Center of Mass Torque

In this proof, I seek to demonstrate that the net torque exerted by an object can be accurately expressed as simply the cross product of the position vector which goes from the point of rotation to the center of mass of the object and the weight vector of the object. In essence, I wish to show that there is no complex integration necessary to account for any oddities in the shape of the object.

Component of Cross Product Perpendicular to Plane

Given $\vec{a},\ \vec{b},\ \vec{c}\ \in \mathbb{R}^3$ and W is a plane in $\mathbb{R}^3$:

$$\vec{a} \times \vec{b} = \vec{c}$$
$$\vec{c} = \text{proj}_W \vec{c} + \vec{c}_\perp$$

$$\begin{aligned}\vec{a} \times \vec{b} &= (\text{proj}_W \vec{a} + \vec{a}_\perp) \times \vec{b} \\ &= \text{proj}_W \vec{a} \times \vec{b} + \vec{a}_\perp \times \vec{b} \\ &= \text{proj}_W \vec{a} \times (\text{proj}_W \vec{b} + \vec{b}_\perp) + \vec{a}_\perp \times (\text{proj}_W \vec{b} + \vec{b}_\perp) \\ &= \text{proj}_W \vec{a} \times \text{proj}_W \vec{b} + \text{proj}_W \vec{a} \times \vec{b}_\perp + \vec{a}_\perp \times \text{proj}_W \vec{b} + \vec{a}_\perp \times \vec{b}_\perp\end{aligned}$$
$$\vec{a}_\perp \times \vec{b}_\perp = \vec{0}$$

Since $\vec{b}_\perp \parallel \vec{c}_\perp,\ (\text{proj}_W \vec{a} \times \vec{b}_\perp) \perp \vec{c}_\perp$

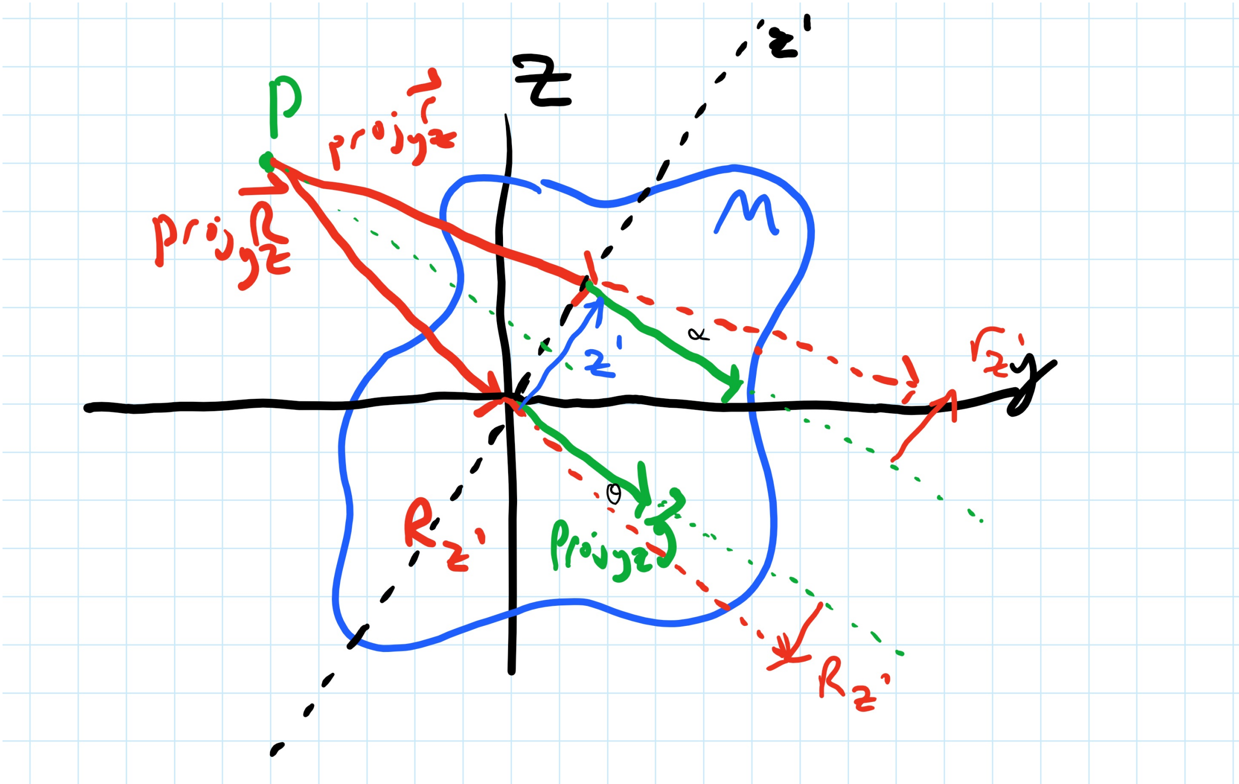
Since $\vec{b}_\perp \parallel \vec{c}_\perp,\ (\vec{a}_\perp \times \text{proj}_W \vec{b}) \perp \vec{c}_\perp$

$$\Rightarrow \text{proj}_W \vec{a} \times \text{proj}_W \vec{b} = \vec{c}_\perp \tag{1}$$

Finding the Torque due to Weight

Let $\vec{\tau}$ be the net torque exerted about a point P by the weight of an object, $\vec{F}$, with the center of mass (cm) located at the origin of our axes such that R is the position vector from P to the cm

$$\begin{aligned}\text{Then: } \vec{\tau} &= \vec{\tau}_x + \vec{\tau}_y + \vec{\tau}_z \\ \vec{R} &= \vec{R}_x + \vec{R}_y + \vec{R}_z\end{aligned} \tag{2}$$



Finding $\vec{\tau}_x$:

By (1) and (2):

$$d\tau_x = \text{proj}_{yz} r d(\text{proj}_{yz} F) \sin\alpha, \qquad \text{proj}_{yz} r \sin\alpha = r_{z'},\ d(\text{proj}_{yz} F) = \text{proj}_{yz} g dm$$

$$d\tau_x = r_{z'} \text{proj}_{yz} g dm$$

$$\int d\tau_x = \int r_{z'} \text{proj}_{yz} g dm$$

$$\tau_x = \text{proj}_{yz} g \int r_{z'} dm \qquad r_{z'} = R_{z'} + z'$$

$$= \text{proj}_{yz} g \int (R_{z'} + z') dm$$

$$= \text{proj}_{yz} g \int R_{z'} dm + \int z' dm \qquad \text{Since } (0,0,0) = \text{cm},\ \int z' dm = 0$$

$$= \text{proj}_{yz} g R_{z'} \int dm$$

$$= \text{proj}_{yz} g R_{z'} M \qquad R_{z'} = \text{proj}_{yz} R \sin\theta$$

$$\tau_x = (\text{proj}_{yz} g)(\text{proj}_{yz} R) M \sin\theta$$

$$\Rightarrow \vec{\tau}_x = \text{proj}_{yz} \vec{R} \times \text{proj}_{yz} \vec{F}$$

By repeating this process, we can find:

$$\begin{aligned}
\vec{\tau}_x &= \mathrm{proj}_{yz}\vec{R} \times \mathrm{proj}_{yz}\vec{F} \\
\vec{\tau}_y &= \mathrm{proj}_{xz}\vec{R} \times \mathrm{proj}_{xz}\vec{F} \\
\vec{\tau}_z &= \mathrm{proj}_{xy}\vec{R} \times \mathrm{proj}_{xy}\vec{F}
\end{aligned} \tag{3}$$

Combining (2) and (3):

$$\vec{\tau} = \mathrm{proj}_{yz}\vec{R} \times \mathrm{proj}_{yz}\vec{F} + \mathrm{proj}_{xz}\vec{R} \times \mathrm{proj}_{xz}\vec{F} + \mathrm{proj}_{xy}\vec{R} \times \mathrm{proj}_{xy}\vec{F} \tag{4}$$

Finding $\vec{R} \times \vec{F}$ **:**

$$\vec{R} \times \vec{F} = (\vec{R} \times \vec{F})_x + (\vec{R} \times \vec{F})_y + (\vec{R} \times \vec{F})_z$$

By (1):

$$\vec{R} \times \vec{F} = (\mathrm{proj}_{yz}\vec{R} \times \mathrm{proj}_{yz}\vec{F}) + (\mathrm{proj}_{xz}\vec{R} \times \mathrm{proj}_{xz}\vec{F}) + (\mathrm{proj}_{xy}\vec{R} \times \mathrm{proj}_{xy}\vec{F})$$

$$\vec{R} \times \vec{F} = \mathrm{proj}_{yz}\vec{R} \times \mathrm{proj}_{yz}\vec{F} + \mathrm{proj}_{xz}\vec{R} \times \mathrm{proj}_{xz}\vec{F} + \mathrm{proj}_{xy}\vec{R} \times \mathrm{proj}_{xy}\vec{F} \tag{5}$$

Conclusion

By (4) and (5), since:

$$\vec{\tau} = \mathrm{proj}_{yz}\vec{R} \times \mathrm{proj}_{yz}\vec{F} + \mathrm{proj}_{xz}\vec{R} \times \mathrm{proj}_{xz}\vec{F} + \mathrm{proj}_{xy}\vec{R} \times \mathrm{proj}_{xy}\vec{F}$$

And

$$\vec{R} \times \vec{F} = \mathrm{proj}_{yz}\vec{R} \times \mathrm{proj}_{yz}\vec{F} + \mathrm{proj}_{xz}\vec{R} \times \mathrm{proj}_{xz}\vec{F} + \mathrm{proj}_{xy}\vec{R} \times \mathrm{proj}_{xy}\vec{F},$$

$$\boxed{\vec{\tau} = \vec{R} \times \vec{F}} \tag{6}$$