Inductive Proof Outline

Theorem 1. *Let S be a set with cardinality $n \geq 1$. S has exactly 2^n unique subsets.*

Proof. Prove that $P(n) = 2^n$, for all $n \geq 1$ holds true.

Base Step: When $n = 1$, a set S has one element $\{a_1\}$. $|P(\{a_1\})| = 2 = 2^1$. This holds as $2^1 \geq 1$.

Inductive Hypothesis: Assume the statement holds for a set S with k elements, that is, S has 2^k subsets for $k \geq 1$.

Inductive Step: Here we will show that it holds for $P(k+1)$. Let S be a set where $|S| = k+1$

$$S = \{a_1, a_2, ..., a_k, a_{k+1}\}$$
$$S = \{a_1, a_2, ..., a_k\} \cup \{a_{k+1}\}$$
$$|P(S)| = 2^k + 2^k$$
$$|P(S)| = 2 * 2^k$$
$$|P(S)| = 2^{k+1}$$

This holds as whenever a new element is added to a set. The unique subsets the set has will double.

Conclusion: By mathematical induction, the statement holds for all $n \geq 1$. Therefore, a set with cardinality n has exactly 2^n unique subsets. □

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Theorem 2. *Let S be a set with cardinality $n \geq 2$. S has exactly $\frac{n(n-1)}{2}$ unique subsets of cardinality 2.*

Proof. We will prove the statement by mathematical induction on n.

Base Step: Let $n = 2$. A set S with two elements, say $S = \{a, b\}$, has exactly one subset of cardinality 2, which is S itself. This is equal to $\frac{2(2-1)}{2} = 1$, which satisfies the base case.

Inductive Hypothesis: Assume that for a set S with k elements, where $k \geq 2$, there are exactly $\frac{2(2-1)}{2}$ subsets of cardinality 2.

Inductive Step: Consider a set S with $k+1$ elements. The original k elements have $\frac{k(k-1)}{2}$ subsets of cardinality 2. When we add the $(k+1)^{th}$ element, we can form new subsets of cardinality 2 by pairing the $(k+1)^{th}$ with each of the k elements of the set. This gives us an additional k subsets.The total number of subsets of cardinality 2 for the set T is $\frac{k(k-1)}{2} + k$. Simplifying this expression gives us $\frac{k(k-1)+2k}{2} = \frac{k^2-k+2k}{2} = \frac{k^2+k}{2} = \frac{k(k+1)}{2}$, which is the formula for $\frac{(k+1)k}{2}$, confirming the inductive step.

Conclusion: By the principle of mathematical induction, the theorem is true for all $n \geq 2$. □

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Theorem 3. *For any n sets $S_1, S_2, \ldots, S_n$, the complement of the union of these sets is equivalent to the intersection of their individual complements.*

$$(S_1 \cup S_2 \cup \ldots \cup S_n)^C = S_1^C \cap S_2^C \cap \ldots \cap S_n^C$$

for all $n \geq 1$.

Proof. We will prove the statement by mathematical induction on n.

Base Step: For $n = 1$, the complement of a single set is itself.

$$S_1^C = S_1^C$$

For $n = 2$, De Morgan's Law

$$(S_1 \cup S_2)^C = S_1^C \cap S_2^C$$

Inductive Hypothesis: Assume that the statement holds for $n = k$, that is

$$(S_1 \cup S_2 \cup \ldots \cup S_k)^C = S_1^C \cap S_2^C \cap \ldots \cap S_k^C.$$

Inductive Step: Consider $n = k+1$ sets. We want to show that

$$(S_1 \cup S_2 \cup \ldots \cup S_k \cup S_{k+1})^C = S_1^C \cap S_2^C \cap \ldots \cap S_k^C \cap S_{k+1}^C.$$

WDe Morgan's Law, we have

$$\begin{aligned}(S_1 \cup S_2 \cup \ldots \cup S_k \cup S_{k+1})^C &= (S_1 \cup S_2 \cup \ldots \cup S_k)^C \cap (S_{k+1})^C \\ &= (S_1^C \cap S_2^C \cap \ldots \cap S_k^C) \cap S_{k+1}^C.\end{aligned}$$

This confirms the inductive step.

Conclusion: By the principle of mathematical induction, the statement is true for all $n \geq 1$. □

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Theorem 4. *Let S be a set with cardinality $n \geq 1$. Then, S has exactly 2^n unique subsets.*

Statement

We want to prove by contradiction that the intersection of any set S_1 with the difference of any set S_2 and S_1 is the empty set, i.e., $S_1 \cap (S_2 \setminus S_1) = \emptyset$.

Proof by Contradiction

Proof. Suppose, for contradiction, that the intersection $S_1 \cap (S_2 \setminus S_1)$ is not empty. Then there exists an element x such that $x \in S_1$ and $x \in (S_2 \setminus S_1)$.

If $x \in (S_2 \setminus S_1)$, by definition of set difference, $x \in S_2$ and $x \notin S_1$. This leads to a contradiction since we have assumed that $x \in S_1$.

Therefore, our initial assumption must be false. $\square$