Decay Proof

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$$
\begin{aligned}
t &= \text{Time} \\
t_o &= \text{The length of the selected time interval} \\
a, b &= \text{Two times such that } t_o = b - a \text{ and } a \geq 0 \\
N &= \text{Number of detected decays over the time interval from } t = a \text{ to } t = b \\
N_o &= \text{Number of detected decays over the first interval, from } t = 0 \text{ to } t_o \\
R &= \text{The instantaneous rate of detected decays at a time } t \\
R_o &= \text{The instantaneous rate of detected decays at time } t = 0 \\
\lambda &= \text{Decay constant}
\end{aligned}
$$

The goal of this proof is to find an exact expression (without approximations) relating the number of detected decays over a regular interval to the decay constant, and thus to the half-life.

First, a function for the number of detected decays over a single time interval will be found:

$$
\begin{aligned}
N &= \int_a^b R(t)dt \\
&= \int_a^b R_o e^{-\lambda t} dt \\
&\boxed{N = \frac{R_o}{\lambda}\left(e^{-\lambda a} - e^{-\lambda b}\right)}
\end{aligned}
$$

Next, the number of detected decays over the very first time interval will be found:

$$
\begin{aligned}
N_o &= \int_0^{t_o} R_o e^{-\lambda t} dt \\
&\boxed{N_o = \frac{R_o}{\lambda}\left(1 - e^{-\lambda t_o}\right)}
\end{aligned}
$$

Since the interval over which decays are counted remains constant, the following relationship exists:

$$\boxed{t_o = b - a}$$

Next, we will divide the number of detected decays over the interval from a to b by the number of detected decays over the first time interval, keeping in mind the above relationship and the laws of exponents:

$$\begin{aligned}
\frac{N}{N_o} &= \frac{\frac{R_o}{\lambda}\left(e^{-\lambda a} - e^{-\lambda b}\right)}{\frac{R_o}{\lambda}\left(1 - e^{-\lambda t_o}\right)} \\
&= \frac{\frac{1}{e^{\lambda a}} - \frac{1}{e^{\lambda b}}}{1 - \frac{1}{e^{\lambda t_o}}} \\
&= \left(\frac{\frac{1}{e^{\lambda a}} - \frac{1}{e^{\lambda b}}}{1 - \frac{1}{e^{\lambda t_o}}}\right)\left(\frac{e^{\lambda(b+a+t_o)}}{e^{\lambda(b+a+t_o)}}\right) \\
&= \frac{e^{\lambda(b+t_o)} - e^{\lambda(a+t_o)}}{e^{\lambda(b+a+t_o)} - e^{\lambda(b+a)}} \\
&= \left(\frac{e^{\lambda t_o}}{e^{\lambda t_o}}\right)\left(\frac{e^{\lambda b} - e^{\lambda a}}{e^{\lambda(b+a) - e^{2\lambda a}}}\right) \\
&= \frac{e^{\lambda b} - e^{\lambda a}}{e^{\lambda(b+a) - e^{2\lambda a}}} \\
&= \left(\frac{e^{\lambda a}}{e^{2\lambda a}}\right)\left(\frac{e^{\lambda t_o} - 1}{e^{\lambda t_o} - 1}\right) \\
&\boxed{\frac{N}{N_o} = e^{-\lambda a}}
\end{aligned}$$

Finally, this equation will be linearized by taking a natural logarithm:

$$\begin{aligned}
\frac{N}{N_o} &= e^{-\lambda a} \\
&\boxed{\ln\frac{N}{N_o} = -\lambda a}
\end{aligned}$$

Continued on next page.

Next, we will note that in the lab, we used the following equation:

$$\ln \frac{R}{R_o} = -\lambda t$$

Where the following relationships were true:

$$R = \frac{N}{\Delta t}$$
$$R_o = \frac{N_o}{\Delta t}$$

And t was specifically the time at the beginning of the interval; thus in our terms:

$$t = a$$

Thus, putting it all together:

$$\ln \frac{\frac{N}{\Delta t}}{\frac{N_o}{\Delta t}} = -\lambda a$$
$$\boxed{\ln \frac{N}{N_o} = -\lambda a}$$

Thus, the lab surprisingly used an exact method which was *not* an approximation, although it used a strange method to get there.