to rehearse with a partner their explanation for why the expression $b \cdot h \div 2$ works for finding the area of any triangle before they share with the whole class or explain it in writing.
Advances: Speaking. Writing

Access for Students with Diverse Abilities (Activity 1, Synthesis)

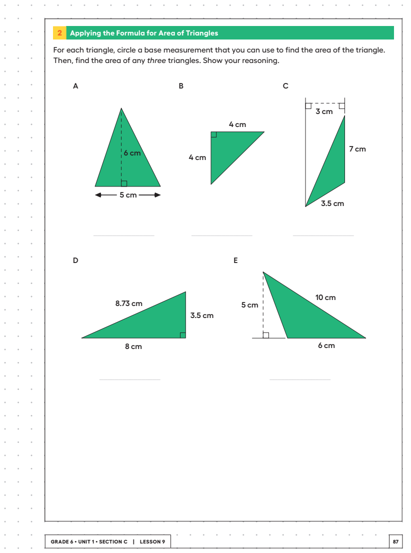
Representation: Develop Language and Symbols. Maintain a display of important terms and vocabulary. Invite students to suggest language or diagrams to include that will support their understanding of finding the area of a triangle. Terms may include: opposite vertex, base (of a triangle), height (of a triangle), and formula for finding area.
Supports accessibility for: Conceptual Processing, Language

Building on Student Thinking

The extra measurement in Triangles C, D, and E may confuse some students. If they are unsure how to decide which measurement to use, ask what they learned must be true about a base and a corresponding height in a triangle. Urge them to review the work from the *Warm-up* activity.

Some students may omit the step of dividing by two when finding the area of a triangle. Remind them of the idea that a triangle takes up half as much space as does a parallelogram with the same base and height, post images for reference, or point out the importance of the $\frac{1}{2}$ in the formula for the area of a triangle.

Student Workbook



Activity 2

Applying the Formula for Area of Triangles

10 min

Activity Narrative

In this activity, students apply the expression they previously generated to find the areas of various triangles. Each diagram is labeled with two or three measurements. Before calculating, students think about which lengths can be used to find the area of each triangle.

As students work, monitor for students who choose different bases for Triangles B and D. Later, invite them to contribute to the discussion about finding the areas of right triangles.

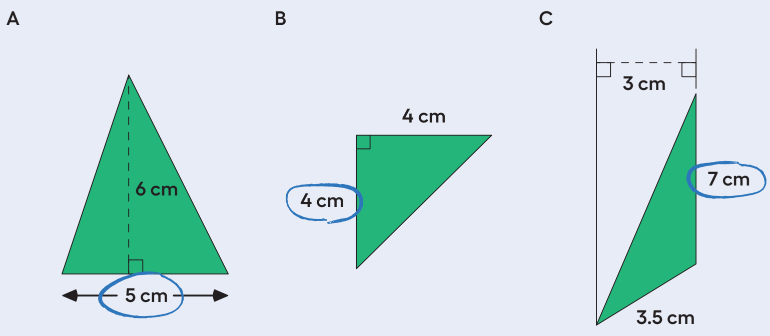
Launch

Explain to students that they will now practice using their expression to find the area of triangles without a grid. For each triangle, ask students to be prepared to explain which measurement they chose for the base and which one for the corresponding height and why.

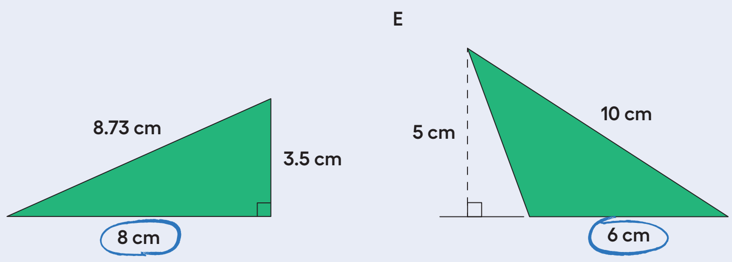
Keep students in groups of 2–4. Give students 3 minutes of quiet think time and 1 minute to discuss their responses with their group.

Student Task Statement

For each triangle, circle a base measurement that you can use to find the area of the triangle. Then, find the area of any *three* triangles. Show your reasoning.



15 square cm | 8 square cm | 10.5 square cm



14 square cm | 15 square cm

In B either of the given pair of measurements can be the base.

Triangle A: 15 square cm, $b = 5$, $h = 6$, $A = 5 \cdot 6 \div 2 = 15$

Triangle B: 8 square cm, $b = 4$, $h = 4$, $A = 4 \cdot 4 \div 2 = 8$

Triangle C: 10.5 square cm, $b = 7$, $h = 3$, $A = 7 \cdot 3 \div 2 = 10.5$

Triangle D: 14 square cm, $b = 8$, $h = 3.5$, $A = 8 \cdot (3.5) \div 2 = 14$

Triangle E: 15 square cm, $b = 6$, $h = 5$, $A = 6 \cdot 5 \div 2 = 15$

Activity Synthesis

The aim of this whole-class discussion is to deepen students' awareness of the base and height of triangles. Discuss questions such as:

- *"For Triangle A, can we say that the 6-cm segment is the base and the 5-cm segment is the height? Why or why not?"*

  No, the base of a triangle is one of its sides.

- *"For Triangle C, can the 3.5-cm segment serve as the base? Why or why not?"*

  Yes, it is a side of the triangle.

- *"What about the 3-cm segment?"*

  No, that segment is not a side of the triangle.

- *"More than two measurements are given for Triangles C, D, and E. Which ones are helpful for finding area?"*

  We need a base and a corresponding height, which means the length of one side of the triangle and the length of a perpendicular segment between that side and the opposite vertex.

Lesson Synthesis

In this lesson, students learned that, similar to the area of a parallelogram, the area of a triangle can also be determined using base and height measurements. Discuss with students:

- *"How do we locate the base of a triangle? How many possible bases are there?"*

  Any side of a triangle can be a base. There are 3 possible bases.

- *"How do we locate the height once we know the base?"*

  Find the length of a perpendicular segment that connects the base and its opposite vertex.

- *"Can both the base and height be sides of the triangle?"*

  Yes

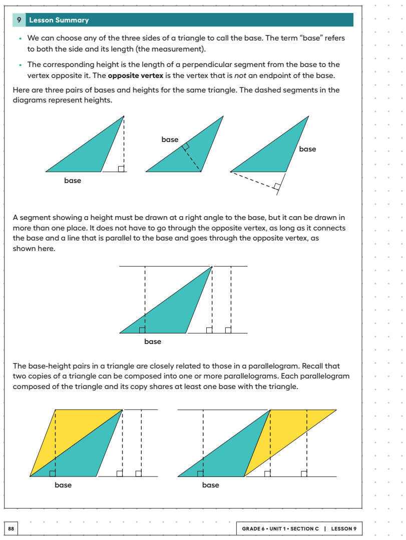
- *"When is that possible?"*

  In a right triangle, both the base and height can be the sides of the triangle.

- *"In the last activity, Triangles C, D, and E each have more than two given measurements. Which ones are helpful for finding area?*

  The length of one side of the triangle and the length of a perpendicular segment between that side and the opposite vertex.

Student Workbook


Lesson Summary

- We can choose any of the three sides of a triangle to call the base. The term "base" refers to both the side and its length (the measurement).
- The corresponding height is the length of a perpendicular segment from the base to the vertex opposite it. The **opposite vertex** is the vertex that is *not* an endpoint of the base.

Here are three pairs of bases and heights for the same triangle. The dashed segments in the diagrams represent heights.

A segment showing a height must be drawn at a right angle to the base, but it can be drawn in more than one place. It does not have to go through the opposite vertex, as long as it connects the base and a line that is parallel to the base and goes through the opposite vertex, as shown here.

The base-height pairs in a triangle are closely related to those in a parallelogram. Recall that two copies of a triangle can be composed into one or more parallelograms. Each parallelogram composed of the triangle and its copy shares at least one base with the triangle.

Next, discuss the formula for finding the area of a triangle. Consider asking students:

"What expression works for finding the area of a triangle?"

$\frac{1}{2} \cdot b \cdot h$ or $\frac{b \cdot h}{2}$

"Can you explain briefly why this expression or formula works?"

The area of a triangle is always half of the area of a related parallelogram that shares the same base and height.

Lesson Summary

- We can choose any of the three sides of a triangle to call the base. The term "base" refers to both the side and its length (the measurement).
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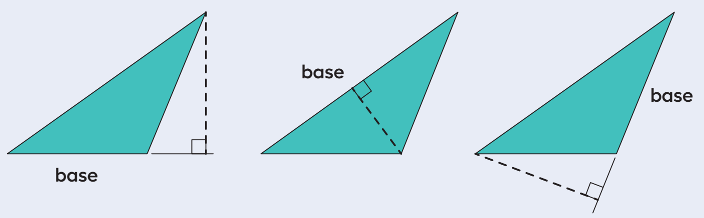

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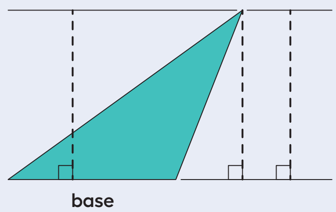

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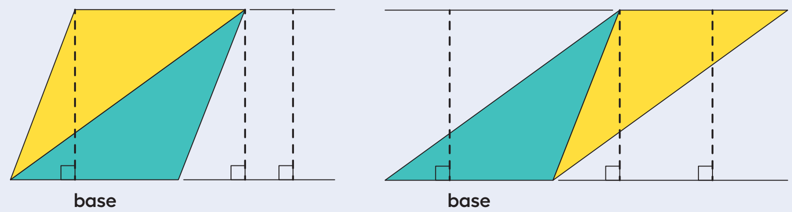
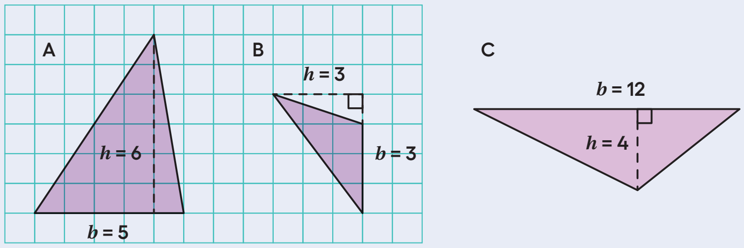


For any base that they share, the corresponding height is also shared, as shown by the dashed segments.

We can use the base-height measurements and our knowledge of parallelograms to find the area of any triangle.

- The formula for the area of a parallelogram with base b and height h is $b \cdot h$.
- A triangle takes up half of the area of a parallelogram with the same base and height. We can therefore express the area, A, of a triangle as: $A = \frac{1}{2} \cdot b \cdot h$

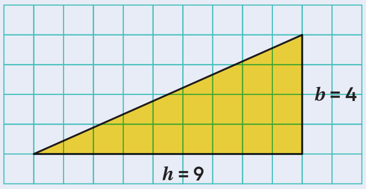


- The area of Triangle A is 15 square units because $\frac{1}{2} \cdot 5 \cdot 6 = 15$
- The area of Triangle B is 4.5 square units because $\frac{1}{2} \cdot 3 \cdot 3 = 4.5$
- The area of Triangle C is 24 square units because $\frac{1}{2} \cdot 12 \cdot 4 = 24$

In each case, one side of the triangle is the base but neither of the other sides is the height. This is because the angle between them is not a right angle.

In right triangles, however, the two sides that are perpendicular can be a base and a height.

The area of this triangle is 18 square units whether we use 4 units or 9 units for the base.



Responding To Student Thinking

More Chances
Students will have more opportunities to understand the mathematical ideas addressed here. There is no need to slow down or add additional work to the next lessons.

Cool-down

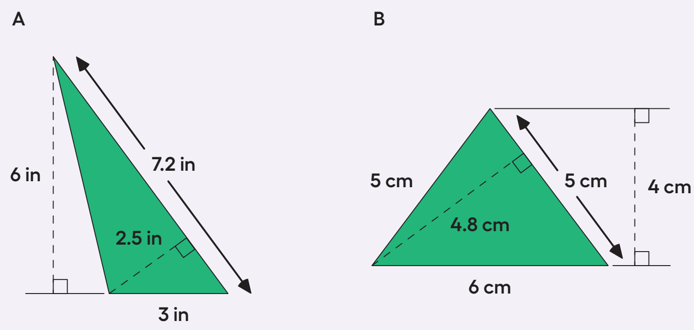
5 min

Two More Triangles

Students apply what they learned about the area formula and about the base and height of a triangle in this *Cool-down*. Multiple measurements are given, so students need to be attentive in choosing the right pair of measurements that would allow them to calculate the area.

Student Task Statement

For each triangle, identify a base and a corresponding height. Use them to find the area. Show your reasoning.



Triangle A: 9 sq in. Sample reasoning:

- $b = 3$, $h = 6$, area: 9 sq in, $\frac{1}{2} \cdot 3 \cdot 6 = 9$
- $b = 7.2$, $h = 2.5$, area: 9 sq in, $\frac{1}{2} \cdot (7.2) \cdot (2.5) = 9$

Triangle B: 12 sq in. Sample reasoning:

- $b = 6$, $h = 4$, area: 12 sq cm, $\frac{1}{2} \cdot 6 \cdot 4 = 12$
- $b = 5$, $h = 4.8$, area: 12 sq cm, $\frac{1}{2} \cdot 5 \cdot (4.8) = 12$

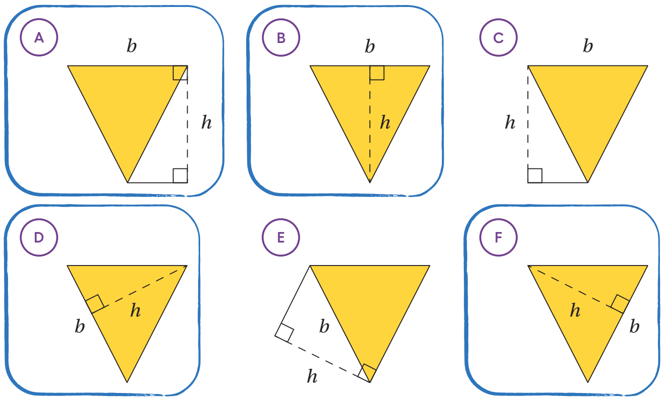
Lesson 9

Practice Problems

6 Problems

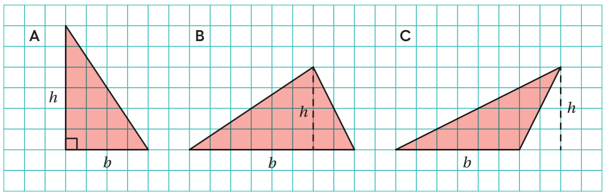
Problem 1

Select **all** drawings in which a corresponding height h for a given base b is correctly identified.



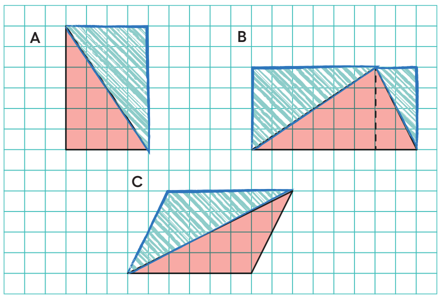
Problem 2

For each triangle, a base and its corresponding height are labeled.



a. Find the area of each triangle.

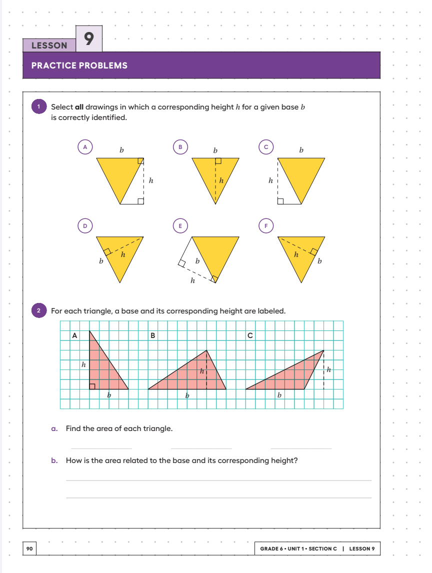
- Triangle A: 12 square units
- Triangle B: 16 square units
- Triangle C: 12 square units



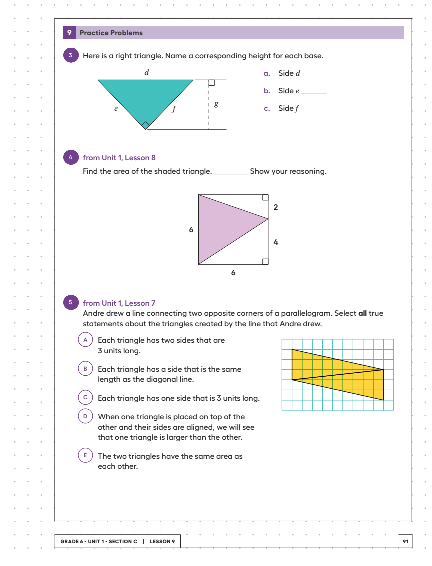
b. How is the area related to the base and its corresponding height?

In each case, the area of the triangle, in square units, is half of the base times its corresponding height, $\frac{b \cdot h}{2}$.

Student Workbook



Student Workbook


9 Practice Problems

3 Here is a right triangle. Name a corresponding height for each base.

a. Side d

b. Side e

c. Side f

4 from Unit 1, Lesson 8

Find the area of the shaded triangle. Show your reasoning.

5 from Unit 1, Lesson 7

Andre drew a line connecting two opposite corners of a parallelogram. Select **all** true statements about the triangles created by the line that Andre drew.

A. Each triangle has two sides that are 3 units long.

B. Each triangle has a side that is the same length as the diagonal line.

C. Each triangle has one side that is 3 units long.

D. When one triangle is placed on top of the other and their sides are aligned, we will see that one triangle is larger than the other.

E. The two triangles have the same area as each other.

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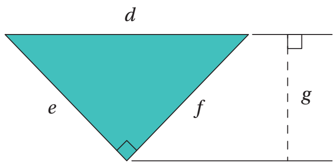
Problem 3

Here is a right triangle. Name a corresponding height for each base.



a. Side d

Segment g

b. Side e

Side f

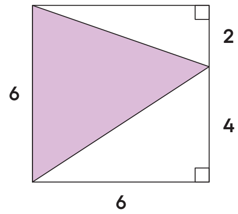
c. Side f

Side e

Problem 4

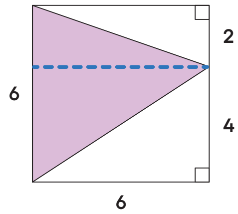
from Unit 1, Lesson 8

Find the area of the shaded triangle. Show your reasoning.



18 square units

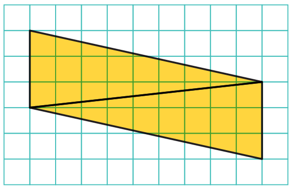
Sample reasoning: Decomposing the triangle with a horizontal line forms two rectangles and splits the triangle into two smaller triangles.
The top triangle is half of the top rectangle, so its area is $\frac{1}{2} \cdot 6 \cdot 2 = 6$.
The bottom triangle is half of the bottom rectangle, so its area is $\frac{1}{2} \cdot 6 \cdot 4 = 12$.
The area of the original triangle is $6 + 12$ or 18 square units.



Problem 5

from Unit 5, Lesson 7

Andre drew a line connecting two opposite corners of a parallelogram. Select **all** true statements about the triangles created by the line that Andre drew.



A. Each triangle has two sides that are 3 units long.

B. Each triangle has a side that is the same length as the diagonal line.

C. Each triangle has one side that is 3 units long.

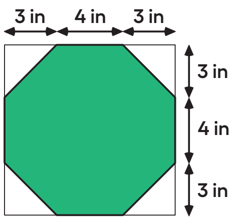
D. When one triangle is placed on top of the other and their sides are aligned, we will see that one triangle is larger than the other.

E. The two triangles have the same area as each other.

Problem 6

from Unit 1, Lesson 3

Here is an octagon. (Note: The diagonal sides of the octagon are *not* 4 inches long.)



a. While estimating the area of the octagon, Lin reasoned that it must be less than 100 square inches. Do you agree? Explain your reasoning.

Yes

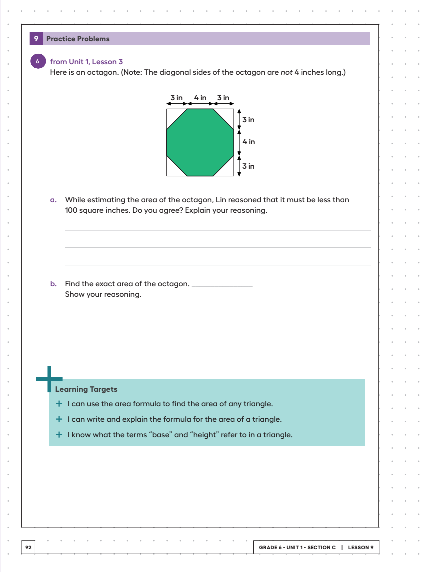
Sample reasoning: The octagon fits in a square that is 10 inches by 10 inches, but with four corners of the square removed. The square has an area of 100 square inches, so the area of the octagon must be less than that.

b. Find the exact area of the octagon. Show your reasoning.

82 square inches

Sample reasoning: A 10-inch-by-10-inch square that encloses the octagon has an area of 100 square inches. Two corner triangles compose a 3 inch-by-3 inch square, so their combined area is 9 square inches. $100 - 2(3 \cdot 3) =$ $100 - 18 = 82$

Student Workbook


Practice Problems

from Unit 1, Lesson 3

Here is an octagon. (Note: The diagonal sides of the octagon are *not* 4 inches long.)

a. While estimating the area of the octagon, Lin reasoned that it must be less than 100 square inches. Do you agree? Explain your reasoning.

b. Find the exact area of the octagon. Show your reasoning.

Learning Targets

+ I can use the area formula to find the area of any triangle.
+ I can write and explain the formula for the area of a triangle.
+ I know what the terms "base" and "height" refer to in a triangle.