Infinite Sequences and Series



Infinite sequences and series were introduced briefly in A Preview of Calculus in connection with Zeno's paradoxes and the decimal representation of numbers. Their importance in calculus stems from Newton's idea of representing functions as sums of infinite series. For instance, in finding areas he often integrated a function by first expressing it as a series and then integrating each term of the series. We will pursue his idea in Section 11.10 in order to integrate such functions as e^{-x^2} . (Recall that we have previously been unable to do this.) Many of the functions that arise in mathematical physics and chemistry, such as Bessel functions, are defined as sums of series, so it is important to be familiar with the basic concepts of convergence of infinite sequences and series.

Physicists also use series in another way, as we will see in Section 11.11. In studying fields as diverse as optics, special relativity, and electromagnetism, they analyze phenomena by replacing a function with the first few terms in the series that represents it.

11.1 Sequences

A sequence can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

The number a_1 is called the *first term*, a_2 is the *second term*, and in general a_n is the *nth term*. We will deal exclusively with infinite sequences and so each term a_n will have a successor a_{n+1} .

Notice that for every positive integer n there is a corresponding number a_n and so a sequence can be defined as a function whose domain is the set of positive integers. But we usually write a_n instead of the function notation f(n) for the value of the function at the number n.

NOTATION The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by

$$\{a_n\}$$
 or $\{a_n\}_{n=1}^{\infty}$

EXAMPLE 1 Some sequences can be defined by giving a formula for the *n*th term. In the following examples we give three descriptions of the sequence: one by using the preceding notation, another by using the defining formula, and a third by writing out the terms of the sequence. Notice that *n* doesn't have to start at 1.

(a)
$$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$$
 $a_n = \frac{n}{n+1}$ $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n}{n+1}, \dots\right\}$

(b)
$$\left\{\frac{(-1)^n(n+1)}{3^n}\right\}$$
 $a_n = \frac{(-1)^n(n+1)}{3^n}$ $\left\{-\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \dots, \frac{(-1)^n(n+1)}{3^n}, \dots\right\}$

(c)
$$\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}$$
 $a_n = \sqrt{n-3}, \ n \ge 3$ $\left\{0, 1, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n-3}, \dots\right\}$

(d)
$$\left\{\cos\frac{n\pi}{6}\right\}^{\infty}$$
 $a_n = \cos\frac{n\pi}{6}, \ n \ge 0 \quad \left\{1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \dots, \cos\frac{n\pi}{6}, \dots\right\}$

V EXAMPLE 2 Find a formula for the general term a_n of the sequence

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \ldots\right\}$$

assuming that the pattern of the first few terms continues.

SOLUTION We are given that

$$a_1 = \frac{3}{5}$$
 $a_2 = -\frac{4}{25}$ $a_3 = \frac{5}{125}$ $a_4 = -\frac{6}{625}$ $a_5 = \frac{7}{3125}$

Notice that the numerators of these fractions start with 3 and increase by 1 whenever we go to the next term. The second term has numerator 4, the third term has numerator 5; in general, the nth term will have numerator n + 2. The denominators are the powers of 5,

so a_n has denominator 5^n . The signs of the terms are alternately positive and negative, so we need to multiply by a power of -1. In Example 1(b) the factor $(-1)^n$ meant we started with a negative term. Here we want to start with a positive term and so we use $(-1)^{n-1}$ or $(-1)^{n+1}$. Therefore

$$a_n = (-1)^{n-1} \frac{n+2}{5^n}$$

EXAMPLE3 Here are some sequences that don't have a simple defining equation.

- (a) The sequence $\{p_n\}$, where p_n is the population of the world as of January 1 in the year n.
- (b) If we let a_n be the digit in the *n*th decimal place of the number e, then $\{a_n\}$ is a well-defined sequence whose first few terms are

(c) The **Fibonacci sequence** $\{f_n\}$ is defined recursively by the conditions

$$f_1 = 1$$
 $f_2 = 1$ $f_n = f_{n-1} + f_{n-2}$ $n \ge 3$

Each term is the sum of the two preceding terms. The first few terms are

$$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$$

This sequence arose when the 13th-century Italian mathematician known as Fibonacci solved a problem concerning the breeding of rabbits (see Exercise 83).

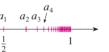


FIGURE 1

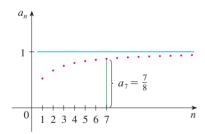


FIGURE 2

A sequence such as the one in Example 1(a), $a_n = n/(n+1)$, can be pictured either by plotting its terms on a number line, as in Figure 1, or by plotting its graph, as in Figure 2. Note that, since a sequence is a function whose domain is the set of positive integers, its graph consists of isolated points with coordinates

$$(1, a_1)$$
 $(2, a_2)$ $(3, a_3)$... (n, a_n) ...

From Figure 1 or Figure 2 it appears that the terms of the sequence $a_n = n/(n+1)$ are approaching 1 as n becomes large. In fact, the difference

$$1 - \frac{n}{n+1} = \frac{1}{n+1}$$

can be made as small as we like by taking n sufficiently large. We indicate this by writing

$$\lim_{n\to\infty}\frac{n}{n+1}=1$$

In general, the notation

$$\lim_{n\to\infty}a_n=L$$

means that the terms of the sequence $\{a_n\}$ approach L as n becomes large. Notice that the following definition of the limit of a sequence is very similar to the definition of a limit of a function at infinity given in Section 2.6.

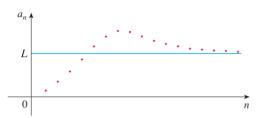
1 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n\to\infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

Figure 3 illustrates Definition 1 by showing the graphs of two sequences that have the limit L.

FIGURE 3 Graphs of two sequences with $\lim_{n \to \infty} a_n = L$





A more precise version of Definition 1 is as follows.

Compare this definition with Definition 2.6.7.

2 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n\to\infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

if
$$n > N$$
 then $|a_n - L| < \varepsilon$

Definition 2 is illustrated by Figure 4, in which the terms a_1, a_2, a_3, \ldots are plotted on a number line. No matter how small an interval $(L - \varepsilon, L + \varepsilon)$ is chosen, there exists an N such that all terms of the sequence from a_{N+1} onward must lie in that interval.

FIGURE 4



Another illustration of Definition 2 is given in Figure 5. The points on the graph of $\{a_n\}$ must lie between the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ if n > N. This picture must be valid no matter how small ε is chosen, but usually a smaller ε requires a larger N.

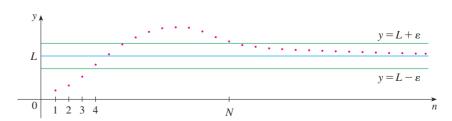


FIGURE 5

If you compare Definition 2 with Definition 2.6.7 you will see that the only difference between $\lim_{n\to\infty} a_n = L$ and $\lim_{x\to\infty} f(x) = L$ is that n is required to be an integer. Thus we have the following theorem, which is illustrated by Figure 6.

3 Theorem If
$$\lim_{x\to\infty} f(x) = L$$
 and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

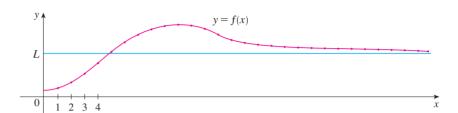


FIGURE 6

In particular, since we know that $\lim_{x\to\infty} (1/x^r) = 0$ when r > 0 (Theorem 2.6.5), we have

$$\lim_{n \to \infty} \frac{1}{n^r} = 0 \quad \text{if } r > 0$$

If a_n becomes large as n becomes large, we use the notation $\lim_{n\to\infty} a_n = \infty$. The following precise definition is similar to Definition 2.6.9.

5 Definition $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

if
$$n > N$$
 then $a_n > M$

If $\lim_{n\to\infty} a_n = \infty$, then the sequence $\{a_n\}$ is divergent but in a special way. We say that $\{a_n\}$ diverges to ∞ .

The Limit Laws given in Section 2.3 also hold for the limits of sequences and their proofs are similar.

Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n$$

$$\lim_{n \to \infty} (a_n - b_n) = \lim_{n \to \infty} a_n - \lim_{n \to \infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty}c=c$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\frac{\lim_{n\to\infty}a_n}{\lim_{n\to\infty}b_n}\quad\text{if }\lim_{n\to\infty}b_n\neq0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \text{ if } p>0 \text{ and } a_n>0$$

The Squeeze Theorem can also be adapted for sequences as follows (see Figure 7).

Squeeze Theorem for Sequences

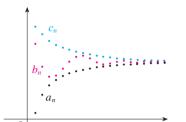


FIGURE 7

The sequence $\{b_n\}$ is squeezed between the sequences $\{a_n\}$ and $\{c_n\}$.

This shows that the guess we made earlier from Figures 1 and 2 was correct.

If
$$a_n \le b_n \le c_n$$
 for $n \ge n_0$ and $\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L$, then $\lim_{n \to \infty} b_n = L$.

Another useful fact about limits of sequences is given by the following theorem, whose proof is left as Exercise 87.

If
$$\lim_{n\to\infty} |a_n| = 0$$
, then $\lim_{n\to\infty} a_n = 0$.

EXAMPLE 4 Find
$$\lim_{n\to\infty} \frac{n}{n+1}$$
.

SOLUTION The method is similar to the one we used in Section 2.6: Divide numerator and denominator by the highest power of n that occurs in the denominator and then use the Limit Laws.

$$\lim_{n \to \infty} \frac{n}{n+1} = \lim_{n \to \infty} \frac{1}{1+\frac{1}{n}} = \frac{\lim_{n \to \infty} 1}{\lim_{n \to \infty} 1 + \lim_{n \to \infty} \frac{1}{n}}$$
$$= \frac{1}{1+0} = 1$$

Here we used Equation 4 with r = 1.

EXAMPLE 5 Is the sequence $a_n = \frac{n}{\sqrt{10 + n}}$ convergent or divergent?

SOLUTION As in Example 4, we divide numerator and denominator by n:

$$\lim_{n \to \infty} \frac{n}{\sqrt{10 + n}} = \lim_{n \to \infty} \frac{1}{\sqrt{\frac{10}{n^2} + \frac{1}{n}}} = \infty$$

because the numerator is constant and the denominator approaches 0. So $\{a_n\}$ is divergent.

EXAMPLE 6 Calculate $\lim_{n\to\infty} \frac{\ln n}{n}$.

SOLUTION Notice that both numerator and denominator approach infinity as $n \to \infty$. We can't apply l'Hospital's Rule directly because it applies not to sequences but to functions of a real variable. However, we can apply l'Hospital's Rule to the related function $f(x) = (\ln x)/x$ and obtain

$$\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = 0$$

Therefore, by Theorem 3, we have

$$\lim_{n\to\infty}\frac{\ln n}{n}=0$$

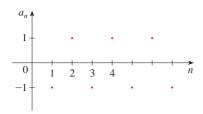


FIGURE 8

The graph of the sequence in Example 8 is shown in Figure 9 and supports our answer.

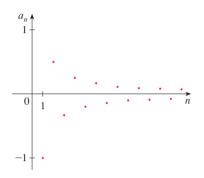


FIGURE 9

Creating Graphs of Sequences

Some computer algebra systems have special commands that enable us to create sequences and graph them directly. With most graphing calculators, however, sequences can be graphed by using parametric equations. For instance, the sequence in Example 10 can be graphed by entering the parametric equations

$$x = t$$
 $y = t!/t^t$

and graphing in dot mode, starting with t=1 and setting the t-step equal to 1. The result is shown in Figure 10.

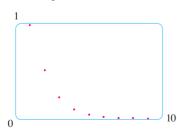


FIGURE 10

EXAMPLE7 Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

SOLUTION If we write out the terms of the sequence, we obtain

$$\{-1, 1, -1, 1, -1, 1, -1, \ldots\}$$

The graph of this sequence is shown in Figure 8. Since the terms oscillate between 1 and -1 infinitely often, a_n does not approach any number. Thus $\lim_{n\to\infty} (-1)^n$ does not exist; that is, the sequence $\{(-1)^n\}$ is divergent.

EXAMPLE 8 Evaluate $\lim_{n\to\infty}\frac{(-1)^n}{n}$ if it exists.

SOLUTION We first calculate the limit of the absolute value:

$$\lim_{n\to\infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n\to\infty} \frac{1}{n} = 0$$

Therefore, by Theorem 6,

$$\lim_{n\to\infty}\frac{(-1)^n}{n}=0$$

The following theorem says that if we apply a continuous function to the terms of a convergent sequence, the result is also convergent. The proof is left as Exercise 88.

7 Theorem If $\lim_{n \to \infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

EXAMPLE 9 Find $\lim_{n\to\infty} \sin(\pi/n)$

SOLUTION Because the sine function is continuous at 0, Theorem 7 enables us to write

$$\lim_{n \to \infty} \sin(\pi/n) = \sin\left(\lim_{n \to \infty} (\pi/n)\right) = \sin 0 = 0$$

EXAMPLE 10 Discuss the convergence of the sequence $a_n = n!/n^n$, where $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$.

SOLUTION Both numerator and denominator approach infinity as $n \to \infty$ but here we have no corresponding function for use with l'Hospital's Rule (x! is not defined when x is not an integer). Let's write out a few terms to get a feeling for what happens to a_n as n gets large:

$$a_1 = 1$$
 $a_2 = \frac{1 \cdot 2}{2 \cdot 2}$ $a_3 = \frac{1 \cdot 2 \cdot 3}{3 \cdot 3 \cdot 3}$

$$a_n = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot n \cdot \cdots \cdot n}$$

It appears from these expressions and the graph in Figure 10 that the terms are decreasing and perhaps approach 0. To confirm this, observe from Equation 8 that

$$a_n = \frac{1}{n} \left(\frac{2 \cdot 3 \cdot \dots \cdot n}{n \cdot n \cdot \dots \cdot n} \right)$$

$$0 < a_n \le \frac{1}{n}$$

We know that $1/n \to 0$ as $n \to \infty$. Therefore $a_n \to 0$ as $n \to \infty$ by the Squeeze Theorem.

V EXAMPLE 11 For what values of r is the sequence $\{r^n\}$ convergent?

SOLUTION We know from Section 2.6 and the graphs of the exponential functions in Section 1.5 that $\lim_{x\to\infty} a^x = \infty$ for a>1 and $\lim_{x\to\infty} a^x = 0$ for 0< a<1. Therefore, putting a=r and using Theorem 3, we have

$$\lim_{n \to \infty} r^n = \begin{cases} \infty & \text{if } r > 1\\ 0 & \text{if } 0 < r < 1 \end{cases}$$

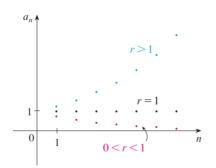
It is obvious that

$$\lim_{n\to\infty} 1^n = 1 \qquad \text{and} \qquad \lim_{n\to\infty} 0^n = 0$$

If -1 < r < 0, then 0 < |r| < 1, so

$$\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n = 0$$

and therefore $\lim_{n\to\infty} r^n=0$ by Theorem 6. If $r\leqslant -1$, then $\{r^n\}$ diverges as in Example 7. Figure 11 shows the graphs for various values of r. (The case r=-1 is shown in Figure 8.)



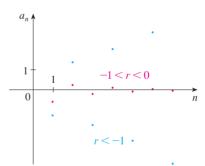


FIGURE 11

The sequence $a_n = r^n$

The results of Example 11 are summarized for future use as follows.

9 The sequence $\{r^n\}$ is convergent if $-1 < r \le 1$ and divergent for all other values of r.

$$\lim_{n \to \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

10 Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \ge 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \ge 1$. A sequence is **monotonic** if it is either increasing or decreasing.

EXAMPLE 12 The sequence $\left\{\frac{3}{n+5}\right\}$ is decreasing because

 $\frac{3}{n+5} > \frac{3}{(n+1)+5} = \frac{3}{n+6}$

and so $a_n > a_{n+1}$ for all $n \ge 1$.

The right side is smaller because it has a

larger denominator.

EXAMPLE 13 Show that the sequence $a_n = \frac{n}{n^2 + 1}$ is decreasing.

SOLUTION 1 We must show that $a_{n+1} < a_n$, that is,

$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$$

This inequality is equivalent to the one we get by cross-multiplication:

$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} \iff (n+1)(n^2+1) < n[(n+1)^2+1]$$

$$\iff n^3+n^2+n+1 < n^3+2n^2+2n$$

$$\iff 1 < n^2+n$$

Since $n \ge 1$, we know that the inequality $n^2 + n > 1$ is true. Therefore $a_{n+1} < a_n$ and so $\{a_n\}$ is decreasing.

SOLUTION 2 Consider the function $f(x) = \frac{x}{x^2 + 1}$:

$$f'(x) = \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} < 0$$
 whenever $x^2 > 1$

Thus f is decreasing on $(1, \infty)$ and so f(n) > f(n+1). Therefore $\{a_n\}$ is decreasing.

11 Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \le M$$
 for all $n \ge 1$

It is **bounded below** if there is a number m such that

$$m \le a_n$$
 for all $n \ge 1$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

For instance, the sequence $a_n = n$ is bounded below $(a_n > 0)$ but not above. The sequence $a_n = n/(n + 1)$ is bounded because $0 < a_n < 1$ for all n.

We know that not every bounded sequence is convergent [for instance, the sequence $a_n = (-1)^n$ satisfies $-1 \le a_n \le 1$ but is divergent from Example 7] and not every mono-

tonic sequence is convergent $(a_n = n \to \infty)$. But if a sequence is both bounded *and* monotonic, then it must be convergent. This fact is proved as Theorem 12, but intuitively you can understand why it is true by looking at Figure 12. If $\{a_n\}$ is increasing and $a_n \le M$ for all n, then the terms are forced to crowd together and approach some number L.

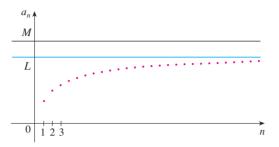


FIGURE 12

The proof of Theorem 12 is based on the **Completeness Axiom** for the set \mathbb{R} of real numbers, which says that if *S* is a nonempty set of real numbers that has an upper bound M ($x \le M$ for all x in S), then S has a **least upper bound** b. (This means that b is an upper bound for S, but if M is any other upper bound, then $b \le M$.) The Completeness Axiom is an expression of the fact that there is no gap or hole in the real number line.

12 Monotonic Sequence Theorem Every bounded, monotonic sequence is convergent.

PROOF Suppose $\{a_n\}$ is an increasing sequence. Since $\{a_n\}$ is bounded, the set $S = \{a_n \mid n \ge 1\}$ has an upper bound. By the Completeness Axiom it has a least upper bound L. Given $\varepsilon > 0$, $L - \varepsilon$ is *not* an upper bound for S (since L is the *least* upper bound). Therefore

$$a_N > L - \varepsilon$$
 for some integer N

But the sequence is increasing so $a_n \ge a_N$ for every n > N. Thus if n > N, we have

$$a_n > L - \varepsilon$$

SO

$$0 \le L - a_n < \varepsilon$$

since $a_n \leq L$. Thus

$$|L - a_n| < \varepsilon$$
 whenever $n > N$

so $\lim_{n\to\infty} a_n = L$.

A similar proof (using the greatest lower bound) works if $\{a_n\}$ is decreasing.

The proof of Theorem 12 shows that a sequence that is increasing and bounded above is convergent. (Likewise, a decreasing sequence that is bounded below is convergent.) This fact is used many times in dealing with infinite series.

$$a_1 = 2$$
 $a_{n+1} = \frac{1}{2}(a_n + 6)$ for $n = 1, 2, 3, ...$

SOLUTION We begin by computing the first several terms:

$$a_1 = 2$$
 $a_2 = \frac{1}{2}(2+6) = 4$ $a_3 = \frac{1}{2}(4+6) = 5$
 $a_4 = \frac{1}{2}(5+6) = 5.5$ $a_5 = 5.75$ $a_6 = 5.875$
 $a_7 = 5.9375$ $a_8 = 5.96875$ $a_9 = 5.984375$

Mathematical induction is often used in dealing with recursive sequences. See page 76 for a discussion of the Principle of Mathematical Induction.

These initial terms suggest that the sequence is increasing and the terms are approaching 6. To confirm that the sequence is increasing, we use mathematical induction to show that $a_{n+1} > a_n$ for all $n \ge 1$. This is true for n = 1 because $a_2 = 4 > a_1$. If we assume that it is true for n = k, then we have

$$a_{k+1} > a_k$$
 so
$$a_{k+1} + 6 > a_k + 6$$
 and
$$\frac{1}{2}(a_{k+1} + 6) > \frac{1}{2}(a_k + 6)$$
 Thus
$$a_{k+2} > a_{k+1}$$

We have deduced that $a_{n+1} > a_n$ is true for n = k + 1. Therefore the inequality is true for all n by induction.

Next we verify that $\{a_n\}$ is bounded by showing that $a_n < 6$ for all n. (Since the sequence is increasing, we already know that it has a lower bound: $a_n \ge a_1 = 2$ for all n.) We know that $a_1 < 6$, so the assertion is true for n = 1. Suppose it is true for n = k. Then

$$a_{k} < 6$$
 so
$$a_{k} + 6 < 12$$
 and
$$\frac{1}{2}(a_{k} + 6) < \frac{1}{2}(12) = 6$$
 Thus
$$a_{k+1} < 6$$

This shows, by mathematical induction, that $a_n < 6$ for all n.

Since the sequence $\{a_n\}$ is increasing and bounded, Theorem 12 guarantees that it has a limit. The theorem doesn't tell us what the value of the limit is. But now that we know $L = \lim_{n \to \infty} a_n$ exists, we can use the given recurrence relation to write

$$\lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \frac{1}{2} (a_n + 6) = \frac{1}{2} \left(\lim_{n \to \infty} a_n + 6 \right) = \frac{1}{2} (L + 6)$$

A proof of this fact is requested in Exercise 70. Since $a_n \to L$, it follows that $a_{n+1} \to L$ too (as $n \to \infty$, $n+1 \to \infty$ also). So we have

$$L = \frac{1}{2}(L+6)$$

Solving this equation for L, we get L = 6, as we predicted.

11.1 Exercises

- 1. (a) What is a sequence?
 - (b) What does it mean to say that $\lim_{n\to\infty} a_n = 8$?
 - (c) What does it mean to say that $\lim_{n\to\infty} a_n = \infty$?
- **2.** (a) What is a convergent sequence? Give two examples.
 - (b) What is a divergent sequence? Give two examples.
- 3-12 List the first five terms of the sequence.

3.
$$a_n = \frac{2n}{n^2 + 1}$$

4.
$$a_n = \frac{3^n}{1+2^n}$$

5.
$$a_n = \frac{(-1)^{n-1}}{5^n}$$

6.
$$a_n = \cos \frac{n\pi}{2}$$

7.
$$a_n = \frac{1}{(n+1)!}$$

8.
$$a_n = \frac{(-1)^n n}{n! + 1}$$

9.
$$a_1 = 1$$
, $a_{n+1} = 5a_n - 3$

10.
$$a_1 = 6$$
, $a_{n+1} = \frac{a_n}{n}$

11.
$$a_1 = 2$$
, $a_{n+1} = \frac{a_n}{1 + a_n}$

12.
$$a_1 = 2$$
, $a_2 = 1$, $a_{n+1} = a_n - a_{n-1}$

13–18 Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

13.
$$\{1, \frac{1}{2}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \dots\}$$

14.
$$\left\{1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \ldots\right\}$$

15.
$$\left\{-3, 2, -\frac{4}{3}, \frac{8}{9}, -\frac{16}{27}, \ldots\right\}$$

17.
$$\left\{\frac{1}{2}, -\frac{4}{3}, \frac{9}{4}, -\frac{16}{5}, \frac{25}{6}, \ldots\right\}$$

18.
$$\{1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$$

19–22 Calculate, to four decimal places, the first ten terms of the sequence and use them to plot the graph of the sequence by hand. Does the sequence appear to have a limit? If so, calculate it. If not, explain why.

19.
$$a_n = \frac{3n}{1+6n}$$

20.
$$a_n = 2 + \frac{(-1)^n}{n}$$

21.
$$a_n = 1 + \left(-\frac{1}{2}\right)^n$$

22.
$$a_n = 1 + \frac{10^n}{9^n}$$

23-56 Determine whether the sequence converges or diverges. If it converges, find the limit.

23.
$$a_n = 1 - (0.2)^n$$

24.
$$a_n = \frac{n^3}{n^3 + 1}$$

25.
$$a_n = \frac{3 + 5n^2}{n + n^2}$$

26.
$$a_n = \frac{n^3}{n+1}$$

27.
$$a_n = e^{1/n}$$

28.
$$a_n = \frac{3^{n+2}}{5^n}$$

$$29. \ a_n = \tan\left(\frac{2n\pi}{1+8n}\right)$$

30.
$$a_n = \sqrt{\frac{n+1}{9n+1}}$$

31.
$$a_n = \frac{n^2}{\sqrt{n^3 + 4n}}$$

32.
$$a_n = e^{2n/(n+2)}$$

33.
$$a_n = \frac{(-1)^n}{2\sqrt{n}}$$

34.
$$a_n = \frac{(-1)^{n+1}n}{n+\sqrt{n}}$$

35.
$$a_n = \cos(n/2)$$

36.
$$a_n = \cos(2/n)$$

37.
$$\left\{ \frac{(2n-1)!}{(2n+1)!} \right\}$$

$$38. \left\{ \frac{\ln n}{\ln 2n} \right\}$$

39.
$$\left\{ \frac{e^n + e^{-n}}{e^{2n} - 1} \right\}$$

40.
$$a_n = \frac{\tan^{-1} n}{n}$$

41.
$$\{n^2e^{-n}\}$$

42.
$$a_n = \ln(n+1) - \ln n$$

43.
$$a_n = \frac{\cos^2 n}{2^n}$$

44.
$$a_n = \sqrt[n]{2^{1+3n}}$$

45.
$$a_n = n \sin(1/n)$$

46.
$$a_n = 2^{-n} \cos n\pi$$

47.
$$a_n = \left(1 + \frac{2}{n}\right)^n$$

48.
$$a_n = \frac{\sin 2n}{1 + \sqrt{n}}$$

49.
$$a_n = \ln(2n^2 + 1) - \ln(n^2 + 1)$$

50.
$$a_n = \frac{(\ln n)^2}{n}$$

51.
$$a_n = \arctan(\ln n)$$

52.
$$a_n = n - \sqrt{n+1} \sqrt{n+3}$$

54.
$$\left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}, \frac{1}{6}, \ldots\right\}$$

56.
$$a_n = \frac{(-3)^n}{n!}$$

₱ 57-63 Use a graph of the sequence to decide whether the sequence is convergent or divergent. If the sequence is convergent, guess the value of the limit from the graph and then prove your guess. (See the margin note on page 695 for advice on graphing sequences.)

57.
$$a_n = 1 + (-2/e)^n$$

58.
$$a_n = \sqrt{n} \sin(\pi/\sqrt{n})$$

59.
$$a_n = \sqrt{\frac{3 + 2n^2}{8n^2 + n}}$$

60.
$$a_n = \sqrt[n]{3^n + 5^n}$$

61.
$$a_n = \frac{n^2 \cos n}{1 + n^2}$$

62.
$$a_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!}$$

63.
$$a_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n)^n}$$

64. (a) Determine whether the sequence defined as follows is convergent or divergent:

$$a_1 = 1$$
 $a_{n+1} = 4 - a_n$ for $n \ge 1$

- (b) What happens if the first term is $a_1 = 2$?
- **65.** If \$1000 is invested at 6% interest, compounded annually, then after n years the investment is worth $a_n = 1000(1.06)^n$ dollars
 - (a) Find the first five terms of the sequence $\{a_n\}$.
 - (b) Is the sequence convergent or divergent? Explain.
- **66.** If you deposit \$100 at the end of every month into an account that pays 3% interest per year compounded monthly, the amount of interest accumulated after *n* months is given by the sequence

$$I_n = 100 \left(\frac{1.0025^n - 1}{0.0025} - n \right)$$

- (a) Find the first six terms of the sequence.
- (b) How much interest will you have earned after two years?
- **67.** A fish farmer has 5000 catfish in his pond. The number of catfish increases by 8% per month and the farmer harvests 300 catfish per month.
 - (a) Show that the catfish population P_n after n months is given recursively by

$$P_n = 1.08P_{n-1} - 300$$
 $P_0 = 5000$

(b) How many catfish are in the pond after six months?

68. Find the first 40 terms of the sequence defined by

$$a_{n+1} = \begin{cases} \frac{1}{2}a_n & \text{if } a_n \text{ is an even number} \\ 3a_n + 1 & \text{if } a_n \text{ is an odd number} \end{cases}$$

and $a_1 = 11$. Do the same if $a_1 = 25$. Make a conjecture about this type of sequence.

- **69.** For what values of r is the sequence $\{nr^n\}$ convergent?
- **70.** (a) If $\{a_n\}$ is convergent, show that

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$$

- (b) A sequence $\{a_n\}$ is defined by $a_1 = 1$ and $a_{n+1} = 1/(1 + a_n)$ for $n \ge 1$. Assuming that $\{a_n\}$ is convergent, find its limit.
- **71.** Suppose you know that $\{a_n\}$ is a decreasing sequence and all its terms lie between the numbers 5 and 8. Explain why the sequence has a limit. What can you say about the value of the limit?

72–78 Determine whether the sequence is increasing, decreasing, or not monotonic. Is the sequence bounded?

72.
$$a_n = (-2)^{n+1}$$

73.
$$a_n = \frac{1}{2n+3}$$

74.
$$a_n = \frac{2n-3}{3n+4}$$

75.
$$a_n = n(-1)^n$$

76.
$$a_n = ne^{-n}$$

77.
$$a_n = \frac{n}{n^2 + 1}$$

78.
$$a_n = n + \frac{1}{n}$$

79. Find the limit of the sequence

$$\left\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\right\}$$

- **80.** A sequence $\{a_n\}$ is given by $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$.
 - (a) By induction or otherwise, show that $\{a_n\}$ is increasing and bounded above by 3. Apply the Monotonic Sequence Theorem to show that $\lim_{n\to\infty} a_n$ exists.
 - (b) Find $\lim_{n\to\infty} a_n$.
- 81. Show that the sequence defined by

$$a_1 = 1 a_{n+1} = 3 - \frac{1}{a_n}$$

is increasing and $a_n < 3$ for all n. Deduce that $\{a_n\}$ is convergent and find its limit.

82. Show that the sequence defined by

$$a_1 = 2$$
 $a_{n+1} = \frac{1}{3 - a_n}$

satisfies $0 < a_n \le 2$ and is decreasing. Deduce that the sequence is convergent and find its limit.

- **83.** (a) Fibonacci posed the following problem: Suppose that rabbits live forever and that every month each pair produces a new pair which becomes productive at age 2 months. If we start with one newborn pair, how many pairs of rabbits will we have in the *n*th month? Show that the answer is f_n , where $\{f_n\}$ is the Fibonacci sequence defined in Example 3(c).
 - (b) Let $a_n = f_{n+1}/f_n$ and show that $a_{n-1} = 1 + 1/a_{n-2}$. Assuming that $\{a_n\}$ is convergent, find its limit.
- **84.** (a) Let $a_1 = a$, $a_2 = f(a)$, $a_3 = f(a_2) = f(f(a))$, ..., $a_{n+1} = f(a_n)$, where f is a continuous function. If $\lim_{n \to \infty} a_n = L$, show that f(L) = L.
 - (b) Illustrate part (a) by taking $f(x) = \cos x$, a = 1, and estimating the value of L to five decimal places.
- **85.** (a) Use a graph to guess the value of the limit

$$\lim_{n\to\infty}\frac{n^5}{n!}$$

- (b) Use a graph of the sequence in part (a) to find the smallest values of N that correspond to $\varepsilon=0.1$ and $\varepsilon=0.001$ in Definition 2.
- **86.** Use Definition 2 directly to prove that $\lim_{n\to\infty} r^n = 0$ when |r| < 1.
- **87.** Prove Theorem 6. [*Hint:* Use either Definition 2 or the Squeeze Theorem.]
- **88.** Prove Theorem 7.
- **89.** Prove that if $\lim_{n\to\infty} a_n = 0$ and $\{b_n\}$ is bounded, then $\lim_{n\to\infty} (a_n b_n) = 0$.
- **90.** Let $a_n = \left(1 + \frac{1}{n}\right)^n$.
 - (a) Show that if $0 \le a < b$, then

$$\frac{b^{n+1} - a^{n+1}}{b - a} < (n+1)b^n$$

- (b) Deduce that $b^n \lceil (n+1)a nb \rceil < a^{n+1}$.
- (c) Use a = 1 + 1/(n + 1) and b = 1 + 1/n in part (b) to show that $\{a_n\}$ is increasing.
- (d) Use a = 1 and b = 1 + 1/(2n) in part (b) to show that $a_{2n} < 4$.
- (e) Use parts (c) and (d) to show that $a_n < 4$ for all n.
- (f) Use Theorem 12 to show that $\lim_{n\to\infty} (1 + 1/n)^n$ exists. (The limit is e. See Equation 3.6.6.)

91. Let a and b be positive numbers with a > b. Let a_1 be their arithmetic mean and b_1 their geometric mean:

$$a_1 = \frac{a+b}{2} \qquad b_1 = \sqrt{ab}$$

Repeat this process so that, in general,

$$a_{n+1} = \frac{a_n + b_n}{2}$$
 $b_{n+1} = \sqrt{a_n b_n}$

(a) Use mathematical induction to show that

$$a_n > a_{n+1} > b_{n+1} > b_n$$

- (b) Deduce that both $\{a_n\}$ and $\{b_n\}$ are convergent.
- (c) Show that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. Gauss called the common value of these limits the **arithmetic-geometric mean** of the numbers a and b.
- **92.** (a) Show that if $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$, then $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$.
 - (b) If $a_1 = 1$ and

$$a_{n+1} = 1 + \frac{1}{1 + a_n}$$

find the first eight terms of the sequence $\{a_n\}$. Then use part (a) to show that $\lim_{n\to\infty}a_n=\sqrt{2}$. This gives the **continued fraction expansion**

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \dots}}$$

93. The size of an undisturbed fish population has been modeled by the formula

$$p_{n+1} = \frac{bp_n}{a + p_n}$$

where p_n is the fish population after n years and a and b are positive constants that depend on the species and its environment. Suppose that the population in year 0 is $p_0 > 0$.

- (a) Show that if $\{p_n\}$ is convergent, then the only possible values for its limit are 0 and b-a.
- (b) Show that $p_{n+1} < (b/a)p_n$.
- (c) Use part (b) to show that if a > b, then $\lim_{n \to \infty} p_n = 0$; in other words, the population dies out.
- (d) Now assume that a < b. Show that if $p_0 < b a$, then $\{p_n\}$ is increasing and $0 < p_n < b a$. Show also that if $p_0 > b a$, then $\{p_n\}$ is decreasing and $p_n > b a$. Deduce that if a < b, then $\lim_{n \to \infty} p_n = b a$.

LABORATORY PROJECT AS LOGISTIC SEQUENCES

A sequence that arises in ecology as a model for population growth is defined by the **logistic difference equation**

$$p_{n+1} = kp_n(1 - p_n)$$

where p_n measures the size of the population of the *n*th generation of a single species. To keep the numbers manageable, p_n is a fraction of the maximal size of the population, so $0 \le p_n \le 1$. Notice that the form of this equation is similar to the logistic differential equation in Section 9.4. The discrete model—with sequences instead of continuous functions—is preferable for modeling insect populations, where mating and death occur in a periodic fashion.

An ecologist is interested in predicting the size of the population as time goes on, and asks these questions: Will it stabilize at a limiting value? Will it change in a cyclical fashion? Or will it exhibit random behavior?

Write a program to compute the first n terms of this sequence starting with an initial population p_0 , where $0 < p_0 < 1$. Use this program to do the following.

- **1.** Calculate 20 or 30 terms of the sequence for $p_0 = \frac{1}{2}$ and for two values of k such that 1 < k < 3. Graph each sequence. Do the sequences appear to converge? Repeat for a different value of p_0 between 0 and 1. Does the limit depend on the choice of p_0 ? Does it depend on the choice of k?
- **2.** Calculate terms of the sequence for a value of *k* between 3 and 3.4 and plot them. What do you notice about the behavior of the terms?
- **3.** Experiment with values of *k* between 3.4 and 3.5. What happens to the terms?
- **4.** For values of k between 3.6 and 4, compute and plot at least 100 terms and comment on the behavior of the sequence. What happens if you change p_0 by 0.001? This type of behavior is called *chaotic* and is exhibited by insect populations under certain conditions.

CAS Computer algebra system required

11.2 Series

What do we mean when we express a number as an infinite decimal? For instance, what does it mean to write

 $\pi = 3.14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288\dots$

The convention behind our decimal notation is that any number can be written as an infinite sum. Here it means that

 $\pi = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \frac{6}{10^7} + \frac{5}{10^8} + \cdots$

where the three dots (\cdots) indicate that the sum continues forever, and the more terms we add, the closer we get to the actual value of π .

The current record is that π has been computed to 2,576,980,370,000 (more than two trillion)

decimal places by T. Daisuke and his team.

In general, if we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an **infinite series** (or just a **series**) and is denoted, for short, by the symbol

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n$$

Does it make sense to talk about the sum of infinitely many terms? It would be impossible to find a finite sum for the series

$$1 + 2 + 3 + 4 + 5 + \cdots + n + \cdots$$

because if we start adding the terms we get the cumulative sums 1, 3, 6, 10, 15, 21, . . . and, after the *n*th term, we get n(n + 1)/2, which becomes very large as *n* increases.

However, if we start to add the terms of the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \dots + \frac{1}{2^n} + \dots$$

we get $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$, $\frac{15}{16}$, $\frac{31}{32}$, $\frac{63}{64}$, ..., $1-1/2^n$, The table shows that as we add more and more terms, these *partial sums* become closer and closer to 1. (See also Figure 11 in *A Preview of Calculus*, page 6.) In fact, by adding sufficiently many terms of the series we can make the partial sums as close as we like to 1. So it seems reasonable to say that the sum of this infinite series is 1 and to write

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{2^n} + \dots = 1$$

We use a similar idea to determine whether or not a general series 1 has a sum. We consider the **partial sums**

$$s_1 = a_1$$

 $s_2 = a_1 + a_2$
 $s_3 = a_1 + a_2 + a_3$
 $s_4 = a_1 + a_2 + a_3 + a_4$

and, in general,

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^n a_i$$

These partial sums form a new sequence $\{s_n\}$, which may or may not have a limit. If $\lim_{n\to\infty} s_n = s$ exists (as a finite number), then, as in the preceding example, we call it the sum of the infinite series $\sum a_n$.

$$s_n = \sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called **convergent** and we write

$$a_1 + a_2 + \dots + a_n + \dots = s$$
 or $\sum_{n=1}^{\infty} a_n = s$

The number s is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

Thus the sum of a series is the limit of the sequence of partial sums. So when we write $\sum_{n=1}^{\infty} a_n = s$, we mean that by adding sufficiently many terms of the series we can get as close as we like to the number s. Notice that

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} \sum_{i=1}^{n} a_i$$

EXAMPLE 1 Suppose we know that the sum of the first *n* terms of the series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = a_1 + a_2 + \dots + a_n = \frac{2n}{3n+5}$$

Then the sum of the series is the limit of the sequence $\{s_n\}$:

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{2n}{3n+5} = \lim_{n \to \infty} \frac{2}{3+\frac{5}{n}} = \frac{2}{3}$$

In Example 1 we were *given* an expression for the sum of the first n terms, but it's usually not easy to *find* such an expression. In Example 2, however, we look at a famous series for which we *can* find an explicit formula for s_n .

EXAMPLE 2 An important example of an infinite series is the **geometric series**

$$a + ar + ar^{2} + ar^{3} + \dots + ar^{n-1} + \dots = \sum_{n=1}^{\infty} ar^{n-1}$$
 $a \neq 0$

Each term is obtained from the preceding one by multiplying it by the **common ratio** r. (We have already considered the special case where $a = \frac{1}{2}$ and $r = \frac{1}{2}$ on page 704.)

If r=1, then $s_n=a+a+\cdots+a=na\to\pm\infty$. Since $\lim_{n\to\infty}s_n$ doesn't exist, the geometric series diverges in this case.

If $r \neq 1$, we have

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

and $rs_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$

$$\int_{1}^{\infty} f(x) \, dx = \lim_{t \to \infty} \int_{1}^{t} f(x) \, dx$$

To find this integral we integrate from 1 to t and then let $t \to \infty$. For a series, we sum from 1 to n and then let $n \to \infty$.

Figure 1 provides a geometric demonstration of the result in Example 2. If the triangles are constructed as shown and *s* is the sum of the series, then, by similar triangles,

$$\frac{s}{a} = \frac{a}{a - ar}$$
 so $s = \frac{a}{1 - r}$

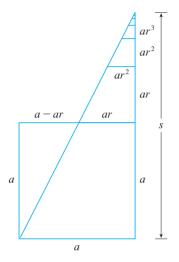


FIGURE 1

In words: The sum of a convergent geometric series is

$$\frac{\text{first term}}{1 - \text{common ratio}}$$

What do we really mean when we say that the sum of the series in Example 3 is 3? Of course, we can't literally add an infinite number of terms, one by one. But, according to Definition 2, the total sum is the limit of the sequence of partial sums. So, by taking the sum of sufficiently many terms, we can get as close as we like to the number 3. The table shows the first ten partial sums s_n and the graph in Figure 2 shows how the sequence of partial sums approaches 3.

Subtracting these equations, we get

$$s_n - rs_n = a - ar^n$$

3

$$s_n = \frac{a(1-r^n)}{1-r}$$

If -1 < r < 1, we know from (11.1.9) that $r^n \to 0$ as $n \to \infty$, so

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{a}{1 - r} \lim_{n \to \infty} r^n = \frac{a}{1 - r}$$

Thus when |r| < 1 the geometric series is convergent and its sum is a/(1-r). If $r \le -1$ or r > 1, the sequence $\{r^n\}$ is divergent by (11.1.9) and so, by Equation 3, $\lim_{n \to \infty} s_n$ does not exist. Therefore the geometric series diverges in those cases.

We summarize the results of Example 2 as follows.

4 The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if |r| < 1 and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

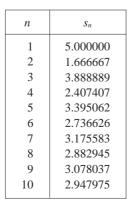
If $|r| \ge 1$, the geometric series is divergent.

V EXAMPLE 3 Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

SOLUTION The first term is a = 5 and the common ratio is $r = -\frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series is convergent by $\boxed{4}$ and its sum is

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \dots = \frac{5}{1 - (-\frac{2}{3})} = \frac{5}{\frac{5}{3}} = 3$$



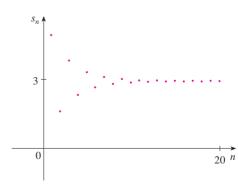


FIGURE 2

SOLUTION Let's rewrite the *n*th term of the series in the form ar^{n-1} :

Another way to identify a and r is to write out the first few terms:

$$4 + \frac{16}{3} + \frac{64}{9} + \cdots$$

$$\sum_{n=1}^{\infty} 2^{2n} 3^{1-n} = \sum_{n=1}^{\infty} (2^2)^n 3^{-(n-1)} = \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} 4 \left(\frac{4}{3}\right)^{n-1}$$

We recognize this series as a geometric series with a=4 and $r=\frac{4}{3}$. Since r>1, the series diverges by $\boxed{4}$.

EXAMPLE 5 Write the number $2.3\overline{17} = 2.3171717...$ as a ratio of integers.

SOLUTION

$$2.3171717... = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$$

After the first term we have a geometric series with $a = 17/10^3$ and $r = 1/10^2$. Therefore

$$2.3\overline{17} = 2.3 + \frac{\frac{17}{10^3}}{1 - \frac{1}{10^2}} = 2.3 + \frac{\frac{17}{1000}}{\frac{99}{100}}$$
$$= \frac{23}{10} + \frac{17}{990} = \frac{1147}{495}$$

EXAMPLE 6 Find the sum of the series $\sum_{n=0}^{\infty} x^n$, where |x| < 1.

SOLUTION Notice that this series starts with n = 0 and so the first term is $x^0 = 1$. (With series, we adopt the convention that $x^0 = 1$ even when x = 0.) Thus

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

This is a geometric series with a = 1 and r = x. Since |r| = |x| < 1, it converges and $\boxed{4}$ gives

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

EXAMPLE 7 Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum.

SOLUTION This is not a geometric series, so we go back to the definition of a convergent series and compute the partial sums.

$$s_n = \sum_{i=1}^n \frac{1}{i(i+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}$$

We can simplify this expression if we use the partial fraction decomposition

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

TEC Module 11.2 explores a series that depends on an angle θ in a triangle and enables you to see how rapidly the series converges when θ varies.

Notice that the terms cancel in pairs. This is an example of a **telescoping sum**: Because of all the cancellations, the sum collapses (like a pirate's collapsing

Figure 3 illustrates Example 7 by showing the graphs of the sequence of terms $a_n = 1/[n(n+1)]$ and the sequence $\{s_n\}$ of partial sums. Notice that $a_n \to 0$ and $s_n \rightarrow 1$. See Exercises 76 and 77 for two geometric interpretations of Example 7.

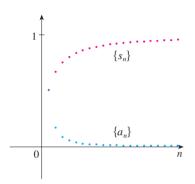


FIGURE 3

telescope) into just two terms.

(see Section 7.4). Thus we have

 $s_n = \sum_{i=1}^n \frac{1}{i(i+1)} = \sum_{i=1}^n \left(\frac{1}{i} - \frac{1}{i+1}\right)$

 $=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$

$$=1-\frac{1}{n+1}$$

and so

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \left(1 - \frac{1}{n+1} \right) = 1 - 0 = 1$$

Therefore the given series is convergent and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

EXAMPLE 8 Show that the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent.

SOLUTION For this particular series it's convenient to consider the partial sums s_2 , s_4 , s_8 , s_{16}, s_{32}, \ldots and show that they become large.

$$s_{2} = 1 + \frac{1}{2}$$

$$s_{4} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$$

$$s_{8} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2}$$

$$s_{16} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \dots + \frac{1}{8}\right) + \left(\frac{1}{9} + \dots + \frac{1}{16}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \dots + \frac{1}{8}\right) + \left(\frac{1}{16} + \dots + \frac{1}{16}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{4}{2}$$

Similarly, $s_{32} > 1 + \frac{5}{2}$, $s_{64} > 1 + \frac{6}{2}$, and in general

$$s_{2^n} > 1 + \frac{n}{2}$$

This shows that $s_{2^n} \to \infty$ as $n \to \infty$ and so $\{s_n\}$ is divergent. Therefore the harmonic series diverges.

The method used in Example 8 for showing that the harmonic series diverges is due to the French scholar Nicole Oresme (1323-1382).

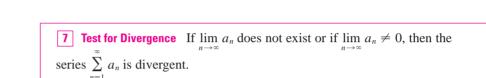
6 Theorem If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

PROOF Let $s_n = a_1 + a_2 + \cdots + a_n$. Then $a_n = s_n - s_{n-1}$. Since Σ a_n is convergent, the sequence $\{s_n\}$ is convergent. Let $\lim_{n\to\infty} s_n = s$. Since $n-1\to\infty$ as $n\to\infty$, we also have $\lim_{n\to\infty} s_{n-1} = s$. Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1}$$
$$= s - s = 0$$

NOTE 1 With any *series* Σ a_n we associate two *sequences:* the sequence $\{s_n\}$ of its partial sums and the sequence $\{a_n\}$ of its terms. If Σ a_n is convergent, then the limit of the sequence $\{s_n\}$ is s (the sum of the series) and, as Theorem 6 asserts, the limit of the sequence $\{a_n\}$ is 0.

NOTE 2 The converse of Theorem 6 is not true in general. If $\lim_{n\to\infty} a_n = 0$, we cannot conclude that $\sum a_n$ is convergent. Observe that for the harmonic series $\sum 1/n$ we have



 $a_n = 1/n \to 0$ as $n \to \infty$, but we showed in Example 8 that $\sum 1/n$ is divergent.

The Test for Divergence follows from Theorem 6 because, if the series is not divergent, then it is convergent, and so $\lim_{n\to\infty} a_n = 0$.

EXAMPLE 9 Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2 + 4}$ diverges.

SOLUTION

Let

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n^2}{5n^2 + 4} = \lim_{n \to \infty} \frac{1}{5 + 4/n^2} = \frac{1}{5} \neq 0$$

So the series diverges by the Test for Divergence.

NOTE 3 If we find that $\lim_{n\to\infty} a_n \neq 0$, we know that $\sum a_n$ is divergent. If we find that $\lim_{n\to\infty} a_n = 0$, we know *nothing* about the convergence or divergence of $\sum a_n$. Remember the warning in Note 2: If $\lim_{n\to\infty} a_n = 0$, the series $\sum a_n$ might converge or it might diverge.

8 Theorem If Σ a_n and Σ b_n are convergent series, then so are the series Σ ca_n (where c is a constant), Σ $(a_n + b_n)$, and Σ $(a_n - b_n)$, and

(i)
$$\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$$

(ii)
$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

(iii)
$$\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

These properties of convergent series follow from the corresponding Limit Laws for Sequences in Section 11.1. For instance, here is how part (ii) of Theorem 8 is proved:

$$s_n = \sum_{i=1}^n a_i$$
 $s = \sum_{i=1}^\infty a_i$ $t_n = \sum_{i=1}^n b_i$ $t = \sum_{i=1}^\infty b_i$

The *n*th partial sum for the series $\sum (a_n + b_n)$ is

$$u_n = \sum_{i=1}^n \left(a_i + b_i \right)$$

and, using Equation 5.2.10, we have

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \sum_{i=1}^n (a_i + b_i) = \lim_{n \to \infty} \left(\sum_{i=1}^n a_i + \sum_{i=1}^n b_i \right)$$

$$= \lim_{n \to \infty} \sum_{i=1}^n a_i + \lim_{n \to \infty} \sum_{i=1}^n b_i$$

$$= \lim_{n \to \infty} s_n + \lim_{n \to \infty} t_n = s + t$$

Therefore $\sum (a_n + b_n)$ is convergent and its sum is

$$\sum_{n=1}^{\infty} (a_n + b_n) = s + t = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

EXAMPLE 10 Find the sum of the series $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right)$.

SOLUTION The series $\sum 1/2^n$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$, so

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

In Example 7 we found that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

So, by Theorem 8, the given series is convergent and

$$\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right) = 3 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{2^n}$$
$$= 3 \cdot 1 + 1 = 4$$

NOTE 4 A finite number of terms doesn't affect the convergence or divergence of a series. For instance, suppose that we were able to show that the series

$$\sum_{n=4}^{\infty} \frac{n}{n^3 + 1}$$

is convergent. Since

$$\sum_{n=1}^{\infty} \frac{n}{n^3 + 1} = \frac{1}{2} + \frac{2}{9} + \frac{3}{28} + \sum_{n=1}^{\infty} \frac{n}{n^3 + 1}$$

it follows that the entire series $\sum_{n=1}^{\infty} n/(n^3+1)$ is convergent. Similarly, if it is known that the series $\sum_{n=N+1}^{\infty} a_n$ converges, then the full series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

is also convergent.

- 1. (a) What is the difference between a sequence and a series?
 - (b) What is a convergent series? What is a divergent series?
- **2.** Explain what it means to say that $\sum_{n=1}^{\infty} a_n = 5$.
- **3–4** Calculate the sum of the series $\sum_{n=1}^{\infty} a_n$ whose partial sums are given.
- 3. $s_n = 2 3(0.8)^n$
- **4.** $s_n = \frac{n^2 1}{4n^2 + 1}$
- 5-8 Calculate the first eight terms of the sequence of partial sums correct to four decimal places. Does it appear that the series is convergent or divergent?
- **5.** $\sum_{n=1}^{\infty} \frac{1}{n^3}$

6. $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$

7. $\sum_{n=1}^{\infty} \frac{n}{1 + \sqrt{n}}$

- **8.** $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$
- 9-14 Find at least 10 partial sums of the series. Graph both the sequence of terms and the sequence of partial sums on the same screen. Does it appear that the series is convergent or divergent? If it is convergent, find the sum. If it is divergent, explain why.
 - **9.** $\sum_{n=1}^{\infty} \frac{12}{(-5)^n}$

- **10.** $\sum_{n=1}^{\infty} \cos n$
- 11. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+4}}$
- **12.** $\sum_{n=1}^{\infty} \frac{7^{n+1}}{10^n}$
- **13.** $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} \frac{1}{\sqrt{n+1}} \right)$ **14.** $\sum_{n=2}^{\infty} \frac{1}{n(n+2)}$
- **15.** Let $a_n = \frac{2n}{3n+1}$.
 - (a) Determine whether $\{a_n\}$ is convergent.
 - (b) Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.
- 16. (a) Explain the difference between

$$\sum_{i=1}^{n} a_i \quad \text{and} \quad \sum_{j=1}^{n} a_j$$

(b) Explain the difference between

$$\sum_{i=1}^{n} a_i \quad \text{and} \quad \sum_{i=1}^{n} a_j$$

- 17–26 Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.
- **17.** $3-4+\frac{16}{3}-\frac{64}{9}+\cdots$ **18.** $4+3+\frac{9}{4}+\frac{27}{16}+\cdots$
- **19.** $10 2 + 0.4 0.08 + \cdots$
- **20.** $2 + 0.5 + 0.125 + 0.03125 + \cdots$

- **21.** $\sum_{n=0}^{\infty} 6(0.9)^{n-1}$
- **22.** $\sum_{n=0}^{\infty} \frac{10^n}{(-9)^{n-1}}$
- **23.** $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{\Delta^n}$
- **24.** $\sum_{n=0}^{\infty} \frac{1}{(\sqrt{2})^n}$

25. $\sum_{n=0}^{\infty} \frac{\pi^n}{3^{n+1}}$

- **26.** $\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}}$
- 27–42 Determine whether the series is convergent or divergent. If it is convergent, find its sum.

27.
$$\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \frac{1}{15} + \cdots$$

- **28.** $\frac{1}{3} + \frac{2}{9} + \frac{1}{27} + \frac{2}{81} + \frac{1}{243} + \frac{2}{729} + \cdots$
- **29.** $\sum_{n=1}^{\infty} \frac{n-1}{3n-1}$

30. $\sum_{k=1}^{\infty} \frac{k(k+2)}{(k+3)^2}$

31. $\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}$

32. $\sum_{n=0}^{\infty} \frac{1+3^n}{2^n}$

33. $\sum_{n=0}^{\infty} \sqrt[n]{2}$

- **34.** $\sum_{n=1}^{\infty} \left[(0.8)^{n-1} (0.3)^n \right]$
- **35.** $\sum_{n=1}^{\infty} \ln \left(\frac{n^2 + 1}{2n^2 + 1} \right)$
- **36.** $\sum_{n=1}^{\infty} \frac{1}{1+(\frac{2}{n})^n}$

37. $\sum_{k=0}^{\infty} \left(\frac{\pi}{3}\right)^k$

38. $\sum_{k=1}^{\infty} (\cos 1)^k$

39. $\sum_{n=0}^{\infty} \arctan n$

- **40.** $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n} \right)$
- **41.** $\sum_{n=1}^{\infty} \left(\frac{1}{e^n} + \frac{1}{n(n+1)} \right)$
- **42.** $\sum_{n=0}^{\infty} \frac{e^n}{2}$
- 43–48 Determine whether the series is convergent or divergent by expressing s_n as a telescoping sum (as in Example 7). If it is convergent, find its sum.
- **43.** $\sum_{n=1}^{\infty} \frac{2}{n^2 + 1}$

- 44. $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$
- **45.** $\sum_{n=0}^{\infty} \frac{3}{n(n+3)}$
- **46.** $\sum_{n=1}^{\infty} \left(\cos \frac{1}{n^2} \cos \frac{1}{(n+1)^2} \right)$
- **47.** $\sum^{\infty} \left(e^{1/n} e^{1/(n+1)} \right)$
- **48.** $\sum_{n=1}^{\infty} \frac{1}{n^3}$

- **49.** Let $x = 0.99999 \dots$
 - (a) Do you think that x < 1 or x = 1?
 - (b) Sum a geometric series to find the value of x.
 - (c) How many decimal representations does the number 1 have?
 - (d) Which numbers have more than one decimal representation?
- **50.** A sequence of terms is defined by

$$a_1 = 1$$
 $a_n = (5 - n)a_{n-1}$

Calculate $\sum_{n=1}^{\infty} a_n$.

51–56 Express the number as a ratio of integers.

- **51.** 0.8 = 0.8888...
- **52.** $0.\overline{46} = 0.46464646...$
- **53.** $2.\overline{516} = 2.516516516...$
- **54.** $10.1\overline{35} = 10.135353535...$
- **55.** 1.5342

56. 7.12345

57-63 Find the values of x for which the series converges. Find the sum of the series for those values of x.

57. $\sum_{n=0}^{\infty} (-5)^n x^n$

- **58.** $\sum_{n=1}^{\infty} (x+2)^n$
- **59.** $\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}$
- **60.** $\sum_{n=0}^{\infty} (-4)^n (x-5)^n$

 $\mathbf{61.} \ \sum_{n=0}^{\infty} \frac{2^n}{x^n}$

62. $\sum_{n=0}^{\infty} \frac{\sin^n x}{3^n}$

- **63.** $\sum_{n=0}^{\infty} e^{nx}$
- 64. We have seen that the harmonic series is a divergent series whose terms approach 0. Show that

$$\sum_{n=1}^{\infty} \ln \left(1 + \frac{1}{n} \right)$$

is another series with this property.

- CAS 65-66 Use the partial fraction command on your CAS to find a convenient expression for the partial sum, and then use this expression to find the sum of the series. Check your answer by using the CAS to sum the series directly.

 - **65.** $\sum_{n=1}^{\infty} \frac{3n^2 + 3n + 1}{(n^2 + n)^3}$ **66.** $\sum_{n=3}^{\infty} \frac{1}{n^5 5n^3 + 4n}$
 - **67.** If the *n*th partial sum of a series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = \frac{n-1}{n+1}$$

find a_n and $\sum_{n=1}^{\infty} a_n$.

- **68.** If the *n*th partial sum of a series $\sum_{n=1}^{\infty} a_n$ is $s_n = 3 n2^{-n}$, find a_n and $\sum_{n=1}^{\infty} a_n$.
- 69. A patient takes 150 mg of a drug at the same time every day. Just before each tablet is taken, 5% of the drug remains in the
 - (a) What quantity of the drug is in the body after the third tablet? After the *n*th tablet?
 - (b) What quantity of the drug remains in the body in the long run?
- **70.** After injection of a dose D of insulin, the concentration of insulin in a patient's system decays exponentially and so it can be written as De^{-at} , where t represents time in hours and a is a positive constant.
 - (a) If a dose D is injected every T hours, write an expression for the sum of the residual concentrations just before the (n + 1)st injection.
 - (b) Determine the limiting pre-injection concentration.
 - (c) If the concentration of insulin must always remain at or above a critical value C, determine a minimal dosage D in terms of C, a, and T.
- 71. When money is spent on goods and services, those who receive the money also spend some of it. The people receiving some of the twice-spent money will spend some of that, and so on. Economists call this chain reaction the multiplier effect. In a hypothetical isolated community, the local government begins the process by spending D dollars. Suppose that each recipient of spent money spends 100c% and saves 100s% of the money that he or she receives. The values c and s are called the marginal propensity to consume and the marginal propensity to save and, of course, c + s = 1.
 - (a) Let S_n be the total spending that has been generated after n transactions. Find an equation for S_n .
 - (b) Show that $\lim_{n\to\infty} S_n = kD$, where k = 1/s. The number k is called the *multiplier*. What is the multiplier if the marginal propensity to consume is 80%?

Note: The federal government uses this principle to justify deficit spending. Banks use this principle to justify lending a large percentage of the money that they receive in deposits.

- 72. A certain ball has the property that each time it falls from a height h onto a hard, level surface, it rebounds to a height rh, where 0 < r < 1. Suppose that the ball is dropped from an initial height of H meters.
 - (a) Assuming that the ball continues to bounce indefinitely, find the total distance that it travels.
 - (b) Calculate the total time that the ball travels. (Use the fact that the ball falls $\frac{1}{2}gt^2$ meters in t seconds.)
 - (c) Suppose that each time the ball strikes the surface with velocity v it rebounds with velocity -kv, where 0 < k < 1. How long will it take for the ball to come to rest?
- **73.** Find the value of c if

$$\sum_{n=0}^{\infty} (1+c)^{-n} = 2$$

74. Find the value of c such that

$$\sum_{n=0}^{\infty} e^{nc} = 10$$

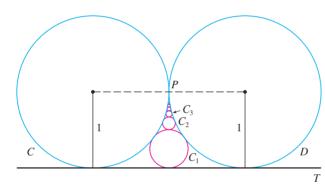
75. In Example 8 we showed that the harmonic series is divergent. Here we outline another method, making use of the fact that $e^x > 1 + x$ for any x > 0. (See Exercise 4.3.78.)

If s_n is the *n*th partial sum of the harmonic series, show that $e^{s_n} > n + 1$. Why does this imply that the harmonic series is divergent?

76. Graph the curves $y = x^n$, $0 \le x \le 1$, for $n = 0, 1, 2, 3, 4, \dots$ on a common screen. By finding the areas between successive curves, give a geometric demonstration of the fact, shown in Example 7, that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

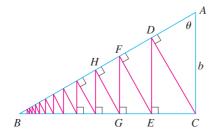
77. The figure shows two circles C and D of radius 1 that touch at P. T is a common tangent line; C_1 is the circle that touches C, D, and T; C_2 is the circle that touches C, D, and C_1 ; C_3 is the circle that touches C, D, and C_2 . This procedure can be continued indefinitely and produces an infinite sequence of circles $\{C_n\}$. Find an expression for the diameter of C_n and thus provide another geometric demonstration of Example 7.



78. A right triangle *ABC* is given with $\angle A = \theta$ and |AC| = b. *CD* is drawn perpendicular to *AB*, *DE* is drawn perpendicular to *BC*, $EF \perp AB$, and this process is continued indefinitely, as shown in the figure. Find the total length of all the perpendiculars

$$|CD| + |DE| + |EF| + |FG| + \cdots$$

in terms of b and θ .



79. What is wrong with the following calculation?

$$0 = 0 + 0 + 0 + \cdots$$

$$= (1 - 1) + (1 - 1) + (1 - 1) + \cdots$$

$$= 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

$$= 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \cdots$$

$$= 1 + 0 + 0 + 0 + \cdots = 1$$

(Guido Ubaldus thought that this proved the existence of God because "something has been created out of nothing.")

- **80.** Suppose that $\sum_{n=1}^{\infty} a_n (a_n \neq 0)$ is known to be a convergent series. Prove that $\sum_{n=1}^{\infty} 1/a_n$ is a divergent series.
- **81.** Prove part (i) of Theorem 8.
- **82.** If Σa_n is divergent and $c \neq 0$, show that Σca_n is divergent.
- **83.** If $\sum a_n$ is convergent and $\sum b_n$ is divergent, show that the series $\sum (a_n + b_n)$ is divergent. [*Hint:* Argue by contradiction.]
- **84.** If $\sum a_n$ and $\sum b_n$ are both divergent, is $\sum (a_n + b_n)$ necessarily divergent?
- **85.** Suppose that a series $\sum a_n$ has positive terms and its partial sums s_n satisfy the inequality $s_n \le 1000$ for all n. Explain why $\sum a_n$ must be convergent.
- **86.** The Fibonacci sequence was defined in Section 11.1 by the equations

$$f_1 = 1$$
, $f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$ $n \ge 3$

Show that each of the following statements is true.

(a)
$$\frac{1}{f_{n-1}f_{n+1}} = \frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}}$$

(b)
$$\sum_{n=2}^{\infty} \frac{1}{f_{n-1} f_{n+1}} = 1$$

(c)
$$\sum_{n=2}^{\infty} \frac{f_n}{f_{n-1} f_{n+1}} = 2$$

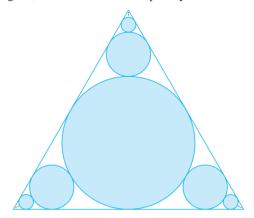
- 87. The Cantor set, named after the German mathematician Georg Cantor (1845–1918), is constructed as follows. We start with the closed interval [0, 1] and remove the open interval $(\frac{1}{3}, \frac{2}{3})$. That leaves the two intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ and we remove the open middle third of each. Four intervals remain and again we remove the open middle third of each of them. We continue this procedure indefinitely, at each step removing the open middle third of every interval that remains from the preceding step. The Cantor set consists of the numbers that remain in [0, 1] after all those intervals have been removed.
 - (a) Show that the total length of all the intervals that are removed is 1. Despite that, the Cantor set contains infinitely many numbers. Give examples of some numbers in the Cantor set.
 - (b) The **Sierpinski carpet** is a two-dimensional counterpart of the Cantor set. It is constructed by removing the center one-ninth of a square of side 1, then removing the centers

of the eight smaller remaining squares, and so on. (The figure shows the first three steps of the construction.) Show that the sum of the areas of the removed squares is 1. This implies that the Sierpinski carpet has area 0.



- **88.** (a) A sequence $\{a_n\}$ is defined recursively by the equation $a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$ for $n \ge 3$, where a_1 and a_2 can be any real numbers. Experiment with various values of a_1 and a_2 and use your calculator to guess the limit of the sequence.
 - (b) Find $\lim_{n\to\infty} a_n$ in terms of a_1 and a_2 by expressing $a_{n+1} a_n$ in terms of $a_2 a_1$ and summing a series.
- **89.** Consider the series $\sum_{n=1}^{\infty} n/(n+1)!$.
 - (a) Find the partial sums s_1 , s_2 , s_3 , and s_4 . Do you recognize the denominators? Use the pattern to guess a formula for s_n .

- (b) Use mathematical induction to prove your guess.
- (c) Show that the given infinite series is convergent, and find its sum.
- **90.** In the figure there are infinitely many circles approaching the vertices of an equilateral triangle, each circle touching other circles and sides of the triangle. If the triangle has sides of length 1, find the total area occupied by the circles.



The Integral Test and Estimates of Sums

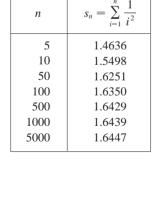
In general, it is difficult to find the exact sum of a series. We were able to accomplish this for geometric series and the series $\sum 1/[n(n+1)]$ because in each of those cases we could find a simple formula for the nth partial sum s_n . But usually it isn't easy to discover such a formula. Therefore, in the next few sections, we develop several tests that enable us to determine whether a series is convergent or divergent without explicitly finding its sum. (In some cases, however, our methods will enable us to find good estimates of the sum.) Our first test involves improper integrals.

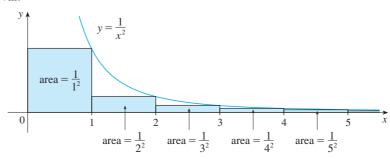
We begin by investigating the series whose terms are the reciprocals of the squares of the positive integers:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

There's no simple formula for the sum s_n of the first n terms, but the computer-generated table of approximate values given in the margin suggests that the partial sums are approaching a number near 1.64 as $n \to \infty$ and so it looks as if the series is convergent.

We can confirm this impression with a geometric argument. Figure 1 shows the curve $y = 1/x^2$ and rectangles that lie below the curve. The base of each rectangle is an interval of length 1; the height is equal to the value of the function $y = 1/x^2$ at the right endpoint of the interval.





So the sum of the areas of the rectangles is

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

If we exclude the first rectangle, the total area of the remaining rectangles is smaller than the area under the curve $y = 1/x^2$ for $x \ge 1$, which is the value of the integral $\int_1^{\infty} (1/x^2) dx$. In Section 7.8 we discovered that this improper integral is convergent and has value 1. So the picture shows that all the partial sums are less than

$$\frac{1}{1^2} + \int_1^\infty \frac{1}{x^2} \, dx = 2$$

Thus the partial sums are bounded. We also know that the partial sums are increasing (because all the terms are positive). Therefore the partial sums converge (by the Monotonic Sequence Theorem) and so the series is convergent. The sum of the series (the limit of the partial sums) is also less than 2:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots < 2$$

[The exact sum of this series was found by the Swiss mathematician Leonhard Euler (1707–1783) to be $\pi^2/6$, but the proof of this fact is quite difficult. (See Problem 6 in the Problems Plus following Chapter 15.)]

Now let's look at the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots$$

The table of values of s_n suggests that the partial sums aren't approaching a finite number, so we suspect that the given series may be divergent. Again we use a picture for confirmation. Figure 2 shows the curve $y = 1/\sqrt{x}$, but this time we use rectangles whose tops lie *above* the curve.

n	$s_n = \sum_{i=1}^n \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

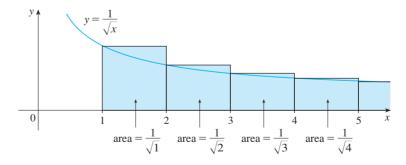


FIGURE 2

The base of each rectangle is an interval of length 1. The height is equal to the value of the function $y = 1/\sqrt{x}$ at the *left* endpoint of the interval. So the sum of the areas of all the rectangles is

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \dots = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

This total area is greater than the area under the curve $y = 1/\sqrt{x}$ for $x \ge 1$, which is equal

to the integral $\int_{1}^{\infty} (1/\sqrt{x}) dx$. But we know from Section 7.8 that this improper integral is divergent. In other words, the area under the curve is infinite. So the sum of the series must be infinite; that is, the series is divergent.

The same sort of geometric reasoning that we used for these two series can be used to prove the following test. (The proof is given at the end of this section.)

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_{1}^{\infty} f(x) dx$ is convergent. In other words:

- (i) If $\int_{1}^{\infty} f(x) dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
- (ii) If $\int_{1}^{\infty} f(x) dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

NOTE When we use the Integral Test, it is not necessary to start the series or the integral at n = 1. For instance, in testing the series

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2}$$
 we use $\int_{4}^{\infty} \frac{1}{(x-3)^2} dx$

Also, it is not necessary that f be always decreasing. What is important is that f be ultimately decreasing, that is, decreasing for x larger than some number N. Then $\sum_{n=N}^{\infty} a_n$ is convergent, so $\sum_{n=1}^{\infty} a_n$ is convergent by Note 4 of Section 11.2.

EXAMPLE 1 Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$ for convergence or divergence.

SOLUTION The function $f(x) = 1/(x^2 + 1)$ is continuous, positive, and decreasing on $[1, \infty)$ so we use the Integral Test:

$$\int_{1}^{\infty} \frac{1}{x^{2} + 1} dx = \lim_{t \to \infty} \int_{1}^{t} \frac{1}{x^{2} + 1} dx = \lim_{t \to \infty} \tan^{-1}x \Big]_{1}^{t}$$
$$= \lim_{t \to \infty} \left(\tan^{-1}t - \frac{\pi}{4} \right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Thus $\int_{1}^{\infty} 1/(x^2+1) dx$ is a convergent integral and so, by the Integral Test, the series $\sum 1/(n^2+1)$ is convergent.

V EXAMPLE 2 For what values of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

SOLUTION If p < 0, then $\lim_{n\to\infty} (1/n^p) = \infty$. If p = 0, then $\lim_{n\to\infty} (1/n^p) = 1$. In either case $\lim_{n\to\infty} (1/n^p) \neq 0$, so the given series diverges by the Test for Divergence (11.2.7).

If p > 0, then the function $f(x) = 1/x^p$ is clearly continuous, positive, and decreasing on $[1, \infty)$. We found in Chapter 7 [see (7.8.2)] that

 $\int_{1}^{\infty} \frac{1}{x^{p}} dx$ converges if p > 1 and diverges if $p \le 1$

In order to use the Integral Test we need to be able to evaluate $\int_1^\infty f(x) \, dx$ and therefore we have to be able to find an antiderivative of f. Frequently this is difficult or impossible, so we need other tests for convergence too.

It follows from the Integral Test that the series $\sum 1/n^p$ converges if p > 1 and diverges if 0 . (For <math>p = 1, this series is the harmonic series discussed in Example 8 in Section 11.2.)

The series in Example 2 is called the *p*-series. It is important in the rest of this chapter, so we summarize the results of Example 2 for future reference as follows.

The *p*-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if p > 1 and divergent if $p \le 1$.

EXAMPLE 3

(a) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$$

is convergent because it is a p-series with p = 3 > 1.

(b) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots$$

is divergent because it is a *p*-series with $p = \frac{1}{3} < 1$.

NOTE We should *not* infer from the Integral Test that the sum of the series is equal to the value of the integral. In fact,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{whereas} \quad \int_1^{\infty} \frac{1}{x^2} dx = 1$$

Therefore, in general,

$$\sum_{n=1}^{\infty} a_n \neq \int_{1}^{\infty} f(x) \ dx$$

EXAMPLE 4 Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

SOLUTION The function $f(x) = (\ln x)/x$ is positive and continuous for x > 1 because the logarithm function is continuous. But it is not obvious whether or not f is decreasing, so we compute its derivative:

$$f'(x) = \frac{(1/x)x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

Thus f'(x) < 0 when $\ln x > 1$, that is, x > e. It follows that f is decreasing when x > e and so we can apply the Integral Test:

$$\int_{1}^{\infty} \frac{\ln x}{x} dx = \lim_{t \to \infty} \int_{1}^{t} \frac{\ln x}{x} dx = \lim_{t \to \infty} \frac{(\ln x)^{2}}{2} \bigg]_{1}^{t}$$
$$= \lim_{t \to \infty} \frac{(\ln t)^{2}}{2} = \infty$$

Since this improper integral is divergent, the series $\Sigma (\ln n)/n$ is also divergent by the Integral Test.

Estimating the Sum of a Series

Suppose we have been able to use the Integral Test to show that a series $\sum a_n$ is convergent and we now want to find an approximation to the sum s of the series. Of course, any partial sum s_n is an approximation to s because $\lim_{n\to\infty} s_n = s$. But how good is such an approximation? To find out, we need to estimate the size of the **remainder**

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

The remainder R_n is the error made when s_n , the sum of the first n terms, is used as an approximation to the total sum.

We use the same notation and ideas as in the Integral Test, assuming that f is decreasing on $[n, \infty)$. Comparing the areas of the rectangles with the area under y = f(x) for x > n in Figure 3, we see that

$$R_n = a_{n+1} + a_{n+2} + \cdots \leq \int_n^\infty f(x) \, dx$$

Similarly, we see from Figure 4 that

$$R_n = a_{n+1} + a_{n+2} + \cdots \ge \int_{n+1}^{\infty} f(x) dx$$

So we have proved the following error estimate.

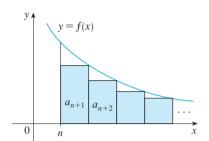


FIGURE 3

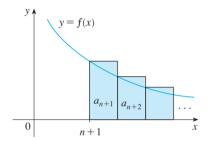


FIGURE 4

2 Remainder Estimate for the Integral Test Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \ge n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x) \, dx \le R_n \le \int_{n}^{\infty} f(x) \, dx$$

V EXAMPLE 5

- (a) Approximate the sum of the series $\sum 1/n^3$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.
- (b) How many terms are required to ensure that the sum is accurate to within 0.0005?

SOLUTION In both parts (a) and (b) we need to know $\int_n^\infty f(x) dx$. With $f(x) = 1/x^3$, which satisfies the conditions of the Integral Test, we have

$$\int_{n}^{\infty} \frac{1}{x^{3}} dx = \lim_{t \to \infty} \left[-\frac{1}{2x^{2}} \right]_{n}^{t} = \lim_{t \to \infty} \left(-\frac{1}{2t^{2}} + \frac{1}{2n^{2}} \right) = \frac{1}{2n^{2}}$$

(a) Approximating the sum of the series by the 10th partial sum, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx s_{10} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \dots + \frac{1}{10^3} \approx 1.1975$$

According to the remainder estimate in 2, we have

$$R_{10} \le \int_{10}^{\infty} \frac{1}{x^3} dx = \frac{1}{2(10)^2} = \frac{1}{200}$$

So the size of the error is at most 0.005.

$$R_n \leqslant \int_n^\infty \frac{1}{x^3} \, dx = \frac{1}{2n^2}$$

we want

$$\frac{1}{2n^2} < 0.0005$$

Solving this inequality, we get

$$n^2 > \frac{1}{0.001} = 1000$$
 or $n > \sqrt{1000} \approx 31.6$

We need 32 terms to ensure accuracy to within 0.0005.

If we add s_n to each side of the inequalities in $\boxed{2}$, we get

$$s_n + \int_{n+1}^{\infty} f(x) dx \le s \le s_n + \int_{n}^{\infty} f(x) dx$$

because $s_n + R_n = s$. The inequalities in $\boxed{3}$ give a lower bound and an upper bound for s. They provide a more accurate approximation to the sum of the series than the partial sum s_n does.

Although Euler was able to calculate the exact sum of the p-series for p=2, nobody has been able to find the exact sum for p=3. In Example 6, however, we show how to *estimate* this sum.

EXAMPLE 6 Use 3 with n = 10 to estimate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

SOLUTION The inequalities in [3] become

$$s_{10} + \int_{11}^{\infty} \frac{1}{x^3} dx \le s \le s_{10} + \int_{10}^{\infty} \frac{1}{x^3} dx$$

From Example 5 we know that

$$\int_n^\infty \frac{1}{x^3} \, dx = \frac{1}{2n^2}$$

so

$$s_{10} + \frac{1}{2(11)^2} \le s \le s_{10} + \frac{1}{2(10)^2}$$

Using $s_{10} \approx 1.197532$, we get

$$1.201664 \le s \le 1.202532$$

If we approximate *s* by the midpoint of this interval, then the error is at most half the length of the interval. So

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.2021 \quad \text{with error} < 0.0005$$

If we compare Example 6 with Example 5, we see that the improved estimate in $\boxed{3}$ can be much better than the estimate $s \approx s_n$. To make the error smaller than 0.0005 we had to use 32 terms in Example 5 but only 10 terms in Example 6.

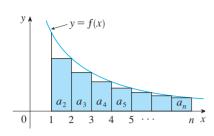


FIGURE 5

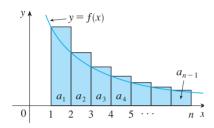


FIGURE 6

Proof of the Integral Test

We have already seen the basic idea behind the proof of the Integral Test in Figures 1 and 2 for the series $\sum 1/n^2$ and $\sum 1/\sqrt{n}$. For the general series $\sum a_n$, look at Figures 5 and 6. The area of the first shaded rectangle in Figure 5 is the value of f at the right endpoint of [1, 2], that is, $f(2) = a_2$. So, comparing the areas of the shaded rectangles with the area under v = f(x) from 1 to n, we see that

$$a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x) \, dx$$

(Notice that this inequality depends on the fact that f is decreasing.) Likewise, Figure 6 shows that

$$\int_{1}^{n} f(x) dx \le a_{1} + a_{2} + \dots + a_{n-1}$$

(i) If $\int_{1}^{\infty} f(x) dx$ is convergent, then 4 gives

$$\sum_{i=2}^{n} a_i \le \int_1^n f(x) \, dx \le \int_1^\infty f(x) \, dx$$

since $f(x) \ge 0$. Therefore

$$s_n = a_1 + \sum_{i=2}^n a_i \le a_1 + \int_1^\infty f(x) \, dx = M$$
, say

Since $s_n \leq M$ for all n, the sequence $\{s_n\}$ is bounded above. Also

$$s_{n+1} = s_n + a_{n+1} \geqslant s_n$$

since $a_{n+1} = f(n+1) \ge 0$. Thus $\{s_n\}$ is an increasing bounded sequence and so it is convergent by the Monotonic Sequence Theorem (11.1.12). This means that $\sum a_n$ is convergent.

(ii) If $\int_{1}^{\infty} f(x) dx$ is divergent, then $\int_{1}^{n} f(x) dx \to \infty$ as $n \to \infty$ because $f(x) \ge 0$. But 5gives

$$\int_{1}^{n} f(x) \ dx \le \sum_{i=1}^{n-1} a_{i} = s_{n-1}$$

and so $s_{n-1} \to \infty$. This implies that $s_n \to \infty$ and so Σ a_n diverges.

Exercises

1. Draw a picture to show that

$$\sum_{n=0}^{\infty} \frac{1}{n^{1.3}} < \int_{1}^{\infty} \frac{1}{r^{1.3}} dx$$

What can you conclude about the series?

2. Suppose f is a continuous positive decreasing function for $x \ge 1$ and $a_n = f(n)$. By drawing a picture, rank the following three quantities in increasing order:

$$\int_{1}^{6} f(x) \, dx \qquad \sum_{i=1}^{5} a_{i} \qquad \sum_{i=2}^{6} a_{i}$$

$$\sum_{i=1}^{5} a_i$$

$$\sum_{i=2}^{6} a_i$$

3-8 Use the Integral Test to determine whether the series is convergent or divergent.

3.
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$$

4.
$$\sum_{n=1}^{\infty} \frac{1}{n^5}$$

5.
$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^3}$$

6.
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$$

7.
$$\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$$

8.
$$\sum_{n=1}^{\infty} n^2 e^{-n^3}$$

9.
$$\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$$

10.
$$\sum_{n=0.9999}^{\infty} n^{-0.9999}$$

11.
$$1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots$$

12.
$$1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots$$

13.
$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots$$

14.
$$\frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \frac{1}{14} + \frac{1}{17} + \cdots$$

15.
$$\sum_{n=1}^{\infty} \frac{\sqrt{n} + 4}{n^2}$$

16.
$$\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$$

17.
$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 4}$$

18.
$$\sum_{n=3}^{\infty} \frac{3n-4}{n^2-2n}$$

19.
$$\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$$

20.
$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 6n + 13}$$

21.
$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

22.
$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

23.
$$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$$

24.
$$\sum_{n=3}^{\infty} \frac{n^2}{e^n}$$

25.
$$\sum_{n=1}^{\infty} \frac{1}{n^2 + n^3}$$

26.
$$\sum_{n=1}^{\infty} \frac{n}{n^4 + 1}$$

27–28 Explain why the Integral Test can't be used to determine whether the series is convergent.

27.
$$\sum_{n=1}^{\infty} \frac{\cos \pi n}{\sqrt{n}}$$

28.
$$\sum_{n=1}^{\infty} \frac{\cos^2 n}{1 + n^2}$$

29–32 Find the values of p for which the series is convergent.

29.
$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

30.
$$\sum_{n=3}^{\infty} \frac{1}{n \ln n \left[\ln(\ln n) \right]^p}$$

31.
$$\sum_{n=1}^{\infty} n(1+n^2)^p$$

$$32. \sum_{n=1}^{\infty} \frac{\ln n}{n^p}$$

33. The Riemann zeta-function ζ is defined by

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

and is used in number theory to study the distribution of prime numbers. What is the domain of ζ ?

34. Leonhard Euler was able to calculate the exact sum of the p-series with p = 2:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

(See page 715.) Use this fact to find the sum of each series.

(a)
$$\sum_{n=2}^{\infty} \frac{1}{n^2}$$

(b)
$$\sum_{n=3}^{\infty} \frac{1}{(n+1)^2}$$

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(c)
$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2}$$

35. Euler also found the sum of the *p*-series with p = 4:

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Use Euler's result to find the sum of the series.

(a)
$$\sum_{n=1}^{\infty} \left(\frac{3}{n} \right)^4$$

(b)
$$\sum_{k=5}^{\infty} \frac{1}{(k-2)^4}$$

36. (a) Find the partial sum s_{10} of the series $\sum_{n=1}^{\infty} 1/n^4$. Estimate the error in using s_{10} as an approximation to the sum of the series.

(b) Use $\boxed{3}$ with n = 10 to give an improved estimate of the sum.

(c) Compare your estimate in part (b) with the exact value given in Exercise 35.

(d) Find a value of n so that s_n is within 0.00001 of the sum.

37. (a) Use the sum of the first 10 terms to estimate the sum of the series $\sum_{n=1}^{\infty} 1/n^2$. How good is this estimate?

(b) Improve this estimate using $\boxed{3}$ with n = 10.

(c) Compare your estimate in part (b) with the exact value given in Exercise 34.

(d) Find a value of *n* that will ensure that the error in the approximation $s \approx s_n$ is less than 0.001.

38. Find the sum of the series $\sum_{n=1}^{\infty} 1/n^5$ correct to three decimal places.

39. Estimate $\sum_{n=1}^{\infty} (2n+1)^{-6}$ correct to five decimal places.

40. How many terms of the series $\sum_{n=2}^{\infty} 1/[n(\ln n)^2]$ would you need to add to find its sum to within 0.01?

41. Show that if we want to approximate the sum of the series $\sum_{n=1}^{\infty} n^{-1.001}$ so that the error is less than 5 in the ninth decimal place, then we need to add more than $10^{11,301}$ terms!

CAS 42. (a) Show that the series $\sum_{n=1}^{\infty} (\ln n)^2/n^2$ is convergent.

(b) Find an upper bound for the error in the approximation $s \approx s_n$.

(c) What is the smallest value of *n* such that this upper bound is less than 0.05?

(d) Find s_n for this value of n.

43. (a) Use $\boxed{4}$ to show that if s_n is the *n*th partial sum of the harmonic series, then

$$s_n \leq 1 + \ln n$$

- (b) The harmonic series diverges, but very slowly. Use part (a) to show that the sum of the first million terms is less than 15 and the sum of the first billion terms is less than 22.
- **44.** Use the following steps to show that the sequence

$$t_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n$$

has a limit. (The value of the limit is denoted by γ and is called Euler's constant.)

(a) Draw a picture like Figure 6 with f(x) = 1/x and interpret t_n as an area [or use 5] to show that $t_n > 0$ for all n.

(b) Interpret

$$t_n - t_{n+1} = [\ln(n+1) - \ln n] - \frac{1}{n+1}$$

as a difference of areas to show that $t_n - t_{n+1} > 0$. Therefore $\{t_n\}$ is a decreasing sequence.

- (c) Use the Monotonic Sequence Theorem to show that $\{t_n\}$ is convergent.
- **45.** Find all positive values of *b* for which the series $\sum_{n=1}^{\infty} b^{\ln n}$ converges.
- **46.** Find all values of c for which the following series converges.

$$\sum_{n=1}^{\infty} \left(\frac{c}{n} - \frac{1}{n+1} \right)$$

The Comparison Tests

In the comparison tests the idea is to compare a given series with a series that is known to be convergent or divergent. For instance, the series

1

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$$

reminds us of the series $\sum_{n=1}^{\infty} 1/2^n$, which is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$ and is therefore convergent. Because the series $\boxed{1}$ is so similar to a convergent series, we have the feeling that it too must be convergent. Indeed, it is. The inequality

$$\frac{1}{2^n + 1} < \frac{1}{2^n}$$

shows that our given series
1 has smaller terms than those of the geometric series and therefore all its partial sums are also smaller than 1 (the sum of the geometric series). This means that its partial sums form a bounded increasing sequence, which is convergent. It also follows that the sum of the series is less than the sum of the geometric series:

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1} < 1$$

Similar reasoning can be used to prove the following test, which applies only to series whose terms are positive. The first part says that if we have a series whose terms are *smaller* than those of a known *convergent* series, then our series is also convergent. The second part says that if we start with a series whose terms are *larger* than those of a known *divergent* series, then it too is divergent.

The Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

- (i) If Σb_n is convergent and $a_n \le b_n$ for all n, then Σa_n is also convergent.
- (ii) If Σb_n is divergent and $a_n \ge b_n$ for all n, then Σa_n is also divergent.

It is important to keep in mind the distinction between a sequence and a series. A sequence is a list of numbers, whereas a series is a sum. With every series Σ a_n there are associated two sequences: the sequence $\{a_n\}$ of terms and the sequence $\{s_n\}$ of partial sums.

PROOF

(i) Let
$$s_n = \sum_{i=1}^n a_i$$
 $t_n = \sum_{i=1}^n b_i$ $t = \sum_{n=1}^\infty b_n$

Since both series have positive terms, the sequences $\{s_n\}$ and $\{t_n\}$ are increasing $(s_{n+1} = s_n + a_{n+1} \ge s_n)$. Also $t_n \to t$, so $t_n \le t$ for all n. Since $a_i \le b_i$, we have $s_n \le t_n$. Thus $s_n \le t$ for all n. This means that $\{s_n\}$ is increasing and bounded above and therefore converges by the Monotonic Sequence Theorem. Thus $\sum a_n$ converges.

(ii) If Σ b_n is divergent, then $t_n \to \infty$ (since $\{t_n\}$ is increasing). But $a_i \ge b_i$ so $s_n \ge t_n$. Thus $s_n \to \infty$. Therefore Σ a_n diverges.

Standard Series for Use with the Comparison Test

In using the Comparison Test we must, of course, have some known series $\sum b_n$ for the purpose of comparison. Most of the time we use one of these series:

- A p-series $\left[\sum 1/n^p \text{ converges if } p > 1 \text{ and diverges if } p \le 1; \text{ see } (11.3.1)\right]$
- A geometric series $\left[\sum ar^{n-1} \text{ converges if } |r| < 1 \text{ and diverges if } |r| \ge 1;$ see (11.2.4)

V EXAMPLE 1 Determine whether the series $\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$ converges or diverges.

SOLUTION For large n the dominant term in the denominator is $2n^2$, so we compare the given series with the series $\sum 5/(2n^2)$. Observe that

$$\frac{5}{2n^2 + 4n + 3} < \frac{5}{2n^2}$$

because the left side has a bigger denominator. (In the notation of the Comparison Test, a_n is the left side and b_n is the right side.) We know that

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent because it's a constant times a p-series with p = 2 > 1. Therefore

$$\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$$

is convergent by part (i) of the Comparison Test.

NOTE 1 Although the condition $a_n \le b_n$ or $a_n \ge b_n$ in the Comparison Test is given for all n, we need verify only that it holds for $n \ge N$, where N is some fixed integer, because the convergence of a series is not affected by a finite number of terms. This is illustrated in the next example.

EXAMPLE 2 Test the series $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ for convergence or divergence.

SOLUTION We used the Integral Test to test this series in Example 4 of Section 11.3, but we can also test it by comparing it with the harmonic series. Observe that $\ln k > 1$ for $k \ge 3$ and so

$$\frac{\ln k}{k} > \frac{1}{k} \qquad k \ge 3$$

We know that $\sum 1/k$ is divergent (*p*-series with p=1). Thus the given series is divergent by the Comparison Test.

NOTE 2 The terms of the series being tested must be smaller than those of a convergent series or larger than those of a divergent series. If the terms are larger than the terms of a convergent series or smaller than those of a divergent series, then the Comparison Test doesn't apply. Consider, for instance, the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

The inequality

$$\frac{1}{2^n-1} > \frac{1}{2^n}$$

is useless as far as the Comparison Test is concerned because Σ $b_n = \Sigma \left(\frac{1}{2}\right)^n$ is convergent and $a_n > b_n$. Nonetheless, we have the feeling that $\Sigma 1/(2^n - 1)$ ought to be convergent because it is very similar to the convergent geometric series $\Sigma \left(\frac{1}{2}\right)^n$. In such cases the following test can be used.

The Limit Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty}\frac{a_n}{b_n}=c$$

where c is a finite number and c > 0, then either both series converge or both diverge.

PROOF Let m and M be positive numbers such that m < c < M. Because a_n/b_n is close to c for large n, there is an integer N such that

$$m < \frac{a_n}{b_n} < M$$
 when $n > N$

and so

$$mb_n < a_n < Mb_n$$
 when $n > N$

If Σ b_n converges, so does Σ Mb_n . Thus Σ a_n converges by part (i) of the Comparison Test. If Σ b_n diverges, so does Σ mb_n and part (ii) of the Comparison Test shows that Σ a_n diverges.

EXAMPLE 3 Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.

SOLUTION We use the Limit Comparison Test with

$$a_n = \frac{1}{2^n - 1} \qquad b_n = \frac{1}{2^n}$$

and obtain

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{1/(2^n - 1)}{1/2^n} = \lim_{n \to \infty} \frac{2^n}{2^n - 1} = \lim_{n \to \infty} \frac{1}{1 - 1/2^n} = 1 > 0$$

Exercises 40 and 41 deal with the cases c = 0 and $c = \infty$.

Since this limit exists and $\sum 1/2^n$ is a convergent geometric series, the given series converges by the Limit Comparison Test.

EXAMPLE 4 Determine whether the series $\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ converges or diverges.

SOLUTION The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$. This suggests taking

$$a_n = \frac{2n^2 + 3n}{\sqrt{5 + n^5}} \qquad b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}$$

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}} \cdot \frac{n^{1/2}}{2} = \lim_{n \to \infty} \frac{2n^{5/2} + 3n^{3/2}}{2\sqrt{5 + n^5}}$$

$$= \lim_{n \to \infty} \frac{2 + \frac{3}{n}}{2\sqrt{\frac{5}{n^5} + 1}} = \frac{2 + 0}{2\sqrt{0 + 1}} = 1$$

Since $\sum b_n = 2 \sum 1/n^{1/2}$ is divergent (*p*-series with $p = \frac{1}{2} < 1$), the given series diverges by the Limit Comparison Test.

Notice that in testing many series we find a suitable comparison series $\sum b_n$ by keeping only the highest powers in the numerator and denominator.

Estimating Sums

If we have used the Comparison Test to show that a series Σ a_n converges by comparison with a series Σ b_n , then we may be able to estimate the sum Σ a_n by comparing remainders. As in Section 11.3, we consider the remainder

$$R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots$$

For the comparison series $\sum b_n$ we consider the corresponding remainder

$$T_n = t - t_n = b_{n+1} + b_{n+2} + \cdots$$

Since $a_n \le b_n$ for all n, we have $R_n \le T_n$. If Σb_n is a p-series, we can estimate its remainder T_n as in Section 11.3. If Σb_n is a geometric series, then T_n is the sum of a geometric series and we can sum it exactly (see Exercises 35 and 36). In either case we know that R_n is smaller than T_n .

EXAMPLE 5 Use the sum of the first 100 terms to approximate the sum of the series $\sum 1/(n^3 + 1)$. Estimate the error involved in this approximation.

SOLUTION Since

$$\frac{1}{n^3+1} < \frac{1}{n^3}$$

the given series is convergent by the Comparison Test. The remainder T_n for the comparison series $\sum 1/n^3$ was estimated in Example 5 in Section 11.3 using the Remainder Estimate for the Integral Test. There we found that

$$T_n \leqslant \int_n^\infty \frac{1}{x^3} \, dx = \frac{1}{2n^2}$$

Therefore the remainder R_n for the given series satisfies

$$R_n \leqslant T_n \leqslant \frac{1}{2n^2}$$

With n = 100 we have

$$R_{100} \le \frac{1}{2(100)^2} = 0.00005$$

Using a programmable calculator or a computer, we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + 1} \approx \sum_{n=1}^{100} \frac{1}{n^3 + 1} \approx 0.6864538$$

with error less than 0.00005.

11.4 Exercises

- **1.** Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is known to be convergent.
 - (a) If $a_n > b_n$ for all n, what can you say about $\sum a_n$? Why?
 - (b) If $a_n < b_n$ for all n, what can you say about $\sum a_n$? Why?
- **2.** Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is known to be divergent.
 - (a) If $a_n > b_n$ for all n, what can you say about $\sum a_n$? Why?
 - (b) If $a_n < b_n$ for all n, what can you say about $\sum a_n$? Why?
- 3-32 Determine whether the series converges or diverges.

3.
$$\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$$

4.
$$\sum_{n=3}^{\infty} \frac{n^3}{n^4 - 1}$$

5.
$$\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$$

$$6. \sum_{n=1}^{\infty} \frac{n-1}{n^2 \sqrt{n}}$$

7.
$$\sum_{n=1}^{\infty} \frac{9^n}{3 + 10^n}$$

8.
$$\sum_{n=1}^{\infty} \frac{6^n}{5^n - 1}$$

$$9. \sum_{k=1}^{\infty} \frac{\ln k}{k}$$

10.
$$\sum_{k=1}^{\infty} \frac{k \sin^2 k}{1 + k^3}$$

11.
$$\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}}{\sqrt{k^3 + 4k + 3}}$$

12.
$$\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$$

13.
$$\sum_{n=1}^{\infty} \frac{\arctan n}{n^{1.2}}$$

14.
$$\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$$

15.
$$\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n - 2}$$

16.
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4+1}}$$

17.
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$$

18.
$$\sum_{n=1}^{\infty} \frac{1}{2n+3}$$

19.
$$\sum_{n=1}^{\infty} \frac{1+4^n}{1+3^n}$$

20.
$$\sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$$

21. $\sum_{n=1}^{\infty} \frac{\sqrt{n+2}}{2n^2+n+1}$

22.
$$\sum_{n=3}^{\infty} \frac{n+2}{(n+1)^3}$$

23.
$$\sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$$

24.
$$\sum_{n=1}^{\infty} \frac{n^2 - 5n}{n^3 + n + 1}$$

25.
$$\sum_{n=1}^{\infty} \frac{\sqrt{n^4 + 1}}{n^3 + n^2}$$

26.
$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$$

27.
$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^2 e^{-n}$$

28.
$$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$$

29.
$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

30.
$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

31.
$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

32.
$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$$

33–36 Use the sum of the first 10 terms to approximate the sum of the series. Estimate the error.

33.
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4 + 1}}$$

34.
$$\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3}$$

35.
$$\sum_{n=1}^{\infty} 5^{-n} \cos^2 n$$

36.
$$\sum_{n=1}^{\infty} \frac{1}{3^n + 4^n}$$

37. The meaning of the decimal representation of a number $0.d_1d_2d_3...$ (where the digit d_i is one of the numbers 0, 1, 2, ..., 9) is that

$$0.d_1d_2d_3d_4... = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{d_4}{10^4} + \cdots$$

Show that this series always converges.

39. Prove that if $a_n \ge 0$ and $\sum a_n$ converges, then $\sum a_n^2$ also converges.

40. (a) Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is convergent. Prove that if

$$\lim_{n\to\infty}\frac{a_n}{b_n}=0$$

then $\sum a_n$ is also convergent.

(b) Use part (a) to show that the series converges.

(i)
$$\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$$

(i)
$$\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$$
 (ii)
$$\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n} e^n}$$

41. (a) Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is divergent. Prove that if

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\infty$$

then $\sum a_n$ is also divergent.

(b) Use part (a) to show that the series diverges.

(i)
$$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$

(i)
$$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$
 (ii)
$$\sum_{n=1}^{\infty} \frac{\ln n}{n}$$

42. Give an example of a pair of series $\sum a_n$ and $\sum b_n$ with positive terms where $\lim_{n\to\infty} (a_n/b_n) = 0$ and $\sum b_n$ diverges, but $\sum a_n$ converges. (Compare with Exercise 40.)

43. Show that if $a_n > 0$ and $\lim_{n \to \infty} n a_n \neq 0$, then $\sum a_n$ is divergent.

44. Show that if $a_n > 0$ and $\sum a_n$ is convergent, then $\sum \ln(1 + a_n)$ is convergent.

45. If $\sum a_n$ is a convergent series with positive terms, is it true that $\sum \sin(a_n)$ is also convergent?

46. If $\sum a_n$ and $\sum b_n$ are both convergent series with positive terms, is it true that $\sum a_n b_n$ is also convergent?

Alternating Series

The convergence tests that we have looked at so far apply only to series with positive terms. In this section and the next we learn how to deal with series whose terms are not necessarily positive. Of particular importance are alternating series, whose terms alternate

An alternating series is a series whose terms are alternately positive and negative. Here are two examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \dots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$$

We see from these examples that the *n*th term of an alternating series is of the form

$$a_n = (-1)^{n-1}b_n$$
 or $a_n = (-1)^n b_n$

where b_n is a positive number. (In fact, $b_n = |a_n|$.)

The following test says that if the terms of an alternating series decrease toward 0 in absolute value, then the series converges.

Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies

(i)
$$b_{n+1} \leq b_n$$
 for all n

(ii)
$$\lim_{n\to\infty} b_n = 0$$

then the series is convergent.

Before giving the proof let's look at Figure 1, which gives a picture of the idea behind the proof. We first plot $s_1 = b_1$ on a number line. To find s_2 we subtract b_2 , so s_2 is to the left of s_1 . Then to find s_3 we add b_3 , so s_3 is to the right of s_2 . But, since $b_3 < b_2$, s_3 is to the left of s_1 . Continuing in this manner, we see that the partial sums oscillate back and forth. Since $b_n \rightarrow 0$, the successive steps are becoming smaller and smaller. The even partial sums s_2 , s_4 , s_6 , . . . are increasing and the odd partial sums s_1 , s_3 , s_5 , . . . are decreasing. Thus it seems plausible that both are converging to some number s, which is the sum of the series. Therefore we consider the even and odd partial sums separately in the following proof.

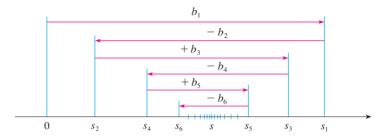


FIGURE 1

PROOF OF THE ALTERNATING SERIES TEST We first consider the even partial sums:

$$s_2 = b_1 - b_2 \ge 0 \qquad \text{since } b_2 \le b_1$$

$$s_4 = s_2 + (b_3 - b_4) \ge s_2 \qquad \text{since } b_4 \le b_3$$
In general
$$s_{2n} = s_{2n-2} + (b_{2n-1} - b_{2n}) \ge s_{2n-2} \qquad \text{since } b_{2n} \le b_{2n-1}$$
Thus
$$0 \le s_2 \le s_4 \le s_6 \le \dots \le s_{2n} \le \dots$$

But we can also write

$$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n}$$

Every term in brackets is positive, so $s_{2n} \le b_1$ for all n. Therefore the sequence $\{s_{2n}\}$ of even partial sums is increasing and bounded above. It is therefore convergent by the Monotonic Sequence Theorem. Let's call its limit s, that is,

$$\lim_{n\to\infty} s_{2n} = s$$

Now we compute the limit of the odd partial sums:

$$\lim_{n \to \infty} s_{2n+1} = \lim_{n \to \infty} (s_{2n} + b_{2n+1})$$

$$= \lim_{n \to \infty} s_{2n} + \lim_{n \to \infty} b_{2n+1}$$

$$= s + 0$$
 [by condition (ii)]
$$= s$$

Since both the even and odd partial sums converge to s, we have $\lim_{n\to\infty} s_n = s$ [see Exercise 92(a) in Section 11.1] and so the series is convergent.

Figure 2 illustrates Example 1 by showing the graphs of the terms $a_n = (-1)^{n-1}/n$ and the partial sums s_n . Notice how the values of s_n zigzag across the limiting value, which appears to be about 0.7. In fact, it can be proved that the exact sum of the series is $\ln 2 \approx 0.693$ (see Exercise 36).

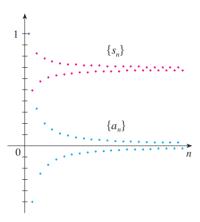


FIGURE 2

Instead of verifying condition (i) of the Alternating Series Test by computing a derivative, we could verify that $b_{n+1} < b_n$ directly by using the technique of Solution 1 of Example 13 in Section 11.1.

EXAMPLE 1 The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

satisfies

(i)
$$b_{n+1} < b_n$$
 because $\frac{1}{n+1} < \frac{1}{n}$

(ii)
$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

so the series is convergent by the Alternating Series Test.

EXAMPLE 2 The series $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ is alternating, but

$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{3n}{4n - 1} = \lim_{n \to \infty} \frac{3}{4 - \frac{1}{n}} = \frac{3}{4}$$

so condition (ii) is not satisfied. Instead, we look at the limit of the *n*th term of the series:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n 3n}{4n-1}$$

This limit does not exist, so the series diverges by the Test for Divergence.

EXAMPLE 3 Test the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3 + 1}$ for convergence or divergence.

SOLUTION The given series is alternating so we try to verify conditions (i) and (ii) of the Alternating Series Test.

Unlike the situation in Example 1, it is not obvious that the sequence given by $b_n = n^2/(n^3 + 1)$ is decreasing. However, if we consider the related function $f(x) = x^2/(x^3 + 1)$, we find that

$$f'(x) = \frac{x(2-x^3)}{(x^3+1)^2}$$

Since we are considering only positive x, we see that f'(x) < 0 if $2 - x^3 < 0$, that is, $x > \sqrt[3]{2}$. Thus f is decreasing on the interval $(\sqrt[3]{2}, \infty)$. This means that f(n+1) < f(n) and therefore $b_{n+1} < b_n$ when $n \ge 2$. (The inequality $b_2 < b_1$ can be verified directly but all that really matters is that the sequence $\{b_n\}$ is eventually decreasing.)

Condition (ii) is readily verified:

$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{n^2}{n^3 + 1} = \lim_{n \to \infty} \frac{\frac{1}{n}}{1 + \frac{1}{n^3}} = 0$$

Thus the given series is convergent by the Alternating Series Test.

Estimating Sums

A partial sum s_n of any convergent series can be used as an approximation to the total sum s, but this is not of much use unless we can estimate the accuracy of the approximation. The error involved in using $s \approx s_n$ is the remainder $R_n = s - s_n$. The next theorem says that for series that satisfy the conditions of the Alternating Series Test, the size of the error is smaller than b_{n+1} , which is the absolute value of the first neglected term.

You can see geometrically why the Alternating Series Estimation Theorem is true by looking at Figure 1 (on page 728). Notice that $s-s_4 < b_5$, $|s-s_5| < b_6$, and so on. Notice also that s lies between any two consecutive partial sums.

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Alternating Series Estimation Theorem If $s = \sum (-1)^{n-1}b_n$ is the sum of an alternating series that satisfies

(i)
$$b_{n+1} \leq b_n$$
 and (ii) $\lim_{n \to \infty} b_n = 0$

then

$$|R_n| = |s - s_n| \leqslant b_{n+1}$$

PROOF We know from the proof of the Alternating Series Test that s lies between any two consecutive partial sums s_n and s_{n+1} . (There we showed that s is larger than all even partial sums. A similar argument shows that s is smaller than all the odd sums.) It follows that

$$|s - s_n| \le |s_{n+1} - s_n| = b_{n+1}$$

By definition, 0! = 1.

EXAMPLE 4 Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

SOLUTION We first observe that the series is convergent by the Alternating Series Test because

(i)
$$\frac{1}{(n+1)!} = \frac{1}{n!(n+1)} < \frac{1}{n!}$$

(ii)
$$0 < \frac{1}{n!} < \frac{1}{n} \to 0$$
 so $\frac{1}{n!} \to 0$ as $n \to \infty$

To get a feel for how many terms we need to use in our approximation, let's write out the first few terms of the series:

$$s = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \cdots$$
$$= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040} + \cdots$$

Notice that

$$b_7 = \frac{1}{5040} < \frac{1}{5000} = 0.0002$$

and

$$s_6 = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \approx 0.368056$$

By the Alternating Series Estimation Theorem we know that

$$|s - s_6| \le b_7 < 0.0002$$

This error of less than 0.0002 does not affect the third decimal place, so we have $s \approx 0.368$ correct to three decimal places.

In Section 11.10 we will prove that $e^x = \sum_{n=0}^{\infty} x^n/n!$ for all x, so what we have obtained in Example 4 is actually an approximation to the number e^{-1} .

NOTE The rule that the error (in using s_n to approximate s) is smaller than the first neglected term is, in general, valid only for alternating series that satisfy the conditions of the Alternating Series Estimation Theorem. The rule does not apply to other types of series.

Exercises 11.5

- **1.** (a) What is an alternating series?
 - (b) Under what conditions does an alternating series
 - (c) If these conditions are satisfied, what can you say about the remainder after *n* terms?
- **2–20** Test the series for convergence or divergence.

2.
$$\frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9} + \frac{2}{11} - \cdots$$

3.
$$-\frac{2}{5} + \frac{4}{6} - \frac{6}{7} + \frac{8}{8} - \frac{10}{9} + \cdots$$

4.
$$\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} - \cdots$$

5.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$$

6.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+4)}$$

7.
$$\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$$

8.
$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$$

9.
$$\sum_{n=1}^{\infty} (-1)^n e^{-n}$$

10.
$$\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{2n+3}$$

11.
$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3 + 4}$$
 12. $\sum_{n=1}^{\infty} (-1)^{n+1} n e^{-n}$

12.
$$\sum_{n=1}^{\infty} (-1)^{n+1} n e^{-n}$$

13.
$$\sum_{n=1}^{\infty} (-1)^{n-1} e^{2/n}$$

14.
$$\sum_{n=1}^{\infty} (-1)^{n-1} \arctan n$$

15.
$$\sum_{n=0}^{\infty} \frac{\sin(n+\frac{1}{2})\pi}{1+\sqrt{n}}$$

$$16. \sum_{n=1}^{\infty} \frac{n \cos n\pi}{2^n}$$

$$17. \sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right)$$

$$18. \sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$$

19.
$$\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$$

20.
$$\sum_{n=1}^{\infty} (-1)^n (\sqrt{n+1} - \sqrt{n})$$

21-22 Graph both the sequence of terms and the sequence of partial sums on the same screen. Use the graph to make a rough estimate of the sum of the series. Then use the Alternating Series Estimation Theorem to estimate the sum correct to four decimal

21.
$$\sum_{n=1}^{\infty} \frac{(-0.8)^n}{n!}$$

22.
$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{8^n}$$

23-26 Show that the series is convergent. How many terms of the series do we need to add in order to find the sum to the indicated accuracy?

23.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6} \quad (|\operatorname{error}| < 0.00005)$$

24.
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n \, 5^n} \quad (|\operatorname{error}| < 0.0001)$$

25.
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{10^n n!} \quad (|\operatorname{error}| < 0.000005)$$

26.
$$\sum_{n=1}^{\infty} (-1)^{n-1} n e^{-n} \quad (|\operatorname{error}| < 0.01)$$

27–30 Approximate the sum of the series correct to four decimal places.

27.
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$$

28.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$$

29.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{10^n}$$

30.
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n n!}$$

- **31.** Is the 50th partial sum s_{50} of the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1}/n$ an overestimate or an underestimate of the total sum? Explain.
- **32–34** For what values of p is each series convergent?

32.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$$

33.
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+p}$$

34.
$$\sum_{n=2}^{\infty} (-1)^{n-1} \frac{(\ln n)^p}{n}$$

35. Show that the series $\sum (-1)^{n-1}b_n$, where $b_n = 1/n$ if n is odd and $b_n = 1/n^2$ if n is even, is divergent. Why does the Alternating Series Test not apply?

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2$$

Let h_n and s_n be the partial sums of the harmonic and alternating harmonic series.

(a) Show that $s_{2n} = h_{2n} - h_n$.

(b) From Exercise 44 in Section 11.3 we have

$$h_n - \ln n \rightarrow \gamma$$
 as $n \rightarrow \infty$

and therefore

$$h_{2n} - \ln(2n) \rightarrow \gamma$$
 as $n \rightarrow \infty$

Use these facts together with part (a) to show that $s_{2n} \rightarrow \ln 2$ as $n \rightarrow \infty$.

Absolute Convergence and the Ratio and Root Tests

Given any series $\sum a_n$, we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

whose terms are the absolute values of the terms of the original series.

We have convergence tests for series with positive terms and for alternating series. But what if the signs of the terms switch back and forth irregularly? We will see in Example 3 that the idea of absolute convergence sometimes helps in such cases.

1 Definition A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

Notice that if $\sum a_n$ is a series with positive terms, then $|a_n| = a_n$ and so absolute convergence is the same as convergence in this case.

EXAMPLE 1 The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is absolutely convergent because

$$\left| \sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \right|$$

is a convergent p-series (p = 2).

EXAMPLE 2 We know that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is convergent (see Example 1 in Section 11.5), but it is not absolutely convergent because the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

which is the harmonic series (p-series with p = 1) and is therefore divergent.

2 Definition A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

Example 2 shows that the alternating harmonic series is conditionally convergent. Thus it is possible for a series to be convergent but not absolutely convergent. However, the next theorem shows that absolute convergence implies convergence.

3 Theorem If a series $\sum a_n$ is absolutely convergent, then it is convergent.

PROOF Observe that the inequality

$$0 \le a_n + |a_n| \le 2|a_n|$$

is true because $|a_n|$ is either a_n or $-a_n$. If $\sum a_n$ is absolutely convergent, then $\sum |a_n|$ is convergent, so $\sum 2|a_n|$ is convergent. Therefore, by the Comparison Test, $\sum (a_n + |a_n|)$ is convergent. Then

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$$

is the difference of two convergent series and is therefore convergent.

V EXAMPLE 3 Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$$

is convergent or divergent.

SOLUTION This series has both positive and negative terms, but it is not alternating. (The first term is positive, the next three are negative, and the following three are positive: The signs change irregularly.) We can apply the Comparison Test to the series of absolute values

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{\left| \cos n \right|}{n^2}$$

Since $|\cos n| \le 1$ for all n, we have

$$\frac{|\cos n|}{n^2} \leqslant \frac{1}{n^2}$$

We know that $\Sigma 1/n^2$ is convergent (*p*-series with p=2) and therefore $\Sigma |\cos n|/n^2$ is convergent by the Comparison Test. Thus the given series $\Sigma (\cos n)/n^2$ is absolutely convergent and therefore convergent by Theorem 3.

The following test is very useful in determining whether a given series is absolutely convergent.

Figure 1 shows the graphs of the terms a_n and partial sums s_n of the series in Example 3. Notice that the series is not alternating but has positive and negative terms.

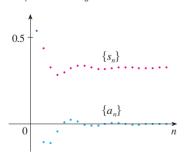


FIGURE 1

The Ratio Test

- (i) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
- (ii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- (iii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

PROOF

(i) The idea is to compare the given series with a convergent geometric series. Since L < 1, we can choose a number r such that L < r < 1. Since

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \quad \text{and} \quad L < r$$

the ratio $|a_{n+1}/a_n|$ will eventually be less than r; that is, there exists an integer N such that

$$\left| \frac{a_{n+1}}{a_n} \right| < r$$
 whenever $n \ge N$

or, equivalently,

$$|a_{n+1}| < |a_n|r$$
 whenever $n \ge N$

Putting *n* successively equal to N, N + 1, N + 2, . . . in $\boxed{4}$, we obtain

$$|a_{N+1}| < |a_N|r$$

 $|a_{N+2}| < |a_{N+1}|r < |a_N|r^2$
 $|a_{N+3}| < |a_{N+2}|r < |a_N|r^3$

and, in general,

$$|a_{N+k}| < |a_N| r^k \qquad \text{for all } k \ge 1$$

Now the series

$$\sum_{k=1}^{\infty} |a_N| r^k = |a_N| r + |a_N| r^2 + |a_N| r^3 + \cdots$$

is convergent because it is a geometric series with 0 < r < 1. So the inequality $\boxed{5}$ together with the Comparison Test, shows that the series

$$\sum_{n=N+1}^{\infty} |a_n| = \sum_{k=1}^{\infty} |a_{N+k}| = |a_{N+1}| + |a_{N+2}| + |a_{N+3}| + \cdots$$

(ii) If $|a_{n+1}/a_n| \to L > 1$ or $|a_{n+1}/a_n| \to \infty$, then the ratio $|a_{n+1}/a_n|$ will eventually be greater than 1; that is, there exists an integer N such that

$$\left| \frac{a_{n+1}}{a_n} \right| > 1$$
 whenever $n \ge N$

This means that $|a_{n+1}| > |a_n|$ whenever $n \ge N$ and so

$$\lim_{n\to\infty}a_n\neq 0$$

Therefore $\sum a_n$ diverges by the Test for Divergence.

NOTE Part (iii) of the Ratio Test says that if $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$, the test gives no information. For instance, for the convergent series $\sum 1/n^2$ we have

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} = \frac{1}{\left(1 + \frac{1}{n}\right)^2} \to 1 \quad \text{as } n \to \infty$$

whereas for the divergent series $\sum 1/n$ we have

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to 1$$
 as $n \to \infty$

Therefore, if $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$, the series $\sum a_n$ might converge or it might diverge. In this case the Ratio Test fails and we must use some other test.

EXAMPLE 4 Test the series $\sum_{n=0}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolute convergence.

SOLUTION We use the Ratio Test with $a_n = (-1)^n n^3 / 3^n$:

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}} \right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$
$$= \frac{1}{3} \left(\frac{n+1}{n} \right)^3 = \frac{1}{3} \left(1 + \frac{1}{n} \right)^3 \to \frac{1}{3} < 1$$

Thus, by the Ratio Test, the given series is absolutely convergent and therefore convergent.

The Ratio Test is usually conclusive if the nth term of the series contains an exponential or a factorial, as we will see in Examples 4 and 5.

Estimating Sums

In the last three sections we used various methods for estimating the sum of a series—the method depended on which test was used to prove convergence. What about series for which the Ratio Test works? There are two possibilities: If the series happens to be an alternating series, as in Example 4, then it is best to use the methods of Section 11.5. If the terms are all positive, then use the special methods explained in Exercise 38.

SOLUTION Since the terms $a_n = n^n/n!$ are positive, we don't need the absolute value signs.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}$$

$$= \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n}$$

$$= \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n \to e \quad \text{as } n \to \infty$$

(see Equation 3.6.6). Since e > 1, the given series is divergent by the Ratio Test.

NOTE Although the Ratio Test works in Example 5, an easier method is to use the Test for Divergence. Since

$$a_n = \frac{n^n}{n!} = \frac{n \cdot n \cdot n \cdot \dots \cdot n}{1 \cdot 2 \cdot 3 \cdot \dots \cdot n} \ge n$$

it follows that a_n does not approach 0 as $n \to \infty$. Therefore the given series is divergent by the Test for Divergence.

The following test is convenient to apply when *n*th powers occur. Its proof is similar to the proof of the Ratio Test and is left as Exercise 41.

The Root Test

- (i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
- (ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- (iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then part (iii) of the Root Test says that the test gives no information. The series $\sum a_n$ could converge or diverge. (If L=1 in the Ratio Test, don't try the Root Test because L will again be 1. And if L=1 in the Root Test, don't try the Ratio Test because it will fail too.)

EXAMPLE 6 Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2} \right)^n$. SOLUTION

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2+\frac{3}{n}}{3+\frac{2}{n}} \to \frac{2}{3} < 1$$

Thus the given series converges by the Root Test.

The question of whether a given convergent series is absolutely convergent or conditionally convergent has a bearing on the question of whether infinite sums behave like finite sums.

If we rearrange the order of the terms in a finite sum, then of course the value of the sum remains unchanged. But this is not always the case for an infinite series. By a rearrange**ment** of an infinite series $\sum a_n$ we mean a series obtained by simply changing the order of the terms. For instance, a rearrangement of $\sum a_n$ could start as follows:

$$a_1 + a_2 + a_5 + a_3 + a_4 + a_{15} + a_6 + a_7 + a_{20} + \cdots$$

It turns out that

if $\sum a_n$ is an absolutely convergent series with sum s, then any rearrangement of $\sum a_n$ has the same sum s.

However, any conditionally convergent series can be rearranged to give a different sum. To illustrate this fact let's consider the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = \ln 2$$

(See Exercise 36 in Section 11.5.) If we multiply this series by $\frac{1}{2}$, we get

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots = \frac{1}{2} \ln 2$$

Inserting zeros between the terms of this series, we have

$$0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \dots = \frac{1}{2} \ln 2$$

Now we add the series in Equations 6 and 7 using Theorem 11.2.8:

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \dots = \frac{3}{2} \ln 2$$

Notice that the series in 8 contains the same terms as in 6, but rearranged so that one negative term occurs after each pair of positive terms. The sums of these series, however, are different. In fact, Riemann proved that

if $\sum a_n$ is a conditionally convergent series and r is any real number whatsoever, then there is a rearrangement of $\sum a_n$ that has a sum equal to r.

A proof of this fact is outlined in Exercise 44.

Adding these zeros does not affect the sum of the series: each term in the sequence of partial sums is repeated, but the limit is the same.

Exercises 11.6

1. What can you say about the series $\sum a_n$ in each of the following cases?

(a)
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 8$$

(a)
$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 8$$
 (b) $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0.8$

(c)
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$$

2–30 Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

2.
$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$$

$$3. \sum_{n=1}^{\infty} \frac{n}{5^n}$$

4.
$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2 + 4}$$

5.
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{5n+1}$$

$$\sum_{n=0}^{\infty} 5n +$$

7.
$$\sum_{k=1}^{\infty} k(\frac{2}{3})^k$$

9.
$$\sum_{n=1}^{\infty} (-1)^n \frac{(1.1)^n}{n^4}$$

11.
$$\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^3}$$

13.
$$\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$$

6.
$$\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$$

8.
$$\sum_{n=1}^{\infty} \frac{n!}{100^n}$$

10.
$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$$

12.
$$\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}$$

14. $\sum_{n=1}^{\infty} \frac{n^{10}}{(-10)^{n+1}}$

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15.
$$\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{n^2}$$

16.
$$\sum_{n=1}^{\infty} \frac{3 - \cos n}{n^{2/3} - 2}$$

17.
$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$$

18.
$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

19.
$$\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$$

20.
$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$$

21.
$$\sum_{n=1}^{\infty} \left(\frac{n^2 + 1}{2n^2 + 1} \right)^n$$

$$22. \sum_{n=2}^{\infty} \left(\frac{-2n}{n+1} \right)^{5n}$$

23.
$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2}$$

24.
$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$$

25.
$$\sum_{n=1}^{\infty} \frac{n^{100} 100^n}{n!}$$

26.
$$\sum_{n=1}^{\infty} \frac{2^{n^2}}{n!}$$

27.
$$1 - \frac{1 \cdot 3}{3!} + \frac{1 \cdot 3 \cdot 5}{5!} - \frac{1 \cdot 3 \cdot 5 \cdot 7}{7!} + \cdots + (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n-1)!} + \cdots$$

28.
$$\frac{2}{5} + \frac{2 \cdot 6}{5 \cdot 8} + \frac{2 \cdot 6 \cdot 10}{5 \cdot 8 \cdot 11} + \frac{2 \cdot 6 \cdot 10 \cdot 14}{5 \cdot 8 \cdot 11 \cdot 14} + \cdots$$

29.
$$\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}{n!}$$

30.
$$\sum_{n=1}^{\infty} (-1)^n \frac{2^n n!}{5 \cdot 8 \cdot 11 \cdot \cdots \cdot (3n+2)}$$

31. The terms of a series are defined recursively by the equations

$$a_1 = 2 a_{n+1} = \frac{5n+1}{4n+3} a_n$$

Determine whether $\sum a_n$ converges or diverges.

32. A series $\sum a_n$ is defined by the equations

$$a_1 = 1$$
 $a_{n+1} = \frac{2 + \cos n}{\sqrt{n}} a_n$

Determine whether $\sum a_n$ converges or diverges.

33-34 Let $\{b_n\}$ be a sequence of positive numbers that converges to $\frac{1}{2}$. Determine whether the given series is absolutely convergent.

$$33. \sum_{n=1}^{\infty} \frac{b_n^n \cos n\pi}{n}$$

34.
$$\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n b_1 b_2 b_3 \cdots b_n}$$

35. For which of the following series is the Ratio Test inconclusive (that is, it fails to give a definite answer)?

(a)
$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

(b)
$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

(c)
$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{\sqrt{n}}$$
 (d) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}$

(d)
$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n}$$

36. For which positive integers k is the following series convergent?

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$$

37. (a) Show that $\sum_{n=0}^{\infty} x^n/n!$ converges for all x.

(b) Deduce that $\lim_{n\to\infty} x^n/n! = 0$ for all x.

38. Let $\sum a_n$ be a series with positive terms and let $r_n = a_{n+1}/a_n$. Suppose that $\lim_{n\to\infty} r_n = L < 1$, so $\sum a_n$ converges by the Ratio Test. As usual, we let R_n be the remainder after n terms, that is.

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

(a) If $\{r_n\}$ is a decreasing sequence and $r_{n+1} < 1$, show, by summing a geometric series, that

$$R_n \leqslant \frac{a_{n+1}}{1 - r_{n+1}}$$

(b) If $\{r_n\}$ is an increasing sequence, show that

$$R_n \leq \frac{a_{n+1}}{1 - I}$$

39. (a) Find the partial sum s_5 of the series $\sum_{n=1}^{\infty} 1/(n2^n)$. Use Exercise 38 to estimate the error in using s_5 as an approximation to the sum of the series.

(b) Find a value of n so that s_n is within 0.00005 of the sum. Use this value of n to approximate the sum of the series.

40. Use the sum of the first 10 terms to approximate the sum of the series

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

Use Exercise 38 to estimate the error.

41. Prove the Root Test. [Hint for part (i): Take any number r such that L < r < 1 and use the fact that there is an integer N such that $\sqrt[n]{|a_n|} < r$ whenever $n \ge N$.]

42. Around 1910, the Indian mathematician Srinivasa Ramanujan discovered the formula

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}}$$

William Gosper used this series in 1985 to compute the first 17 million digits of π .

(a) Verify that the series is convergent.

(b) How many correct decimal places of π do you get if you use just the first term of the series? What if you use two terms?

43. Given any series $\sum a_n$, we define a series $\sum a_n^+$ whose terms are all the positive terms of $\sum a_n$ and a series $\sum a_n^-$ whose terms are all the negative terms of $\sum a_n$. To be specific, we let

$$a_n^+ = \frac{a_n + |a_n|}{2}$$
 $a_n^- = \frac{a_n - |a_n|}{2}$

Notice that if $a_n > 0$, then $a_n^+ = a_n$ and $a_n^- = 0$, whereas if $a_n < 0$, then $a_n^- = a_n$ and $a_n^+ = 0$.

- (a) If $\sum a_n$ is absolutely convergent, show that both of the series $\sum a_n^+$ and $\sum a_n^-$ are convergent.
- (b) If Σ a_n is conditionally convergent, show that both of the series Σ a_n^+ and Σ a_n^- are divergent.
- **44.** Prove that if $\sum a_n$ is a conditionally convergent series and r is any real number, then there is a rearrangement of $\sum a_n$ whose sum is r. [Hints: Use the notation of Exercise 43.

Take just enough positive terms a_n^+ so that their sum is greater than r. Then add just enough negative terms a_n^- so that the cumulative sum is less than r. Continue in this manner and use Theorem 11.2.6.]

- **45.** Suppose the series $\sum a_n$ is conditionally convergent.
 - (a) Prove that the series $\sum n^2 a_n$ is divergent.
 - (b) Conditional convergence of $\sum a_n$ is not enough to determine whether $\sum na_n$ is convergent. Show this by giving an example of a conditionally convergent series such that $\sum na_n$ converges and an example where $\sum na_n$ diverges.

11.7

Strategy for Testing Series

We now have several ways of testing a series for convergence or divergence; the problem is to decide which test to use on which series. In this respect, testing series is similar to integrating functions. Again there are no hard and fast rules about which test to apply to a given series, but you may find the following advice of some use.

It is not wise to apply a list of the tests in a specific order until one finally works. That would be a waste of time and effort. Instead, as with integration, the main strategy is to classify the series according to its *form*.

- **1.** If the series is of the form $\sum 1/n^p$, it is a *p*-series, which we know to be convergent if p > 1 and divergent if $p \le 1$.
- **2.** If the series has the form $\sum ar^{n-1}$ or $\sum ar^n$, it is a geometric series, which converges if |r| < 1 and diverges if $|r| \ge 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.
- 3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or an algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series. Notice that most of the series in Exercises 11.4 have this form. (The value of p should be chosen as in Section 11.4 by keeping only the highest powers of p in the numerator and denominator.) The comparison tests apply only to series with positive terms, but if p and test for absolute terms, then we can apply the Comparison Test to p and test for absolute convergence.
- **4.** If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then the Test for Divergence should be used.
- **5.** If the series is of the form $\Sigma (-1)^{n-1}b_n$ or $\Sigma (-1)^nb_n$, then the Alternating Series Test is an obvious possibility.
- 6. Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test. Bear in mind that | a_{n+1}/a_n| → 1 as n → ∞ for all p-series and therefore all rational or algebraic functions of n. Thus the Ratio Test should not be used for such series.
- 7. If a_n is of the form $(b_n)^n$, then the Root Test may be useful.
- **8.** If $a_n = f(n)$, where $\int_1^\infty f(x) dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

In the following examples we don't work out all the details but simply indicate which tests should be used.

EXAMPLE 1 $\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$

Since $a_n \to \frac{1}{2} \neq 0$ as $n \to \infty$, we should use the Test for Divergence.

EXAMPLE 2
$$\sum_{n=1}^{\infty} \frac{\sqrt{n^3 + 1}}{3n^3 + 4n^2 + 2}$$

Since a_n is an algebraic function of n, we compare the given series with a p-series. The comparison series for the Limit Comparison Test is $\sum b_n$, where

$$b_n = \frac{\sqrt{n^3}}{3n^3} = \frac{n^{3/2}}{3n^3} = \frac{1}{3n^{3/2}}$$

EXAMPLE 3
$$\sum_{n=1}^{\infty} ne^{-n^2}$$

Since the integral $\int_{1}^{\infty} xe^{-x^2} dx$ is easily evaluated, we use the Integral Test. The Ratio Test also works.

EXAMPLE 4
$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4 + 1}$$

Since the series is alternating, we use the Alternating Series Test.

V EXAMPLE 5
$$\sum_{k=1}^{\infty} \frac{2^k}{k!}$$

Since the series involves k!, we use the Ratio Test.

EXAMPLE 6
$$\sum_{n=1}^{\infty} \frac{1}{2+3^n}$$

Since the series is closely related to the geometric series $\sum 1/3^n$, we use the Comparison Test.

Exercises

1–38 Test the series for convergence or divergence.

1.
$$\sum_{n=1}^{\infty} \frac{1}{n+3^n}$$

2.
$$\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$$

11.
$$\sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{1}{3^n} \right)$$

12.
$$\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k^2+1}}$$

3.
$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$$

4.
$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + 2}$$
 13. $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

13.
$$\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$$

14.
$$\sum_{n=1}^{\infty} \frac{\sin 2n}{1 + 2^n}$$

5.
$$\sum_{n=1}^{\infty} \frac{n^2 2^{n-1}}{(-5)^n}$$

6.
$$\sum_{n=1}^{\infty} \frac{1}{2n+1}$$

15.
$$\sum_{k=1}^{\infty} \frac{2^{k-1}3^{k+1}}{k^k}$$

16.
$$\sum_{n=1}^{\infty} \frac{n^2 + 1}{n^3 + 1}$$

7.
$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$$

8.
$$\sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)!}$$

17.
$$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n-1)}$$

9.
$$\sum_{k=1}^{\infty} k^2 e^{-k}$$

10.
$$\sum_{n=1}^{\infty} n^2 e^{-n^3}$$

18.
$$\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$$