Homework 8: Convex Optimization

Due date: Friday April 15, 2022
See the course website for instructions and submission details.

1. ~~**ABC Investments.** ABC Inc. is considering several investment options. Each option has a minimum and maximum investment allowed (only if the option is chosen). These restrictions, along with the expected return are summarized in the following table (figures are in millions of dollars):~~

Do not solve Q1!!!

~~Option~~	~~Minimum investment~~	~~Maximum investment~~	~~Expected return (%)~~
~~1~~	~~3~~	~~27~~	~~13~~
~~2~~	~~2~~	~~12~~	~~9~~
~~3~~	~~9~~	~~35~~	~~17~~
~~4~~	~~5~~	~~15~~	~~10~~
~~5~~	~~12~~	~~46~~	~~22~~
~~6~~	~~4~~	~~18~~	~~12~~

~~Because of the high-risk nature of Option 5, company policy requires that the total amount invested in Option 5 be no more that the combined amount invested in Options 2, 4 and 6. In addition, if an investment is made in Option 3, it is required that at least a minimum investment be made in Option 6. ABC has $80 million to invest and obviously wants to maximize its total expected return on investment. Which options should ABC invest in, and how much should be invested?~~

2. **Heat pipe design.** A heated fluid at temperature T (degrees above ambient temperature) flows in a pipe with fixed length and circular cross section with radius r. A layer of insulation, with thickness w, surrounds the pipe to reduce heat loss through the pipe walls (w is much smaller than r). The design variables in this problem are T, r, and w.

The energy cost due to heat loss is roughly equal to $\alpha_1 Tr/w$. The cost of the pipe, which has a fixed wall thickness, is approximately proportional to the total material, i.e., it is given by $\alpha_2 r$. The cost of the insulation is also approximately proportional to the total insulation material, i.e., roughly $\alpha_3 rw$. The total cost is the sum of these three costs.

The heat flow down the pipe is entirely due to the flow of the fluid, which has a fixed velocity, i.e., it is given by $\alpha_4 Tr^2$. The constants α_i are all positive, as are the variables T, r, and w.

Now the problem: maximize the total heat flow down the pipe, subject to an upper limit $C_{\max}$ on total cost, and the constraints

$$T_{\min} \le T \le T_{\max}, \qquad r_{\min} \le r \le r_{\max} \qquad w_{\min} \le w \le w_{\max}, \qquad w \le 0.1r$$

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a) Express this problem as a geometric program, and convert it into a convex optimization problem. Recall that a generic geometric program has the following form

$$\begin{aligned}
\min_{x} \quad & \sum_j c_{j0} x_1^{\beta_{j01}} x_2^{\beta_{j02}} \cdots x_n^{\beta_{j0n}} \\
s.t. \quad & \sum_j c_{ji} x_1^{\beta_{ji1}} x_2^{\beta_{ji2}} \cdots x_n^{\beta_{ji3}} \le 1, \quad i = 1, \cdots, m \\
& \sum_j d_{jk} x_1^{\beta_{jk1}} x_2^{\beta_{jk2}} \cdots x_n^{\beta_{jk3}} = 1, \quad k = 1, \cdots, p \\
& x_q > 0, \quad q = 1, ..., n \qquad c_{ji} > 0, d_{jk} > 0, \quad \forall ji, jk
\end{aligned}$$

b) Consider a simple instance of this problem, where $C_{\max} = 500$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$. Also assume for simplicity that each variable has a lower bound of zero and no upper bound. Solve this problem using JuMP. Use the `Ipopt` solver and the command `@NLconstraint(...)` to specify nonlinear constraints such as log-sum-exp functions. What is the optimal T, r, and w?