A comprehensive open-source quasi-static mooring system model in Python

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ABSTRACT

MoorPy is a quasi-static mooring model and a suite of associated functions for mooring system analysis. The core model supports quasi-static analysis of moored floating systems including any arrangement of mooring lines and floating platforms. It solves the distributed position and tension of each mooring line segment using standard catenary equations. Floating platforms can be represented with linear hydrostatic characteristics. MoorPy automatically computes a floating system's equilibrium state and can be queried to identify a mooring system's nonlinear force-displacement relationships. Linearized stiffness matrices are efficiently computed using semi-analytic Jacobians. MoorPy also includes plotting functions and a library of mooring component property and cost coefficients. MoorPy can be used directly from Python scripts to perform mooring design and analysis tasks, or it can be coupled with other tools to compute quasi-static mooring reactions as part of a larger simulation.

mooring system linear model shared mooring fl

1. Draft Standard Nomenclature for NREL Moorings Work

These are a work in progress, trying to aim for standard variables in future work.

τ - tension, or horizontal tension component for linear analysis

$\mathbf{f}$ - vector of forces (or forces and moments) on a point (or body)

$\mathbf{r}$ - vector of positions (or positions and orientations) in the global frame

$\mathbf{r}$ - vector of positions (or positions and orientations) in the global frame

ξ - vector of degrees of freedom (in whatever coordinate systems they have)

m - a point mass

$\mathbf{M}$ - a mass matrix (3 or 6 DOF)

NOTE: the various quasi-static derivations don't need to follow these - it's fine to leave them as they are for simplicity and comparison with previous publications.

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2. Introduction

This paper presents a formulation for robust and efficient quasi-static modeling of mooring systems that is the theory basis for the open-source mooring model MoorPy.

...a new quasi-static mooring system analysis tool with more comprehensive capabilities than previous quasi-static mooring models. It has a general formulation, allowing simulation of any arrangement of mooring lines, including...

Quasi-static mooring models have long been a staple of offshore system design. These models calculate the instantaneous state of a mooring line assuming that it is under static equilibrium and free of hydrodynamic loads. In other words, the forces involved are limited to wet weight and axial stiffness. With these simplifications, each homogeneous mooring line segment will have the shape of a catenary, for which a pair of nonlinear analytic equations exists. This makes it extremely efficient to calculate the mooring line behavior by simply solving the system of equations.

The catenary equations are often extended to include elastic stretching of the mooring line, to account for a portion of the line resting on the seabed, and even to account for static friction of the line on the seabed, which can reduce the line tension as it approaches the anchor point. [cite MIT swim-motion-lines, FASTv7 from Jason's thesis, etc.]

Quasi-static mooring models have also been equipped with capabilities for multisegmented mooring lines–assemblies of more than one mooring lines, potentially with different properties. Al Solihat [cite] developed one such model that specifically supports bridle mooring systems, in which one main mooring line is split into two smaller mooring lines before attaching to a floating platform. Masciola created the Mooring Analysis Program (MAP) [cite], which implements a general multisegmented approach, meaning it supports any arrangement of mooring lines connected together. Quasi-static mooring models are also common in various commercial and in-house offshore strucutures tools. For example, OrcaFlex has an option to simulate mooring line segments with a quasi-static model [cite]. [Are there any other quasi static models worth mentioning?]

In general, existing publicly available quasi-static mooring models have limitations in their usability and capabilities. MAP, which is arguably the most readily available and versatile of these models, has several important limitations. First, its model for individual mooring lines does not support situations where a mooring line with both ends suspended makes contact with the seabed, or when a fully slack mooring line rests on the seabed such that the resting portion has zero tension. These limitations prevent the model from simulating some situations that can occur in the course of a simulation. Second, MAP is not set up to support multiple floating bodies, meaning it cannot simulate mooring behaviors when multiple floating systems are coupled together. Third, it uses numerical, finite-difference derivatives, adding overhead and approximations to its solution process. And fourth, its solver operation is not robust, in that some configurations cannot be solved with the default settings and various adjustments to the solver settings may be needed for the model to run.

In response to these limitations, we created a model that addresses each one of them. The model, called MoorPy, is written in Python for versatility. It is a quasi-static mooring system model that solves the catenary equations for individual mooring line segments and solves for the static equilibrium of entire moored floating systems. Its modeling of individual mooring line segments supports less-common situations such as when lines on the seabed become full slack or rest entirely on the seabed, making it more robust than previous tools. It also has a fully general mooring system formulation such that any number of floating bodies can be specified. Perhaps most notably, MoorPy calculates the system derivatives analytically, making its solution process more efficient and robust.

This paper presents the theory and implementation of these capabilities in MoorPy. Section 3 describes the quasi-static formulation for individual mooring line segments in a range of different situations. Section **??** describes the formulation for connecting line segments with point or rigid body objects to form a complete mooring system, along with how the system properties are computed and how the equilibrium state is solved for. Section **??** presents demonstration results from MoorPy for scenarios of increasing complexity and compares these results against equilibrium solutions from MoorDyn for verification.

3. Quasi-Static Line Segment Model

The first step of a quasi-static model of a mooring system is modeling each individual mooring line segment. Here, mooring line segment refers to a length of mooring line over which the mechanical properties are uniform and there are no applied loads aside from weight, buoyancy, and seabed contact. In other words, there are no attachments or changes in properties over the segment length. Instead, attachments or changes in properties would be handled by connecting multiple segments together, as will be described in Section **??**.

A line segment is given cross sectional properties of wet weight and axial elasticity. Specifically, the wet weight per unit length is $w = (\rho_{line} - \rho_{water})gA$, where ρ is density, g is gravitational constant, and A is the line's cross sectional area. The elasticity is represented by the cross sectional stiffness coefficient EA, where E is the effective elasticity modulus.

To further simplify the modeling, hydrodynamics and seabed friction are neglected, and the seabed is assumed to be flat and horizontal. Including hydrodynamic loads would significantly complicate the modeling and prevent an efficient, generalizable approach. Seabed friction, while often included in quasi-static models, is not discussed in the following formulation because it introduces a causality or path-dependence to the quasi-static solution, which is at odds with generalized approach developed in this paper. A separate model development effort for seabed friction and bathymetry is planned for future work to give these interrelated topics proper treatment.

With these assumptions, the only applied loads along a mooring line segment are weight, buoyancy, and a vertical force from seabed contact. The presence of only vertical loads along the segment combined with the definition of uniform material properties allows for a quasi-static model to be built up based on several key physical principles:

- The line segment profile falls along a vertical plane and can therefore be analyzed in only two dimensions.
- Because only vertical loads are applied along the segment length, the horizontal component of tension along the segment will be uniform.
- Because the horizontal tension component is uniform, the local incline angle and the total tension magnitude are interrelated at any point on the line by $T = T_h / \cos\varphi$.
- For suspended portions of the line segment, the change in vertical tension component is equal to the wet weight of the line segment – per unit length.

These principles result in the characteristic catenary curve that describes the profile of a suspended cable hanging under its own weight. To illustrate, Figure 1 shows a small element of length ds along a free-hanging catenary mooring line. The tension at one end of the element is denoted T, resulting in an axial strain of T/EA. The element has weight wds, which must be equal to the difference in the vertical component of tension at each end of the element. Because the only force in the horizontal direction is tension, the horizontal tension component does not vary over the element length.

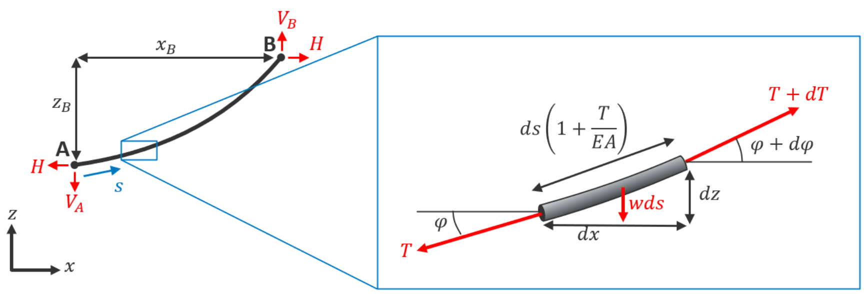

Figure 1: Differential portion of a mooring line segment

Solving the differential relations illustrated in Figure 1 over the length of a suspended line (see [refs]) gives rise to the well-known catenary equations, which describe the profile of a suspended cable hanging under its own weight. The following equations describe the horizontal (x) and vertical (z) coordinates along the profile as a function of the unstretched arc length along the line, s:

$$x(s) = \frac{H}{w}\left\{\ln\left[\frac{V_A + ws}{H} + \sqrt{1 + \left(\frac{V_A + ws}{H}\right)^2}\right] - \ln\left[\frac{V_A}{H} + \sqrt{1 + \left(\frac{V_A}{H}\right)^2}\right]\right\} + \frac{Hs}{EA} \tag{1}$$

$$z(s) = \frac{H}{w}\left[\sqrt{1 + \left(\frac{V_A + ws}{H}\right)^2} - \sqrt{1 + \left(\frac{V_A}{H}\right)^2}\right] + \frac{1}{EA}\left(V_A s + \frac{ws^2}{2}\right) \tag{2}$$

where H is the horizontal component of tension in the line, V_A is the vertical component of tension at end A of the line, and s is the location along the line measured in terms of the unstretched length from end A. The first term in each equation comes from the effect of the weight of the line and the second term in each equation is due to the elasticity of the line.

From a balance of forces in the vertical direction, the vertical component of tension is

$$V(s) = V_A + ws \tag{3}$$

and the tension magnitude along the line is

$$T(s) = \sqrt{H_F^2 + (V_A + ws)^2} \tag{4}$$

When applying equations (1-4) to model the shape and tension of a mooring line segment, the end positions of the segment are usually known while the tensions H and V_A are typically unknown. Equations 1 and **??** need to be solved numerically to determine these unknowns. This can be done by setting s to the total unstretched segment length, L, which results in (1) and (**??**) being equal to the horizontal and vertical spacing of the segment end points, $x_B = x(L)$ and $z_B = z(L)$, respectively. Jonkman [] describes how the equations can then be numerically solved with an iterative Newton-Raphson method to find the tension components, after which the full description of line profile and tensions can be calculated directly from (1-4).

The following subsections detail the quasi-static modeling of a mooring line segment in several distinct cases depending on the degree of seabed contact and tension in the line. These cases apply the catenary approach overviewed above along with simpler models for when lines are slack or lying along the seabed to fully describe the mooring segment profile and tension distribution across the range of likely scenarios that could be encountered.

H_F and V_F are the horizontal and vertical forces at the fairlead, which is when $s = L$. Also, it can be shown using a balance of forces:

3.1. Fully suspended line segment

The most fundamental case is a mooring line segment that does not contact the seabed at any part of its length. As such, the segment has a catenary shape over its entire length and (1-4) apply to its entirety. Figure 2 shows three supported scenarios in this case.

Substituting $s = L$ and $V_B = V_A + wL$ into (1) and (**??**) gives the equations for catenary horizontal and vertical dimensions as a function of horizontal tension, H, and vertical tension at end B, V_B:

$$x_B(H, V_B) = \frac{H}{w}\left\{\ln\left[\frac{V_B}{H} + \sqrt{1 + \left(\frac{V_B}{H}\right)^2}\right] - \ln\left[\frac{V_B - wL}{H} + \sqrt{1 + \left(\frac{V_B - wL}{H}\right)^2}\right]\right\} + \frac{HL}{EA} \tag{5}$$

$$z_B(H, V_B) = \frac{H}{w}\left[\sqrt{1 + \left(\frac{V_B}{H}\right)^2} - \sqrt{1 + \left(\frac{V_B - wL}{H}\right)^2}\right] + \frac{1}{EA}\left(V_B L - \frac{wL^2}{2}\right) \tag{6}$$

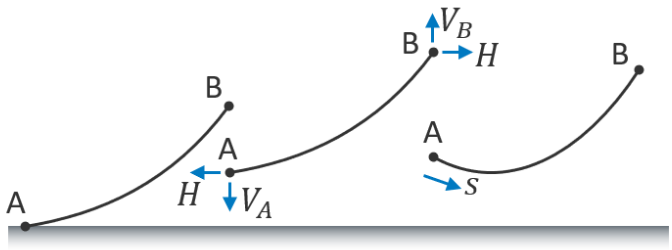

Figure 2: Fully suspended segment profiles

where x_B and z_B represent the horizontal and vertical position of end B relative to end A.

Using these equations and their partial derivatives, the tension components at end B can solved...

3.2. Line fully on the seabed

A very simple case is when a mooring line segment lies entirely along the seabed. With friction on the seabed neglected, tension through the line segment is uniform:

$$T(s) = \text{Maximum}\left(0, \left(\frac{X}{L} - 1\right) EA\right) \tag{7}$$

The segment profile's z coordinates are uniform at the seabed depth and the x coordinates are uniformly stretched:

$$x(s) = \frac{X_F}{L} s \tag{8}$$

As a result, the end tensions are $V_B = 0$ and $H = T$, for which no iterative solution is required.

3.3. Portion of the line rests on the seabed

A common scenario is for a mooring line segment to lie partly along the seabed (Figure 3). This case combines the two previous cases using a piecewise approach. The portion of the line resting on the seabed is denoted L_b, and the suspended portion is denoted L_s. These unstretched lengths sum to give the total unstretched line length, L.

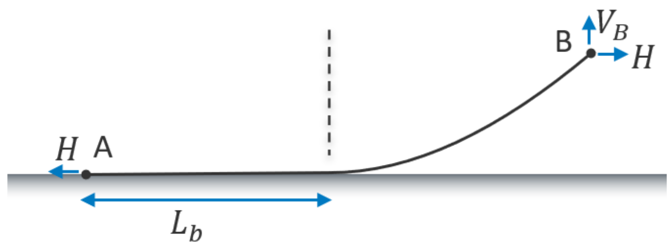

Figure 3: Segment with seabed contact profile

As with the fully suspended case, determining the line profile requires solving nonlinear profile shape equations for the end tension components. Because the end of the line that rests on the seabed will have no vertical tensions, the length of line not in contact with the seabed can expressed from (3) as

$$L_s = \frac{V_B}{w}. \tag{9}$$

If the vertical force at the fairlead happens to be greater than the total weight of the line ($L_s > L$ or $V_B > Lw$), then the line will be fully suspended and have no contact with the seabed–this case was discussed in Section 3.1. Otherwise (when $V_B < wL$), the vertical force at end B is less than the weight of the line, meaning that a length of line will rest on the seabed:

$$L_b = L - L_S = L - \frac{V_B}{w} \tag{10}$$

As long as this resting length of line is less than the horizontal dimension of the profile ($L_b < x_B$), the line will have some horizontal tension and can be solved with the catenary equations. Otherwise, the next case in Section 3.4 applies.

The equations governing the profile and tension distribution of a line that is partially on the seabed consist of a combination of those for a suspended line and a line on the seabed. For the portion on the seabed, the equations are identical to those in Section 3.2. For the suspended portion, as long as there is horizontal tension in the line, the equations are based on those of Section 3.1 with $V_A = 0$ and s replaced with $s - L_B$ to account for the portion of line on the seabed. These substitutions result in the following equations for the horizontal position, vertical position, and tension along the line:

$$x(s) = \begin{cases} s\left(1 + \frac{H}{EA}\right) & 0 \le s \le L_b \\ L_b + \frac{H}{w}\ln\left[\frac{w(s-L_b)}{H} + \sqrt{1 + \left(\frac{w(s-L_b)}{H}\right)^2}\right] + \frac{Hs}{EA} & L_b \le s \le L \end{cases} \tag{11}$$

$$z(s) = \begin{cases} 0 & 0 \le s \le L_b \\ \frac{H}{w}\ln\left[\sqrt{1 + \left(\frac{w(s-L_b)}{H}\right)^2} - 1\right] + \frac{w(s-L_b)^2}{2EA} & L_b \le s \le L \end{cases} \tag{12}$$

The tensions along the line are:

$$T(s) = \begin{cases} \mathbb{M}(H & 0 \le s \le L_b \\ \sqrt{H^2 + (w(s-L_b))^2} & L_b \le s \le L \end{cases} \tag{13}$$

Similarly to Section 3.1, the tension components H and V_F can found by a numerical solution of the equations for the known end coordinates (x_B and z_B). These equations are obtained by substituting L in for s in (**??**) and (**??**):

$$x_B(H, V_B) = L_b + \frac{H}{w}\ln\left[\frac{V_B}{H} + \sqrt{1 + \left(\frac{V_B}{H}\right)^2}\right] + \frac{HL}{EA} \tag{14}$$

$$z_B(H, V_B) = \frac{H}{w}\left[\sqrt{1 + \left(\frac{V_B}{H}\right)^2} - 1\right] + \frac{V^2}{2EAw} \tag{15}$$

...derivatives?

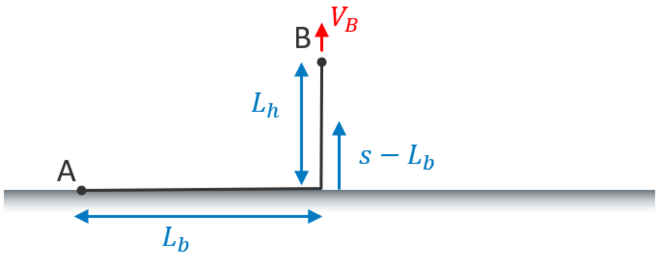

Figure 4: Segment with seabed contact profile

3.4. Fully slack line

If the line segment from the previous case is so slack that the length lying on the seabed equals of exceeds the horizontal spacing of the endpoints ($L_b >= x_B$), then there will be no horizontal tension in the line. In this case, the suspended portion of the line will hang vertically from end B (Figure 4) and the catenary equations do not apply.

The tension where the vertically hanging portion contacts the seabed will be zero, and the tension will increase up the length of the suspended portion according to

$$T(s) = w(s - L_b). \tag{16}$$

The strain along the vertical line can then be expressed as

$$\frac{dz}{ds} = 1 + \frac{T(s)}{EA} = 1 + \frac{w(s - L_b)}{EA}, \tag{17}$$

which can be integrated to give the z coordinate of a given point along the unstretched length of the suspended portion:

$$z(s) = (s - L_b) + \frac{w(s - L_b)^2}{2EA}, \; s \geq L_b. \tag{18}$$

Setting $s = L$ gives an equality that relates the height of end B off the seabed, z_B, to the suspended length, $L_h = L - L_b$. Solving using quadratic formula gives the following equation for the suspended length of the line:

$$L_H = L - L_b = \frac{EA}{w}\left(\sqrt{1 + \frac{2wz_B}{EA}} - 1\right) \tag{19}$$

In a typical application, this equation can be used immediately to determine how much of the line is suspended versus laying on the seabed, at which point (18) and (16) can be used determine the positions and tensions of the suspended portion. The tension at end B is

$$V_B = wL_h = EA\left(\sqrt{1 + \frac{2wz_B}{EA}} - 1\right) \tag{20}$$

No iterative solution process is required in this case. Along the horizontal portion of the line that sits on the seabed, tensions are zero and node positions are arbitrarily assumed to be evenly spaced.

3.5. Vertical line segment

If there is no horizontal separation between the segment end points, then the catenary equations do not apply and instead the segment behavior needs to be modeled purely in the vertical direction. A key calculation for this case is the stretched length of an elastic line segment hanging freely from one end. Similar to (18),this stretched length, or height, of a line segment of length L_i is

$$h_i = L_i + \frac{1}{2EA}L_i^2. \tag{21}$$

A vertical line segments attached at both ends can be either taut or slack. Applying (21) to the full unstretched length of the segment gives h_L, a measure of the vertical extent over the segment if it was right on the margin between taut and slack, with zero tension at the lower end. If this distance is greater than the actual distance between end points ($h_L > z_B$), then the line is slack; otherwise, it is taut.

If the segment is slack, it will have two portions hanging vertically from the respective end points, which come together at a low point of zero tension. Labelling these two portions A and B, by applying (21) to each of them and then considering that $L_A + L_B = L$ and $h_B - h_A = z_F$, one of the side lengths can be solved for as follows:

$$L_B = \frac{z_F + L + \frac{WL^2}{2EA}}{2 + \frac{WL}{EA}} \tag{22}$$

Doing similarly for the other side, the tension on each end point can then be calculated from $V_A = wL_A$ and $V_B = wL_B$. Equations similar to () and () can be applied to compute the vertical location and tension at any point along the segment's unstretched length.

If the segment is taut ($h_L < z_B$), then a uniform additional strain term can be calculated that represents how much the segment is stretched beyond a free-hanging state due to the end point locations:

$$\epsilon_u = \frac{L_B - h_L}{L} \tag{23}$$

This strain is superimposed with the strain implied by (21) to give the true total strain and tension throughout the line:

$$T(s) = \epsilon_u EA + ws \tag{24}$$

3.6. Line segment suspended at both ends with seabed contact

A special case arises if a line segment running between two points that are off the seabed hangs in such a way that it contacts the seabed. This situation does not lend itself to the direct or iterative solving techniques already discussed because it will have two catenary portions–one at each end. Determining the suspended lengths of these catenaries and the forces at the line ends is more difficult than previous cases because the system of equations is larger. This scenario is solved using an iterative approach that performs two separate catenary solutions–one for each end of the line.

To begin with, the special case is detected when the lowest point of a suspended line is found to be below the seabed. The lowest point is located where the vertical component of the line tension crosses zero, $s_{low} = L - V_B/w$, which is substituted into the equation for z coordinate (6) to get:

$$z_{low} = z\left(s = L - \frac{V_B}{w}\right) = \left[1 - \sqrt{1 + \frac{(V_B - wL)^2}{H^2}}\right]\frac{H}{w} - \frac{(V_B - wL)^2}{2EAw} \tag{25}$$

When $z_{low} < -hAfromBot$, some portion of the line must lie along the seabed. The segment is divided into segments at this hypothetical low point, s_{low}, and each portion undergoes its own catenary solve following the approach described in Section 3.3. However, an outer-loop iteration is applied to couple the two line portions together. In order for the two line portion solutions to represent the full U-shaped line profile, the horizontal tension component in each portion must be equal. The outer-loop iteration achieves this by adjusting the connection point location laterally on the seabed. A Newton-Raphson iteration method is used to make the adjustments until the line tensions are equal. Figure 5 illustrates the process.

If the line is fully slack ($H = 0$), then the above process is not required and instead the equations of Section 3.4 can be applied to both hanging ends of the line.

In either case, the solution of line profile coordinates and tensions from each side of the U-shaped line is combined, with appropriate coordinate transformations, to give the final coordinate and tension results for the U-shaped line.

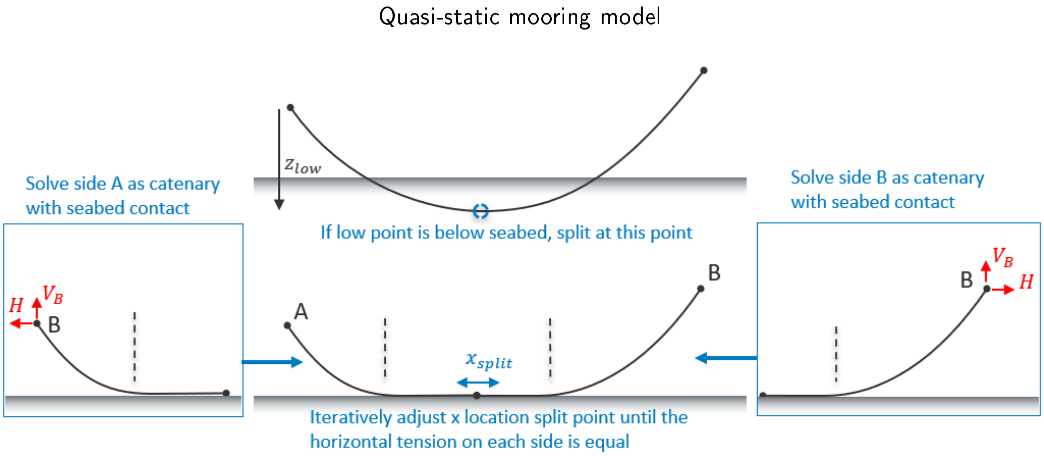

Figure 5: Segment with seabed contact profile

3.7. Variations to Cover All Cases

The above methods cover the range of anticipated scenarios for a uniform mooring line segment with axial elasticity over an even seabed with no friction or external applied loads. However, they assume that the segment is negatively buoyant and its end B is higher than end A. In the case of positively buoyant lines, the fully suspended line calculations of Section 3.1 are applied with the buoyancy reversed. Then, the results in terms of vertical forces and z coordinates are flipped.

In practice, any mooring line segment connecting two points in three dimensions can be looked at from a two-dimensional view that labels the higher end as end B and the lower end as end A. The specifics of going between two-dimensional and three-dimensional views are discussed in the next section.

4. Mooring System Assembly and Solution

A general quasi-static mooring system model requires the ability to connect individual mooring line segments together in various arrangements to represent different types of mooring systems. MoorPy does this in a general way by using three distinct object types–Lines, Points, and Bodies–which are also used in MoorDyn. (Capitalization is used to indicate these specific objects.)

A Line describes a single mooring line segment with uniform distributed properties.

A Point is an entity with three translational degrees of freedom (DOFs) that can have weight and buoyancy properties and that serves as the attachment mechanism at the end of Lines. Any number of line ends can be attached to a point, but every line end must be assigned to a point.

A Body is a representation of a rigid body that can both translate and rotate. It can have weight, buoyancy, and hydrostatic properties such that it can represent a floating platform. Points can be attached to a body, which then allows any lines attached to those points to impart both forces and moments on the body's six degrees of freedom (DOFs).

Following the same convention as used in MoorDyn, the Bodies and Points in MoorPy can be fixed, coupled, or free. Fixed objects cannot move. Coupled objects are expected to be manipulated by an external program. Free objects have degrees of freedom and their equilibrium positions are solved within the model.

MoorPy uses hierarchy in its objects when assigning positions or computing mooring system reaction forces. Bodies are at the highest level, followed by points and then lines. Positions are assigned top-down–first the body positions are set, which then determines the positions of any attached Points, which then determines the positions of any attached Line ends. Points not attached to a Body have their positions set separately. Once all point positions are known, these define the Line end coordinates so the Line profiles and forces can be computed. Then, the forces are assigned bottom-up–the Line end forces are summed to compute the net forces on the Points, and then any attached Point forces are summed to compute the total force on each Body.

This process is visualized in FIGURE...

[add figure of heirarchy]

For greater efficiency, MoorPy features a semi-analytic approach to calculating the system stiffness matrix. This avoids the computational expense of finite-difference calculations and allows for more reliable solving of system equilibrium states. The semi-analytic stiffness computation builds on the two-dimensional stiffness matrices that are computed from each Line.

4.1. Line forces and stiffnesses

In MoorPy, a Line object represents a single line segment in three dimensions. For given end positions, the lines profile and end forces are solved using the approach discussed in Section 3. Rotational transformations are used to go between a three-dimensional representation of the line segment in the context of a larger mooring system and the two-dimensional representation in which the profile and tensions are solved (Figure 6).

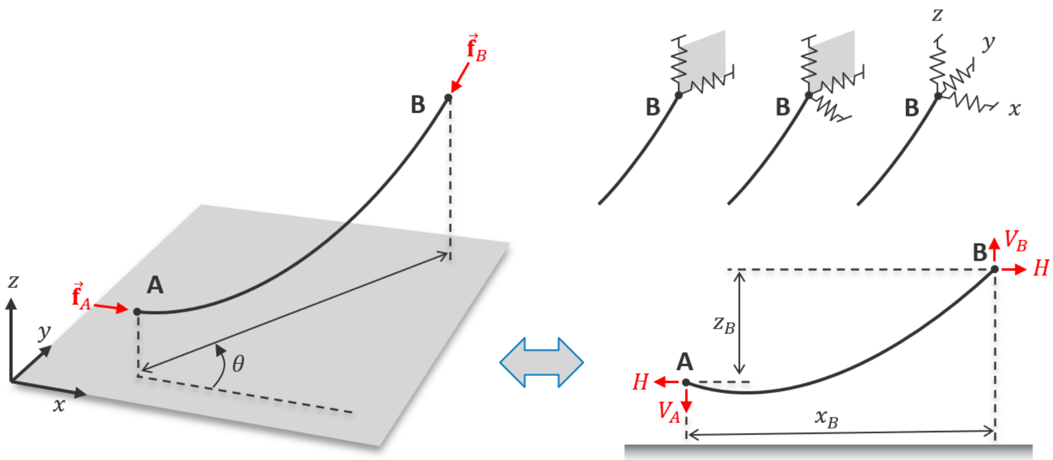

Figure 6: Transforming between a three-dimensional and two-dimensional representation of a Line, and three stages of end stiffness matrices.

We describe a mooring line's heading, θ, as the direction from its end A to end B as measured as a rotation about the z axis relative to the direction of the x axis. The heading rotation matrix is then the direction cosines matrix that would rotated a line's orientation from along the x axis to along its true heading, θ:

$$\mathbf{R} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{26}$$

Based on a Line's end coordinates, r_A and r_B, the line segment horizontal and vertical distances are

$$x_B = \sqrt{(r_{xB} - r_{xA})^2 + ((r_{yB} - r_{yA})^2} \tag{27}$$

$$z_B = r_{zB} - r_{zA} \tag{28}$$

Solved line profiles in two dimensions are converted back to three dimensions with

$$\mathbf{r}(s) = \begin{bmatrix} x(s)\cos(\theta) + r_{xA} \\ x(s)\sin(\theta) + r_{yA} \\ z(s) + r_{zA} \end{bmatrix} \tag{29}$$

Similarly, the tension force vectors at the line ends are

$$\mathbf{F}_A = \begin{bmatrix} H\cos(\theta) \\ H\sin(\theta) \\ V_A \end{bmatrix} \tag{30}$$

$$\mathbf{F}_B = \begin{bmatrix} -H\cos(\theta) \\ -H\sin(\theta) \\ -V_B \end{bmatrix} \tag{31}$$

Solving for the equilibrium and restoring characteristics of a mooring system requires knowing the stiffness matrix of the Line in terms of the reaction forces to displacements of the end points. This matrix can be generated with a finite difference approach by perturbing each line end coordinate and measuring how the reaction forces change. However, a more efficient analytic approach is possible based on the jacobians used in the line segment solution. Each of the line profiles described in Section 3 has expressions for tension components as a function of end coordinates, or vice versa. These can be used to calculate two-dimensional stiffness matrices for each end of the line. For an end of a line, this planar stiffness matrix has the form

$$\mathbf{K2}_A = \mathbf{J}_A^{-1} = \begin{bmatrix} \frac{\partial H}{\partial x_A} & \frac{\partial H}{\partial z_A} \\ \frac{\partial V_A}{\partial x_A} & \frac{\partial V_A}{\partial z_A} \end{bmatrix} \tag{32}$$

$$\mathbf{K2}_B = \mathbf{J}_B^{-1} = \begin{bmatrix} \frac{\partial H}{\partial x_B} & \frac{\partial H}{\partial z_B} \\ \frac{\partial V_B}{\partial x_B} & \frac{\partial V_b}{\partial z_B} \end{bmatrix} \tag{33}$$

This matrix only considers motion in the plane, when in fact a mooring line will also apply a reaction force opposing motions of an end point that are normal to the plane. As discussed in **?**, the out-of-plane stiffness term is equal to the ratio of line tension to horizontal length: $K_t = \frac{T}{L}$, where T is the horizontal component of line tension and l is the horizontal spacing between the line end points. The other terms coupled with that degree of freedom are assumed zero in a linear analysis, as shown. The 3x3 matrix for a mooring line end with the addition of the "out-of-plane" stiffness term is

$$\mathbf{K3}' = \begin{bmatrix} \frac{dF_L}{dL} & 0 & \frac{dF_L}{dz} \\ 0 & \frac{T}{x_b**} & 0 \\ \frac{dF_z}{dL} & 0 & \frac{dF_z}{dz} \end{bmatrix} \tag{34}$$

The next step is to transform this matrix, which relative to the line's local heading, into the global coordinates of the mooring system using the heading rotation matrix:

$$[K] = [R][K3'][R]^T \tag{35}$$

This provides the stiffness matrix in global coordinates for the equation $[F] = [K][X]$.

The approach of (34) and (35) is used to make three stiffness matrices for every line: for end A, end B, and the coupling between ends A and B:

$$\mathbf{K}_A = \frac{\partial \mathbf{f}_A}{\partial \mathbf{r}_A}\mathbf{K}_B = \frac{\partial \mathbf{f}_B}{\partial \mathbf{r}_B}\mathbf{K}_A B = \frac{\partial \mathbf{f}_B}{\partial \mathbf{r}_A} \tag{36}$$

When a line is fully suspended, $\mathbf{K}_A = \mathbf{K}_B$ and $\mathbf{K}_{AB} = -\mathbf{K}_A$. However, when a line contacts the seabed the matrices differ because some of the reaction force is through a change in how much line weight is supported by the seabed. These differences are captured in the Jacobians for each segment case.

4.2. Point forces and stiffness

Point objects primarily serve to represent the end points for Line objects, allowing line ends to be fixed, moved, or connected together. Points can also have their own mass and volume properties, so that they can add weight or buoyancy forces to the mooring system. The forces on a Point are then the sum of its own weight and buoyancy plus the summed tension vectors from any attached line ends, plus any specified external applied force ($\mathbf{f}_{ext}$):

$$\mathbf{f}_{point} = \begin{bmatrix} 0 \\ 0 \\ (\rho v - m)g \end{bmatrix} + \sum \mathbf{f}_{lineend} + \mathbf{f}_{ext} \tag{37}$$

The total stiffness from mooring lines attached to a 3-DOF point is simply the sum of the individual stiffness matrices of each attached line end.

4.3. Body forces and stiffness

Body objects receive forces and moments from any attached Points, and from any line ends attached to those points.

Each point attached to the body will apply a moment of

$$\mathbf{M} = \mathbf{r} \times \mathbf{f}, \tag{38}$$

where $\mathbf{r}$ is the position vector from the body reference point to the Point, and $\mathbf{f}$ is the total force on that point.

Bodies also have mass, buoyancy, and hydrostatic properties defined by a mass, m, centered at position $\mathbf{r}_{CG}$ in the body reference frame, a volume v assumed to act at a metacentre, $\mathbf{r}_M$, and a waterplane area A. Combining these factors result in the following equation for the total force and moment vector acting on a body:

$$\mathbf{f}_{body} = \left\{ \begin{array}{c} \begin{Bmatrix} 0 \\ 0 \\ -mg \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \rho g v \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \rho g A_{WP} r_z \end{Bmatrix} \\ (\mathbf{R}\mathbf{r}_{CG}) \times \begin{Bmatrix} 0 \\ 0 \\ -mg \end{Bmatrix} + (\mathbf{R}\mathbf{r}_M) \times \begin{Bmatrix} 0 \\ 0 \\ \rho g v \end{Bmatrix} \end{array} \right\} + \sum \begin{Bmatrix} \mathbf{f}_{point} \\ (\mathbf{R}\mathbf{r}_{point}) \times \mathbf{f}_{point} \end{Bmatrix} + \mathbf{f}_{ext} \tag{39}$$

The 6-by-6 stiffness matrix on a 6-DOF body can be represented by four 3-by-3 submatrices.

$$\mathbf{K} = \begin{bmatrix} -\frac{\partial \mathbf{F}}{\partial \mathbf{x}} & -\frac{\partial \mathbf{F}}{\partial \theta} \\ -\frac{\partial \mathbf{M}}{\partial \mathbf{x}} & -\frac{\partial \mathbf{M}}{\partial \theta} \end{bmatrix} \tag{40}$$

This matrix is the summation of equivalent-format matrices that describe the stiffness contribution of each individual mooring line on the body. The following explanation concerns one of the matrices for an individual line.

The upper-left submatrix is the same as the stiffness matrix of the attached line end (35) because it is the Jacobian of line force on the body with respect to attachment point translation:

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = -\mathbf{K}_{line}. \tag{41}$$

The bottom-right sub-matrix relates body rotations to resulting moments. It is the most involved to calculate. Recalling that (38) describes the moment on a body due to a line, the rotational stiffness matrix is the Jacobian of the moments with respect to the rotation angles,

$$\frac{\partial \mathbf{M}}{\partial \theta} = \frac{\partial}{\partial \theta}(\mathbf{r} \times \mathbf{f}) \tag{42}$$

Considering only one rotational direction, θ_i, at a time allows the use of cross products when applying the product rule to the cross product:

$$\frac{\partial \mathbf{M}}{\partial \theta_i} = \frac{\partial}{\partial \theta_i}(\mathbf{r} \times \mathbf{f}) = \frac{\partial \mathbf{r}}{\partial \theta_i} \times \mathbf{f} + \mathbf{r} \times \frac{\partial \mathbf{f}}{\partial \theta_i}. \tag{43}$$

This shows how the stiffness is composed of two terms: one related to the change in moment arm of the line tension force, and one related to the change in line tension force.

Calculating these terms relies on producing the alternator matrix for a given vector, which is defined as

$$\mathbf{H}(\mathbf{r}) = \begin{bmatrix} 0 & r_3 & -r_2 \\ -r_3 & 0 & r_1 \\ r_2 & -r_1 & 0 \end{bmatrix} \tag{44}$$

The alternator matrix has two roles. First, it provides the partial derivative of the position of a point attached to a body with respect to rotation angles of the body:

$$\frac{\partial \mathbf{r}}{\partial \theta} = \mathbf{H}(\mathbf{r}). \tag{45}$$

Second, it provides a matrix that can be multiplied by a vector to perform the same operation as the cross product, for example:

$$\mathbf{r} \times \mathbf{f} = -\mathbf{H}(\mathbf{r})\mathbf{f}. \tag{46}$$

When considering all rotation directions, the first term in (43) is equivalent to

$$\mathbf{H}(-\mathbf{H}(\mathbf{r}))\mathbf{f} = \mathbf{H}(\mathbf{f})\mathbf{H}(\mathbf{r}) = -\mathbf{H}(\mathbf{f})\mathbf{H}(\mathbf{r})^T \tag{47}$$

The derivation for this involves third-order tensors and is shown in Appendix [XXX].

The second term is (change in tension due to rotation):

$$\mathbf{r} \times \frac{\partial \mathbf{f}}{\partial \theta} = \mathbf{r} \times \frac{\partial \mathbf{f}}{\partial \mathbf{r}} \frac{\partial \mathbf{r}}{\partial \theta} \tag{48}$$

$$= \mathbf{r} \times [-K_3][H(r)] \tag{49}$$

$$= -[H(r)][-K_3][H(r)] \tag{50}$$

$$= -[H(r)][K_3][H(r)]^{\intercal} \tag{51}$$

Combining, this amounts to

$$\frac{\partial \mathbf{M}}{\partial \theta} = -\mathbf{H}(\mathbf{f})\mathbf{H}(\mathbf{r})^T - \mathbf{H}(\mathbf{r})\mathbf{K}\mathbf{H}^T(\mathbf{r}) \tag{52}$$

The upper-right sub-matrix represents the change in mooring line force with respect to body rotation.

$$\frac{\partial \mathbf{F}}{\partial \theta} = \frac{\partial \mathbf{F}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \theta} = -\mathbf{K}\mathbf{H}(\mathbf{r}) \tag{53}$$

The bottom-left sub-matrix represents how the moment from the mooring line changes with body translation.

$$\frac{\partial \mathbf{M}}{\partial \mathbf{x}} = \mathbf{r} \times \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \mathbf{H}(\mathbf{r})\mathbf{K} = -\mathbf{H}^T(\mathbf{r})\mathbf{K} \tag{54}$$

The end result is

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{line} & \mathbf{K}\mathbf{H}(\mathbf{r}) \\ \mathbf{H}(\mathbf{r})\mathbf{K} & +\mathbf{H}(\mathbf{f})\mathbf{H}(\mathbf{r})^T + \mathbf{H}(\mathbf{r})\mathbf{K}\mathbf{H}(\mathbf{r})^T. \end{bmatrix} \tag{55}$$

Because $\mathbf{K}_{line}$ is symmetric and $\mathbf{H}$ is anti-symmetric, all of the matrix except for the bottom-right quadrant must be symmetric.

4.4. Cross coupling stiffness terms

Forming a full mooring system stiffness matrix for a system that has more than one free object requires combining the individual point or body stiffness matrices with cross-coupling terms that represent how displacements of one object will generate forces on another object. These cross-coupling terms only need to be calculated when a mooring line goes from one free object to another free object.

As mentioned earlier, a cross-coupling stiffness matrix, $\mathbf{K}_{AB}$, is defined to represent the reaction forces on one end of the line due to motions of the other end. Effects from seabed contact will be represented by that matrix, so it can be used to keep simple expressions for the cross-coupling effects between objects.

The cross-coupling stiffness between two points connected by a line is simply $\mathbf{K}_{AB}$, and it is the same for either direction.

A fully suspended mooring line attached to two bodies creates a 6-by-6 cross-coupling matrix between the bodies:

$$\mathbf{K}_{1,2} = \begin{bmatrix} \mathbf{K}_{AB} & \mathbf{K}_{AB}\mathbf{H}(\mathbf{r}_1) \\ \mathbf{H}(\mathbf{r}_2)^T\mathbf{K}_{AB} & \mathbf{H}(\mathbf{r}_2)\mathbf{K}_{AB}\mathbf{H}(\mathbf{r}_1)^T \end{bmatrix} \tag{56}$$

This is similar to (55) except that it omits the term for rotation causing change in moment arm because in nearly all cases the rotation of one body will cause negligible change in the moment arm of the other body.

A fully suspended mooring line going from a body to a point creates a 6-by-3 cross-coupling matrix:

$$\mathbf{K}_{1,2} = \begin{bmatrix} \mathbf{K}_{AB} \\ \mathbf{H}(\mathbf{r}_1)^T\mathbf{K}_{AB} \end{bmatrix} \tag{57}$$

In the above, the stiffness is considered to be the response at end B to motion at end A of the line. In the other direction, the matrix $\mathbf{K}_{AB}$ must be transposed.

4.5. System Equilibrium Solution

The full system stiffness matrix is generated by adding the above stiffness components in the corresponding locations in the matrix. Each of the cross-coupling matrices described above will also be added in transposed form in the symmetrical position in the system matrix.

[provide illustrations/example for stiffness components for a simple system...]

Aside from the individual line segment computations discussed in Section 3, the main challenge in creating a quasi-static mooring system model is solving for the system's equilibrium state. If $\mathbf{r}$ is the vector of system DOFs and $\mathbf{F}$ is the vector of net forces in each DOF, then the solving for equilibrium means determines the vector $\mathbf{r}$ that will achieve

$$\mathbf{F}(\mathbf{r}) = \mathbf{0} \tag{58}$$

Because the individual mooring line force-displacement relations are nonlinear, determining what values of $\mathbf{r}$ satisfies (58) requires an iterative solution process over top of the individual mooring lines' iterative solutions. MoorPy, like other models (cite MAP and Al Solihat), uses a Netwon-Raphson approach. A given iteration i is as follows:

$$\mathbf{r}_{n+1} = \mathbf{r}_n - \mathbf{K}(\mathbf{r}_n)^{-1}\mathbf{F}(\mathbf{r}_n) \tag{59}$$

where $\mathbf{K}$ is the system stiffness matrix, or the partial derivatives of system forces with respect to displacements about the current position.

The simplest way to calculate the system stiffness matrix is by finite differencing. Each system DOF i can be perturbed in turn to compute column i of the stiffness matrix, $\mathbf{k}_i$. The following is an example using a forward difference:

$$\mathbf{k}_i = \frac{\mathbf{F}(\mathbf{r}^{i+}) - \mathbf{F}(\mathbf{r}^0)}{dr_i} \tag{60}$$

where $\mathbf{r}^0$ is the unperturbed DOF vector, $\mathbf{r}^{i+}$ is the DOF vector with DOF i perturbed, and dr_i is the size of the perturbation.

Unlike other models that have been cited, MoorPy is general in that it supports solving equilibrium for multiple bodies and any combination of attachments in the mooring system. The DOF vector includes any enabled DOFs, and select DOFs can be disabled to improve efficiency (e.g., the y DOFs in a two-dimensional scenario).

When considering the computational expense of these calculations, a given calculation of the system forces, $\mathbf{F}(\mathbf{r})$, can be considered roughly proportional in expense to the number of DOFs in the system, m. From (60), populating the stiffness matrix using forward differencing requires $m + 1$ evaluations, so the expense is of order m^2. Considering also that the stiffness matrix must be inverted, there is a clear nonlinear increase in computational complexity with system size.

5. Verification and Demonstration

The quasi-static model has been verified by comparing against a dynamic mooring model's results after they have equilibrated. The implementation of individual mooring line segment models was original verified against results from other quasi-static models; however, the most important new capabilities of MoorPy are its handling of complex mooring systems, which are most effortlessly represented by a model such as MoorDyn, which is fully general to the mooring system configuration.

The following cases provide a comparison between MoorPy, (MAP++??), and equivalent static results computed by MoorDyn-C, which has been validated and verified in a number of previous studies (cite a bunch). MoorDyn has a general formulation, which allows it to simulate simple and highly intricate mooring configurations equally well. It is therefore well suited to testing the validity of MoorPy's implementation of different types of mooring line profiles and interconnections.

In the comparisons that follow, MoorPy is given a convergence tolerance of 1 mm. Meanwhile, MoorDyn simulations are run for 600 s with elevated damping to ensure static equilibrium. Unless otherwise noted, each mooring line segment in MoorDyn is subdivided into at least 20 internal segments. In addition to presenting and comparing the position and tension predictions, the results also include the number of system stiffness evaluations and the amount of time it took for each equilibrium to be calculated by MoorPy.

5.1. Comparison of individual mooring line asssemblies

The first stage of verification is looking at individual mooring legs–i.e., an assembly of mooring line segments going between and anchor point and a floating platform. Such an assembly involves only three-DOF points at the connections. If the platform is assumed fixed, then the system has $3n$ degrees of freedom, where n is the number of connecting points along the line assembly.

Case 1 is a basic catenary mooring line with a portion lying along the seabed. The comparison with equilibrated MoorDyn results is shown in Figure 7.

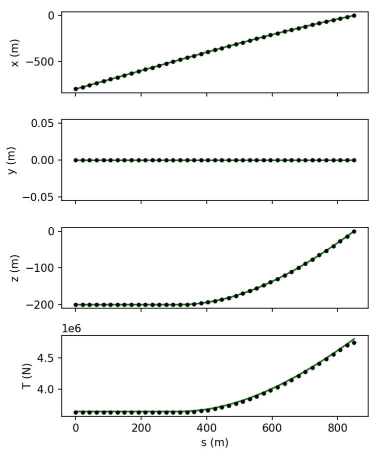

Figure 7: case1 comparison

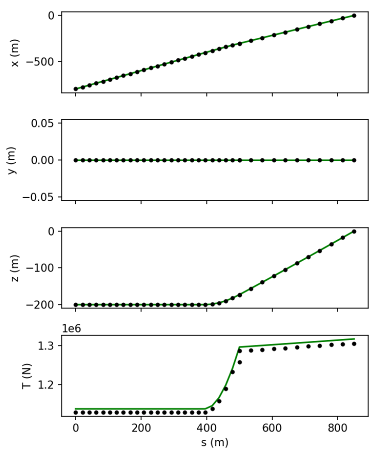

Figure 8: case2 comparison

Case 2 is a two-segment mooring leg that corresponds to catenary chain at the seabed and a rope segment going up to the platform. Its results are plotted in Figure 8.

Case 3 comprises three chain segments connected by a float and a clump weight. This demonstrates finding equilibrium with multiple catenary segments and the presences of point forces. The results are plotted in Figure 9.

Case 4 features a mooring leg with two different portions making contact with the seabed, and a float between them. There is also a massless point part-way along the second portion that is on the seabed, testing the ability for different seabed contact scenarios. Additionally, this mooring system is given a 45 degree heading so that it has both x and y axis components, and in both MoorPy and MoorDyn the system is initialized with the floating point at 0, 0, so that solving for the equilibrium of the floating point involves motion in all three dimensions. The results are in Figure 11.

NEED TO DEBUG STIFFNESS/JACOBIANs IN THIS SCENARIO.

Case 5 is a fully slack mooring line, with the suspended portion hanging vertically and the portion on the seabed having no tension. This represents an extreme case that is often skipped in catenary models but is useful to include when simulating unconventional designs or mooring line failures. The results are shown in Figure 11.

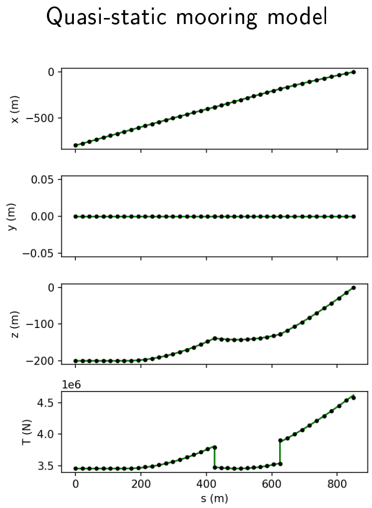

Figure 9: case3 comparison

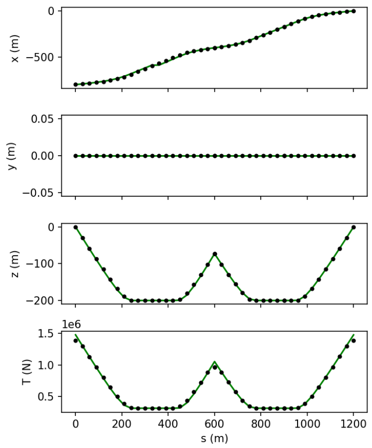

Figure 10: case4 comparison

Case 6 represents a dynamic power cable with a catenary section rising from the seabed, a buoyant section (akin to the averaged sectional properties of a dynamic cable's buoyancy section) with an inverse catenary, and another non-buoyant catenary section rising up to the platform. The results are shown in Figure 13.

[make summary comparison table]

5.2. Mooring system of a floating platform

Floating platform with some catenary bridle lines

Show body stiffness matrix (other points free) and full system stiffness matrix.

5.3. Shared mooring system with multiple floating platforms

Two floating platforms, no bridles, one with a suspended weight tetraspar style except with one line too short/tight!

(demo, not verified)

show stiffness matrix

6. Conclusions

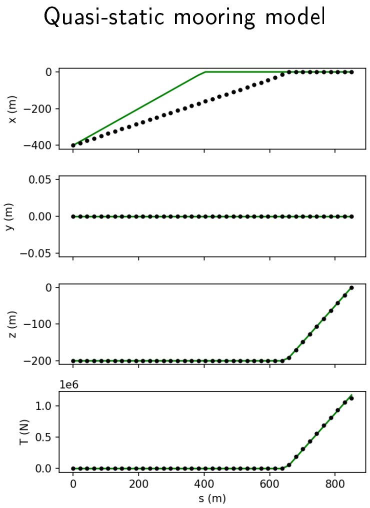

Figure 11: case5 comparison

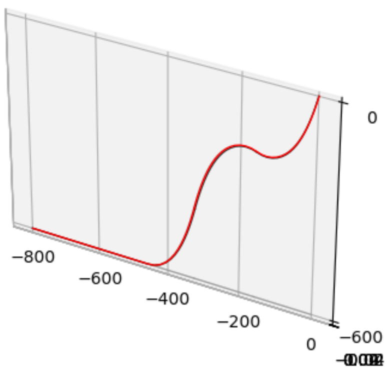

Figure 12: Cable profile

7. Misc MoorPy theory...

7.1. checking for seabed collision in fully suspended line

Note, this only needs to be done for slack lines.

$$Z(s) = \left(\sqrt{1 + \frac{(V_F - WL + Ws)^2}{H^2}} - \sqrt{1 + \frac{(V_F - WL)^2}{H^2}} \right) \frac{H}{W} + \frac{s}{EA}(V_F - WL + Ws/2) \tag{61}$$

rearranging:

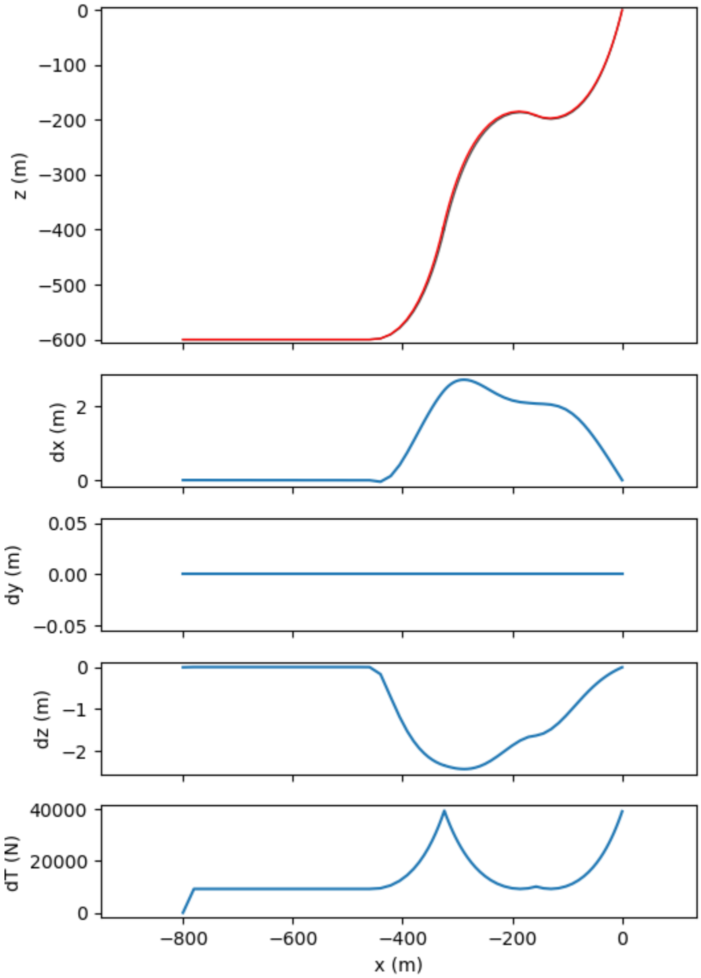

Figure 13: Cable results

$$Z(s) = \left(\sqrt{1 + \frac{[V_F - W(L-s)]^2}{H^2}} - \sqrt{1 + \frac{(V_F - WL)^2}{H^2}} \right) \frac{H}{W} + \frac{s}{EA}[V_F - W(L - s/2)] \tag{62}$$

Differentiating wrt s (using sympy)

$$\frac{dZ(s)}{ds} = \frac{V_F - W(L-s)}{H\sqrt{1 + \frac{[V_F - W(L-s)]^2}{H^2}}} + \frac{V_F - WL + Ws}{EA} \tag{63}$$

Above derivative is zero when

$$V_F - WL + Ws = 0 \tag{64}$$

$$s = \frac{Wl - V_F}{W} = L - V_f/W \tag{65}$$

Subbing into Z(s) gives

$$Z(min) = \left(1 - \sqrt{1 + \frac{(V_f - WL)^2}{H^2}}\right)\frac{H}{W} - \frac{(V_f - WL)^2}{2EAW} \tag{66}$$

Similarly, for the x location:

$$x(s) = \frac{H}{w}\left\{\ln\left[\frac{V_F - w(L-s)}{H} + \sqrt{1 + \left(\frac{V_F - w(L-s)}{H}\right)^2}\right] - \ln\left[\frac{V_F - wL}{H} + \sqrt{1 + \left(\frac{V_F - wL}{H}\right)^2}\right]\right\} + \frac{Hs}{EA} \tag{67}$$

$$= \frac{H}{w}\left\{-\ln\left[\frac{V_A}{H} + \sqrt{1 + \left(\frac{V_A}{H}\right)^2}\right]\right\} + \frac{HL}{EA} - \frac{HV_F}{wEA} \tag{68}$$

$$z(s) = \frac{H}{w}\left[\sqrt{1 + \left(\frac{V_A + ws}{H}\right)^2} - \sqrt{1 + \left(\frac{V_A}{H}\right)^2}\right] + \frac{1}{EA}\left(V_A s + \frac{ws^2}{2}\right)$$

8. moorsolve notes

8.1. dopt constraint handling

If a constraint is violated, $g_i < 0$

Assuming we will step in direction s scaled by α such that

$$g_i + \alpha \vec{s} \vec{\nabla} g = 0 \tag{69}$$

We set $\vec{s}$ to be the gradient but with any bound-impeded directions set to zero.

We find α as

$$\alpha = \frac{-g_i}{\vec{s} \vec{\nabla} g} \tag{70}$$

and then take a step of $\alpha \vec{s}$

Originally, this would be done for each constraint, and the computed steps would then be summed. This runs the risk of overshooting or undershooting when multiple constraints are violated, depending on the orientation of the constraint surfaces.

A better solution is to sum the gradients of all violated constraints

$$\hat{s} = \sum_j \vec{\nabla} g_j \tag{71}$$

and then move along that direction until all constraints are satisfied (based on the current linear approximation). We can do this by calculating an α_j for each constraint and then taking the largest one for how far to move. This also has problems.

Maybe the best approach is more complicated... This assumes there are two constraints active.

$$\mathbf{a}_j = \frac{-g_j(0) \vec{\nabla} g_j}{\vec{\nabla} g_j \cdot \vec{\nabla} g_j} \tag{72}$$

$$\mathbf{g}_1(a_2) = g_1(0) + \mathbf{a}_2 \cdot \nabla \mathbf{g_1} = g_1(0) - g_2(0) \frac{\vec{\nabla} g_2 \cdot \vec{\nabla} g_1}{\vec{\nabla} g_2 \cdot \vec{\nabla} g_2} \tag{73}$$

$$\mathbf{g}_1(b_2) = g_1(0) = g_1(a_2) + \beta \nabla \mathbf{g_1} \cdot \nabla \mathbf{g_1} \tag{74}$$

where $b_2 = \beta \nabla \mathbf{g_1}$.

$$\beta = \frac{g_1(0) - g_1(a_2)}{\vec{\nabla} g_1 \cdot \vec{\nabla} g_1} \tag{75}$$

$$\mathbf{b}_2 = \beta \nabla \mathbf{g_1} = \frac{[g_1(0) - g_1(a_2)]\nabla \vec{g}_1}{\nabla \vec{g}_1 \cdot \nabla \vec{g}_1} \tag{76}$$

$$\mathbf{c}_2 = a_2 + b_2 \tag{77}$$

$$= \frac{-g_2(0)\nabla \vec{g}_2}{\nabla \vec{g}_2 \cdot \nabla \vec{g}_2} + \frac{[g_1(0) - g_1(a_2)]\nabla \vec{g}_1}{\nabla \vec{g}_1 \cdot \nabla \vec{g}_1} \tag{78}$$

$$= \frac{-g_2(0)\nabla \vec{g}_2}{\nabla \vec{g}_2 \cdot \nabla \vec{g}_2} + \frac{[g_1(0) - [g_1(0) + \mathbf{a}_2 \cdot \nabla \mathbf{g_1}]]\nabla \vec{g}_1}{\nabla \vec{g}_1 \cdot \nabla \vec{g}_1} \tag{79}$$

$$= \frac{-g_2(0)\nabla \vec{g}_2}{\nabla \vec{g}_2 \cdot \nabla \vec{g}_2} + \frac{[g_1(0) - [g_1(0) - g_2(0)\frac{\nabla \vec{g}_2 \cdot \nabla \vec{g}_1}{\nabla \vec{g}_2 \cdot \nabla \vec{g}_2}]]\nabla \vec{g}_1}{\nabla \vec{g}_1 \cdot \nabla \vec{g}_1} \tag{80}$$

$$= \frac{-g_2(0)\nabla \vec{g}_2}{\nabla \vec{g}_2 \cdot \nabla \vec{g}_2} + \frac{g_2(0)[\nabla \vec{g}_2 \cdot \nabla \vec{g}_1]\nabla \vec{g}_1}{(\nabla \vec{g}_1 \cdot \nabla \vec{g}_1)(\nabla \vec{g}_2 \cdot \nabla \vec{g}_2)} \tag{81}$$

$$= \frac{g_2}{\nabla \vec{g}_2 \cdot \nabla \vec{g}_2}\left((\nabla \vec{g}_2 \cdot \nabla \vec{g}_1)\left(\frac{\nabla \vec{g}_1}{\nabla \vec{g}_1 \cdot \nabla \vec{g}_1}\right) - \nabla \vec{g}_2 +\right) \tag{82}$$

Inportant note: if some directions are disabled, they should be excluded from the above calculations, crucially including the computation of vector lenghts, e.g. $\nabla \vec{g}_1 \cdot \nabla \vec{g}_1$.

9. Simple System

9.1. getStructureMatrix

Create a Structure Matrix from the shared mooring configuration. The Structure Matrix **S** is used to calculate the force contribution of each degree of freedom called the wrench **w** from each line Tension **f**.

$$\mathbf{w} = \mathbf{S}\mathbf{f} \tag{83}$$

To determine the Structure matrix....

9.2. getTensionMatrix

$$\mathbf{w} = \mathbf{T}\mathbf{k} \tag{84}$$

To determine the Tension matrix....

9.3. getKnobs

$$\mathbf{S}\mathbf{f} = 0 \tag{85}$$

First, we find the Null Space N_1 of the Structure Matrix S.

$$Null(\mathbf{S}) = N\_1 = c_1\vec{v}_1 + c_2\vec{v}_2 + ... + c_n\vec{v}_n$$

n is the nullity of S.$\vec{v}_i$ is an ortho-normal basis vector c_i is any integer. The space N_1 is defined by all possible linear combinations of these ortho-normal basis vectors. The Tension Matrix can broken into it's linear components where m is the number of linetypes used:

$$\mathbf{T} = k_1\vec{u}_1 + k_2\vec{u}_2 + ... + k_m\vec{u}_m$$

We wish to describe a system who's Tensions are in equilibrium and who's types are introduced in a systematic way. To achieve this, we say The Tension Matrix (**T**) and the Null Space of the Structure Matrix (N_1) must be equivalent.

$$N\_1 = \mathbf{T}$$

Setting both variables to one side.

$$N\_1 - \mathbf{T} = 0$$

Writing the variables out:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + ... + c_n\vec{v}_n - k_1\vec{u}_1 - k_2\vec{u}_2 - ... - k_m\vec{u}_m = 0$$

Combining the linear components into matrix form:

$$\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n & -\vec{u}_1 & -\vec{u}_2 & \cdots & -\vec{u}_m\end{bmatrix}\begin{bmatrix}c_1\\ c_2\\ \vdots\\ c_n\\ k_1\\ k_2\\ \vdots\\ k_m\end{bmatrix} = 0$$

Again, we find the Null Space of of the matrix on the left-hand side:

$$N\_2 = j_1\vec{w}_1 + ... + j_l\vec{w}_l$$

l is the nullity of this left-hand side matrix.j_1 ... j_l are the knobs which one can use to explore this space.$\vec{w}_1$ ... $\vec{w}_l$ are the orthogonal normal vectors which define the Null space. All linear combination of these vectors satisfy the equations (85) and (84).

9.4. getSystemStiffness

To determine Stiffness Matrix...

9.4.1. Line.getLineStiffness

Determine Line Stiffness equation

$$\boldsymbol{Kl_x} = k_x \left[\begin{array}{cc|cc} l_x l_x & l_x l_y & -l_x L_x & -l_x l_y \\ l_x l_y & l_y l_y & -l_x l_y & -l_y l_y \\ \hline -l_x l_x & -l_x l_y & l_x l_x & l_x l_y \\ -l_x l_y & -l_y l_y & l_x l_y & l_y l_y \end{array}\right]$$

$$\boldsymbol{Kl_t} = k_t \left[\begin{array}{cc|cc} l_y l_y & l_x l_y & -l_y L_y & -l_x l_y \\ l_x l_y & l_x l_x & -l_x l_y & -l_x l_x \\ \hline -l_y l_y & -l_x l_y & l_y l_y & l_x l_y \\ -l_x l_y & -l_x l_x & l_x l_y & l_x l_x \end{array}\right]$$

where l_x and l_y are the horizontal and vertical components of the line normal vector.

$$\boldsymbol{F_l} = [\boldsymbol{Kl_x} + \boldsymbol{Kl_t}]\boldsymbol{u}$$

The Global Stiffness Matrix is the summation of the Line Stiffness Matricies for each Line is the System. (I can't think of a good equation to explain this)

9.5. get_x

K = SystemStiffnessMatrix, F = $\begin{bmatrix} w_x \\ w_y \\ w_x \\ w_y \\ \vdots \end{bmatrix}$ * 2000KN

where w_x and w_y are the horizontal and vertical components of the wind normal vector.

$$F = KX$$

$$X = K^{-1}F$$

9.6. getCost

$$Cost = K\_s * LineLengths$$

9.7. solveSystemStiffness

To optimize System Stiffness...

10. Results

10.1. Model Verification

10.2. Analytic Stiffness Verification

10.3. Example Mooring Systems (Demonstrations)

References

slack hanging line derivation

THIS IS REDUNDANT!!!

Consider a free-hanging vertical line of length L_H. From the bottom (free) end of the up, the tension is $T(s) = ws$. Considering a differential element of the line with unstretched length ds, its stretched length is

$$dz = (1 + \frac{T}{EA})ds = (1 + \frac{ws}{EA})ds \tag{86}$$

Integrating from the bottom of the line up, we can solve for the stretched length:

$$z(s) = \int_0^s = (1 + \frac{ws}{EA})ds = s + \frac{w}{2EA}s^2 \tag{87}$$

Substituting $s = L_H$ and solving using the quadratic formula gives

$$L_H = \sqrt{2z_F\frac{EA}{w} + \left(\frac{EA}{w}\right)^2} - \frac{EA}{w} = \frac{EA}{w}\left(\sqrt{2z_F\frac{w}{EA} + 1} - 1\right) \tag{88}$$

At the line top, the horizontal tension is zero and the vertical tension is $V_F = wL_H$. The change in tension with respect to line length (stretched length, as if the line is slack on the seabed and more can be pulled up) is:

$$\frac{dV_F}{dz_F} = \frac{d}{dZ_F}\left(\sqrt{2z_F EAw + (EA)^2} - EA\right) = \frac{w}{\sqrt{2z_F\frac{w}{EA} + 1}} \tag{89}$$

suspended line derivatives

Equations to solve:

$$x_F(H_F, V_F) = \frac{H_F}{w}\left\{\ln\left[\frac{V_F}{H_F} + \sqrt{1 + \left(\frac{V_F}{H_F}\right)^2}\right] - \ln\left[\frac{V_F - wL}{H_F} + \sqrt{1 + \left(\frac{V_F - wL}{H_F}\right)^2}\right]\right\} + \frac{H_F L}{EA}$$

$$z_F(H_F, Z_F) = \frac{H_F}{w}\left[\sqrt{1 + \left(\frac{V_F}{H_F}\right)^2} - \sqrt{1 + \left(\frac{V_F - wL}{H_F}\right)^2}\right] + \frac{1}{EA}\left(V_F L - \frac{wL^2}{2}\right)$$

Derivatives:

$$E_{X_F} = \left[\ln\left(\frac{V_F}{H_F} + \sqrt{1 + \left(\frac{V_F}{H_F}\right)^2}\right) - \ln\left(\left(\frac{V_F - wL}{H_F}\right) + \sqrt{1 + \left(\frac{V_F - wL}{H_F}\right)^2}\right)\right]\frac{H_F}{w} + \frac{L}{EA}H_F - X_F$$

$$E_{Z_F} = \left(\sqrt{1 + \left(\frac{V_F}{H_F}\right)^2} - \sqrt{1 + \left(\frac{V_F - WL}{H_F}\right)^2}\right)\frac{H_F}{W} + \frac{L}{EA}\left(V_F - \frac{1}{2}WL\right) - Z_F$$

$$\frac{dX_F}{dH_F} = \left[\ln\left(\frac{V_F}{H_F} + \sqrt{1+\left(\frac{V_F}{H_F}\right)^2} \right) - \ln\left(\left(\frac{V_F - WL}{H_F}\right) + \sqrt{1+\left(\frac{V_F - WL}{H_F}\right)^2} \right) \right] \frac{1}{W}$$

$$-\left[\frac{\frac{V_F}{H_F} + \frac{\left(\frac{V_F}{H_F}\right)^2}{\sqrt{1+\left(\frac{V_F}{H_F}\right)^2}}}{\frac{V_F}{H_F} + \sqrt{1+\left(\frac{V_F}{H_F}\right)^2}} - \frac{\left(\frac{V_F-WL}{H_F}\right) + \frac{\left(\frac{V_F-WL}{H_F}\right)^2}{\sqrt{1+\left(\frac{V_F-WL}{H_F}\right)^2}}}{\left(\frac{V_F-WL}{H_F}\right) + \sqrt{1+\left(\frac{V_F-WL}{H_F}\right)^2}} \right] \frac{1}{W} + \frac{L}{EA}$$

$$\frac{dX_F}{dV_F} = \left[\frac{1 + \frac{\frac{V_F}{H_F}}{\sqrt{1+\left(\frac{V_F}{H_F}\right)^2}}}{\frac{V_F}{H_F} + \sqrt{1+\left(\frac{V_F}{H_F}\right)^2}} - \frac{1 + \frac{\left(\frac{V_F-WL}{H_F}\right)}{\sqrt{1+\left(\frac{V_F-WL}{H_F}\right)^2}}}{\left(\frac{V_F-WL}{H_F}\right) + \sqrt{1+\left(\frac{V_F-WL}{H_F}\right)^2}} \right] \frac{1}{W}$$

$$\frac{dZ_F}{dH_F} = \left[\sqrt{1+\left(\frac{V_F}{H_F}\right)^2} - \sqrt{1+\left(\frac{V_F - WL}{H_F}\right)^2} \right] \frac{1}{W} - \left[\frac{\left(\frac{V_F}{H_F}\right)^2}{\sqrt{1+\left(\frac{V_F}{H_F}\right)^2}} - \frac{\left(\frac{V_F-WL}{H_F}\right)^2}{\sqrt{1+\left(\frac{V_F-WL}{H_F}\right)^2}} \right] \frac{1}{W}$$

$$\frac{dZ_F}{dV_F} = \left[\frac{\frac{V_F}{H_F}}{\sqrt{1+\left(\frac{V_F}{H_F}\right)^2}} - \frac{\left(\frac{V_F-WL}{H_F}\right)}{\sqrt{1+\left(\frac{V_F-WL}{H_F}\right)^2}} \right] \frac{1}{W} + \frac{L}{EA}$$

catenary line with seabed contact derivatives

.0.1. ProfileType=2 - anchor tension is nonzero

Rewriting the above equations for when x_B is 0 or negative and there is a horizontal force on the anchor.

$$x(s) = \begin{cases} s + \frac{C_B w}{2EA}\left[s^2 - 2\left(L_B - \frac{H_F}{C_B w}\right)s\right] & 0 \le s \le L_B \\ L_B + \frac{H_F}{w}\ln\left[\frac{w(s-L_B)}{H_F} + \sqrt{1+\left(\frac{w(s-L_B)}{H_F}\right)^2}\right] + \frac{H_F s}{EA} + \frac{C_B w}{2EA}\left[-L_B^2\right] & L_B \le s \le L \end{cases}$$

(In Catenary, same thing but written differently for 0 s Lb)

$$x(s) = s + \frac{s}{EA}(H_F + C_B(V_F - wL) + 0.5 w s C_B$$

$$T(s) = H_F + C_B(V_F - wL + ws)$$

(for Lb s L, it all checks out)

$$x(s) = \ln\left(\frac{V_F - wL + ws}{H_F} + \sqrt{1 + \left(\frac{V_F - wL + ws}{H_F}\right)^2}\right)\frac{H_F}{w} + \frac{s}{EA}H_F + L_{Bot} - \frac{C_B}{2}(V_F - wL)\frac{V_F - wL}{wEA}$$

$$z(s) = \left(-1 + \sqrt{1 + \left(\frac{V_F - wL + ws}{H_F}\right)^2}\right) + \frac{s}{EA}(V_F - wL + ws/2) + \frac{1}{2wEA}(V_F - wL)^2$$

$$T(s) = \sqrt{H_F^2 + (V_F - wL - ws)^2}$$

Back to Theory

$$z(s) = \begin{cases} 0 & 0 \le s \le L_B \\ \frac{H_F}{w}\ln\left[\sqrt{1 + \left(\frac{w(s-L_B)}{H_F}\right)^2} - 1\right] + \frac{w(s-L_B)^2}{2EA} & L_B \le s \le L \end{cases}$$

$$T(s) = \begin{cases} H_F + C_B w(s - L_B) & 0 \le s \le L_B \\ \sqrt{H_F^2 + (w(s - L_B))^2} & L_B \le s \le L \end{cases}$$

$$H_A = H_F - C_B w L_B$$

or

$$H_A = H_F + C_B(V_F - wL)$$

$$V_A = 0$$

$$x_F(H_F, V_F) = L - \frac{V_F}{w} + \frac{H_F}{w}\ln\left[\frac{V_F}{H_F} + \sqrt{1 + \left(\frac{V_F}{H_F}\right)^2}\right] + \frac{H_F L}{EA} + \frac{C_B w}{2EA}\left[-\left(L - \frac{V_F}{w}\right)^2\right]$$

$$z_F(H_F, V_F) = \frac{H_F}{w}\left[\sqrt{1 + \left(\frac{V_F}{H_F}\right)^2} - 1\right] + \frac{V_F^2}{2EAw}$$

Going back to the Catenary function, the derivatives need to be taken in order to use the Newton-Raphson iteration scheme.

$$E_{X_F} = \ln\left(\frac{V_F}{H_F} + \sqrt{1 + \left(\frac{V_F}{H_F}\right)^2}\right)\frac{H_F}{W} - \frac{1}{2}\frac{C_B}{EA}WL_{Bot}^2 + \frac{L}{EA}H_F + L_{Bot} - X_F$$

$$E_{Z_F} = \left(\sqrt{1 + \left(\frac{V_F}{H_F} \right)^2} - 1 \right) \frac{H_F}{W} + \frac{1}{2} V_F \frac{V_F}{WEA} - Z_F$$

$$\frac{dX_F}{dH_F} = \ln\left(\frac{V_F}{H_F} + \sqrt{1 + \left(\frac{V_F}{H_F} \right)^2} \right) \frac{1}{W} - \left[\frac{ \frac{V_F}{H_F} + \frac{\left(\frac{V_F}{H_F}\right)^2}{\sqrt{1+\left(\frac{V_F}{H_F}\right)^2}} }{ \frac{V_F}{H_F} + \sqrt{1 + \left(\frac{V_F}{H_F} \right)^2} } \right] \frac{1}{W} + \frac{L}{EA}$$

$$\frac{dX_F}{dV_F} = \left[\frac{ 1 + \frac{\frac{V_F}{H_F}}{\sqrt{1+\left(\frac{V_F}{H_F}\right)^2}} }{ \frac{V_F}{H_F} + \sqrt{1 + \left(\frac{V_F}{H_F} \right)^2} } \right] \frac{1}{W} + \frac{C_B}{EA} L_{Bot} - \frac{1}{W}$$

$$\frac{dZ_F}{dH_F} = \left[\sqrt{1 + \left(\frac{V_F}{H_F} \right)^2} - 1 - \frac{\left(\frac{V_F}{H_F}\right)^2}{\sqrt{1 + \left(\frac{V_F}{H_F}\right)^2}} \right] \frac{1}{W}$$

$$\frac{dZ_F}{dV_F} = \left[\frac{\frac{V_F}{H_F}}{\sqrt{1 + \left(\frac{V_F}{H_F}\right)^2}} \right] \frac{1}{W} + \frac{V_F}{WEA}$$

.0.2. ProfileType=3 - anchor tension is zero

Rewriting the above equations for when x_B is positive and there is zero tension on the anchor.

$$x(s) = \begin{cases} s & 0 \le s \le L_B - \frac{H_F}{C_B w} \\ s + \frac{C_B w}{2EA}\left[s^2 - 2\left(L_B - \frac{H_F}{C_B w}\right) s + \left(L_B - \frac{H_F}{C_B w}\right)^2 \right] & L_B - \frac{H_F}{C_B w} \le s \le L_B \\ L_B + \frac{H_F}{w} \ln\left[\frac{w(s-L_B)}{H_F} + \sqrt{1 + \left(\frac{w(s-L_B)}{H_F}\right)^2} \right] + \frac{H_F s}{EA} + & \\ \frac{C_B w}{2EA}\left[-L_B^2 + \left(L_B - \frac{H_F}{C_B w}\right)^2 \right] & L_B \le s \le L \end{cases}$$

(In Catenary, same thing but written differently for xB s Lb)

$$x(s) = s - (L_{Bot} - 0.5\frac{H_F}{wC_B})\frac{H_F}{EA} + \frac{s}{EA}(H_F + C_B(V_F - wL) + 0.5wsC_B) + 0.5C_B(V_F - wL)\frac{V_F - wL}{wEA}$$

$$T(s) = H_F + C_B(V_F - wL + ws)$$

(for Lb s L, it all checks out)

$$x(s) = \ln\left(\frac{V_F - wL + ws}{H_F} + \sqrt{1 + \left(\frac{V_F - wL + ws}{H_F}\right)^2}\right)\frac{H_F}{w} + \frac{s}{EA}H_F + L_{Bot} - (L_{Bot} - 0.5\frac{H_F}{wC_B})\frac{H_F}{EA}$$

$$z(s) = \left(-1 + \sqrt{1 + \left(\frac{V_F - wL + ws}{H_F}\right)^2}\right) + \frac{s}{EA}(V_F - wL + ws/2) + \frac{1}{2wEA}(V_F - wL)^2$$

$$T(s) = \sqrt{H_F^2 + (V_F - wL + ws)^2}$$

$$z(s) = \begin{cases} 0 & 0 \le s \le L_B \\ \frac{H_F}{w}\ln\left[\sqrt{1 + \left(\frac{w(s-L_B)}{H_F}\right)^2} - 1\right] + \frac{w(s-L_B)^2}{2EA} & L_B \le s \le L \end{cases}$$

$$T(s) = \begin{cases} 0 & 0 \le s \le L_B - \frac{H_F}{C_B w} \\ H_F + C_B w(s - L_B) & L_B - \frac{H_F}{C_B w} \le s \le L_B \\ \sqrt{H_F^2 + (w(s - L_B))^2} & L_B \le s \le L \end{cases}$$

$$H_A = 0$$

$$V_A = 0$$

$$x_F(H_F, V_F) = L - \frac{V_F}{w} + \frac{H_F}{w}\ln\left[\frac{V_F}{H_F} + \sqrt{1 + \left(\frac{V_F}{H_F}\right)^2}\right] + \frac{H_F L}{EA} + \frac{C_B w}{2EA}\left[-\left(L - \frac{V_F}{w}\right)^2 + \left(L - \frac{V_F}{w} - \frac{H_F}{C_B w}\right)^2\right]$$

$$z_F(H_F, V_F) = \frac{H_F}{w}\left[\sqrt{1 + \left(\frac{V_F}{H_F}\right)^2} - 1\right] + \frac{V_F^2}{2EAw}$$

Going back to the Catenary function, the derivatives need to be taken in order to use the Newton-Raphson iteration scheme.

$$E_{X_F} = \ln\left(\frac{V_F}{H_F} + \sqrt{1+\left(\frac{V_F}{H_F}\right)^2}\right)\frac{H_F}{W} - \frac{1}{2}\frac{C_B}{EA}W\left(L_{Bot}^2 - \left(L_{Bot} - \frac{\frac{H_F}{W}}{C_B}\right)^2\right) + \frac{L}{EA}H_F + L_{Bot} - X_F$$

$$E_{Z_F} = \left(\sqrt{1+\left(\frac{V_F}{H_F}\right)^2} - 1\right)\frac{H_F}{W} + \frac{1}{2}V_F\frac{V_F}{WEA} - Z_F$$

$$\frac{dX_F}{dH_F} = \ln\left(\frac{V_F}{H_F} + \sqrt{1+\left(\frac{V_F}{H_F}\right)^2}\right)\frac{1}{W} - \left[\frac{\frac{V_F}{H_F} + \frac{\left(\frac{V_F}{H_F}\right)^2}{\sqrt{1+\left(\frac{V_F}{H_F}\right)^2}}}{\frac{V_F}{H_F} + \sqrt{1+\left(\frac{V_F}{H_F}\right)^2}}\right]\frac{1}{W} + \frac{L}{EA} - \frac{(L_{Bot} - \frac{\frac{H_F}{W}}{C_B})}{EA}$$

$$\frac{dX_F}{dV_F} = \left[\frac{1 + \frac{\frac{V_F}{H_F}}{\sqrt{1+\left(\frac{V_F}{H_F}\right)^2}}}{\frac{V_F}{H_F} + \sqrt{1+\left(\frac{V_F}{H_F}\right)^2}}\right]\frac{1}{W} + \frac{H_F}{WEA} - \frac{1}{W}$$

$$\frac{dZ_F}{dH_F} = \left[\sqrt{1+\left(\frac{V_F}{H_F}\right)^2} - 1 - \frac{\left(\frac{V_F}{H_F}\right)^2}{\sqrt{1+\left(\frac{V_F}{H_F}\right)^2}}\right]\frac{1}{W}$$

$$\frac{dZ_F}{dV_F} = \left[\frac{\frac{V_F}{H_F}}{\sqrt{1+\left(\frac{V_F}{H_F}\right)^2}}\right]\frac{1}{W} + \frac{V_F}{WEA}$$

.1. Z extreme derivation

For a fully suspend U shaped line, the minimum elevation point occurs where the slope is zero or where the vertical component of tension is zero. This condition can be expressed as

$$V_B - wL + ws = 0 \longrightarrow s = L - \frac{V_B}{w} \tag{90}$$

Substituting into the equation for z of a suspended line gives:

$$z\left(s = L - \frac{V_B}{w}\right) = \left[1 - \sqrt{1 + \frac{(V_F - wL)^2}{H_F^2}}\right]\frac{H_F}{w} + \frac{L - V_F/w}{EA}\left[V_F - wL + \frac{1}{2}w\left(L - \frac{V_f}{w}\right)\right] \tag{91}$$

$$z\left(s = L - \frac{V_B}{w}\right) = \left[1 - \sqrt{1 + \frac{(V_B - wL)^2}{H^2}}\right]\frac{H}{w} - \frac{(V_B - wL)^2}{2EAw} \tag{92}$$

A. Step 2 - Rotate matrix into global coordinates

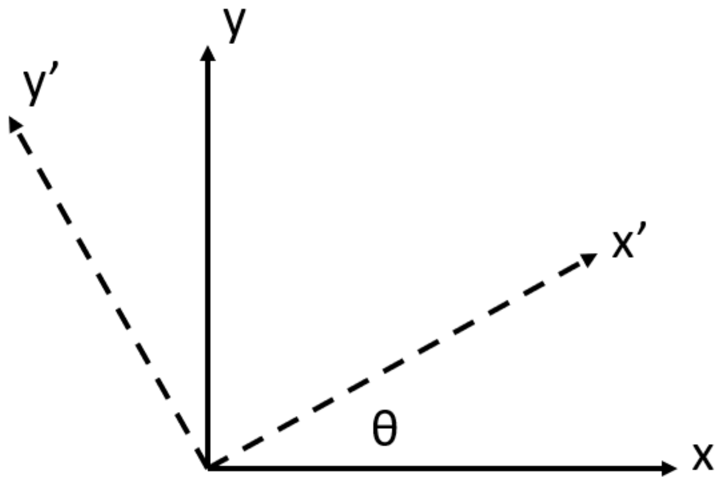

Figure 14:

A.0.1. Rotation Matrix Review

$$RegularRotationMatrix = R1 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$BackwardsRotationMatrix = R2 = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

In general, the R1 matrix is used to rotate the object’s axes from the point that they are in currently to the axes that you want. Note that the angle theta then becomes the angle starting from the object’s axes to the global axes. For example, in the figure, the angle to plug into the rotation matrix should be negative theta.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

The R2 matrix, or the transpose of the regular rotation matrix, is used to rotate the axes that you want the system to be in (global cartesian) to the axes that the object is currently in. Note that the angle theta then becomes the angle starting from the global axes to the object's axes. For example, in the figure, the angle to plug into R2 should be positive theta.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Another note is that this is for the matrix multiplication of the rotation matrix by the axes. Matrix multiplication is not commutative so if you were to multiply the axes by the rotation matrix ([u][R]), things would become backwards, and you would need to flip the signs of the angles.

$$LineStiffnessMatrixinGlobalCoords = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{dF_L}{dL} & \frac{dF_L}{dt} & \frac{dF_L}{dz} \\ \frac{dF_t}{dL} & \frac{dF_t}{dt} & \frac{dF_t}{dz} \\ \frac{dF_z}{dL} & \frac{dF_z}{dt} & \frac{dF_z}{dz} \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Each MoorPy mooring line has a heading attribute to describe the angle between ends A and B of the line (i.e. where end B is in relation to end A). For example, if the x' axis in Figure 14 was a mooring line in the x-y frame with its end A at the tip of the x' axis arrow and end B at the origin, it would have a heading of -135 degrees, since that's the angle it makes with the horizontal.

Other hypotheses are that the equation for Hooke's law is actually

$$[F] = -[K]X$$

B. Proof of body rotational stiffness terms

This section provides a proof for the calculation of the first term in the following:

$$\begin{aligned} \frac{\partial \mathbf{M}}{\partial \theta} &= \frac{\partial}{\partial \theta}(\mathbf{r} \times \mathbf{f}) && (93) \\ &= \frac{\partial}{\partial \theta}(-\mathbf{H}(\mathbf{r})\mathbf{f}) && (94) \\ &= \frac{\partial \mathbf{H}(\mathbf{r})}{\partial \theta}\mathbf{f} + \mathbf{H}(\mathbf{r})\frac{\partial \mathbf{f}}{\partial \theta} && (95) \\ &= \mathbf{H}(-\mathbf{H}(\mathbf{r}))\mathbf{f} + \mathbf{H}(\mathbf{r})\frac{\partial \mathbf{f}}{\partial \theta} && (96) \\ & && (97) \end{aligned}$$

The meaning of successive H operators is most conveniently expressed by considering the alternator matrix with Einstein notation, which is

$$\mathbf{H}(\mathbf{r}) = H(r)_{ij} = \epsilon_{ijk} r_k \tag{98}$$

From this, the meaning of H(-H(r)) can be understood using tensor math:

$$\begin{aligned} \mathbf{H}(-\mathbf{H}(\mathbf{r})) &= -\epsilon_{ijk}\epsilon_{lkm} r_m && (99) \\ &= \epsilon_{kij}\epsilon_{klm} r_m && (100) \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) r_m && (101) \end{aligned}$$

$$= \delta_{il} r_j - \delta_{jl} r_i \tag{102}$$

$$\tag{103}$$

Multiplying by the force vector gives

$$\begin{aligned}
\mathbf{H}(-\mathbf{H}(\mathbf{r}))\mathbf{f} &= (\delta_{il} r_j - \delta_{jl} r_i) f_i && (104)\\
&= r_j f_l - \delta_{jl} r_i f_i && (105)\\
&= \begin{bmatrix} -r_3 f_3 - r_2 f_2 & r_1 f_2 & r_1 f_3 \\ r_2 f_1 & -r_1 f_1 - r_3 f_3 & r_2 f_3 \\ r_3 f_1 & r_3 f_2 & -r_1 f_1 - r_2 f_2 \end{bmatrix} && (106)\\
&= \mathbf{H}(\mathbf{f})\mathbf{H}(\mathbf{r}) && (107)
\end{aligned}$$

Multiplying by the force vector givesXXXX

$$\begin{aligned}
\mathbf{H}(-\mathbf{H}(\mathbf{r}))\mathbf{f} &= (\delta_{il} r_j - \delta_{jl} r_i) f_j && (108)\\
&= \delta_{il} r_j f_j - r_i f_l && (109)\\
&= \begin{bmatrix} r_3 f_3 + r_2 f_2 & -r_1 f_2 & -r_1 f_3 \\ -r_2 f_1 & r_1 f_1 + r_3 f_3 & -r_2 f_3 \\ -r_3 f_1 & -r_3 f_2 & r_1 f_1 + r_2 f_2 \end{bmatrix} && (110)\\
&= -\mathbf{H}(\mathbf{f})\mathbf{H}(\mathbf{r}) && (111)
\end{aligned}$$

C. seabed contact stiffness matrices...

If there is a line with seabed contact between two free objects... (things are tricky)

Friction not allowed! Normal catenary line with no friction:

$$\mathbf{K}_{line} = \begin{Bmatrix} \frac{\partial H_B}{\partial x_B} & ? & -\frac{\partial H_B}{\partial x_B} & \frac{\partial H_B}{\partial z_B} \\ ? & ? & 0 & 0 \\ -\frac{\partial H_B}{\partial x_B} + & 0ish & \frac{\partial H_B}{\partial x_B} & \frac{\partial H_B}{\partial z_B} \\ -\frac{\partial V_B}{\partial x_B} + & 0ish & \frac{\partial V_B}{\partial x_B} & \frac{\partial V_B}{\partial z_B} \end{Bmatrix} \tag{112}$$

U shaped line:

$$
\mathbf{K}_{line} = \begin{bmatrix}
\frac{\frac{\partial H_A}{\partial x_A}\frac{\partial H_B}{\partial x_B}}{\frac{\partial H_A}{\partial x_A}+\frac{\partial H_B}{\partial x_B}} & \frac{\frac{\partial H_A}{\partial z_A}\frac{\partial H_B}{\partial x_B}}{\frac{\partial H_A}{\partial x_A}+\frac{\partial H_B}{\partial x_B}} & -\frac{\frac{\partial H_A}{\partial x_A}\frac{\partial H_B}{\partial x_B}}{\frac{\partial H_A}{\partial x_A}+\frac{\partial H_B}{\partial x_B}} & -\frac{\frac{\partial H_A}{\partial x_A}\frac{\partial H_B}{\partial z_B}}{\frac{\partial H_A}{\partial x_A}+\frac{\partial H_B}{\partial x_B}} \\
\frac{\frac{\partial V_A}{\partial x_A}\frac{\partial H_B}{\partial x_B}}{\frac{\partial H_A}{\partial x_A}+\frac{\partial H_B}{\partial x_B}} & \frac{\partial V_A}{\partial z_A} - \frac{\frac{\partial H_A}{\partial z_A}\frac{\partial V_A}{\partial x_A}}{\frac{\partial H_A}{\partial x_A}+\frac{\partial H_B}{\partial x_B}} & -\frac{\frac{\partial V_A}{\partial x_A}\frac{\partial H_B}{\partial x_B}}{\frac{\partial H_A}{\partial x_A}+\frac{\partial H_B}{\partial x_B}} & -\frac{\frac{\partial V_A}{\partial x_A}\frac{\partial H_B}{\partial z_B}}{\frac{\partial H_A}{\partial x_A}+\frac{\partial H_B}{\partial x_B}} \\
-\frac{\frac{\partial H_A}{\partial x_A}\frac{\partial H_B}{\partial x_B}}{\frac{\partial H_A}{\partial x_A}+\frac{\partial H_B}{\partial x_B}} & -\frac{\frac{\partial H_A}{\partial z_A}\frac{\partial H_B}{\partial x_B}}{\frac{\partial H_A}{\partial x_A}+\frac{\partial H_B}{\partial x_B}} & \frac{\frac{\partial H_A}{\partial x_A}\frac{\partial H_B}{\partial x_B}}{\frac{\partial H_A}{\partial x_A}+\frac{\partial H_B}{\partial x_B}} & \frac{\frac{\partial H_A}{\partial x_A}\frac{\partial H_B}{\partial z_B}}{\frac{\partial H_A}{\partial x_A}+\frac{\partial H_B}{\partial x_B}} \\
-\frac{\frac{\partial H_A}{\partial x_A}\frac{\partial V_B}{\partial x_B}}{\frac{\partial H_A}{\partial x_A}+\frac{\partial H_B}{\partial x_B}} & -\frac{\frac{\partial H_A}{\partial z_A}\frac{\partial V_B}{\partial x_B}}{\frac{\partial H_A}{\partial x_A}+\frac{\partial H_B}{\partial x_B}} & \frac{\frac{\partial H_A}{\partial x_A}\frac{\partial V_B}{\partial x_B}}{\frac{\partial H_A}{\partial x_A}+\frac{\partial H_B}{\partial x_B}} & \frac{\partial V_B}{\partial z_B} - \frac{\frac{\partial H_B}{\partial z_B}\frac{\partial V_B}{\partial x_B}}{\frac{\partial H_A}{\partial x_A}+\frac{\partial H_B}{\partial x_B}}
\end{bmatrix} \tag{113}
$$