For return on 24 January 2014 (late submission: 7 February 2014)

Electronic submission: **pdf** *files only*

1. **(3%)** Construct the truth-table for the Boolean function given by the Boolean formula

$$\neg\big(A \wedge (B \to C)\big) \wedge \neg B.$$

Use the truth-table to realise this function by a formula with the connectives $\neg$, $\vee$, $\wedge$ only. Simplify the formula in such a way that the corresponding Boolean circuit contains a minimal number of gates. Show the Boolean circuit.

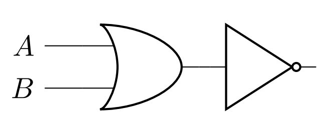
Truth Table

A	B	C	$\neg$	$(A$	$\wedge$	$(B$	$\to$	$C))$	$\wedge$	$\neg B)$
0	0	0	1		0		1		1	1
0	0	1	1		0		1		1	1
0	1	0	1		0		0		0	0
0	1	1	1		0		1		0	0
1	0	0	0		1		1		0	1
1	0	1	0		1		1		0	1
1	1	0	1		0		0		0	0
1	1	1	0		1		1		0	0

Simplification

1. $(\neg A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge \neg B \wedge C)$
2. $(\neg(A \vee B) \wedge \neg C) \vee (\neg(A \vee B) \wedge C)$
3. $\neg(A \vee B) \vee (\neg C \wedge C)$
4. $\neg(A \vee B) \vee 0$
5. $\neg(A \vee B)$

Boolean Circuit



2. **(3%)** Are the following Boolean formulas equivalent? Explain your answer.

(a) $A \to (B \land C)$ and $(A \to B) \land (A \to C)$

A	B	C	A	$\to$	$(B$	$\land$	$C)$
0	0	0		1		0	
0	0	1		1		0	
0	1	0		1		0	
0	1	1		1		1	
1	0	0		0		0	
1	0	1		0		0	
1	1	0		0		0	
1	1	1		1		1	

A	B	C	$(A$	$\to$	$B)$	$\land$	$(A$	$\to$	$C)$
0	0	0		1		1		1	
0	0	1		1		1		1	
0	1	0		1		1		1	
0	1	1		1		1		1	
1	0	0		0		0		0	
1	0	1		0		0		1	
1	1	0		1		0		0	
1	1	1		1		1		1	

Yes they are equivalent

(b) $(A \land B) \to C$ and $(\neg C \to \neg A) \land (\neg C \to \neg B)$

A	B	C	$(A$	$\land$	$B)$	$\to$	C
0	0	0		0		1	
0	0	1		0		1	
0	1	0		0		1	
0	1	1		0		1	
1	0	0		0		1	
1	0	1		0		1	
1	1	0		1		0	
1	1	1		1		1	

A	B	C	$(\neg C$	$\to$	$\neg A)$	$\land$	$(\neg C$	$\to$	$\neg B)$
0	0	0	1	1	1	1	1	1	1
0	0	1	0	1	1	1	0	1	1
0	1	0	1	1	1	0	1	0	0
0	1	1	0	1	1	1	0	1	0
1	0	0	1	0	0	0	1	1	1
1	0	1	0	1	0	1	0	1	1
1	1	0	1	0	0	0	1	0	0
1	1	1	0	1	0	1	0	1	0

No they are not equivalent

(c) $(A \lor B) \to C$ and $(A \to B) \lor (A \to C)$

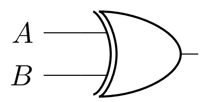
A	B	C	$(A$	$\lor$	$B)$	$\to$	C
0	0	0		0		1	
0	0	1		0		1	
0	1	0		1		0	
0	1	1		1		1	
1	0	0		1		0	
1	0	1		1		1	
1	1	0		1		0	
1	1	1		1		1	

A	B	C	$(A$	$\to$	$B)$	$\lor$	$(A$	$\to$	$C)$
0	0	0		1		1		1	
0	0	1		1		1		1	
0	1	0		1		1		1	
0	1	1		1		1		1	
1	0	0		0		0		0	
1	0	1		0		1		1	
1	1	0		1		1		0	
1	1	1		1		1		1	

No they are not equivalent

3. **(3%)** A parity function is a Boolean function whose value is 1 if the input has an odd number of ones. Design a Boolean circuit for the 2-bit parity function. Show your working. (Hint: you may find XOR gates useful.)

A	B	
0	0	0
0	1	1
1	0	1
1	1	0



4. **(3%)** Suppose $\alpha = a_{31}a_{30}\ldots a_1a_0$ is a 32-bit binary word. Consider the 32-bit binary word $\beta = b_{31}b_{30}\ldots b_1b_0$ computed by the following algorithm: scan α from right to left and copy its bits to β until the first 1 is found (which is also copied to β); after that, copy the Boolean negations of the bits in α. For example, $\alpha = 10100\ldots00$ is transformed to $\beta = 01100\ldots00$. Explain what this algorithm computes if α and β are interpreted as binary numbers.

Given a two's compliment binary number, the algorithm computes the conversion into a binary number.

5. **(6%)** Given the machine 32-bit word

1100 0001 0011 0000 0000 0000 0000 0000

find the decimal number represented by this word assuming that it is

(a) a two's complement integer;

1. Read the first character from the left of the word to get the sign of the integer it represents. In this case it is a one so the number is negative.
2. Use the two's compliment algorithm to convert it into a binary number. Starting from the right of the word, move to the first one :
   - 1100 0001 001$\underline{1}$ 00...00
3. Move left, from the first on flipping the bit after the first one until you reach the far left:
   - 1100 0001 00$\underline{0}$1 00...00
   - 1100 0001 0$\underline{1}$01 00...00
   - 1100 0001 $\underline{1}$101 00...00
   - 1100 000$\underline{0}$ 1101 00...00
   - 1100 00$\underline{1}$0 1101 00...00
   - 1100 0$\underline{1}$10 1101 00...00
   - 1100 $\underline{1}$110 1101 00...00
   - 110$\underline{1}$ 1110 1101 00...00
   - 11$\underline{1}$1 1110 1101 00...00
   - 1$\underline{0}$11 1110 1101 00...00
   - $\underline{0}$011 1110 1101 00...00
4. Then add all the base two 1's together to get the integer:
   - $2^{29} + 2^{28} + 2^{27} + 2^{26} + 2^{25} + 2^{23} + 2^{22} + 2^{20} = 1053818880$
5. And multiply it by the sign, which is -1:
   - -1(1053818880) = -1053818880

(b) an unsigned integer;

As an unsigned integer, add the base two ones together:
$2^{31} + 2^{30} + 2^{24} + 2^{21} + 2^{20} = 3241148416$

(c) a single precision IEEE 754 floating-point number.

$(-1)^S \times (1 + F) \times 2^E$
S = Sign
F = fraction (0 < F <1)
E = Exponent - Bias
Bias = 127 for single precision

If the first character of the word is a one then the sign is negative:
S = 1

The next eight bits make up the exponent:
E = 100 0001 0 = $2^7 + 2^1$ = 128 + 2 = 130 - Bias = 130 - 127 = 3

F = 011 0000 0000 0000 0000 0000
Going from left to right, with left as the zero point and going into the negative:
$2^{-2} + 2^{-3} = 0.25 + 0.125 = 0.375$

$(-1)^1 \times (1 + 0.375) \times 2^3$
$-1 \times 1.375 \times 8$
The answer is -11

6. **(6%)** Find computer representations of the following numbers:

(a) -1022 as a two's complement 32-bit binary number;

First turn it into a positive binary number:
S = -1
N = 1022

$1022/2 = 511 \rightarrow 0$
$511/2 = 255.5 \rightarrow 1$
$255/2 = 127.5 \rightarrow 1$
$127/2 = 63.5 \rightarrow 1$
$63/2 = 31.5 \rightarrow 1$
$31/2 = 15.5 \rightarrow 1$
$15/2 = 7.5 \rightarrow 1$
$7/2 = 3.5 \rightarrow 1$
$3/2 = 1.5 \rightarrow 1$
$1/2 = 0.5 \rightarrow 1$

Now we have 0...0011 1111 1110
Since it needs to be a two's compliment number we need to use the algorithm from before to flip the bits.
Which becomes: 1...1100 0000 0010 Since it is a negative number we leave the first character as a one:

Answer = 1111 1111 1111 1111 1111 1100 0000 0010

(b) -32.75 as an IEEE 754 32-bit floating-point number;

S	E (Exponent)	F (Fraction)
1bit	8 bits	23 bits

Bias = 127
S = 1 since it is a negative number.

32.75
32 = 10000 in binary
$0.75 \times 2 = 1.5 \rightarrow 1$
$0.5 \times 2 = 1 \rightarrow 1$
1.1

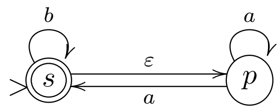
100001.1 $1.000011 \times 2^5 = 100001.1$
E = 5 + 127 = 132 = 10000100
F = 000011

1	100 0010 0	000 0110 0000 0000 0000 0000
S	E (Exponent)	F (Fraction)

(c) 77 as a two's complement 32-bit binary number.

$77/2 = 38.5 \rightarrow 1$
$38/2 = 19 \rightarrow 0$
$19/2 = 9.5 \rightarrow 1$
$9/2 = 4.5 \rightarrow 1$
$4/2 = 2 \rightarrow 0$
$2/2 = 1 \rightarrow 0$
$1/2 = 0.5 \rightarrow 1$
Answer = 0000 0000 0000 0000 0000 0000 0100 1101

7. **(6%)** Consider the following finite automaton:



(a) Give all the computations of the automaton on the input strings *aab*, *aba*, and ε, and determine if the strings are accepted.

> Input: *aab*
> Computation:
> · (s, aab), (p, aab), (p, ab), (s, b), (s, ε) Accepted
> · (s, aab), (p, aab), (s, ab), (p, ab), (s, b), (s, ε) Accepted
> · (s, aab), (p, aab), (p, ab), (p, b) stuck
> · (s, aab), (p, aab), (p, ab), (s, b), (p, b) stuck
> · (s, aab), (p, aab), (s, ab), (p, ab), (s, b), (s, ε) Accepted
>
> word *aab* is Accepted.
>
> Input: *aba*
> Computation:
> · (s, aba), (p, aba), (p, ba) stuck
> · (s, aba), (p, aba), (s, ba), (s, a), (p, a), (p, ε) stuck
> · (s, aba), (p, aba), (s, ba), (s, a), (p, a), (s, ε) Accepted
> · (s, aba), (p, aba), (s, ba), (p, ba) stuck
>
> word *aba* is Accepted.
>
> Input: ε
> Computation:
> · (s, ε), (p, ε) stuck
> · (s, ε) Accepted
>
> word ε is Accepted.

(b) Describe the language accepted by the automaton in English.

> The language accepts any amount of a's and b's and the empty word.

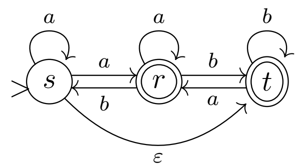
(c) Describe the language accepted by the automaton by means of a regular expression.

> (a ∪ b)*

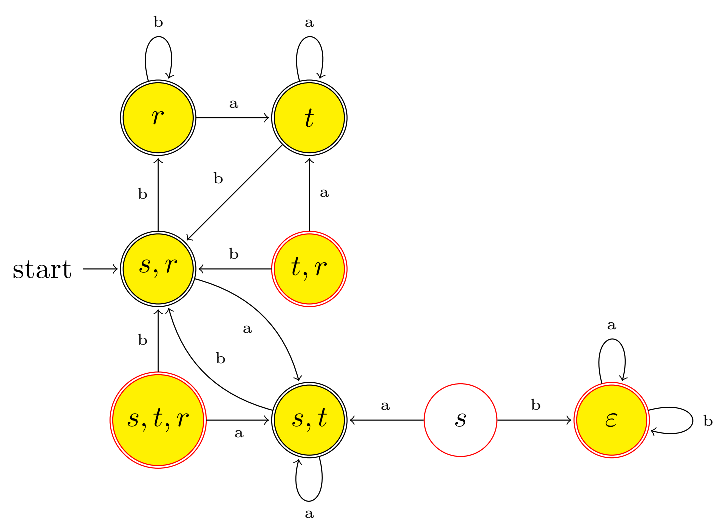
(d) Describe the language accepted by the automaton by means of a context-free grammar.

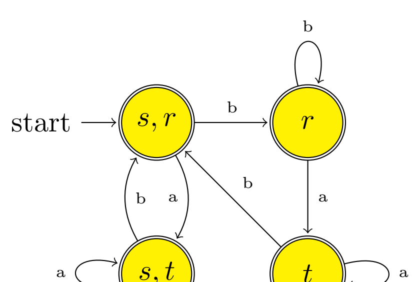
$$S \to \varepsilon$$
$$S \to aS$$
$$S \to bS$$

8. **(10%)** Transform, using the subset construction, the following nondeterministic finite automaton into an equivalent deterministic finite automaton. Show your working.



	A	B
s	s,t	ε
t	t	s,r
r	t	r
s,t	s,t	s,r
s,r	s,t	r
t,r	t	s,r
s,t,r	s,t	s,r
ε	ε	ε

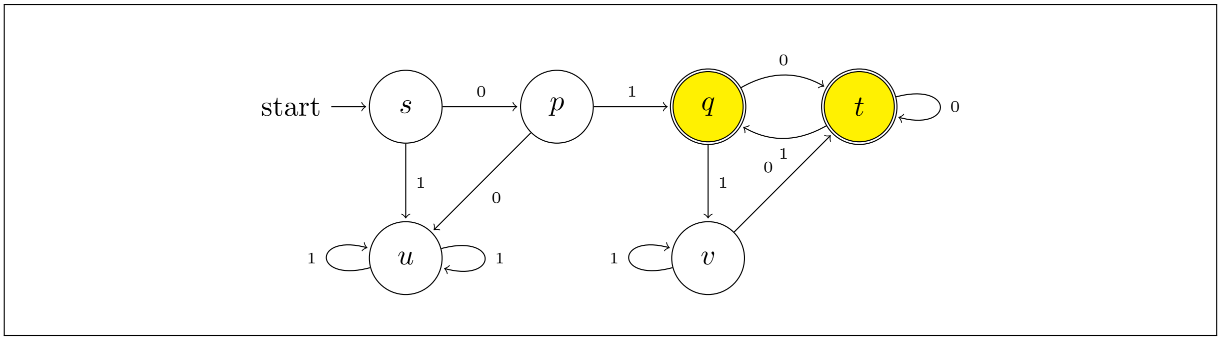




What is the language of this automaton.

This language accepts any number of a's and b's.

9. **(7%)** Design a (deterministic or nondeterministic) finite automaton A such that $L(A)$ consists of all strings over the alphabet $\{0,1\}$ that begin with 01 and do not end with 11. Find a regular expression representing the language $L(A)$.



01((0 ∪ 1)*(0 ∪ 01))*

10. **(6%)** Design a (deterministic or nondeterministic) finite automaton A such that $L(A)$ consists of all strings over the alphabet $\{0,1\}$ whose third symbol from the right end is 1 (for example, 100101 is in $L(A)$, but 100011 is not). Find a regular expression representing $L(A)$.

11. **(8%)** Convert the regular language $L\big[\,x\big((y \cup x)^{*}x\big)^{*}y\,\big]$ to a finite automaton accepting it.

12. **(4%)** Consider the following context free grammar:

$$S \to SS, \quad S \to L0L0L, \qquad L \to \varepsilon, \quad L \to 1L, \quad L \to 0L.$$

(a) Give a derivation for the string 101101.

(b) Describe in English the language of this grammar.

13. **(6%)** Construct context free grammars for the following languages

(a) $\{w \in \{0,1\}^{*} \mid w \text{ starts and ends with different symbols}\}$,

(b) $\{w \in \{0,1\}^{*} \mid \text{ the length of } w \text{ is even}\}$

14. **(15%)** Construct a context free grammar and a pushdown automaton for the language of words over the alphabet $\{0,1\}$ that start and end with the same symbol and have the same number of 0s as 1s.

15. **(5%)** Consider the following transition table of a Turing machine:

s	0	s	$\to$
s	1	s	$\to$
s	$\sqcup$	p	$\leftarrow$
s	$\triangleright$	s	$\to$
p	0	h	1
p	1	p	$\to$
p	$\sqcup$	h	0
p	$\triangleright$	s	$\to$

(i) Give the computations of the machine starting with the configurations

– $(s, \triangleright\underline{0})$,

– $(s, \triangleright\underline{1}11)$,

– $(s, \triangleright\underline{1}00)$.

(ii) Describe in English what this Turing machine does.

16. **(9%)** Consider the following $\mathbb{N} \to \mathbb{N}$ function f:

$$f(n) = \begin{cases} 4n & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$$

(Don't forget that all numbers are represented in binary.)

(i) Explain what it means to say that a Turing machine *computes* this function f.

(ii) Give an implementation level description in English of a Turing machine that computes this f.

(iii) Give the complete transition table of this Turing machine.

(iv) Give the computations of your Turing machine on inputs 0, 11 and 100.