Lotka Volterra Model Analysis

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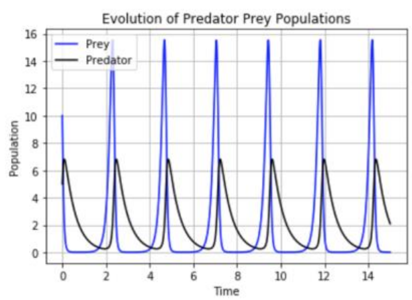
The Lotka-Volterra Predator Prey Model is widely used in the fields of Biology and Ecology to describe the population dynamics between predators and their prey. It is a simple model consisting of two differential equations, one of which maps the change of the population of prey with respects to time (denoted as x), the other which maps the change of the population of predator with respects to time (denoted as y).

$$\frac{dx}{dt} = \alpha x - \beta xy,$$
$$\frac{dy}{dt} = \delta xy - \gamma y,$$

The four parameters (alpha, beta, gamma, and delta) represent various rates, specific to individual species and populations, that help tailor the model for case by case usage. Alpha represents the growth rate of the species of the prey in the absence of predation, beta represents the death rate of our prey due to encounters with the predator, delta represents the rate of growth of the predator population in response to the relationship with the prey, and lastly gamma represents the natural death rate of the predator in the absence of the prey. Graphically, the population dynamics that are created from this system have the following periodic relationship. With the population of predator mirroring the population of prey, yet delayed slightly as it has to take time to interpret and respond to the prey population changes. This can be seen in the graph below.

1)



In order to better understand the dynamics of the predator prey relationship, we have chosen to derive solutions to the two differential equations stated above, as well as to construct the periodic graph shown in the paragraph above (1), the first of which will serve as the main mathematical model for this paper, and the second of which will serve as the main graphical model for this paper.

First, we must validate that a solution to these differential equations exists and is relevant. The solutions to the Lotka-Volterra differential equations can be derived algebraically through the process demonstrated below by first setting the rates of change with respects to time equal to zero for both the predator and the prey equation, and then solving for x and for y.

$$\begin{array}{ll} \frac{dx}{dt} = \alpha x - \beta xy & \frac{dy}{dt} = \delta xy - \gamma y \\ 0 = \alpha x - \beta xy & 0 = \delta xy - \gamma y \\ 0 = x(\alpha - \beta y) & 0 = y(\delta x - \gamma) \\ 0 = \alpha - \beta y & 0 = \delta x - \gamma \\ x = 0 \quad y = \frac{\alpha}{\beta} & y = 0 \quad x = \frac{\gamma}{\delta} \end{array}$$

Where

$$\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) \quad (0,0)$$

Will be the two solutions to our differential equations

The latter of the two solutions "(0,0)" being the state at which our populations enter complete extinction, thus they both will not be changing with respects to time, and the former representing a non-trivial steady state equilibrium point that will be of interest to us throughout this paper. At this point both our populations will be stable at the same time, thus their rates of change with respects to time will both equal zero.

Despite a simple algebraic solution to this problem existing, we can however, still use numerical methods to derive these same solutions, and do so with high degrees of success and accuracy. However, before we do so, we must first establish the validity of the methods that we plan on implementing. In order to this, we begin with the following.

We start by doing some simple mathematical rearrangement.

$$dt = \frac{dx}{\alpha x - \beta xy} \quad dt = \frac{dy}{\delta xy - \gamma y}$$

$$\frac{dx}{\alpha x - \beta xy} = \frac{dy}{\delta xy - \gamma y}$$
$$dx(\delta xy - \gamma y) = dy(\alpha x - \beta xy)$$

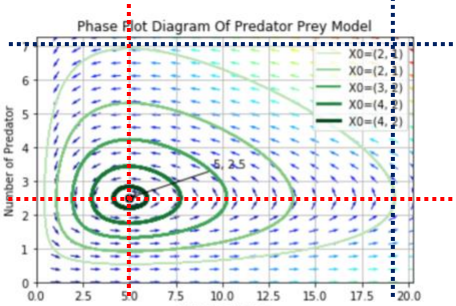
Then using integration, we are able to find "V" our constant of motion for our system of differential equations.

$$0 = \int \frac{\delta x - \gamma}{x} dx + \int \frac{\beta y - \alpha}{y} dy$$
$$0 = \delta x - \gamma \ln(x) + \beta y - \alpha \ln(y)$$
$$V = \delta x - \gamma \ln(x) + \beta y - \alpha \ln(y)$$

It is from this V, that we are able to have validation of the convergence of the Fixed-Point Iteration Method that we will be using later on in this paper. We have this result due to the equation V leading to the creation of the orbital shape as seen in phase space plot diagram shown below (2).

To observe this further, we can create a new equation P, where P = exp(V). Our solution will lie in the center of this orbital shape and thus its existence is validated. This can be seen visually by observing a phase space plot diagram of our system, and noticing that our solution (the black dot) lies at the point where our partial derivatives of the x and y directions are zero, in other words where our plot "turns". The blue lines represent dP/dx and dP/dy equaling zero, the red lines show intersection at the solution.

2)



Thus, from this observation we are able to conclude that the first derivative of P, when solved for with respects to “x” and “y”, will yield the non-trivial steady state solution of our model, so the existence of our solution and justification of the use of our numerical iterative method is verified. From this we can conclude that Fixed Point Iteration will converge to one of our two solutions listed previously, dependent on the starting values that they are given for the predator and prey populations “x” and “y”.

It is also important to note the uniqueness of our non-trivial solution point (the black dot). At this location, we are unable to see the fluctuations of predator and prey that we see on the graphical model from page one (1), since both these rates are equal to zero. Thus, we would not be able to observe this point in the real world, or on our periodic graph from earlier (1), since there is no way to ensure that both populations will cease to change simultaneously and do so indefinitely.

Since we are using iterative methods with two variables, the process becomes a bit trickier than would be in one variable. For the Fixed-Point method, we are able to set up the equation as normal, rearranging the first derivative of our equation P with respects to “x” and “y” such that we result in

$$x_{n+1} = \left(\frac{\gamma x_n^{\gamma-1}}{\delta}\right)^{1/\gamma} \qquad y_{n+1} = \left(\frac{\alpha y_n^{\alpha-1}}{\beta}\right)^{1/\alpha}$$

We then proceeded to do Fixed-Point Iteration on these equations simultaneously, and eventually approached our non-trivial solution.

Now we must verify the use of the graphical method we wish to use, which is Natural Cubic Spline Interpolation. This is less rigorous, since Cubic Splines allow for the most “natural” looking fit to data, and handle data the cleanest with regards to the points that lie in between our known data values, it is the obvious choice. There are no conditions that need be satisfied in order to successfully use this method, aside from sufficiency of data points, and that the data behaves as a function, which ours does with respects to time, so we know that it will work.

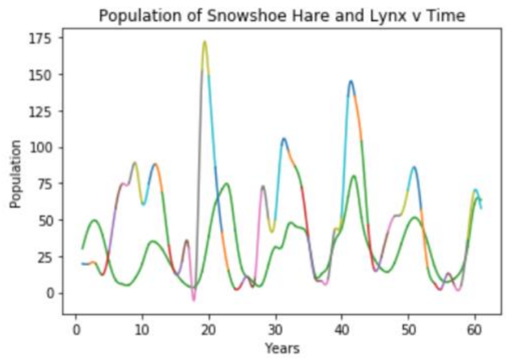
Now that we have validated the use of our numerical method (Fixed-Point Iteration) as well as our graphical method (Natural Cubic Spline), we are ready to run our code and discuss the results.

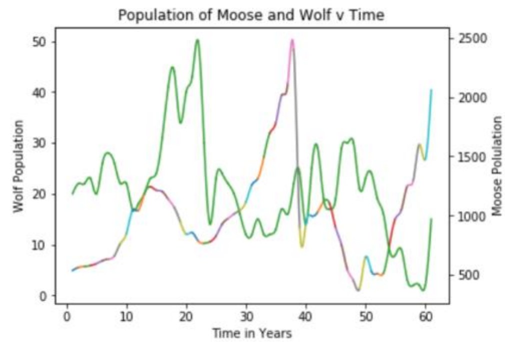
Fixed Point Results

```
In [1]: import numpy as np
        import matplotlib.pyplot as plt
        def FixedPointMethod(X,Y):
            a = 12
            B = 4
            g = 5
            d = 1
            arr = np.array([X,Y])
            acc= np.zeros((100,2))
            for i in range (100):
                Xn1 = ((g*(arr[0])**(g-1))/d)**(1/g)
                Yn1 = ((a*(arr[1])**(a-1))/B)**(1/a)
                arr = np.array([Xn1,Yn1])
                acc[i] = arr
            print ( "Our Fixed Point solution is", arr, "\nOur Iterations can be stored and viewed as\n", acc)
        FixedPointMethod(20,20)

          Our Fixed Point solution is [5.         3.00094717]
          Our Iterations can be stored and viewed as
```

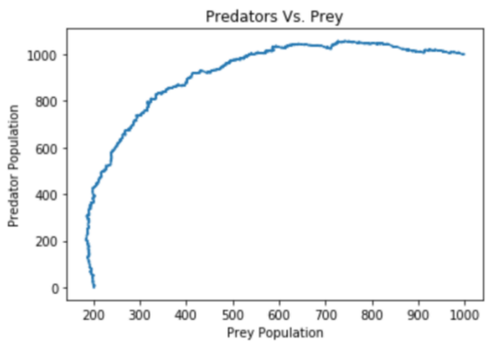
Now looking at the Natural Cubic Spline results for two separate predator prey relationships and population datasets. The leftmost being data from the Hudson Bay Company, inadvertently collected over the last few hundred years (here shown is only the years 1845-1905) due to the collection of hare and lynx pelts as a part of their business. The rightmost dataset is from a study conducted on Isle Royale on the moose and wolf population dynamics, as part of a scientific study that lasted multiple decades from the years 1959-2019.

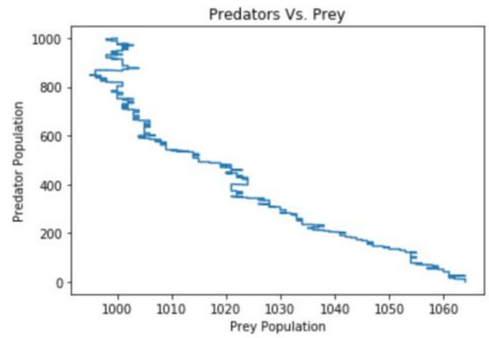




In each of these graphs, the predator population function is colored solid green while the prey population function is rainbow colored (each colored segment in the rainbow-colored function represents one of the 60 unique cubic polynomials that combine to make the cubic spline function). There are quite a few interesting things to note here, the first being how much more "accurate" or true to our expected result (as seen on page 1 graph (1)) our graph of the hare and lynx looks. Here we see the staggered predator population following the prey with a time delay, and remaining in a type of equilibrium that we would expect to see. Observing the graph of our moose and wolf, we do not see nearly the relationship we were hoping for, which brings up a good time to discuss the flaws of the model. It is unable to account for disease, excess predation, immigration and emigration, or any other sort of stimulus that might affect either the predator or prey populations aside from the parameters (alpha, beta, delta, gamma) that are being used. This makes for a pretty unrealistic model of the world, and after further research, our wolf population and moose populations both experienced extreme affects from disease that contributed to their population declines at separate times over the course of this study, at times when they should have been theoretically increasing in number, had only their relationship with the other been the only factor at play.

We now attempt to model this variation between what would be expected theoretically in the first graph of this paper (1) versus what is seen in practice with our Cubic Spline Graphs (shown above). We did this through the use of statistical analysis and by implementing Continuous Time Markov Chains. Below are the results of our process.





Here it can be seen that the populations do not move in a perfectly smooth fashion, as theoretically assumed in our graphical model and in the phase-space plot diagram. Rather, there is a stochastic element of variation that accounts for the imperfections that we see in the Cubic Spline Interpolation and in the real world, as these populations evolve and change over time.

Despite the fact that in application, we see pretty different results than we expected to see theoretically, there is still much usefulness behind these results. These results verify that there is a balance in our ecosystem, and while not the only factor at play, interspecies relationships are one of if not the main things to observe when attempting to preserve the ecosystem. In the disappearing environmental world that we live in today, it is important that we are aware of our impact on nature so that we are able to preserve it for generations to come. Hunting license restrictions, backpacking permits, construction permits, and many other things can be better implemented with a strong understanding of ecological population dynamics, and while the Lotka-Volterra Model is not perfect, it is a fantastic starting point.