Lesson 11

Rectangle Fractions

Goals

- Generalize that decomposing rectangles into squares is a geometric way to determine the greatest common factor of two numbers.
- Interpret and create diagrams involving a rectangle decomposed into squares.

Lesson Narrative

This optional lesson continues and builds on the work of the previous one. Students represent fractions geometrically by decomposing rectangles into squares. The activities in this lesson build on each other, providing students an opportunity to express the relationship between the greatest common factor of two numbers and related fractions through repeated reasoning. It is not necessary to do the entire set of problems to get some benefit from the activities in this lesson, although more connections are made the farther one gets.

Student Learning Goal

Let's compare fractions and rectangles.

Lesson Timeline

20 min	25 min
Activity 1	Activity 2

Access for Students with Diverse Abilities

- Representation (Activity 2)

Required Materials

Materials to Gather

- Graph paper: Activity 1

Activity 1

Finding Equivalent Fractions

20 min

Activity Narrative

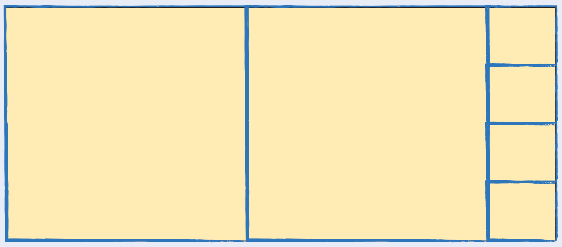
In this activity, students make the connection between the fraction determined by the original rectangle and the resulting more precise fraction. The two rectangles taken together are designed to help students notice that decomposing rectangles is a geometric way to determine the greatest common factor of two numbers. (This is a geometric version of Euclid's algorithm for finding the greatest common factor.) Students reason abstractly and quantitatively when switching between numeric and geometric representations.

Launch

Arrange students in groups of 2. Provide access to graph paper. Students work on problems alone and compare work with a partner.

Student Task Statement

1. Use this rectangle to answer the questions. Suppose this rectangle is 9 units by 4 units.



   a. In the rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until your original rectangle has been entirely decomposed into squares.

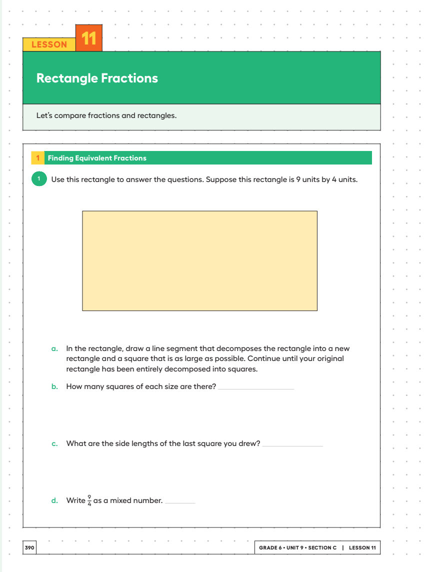
   b. How many squares of each size are there?

   Two 4-by-4 squares and four 1-by-1 squares

   c. What are the side lengths of the last square you drew? 1 by 1

   d. Write $\frac{9}{4}$ as a mixed number. $2\frac{1}{4}$

Student Workbook


LESSON 11

Rectangle Fractions

Let's compare fractions and rectangles.

1 Finding Equivalent Fractions

1 Use this rectangle to answer the questions. Suppose this rectangle is 9 units by 4 units.

a. In the rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until your original rectangle has been entirely decomposed into squares.

b. How many squares of each size are there?

c. What are the side lengths of the last square you drew?

d. Write $\frac{9}{4}$ as a mixed number.

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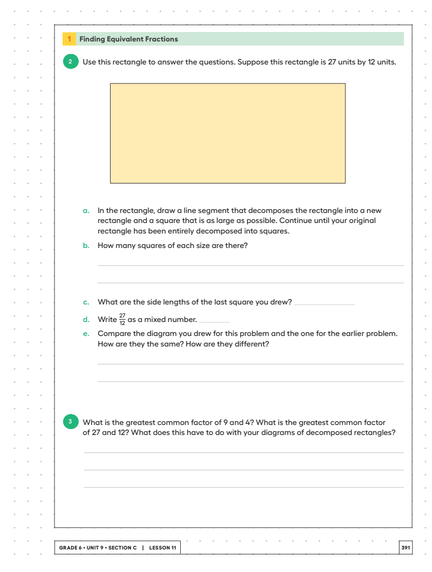
Student Workbook


1 Finding Equivalent Fractions

2 Use this rectangle to answer the questions. Suppose this rectangle is 27 units by 12 units.

a. In the rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until your original rectangle has been entirely decomposed into squares.

b. How many squares of each size are there?

c. What are the side lengths of the last square you drew?

d. Write $\frac{27}{12}$ as a mixed number.

e. Compare the diagram you drew for this problem and the one for the earlier problem. How are they the same? How are they different?

3 What is the greatest common factor of 9 and 4? What is the greatest common factor of 27 and 12? What does this have to do with your diagrams of decomposed rectangles?

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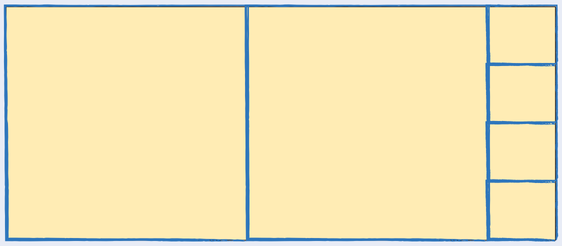
2. Use this rectangle to answer the questions. Suppose this rectangle is 27 units by 12 units.



a. In the rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until your original rectangle has been entirely decomposed into squares.

b. How many squares of each size are there?

Two 12-by-12 squares and four 3-by-3 squares

c. What are the side lengths of the last square you drew? 3 by 3

d. Write $\frac{27}{12}$ as a mixed number. $\frac{27}{12} = 2\frac{3}{12} = 2\frac{1}{4}$

e. Compare the diagram you drew for this problem and the one for the earlier problem. How are they the same? How are they different?

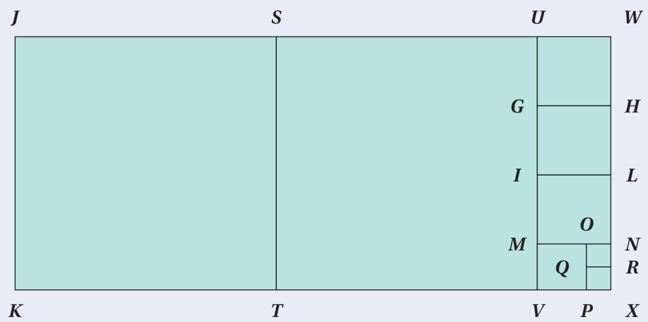
It is similar because there are 2 large squares and 4 small squares. It is different because the squares are not the same size. The larger rectangle split into larger squares is a scaled up version of the smaller rectangle split into smaller squares.

3. What is the greatest common factor of 9 and 4? What is the greatest common factor of 27 and 12? What does this have to do with your diagrams of decomposed rectangles?

The greatest common factor of 9 and 4 is 1. The greatest common factor of 27 and 12 is 3. The greatest common factor is the same as the side length of the smallest square.

Are You Ready for More?

We have seen some examples of rectangle tilings. A *tiling* means a way to completely cover a shape with other shapes, without any gaps or overlaps. For example, here is a tiling of rectangle $KXWJ$ with 2 large squares, 3 medium squares, 1 small square, and 2 tiny squares.



Some of the squares used to tile this rectangle have the same size.

Is it possible to tile a rectangle with squares where the squares are all different sizes? Yes

If you think it is possible, find an example that works. If you think it is not possible, explain why it is not possible.

Sample response: It is possible to tile a rectangle with only differently-sized squares. One example is a 32-by-33 rectangle that can be tiled with squares of side length 18, 15, 14, 10, 9, 8, 7, 4, and 1. (For more examples of solutions and more history on the matter, research Martin Gardner's November 1958 column in Scientific American.)

Activity Synthesis

The purpose of this discussion is to make sure students understand that using a decomposition model can help turn fractions into mixed numbers and find the greatest common factor of the numerator and denominator. It is not necessary for students to understand a general argument for why chopping rectangles can help you know the greatest common factor of two numbers. Here are some questions for discussion:

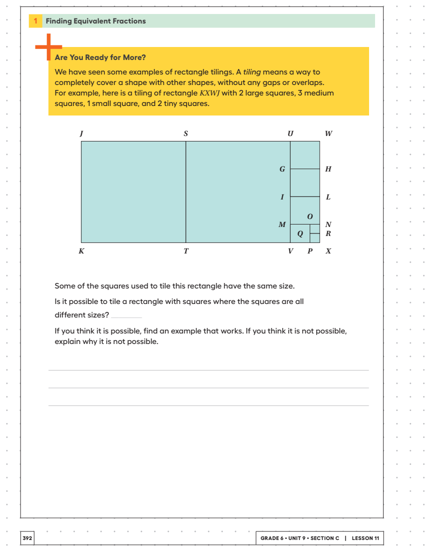
- *"We call fractions like $\frac{9}{4}$ and $\frac{27}{12}$ equivalent fractions. What would be true about the rectangles we could draw about other equivalent fractions?"*

  Equivalent fractions would have rectangles that are decomposed in the same way, but each square would be larger or smaller by the same amount.

- *"Suppose there is a pair of equivalent fractions where the numbers in one of the fractions are 5 times the numbers on the other fractions. If both fractions were represented with rectangles decomposed into squares, how would the side lengths of the squares compare?"*

  The rectangle representing the fraction with the larger numbers would have squares whose side lengths are 5 times those of the smaller rectangle.

Student Workbook


1 Finding Equivalent Fractions

Are You Ready for More?

We have seen some examples of rectangle tilings. A *tiling* means a way to completely cover a shape with other shapes, without any gaps or overlaps. For example, here is a tiling of rectangle $KXWJ$ with 2 large squares, 3 medium squares, 1 small square, and 2 tiny squares.

Some of the squares used to tile this rectangle have the same size.

Is it possible to tile a rectangle with squares where the squares are all different sizes?

If you think it is possible, find an example that works. If you think it is not possible, explain why it is not possible.

Activity 2

It's All About Fractions

25 min

Activity Narrative

This activity extends the work with rectangles and fractions to continued fractions. Continued fractions are not a part of grade-level work, but they can be reasoned about and rewritten using grade-level skills for operating on fractions. This repeated reasoning allows students to generalize the relationship between the geometric representation and the continued fraction. In particular, the insight that $\frac{1}{\frac{a}{b}} = \frac{b}{a}$ (a special case of invert and multiply) is helpful.

In this activity, students consolidate their understanding about how the greatest common factor of the numerator and denominator of a fraction can help them write an equivalent fraction whose numerator and denominator have greatest common factor 1—sometimes called "lowest terms."

Launch

Arrange students in groups of 2.

Begin by displaying these expressions and giving partners 2–3 minutes to write a fraction that is equal to each expression:

1. $3 + \frac{1}{5}$ $\frac{16}{5}$
2. $\frac{1}{3 + \frac{1}{5}}$ $\frac{5}{16}$
3. $2 + \frac{1}{3 + \frac{1}{5}}$ $\frac{37}{16}$

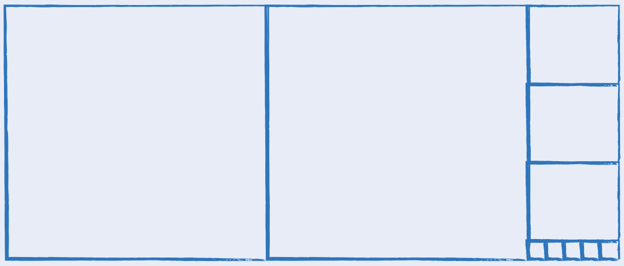
Invite 1–2 students to share their thinking for each expression.

Give students 5–10 minutes to begin work on the activity before pausing to compare their work with their partner and then completing any remaining problems.

Student Task Statement

1. Draw a 37-by-16 rectangle. (Use graph paper, if possible.)

   Sample diagram:



   a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until your original rectangle has been entirely decomposed into squares.

   b. How many squares of each size are there?

   Two 16-by-16 squares, three 5-by-5 squares, five 1-by-1 squares

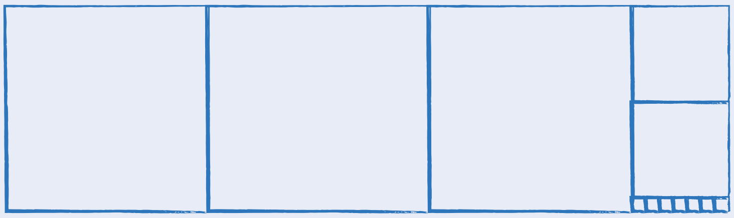
   c. What are the dimensions of the last square you drew? 1

   d. How does your decomposition relate to $2+\frac{1}{3+\frac{1}{5}}$?

   2, 3, and 5 appear

2. Draw a 52-by-15 rectangle.

   Sample diagram:



   a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until your original rectangle has been entirely decomposed into squares.

   b. Write a fraction equal to this expression: $3+\frac{1}{2+\frac{1}{7}}$. $\frac{52}{15}$

   c. What are some connections between the rectangle and the fraction?

   Sample response: When the continued fraction is rewritten, the numerator and denominator equal the side lengths of the rectangle. When the rectangle was partitioned into squares, there were 3, 2, and 7 squares of different sizes. These match the numbers in the continued fraction that was given.

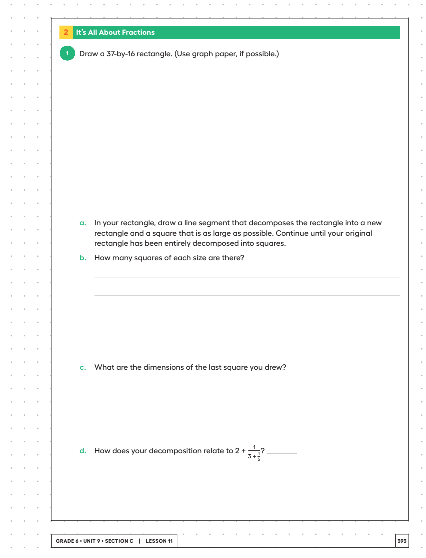
   d. What is the greatest common factor of 52 and 15? 1

Access for Students with Diverse Abilities (Activity 2, Student Task)

Representation: Develop Language and Symbols.
Use virtual or concrete manipulatives to connect symbols to concrete objects or values. For example, provide students with access to the four large rectangles prepared on graph paper.

Supports accessibility for: Visual-Spatial Processing, Conceptual Processing

Student Workbook



2 It's All About Fractions

1 Draw a 37-by-16 rectangle. (Use graph paper, if possible.)

a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until your original rectangle has been entirely decomposed into squares.

b. How many squares of each size are there?

c. What are the dimensions of the last square you drew?

d. How does your decomposition relate to $2+\frac{1}{3+\frac{1}{5}}$?

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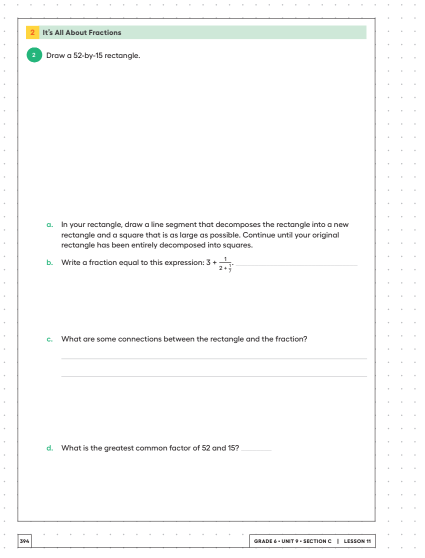
Student Workbook



2 It's All About Fractions

2 Draw a 52-by-15 rectangle.

a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until your original rectangle has been entirely decomposed into squares.

b. Write a fraction equal to this expression: $3+\frac{1}{2+\frac{1}{7}}$

c. What are some connections between the rectangle and the fraction?

d. What is the greatest common factor of 52 and 15?

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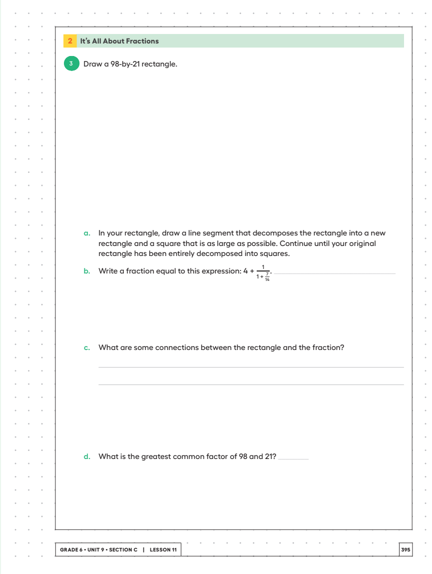
Student Workbook


2 It's All About Fractions

3 Draw a 98-by-21 rectangle.

a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until your original rectangle has been entirely decomposed into squares.

b. Write a fraction equal to this expression: $4+\frac{1}{1+\frac{7}{14}}$.

c. What are some connections between the rectangle and the fraction?

d. What is the greatest common factor of 98 and 21?

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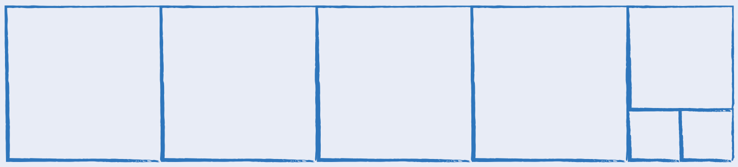
3. Draw a 98-by-21 rectangle.

Sample diagram:



a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until your original rectangle has been entirely decomposed into squares.

b. Write a fraction equal to this expression: $4+\frac{1}{1+\frac{7}{14}}$. $\frac{98}{21}$ or $\frac{14}{3}$

c. What are some connections between the rectangle and the fraction?

Sample response: The side lengths of the rectangle were 98 and 21, and the fraction is $\frac{98}{21}$ (or equivalent). The fraction written with the smallest possible numbers is $\frac{14}{3}$, and both 14 and 3 are multiplied by 7, the result is $\frac{98}{21}$.

d. What is the greatest common factor of 98 and 21?

7

Activity Synthesis

The goal of this discussion is to make sure students understand the connection between decomposition of rectangles, equivalent fractions, and greatest common factor. Discuss this problem with students:

Consider a 121-by-38 rectangle.

1. Use the decomposition-into-squares process to write a continued fraction for $\frac{121}{38}$.

$3+\frac{1}{5+\frac{1}{2+\frac{1}{3}}}$

2. What is the greatest common factor of 121 and 38?

1