11. *(a)* Let $f(z)$ denote a function which is analytic in some annular domain about the origin that includes the unit circle $z = e^{i\phi}$ $(-\pi \le \phi \le \pi)$. By taking that circle as the path of integration in expressions (2) and (3), Sec. 60, for the coefficients a_n and b_n in a Laurent series in powers of z, show that

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\,d\phi + \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})\left[\left(\frac{z}{e^{i\phi}}\right)^n + \left(\frac{e^{i\phi}}{z}\right)^n\right] d\phi$$

when z is any point in the annular domain.

(b) Write $u(\theta) = \operatorname{Re}[f(e^{i\theta})]$ and show how it follows from the expansion in part *(a)* that

$$u(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\phi)\,d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} u(\phi)\cos[n(\theta-\phi)]\,d\phi.$$

This is one form of the *Fourier series* expansion of the real-valued function $u(\theta)$ on the interval $-\pi \le \theta \le \pi$. The restriction on $u(\theta)$ is more severe than is necessary in order for it to be represented by a Fourier series.*

63. ABSOLUTE AND UNIFORM CONVERGENCE OF POWER SERIES

This section and the three following it are devoted mainly to various properties of power series. A reader who wishes to simply accept the theorems and the corollary in these sections can easily skip the proofs in order to reach Sec. 67 more quickly.

We recall from Sec. 56 that a series of complex numbers converges *absolutely* if the series of absolute values of those numbers converges. The following theorem concerns the absolute convergence of power series.

Theorem 1. *If a power series*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{1}$$

converges when $z = z_1$ $(z_1 \ne z_0)$*, then it is absolutely convergent at each point* z *in the open disk* $|z - z_0| < R_1$ *where* $R_1 = |z_1 - z_0|$ (Fig. 79).

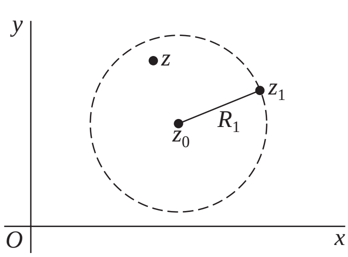

FIGURE 79

*For other sufficient conditions, see Secs. 12 and 13 of the book cited in the footnote to Exercise 10.

We start the proof by assuming that the series

$$\sum_{n=0}^{\infty} a_n(z_1 - z_0)^n \quad (z_1 \neq z_0)$$

converges. The terms $a_n(z_1 - z_0)^n$ are thus bounded; that is,

$$|a_n(z_1 - z_0)^n| \leq M \quad (n = 0, 1, 2, \ldots)$$

for some positive constant M (see Sec. 56). If $|z - z_0| < R_1$ and if we write

$$\rho = \frac{|z - z_0|}{|z_1 - z_0|},$$

we can see that

$$|a_n(z - z_0)^n| = |a_n(z_1 - z_0)^n| \left(\frac{|z - z_0|}{|z_1 - z_0|}\right)^n \leq M\rho^n \quad (n = 0, 1, 2, \ldots).$$

Now the series

$$\sum_{n=0}^{\infty} M\rho^n$$

is a geometric series, which converges since $\rho < 1$. Hence, by the comparison test for series of real numbers,

$$\sum_{n=0}^{\infty} |a_n(z - z_0)^n|$$

converges in the open disk $|z - z_0| < R_1$. This completes the proof.

The theorem tells us that the set of all points inside some circle centered at z_0 is a region of convergence for the power series (1), provided it converges at some point other than z_0. The greatest circle centered at z_0 such that series (1) converges at each point inside is called the *circle of convergence* of series (1). The series cannot converge at any point z_2 outside that circle, according to the theorem; for if it did, it would converge everywhere inside the circle centered at z_0 and passing through z_2. The first circle could not, then, be the circle of convergence.

Our next theorem involves terminology that we must first define. Suppose that the power series (1) has circle of convergence $|z - z_0| = R$, and let $S(z)$ and $S_N(z)$ represent the sum and partial sums, respectively, of that series:

$$S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \quad S_N(z) = \sum_{n=0}^{N-1} a_n(z - z_0)^n \qquad (|z - z_0| < R).$$

Then write the remainder function (see Sec. 56)

$$\rho_N(z) = S(z) - S_N(z) \qquad (|z - z_0| < R). \tag{2}$$

Since the power series converges for any fixed value of z when $|z - z_0| < R$, we know that the remainder $\rho_N(z)$ approaches zero for any such z as N tends to infinity. According to definition (2), Sec. 55, of the limit of a sequence, this means that corresponding to each positive number ε, there is a positive integer N_ε such that

$$|\rho_N(z)| < \varepsilon \quad \text{whenever} \quad N > N_\varepsilon. \tag{3}$$

When the choice of N_ε depends only on the value of ε and is independent of the point z taken in a specified region within the circle of convergence, the convergence is said to be *uniform* in that region.

Theorem 2. *If z_1 is a point inside the circle of convergence $|z - z_0| = R$ of a power series*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{4}$$

then that series must be uniformly convergent in the closed disk $|z - z_0| \le R_1$, where $R_1 = |z_1 - z_0|$ (Fig. 80).

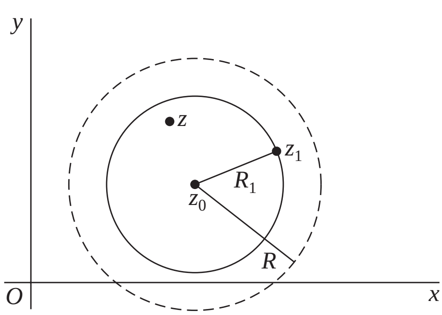

FIGURE 80

Our proof of this theorem depends on Theorem 1. Given that z_1 is a point lying inside the circle of convergence of series (4), we note that there are points inside that circle and farther from z_0 than z_1 for which the series converges. So, according to Theorem 1,

$$\sum_{n=0}^{\infty} |a_n (z_1 - z_0)^n| \tag{5}$$

converges. Letting m and N denote positive integers, where $m > N$, one can write the remainders of series (4) and (5) as

$$\rho_N(z) = \lim_{m\to\infty} \sum_{n=N}^{m} a_n (z - z_0)^n \tag{6}$$

and

$$\sigma_N = \lim_{m\to\infty} \sum_{n=N}^{m} |a_n(z_1 - z_0)^n|, \tag{7}$$

respectively.

Now, in view of Exercise 3, Sec. 56,

$$|\rho_N(z)| = \lim_{m\to\infty} \left| \sum_{n=N}^{m} a_n(z - z_0)^n \right|;$$

and, when $|z - z_0| \leq |z_1 - z_0|$,

$$\left| \sum_{n=N}^{m} a_n(z - z_0)^n \right| \leq \sum_{n=N}^{m} |a_n||z - z_0|^n \leq \sum_{n=N}^{m} |a_n||z_1 - z_0|^n = \sum_{n=N}^{m} |a_n(z_1 - z_0)^n|.$$

Consequently,

$$|\rho_N(z)| \leq \sigma_N \quad \text{when} \quad |z - z_0| \leq R_1. \tag{8}$$

Since σ_N are the remainders of a convergent series, they tend to zero as N tends to infinity. That is, for each positive number ε, an integer N_ε exists such that

$$\sigma_N < \varepsilon \quad \text{whenever} \quad N > N_\varepsilon. \tag{9}$$

Because of conditions (8) and (9), then, condition (3) holds for all points z in the disk $|z - z_0| \leq R_1$; and the value of N_ε is independent of the choice of z. Hence the convergence of series (4) is uniform in that disk.

64. CONTINUITY OF SUMS OF POWER SERIES

Our next theorem is an important consequence of uniform convergence, discussed in the previous section.

Theorem. *A power series*

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n \tag{1}$$

represents a continuous function $S(z)$ at each point inside its circle of convergence $|z - z_0| = R$.

Another way to state this theorem is to say that if $S(z)$ denotes the sum of series (1) within its circle of convergence $|z - z_0| = R$ and if z_1 is a point inside that circle, then for each positive number ε there is a positive number δ such that

$$|S(z) - S(z_1)| < \varepsilon \quad \text{whenever} \quad |z - z_1| < \delta. \tag{2}$$

[See definition (4), Sec. 18, of continuity.] The number δ here is small enough so that z lies in the domain of definition $|z - z_0| < R$ of $S(z)$ (Fig. 81).

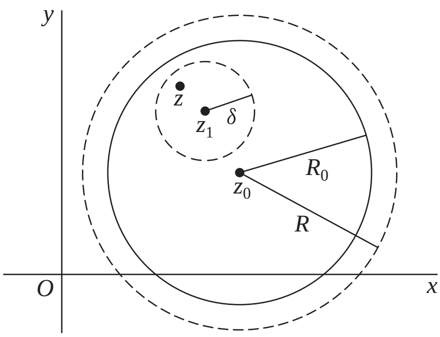

FIGURE 81

To prove the theorem, we let $S_n(z)$ denote the sum of the first N terms of series (1) and write the remainder function

$$\rho_N(z) = S(z) - S_N(z) \qquad (|z - z_0| < R).$$

Then, because

$$S(z) = S_N(z) + \rho_N(z) \qquad (|z - z_0| < R),$$

one can see that

$$|S(z) - S(z_1)| = |S_N(z) - S_N(z_1) + \rho_N(z) - \rho_N(z_1)|,$$

or

$$|S(z) - S(z_1)| \le |S_N(z) - S_N(z_1)| + |\rho_N(z)| + |\rho_N(z_1)|. \tag{3}$$

If z is any point lying in some closed disk $|z - z_0| \le R_0$ whose radius R_0 is greater than $|z_1 - z_0|$ but less than the radius R of the circle of convergence of series (1) (see Fig. 81), the uniform convergence stated in Theorem 2, Sec. 63, ensures that there is a positive integer N_ε such that

$$|\rho_N(z)| < \frac{\varepsilon}{3} \quad \text{whenever} \quad N > N_\varepsilon. \tag{4}$$

In particular, condition (4) holds for each point z in some neighborhood $|z - z_1| < \delta$ of z_1 that is small enough to be contained in the disk $|z - z_0| \le R_0$.

Now the partial sum $S_N(z)$ is a polynomial and is, therefore, continuous at z_1 for each value of N. In particular, when $N = N_\varepsilon + 1$, we can choose our δ so small that

$$|S_N(z) - S_N(z_1)| < \frac{\varepsilon}{3} \quad \text{whenever} \quad |z - z_1| < \delta. \tag{5}$$

By writing $N = N_\varepsilon + 1$ in inequality (3) and using the fact that statements (4) and (5) are true when $N = N_\varepsilon + 1$, we now find that

$$|S(z) - S(z_1)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \quad \text{whenever} \quad |z - z_1| < \delta.$$

This is statement (2), and the theorem is now established.

By writing $w = 1/(z - z_0)$, one can modify the two theorems in the previous section and the theorem here so as to apply to series of the type

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}. \tag{6}$$

If, for instance, series (6) converges at a point z_1 $(z_1 \neq z_0)$, the series

$$\sum_{n=1}^{\infty} b_n w^n$$

must converge absolutely to a continuous function when

$$|w| < \frac{1}{|z_1 - z_0|}. \tag{7}$$

Thus, since inequality (7) is the same as $|z - z_0| > |z_1 - z_0|$, series (6) must converge absolutely to a continuous function in the domain *exterior to* the circle $|z - z_0| = R_1$, where $R_1 = |z_1 - z_0|$. Also, we know that if a Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

is valid in an annulus $R_1 < |z - z_0| < R_2$, then *both* of the series on the right converge uniformly in any closed annulus which is concentric to and interior to that region of validity.

65. INTEGRATION AND DIFFERENTIATION OF POWER SERIES

We have just seen that a power series

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{1}$$

represents a continuous function at each point interior to its circle of convergence. In this section, we prove that the sum $S(z)$ is actually analytic within that circle. Our proof depends on the following theorem, which is of interest in itself.

Theorem 1. *Let C denote any contour interior to the circle of convergence of the power series* (1)*, and let $g(z)$ be any function that is continuous on C. The series formed by multiplying each term of the power series by $g(z)$ can be integrated term by term over C; that is,*

$$\int_C g(z)S(z)\,dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z-z_0)^n\,dz. \tag{2}$$

To prove this theorem, we note that since both $g(z)$ and the sum $S(z)$ of the power series are continuous on C, the integral over C of the product

$$g(z)S(z) = \sum_{n=0}^{N-1} a_n\, g(z)(z-z_0)^n + g(z)\rho_N(z),$$

where $\rho_N(z)$ is the remainder of the given series after N terms, exists. The terms of the finite sum here are also continuous on the contour C, and so their integrals over C exist. Consequently, the integral of the quantity $g(z)\rho_N(z)$ must exist; and we may write

$$\int_C g(z)S(z)\,dz = \sum_{n=0}^{N-1} a_n \int_C g(z)(z-z_0)^n\,dz + \int_C g(z)\rho_N(z)\,dz. \tag{3}$$

Now let M be the maximum value of $|g(z)|$ on C, and let L denote the length of C. In view of the uniform convergence of the given power series (Sec. 63), we know that for each positive number ε there exists a positive integer N_ε such that, for all points z on C,

$$|\rho_N(z)| < \varepsilon \quad \text{whenever} \quad N > N_\varepsilon.$$

Since N_ε is independent of z, we find that

$$\left|\int_C g(z)\rho_N(z)\,dz\right| < M\varepsilon L \quad \text{whenever} \quad N > N_\varepsilon;$$

that is,

$$\lim_{N\to\infty} \int_C g(z)\rho_N(z)\,dz = 0.$$

It follows, therefore, from equation (3) that

$$\int_C g(z)S(z)\,dz = \lim_{N\to\infty} \sum_{n=0}^{N-1} a_n \int_C g(z)(z-z_0)^n\,dz.$$

This is the same as equation (2), and Theorem 1 is proved.

If $g(z) = 1$ for each value of z in the open disk bounded by the circle of convergence of power series (1), the fact that $(z - z_0)^n$ is entire when $n = 0, 1, 2, \ldots$ ensures that

$$\int_C g(z)(z - z_0)^n \, dz = \int_C (z - z_0)^n \, dz = 0 \qquad (n = 0, 1, 2, \ldots)$$

for every *closed* contour C lying in that domain. According to equation (2), then,

$$\int_C S(z) \, dz = 0$$

for every such contour; and, by Morera's theorem (Sec. 52), the function $S(z)$ is analytic throughout the domain. We state this result as a corollary.

Corollary. *The sum $S(z)$ of power series* (1) *is analytic at each point z interior to the circle of convergence of that series.*

This corollary is often helpful in establishing the analyticity of functions and in evaluating limits.

EXAMPLE 1. To illustrate, let us show that the function defined by means of the equations

$$f(z) = \begin{cases} (e^z - 1)/z & \text{when } z \neq 0, \\ 1 & \text{when } z = 0 \end{cases}$$

is entire. Since the Maclaurin series expansion

$$e^z - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n!} \tag{4}$$

represents $e^z - 1$ for every value of z, the representation

$$f(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!} = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \cdots, \tag{5}$$

obtained by dividing each side of equation (4) by z, is valid when $z \neq 0$. But series (5) clearly converges to $f(0)$ when $z = 0$. Hence representation (5) is valid for all z; and f is, therefore, an entire function. Note that since $(e^z - 1)/z = f(z)$ when $z \neq 0$ and since f is continuous at $z = 0$,

$$\lim_{z \to 0} \frac{e^z - 1}{z} = \lim_{z \to 0} f(z) = f(0) = 1.$$

The first limit here is, of course, also evident if we write it in the form

$$\lim_{z\to 0} \frac{(e^z - 1) - 0}{z - 0},$$

which is the definition of the derivative of $e^z - 1$ at $z = 0$.

We observed in Sec. 57 that the Taylor series for a function f about a point z_0 converges to $f(z)$ at each point z interior to the circle centered at z_0 and passing through the nearest point z_1 where f fails to be analytic. In view of our corollary to Theorem 1, we now know that *there is no larger circle* about z_0 such that at each point z interior to it the Taylor series converges to $f(z)$. For if there were such a circle, f would be analytic at z_1; but f is not analytic at z_1.

We now present a companion to Theorem 1.

Theorem 2. *The power series* (1) *can be differentiated term by term. That is, at each point z interior to the circle of convergence of that series,*

$$S'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}. \tag{6}$$

To prove this, let z denote any point interior to the circle of convergence of series (1). Then let C be some positively oriented simple closed contour surrounding z and interior to that circle. Also, define the function

$$g(s) = \frac{1}{2\pi i} \cdot \frac{1}{(s - z)^2} \tag{7}$$

at each point s on C. Since $g(s)$ is continuous on C, Theorem 1 tells us that

$$\int_C g(s)S(s)\, ds = \sum_{n=0}^{\infty} a_n \int_C g(s)(s - z_0)^n\, ds. \tag{8}$$

Now $S(z)$ is analytic inside and on C, and this enables us to write

$$\int_C g(s)S(s)\, ds = \frac{1}{2\pi i} \int_C \frac{S(s)\, ds}{(s - z)^2} = S'(z)$$

with the aid of the integral representation for derivatives in Sec. 51. Furthermore,

$$\int_C g(s)(s - z_0)^n\, ds = \frac{1}{2\pi i} \int_C \frac{(s - z_0)^n}{(s - z)^2}\, ds = \frac{d}{dz}(z - z_0)^n \qquad (n = 0, 1, 2, \ldots).$$

Thus equation (8) reduces to

$$S'(z) = \sum_{n=0}^{\infty} a_n \frac{d}{dz}(z - z_0)^n,$$

which is the same as equation (6). This completes the proof.

EXAMPLE 2. In Example 4, Sec. 59, we saw that

$$\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n \qquad (|z-1| < 1).$$

Differentiation of each side of this equation reveals that

$$-\frac{1}{z^2} = \sum_{n=1}^{\infty} (-1)^n n (z-1)^{n-1} \qquad (|z-1| < 1),$$

or

$$\frac{1}{z^2} = \sum_{n=0}^{\infty} (-1)^n (n+1)(z-1)^n \qquad (|z-1| < 1).$$

66. UNIQUENESS OF SERIES REPRESENTATIONS

The uniqueness of Taylor and Laurent series representations, anticipated in Secs. 59 and 62, respectively, follows readily from Theorem 1 in Sec. 65. We consider first the uniqueness of Taylor series representations.

Theorem 1. *If a series*

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \tag{1}$$

converges to $f(z)$ at all points interior to some circle $|z - z_0| = R$, then it is the Taylor series expansion for f in powers of $z - z_0$.

To start the proof, we write the series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad (|z-z_0| < R) \tag{2}$$

in the hypothesis of the theorem using the index of summation m:

$$f(z) = \sum_{m=0}^{\infty} a_m (z-z_0)^m \qquad (|z-z_0| < R).$$

Then, by appealing to Theorem 1 in Sec. 65, we may write

$$\int_C g(z) f(z)\, dz = \sum_{m=0}^{\infty} a_m \int_C g(z)(z-z_0)^m\, dz, \tag{3}$$