We then enumerate this array using the "diagonal procedure":

$$a_{11},\ a_{12},\ a_{21},\ a_{13},\ a_{22},\ a_{31},\ a_{14}, \ldots,$$

as was displayed in Figure 1.3.1.

Georg Cantor

Georg Cantor (1845–1918) was born in St. Petersburg, Russia. His father, a Danish businessman working in Russia, moved the family to Germany several years later. Cantor studied briefly at Zurich, then went to the University of Berlin, the best in mathematics at the time. He received his doctorate in 1869, and accepted a position at the University of Halle, where he worked alone on his research, but would occasionally travel the seventy miles to Berlin to visit colleagues.

Cantor is known as the founder of modern set theory and he was the first to study the concept of infinite set in rigorous detail. In 1874 he proved that $\mathbb{Q}$ is countable and, in contrast, that $\mathbb{R}$ is uncountable (see Section 2.5), exhibiting two kinds of infinity. In a series of papers he developed a general theory of infinite sets, including some surprising results. In 1877 he proved that the two-dimensional unit square in the plane could be put into one-one correspondence with the unit interval on the line, a result he sent in a letter to his colleague Richard Dedekind in Berlin, writing "I see it, but I do not believe it." Cantor's Theorem on sets of subsets shows there are many different orders of infinity and this led him to create a theory of "transfinite" numbers that he published in 1895 and 1897. His work generated considerable controversy among mathematicians of that era, but in 1904, London's Royal Society awarded Cantor the Sylvester Medal, its highest honor.

Beginning in 1884, he suffered from episodes of depression that increased in severity as the years passed. He was hospitalized several times for nervous breakdowns in the Halle Nervenklinik and spent the last seven months of his life there.

We close this section with one of Cantor's more remarkable theorems.

1.3.13 Cantor's Theorem *If A is any set, then there is no surjection of A onto the set $\mathcal{P}(A)$ of all subsets of A.*

Proof. Suppose that $\varphi : A \to \mathcal{P}(A)$ is a surjection. Since $\varphi(a)$ is a subset of A, either a belongs to $\varphi(a)$ or it does not belong to this set. We let

$$D := \{a \in A : a \notin \varphi(a)\}.$$

Since D is a subset of A, if φ is a surjection, then $D = \varphi(a_0)$ for some $a_0 \in A$.

We must have either $a_0 \in D$ or $a_0 \notin D$. If $a_0 \in D$, then since $D = \varphi(a_0)$, we must have $a_0 \in \varphi(a_0)$, contrary to the definition of D. Similarly, if $a_0 \notin D$, then $a_0 \notin \varphi(a_0)$ so that $a_0 \in D$, which is also a contradiction.

Therefore, φ cannot be a surjection. Q.E.D.

Cantor's Theorem implies that there is an unending progression of larger and larger sets. In particular, it implies that the collection $\mathcal{P}(\mathbb{N})$ of all subsets of the natural numbers $\mathbb{N}$ is uncountable.