(03) Series

Wk 4 Material; Topic 3; Due 28 March

The Cauchy Criterion (3.5)

The Cauchy Convergence Criterion

A sequence is convergent if and only if it is a Cauchy sequence

- **Cauchy Sequence** implies **Convergence**
  - Every Cauchy sequence of real numbers is bounded, hence by the Bolzano-Weierstrass theorem the sequence has a convergent subsequence, hence is itself convergent.
- **Convergence** imples **Cauchy Sequence**
  - If two terms can be made arbitrarily close then any term can be made arbitrarily close to another term in the set (which will be the limit point).

Properly Divergent

A series (x_n) is said to be properly divergent if $\lim_{n\to\infty}(x_n) = \pm\infty$

Definition of a Series [3.7.1]

if x_n is a sequence, then the **series** generated by the sequence is $S = (s_k)$:

- The terms of the sequence are $x_n) = (x_1, x_2, x_3, x_4, \dots s_n)$

The terms of the series are $(s_n) = (s_1, s_2, s_3, s_4, \dots s_n)$

The terms of the series are called the **partial sums** and are defined as such:

$$
\begin{aligned}
S_1 &= u_1 &&= u_1 \\
S_2 &= S_1 + u_2 &&= u_1 + u_2 \\
S_3 &= s_2 + u_3 &&= u_1 + u_2 + u_3 \\
S_4 &= S_4 + u_4 &&= u_1 + u_2 + u_3 + u_4 \\
&\dots \\
S_n &= S_n + u_n &&= u_1 + u_2 + u_3 + \dots u_n
\end{aligned}
$$

This was the old scale

$$
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S_2 &= S_1 + u_2 &&= u_1 + u_2 \\
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&\dots \\
S_n &= S_n + u_n &&= u_1 + u_2 + u_3 + \dots u_n
\end{aligned}
$$

Common Series Types

These are series that we are expected to memorise because they so often appear in series problems (and moreover we we will need them for the exam).

Geometric Series (3.7.6 (a))

The Geometric Series is Convergent if and only if $|r| <$:

$$\sum_{n=1}^{\infty}[r^n] = 1 + r + r^2 + r^3 + \dots r^n$$

iff $|r|<1$ then this is convergent

$$|r|<1 \Rightarrow \sum_{n=1}^{\infty}[r^n] = \frac{1}{1-r}$$

$$r \geqslant 1 \Rightarrow \lim(r^n) > 0$$
$$\Rightarrow \text{divergence}$$

Harmonic Series (3.7.6(b))

The Harmonic series $\sum_{n=1}^{\infty}[1/n]$ is divergent:

assume S converges to a number:

$$\begin{aligned} S &= (1 + 1/2) + (1/3 + 1/4) + (1/5 + 1/6) \cdots + \left(\tfrac{1}{2n-1} + 1/2n\right) \\ &> (1/2 + 1/2) + (1/4 + 1/4) + (1/6 + 1/6) \cdots + (1/2n + 1/2n) \\ &= (1) + (2/4) + (2/6) \cdots + (1/n) \\ &= 1 + (1/2) + (1/3) \cdots + (1/n) \\ &= S \end{aligned}$$

$\therefore$ the assumption that $\sum_{n=1}^{\infty}[1/n] = S$ implies $S > S$
hence S DNE and the series diverges.

P-Series

The P-Series is convergent for $P>1$:

$$\sum_{n=1}^{\infty}[1/n^P] \text{ is convergent}$$

For $0 < P \leq 1$ this is divergent.

For $P=1$ this is the harmonic sequence
For $P=-1$ this is the geometric sequence.