Seperation of Variables

Contents

Differentiation Rules

The Chain Rule and Product Rule can be established visually fairly easily [1] and the proofs are fairly straightforward. [2]

Product Rule

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left(u\cdot v\right)=\frac{\mathrm{d}\,u}{\mathrm{d}\,v}\cdot v+u\cdot\frac{\mathrm{d}\,v}{\mathrm{d}\,x} \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left(f\left(x\right)\cdot g\left(x\right)\right)=f'\left(x\right)\cdot g\left(x\right)+f\left(x\right)\cdot g'\left(x\right) \tag{2}$$

$$\tag{3}$$

Chain Rule

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x}=\frac{\mathrm{d}\,y}{\mathrm{d}\,u}\cdot\frac{\mathrm{d}\,u}{\mathrm{d}\,x} \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left[f\left(g\left(x\right)\right)\right]=f'\left(g\left(x\right)\right)\cdot g\left(x\right) \tag{5}$$

$$\tag{6}$$

Integration Rules

Integration by Substitution

The chain rule can be used for integration with some clever substitution:

Let:

[1] Visualizing the chain rule and product rule * test
[2] Differentiation Rules Proof

$$
\begin{aligned}
u &= g(x) \qquad F(x): \quad F'(x) = f(x) = y \\
\tfrac{du}{dx} &= g'(x)
\end{aligned}
\tag{7}
$$

Now by direct substitution into the chain rule:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\,x}\left[F'\left(u\right)\right] &= F'\left(g\left(x\right)\right)\cdot g'\left(x\right) \\
&= f\left(g\left(x\right)\right)\cdot g'\left(x\right) \\
\implies f\left(g\left(x\right)\right)\cdot g'\left(x\right) &= \frac{\mathrm{d}}{\mathrm{d}\,x}\left[F\left(u\right)\right] \\
f\left(g\left(x\right)\right)\cdot g'\left(x\right) &= \frac{\mathrm{d}}{\mathrm{d}\,x}\left[F\left(u\right)+C\right]
\end{aligned}
\tag{8}
$$

Now by integrating both sides:

$$
\begin{aligned}
\int f\left(g\left(x\right)\right)\cdot g'\left(x\right)\mathrm{d}\,x &= \int \frac{\mathrm{d}}{\mathrm{d}\,x}\left[F\left(u\right)+C\right]\mathrm{d}\,x \\
&= F\left(u\right)+C \\
&= \int f\left(u\right)\mathrm{d}\,u
\end{aligned}
\tag{9}
$$

So what we have is integration by substitution:

$$
\int f\left(g\left(x\right)\right)\cdot g'\left(x\right)\mathrm{d}\,x = \int f\left(u\right)\mathrm{d}\,u \tag{10}
$$

$$
\int f\left(u\right)\cdot \frac{\mathrm{d}\,u}{\mathrm{d}\,x}\mathrm{d}\,x = \int f\left(u\right)\mathrm{d}\,u \tag{11}
$$

$$
\tag{12}
$$

This basically means that if an integral looks like the differentials could cancel out, they do, making the *Leibniz* notation particularly useful.

Integration by Parts

The product rule can be use for integration, but it's only fruitful when:

1. you can choose $u = f(x)$:

1. it simplifies as you differentiate
    1. Or at least stays the same, e.g. $\frac{\mathrm{d}}{\mathrm{d}\,x}\left[\sin\left(x\right)\right]=\cos\left(x\right)$
    2. $\mathrm{d}\,v=g'\left(x\right)\ \mathrm{d}\,x$ can be chosen such the differential can be intergrated to give v.

    Consider the Product Rule (1):

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left[f\left(x\right)\cdot g\left(x\right)\right]=f'\left(x\right)\cdot g\left(x\right)+f\left(x\right)\cdot g'\left(x\right)$$

Let,

$$\begin{array}{cc} u=f\left(x\right) & v=g\left(x\right) \\ \frac{\mathrm{d}\,u}{\mathrm{d}\,x}=f'\left(x\right) & \frac{\mathrm{d}\,v}{\mathrm{d}\,x}=g'\left(x\right) \end{array} \tag{13}$$

Now we have:

$$\int\left(\frac{\mathrm{d}\,u}{\mathrm{d}\,x}\cdot v+u\cdot\frac{\mathrm{d}\,v}{\mathrm{d}\,x}\right)\mathrm{d}\,x=u\cdot v$$

$$\int\left(v\cdot\frac{\mathrm{d}\,u}{\mathrm{d}\,x}\right)\mathrm{d}\,x+\int\left(u\cdot\frac{\mathrm{d}\,v}{\mathrm{d}\,x}\right)\mathrm{d}\,x=u\cdot v \tag{14}$$

By Rule (11) we have:

$$\int v\,\mathrm{d}\,u+\int u\,\mathrm{d}\,v=u\cdot v$$

$$\int u\,\mathrm{d}\,v=u\cdot v-\int v\,\mathrm{d}\,u \tag{15}$$

$$\tag{16}$$

Application These are really the only two rules we've got (other than manipulation with partial fractions if possible) so the only trick is choosing when to use which one:

- Look at the intergrand $\int\left[\ \right]\mathrm{d}$:
    - If it's of the form $\left[f\left(u\right)\cdot\frac{\mathrm{d}\,u}{\mathrm{d}\,x}\right]=\left[f\left(g\left(x\right)\right)\cdot g'\left(x\right)\right]$

    * Use Integration by Substitution
- If it's of the form: $\left[f\left(x\right)\cdot\frac{\mathrm{d}\,u}{\mathrm{d}\,x}\right]=\left[f\left(x\right)\cdot g'\left(x\right)\right]$
    * Use Integration by Parts