Contents

1 Relating the Power Walk to the Random Surfer

1.1 Introduction

These are notes relating to [3, §3.3]

So if a term in the Power Walk can be related to α in the random surfer, which is in turn ξ_2, I'll be able to understand it better. [1]

Consider the equation:

$$\begin{aligned}\mathbf{T} &= \mathbf{B}\mathbf{D}_{\mathbf{B}}^{-1} \\ &= (\mathbf{B} + \mathbf{O} - \mathbf{O})\,\mathbf{D}_{\mathbf{B}}^{-1}\end{aligned}$$

Break this into to terms so that we can simplify it a bit:

$$\mathbf{T} = \left[(\mathbf{B} - \mathbf{O})\,\mathbf{D}_{\mathbf{B}}^{-1} \right] + \left\{ \mathbf{O}\mathbf{D}_{\mathbf{B}}^{-1} \right\}$$

1.2 Value of [1st Term]

Observe that for all $\forall i, j \in \mathbb{Z}^+$:

[1] Although I'm not quite sure why α is ξ_2 either

$$
\begin{aligned}
\mathbf{A}_{i,j} &\in \{0,1\} \\
\implies \mathbf{B}^{\mathbf{A}_{i,j}} &\in \left\{\beta^0, \beta^1\right\} \\
&= \{1, \beta\} \\
\implies \beta\mathbf{A} &= \{1, \beta\}
\end{aligned}
$$

Using this property we get the following

$$
\begin{aligned}
\mathbf{B}_{i,j} - \mathbf{O}_{i,j} = \left(\beta^{\mathbf{A}_{i,j}} - 1\right) &= \begin{cases} 0, & \mathbf{A}_{i,j} = 0 \\ \beta - 1, & \mathbf{A}_{i,j} = 1 \end{cases} \\
(\beta - 1)\, \mathbf{A}_{i,j} &= \begin{cases} 0, & \mathbf{A}_{i,j} = 0 \\ \beta - 1, & \mathbf{A}_{i,j} = 1 \end{cases}
\end{aligned}
$$

This means we have

$$
\mathbf{A} \in \{0,1\}\, \forall i, j \implies \mathbf{B}_{i,j} - \mathbf{O}_{i,j} = (\beta - 1)\, \mathbf{A}_{i,j}
$$

$$
\begin{aligned}
\mathbf{B} &= (\mathbf{B} + \mathbf{O} - \mathbf{O}) \\
&= (\mathbf{B} - 1)
\end{aligned}
$$

1.3 Value of {2nd Term}

$$
\begin{aligned}
\mathbf{O}\mathbf{D}_{\mathbf{B}}^{-\mathbf{1}} &= \begin{pmatrix} 1 & 1 & 1 & \\ 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & \\ & \vdots & & \ddots \end{pmatrix} \begin{pmatrix} \frac{1}{\delta_1} & 1 & 1 & \\ 1 & \frac{1}{\delta_2} & 1 \cdots & \\ 1 & 1 & \frac{1}{\delta_3} & \\ & \vdots & & \ddots \end{pmatrix} \\
&= n \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \\ & \vdots & & \ddots \end{pmatrix} \begin{pmatrix} \frac{1}{\delta_1} & 1 & 1 & \\ 1 & \frac{1}{\delta_2} & 1 & \cdots \\ 1 & 1 & \frac{1}{\delta_3} & \\ & \vdots & & \ddots \end{pmatrix} \\
&= n\mathbf{E}\mathbf{D}_{\mathbf{B}}{}^{-1}
\end{aligned}
$$

where the following definitions hold ($\forall i, j \in \mathbb{Z}^+$):

- $\mathbf{E}_{i,j} = \frac{1}{n}$
- $\mathbf{D_B}_{k,k}^{-1} = \frac{1}{\delta_k}$
- The value of δ is value that each term in a column must be divided by to become zero, in the case of the power walk that is just $\frac{1}{\texttt{colSums}(\mathbf{B})} = \vec{1}\mathbf{B}$, but if there were zeros in a column, it would be necessary to swap out the \$0\$s for \$1\$s and then sum in order to prevent a division by zero issue and because the 0s should be left.

- $\mathbf{A} \in \{0,1\} \, \forall i,j$ is the unweighted adjacency matrix of the relevant graph.

putting this all together we can do the following:

$$\begin{aligned}\mathbf{T} &= \mathbf{B}\mathbf{D}_{\mathbf{B}}^{-1} \\ &= (\mathbf{B}+\mathbf{O}-\mathbf{O})\,\mathbf{D}_{\mathbf{B}}^{-1} \\ &= (\mathbf{B}-\mathbf{O})\,\mathbf{D}_{B}^{-1} + \mathbf{O}\mathbf{D}_{\mathbf{B}}^{-1}\end{aligned}$$

From above:

$$\begin{aligned}&= (\beta-1)\,\mathbf{A}_{i,j} + n\mathbf{E}\mathbf{D}_{\mathbf{B}}^{-1} \\ &= \mathbf{A}_{i,j}\,(\beta-1) + n\mathbf{E}\mathbf{D}_{\mathbf{B}}^{-1}\end{aligned}$$

because $\mathbf{D}\mathbf{D}^{-1} = \mathbf{I}$ we can multiply one side through:

$$= \mathbf{D}_{\mathbf{A}}\mathbf{D}_{\mathbf{A}}^{-1}\mathbf{A}_{i,j}\,(\beta-1) + n\mathbf{E}\mathbf{D}_{\mathbf{B}}^{-1}$$

But the next step requires showing that:

$$(\beta-1)\,\mathbf{D}_{\mathbf{A}}\mathbf{D}_{\mathbf{B}}^{-1} = \mathbf{I} - n\mathbf{D}_{B}^{-1}$$

1.4 Equate the Power Walk to the Random Surfer

Define the matrix $\mathbf{D}_{\mathbf{M}}$:

$$\mathbf{D}_{\mathbf{M}} = \operatorname{diag}\left(\texttt{colSum}\,(\mathbf{M})\right) = \operatorname{diag}\left(\vec{1}\mathbf{M}\right) \tag{1}$$

To scale each column of that matrix to 1, each column will need to be divieded by the column sum, unless the column is already zero, this needs to be done to turn an adjacency matrix into a matrix of probabilities:

$$\mathbf{D}_{\mathbf{A}}^{-1} : \left[\mathbf{D}_{\mathbf{A}}^{-1}\right]_i = \begin{cases} 0, & [\mathbf{D}_{\mathbf{A}}]_i = 0 \\ \left[\frac{1}{\mathbf{D}_{\mathbf{A}}}\right], & [\mathbf{D}_{\mathbf{A}}]_i \neq 0 \end{cases} \tag{2}$$

In the case of the power walk $\mathbf{B} = \beta^{\mathbf{A}} \neq 0$ so it is sufficient:

$$\mathbf{D}_{\mathbf{B}}^{-1} = \frac{1}{\operatorname{diag}\left(\vec{1}\,(\beta^{\mathbf{A}})\right)} \tag{3}$$

Recall that the *power walk* gives a transition probability matrix:

Power Walk

$$\mathbf{T} = \boxed{\mathbf{A}\mathbf{D}_{\mathbf{A}}^{-1}} \mathbf{D}_{\mathbf{A}} \left(\beta - 1\right) \mathbf{D}_{\mathbf{B}}^{-1} + \boxed{\mathbf{E}} n \mathbf{D}_{\mathbf{B}}^{-1} \tag{4}$$

Random Surfer

$$\mathbf{T} = \alpha \boxed{\mathbf{A}\mathbf{D}_{\mathbf{A}}^{-1}} + \left(1 - \alpha\right) \boxed{\mathbf{E}} \tag{5}$$

So these are equivalent when:

$$\mathbf{D}_{\mathbf{A}} \left(\beta - 1\right) \mathbf{D}_{\mathbf{B}^{-1}} = \mathbf{I}\alpha \tag{6}$$

$$\begin{aligned} \vec{1} \left(1 - \alpha\right) &= -n \mathbf{D}_{\mathbf{B}}^{-1} \\ \implies \vec{1}\alpha &= \vec{1} - n \mathbf{D}_{\mathbf{B}}^{-1} \end{aligned} \tag{7}$$

Hence we have:

$$\mathbf{D}_{\mathbf{A}} \left(\beta - 1\right) \mathbf{D}_{\mathbf{B}}^{-1} = \vec{1}\alpha = \mathbf{I} - n \mathbf{D}_{\mathbf{B}}^{-1} \tag{8}$$

Solving for β with (6) :

$$\beta = \frac{1 - \Theta}{\Theta} \tag{9}$$

$$\tag{10}$$

where: [2]

- $\Theta = \mathbf{D}_{\mathbf{A}} \mathbf{D}_{\mathbf{B}}^{-1}$

but we can't really do this so instead:

$$\beta \mathbf{1}_{[n,n]} = \left(1 - \Theta\right) \Theta^{-1}$$

If β is set accordingly then by (8):

$$\begin{aligned} \mathbf{A} \left(\beta - 1\right) \mathbf{D}_{\mathbf{B}}^{-1} &= \alpha = \mathbf{I} - n \mathbf{D}_{\mathbf{B}}^{-1} \\ \implies \mathbf{A} \left(\beta - 1\right) \mathbf{D}_{\mathbf{B}}^{-1} &= \mathbf{I} - n \mathbf{D}_{\mathbf{B}}^{-1} \end{aligned} \tag{11}$$

And setting $\Gamma = \mathbf{I} - n \mathbf{D}_{\mathbf{B}}^{-1}$ from (7) and putting in (4) we have:

$$\begin{aligned} \mathbf{T} &= \boxed{\mathbf{A}\mathbf{D}_{\mathbf{A}}^{-1}} \mathbf{D}_{\mathbf{A}} \left(\beta - 1\right) \mathbf{D}_{\mathbf{B}}^{-1} + \boxed{\mathbf{E}} n \mathbf{D}_{\mathbf{B}}^{-1} \\ \mathbf{T} &= \Gamma \boxed{\mathbf{A}\mathbf{D}_{\mathbf{A}}^{-1}} + \left(1 - \Gamma\right) \boxed{\mathbf{E}} \end{aligned}$$

$$\mathbf{T} = \Gamma \mathbf{A} \mathbf{D}_{\mathbf{A}}^{-1} + \left(1 - \Gamma\right) \mathbf{E} \tag{12}$$

Where $\mathbf{E}$ is square matrix of $\frac{1}{n}$ as in (**??**) (**??**)

[2] NOTE: Similar to a signmoid function, which is a solution to $p \propto p(1 - p)$, I wonder if this provides a connection to the exponential nature of the power walk

1.5 Conclusion

So when the adjacency matrix is stictly boolean, the power walk is equivalent to the random surfer.

1.6 The Second Eigenvalue

1.6.1 The Random Surfer

The Second eigenvalue ξ_2 of the Power Surfer is less than α (See 3.2; Stability and Concvergence, of proposal).

1.6.2 Power Walk

Because the Power Walk relates to the random surfer as demonstrated in section 1 , what can be said about ξ_2

Applying this to Power Walk

Let $\Lambda_{(2)}\left(\mathbf{T}\right) = \lambda_2$ return the second value of a transition, probability Matrix, then observe that:

$$\Lambda_{(2)}\left(\mathbf{T}_{\mathrm{RS}}\right) \leq |\alpha| \implies \Lambda_{(2)}\left(\mathbf{T}_{\mathrm{PW}}\right) \leq \left| \frac{\alpha - \mathbf{D_a}\mathbf{D}_{\mathbf{B}}^{-1}}{\mathbf{D_A}\mathbf{D}_{\mathbf{B}}^{-1}} \right| \tag{13}$$

where:

- $\lambda_{(2)}\left(\mathbf{T}\right)$ refers to the transition probability matrix of the power walk and random surfer approaces as indicated.

My attempt

$$\beta \mathbf{1}_{[n,n]} = \frac{1-\Theta}{\Theta} \tag{14}$$

$$\tag{15}$$

where:

- $\Theta = \mathbf{D_A}\mathbf{D}_{\mathbf{B}}^{-1}$

So I thought maybe if I could find a value of β that satisfied (14) then I could show circumstances under which $|\xi_2| < \alpha$.

Seemingly it's only satisfied where $\beta = 1$ though, using this simulation:

```r
g1 <- igraph::erdos.renyi.game(n = 9, 0.2)
A <- igraph::get.adjacency(g1) # Row to column
A <- t(A)
# plot(g1)

## * Finding beta values to behave like Random Surfer
  beta <- 10
  B <- beta^A

  DA     <- PageRank::create_sparse_diag_sc_inv_mat(A)
  DB_inv <- PageRank::create_sparse_diag_scaling_mat(B)

 THETA <- DA %*% DB_inv

THETA <- function(A, beta) {
  B  <- beta^A
  DA     <- PageRank::create_sparse_diag_sc_inv_mat(A)
  DB_inv <- PageRank::create_sparse_diag_scaling_mat(B)
  return(DA %*% DB_inv)
}

THETA_inv <- function(A, beta) {
  B  <- beta^A
  DB     <- PageRank::create_sparse_diag_sc_inv_mat(B)
  DA_inv <- PageRank::create_sparse_diag_scaling_mat(A)
  return(DA %*% DB_inv)
}

beta_func <- function(A, beta) {
    return(1-THETA(A, beta^A) %*% THETA_inv(A, beta^A))
}

THETA(A, 10) %*% THETA_inv(A, 10)


eta <- 10^-6
beta <- 1.01
while (mean(beta*matrix(1, nrow(A), ncol(A)) - beta_func(A, beta)) > eta) {
    beta <- beta + 0.01
    print(beta)
    print(diag(beta_func(A, beta)))
    print(beta*matrix(1, nrow(A), ncol(A)))
    print(beta_func(A, beta))
#    Sys.sleep(0.1)
}

beta


diag(beta_func(A, beta))
beta


## * blah
```

2 Cauchy Integral Formula

This is from section 54 of the book, isn't it nice that it more or less just works hey? [4]

$$f\left(a\right)\frac{1}{2\pi i}\oint\frac{f\left(z\right)}{z-a}\mathrm{d}z \tag{16}$$

In view of this equation then: [4]

$$\left|\int_{C}\frac{f\left(z\right)}{z-z_{0}}\mathrm{d}z-2\pi if\left(z_{0}\right)\right|<2\pi\varepsilon$$

Some Images: [2]

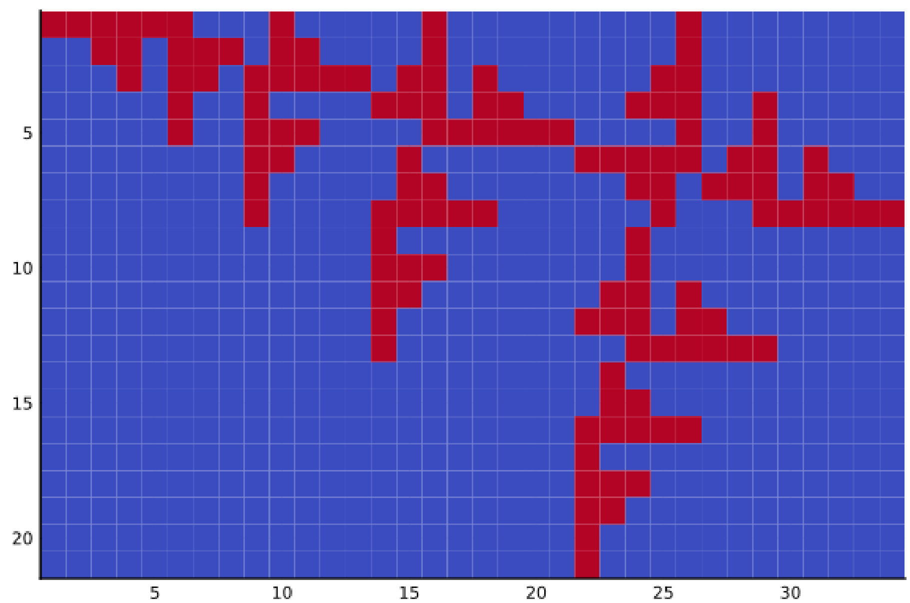


Figure 1: This image is for testing purposes [1]

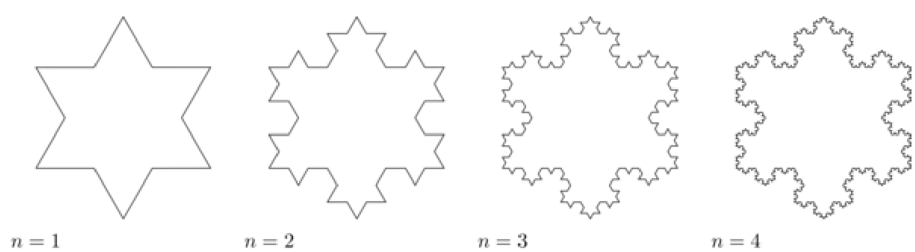
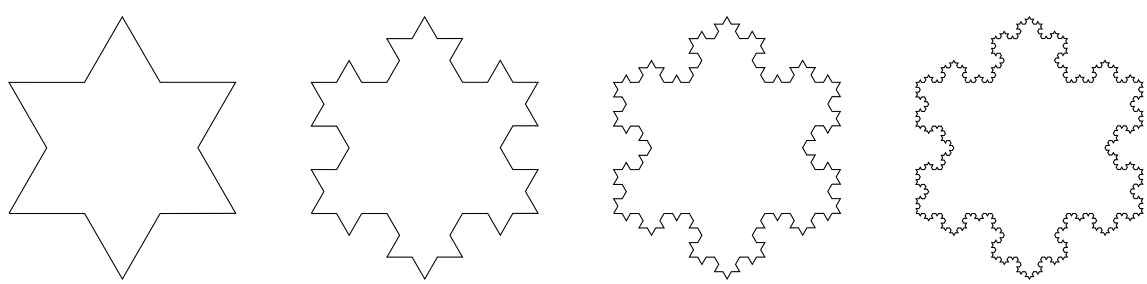


Figure 2: This is a tikz image inserted as a png from imagemagick



@@latex: $n = 1$ $n = 2$ $n = 3$ $n = 4$

2.1 Heading 2

2.1.1 Heading 3

```
echo "Hello World"
```

Heading 4

Heading 5

1. Heading 6 Arbitrary Code:

```
n/bash

# Print Help
if [ "$1" == "-h" ]; then
    echo "Usage: `basename $0` <Format> <CSS>"
    style=~/Dropbox/profiles/Emacs/org-css/github-org.css
    exit 0
fi

# Make a working File from clipboard
filename=lkjdskjjalkjkj392jlkj
xclip -o -selection clipboard >> $filename
LocalFile=$filename.org

pandoc -s  -f org -t gfm $filename -o $filename

echo "
This was converted from `org` to `md` using `pandoc -t gfm` at time:
$(date --utc +%FT%H-%M-%S)
" >> $filename

cat $filename | xclip -selection clipboard
rm $filename

nv & disown
echo "Conversion from Org Successful, MD is in Clipboard"

exit 0
```

References

[1] Paula Moskowitz. *Library Guides: Wikipedia: Should You Use Wikipedia?* URL: https://mville.libguides.com/c.php?g=370066&p=2500344 (visited on 08/19/2020) (cit. on p. 7).

[2] Andrew Y. Ng, Alice X. Zheng, and Michael I. Jordan. "Stable Algorithms for Link Analysis". In: *Proceedings of the 24th Annual International ACM SIGIR Conference on Research and Development in Information Retrieval*. SIGIR '01. New York, NY, USA: Association for Computing Machinery, Sept. 1, 2001, pp. 258–266. ISBN: 978-1-58113-331-8. DOI: 10.1145/383952.384003. URL: http://doi.org/10.1145/383952.384003 (visited on 08/19/2020) (cit. on p. 7).

[3] Laurence A. F. Park and Simeon Simoff. "Power Walk: Revisiting the Random Surfer". In: *Proceedings of the 18th Australasian Document Computing Symposium*. ADCS '13. Brisbane, Queensland, Australia: Association for Computing Machinery, Dec. 5, 2013, pp. 50–57. ISBN: 978-1-4503-2524-0. DOI: 10.1145/2537734.2537749. URL: http://doi.org/10.1145/2537734.2537749 (visited on 07/31/2020) (cit. on p. 1).

[4] Hui Zhang et al. "Making Eigenvector-Based Reputation Systems Robust to Collusion". In: *Algorithms and Models for the Web-Graph*. Ed. by Stefano Leonardi. Lecture Notes in Computer Science. Berlin, Heidelberg: Springer, 2004, pp. 92–104. ISBN: 978-3-540-30216-2. DOI: 10.1007/978-3-540-30216-2_8 (cit. on p. 7).