For return on 24 January 2014 (late submission: 7 February 2014)

Electronic submission: **pdf** *files only*

1. **(3%)** Construct the truth-table for the Boolean function given by the Boolean formula

$$\neg\big(A \wedge (B \to C)\big) \wedge \neg B.$$

Use the truth-table to realise this function by a formula with the connectives $\neg$, $\vee$, $\wedge$ only. Simplify the formula in such a way that the corresponding Boolean circuit contains a minimal number of gates. Show the Boolean circuit.

Truth Table

A	B	C	$\neg$	$(A$	$\wedge$	$(B$	$\to$	$C))$	$\wedge$	$\neg B)$
0	0	0	1		0		1		1	1
0	0	1	1		0		1		1	1
0	1	0	1		0		0		0	0
0	1	1	1		0		1		0	0
1	0	0	0		1		1		0	1
1	0	1	0		1		1		0	1
1	1	0	1		0		0		0	0
1	1	1	0		1		1		0	0

Simplification

1. $(\neg A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge \neg B \wedge C)$

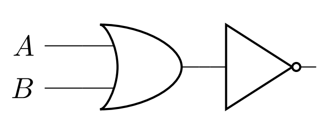
2. $(\neg(A \vee B) \wedge \neg C) \vee (\neg(A \vee B) \wedge C)$

3. $\neg(A \vee B) \vee (\neg C \wedge C)$

4. $\neg(A \vee B) \vee 0$

5. $\neg(A \vee B)$

Boolean Circuit



2. **(3%)** Are the following Boolean formulas equivalent? Explain your answer.

(a) $A \rightarrow (B \wedge C)$ and $(A \rightarrow B) \wedge (A \rightarrow C)$

A	B	C	A	$\rightarrow$	$(B$	$\wedge$	$C)$
0	0	0		1		0	
0	0	1		1		0	
0	1	0		1		0	
0	1	1		1		1	
1	0	0		0		0	
1	0	1		0		0	
1	1	0		0		0	
1	1	1		1		1	

A	B	C	$(A$	$\rightarrow$	$B)$	$\wedge$	$(A$	$\rightarrow$	$C)$
0	0	0		1		1		1	
0	0	1		1		1		1	
0	1	0		1		1		1	
0	1	1		1		1		1	
1	0	0		0		0		0	
1	0	1		0		0		1	
1	1	0		1		0		0	
1	1	1		1		1		1	

Yes they are equivalent

(b) $(A \wedge B) \to C$ and $(\neg C \to \neg A) \wedge (\neg C \to \neg B)$

A	B	C	$(A$	$\wedge$	$B)$	$\to$	C
0	0	0		0		1	
0	0	1		0		1	
0	1	0		0		1	
0	1	1		0		1	
1	0	0		0		1	
1	0	1		0		1	
1	1	0		1		0	
1	1	1		1		1	

A	B	C	$(\neg C$	$\to$	$\neg A)$	$\wedge$	$(\neg C$	$\to$	$\neg B)$
0	0	0	1	1	1	1	1	1	1
0	0	1	0	1	1	1	0	1	1
0	1	0	1	1	1	0	1	0	0
0	1	1	0	1	1	1	0	1	0
1	0	0	1	0	0	0	1	1	1
1	0	1	0	1	0	1	0	1	1
1	1	0	1	0	0	0	1	0	0
1	1	1	0	1	0	1	0	1	0

No they are not equivalent

(c) $(A \vee B) \to C$ and $(A \to B) \vee (A \to C)$

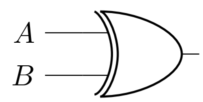
A	B	C	$(A$	$\vee$	$B)$	$\to$	C
0	0	0		0		1	
0	0	1		0		1	
0	1	0		1		0	
0	1	1		1		1	
1	0	0		1		0	
1	0	1		1		1	
1	1	0		1		0	
1	1	1		1		1	

A	B	C	$(A$	$\to$	$B)$	$\vee$	$(A$	$\to$	$C)$
0	0	0		1		1		1	
0	0	1		1		1		1	
0	1	0		1		1		1	
0	1	1		1		1		1	
1	0	0		0		0		0	
1	0	1		0		1		1	
1	1	0		1		1		0	
1	1	1		1		1		1	

No they are not equivalent

3. **(3%)** A parity function is a Boolean function whose value is 1 if the input has an odd number of ones. Design a Boolean circuit for the 2-bit parity function. Show your working. (Hint: you may find XOR gates useful.)

A	B	
0	0	0
0	1	1
1	0	1
1	1	0



4. **(3%)** Suppose $\alpha = a_{31}a_{30}\dots a_1a_0$ is a 32-bit binary word. Consider the 32-bit binary word $\beta = b_{31}b_{30}\dots b_1b_0$ computed by the following algorithm: scan α from right to left and copy its bits to β until the first 1 is found (which is also copied to β); after that, copy the Boolean negations of the bits in α. For example, $\alpha = 10100\dots00$ is transformed to $\beta = 01100\dots00$. Explain what this algorithm computes if α and β are interpreted as binary numbers.

Given a two's compliment binary number, the algorithm computes the conversion into a binary number.

5. **(6%)** Given the machine 32-bit word

1100 0001 0011 0000 0000 0000 0000 0000

find the decimal number represented by this word assuming that it is

(a) a two's complement integer;

1. Read the first character from the left of the word to get the sign of the integer it represents. In this case it is a one so the number is negative.
2. Use the two's compliment algorithm to convert it into a binary number. Starting from the right of the word, move to the first one :
   - $1100\ 0001\ 001\underline{1}\ 00...00$
3. Move left, from the first on flipping the bit after the first one until you reach the far left:
   - $1100\ 0001\ 00\underline{0}1\ 00...00$
   - $1100\ 0001\ 0\underline{1}01\ 00...00$
   - $1100\ 0001\ \underline{1}101\ 00...00$
   - $1100\ 000\underline{0}\ 1101\ 00...00$
   - $1100\ 00\underline{1}0\ 1101\ 00...00$
   - $1100\ 0\underline{1}10\ 1101\ 00...00$
   - $1100\ \underline{1}110\ 1101\ 00...00$
   - $110\underline{1}\ 1110\ 1101\ 00...00$
   - $11\underline{1}1\ 1110\ 1101\ 00...00$
   - $1\underline{0}11\ 1110\ 1101\ 00...00$
   - $\underline{0}011\ 1110\ 1101\ 00...00$
4. Then add all the base two 1's together to get the integer:
   - $2^{29} + 2^{28} + 2^{27} + 2^{26} + 2^{25} + 2^{23} + 2^{22} + 2^{20} = 1053818880$
5. And multiply it by the sign, which is -1:
   - -1(1053818880) = -1053818880

(b) an unsigned integer;

As an unsigned integer, add the base two ones together:
$2^{31} + 2^{30} + 2^{24} + 2^{21} + 2^{20} = 3241148416$

(c) a single precision IEEE 754 floating-point number.

$(-1)^S \times (1 + F) \times 2^E$
S = Sign
F = fraction (0 < F <1)
E = Exponent - Bias
Bias = 127 for single precision

If the first character of the word is a one then the sign is negative:
S = 1

The next eight bits make up the exponent:
E = 100 0001 0 = $2^7 + 2^1$ = 128 + 2 = 130 - Bias = 130 - 127 = 3

F = 011 0000 0000 0000 0000 0000
Going from left to right, with left as the zero point and going into the negative:
$2^{-2} + 2^{-3}$ = 0.25 + 0.125 = 0.375

$(-1)^1 \times (1 + 0.375) \times 2^3$
$-1 \times 1.375 \times 8$
The answer is -11

6. **(6%)** Find computer representations of the following numbers:

(a) -1022 as a two's complement 32-bit binary number;

First turn it into a positive binary number:
S = -1
N = 1022

$1022/2 = 511 \rightarrow 0$
$511/2 = 255.5 \rightarrow 1$
$255/2 = 127.5 \rightarrow 1$
$127/2 = 63.5 \rightarrow 1$
$63/2 = 31.5 \rightarrow 1$
$31/2 = 15.5 \rightarrow 1$
$15/2 = 7.5 \rightarrow 1$
$7/2 = 3.5 \rightarrow 1$
$3/2 = 1.5 \rightarrow 1$
$1/2 = 0.5 \rightarrow 1$

Now we have 0...0011 1111 1110
Since it needs to be a two's compliment number we need to use the algorithm from before to flip the bits.
Which becomes: 1...1100 0000 0010 Since it is a negative number we leave the first character as a one:

Answer = 1111 1111 1111 1111 1111 1100 0000 0010

(b) -32.75 as an IEEE 754 32-bit floating-point number;

S	E (Exponent)	F (Fraction)
1bit	8 bits	23 bits

Bias = 127
S = 1 since it is a negative number.

32.75
32 = 10000 in binary
$0.75 \times 2 = 1.5 \rightarrow 1$
$0.5 \times 2 = 1 \rightarrow 1$
1.1

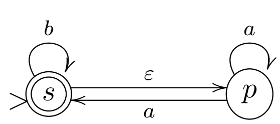
100001.1 $1.000011 \times 2^5 = 100001.1$
E = 5 + 127 = 132 = 10000100
F = 000011

1	100 0010 0	000 0110 0000 0000 0000 0000
S	E (Exponent)	F (Fraction)

(c) 77 as a two's complement 32-bit binary number.

$77/2 = 38.5 \rightarrow 1$
$38/2 = 19 \rightarrow 0$
$19/2 = 9.5 \rightarrow 1$
$9/2 = 4.5 \rightarrow 1$
$4/2 = 2 \rightarrow 0$
$2/2 = 1 \rightarrow 0$
$1/2 = 0.5 \rightarrow 1$
Answer = 0000 0000 0000 0000 0000 0000 0100 1101

7. **(6%)** Consider the following finite automaton:



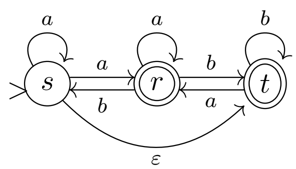
(a) Give all the computations of the automaton on the input strings aab, aba, and ε, and determine if the strings are accepted.

(b) Describe the language accepted by the automaton in English.

(c) Describe the language accepted by the automaton by means of a regular expression.

(d) Describe the language accepted by the automaton by means of a context-free grammar.

8. **(10%)** Transform, using the subset construction, the following nondeterministic finite automaton into an equivalent deterministic finite automaton. Show your working.



What is the language of this automaton.

9. **(7%)** Design a (deterministic or nondeterministic) finite automaton A such that $L(A)$ consists of all strings over the alphabet $\{0, 1\}$ that begin with 01 and do not end with 11. Find a regular expression representing the language $L(A)$.

10. **(6%)** Design a (deterministic or nondeterministic) finite automaton A such that $L(A)$ consists of all strings over the alphabet $\{0, 1\}$ whose third symbol from the right end is 1 (for example, 100101 is in $L(A)$, but 100011 is not). Find a regular expression representing $L(A)$.

11. **(8%)** Convert the regular language $L\big[\,x\big((y \cup x)^*x\big)^*y\,\big]$ to a finite automaton accepting it.

12. **(4%)** Consider the following context free grammar:

$$S \to SS, \quad S \to L0L0L, \qquad L \to \varepsilon, \quad L \to 1L, \quad L \to 0L.$$

(a) Give a derivation for the string 101101.

(b) Describe in English the language of this grammar.

13. **(6%)** Construct context free grammars for the following languages

$$\begin{array}{ll} (a) & \{w \in \{0,1\}^* \mid w \text{ starts and ends with different symbols}\}, \\ (b) & \{w \in \{0,1\}^* \mid \text{ the length of } w \text{ is even}\} \end{array}$$

14. **(15%)** Construct a context free grammar and a pushdown automaton for the language of words over the alphabet $\{0, 1\}$ that start and end with the same symbol and have the same number of 0s as 1s.

15. **(5%)** Consider the following transition table of a Turing machine:

s	0	s	$\rightarrow$
s	1	s	$\rightarrow$
s	$\sqcup$	p	$\leftarrow$
s	$\triangleright$	s	$\rightarrow$
p	0	h	1
p	1	p	$\rightarrow$
p	$\sqcup$	h	0
p	$\triangleright$	s	$\rightarrow$

(i) Give the computations of the machine starting with the configurations

- $(s, \triangleright\underline{0})$,
- $(s, \triangleright\underline{1}11)$,
- $(s, \triangleright\underline{1}00)$.

(ii) Describe in English what this Turing machine does.

16. **(9%)** Consider the following $\mathbb{N} \to \mathbb{N}$ function f:

$$f(n) = \begin{cases} 4n & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$$

(Don't forget that all numbers are represented in binary.)

(i) Explain what it means to say that a Turing machine *computes* this function f.

(ii) Give an implementation level description in English of a Turing machine that computes this f.

(iii) Give the complete transition table of this Turing machine.

(iv) Give the computations of your Turing machine on inputs 0, 11 and 100.