(03) Series

Wk 4 Material; Topic 3; Due 28 March

Contents

The Cauchy Criterion (3.5)

The Cauchy Convergence Criterion

A sequence is convergent if and only if it is a Cauchy sequence

- **Cauchy Sequence** implies **Convergence**
  - Every Cauchy sequence of real numbers is bounded, hence by the Bolzano-Weierstrass theorem the sequence has a convergent subsequence, hence is itself convergent.
- **Convergence** imples **Cauchy Sequence**
  - If two terms can be made arbitrarily close then any term can be made arbitrarily close to another term in the set (which will be the limit point).

Properly Divergent

A series (x_n) is said to be properly divergent if $\lim_{n\to\infty}(x_n) = \pm\infty$

Definition of a Series [3.7.1]

if x_n is a sequence, then the **series** generated by the sequence is $S = (s_k)$:

- The terms of the sequence are $x_n) = (x_1, x_2, x_3, x_4, \dots s_n)$

The terms of the series are $(s_n) = (s_1, s_2, s_3, s_4, \dots s_n)$

The terms of the series are called the **partial sums** and are defined as such:

$$
\begin{aligned}
s_1 &= x_1 &&= x_1 \\
s_2 &= s_1 + x_2 &&= x_1 + x_2 \\
s_3 &= s_3 + x_3 &&= x_1 + x_2 + x_3 \\
s_4 &= s_4 + x_4 &&= x_1 + x_2 + x_3 + x_4 \\
&\dots \\
s_n &= s_n + x_n &&= x_1 + x_2 + x_3 + \dots x_n
\end{aligned}
$$

Figure 1:

Common Series Types

These are series that we are expected to memorise because they so often appear in series problems (and moreover we we will need them for the exam).

Geometric Series (3.7.6 (a))

The Geometric Series is Convergent if and only if $| r |<$:

$$\sum_{n=1}^{\infty} [r^n] = 1 + r + r^2 + r^3 + \dots r^n$$

iff $|r|<1$ then this is convergent

$$|r|<1 \Rightarrow \sum_{n=1}^{\infty} [r^n] = \frac{1}{1-r}$$

$$r \geq 1 \Rightarrow \lim(r^n) > 0$$
$$\Rightarrow \text{divergence}$$

[!h]

Figure 2:

$$\sum_{n=1}^{\infty} \left(ar^n\right) = \frac{a}{1-r}, \text{ for } |r|<1$$

if $|r| \geq 1$, $\sum (ar^n)$ is divergent.

Harmonic Series (3.7.6(b))

[!h]

The Harmonic series $\sum_{n=1}^{\infty}[1/n]$ is divergent:

assume S converges to a number:

$$
\begin{aligned}
S &= (1 + 1/2) + (1/3 + 1/4) + (1/5 + 1/6) \cdots + \left(\tfrac{1}{2n-1} + 1/2n\right) \\
&> (1/2 + 1/2) + (1/4 + 1/4) + (1/6 + 1/6) \cdots + (1/2n + 1/2n) \\
&= (1) + (2/4) + (2/6) \cdots + (1/n) \\
&= 1 + (1/2) + (1/3) \cdots + (1/n) \\
&= S
\end{aligned}
$$

$\therefore$ the assumption that $\sum_{n=1}^{\infty}[1/n] = S$ implies $S > S$
hence S DNE and the series diverges.

[!h]

Figure 3:

P-Series

The P-Series is convergent for $P > 1$:

$\sum_{n=1}^{\infty}\left[1/n^P\right]$ is convergent

for $0 < P \leq 1$ this is divergent.

For $P = 1$ this is the harmonic sequence
For $P = -1$ this is the geometric sequence.

[!h]

Figure 4:

Properties of Series

The n^{th} term test

This is more or less a test for divergence, it is necessary that a sequence (x_n) has a limit of 0 in order for the series to be convergent:

$$\exists L : \quad \sum_{n=1}^{\infty} [x_n] = L \implies \lim(x_n) = 0$$

be careful however because a sequence with a limit of 0 is not sufficient to establish the convergence of a series:

$$\exists L : \quad \sum_{n=1}^{\infty} [x_n] = L \not\Leftarrow \lim(x_n) = 0$$

Cauchy Criterion for series

If a sequence is convergent it must be a Cauchy sequence, hence all convergent series are composed of *Cauchy Sequences* (as a necessary but not sufficient condition).

So to be clear a series converges if and only if it is a *Cauchy Sequence*.

Definitions

- A Cauchy Sequence is:
  - $\forall \varepsilon > 0, \exists M : \ m, n \geq M \implies \mid s_m - s_n \mid = |x_{n+1} + x_{n+2} + x_{n+3} \dots x_m| < \varepsilon$
- A Series Converges (which is an equivalent statement) if:
  - $\forall \varepsilon > 0, \exists M : \ , n \geq N \implies \mid s_n - s \mid = |x_1 + x_2 + x_3 \dots x_n| < \varepsilon$

Convergence Tests

Types of Convergence

A series $\sum[x_n]$ is ***absolutely convergent*** if and only if $\sum[\ |x_n|\ ]$, otherwise the series is said to be conditionally convergent.

This is important because the convergence of $\sum[\ |x_n|\ ] \implies$ the convergence of $\sum[x_n]$

Below the tests have been split into three categories:

- Comparison Tests
  - These establish non-absolute convergence but are broadly applicable and so are introduced early
- Absolute Convergence Tests
  - These establish absolute convergence.
- Non-Absolute Convergence Tests
  - These are useful for *alternating Series* and series that change sign as they progress (e.g. $\frac{sin(n)}{n}$)

Choosing a Test

Choosing the right test can be difficult, hence I have included an appendix with a flow chart [1] that we should probably memorise for want of the exam

Manipulating Series

Sometimes you'll be given a series in an odd way for example:

$$S_n = \sum_{n=1}^{\infty}\left[\frac{1}{(3n-2)\cdot(3n+1)}\right]$$

Now this could be shown to be convergent using the limit comparison test (which is below) but if you are asked to find the value to which the series converges to there is a bit more work involved.

Generally if you are asked to find what value a series converges to it will be either:

- A Geometric Series (3.7.6(a) of TB), or
- A telescoping Series

Geometric Series have already been shown, but a telescoping series is new and not covered in the textbook, basically, it is a series where most of the terms cancel out by way of rearrangement and grouping to leave only one or two terms left.

[1] Strategy for Series, http://tutorial.math.lamar.edu/Classes/CalcII/SeriesStrategy.aspx

Partial Fractions

Often it is necessary to manipulate the terms somewhat in order for them to exhibit the cancelling/telescoping property, often by way of partial fractions (remember from *Mathematics1B*), for an example of this refer to Q3(c) of the corresponding tutorial (tutorial #4 of wk 4 material, due wk. 5, topic 3 from learning guide)

In this case because the provided series is not a geometric series it must be a telescoping series (because otherwise we wouldn't be asked to find the value to which it converges to, we only know how to find the convergence values of those two series, so we know it's telescoping, in order to get it into a form that will work, use partial fractions [2]

$$\frac{1}{(3n-2)\cdot(3n+1)} = \frac{A}{3n-2} \quad \frac{B}{3n+1}$$

$$= \frac{1/3}{3n-2} + \frac{-1/3}{3n+1}$$

Figure 5:

From here we would manipulate the series using grouping and rearrangement

Grouping Series

Grouping terms in a series does not affect the value to which it converges,

- This flows from the associativity of addition, a property exhibited by the $\mathbb{R}$ which is the codomain of the sequence function

So in the above example the regrouping necessary to demonstrate the telescoping nature:

$$\left(1/3 - 1/12\right) + \left(1/12 - 1/21\right) + \left(1/21 - 1/33\right) \cdots \left(1/3/_{3n-2} + 1/3/_{3n+1}\right)$$

$$= 1/3 - 1/12 + 1/12 - 1/21 + 1/21 - 1/33 \cdots 1/3/_{3n-2} + 1/3/_{3n+1}$$

$$= 1/3 + \left(-1/12 + 1/12\right) + \left(-1/21 + 1/21\right) + \left(-1/33 + 1/33\right) \ldots + \left(-1/3/_{3(n-1)+1} + 1/3/_{3n-2}\right) - 1/3/_{3n+1}$$

[!h]

Figure 6:

[2]Partial Fractions, http://tutorial.math.lamar.edu/Classes/CalcII/PartialFractions.aspx

Rearrangements (9.1.5)

If a series is absolutely convergent then you can rearrange the terms and the series will converge to the same value (otherwise you can't so be careful)

- So say you have some series and you rearrange it, if this new series is absolutely convergent thet it's fine.
- However, if you rearrange some series and the new series is only conditionally convergent, then the rearrangement wasn't logically valid and this convergence value is erroneous.

So in our example the series is absolutely convergent so we could rearrange it:

$$= \frac{1}{3} + \frac{-1/3}{3n+1} + \left(\frac{1}{12} - \frac{1}{12}\right) + \left(\frac{1}{21} - \frac{1}{21}\right) + \left(\frac{1}{33} - \frac{1}{33}\right) \dots + \left(\frac{1/3}{3n-2} - \frac{1/3}{3(n-1)+1}\right)$$

$$= \frac{1}{3} - \frac{1}{9n+3} + 0$$

$$\frac{1}{3} > \frac{1}{3} \cdot \frac{1}{3n+1}$$

$$\Rightarrow 0 < \frac{1}{3} - \frac{1}{9n+3}$$

$$\Rightarrow \frac{1}{3} - \frac{1}{9n+3} = \left|\frac{1}{3} - \frac{1}{9n+3}\right|$$

thus the new series is absolutely convergent, hence the rearrangement was logically valid.

[!h]

Figure 7:

Identities to remember

For the exam We need to remember these identities:

Limit of $e^{\frac{1}{n}}$

$$\lim_{n\to\infty}\left(n^{1/n}\right) = \lim_{n\to\infty}\left(e^{\ln(n^{1/n})}\right)$$

because the exponential function is continuous $$= e^{\lim_{n\to\infty}\left(\ln(n^{1/n})\right)}$$

by log laws $$= e^{\lim_{n\to\infty}\left(1/n \cdot \ln(n)\right)}$$

by L'hospital's rule $$= e^{\lim_{n\to\infty}\left(\frac{d/dn\,(\ln(n))}{d/dn\,(n)}\right)}$$

$$= e^{\lim_{n\to\infty}\left(1/n^2\right)}$$

$$= e^{0}$$

$$= 1$$

$$\therefore \quad \lim_{n\to\infty}\left(n^{1/n}\right) = 1$$

[!h]

Figure 8:

Dealing with Inequalities

$$a < x < b \iff \frac{1}{b} < \frac{1}{x} < \frac{1}{c}$$

e.g. $$2 < 5 < 7 \iff \frac{1}{7} < 1/5 < 1/2$$

if $5 > 2$ then $1/5 < 1/2$ $\qquad a < x \Rightarrow x > a \Rightarrow \frac{1}{a} < 1/1$

if $7 > 5$ then $1/7 < 1/5$ $\qquad x < b \Rightarrow b > x \Rightarrow \frac{1}{x} < \frac{1}{b}$

thus $1/7 < 1/5 < 1/2$ $\qquad \Rightarrow \frac{1}{a} < \frac{1}{x} < \frac{1}{b}$

[!h]

Figure 9:

Comparison Tests

Comparison Test (3.7.7)

take positive real sequences and some $k \in \mathbb{N}$:

$$n \geqslant k \implies 0 < x_n < y_n$$

so this order only needs to hold for any tail of the sequence.

a) if $\sum[y_n]$ converges, then $\sum[x_n]$ converges

b) if $\sum[x_n]$ diverges, then $\sum[x_n]$ diverges

take positive real sequences and some $k \in \mathbb{N}$:

$$x_n > 0 \text{ and } y_n > 0 \quad \forall\, n \in \mathbb{N}$$

[!h]

Figure 10:

Limit Comparison Test (3.7.8)

Sometimes it can be difficult to establish the inequealities of the first test and a ratio would be easier to use, in that case this test can be used:

if the following limit exists:

$$r = \lim\left(\frac{x_n}{y_n}\right)$$

then:

a) if $r \neq 0$

$\sum[x_n]$ is convergent $\iff$ $\sum[y_n]$ is convergent

b) if $r = 0$

$\sum[y_n]$ is convergent $\implies$ $\sum[x_n]$ is convergent

be careful, if $\sum[x_n]$ is convergent $\sum[y_n]$ may or may not be convergent.

Absolute Convergence Tests

If these tests are satisfied they will establish that the series is absolutely convergent.

Limit Comparison Test II (9.2.1) (For Absolute Convergence)

This version of the test is useful for establishing absolute convergence, it may be more difficult to establish however.

take positive real sequences and some $k \in \mathbb{N}$:

$$x_n > 0 \quad \text{and} \quad y_n > 0 \quad \forall\ n \in \mathbb{N}$$

if the following limit exists:

$$r = \lim \left| \frac{x_n}{y_n} \right|$$

then:

a) if $r \neq 0$

$\sum[x_n]$ is absolutely convergent $\iff$ $\sum[y_n]$ is absolutely convergent

b) if $r = 0$

$\sum[y_n]$ is absolutely convergent $\implies$ $\sum[x_n]$ is absolutely convergent

be careful, if $\sum[x_n]$ is convergent $\sum[y_n]$ may or may not be convergent.

[!h]

Figure 11:

Ratio Test (9.2.4)

Take a sequence of non-zero real numbers (a_n) and some $K \in \mathbb{N}$

consider:

$L_1 = \left|\frac{a_{n+1}}{a_n}\right| \quad (\forall n \geqslant K)$ | $L_2 = \lim\left(\left|\frac{a_{n+1}}{a_n}\right|\right)$ | $L_3 = \limsup\limits_{n\to\infty}\left(\left|\frac{a_{n+1}}{a_n}\right|\right)$

Any of these three tests is logically valid and will provide the following:

if $L = 1$ the test tells us nothing
if $L < 1$ then $\sum [a_n]$ is absolutely convergent
if $L > 1$ then $\sum [a_n]$ is divergent.

The generalised D'Alambert test is useful for series containing terms like $(-1)^n$.

[!h]

Figure 12:

Generalised D'Alambert

This can be useful where the ratio test fails for want of $(-1)^{n+1}$ because the $\limsup()$ operator will strip that way for a $(+1)$.

It is worth remembering that a sequence (x_n) is convergent if and only if:

$$\liminf(x_n) = \limsup(x_n) = \lim(x_n)$$

In this test however, we simply need to show that the $\limsup$ exists (which it will if the ratio-sequence has an upper bound), it isn't necessary to show that the ratio-sequence is convergent.

- (However, it is necessary that the sequence which generates the series converges to 0, otherwise the series will be divergent)

Root Test

Take a sequence of non-zero real numbers (a_n) and some $K \in \mathbb{N}$

consider:

$q_1 = |a_n|^{1/n} \quad (\forall n \geq K)$ | $q_2 = \lim\left(|a_n|^{1/n}\right)$ | $q_3 = \limsup_{n\to\infty}\left(|a_n|^{1/n}\right)$

Any of these three tests is logically valid and will provide the following:

if $q = 1$ the test tells us nothing
if $q < 1$ then $\sum[a_n]$ is absolutely convergent
if $q > 1$ then $\sum[a_n]$ is divergent.

[!h]

Figure 13:

Generalised *Cauchy* Test

This can be useful where the root test fails for want of $(-1)^n$, the $\limsup()$ operator will strip that away for a $(+1)$.

Integral Test

If the series is of a function that is positive and ecreasint, then the series could converge if and only if the integral converges:

Let $f(k)$ be a positive decreasing function and let k be some natural number:

$$\exists L \in \mathbb{R}: \ \ L = \sum_{n=k}^{\infty}[f(k)] \iff \int_{\infty}^{k} f(x)\,\mathrm{d}x = \lim_{b\to\infty}\left(\int_{b}^{k} f(x)\,\mathrm{d}x\right)$$

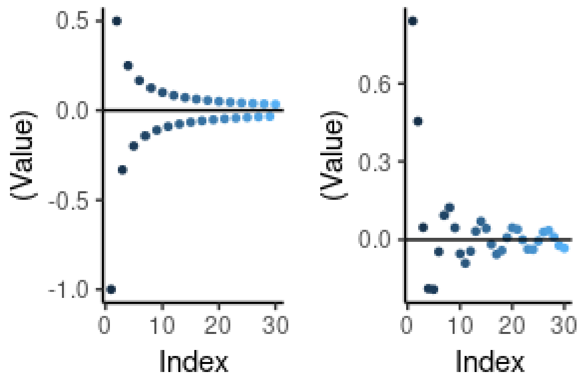
Or basically the series will converge if and only if the corresponding integral converges,

- This flows from the notion that the area under a continuous curve is going to be greater than the various term values, hence by the comparison test it's going to converge.
- this is a test for absolute convergence because the terms of the sequence that generates the series are strictly positive as a prerequisite anyway.

Non-Absolute Convergence Tests

Definition of an Alternating Sequence (9.3.1)

An alternating sequence is a sequence that changes sign at each iteration, so for example $(x_n) = \frac{(-1)^{n+1}}{n}$ is an alternating sequence because at each succession the sequence changes sign $(x_n) = \frac{\sin(n}{n}$ is not an alternating sequence because the terms doesn't alternate at each succession:



Alternating Series Test

Take a decreasing sequence of positive numbers (Z_n), :

- If the sequence is such that:
  - $Z_{n+1} < Z_n \quad \wedge \quad Z_n > 0 \qquad \forall n \in \ltimes$
- Then the series will be convergent:
  - $\exists L \in (R): \ \sum_{n=1}^{\infty} \left[(-1)^{n+1} \cdot Z_n\right]$

So basically if the sequence is decreasing, then the series of the alternating sequence will hence converge.

Partial Summation Formula (Abel's Lemma)

Let $X := (x_n)$ and $Y := (y_n)$ by sequences in $\mathbb{R}$ and let the partial sums of $\sum (y_n)$ be denoted by $\left(s_{)}n\right)$ with $s_0 := 0$

$$\sum_{k=n+1}^{m} [x_k y_k] = (x_m s_m - x_{n+1} s_n) + \sum k = n + 1^{m-1} (x_k - x_k + 1) s_k$$

Dirichlet's Test

if $X = (x_n)$ is a decreasing sequence:

$\lim(x_n) = 0$

and the partial sums (s_n) of $\sum [y_n]$ are bounded

then $\sum [x_n \cdot y_n]$ is convergent.

[!h]

Figure 14:

Abel's Test

if (x_n) is a convergent monotone sequence, and

$\sum [y_n]$ is convergent.

Then $\sum [x_n y_n]$ is convergent.

[!h]

Figure 15:

Section 4-12 : Strategy for Series

Now that we've got all of our tests out of the way it's time to think about organizing all of them into a general set of guidelines to help us determine the convergence of a series.

Note that these are a general set of guidelines and because some series can have more than one test applied to them we will get a different result depending on the path that we take through this set of guidelines. In fact, because more than one test may apply, you should always go completely through the guidelines and identify all possible tests that can be used on a given series. Once this has been done you can identify the test that you feel will be the easiest for you to use.

With that said here is the set of guidelines for determining the convergence of a series.

1. With a quick glance does it look like the series terms don't converge to zero in the limit, *i.e.* does $\lim_{n\to\infty} a_n \neq 0$? If so, use the Divergence Test. Note that you should only do the Divergence Test if a quick glance suggests that the series terms may not converge to zero in the limit.
2. Is the series a *p*-series ($\sum \frac{1}{n^p}$) or a geometric series ($\sum_{n=0}^{\infty} ar^n$ or $\sum_{n=1}^{\infty} ar^{n-1}$)? If so use the fact that *p*-series will only converge if $p > 1$ and a geometric series will only converge if $|r| < 1$. Remember as well that often some algebraic manipulation is required to get a geometric series into the correct form.
3. Is the series similar to a *p*-series or a geometric series? If so, try the Comparison Test.
4. Is the series a rational expression involving only polynomials or polynomials under radicals (*i.e.* a fraction involving only polynomials or polynomials under radicals)? If so, try the Comparison Test and/or the Limit Comparison Test. Remember however, that in order to use the Comparison Test and the Limit Comparison Test the series terms all need to be positive.
5. Does the series contain factorials or constants raised to powers involving *n*? If so, then the Ratio Test may work. Note that if the series term contains a factorial then the only test that we've got that will work is the Ratio Test.
6. Can the series terms be written in the form $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$? If so, then the Alternating Series Test may work.
7. Can the series terms be written in the form $a_n = (b_n)^n$? If so, then the Root Test may work.
8. If $a_n = f(n)$ for some positive, decreasing function and $\int_a^{\infty} f(x)\,dx$ is easy to evaluate then the Integral Test may work.

Again, remember that these are only a set of guidelines and not a set of hard and fast rules to use when trying to determine the best test to use on a series. If more than one test can be used try to use the test that will be the easiest for you to use and remember that what is easy for someone else may not be easy for you!

Also, just so we can put all the tests into one place here is a quick listing of all the tests that we've got.

Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ will diverge

Integral Test

Suppose that $f(x)$ is a positive, decreasing function on the interval $[k,\infty)$ and that $f(n)=a_n$ then,

1. If $\int_k^\infty f(x)\,dx$ is convergent then so is $\sum_{n=k}^{\infty} a_n$.
2. If $\int_k^\infty f(x)\,dx$ is divergent then so is $\sum_{n=k}^{\infty} a_n$.

Comparison Test

Suppose that we have two series $\sum a_n$ and $\sum b_n$ with $a_n, b_n \geq 0$ for all *n* and $a_n \leq b_n$ for all *n*. Then,

1. If $\sum b_n$ is convergent then so is $\sum a_n$.
2. If $\sum a_n$ is divergent then so is $\sum b_n$.

Limit Comparison Test

Suppose that we have two series $\sum a_n$ and $\sum b_n$ with $a_n, b_n \geq 0$ for all *n*. Define,

$$c = \lim_{n\to\infty} \frac{a_n}{b_n}$$

If *c* is positive (*i.e.* $c > 0$) and is finite (*i.e.* $c < \infty$) then either both series converge or both series diverge.

Alternating Series Test

Suppose that we have a series $\sum a_n$ and either $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ where $b_n \geq 0$ for all *n*. Then if,

1. $\lim_{n\to\infty} b_n = 0$ and,
2. $\{b_n\}$ is eventually a decreasing sequence

the series $\sum a_n$ is convergent

Ratio Test

Suppose we have the series $\sum a_n$. Define,

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Then,

1. if $L < 1$ the series is absolutely convergent (and hence convergent).
2. if $L > 1$ the series is divergent.
3. if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.

Root Test

Suppose that we have the series $\sum a_n$. Define,

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

Then,

1. if $L < 1$ the series is absolutely convergent (and hence convergent).
2. if $L > 1$ the series is divergent.
3. if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.