CHAPTER

4

INTEGRALS

Integrals are extremely important in the study of functions of a complex variable. The theory of integration, to be developed in this chapter, is noted for its mathematical elegance. The theorems are generally concise and powerful, and many of the proofs are short.

37. DERIVATIVES OF FUNCTIONS $w(t)$

In order to introduce integrals of $f(z)$ in a fairly simple way, we need to first consider derivatives of complex-valued functions w of a *real* variable t. We write

$$w(t) = u(t) + iv(t), \tag{1}$$

where the functions u and v are *real-valued* functions of t. The derivative

$$w'(t), \text{ or } \frac{d}{dt}w(t),$$

of the function (1) at a point t is defined as

$$w'(t) = u'(t) + iv'(t), \tag{2}$$

provided each of the derivatives u' and v' exists at t.

From definition (2), it follows that for every complex constant $z_0 = x_0 + iy_0$,

$$\begin{aligned}\frac{d}{dt}[z_0 w(t)] &= [(x_0 + iy_0)(u + iv)]' = [(x_0u - y_0v) + i(y_0u + x_0v)]' \\ &= (x_0u - y_0v)' + i(y_0u + x_0v)' = (x_0u' - y_0v') + i(y_0u' + x_0v').\end{aligned}$$

But

$$(x_0u' - y_0v') + i(y_0u' + x_0v') = (x_0 + iy_0)(u' + iv') = z_0w'(t),$$

and so

$$\frac{d}{dt}[z_0w(t)] = z_0w'(t). \tag{3}$$

Another expected rule that we shall often use is

$$\frac{d}{dt}e^{z_0t} = z_0e^{z_0t}, \tag{4}$$

where $z_0 = x_0 + iy_0$. To verify this, we write

$$e^{z_0t} = e^{x_0t}e^{iy_0t} = e^{x_0t}\cos y_0t + ie^{x_0t}\sin y_0t$$

and refer to definition (2) to see that

$$\frac{d}{dt}e^{z_0t} = (e^{x_0t}\cos y_0t)' + i(e^{x_0t}\sin y_0t)'.$$

Familiar rules from calculus and some simple algebra then lead us to the expression

$$\frac{d}{dt}e^{z_0t} = (x_0 + iy_0)(e^{x_0t}\cos y_0t + ie^{x_0t}\sin y_0t),$$

or

$$\frac{d}{dt}e^{z_0t} = (x_0 + iy_0)e^{x_0t}e^{iy_0t}.$$

This is, of course, the same as equation (4).

Various other rules learned in calculus, such as the ones for differentiating sums and products, apply just as they do for real-valued functions of t. As was the case with property (3) and formula (4), verifications may be based on corresponding rules in calculus. It should be pointed out, however, that not every such rule carries over to functions of type (1). The following example illustrates this.

EXAMPLE. Suppose that $w(t)$ is continuous on an interval $a \le t \le b$; that is, its component functions $u(t)$ and $v(t)$ are continuous there. Even if $w'(t)$ exists when $a < t < b$, the mean value theorem for derivatives no longer applies. To be precise, it is not necessarily true that there is a number c in the interval $a < t < b$ such that

$$w'(c) = \frac{w(b) - w(a)}{b - a}.$$

To see this, consider the function $w(t) = e^{it}$ on the interval $0 \le t \le 2\pi$. When that function is used, $|w'(t)| = |ie^{it}| = 1$; and this means that the derivative $w'(t)$ is never zero, while $w(2\pi) - w(0) = 0$.

38. DEFINITE INTEGRALS OF FUNCTIONS $w(t)$

When $w(t)$ is a complex-valued function of a real variable t and is written

$$w(t) = u(t) + iv(t), \tag{1}$$

where u and v are real-valued, the definite integral of $w(t)$ over an interval $a \le t \le b$ is defined as

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt, \tag{2}$$

provided the individual integrals on the right exist. Thus

$$\operatorname{Re}\int_a^b w(t)\,dt = \int_a^b \operatorname{Re}[w(t)]\,dt \quad \text{and} \quad \operatorname{Im}\int_a^b w(t)\,dt = \int_a^b \operatorname{Im}[w(t)]\,dt. \tag{3}$$

EXAMPLE 1. For an illustration of definition (2),

$$\int_0^1 (1+it)^2\,dt = \int_0^1 (1-t^2)\,dt + i\int_0^1 2t\,dt = \frac{2}{3} + i.$$

Improper integrals of $w(t)$ over unbounded intervals are defined in a similar way.

The existence of the integrals of u and v in definition (2) is ensured if those functions are *piecewise continuous* on the interval $a \le t \le b$. Such a function is continuous everywhere in the stated interval except possibly for a finite number of points where, although discontinuous, it has one-sided limits. Of course, only the right-hand limit is required at a; and only the left-hand limit is required at b. When both u and v are piecewise continuous, the function w is said to have that property.

Anticipated rules for integrating a complex constant times a function $w(t)$, for integrating sums of such functions, and for interchanging limits of integration are all valid. Those rules, as well as the property

$$\int_a^b w(t)\,dt = \int_a^c w(t)\,dt + \int_c^b w(t)\,dt,$$

are easy to verify by recalling corresponding results in calculus.

The *fundamental theorem of calculus*, involving antiderivatives, can, moreover, be extended so as to apply to integrals of the type (2). To be specific, suppose that the functions

$$w(t) = u(t) + iv(t) \quad \text{and} \quad W(t) = U(t) + iV(t)$$

are continuous on the interval $a \le t \le b$. If $W'(t) = w(t)$ when $a \le t \le b$, then $U'(t) = u(t)$ and $V'(t) = v(t)$. Hence, in view of definition (2),

$$\int_a^b w(t)\,dt = U(t)\Big]_a^b + iV(t)\Big]_a^b = [U(b) + iV(b)] - [U(a) + iV(a)].$$

That is,

$$\int_a^b w(t)\,dt = W(b) - W(a) = W(t)\Big]_a^b. \tag{4}$$

EXAMPLE 2. Since (see Sec. 37)

$$\frac{d}{dt}\left(\frac{e^{it}}{i}\right) = \frac{1}{i}\frac{d}{dt}e^{it} = \frac{1}{i}\,ie^{it} = e^{it},$$

one can see that

$$\int_0^{\pi/4} e^{it}\,dt = \frac{e^{it}}{i}\Big]_0^{\pi/4} = \frac{e^{i\pi/4}}{i} - \frac{1}{i} = \frac{1}{i}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} - 1\right)$$
$$= \frac{1}{i}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} - 1\right) = \frac{1}{\sqrt{2}} + \frac{1}{i}\left(\frac{1}{\sqrt{2}} - 1\right).$$

Then, because $1/i = -i$,

$$\int_0^{\pi/4} e^{it}\,dt = \frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right).$$

We recall from the example in Sec. 37 how the mean value theorem for derivatives in calculus does not carry over to complex-valued functions $w(t)$. Our final example here shows that the mean value theorem for *integrals* does not carry over either. Thus special care must continue to be used in applying rules from calculus.

EXAMPLE 3. Let $w(t)$ be a continuous complex-valued function of t defined on an interval $a \le t \le b$. In order to show that it is not necessarily true that there is a number c in the interval $a < t < b$ such that

$$\int_a^b w(t)\,dt = w(c)(b-a),$$

we write $a = 0$, $b = 2\pi$ and use the same function $w(t) = e^{it}\,(0 \le t \le 2\pi)$ as in the example in Sec. 37. It is easy to see that

$$\int_a^b w(t)\,dt = \int_0^{2\pi} e^{it}\,dt = \frac{e^{it}}{i}\Big]_0^{2\pi} = 0.$$

But, for any number c such that $0 < c < 2\pi$,

$$|w(c)(b-a)| = |e^{ic}|\,2\pi = 2\pi;$$

and this means that $w(c)(b-a)$ is *not* zero.

EXERCISES

1. Use rules in calculus to establish the following rules when

$$w(t) = u(t) + iv(t)$$

is a complex-valued function of a real variable t and $w'(t)$ exists:

(a) $\dfrac{d}{dt} w(-t) = -w'(-t)$ where $w'(-t)$ denotes the derivative of $w(t)$ with respect to t, evaluated at $-t$;

(b) $\dfrac{d}{dt}[w(t)]^2 = 2\,w(t)w'(t)$.

2. Evaluate the following integrals:

(a) $\displaystyle\int_1^2 \left(\frac{1}{t} - i\right)^2 dt\,;$ *(b)* $\displaystyle\int_0^{\pi/6} e^{i2t}\,dt\,;$ *(c)* $\displaystyle\int_0^{\infty} e^{-zt}\,dt$ $(\operatorname{Re} z > 0)$.

Ans. *(a)* $-\dfrac{1}{2} - i\ln 4\,;$ *(b)* $\dfrac{\sqrt{3}}{4} + \dfrac{i}{4}\,;$ *(c)* $\dfrac{1}{z}$.

3. Show that if m and n are integers,

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta}\,d\theta = \begin{cases} 0 & \text{when } m \neq n, \\ 2\pi & \text{when } m = n. \end{cases}$$

4. According to definition (2), Sec. 38, of definite integrals of complex-valued functions of a real variable,

$$\int_0^{\pi} e^{(1+i)x}\,dx = \int_0^{\pi} e^x \cos x\,dx + i \int_0^{\pi} e^x \sin x\,dx.$$

Evaluate the two integrals on the right here by evaluating the single integral on the left and then using the real and imaginary parts of the value found.

Ans. $-(1 + e^{\pi})/2, \quad (1 + e^{\pi})/2.$

5. Let $w(t) = u(t) + iv(t)$ denote a continuous complex-valued function defined on an interval $-a \le t \le a$.

(a) Suppose that $w(t)$ is *even*; that is, $w(-t) = w(t)$ for each point t in the given interval. Show that

$$\int_{-a}^{a} w(t)\,dt = 2\int_0^{a} w(t)\,dt.$$

(b) Show that if $w(t)$ is an *odd* function, one where $w(-t) = -w(t)$ for each point t in the given interval, then

$$\int_{-a}^{a} w(t)\,dt = 0.$$

Suggestion: In each part of this exercise, use the corresponding property of integrals of *real-valued* functions of t, which is graphically evident.

39. CONTOURS

Integrals of complex-valued functions of a *complex* variable are defined on curves in the complex plane, rather than on just intervals of the real line. Classes of curves that are adequate for the study of such integrals are introduced in this section.

A set of points $z = (x, y)$ in the complex plane is said to be an *arc* if

$$x = x(t), \quad y = y(t) \qquad (a \le t \le b), \tag{1}$$

where $x(t)$ and $y(t)$ are continuous functions of the real parameter t. This definition establishes a continuous mapping of the interval $a \le t \le b$ into the xy, or z, plane; and the image points are ordered according to increasing values of t. It is convenient to describe the points of C by means of the equation

$$z = z(t) \qquad (a \le t \le b), \tag{2}$$

where

$$z(t) = x(t) + iy(t). \tag{3}$$

The arc C is a *simple arc*, or a Jordan arc,* if it does not cross itself; that is, C is simple if $z(t_1) \ne z(t_2)$ when $t_1 \ne t_2$. When the arc C is simple except for the fact that $z(b) = z(a)$, we say that C is a *simple closed curve*, or a Jordan curve. Such a curve is *positively oriented* when it is in the counterclockwise direction.

The geometric nature of a particular arc often suggests different notation for the parameter t in equation (2). This is, in fact, the case in the following examples.

EXAMPLE 1. The polygonal line (Sec. 11) defined by means of the equations

$$z = \begin{cases} x + ix & \text{when } 0 \le x \le 1, \\ x + i & \text{when } 1 \le x \le 2 \end{cases} \tag{4}$$

and consisting of a line segment from 0 to $1 + i$ followed by one from $1 + i$ to $2 + i$ (Fig. 36) is a simple arc.

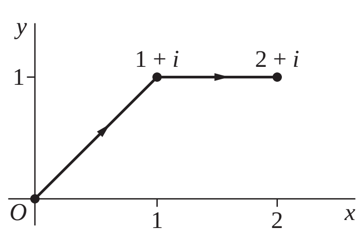

FIGURE 36

*Named for C. Jordan (1838–1922), pronounced *jor-don′*.

EXAMPLE 2. The unit circle

$$(5) \qquad z = e^{i\theta} \qquad (0 \le \theta \le 2\pi)$$

about the origin is a simple closed curve, oriented in the counterclockwise direction. So is the circle

$$(6) \qquad z = z_0 + Re^{i\theta} \qquad (0 \le \theta \le 2\pi),$$

centered at the point z_0 and with radius R (see Sec. 6).

The same set of points can make up different arcs.

EXAMPLE 3. The arc

$$(7) \qquad z = e^{-i\theta} \qquad (0 \le \theta \le 2\pi)$$

is not the same as the arc described by equation (5). The set of points is the same, but now the circle is traversed in the *clockwise* direction.

EXAMPLE 4. The points on the arc

$$(8) \qquad z = e^{i2\theta} \qquad (0 \le \theta \le 2\pi)$$

are the same as those making up the arcs (5) and (7). The arc here differs, however, from each of those arcs since the circle is traversed *twice* in the counterclockwise direction.

The parametric representation used for any given arc C is, of course, not unique. It is, in fact, possible to change the interval over which the parameter ranges to any other interval. To be specific, suppose that

$$(9) \qquad t = \phi(\tau) \qquad (\alpha \le \tau \le \beta),$$

where ϕ is a real-valued function mapping an interval $\alpha \le \tau \le \beta$ onto the interval $a \le t \le b$ in representation (2). (See Fig. 37.) We assume that ϕ is continuous with

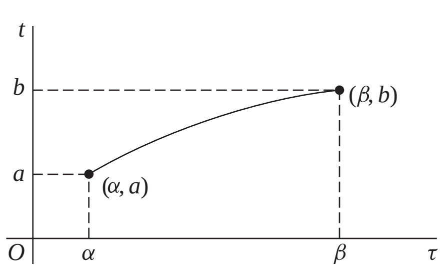

FIGURE 37
$t = \phi\ (\tau)$

a continuous derivative. We also assume that $\phi'(\tau) > 0$ for each τ; this ensures that t increases with τ. Representation (2) is then transformed by equation (9) into

$$z = Z(\tau) \qquad (\alpha \le \tau \le \beta), \tag{10}$$

where

$$Z(\tau) = z[\phi(\tau)]. \tag{11}$$

This is illustrated in Exercise 3.

Suppose now that the components $x'(t)$ and $y'(t)$ of the derivative (Sec. 37)

$$z'(t) = x'(t) + iy'(t) \tag{12}$$

of the function (3), used to represent C, are continuous on the entire interval $a \le t \le b$. The arc is then called a *differentiable arc*, and the real-valued function

$$|z'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

is integrable over the interval $a \le t \le b$. In fact, according to the definition of arc length in calculus, the length of C is the number

$$L = \int_a^b |z'(t)|\,dt. \tag{13}$$

The value of L is invariant under certain changes in the representation for C that is used, as one would expect. More precisely, with the change of variable indicated in equation (9), expression (13) takes the form [see Exercise 1*(b)*]

$$L = \int_\alpha^\beta |z'[\phi(\tau)]|\phi'(\tau)\,d\tau.$$

So, if representation (10) is used for C, the derivative (Exercise 4)

$$Z'(\tau) = z'[\phi(\tau)]\phi'(\tau) \tag{14}$$

enables us to write expression (13) as

$$L = \int_\alpha^\beta |Z'(\tau)|\,d\tau.$$

Thus the same length of C would be obtained if representation (10) were to be used.

If equation (2) represents a differentiable arc and if $z'(t) \ne 0$ anywhere in the interval $a < t < b$, then the unit tangent vector

$$\mathbf{T} = \frac{z'(t)}{|z'(t)|}$$

is well defined for all t in that open interval, with angle of inclination $\arg z'(t)$. Also, when $\mathbf{T}$ turns, it does so continuously as the parameter t varies over the entire interval

$a < t < b$. This expression for $\mathbf{T}$ is the one learned in calculus when $z(t)$ is interpreted as a radius vector. Such an arc is said to be *smooth*. In referring to a smooth arc $z = z(t)$ $(a \le t \le b)$, then, we agree that the derivative $z'(t)$ is continuous on the closed interval $a \le t \le b$ and nonzero throughout the open interval $a < t < b$.

A *contour*, or piecewise smooth arc, is an arc consisting of a finite number of smooth arcs joined end to end. Hence if equation (2) represents a contour, $z(t)$ is continuous, whereas its derivative $z'(t)$ is piecewise continuous. The polygonal line (4) is, for example, a contour. When only the initial and final values of $z(t)$ are the same, a contour C is called a *simple closed contour*. Examples are the circles (5) and (6), as well as the boundary of a triangle or a rectangle taken in a specific direction. The length of a contour or a simple closed contour is the sum of the lengths of the smooth arcs that make up the contour.

The points on any simple closed curve or simple closed contour C are boundary points of two distinct domains, one of which is the interior of C and is bounded. The other, which is the exterior of C, is unbounded. It will be convenient to accept this statement, known as the *Jordan curve theorem*, as geometrically evident; the proof is not easy.*

EXERCISES

1. Show that if $w(t) = u(t) + iv(t)$ is continuous on an interval $a \le t \le b$, then

(a) $\displaystyle\int_{-b}^{-a} w(-t)\,dt = \int_a^b w(\tau)\,d\tau$;

(b) $\displaystyle\int_a^b w(t)\,dt = \int_\alpha^\beta w[\phi(\tau)]\phi'(\tau)\,d\tau$, where $\phi(\tau)$ is the function in equation (9), Sec. 39.

Suggestion: These identities can be obtained by noting that they are valid for *real-valued* functions of t.

2. Let C denote the right-hand half of the circle $|z| = 2$, in the counterclockwise direction, and note that two parametric representations for C are

$$z = z(\theta) = 2\,e^{i\theta} \qquad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$$

and

$$z = Z(y) = \sqrt{4 - y^2} + iy \qquad (-2 \le y \le 2).$$

Verify that $Z(y) = z[\phi(y)]$, where

$$\phi(y) = \arctan \frac{y}{\sqrt{4 - y^2}} \qquad \left(-\frac{\pi}{2} < \arctan t < \frac{\pi}{2}\right).$$

*See pp. 115–116 of the book by Newman or Sec. 13 of the one by Thron, both of which are cited in Appendix 1. The special case in which C is a simple closed polygon is proved on pp. 281–285 of Vol. 1 of the work by Hille, also cited in Appendix 1.

Also, show that this function ϕ has a positive derivative, as required in the conditions following equation (9), Sec. 39.

3. Derive the equation of the line through the points (α, a) and (β, b) in the τt plane that are shown in Fig. 37. Then use it to find the linear function $\phi(\tau)$ which can be used in equation (9), Sec. 39, to transform representation (2) in that section into representation (10) there.

Ans. $\phi(\tau) = \dfrac{b-a}{\beta-\alpha}\,\tau + \dfrac{a\beta - b\alpha}{\beta-\alpha}.$

4. Verify expression (14), Sec. 39, for the derivative of $Z(\tau) = z[\phi(\tau)]$.

Suggestion: Write $Z(\tau) = x[\phi(\tau)] + iy[\phi(\tau)]$ and apply the chain rule for real-valued functions of a real variable.

5. Suppose that a function $f(z)$ is analytic at a point $z_0 = z(t_0)$ lying on a smooth arc $z = z(t)$ $(a \le t \le b)$. Show that if $w(t) = f[z(t)]$, then

$$w'(t) = f'[z(t)]z'(t)$$

when $t = t_0$.

Suggestion: Write $f(z) = u(x, y) + iv(x, y)$ and $z(t) = x(t) + iy(t)$, so that

$$w(t) = u[x(t), y(t)] + iv[x(t), y(t)].$$

Then apply the chain rule in calculus for functions of two real variables to write

$$w' = (u_x x' + u_y y') + i(v_x x' + v_y y'),$$

and use the Cauchy–Riemann equations.

6. Let $y(x)$ be a real-valued function defined on the interval $0 \le x \le 1$ by means of the equations

$$y(x) = \begin{cases} x^3 \sin(\pi/x) & \text{when } 0 < x \le 1, \\ 0 & \text{when } x = 0. \end{cases}$$

(a) Show that the equation

$$z = x + iy(x) \qquad (0 \le x \le 1)$$

represents an arc C that intersects the real axis at the points $z = 1/n$ $(n = 1, 2, \ldots)$ and $z = 0$, as shown in Fig. 38.

(b) Verify that the arc C in part *(a)* is, in fact, a *smooth* arc.

Suggestion: To establish the continuity of $y(x)$ at $x = 0$, observe that

$$0 \le \left| x^3 \sin\left(\frac{\pi}{x}\right) \right| \le x^3$$

when $x > 0$. A similar remark applies in finding $y'(0)$ and showing that $y'(x)$ is continuous at $x = 0$.

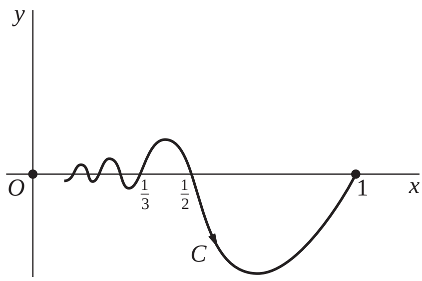

FIGURE 38

40. CONTOUR INTEGRALS

We turn now to integrals of complex-valued functions f of the complex variable z. Such an integral is defined in terms of the values $f(z)$ along a given contour C, extending from a point $z = z_1$ to a point $z = z_2$ in the complex plane. It is, therefore, a line integral; and its value depends, in general, on the contour C as well as on the function f. It is written

$$\int_C f(z)\,dz \quad \text{or} \quad \int_{z_1}^{z_2} f(z)\,dz,$$

the latter notation often being used when the value of the integral is independent of the choice of the contour taken between two fixed end points. While the integral may be defined directly as the limit of a sum, we choose to define it in terms of a definite integral of the type introduced in Sec. 38.

Suppose that the equation

$$z = z(t) \quad (a \le t \le b) \tag{1}$$

represents a contour C, extending from a point $z_1 = z(a)$ to a point $z_2 = z(b)$. We assume that $f[z(t)]$ is *piecewise continuous* (Sec. 38) on the interval $a \le t \le b$ and refer to the function $f(z)$ as being piecewise continuous on C. We then define the line integral, or *contour integral*, of f along C in terms of the parameter t:

$$\int_C f(z)\,dz = \int_a^b f[z(t)]z'(t)\,dt. \tag{2}$$

Note that since C is a contour, $z'(t)$ is also piecewise continuous on $a \le t \le b$; and so the existence of integral (2) is ensured.

The value of a contour integral is invariant under a change in the representation of its contour when the change is of the type (11), Sec. 39. This can be seen by following the same general procedure that was used in Sec. 39 to show the invariance of arc length.

It follows immediately from definition (2) and properties of integrals of complex-valued functions $w(t)$ mentioned in Sec. 38 that

$$\int_C z_0 f(z)\,dz = z_0 \int_C f(z)\,dz, \tag{3}$$

for any complex constant z_0, and

$$\int_C \big[f(z) + g(z)\big]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz. \tag{4}$$

Associated with the contour C used in integral (2) is the contour $-C$, consisting of the same set of points but with the order reversed so that the new contour extends from the point z_2 to the point z_1 (Fig. 39). The contour $-C$ has parametric representation

$$z = z(-t) \qquad (-b \le t \le -a).$$

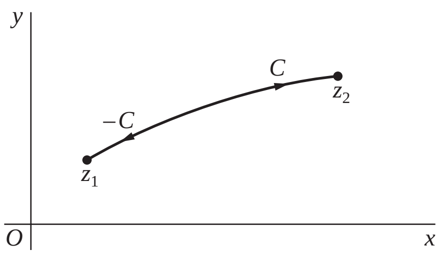

FIGURE 39

Hence, in view of Exercise 1*(a)*, Sec. 38,

$$\int_{-C} f(z)\,dz = \int_{-b}^{-a} f\,[z(-t)]\,\frac{d}{dt}z(-t)\,dt = -\int_{-b}^{-a} f\,[z(-t)]\,z'(-t)\,dt$$

where $z'(-t)$ denotes the derivative of $z(t)$ with respect to t, evaluated at $-t$. Making the substitution $\tau = -t$ in this last integral and referring to Exercise 1*(a)*, Sec. 39, we obtain the expression

$$\int_{-C} f(z)\,dz = -\int_a^b f[z(\tau)]z'(\tau)\,d\tau,$$

which is the same as

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz. \tag{5}$$

Consider now a path C, with representation (1), that consists of a contour C_1 from z_1 to z_2 followed by a contour C_2 from z_2 to z_3, the initial point of C_2 being the final point of C_1 (Fig. 40). There is a value c of t, where $a < c < b$, such that $z(c) = z_2$. Consequently, C_1 is represented by

$$z = z(t) \qquad (a \leq t \leq c)$$

and C_2 is represented by

$$z = z(t) \qquad (c \leq t \leq b).$$

Also, by a rule for integrals of functions $w(t)$ that was noted in Sec. 38,

$$\int_a^b f[z(t)]z'(t)\,dt = \int_a^c f[z(t)]z'(t)\,dt + \int_c^b f[z(t)]z'(t)\,dt.$$

Evidently, then,

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz. \tag{6}$$

Sometimes the contour C is called the *sum* of its legs C_1 and C_2 and is denoted by $C_1 + C_2$. The sum of two contours C_1 and $-C_2$ is well defined when C_1 and C_2 have the same final points, and it is written $C_1 - C_2$.

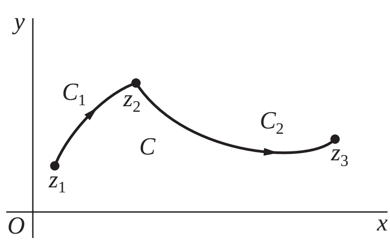

FIGURE 40
$C = C_1 + C_2$

Definite integrals in calculus can be interpreted as areas, and they have other interpretations as well. Except in special cases, no corresponding helpful interpretation, geometric or physical, is available for integrals in the complex plane.

41. SOME EXAMPLES

The purpose of this and the next section is to provide examples of the definition in Sec. 40 of contour integrals and to illustrate various properties that were mentioned there. We defer development of the concept of antiderivatives of the integrands $f(z)$ of contour integrals until Sec. 44.

EXAMPLE 1. Let us find the value of the integral

$$I = \int_C \overline{z}\,dz \tag{1}$$

when C is the right-hand half

$$z = 2\,e^{i\theta} \quad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$$

of the circle $|z| = 2$ from $z = -2i$ to $z = 2i$ (Fig. 41). According to definition (2), Sec. 40,

$$I = \int_{-\pi/2}^{\pi/2} \overline{2\,e^{i\theta}}(2\,e^{i\theta})'\,d\theta = 4\int_{-\pi/2}^{\pi/2} \overline{e^{i\theta}}(e^{i\theta})'\,d\theta;$$

and, since

$$\overline{e^{i\theta}} = e^{-i\theta} \quad \text{and} \quad (e^{i\theta})' = i\,e^{i\theta},$$

this means that

$$I = 4\int_{-\pi/2}^{\pi/2} e^{-i\theta} i\,e^{i\theta}\,d\theta = 4i\int_{-\pi/2}^{\pi/2} d\theta = 4\pi i.$$

Note that $z\overline{z} = |z|^2 = 4$ when z is a point on the semicircle C. Hence the result

(2) $$\int_C \overline{z}\,dz = 4\pi i$$

can also be written

$$\int_C \frac{dz}{z} = \pi i.$$

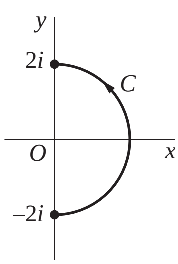

FIGURE 41

If $f(z)$ is given in the form $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, one can sometimes apply definition (2), Sec. 40, using one of the variables x and y as the parameter.

EXAMPLE 2. Here we first let C_1 denote the polygonal line OAB shown in Fig. 42 and evaluate the integral

(3) $$I_1 = \int_{C_1} f(z)\,dz = \int_{OA} f(z)\,dz + \int_{AB} f(z)\,dz,$$

where

$$f(z) = y - x - i3x^2 \quad (z = x + iy).$$

The leg *OA* may be represented parametrically as $z = 0 + iy$ $(0 \le y \le 1)$; and, since $x = 0$ at points on that line segment, the values of f there vary with the parameter y according to the equation $f(z) = y$ $(0 \le y \le 1)$. Consequently,

$$\int_{OA} f(z)\,dz = \int_0^1 yi\,dy = i\int_0^1 y\,dy = \frac{i}{2}.$$

On the leg AB, the points are $z = x + i$ $(0 \le x \le 1)$; and, since $y = 1$ on this segment,

$$\int_{AB} f(z)\,dz = \int_0^1 (1 - x - i3x^2)\cdot 1\,dx = \int_0^1 (1-x)\,dx - 3i\int_0^1 x^2\,dx = \frac{1}{2} - i.$$

In view of equation (3), we now see that

$$I_1 = \frac{1-i}{2}. \tag{4}$$

If C_2 denotes the segment OB of the line $y = x$ in Fig. 42, with parametric representation $z = x + ix$ $(0 \le x \le 1)$, the fact that $y = x$ on *OB* enables us to write

$$I_2 = \int_{C_2} f(z)\,dz = \int_0^1 -i3x^2(1+i)\,dx = 3(1-i)\int_0^1 x^2\,dx = 1 - i.$$

Evidently, then, the integrals of $f(z)$ along the two paths C_1 and C_2 have *different values* even though those paths have the same initial and the same final points.

Observe how it follows that the integral of $f(z)$ over the simple closed contour *OABO*, or $C_1 - C_2$, has the *nonzero value*

$$I_1 - I_2 = \frac{-1+i}{2}.$$

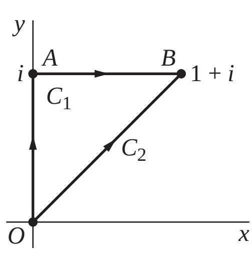

FIGURE 42

EXAMPLE 3. We begin here by letting C denote an arbitrary *smooth* arc (Sec. 39)

$$z = z(t) \quad (a \le t \le b)$$

from a fixed point z_1 to a fixed point z_2 (Fig. 43). In order to evaluate the integral

$$\int_C z\,dz = \int_a^b z(t)z'(t)\,dt,$$

we note that according to Exercise 1*(b)*, Sec. 38,

$$\frac{d}{dt}\frac{[z(t)]^2}{2} = z(t)z'(t).$$

Then, because $z(a) = z_1$ and $z(b) = z_2$, we have

$$\int_C z\,dz = \frac{[z(t)]^2}{2}\Bigg]_a^b = \frac{[z(b)]^2 - [z(a)]^2}{2} = \frac{z_2^2 - z_1^2}{2}.$$

Inasmuch as the value of this integral depends only on the end points of C and is otherwise independent of the arc that is taken, we may write

$$\int_{z_1}^{z_2} z\,dz = \frac{z_2^2 - z_1^2}{2}. \tag{5}$$

(Compare with Example 2, where the value of an integral from one fixed point to another depended on the path that was taken.)

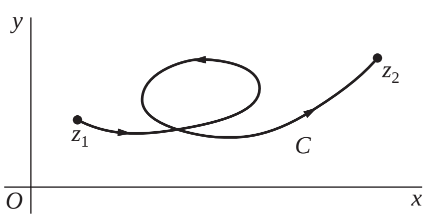

FIGURE 43

Expression (5) is also valid when C is a contour that is not necessarily smooth since a contour consists of a finite number of smooth arcs C_k $(k = 1, 2, \ldots, n)$, joined end to end. More precisely, suppose that each C_k extends from z_k to z_{k+1}. Then

$$\int_C z\,dz = \sum_{k=1}^{n} \int_{C_k} z\,dz = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} z\,dz = \sum_{k=1}^{n} \frac{z_{k+1}^2 - z_k^2}{2} = \frac{z_{n+1}^2 - z_1^2}{2}, \tag{6}$$

where this last summation has telescoped and z_1 is the initial point of C and z_{n+1} is its final point.

It follows from expression (6) that the integral of the function $f(z) = z$ around each closed contour in the plane has value zero. (Once again, compare with Example 2, where the value of the integral of a given function around a closed contour was *not* zero.) The question of predicting when an integral around a closed contour has value zero will be discussed in Secs. 44, 46, and 48.

42. EXAMPLES WITH BRANCH CUTS

The path in a contour integral can contain a point on a branch cut of the integrand involved. The next two examples illustrate this.

EXAMPLE 1. Let C denote the semicircular path

$$z = 3\,e^{i\theta} \quad (0 \le \theta \le \pi)$$

from the point $z = 3$ to the point $z = -3$ (Fig. 44). Although the branch

$$f(z) = z^{1/2} = \exp\left(\frac{1}{2}\log z\right) \quad (|z| > 0, 0 < \arg z < 2\pi)$$

of the multiple-valued function $z^{1/2}$ is not defined at the initial point $z = 3$ of the contour C, the integral

$$I = \int_C z^{1/2}\,dz \tag{1}$$

nevertheless exists. For the integrand is piecewise continuous on C. To see that this is so, we first observe that when $z(\theta) = 3\,e^{i\theta}$,

$$f[z(\theta)] = \exp\left[\frac{1}{2}(\ln 3 + i\theta)\right] = \sqrt{3}\,e^{i\theta/2}.$$

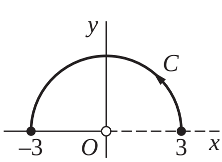

FIGURE 44

Hence the right-hand limits of the real and imaginary components of the function

$$f[z(\theta)]z'(\theta) = \sqrt{3}\,e^{i\theta/2}3ie^{i\theta} = 3\sqrt{3}ie^{i3\theta/2} = -3\sqrt{3}\sin\frac{3\theta}{2} + i3\sqrt{3}\cos\frac{3\theta}{2} \quad (0 < \theta \le \pi)$$

at $\theta = 0$ exist, those limits being 0 and $i3\sqrt{3}$, respectively. This means that $f[z(\theta)]z'(\theta)$ is continuous on the closed interval $0 \le \theta \le \pi$ when its value at $\theta = 0$ is defined as $i3\sqrt{3}$. Consequently,

$$I = 3\sqrt{3}i\int_0^{\pi} e^{i3\theta/2}\,d\theta.$$

Since

$$\int_0^\pi e^{i3\theta/2}\,d\theta = \frac{2}{3i}e^{i3\theta/2}\bigg]_0^\pi = -\frac{2}{3i}(1+i),$$

we now have the value

$$I = -2\sqrt{3}(1+i) \tag{2}$$

of integral (1).

EXAMPLE 2. Suppose that C is the positively oriented circle (Fig. 45)

$$z = Re^{i\theta} \quad (-\pi \le \theta \le \pi)$$

about the origin, and left a denote any nonzero real number. Using the principal branch

$$f(z) = z^{a-1} = \exp[(a-1)\text{Log } z] \quad (|z| > 0, -\pi < \text{Arg } z < \pi)$$

of the power function z^{a-1}, let us evaluate the integral

$$I = \int_C z^{a-1}\,dz. \tag{3}$$

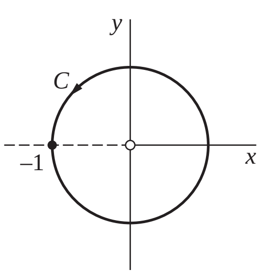

FIGURE 45

When $z(\theta) = Re^{i\theta}$, it is easy to see that

$$f[z(\theta)]z'(\theta) = iR^a e^{ia\theta} = -R^a \sin a\theta + iR^a \cos a\theta,$$

where the positive value of R^a is to be taken. Inasmuch as this function is piecewise continuous on $-\pi < \theta < \pi$, integral (3) exists. In fact,

$$I = iR^a \int_{-\pi}^{\pi} e^{ia\theta}\,d\theta = iR^a\left[\frac{e^{ia\theta}}{ia}\right]_{-\pi}^{\pi} = i\frac{2R^a}{a}\cdot\frac{e^{ia\pi} - e^{-ia\pi}}{2i} = i\frac{2R^a}{a}\sin a\pi. \tag{4}$$

Note that if a is a nonzero integer n, this result tells us that

$$\int_C z^{n-1}\,dz = 0 \quad (n = \pm 1, \pm 2, \ldots). \tag{5}$$

If a is allowed to be zero, we have

$$\int_C \frac{dz}{z} = \int_{-\pi}^{\pi} \frac{1}{Re^{i\theta}} iRe^{i\theta}\,d\theta = i\int_{-\pi}^{\pi} d\theta = 2\pi i. \tag{6}$$

EXERCISES

For the functions f and contours C in Exercises 1 through 7, use parametric representations for C, or legs of C, to evaluate

$$\int_C f(z)\,dz.$$

1. $f(z) = (z+2)/z$ and C is

(a) the semicircle $z = 2\,e^{i\theta}$ $(0 \le \theta \le \pi)$;

(b) the semicircle $z = 2\,e^{i\theta}$ $(\pi \le \theta \le 2\pi)$;

(c) the circle $z = 2\,e^{i\theta}$ $(0 \le \theta \le 2\pi)$.

Ans. *(a)* $-4 + 2\pi i$; *(b)* $4 + 2\pi i$; *(c)* $4\pi i$.

2. $f(z) = z - 1$ and C is the arc from $z = 0$ to $z = 2$ consisting of

(a) the semicircle $z = 1 + e^{i\theta}$ $(\pi \le \theta \le 2\pi)$;

(b) the segment $z = x$ $(0 \le x \le 2)$ of the real axis.

Ans. *(a)* 0; *(b)* 0.

3. $f(z) = \pi \exp(\pi \bar{z})$ and C is the boundary of the square with vertices at the points $0, 1, 1+i$, and i, the orientation of C being in the counterclockwise direction.

Ans. $4(e^{\pi} - 1)$.

4. $f(z)$ is defined by means of the equations

$$f(z) = \begin{cases} 1 & \text{when } y < 0, \\ 4y & \text{when } y > 0, \end{cases}$$

and C is the arc from $z = -1 - i$ to $z = 1 + i$ along the curve $y = x^3$.

Ans. $2 + 3i$.

5. $f(z) = 1$ and C is an arbitrary contour from any fixed point z_1 to any fixed point z_2 in the z plane.

Ans. $z_2 - z_1$.

6. $f(z)$ is the branch

$$z^{-1+i} = \exp[(-1+i)\log z] \qquad (|z| > 0, 0 < \arg z < 2\pi)$$

of the indicated power function, and C is the unit circle $z = e^{i\theta}$ $(0 \le \theta \le 2\pi)$.

Ans. $i(1 - e^{-2\pi})$.

7. $f(z)$ is the principal branch

$$z^i = \exp(i\operatorname{Log} z) \quad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

of this power function, and C is the semicircle $z = e^{i\theta}$ $(0 \le \theta \le \pi)$.

Ans. $-\dfrac{1 + e^{-\pi}}{2}(1 - i)$.

8. With the aid of the result in Exercise 3, Sec. 38, evaluate the integral

$$\int_C z^m \overline{z}^n \, dz,$$

where m and n are integers and C is the unit circle $|z| = 1$, taken counterclockwise.

9. Evaluate the integral I in Example 1, Sec. 41, using this representation for C:

$$z = \sqrt{4 - y^2} + iy \qquad (-2 \le y \le 2).$$

(See Exercise 2, Sec. 39.)

10. Let C_0 and C denote the circles

$$z = z_0 + Re^{i\theta} \ (-\pi \le \theta \le \pi) \quad \text{and} \quad z = Re^{i\theta} \ (-\pi \le \theta \le \pi),$$

respectively.

(a) Use these parametric representations to show that

$$\int_{C_0} f(z - z_0)\, dz = \int_C f(z)\, dz$$

when f is piecewise continuous on C.

(b) Apply the result in part *(a)* to integrals (5) and (6) in Sec. 42 to show that

$$\int_{C_0} (z - z_0)^{n-1}\, dz = 0 \ (n = \pm 1, \pm 2, \ldots) \quad \text{and} \quad \int_{C_0} \frac{dz}{z - z_0} = 2\pi i.$$

11. *(a)* Suppose that a function $f(z)$ is continuous on a smooth arc C, which has a parametric representation $z = z(t)$ $(a \le t \le b)$; that is, $f[z(t)]$ is continuous on the interval $a \le t \le b$. Show that if $\phi(\tau)$ $(\alpha \le \tau \le \beta)$ is the function described in Sec. 39, then

$$\int_a^b f[z(t)]z'(t)\, dt = \int_\alpha^\beta f[Z(\tau)]Z'(\tau)\, d\tau$$

where $Z(\tau) = z[\phi(\tau)]$.

(b) Point out how it follows that the identity obtained in part *(a)* remains valid when C is any contour, not necessarily a smooth one, and $f(z)$ is piecewise continuous on C. Thus show that the value of the integral of $f(z)$ along C is the same when the representation $z = Z(\tau)$ $(\alpha \le \tau \le \beta)$ is used, instead of the original one.

Suggestion: In part *(a)*, use the result in Exercise 1*(b)*, Sec. 39, and then refer to expression (14) in that section.

43. UPPER BOUNDS FOR MODULI OF CONTOUR INTEGRALS

We turn now to an inequality involving contour integrals that is extremely important in various applications. We present the result as a theorem but preface it with a needed lemma involving functions $w(t)$ of the type encountered in Secs. 37 and 38.

Lemma. *If $w(t)$ is a piecewise continuous complex-valued function defined on an interval $a \leq t \leq b$, then*

$$\left| \int_a^b w(t)\,dt \right| \leq \int_a^b |w(t)|\,dt. \tag{1}$$

This inequality clearly holds when the value of the integral on the left is zero. Thus, in the verification we may assume that its value is a *nonzero* complex number and write

$$\int_a^b w(t)\,dt = r_0\,e^{i\theta_0}.$$

Solving for r_0, we have

$$r_0 = \int_a^b e^{-i\theta_0} w(t)\,dt. \tag{2}$$

Now the left-hand side of this equation is a real number, and so the right-hand side is too. Thus, using the fact that the real part of a real number is the number itself, we find that

$$r_0 = \operatorname{Re} \int_a^b e^{-i\theta_0} w(t)\,dt,$$

or

$$r_0 = \int_a^b \operatorname{Re}[e^{-i\theta_0} w(t)]\,dt. \tag{3}$$

But

$$\operatorname{Re}[e^{-i\theta_0} w(t)] \leq |e^{-i\theta_0} w(t)| = |e^{-i\theta_0}|\,|w(t)| = |w(t)|,$$

and it follows from equation (3) that

$$r_0 \leq \int_a^b |w(t)|\,dt.$$

Because r_0 is, in fact, the left-hand side of inequality (1), the verification of the lemma is complete.

Theorem. *Let C denote a contour of length L, and suppose that a function $f(z)$ is piecewise continuous on C. If M is a nonnegative constant such that*

$$|f(z)| \le M \tag{4}$$

for all points z on C at which $f(z)$ is defined, then

$$\left| \int_C f(z)\, dz \right| \le ML. \tag{5}$$

To prove this, let $z = z(t)$ $(a \le t \le b)$ be a parametric representation of C. According to the above lemma,

$$\left| \int_C f(z)\, dz \right| = \left| \int_a^b f[z(t)]z'(t)\, dt \right| \le \int_a^b |f[z(t)]z'(t)|\, dt.$$

Inasmuch as

$$|f[z(t)]z'(t)| = |f[z(t)]|\, |z'(t)| \le M\, |z'(t)|$$

when $a \le t \le b$, it follows that

$$\left| \int_C f(z)\, dz \right| \le M \int_a^b |z'(t)|\, dt.$$

Since the integral on the right here represents the length L of C (see Sec. 39), inequality (5) is established. It is, of course, a strict inequality if inequality (4) is strict.

Note that since C is a contour and f is piecewise continuous on C, a number M such as the one appearing in inequality (4) will always exist. This is because the real-valued function $|f[z(t)]|$ is continuous on the closed bounded interval $a \le t \le b$ when f is continuous on C; and such a function always reaches a maximum value M on that interval.* Hence $|f(z)|$ has a maximum value on C when f is continuous on it. The same is, then, true when f is *piecewise* continuous on C.

EXAMPLE 1. Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant (Fig. 46). Inequality (5) can be used to show that

$$\left| \int_C \frac{z+4}{z^3-1}\, dz \right| \le \frac{6\pi}{7}. \tag{6}$$

This is done by noting first that if z is a point on C, so that $|z| = 2$, then

$$|z+4| \le |z| + 4 = 6$$

*See, for instance A. E. Taylor and W. R. Mann, "Advanced Calculus," 3d ed., pp. 86–90, 1983.

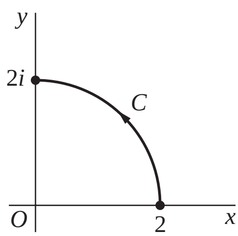

FIGURE 46

and

$$|z^3 - 1| \geq ||z|^3 - 1| = 7.$$

Thus, when z lies on C,

$$\left|\frac{z+4}{z^3-1}\right| = \frac{|z+4|}{|z^3-1|} \leq \frac{6}{7}.$$

Writing $M = 6/7$ and observing that $L = \pi$ is the length of C, we may now use inequality (5) to obtain inequality (6).

EXAMPLE 2. Here C_R is the semicircular path

$$z = Re^{i\theta} \qquad (0 \leq \theta \leq \pi),$$

and $z^{1/2}$ denotes the branch

$$z^{1/2} = \exp\left(\frac{1}{2}\log z\right) = \sqrt{r}e^{i\theta/2} \qquad \left(r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right)$$

of the square root function. (See Fig. 47.) Without actually finding the value of the integral, one can easily show that

$$\lim_{R\to\infty}\int_{C_R}\frac{z^{1/2}}{z^2+1}\,dz = 0. \tag{7}$$

For, when $|z| = R > 1$,

$$|z^{1/2}| = |\sqrt{R}e^{i\theta/2}| = \sqrt{R}$$

and

$$|z^2 + 1| \geq ||z^2| - 1| = R^2 - 1.$$

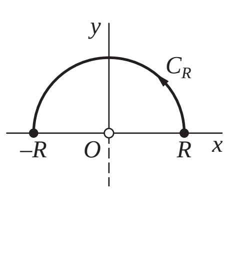

FIGURE 47

Consequently, at points on C_R,

$$\left|\frac{z^{1/2}}{z^2+1}\right| \leq M_R \quad \text{where} \quad M_R = \frac{\sqrt{R}}{R^2-1}.$$

Since the length of C_R is the number $L = \pi R$, it follows from inequality (5) that

$$\left|\int_{C_R} \frac{z^{1/2}}{z^2+1}\, dz\right| \leq M_R L.$$

But

$$M_R L = \frac{\pi R\sqrt{R}}{R^2-1} \cdot \frac{1/R^2}{1/R^2} = \frac{\pi/\sqrt{R}}{1-(1/R^2)},$$

and it is clear that the term on the far right here tends to zero as R tends to infinity. Limit (7) is, therefore, established.

EXERCISES

1. Without evaluating the integral, show that

$$\left|\int_C \frac{dz}{z^2-1}\right| \leq \frac{\pi}{3}$$

when C is the same arc as the one in Example 1, Sec. 43.

2. Let C denote the line segment from $z = i$ to $z = 1$. By observing that of all the points on that line segment, the midpoint is the closest to the origin, show that

$$\left|\int_C \frac{dz}{z^4}\right| \leq 4\sqrt{2}$$

without evaluating the integral.

3. Show that if C is the boundary of the triangle with vertices at the points $0, 3i$, and -4, oriented in the counterclockwise direction (see Fig. 48), then

$$\left|\int_C (e^z - \overline{z})\, dz\right| \leq 60.$$

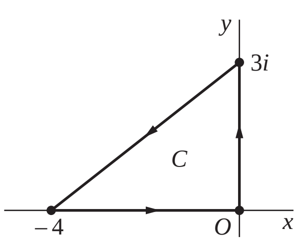

FIGURE 48

4. Let C_R denote the upper half of the circle $|z| = R$ $(R > 2)$, taken in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4}\, dz \right| \le \frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)}.$$

Then, by dividing the numerator and denominator on the right here by R^4, show that the value of the integral tends to zero as R tends to infinity.

5. Let C_R be the circle $|z| = R$ $(R > 1)$, described in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz \right| < 2\pi \left(\frac{\pi + \ln R}{R} \right),$$

and then use l'Hospital's rule to show that the value of this integral tends to zero as R tends to infinity.

6. Let C_ρ denote a circle $|z| = \rho$ $(0 < \rho < 1)$, oriented in the counterclockwise direction, and suppose that $f(z)$ is analytic in the disk $|z| \le 1$. Show that if $z^{-1/2}$ represents any particular branch of that power of z, then there is a nonnegative constant M, *independent of* ρ, such that

$$\left| \int_{C_\rho} z^{-1/2} f(z)\, dz \right| \le 2\pi M \sqrt{\rho}.$$

Thus show that the value of the integral here approaches 0 as ρ tends to 0.

Suggestion: Note that since $f(z)$ is analytic, and therefore continuous, throughout the disk $|z| \le 1$, it is bounded there (Sec. 18).

7. Apply inequality (1), Sec. 43, to show that for all values of x in the interval $-1 \le x \le 1$, the functions*

$$P_n(x) = \frac{1}{\pi} \int_0^\pi (x + i\sqrt{1 - x^2} \cos\theta)^n\, d\theta \qquad (n = 0, 1, 2, \ldots)$$

satisfy the inequality $|P_n(x)| \le 1$.

8. Let C_N denote the boundary of the square formed by the lines

$$x = \pm \left(N + \frac{1}{2} \right) \pi \quad \text{and} \quad y = \pm \left(N + \frac{1}{2} \right) \pi,$$

where N is a positive integer and the orientation of C_N is counterclockwise.

(a) With the aid of the inequalities

$$|\sin z| \ge |\sin x| \quad \text{and} \quad |\sin z| \ge |\sinh y|,$$

obtained in Exercises 8*(a)* and 9*(a)* of Sec. 34, show that $|\sin z| \ge 1$ on the vertical sides of the square and that $|\sin z| > \sinh(\pi/2)$ on the horizontal sides. Thus show that there is a positive constant A, *independent of* N, such that $|\sin z| \ge A$ for all points z lying on the contour C_N.

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*These functions are actually polynomials in x. They are known as *Legendre polynomials* and are important in applied mathematics. See, for example, Chap. 4 of the book by Lebedev that is listed in Appendix 1.

(b) Using the final result in part *(a)*, show that

$$\left|\int_{C_N} \frac{dz}{z^2 \sin z}\right| \le \frac{16}{(2N+1)\pi A}$$

and hence that the value of this integral tends to zero as N tends to infinity.

44. ANTIDERIVATIVES

Although the value of a contour integral of a function $f(z)$ from a fixed point z_1 to a fixed point z_2 depends, in general, on the path that is taken, there are certain functions whose integrals from z_1 to z_2 have values that are *independent of path*. (Recall Examples 2 and 3 in Sec. 41.) The examples just cited also illustrate the fact that the values of integrals around closed paths are sometimes, but not always, zero. Our next theorem is useful in determining when integration is independent of path and, moreover, when an integral around a closed path has value zero.

The theorem contains an extension of the fundamental theorem of calculus that simplifies the evaluation of many contour integrals. The extension involves the concept on an antiderivative of a continuous function $f(z)$ on a domain D, or a function $F(z)$ such that $F'(z) = f(z)$ for all z in D. Note that an antiderivative is, of necessity, an analytic function. Note, too, that *an antiderivative of a given function $f(z)$ is unique except for an additive constant*. This is because the derivative of the difference $F(z) - G(z)$ of any two such antiderivatives is zero; and, according to the theorem in Sec. 24, an analytic function is constant in a domain D when its derivative is zero throughout D.

Theorem. *Suppose that a function $f(z)$ is continuous on a domain D. If any one of the following statements is true, then so are the others:*

(a) *$f(z)$ has an antiderivative $F(z)$ throughout D;*

(b) *the integrals of $f(z)$ along contours lying entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have the same value, namely*

$$\int_{z_1}^{z_2} f(z)\,dz = F(z)\Big]_{z_1}^{z_2} = F(z_2) - F(z_1)$$

where $F(z)$ is the antiderivative in statement (a);

(c) *the integrals of $f(z)$ around closed contours lying entirely in D all have value zero.*

It should be emphasized that the theorem does *not* claim that any of these statements is true for a given function $f(z)$. It says only that all of them are true or that none of them is true. The next section is devoted to the proof of the theorem and can be easily skipped by a reader who wishes to get on with other important aspects of integration theory. But we include here a number of examples illustrating how the theorem can be used.

EXAMPLE 1. The continuous function $f(z) = z^2$ has an antiderivative $F(z) = z^3/3$ throughout the plane. Hence

$$\int_0^{1+i} z^2\,dz = \frac{z^3}{3}\Bigg]_0^{1+i} = \frac{1}{3}(1+i)^3 = \frac{2}{3}(-1+i)$$

for every contour from $z = 0$ to $z = 1 + i$.

EXAMPLE 2. The function $f(z) = 1/z^2$, which is continuous everywhere except at the origin, has an antiderivative $F(z) = -1/z$ in the domain $|z| > 0$, consisting of the entire plane with the origin deleted. Consequently,

$$\int_C \frac{dz}{z^2} = 0$$

when C is the positively oriented circle (Fig. 49)

$$z = 2\,e^{i\theta} \qquad (-\pi \le \theta \le \pi) \tag{1}$$

about the origin.

Note that the integral of the function $f(z) = 1/z$ around the same circle *cannot* be evaluated in a similar way. For, although the derivative of any branch $F(z)$ of $\log z$ is $1/z$ (Sec. 31), $F(z)$ is not differentiable, or even defined, along its branch cut. In particular, if a ray $\theta = \alpha$ from the origin is used to form the branch cut, $F'(z)$ fails to exist at the point where that ray intersects the circle C (see Fig. 49). So C does not lie in any domain throughout which $F'(z) = 1/z$, and one cannot make direct use of an antiderivative. Example 3, just below, illustrates how a combination of *two* different antiderivatives can be used to evaluate $f(z) = 1/z$ around C.

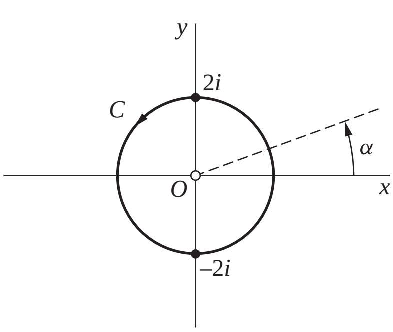

FIGURE 49

EXAMPLE 3. Let C_1 denote the right half

$$z = 2\,e^{i\theta} \qquad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right) \tag{2}$$

of the circle C in Example 2. The principal branch

$$\text{Log}\, z = \ln r + i\Theta \qquad (r > 0, -\pi < \Theta < \pi)$$

of the logarithmic function serves as an antiderivative of the function $1/z$ in the evaluation of the integral of $1/z$ along C_1 (Fig. 50):

$$\int_{C_1} \frac{dz}{z} = \int_{-2i}^{2i} \frac{dz}{z} = \text{Log}\, z\Big]_{-2i}^{2i} = \text{Log}(2i) - \text{Log}(-2i)$$
$$= \left(\ln 2 + i\frac{\pi}{2}\right) - \left(\ln 2 - i\frac{\pi}{2}\right) = \pi i.$$

This integral was evaluated in another way in Example 1, Sec. 41, where representation (2) for the semicircle was used.

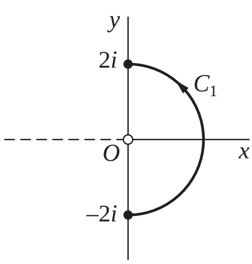

FIGURE 50

Next, let C_2 denote the *left* half

(3) $$z = 2\,e^{i\theta} \qquad \left(\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\right)$$

of the same circle C and consider the branch

$$\log z = \ln r + i\theta \qquad (r > 0, 0 < \theta < 2\pi)$$

of the logarithmic function (Fig. 51). One can write

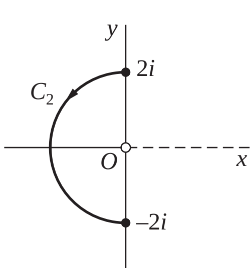

FIGURE 51

$$\int_{C_2} \frac{dz}{z} = \int_{2i}^{-2i} \frac{dz}{z} = \log z \Big]_{2i}^{-2i} = \log(-2i) - \log(2i)$$
$$= \left(\ln 2 + i\frac{3\pi}{2} \right) - \left(\ln 2 + i\frac{\pi}{2} \right) = \pi i.$$

The value of the integral of $1/z$ around the entire circle $C = C_1 + C_2$ is thus obtained:

$$\int_C \frac{dz}{z} = \int_{C_1} \frac{dz}{z} + \int_{C_2} \frac{dz}{z} = \pi i + \pi i = 2\pi i.$$

EXAMPLE 4. Let us use an antiderivative to evaluate the integral

$$\int_{C_1} z^{1/2}\, dz, \tag{4}$$

where the integrand is the branch

$$f(z) = z^{1/2} = \exp\left(\frac{1}{2} \log z \right) = \sqrt{r} e^{i\theta/2} \qquad (r > 0, 0 < \theta < 2\pi) \tag{5}$$

of the square root function and where C_1 is any contour from $z = -3$ to $z = 3$ that, except for its end points, lies above the x axis (Fig. 52). Although the integrand is piecewise continuous on C_1, and the integral therefore exists, the branch (5) of $z^{1/2}$ is not defined on the ray $\theta = 0$, in particular at the point $z = 3$. But another branch,

$$f_1(z) = \sqrt{r} e^{i\theta/2} \qquad \left(r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2} \right),$$

is defined and continuous everywhere on C_1. The values of $f_1(z)$ at all points on C_1 except $z = 3$ coincide with those of our integrand (5); so the integrand can be replaced by $f_1(z)$. Since an antiderivative of $f_1(z)$ is the function

$$F_1(z) = \frac{2}{3} z^{3/2} = \frac{2}{3} r\sqrt{r} e^{i3\theta/2} \qquad \left(r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2} \right),$$

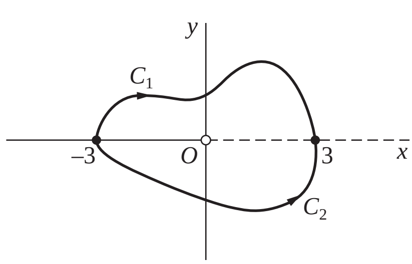

FIGURE 52

we can now write

$$\int_{C_1} z^{1/2}\,dz = \int_{-3}^{3} f_1(z)\,dz = F_1(z)\Big]_{-3}^{3} = 2\sqrt{3}(e^{i0} - e^{i3\pi/2}) = 2\sqrt{3}(1+i).$$

(Compare with Example 1 in Sec. 42.)

The integral

$$\int_{C_2} z^{1/2}\,dz \tag{6}$$

of the function (5) over any contour C_2 that extends from $z = -3$ to $z = 3$ *below* the real axis can be evaluated in a similar way. In this case, we can replace the integrand by the branch

$$f_2(z) = \sqrt{r}e^{i\theta/2} \qquad \left(r > 0, \frac{\pi}{2} < \theta < \frac{5\pi}{2}\right),$$

whose values coincide with those of the integrand at $z = -3$ and at all points on C_2 below the real axis. This enables us to use an antiderivative of $f_2(z)$ to evaluate integral (6). Details are left to the exercises.

45. PROOF OF THE THEOREM

To prove the theorem in the previous section, it is sufficient to show that statement *(a)* implies statement *(b)*, that statement *(b)* implies statement *(c)*, and finally that statement *(c)* implies statement *(a)*.

Let us assume that statement *(a)* is true, or that $f(z)$ has an antiderivative $F(z)$ on the domain D being considered. To show how statement *(b)* follows, we need to show that integration in independent of path in D and that the fundamental theorem of calculus can be extended using $F(z)$. If a contour C from z_1 to z_2 is a *smooth* arc lying in D, with parametric representation $z = z(t)$ $(a \le t \le b)$, we know from Exercise 5, Sec. 39, that

$$\frac{d}{dt}F[z(t)] = F'[z(t)]z'(t) = f[z(t)]z'(t) \quad (a \le t \le b).$$

Because the fundamental theorem of calculus can be extended so as to apply to complex-valued functions of a real variable (Sec. 38), it follows that

$$\int_C f(z)\,dz = \int_a^b f\big[z(t)\big]z'(t)\,dt = F[z(t)]\Big]_a^b = F[z(b)] - F[z(a)].$$

Since $z(b) = z_2$ and $z(a) = z_1$, the value of this contour integral is then

$$F(z_2) - F(z_1);$$

and that value is evidently independent of the contour C as long as C extends from z_1 to z_2 and lies entirely in D. That is,

$$\int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1) = F(z)\Big]_{z_1}^{z_2} \tag{1}$$

when C is smooth. Expression (1) is also valid when C is *any* contour, not necessarily a smooth one, that lies in D. For, if C consists of a finite number of smooth arcs C_k $(k = 1, 2, \ldots, n)$, each C_k extending from a point z_k to a point z_{k+1}, then

$$\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} f(z)\,dz = \sum_{k=1}^{n} [F(z_{k+1}) - F(z_k)].$$

Because the last sum here telescopes to $F(z_{n+1}) - F(z_1)$, we arrive at the expression

$$\int_C f(z)\,dz = F(z_{n+1}) - F(z_1).$$

(Compare with Example 3, Sec. 41.) The fact that statement *(b)* follows from statement *(a)* is now established.

To see that statement *(b)* implies statement *(c)*, we now show that the value of any integral around a closed contour in D is zero when integration is independent of path there. To do this, we let z_1 and z_2 denote two points on any closed contour C lying in D and form two paths C_1 and C_2, each with initial point z_1 and final point z_2, such that $C = C_1 - C_2$ (Fig. 53). Assuming that integration is independent of path in D, one can write

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz, \tag{2}$$

or

$$\int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = 0. \tag{3}$$

That is, the integral of $f(z)$ around the closed contour $C = C_1 - C_2$ has value zero.

It remains to show statement *(c)* implies statement *(a)*. That is, we need to show that if integrals of $f(z)$ around closed contours in D always have value zero,

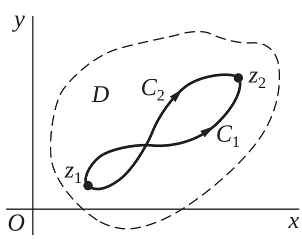

FIGURE 53

then $f(z)$ has an antiderivative on D. Assuming that the values of such integrals are in fact zero, we start by showing that integration is independent of path in D. We let C_1 and C_2 denote any two contours, lying in D, from a point z_1 to a point z_2 and observe that since integrals around closed paths lying in D have value zero, equation (3) holds (see Fig. 53). Thus equation (2) holds. Integration is, therefore, independent of path in D; and we can define the function

$$F(z) = \int_{z_0}^{z} f(s)\,ds$$

on D. The proof of the theorem is complete once we show that $F'(z) = f(z)$ everywhere in D. We do this by letting $z + \Delta z$ be any point distinct from z and lying in some neighborhood of z that is small enough to be contained in D. Then

$$F(z+\Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(s)\,ds - \int_{z_0}^{z} f(s)\,ds = \int_{z}^{z+\Delta z} f(s)\,ds,$$

where the path of integration may be selected as a line segment (Fig. 54). Since

$$\int_{z}^{z+\Delta z} ds = \Delta z$$

(see Exercise 5, Sec. 42), one can write

$$f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)\,ds;$$

and it follows that

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} [f(s) - f(z)]\,ds.$$

But f is continuous at the point z. Hence, for each positive number ε, a positive number δ exists such that

$$|f(s) - f(z)| < \varepsilon \quad \text{whenever} \quad |s - z| < \delta.$$

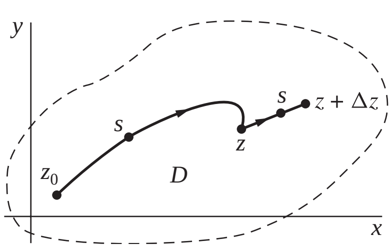

FIGURE 54

Consequently, if the point $z + \Delta z$ is close enough to z so that $|\Delta z| < \delta$, then

$$\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| < \frac{1}{|\Delta z|}\varepsilon|\Delta z| = \varepsilon;$$

that is,

$$\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z),$$

or $F'(z) = f(z)$.

EXERCISES

1. Use an antiderivative to show that for every contour C extending from a point z_1 to a point z_2,

$$\int_C z^n\, dz = \frac{1}{n+1}(z_2^{n+1} - z_1^{n+1}) \qquad (n = 0, 1, 2, \ldots).$$

2. By finding an antiderivative, evaluate each of these integrals, where the path is any contour between the indicated limits of integration:

$$(a)\ \int_i^{i/2} e^{\pi z}\, dz; \qquad (b)\ \int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz; \qquad (c)\ \int_1^3 (z-2)^3\, dz.$$

Ans. (a) $(1+i)/\pi$; *(b)* $e + (1/e)$; *(c)* 0.

3. Use the theorem in Sec. 44 to show that

$$\int_{C_0} (z - z_0)^{n-1}\, dz = 0 \qquad (n = \pm 1, \pm 2, \ldots)$$

when C_0 is any closed contour which does not pass through the point z_0. [Compare with Exercise 10*(b)*, Sec. 42.]

4. Find an antiderivative $F_2(z)$ of the branch $f_2(z)$ of $z^{1/2}$ in Example 4, Sec. 44, to show that integral (6) there has value $2\sqrt{3}(-1+i)$. Note that the value of the integral of the function (5) around the closed contour $C_2 - C_1$ in that example is, therefore, $-4\sqrt{3}$.

5. Show that

$$\int_{-1}^{1} z^i\, dz = \frac{1 + e^{-\pi}}{2}(1 - i),$$

where the integrand denotes the principal branch

$$z^i = \exp(i \operatorname{Log} z) \qquad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

of z^i and where the path of integration is any contour from $z = -1$ to $z = 1$ that, except for its end points, lies above the real axis. (Compare with Exercise 7, Sec. 42.)

Suggestion: Use an antiderivative of the branch

$$z^i = \exp(i \log z) \qquad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

of the same power function.

46. CAUCHY–GOURSAT THEOREM

In Sec. 44, we saw that when a continuous function f has an antiderivative in a domain D, the integral of $f(z)$ around any given closed contour C lying entirely in D has value zero. In this section, we present a theorem giving other conditions on a function f which ensure that the value of the integral of $f(z)$ around a *simple* closed contour (Sec. 39) is zero. The theorem is central to the theory of functions of a complex variable; and some modifications of it, involving certain special types of domains, will be given in Secs. 48 and 49.

We let C denote a simple closed contour $z = z(t)$ $(a \le t \le b)$, described in the *positive sense* (counterclockwise), and we assume that f is analytic at each point interior to and on C. According to Sec. 40,

$$\int_C f(z)\,dz = \int_a^b f[z(t)]z'(t)\,dt; \tag{1}$$

and if

$$f(z) = u(x, y) + iv(x, y) \quad \text{and} \quad z(t) = x(t) + iy(t),$$

the integrand $f[z(t)]z'(t)$ in expression (1) is the product of the functions

$$u[x(t), y(t)] + iv[x(t), y(t)], \qquad x'(t) + iy'(t)$$

of the real variable t. Thus

$$\int_C f(z)\;dz = \int_a^b (ux' - vy')\,dt + i\int_a^b (vx' + uy')\,dt. \tag{2}$$

In terms of line integrals of real-valued functions of two real variables, then,

$$\int_C f(z)\;dz = \int_C u\;dx - v\;dy + i\int_C v\;dx + u\;dy. \tag{3}$$

Observe that expression (3) can be obtained formally by replacing $f(z)$ and dz on the left with the binomials

$$u + iv \quad \text{and} \quad dx + i\,dy,$$

respectively, and expanding their product. Expression (3) is, of course, also valid when C is any contour, not necessarily a simple closed one, and when $f[z(t)]$ is only piecewise continuous on it.

We next recall a result from calculus that enables us to express the line integrals on the right in equation (3) as double integrals. Suppose that two real-valued functions $P(x, y)$ and $Q(x, y)$, together with their first-order partial derivatives, are continuous throughout the closed region R consisting of all points interior to and on the simple closed contour C. According to *Green's theorem*,

$$\int_C P dx + Q\,dy = \iint_R (Q_x - P_y)\,dA.$$

Now f is continuous in R, since it is analytic there. Hence the functions u and v are also continuous in R. Likewise, if the derivative f' of f is continuous in R, so are the first-order partial derivatives of u and v. Green's theorem then enables us to rewrite equation (3) as

$$\int_C f(z)\,dz = \int\!\!\int_R (-v_x - u_y)\,dA + i\int\!\!\int_R (u_x - v_y)\,dA. \tag{4}$$

But, in view of the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x,$$

the integrands of these two double integrals are zero throughout R. So *when f is analytic in R and f' is continuous there,*

$$\int_C f(z)\,dz = 0. \tag{5}$$

This result was obtained by Cauchy in the early part of the nineteenth century.

Note that once it has been established that the value of this integral is zero, the orientation of C is immaterial. That is, statement (5) is also true if C is taken in the clockwise direction, since then

$$\int_C f(z)\,dz = -\int_{-C} f(z)\,dz = 0.$$

EXAMPLE. If C is any simple closed contour, in either direction, then

$$\int_C \exp(z^3)\,dz = 0.$$

This is because the composite function $f(z) = \exp(z^3)$ is analytic everywhere and its derivative $f'(z) = 3z^2\exp(z^3)$ is continuous everywhere.

Goursat* was the first to prove that *the condition of continuity on f' can be omitted*. Its removal is important and will allow us to show, for example, that the derivative f' of an analytic function f is analytic without having to assume the continuity of f', which follows as a consequence. We now state the revised form of Cauchy's result, known as the *Cauchy–Goursat theorem*.

Theorem. *If a function f is analytic at all points interior to and on a simple closed contour C, then*

$$\int_C f(z)\,dz = 0.$$

*E. Goursat (1858–1936), pronounced *gour-sah′*.

The proof is presented in the next section, where, to be specific, we assume that C is positively oriented. The reader who wishes to accept this theorem without proof may pass directly to Sec. 48.

47. PROOF OF THE THEOREM

We preface the proof of the Cauchy–Goursat theorem with a lemma. We start by forming subsets of the region R which consists of the points on a positively oriented simple closed contour C together with the points interior to C. To do this, we draw equally spaced lines parallel to the real and imaginary axes such that the distance between adjacent vertical lines is the same as that between adjacent horizontal lines. We thus form a finite number of closed square subregions, where each point of R lies in at least one such subregion and each subregion contains points of R. We refer to these square subregions simply as *squares*, always keeping in mind that by a square we mean a boundary together with the points interior to it. If a particular square contains points that are not in R, we remove those points and call what remains a *partial square*. We thus *cover* the region R with a finite number of squares and partial squares (Fig. 55), and our proof of the following lemma starts with this covering.

Lemma. *Let f be analytic throughout a closed region R consisting of the points interior to a positively oriented simple closed contour C together with the points on C itself. For any positive number ε, the region R can be covered with a finite number of squares and partial squares, indexed by $j = 1, 2, \ldots, n$, such that in each one there is a fixed point z_j for which the inequality*

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \varepsilon \tag{1}$$

is satisfied by all points other than z_j in that square or partial square.

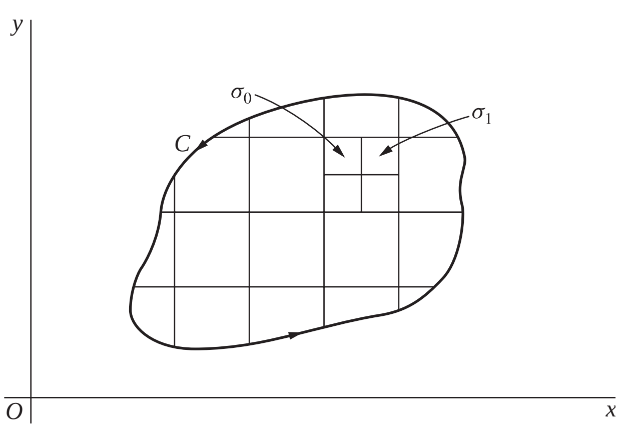

FIGURE 55

To start the proof, we consider the possibility that in the covering constructed just prior to the statement of the lemma, there is some square or partial square in which no point z_j exists such that inequality (1) holds for all other points z in it. If that subregion is a square, we construct four smaller squares by drawing line segments joining the midpoints of its opposite sides (Fig. 55). If the subregion is a partial square, we treat the whole square in the same manner and then let the portions that lie outside of R be discarded. If in any one of these smaller subregions, no point z_j exists such that inequality (1) holds for all other points z in it, we construct still smaller squares and partial squares, etc. When this is done to each of the original subregions that requires it, we find that *after a finite number of steps*, the region R can be covered with a finite number of squares and partial squares such that the lemma is true.

To verify this, we suppose that the needed points z_j do *not* exist after subdividing one of the original subregions a finite number of times and reach a contradiction. We let σ_0 denote that subregion if it is a square; if it is a partial square, we let σ_0 denote the entire square of which it is a part. After we subdivide σ_0, at least one of the four smaller squares, denoted by σ_1, must contain points of R but no appropriate point z_j. We then subdivide σ_1 and continue in this manner. It may be that after a square σ_{k-1} $(k = 1, 2, \ldots)$ has been subdivided, more than one of the four smaller squares constructed from it can be chosen. To make a specific choice, we take σ_k to be the one lowest and then furthest to the left.

In view of the manner in which the nested infinite sequence

$$\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{k-1}, \sigma_k, \ldots \tag{2}$$

of squares is constructed, it is easily shown (Exercise 9, Sec. 49) that there is a point z_0 common to each σ_k; also, each of these squares contains points of R other than possibly z_0. Recall how the sizes of the squares in the sequence are decreasing, and note that any δ neighborhood $|z - z_0| < \delta$ of z_0 contains such squares when their diagonals have lengths less than δ. Every δ neighborhood $|z - z_0| < \delta$ therefore contains points of R distinct from z_0, and this means that z_0 is an accumulation point of R. Since the region R is a closed set, it follows that z_0 is a point in R. (See Sec. 11.)

Now the function f is analytic throughout R and, in particular, at z_0. Consequently, $f'(z_0)$ exists, According to the definition of derivative (Sec. 19), there is, for each positive number ε, a δ neighborhood $|z - z_0| < \delta$ such that the inequality

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$$

is satisfied by all points distinct from z_0 in that neighborhood. But the neighborhood $|z - z_0| < \delta$ contains a square σ_K when the integer K is large enough that the length of a diagonal of that square is less than δ (Fig. 56). Consequently, z_0 serves as the point z_j in inequality (1) for the subregion consisting of the square σ_K or a part of σ_K. Contrary to the way in which the sequence (2) was formed, then, it is not necessary to subdivide σ_K. We thus arrive at a contradiction, and the proof of the lemma is complete.

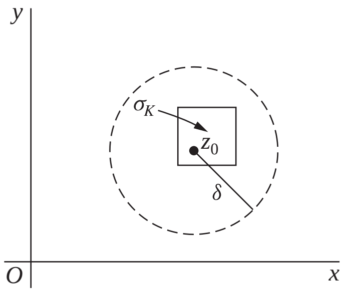

FIGURE 56

Continuing with a function f which is analytic throughout a region R consisting of a positively oriented simple closed contour C and points interior to it, we are now ready to prove the Cauchy–Goursat theorem, namely that

$$\int_C f(z)\,dz = 0. \tag{3}$$

Given an arbitrary positive number ε, we consider the covering of R in the statement of the lemma. We then define on the jth square or partial square a function $\delta_j(z)$ whose values are $\delta_j(z_j) = 0$, where z_j is the fixed point in inequality (1), and

$$\delta_j(z) = \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \quad \text{when } z \neq z_j. \tag{4}$$

According to inequality (1),

$$|\delta_j(z)| < \varepsilon \tag{5}$$

at all points z in the subregion on which $\delta_j(z)$ is defined. Also, the function $\delta_j(z)$ is continuous throughout the subregion since $f(z)$ is continuous there and

$$\lim_{z \to z_j} \delta_j(z) = f'(z_j) - f'(z_j) = 0.$$

Next, we let C_j $(j = 1, 2, \ldots, n)$ denote the positively oriented boundaries of the above squares or partial squares covering R. In view of our definition of $\delta_j(z)$, the value of f at a point z on any particular C_j can be written

$$f(z) = f(z_j) - z_j f'(z_j) + f'(z_j)z + (z - z_j)\delta_j(z);$$

and this means that

$$\int_{C_j} f(z)\ dz = [f(z_j) - z_j f'(z_j)] \int_{C_j} dz + f'(z_j) \int_{C_j} z\ dz + \int_{C_j} (z - z_j)\delta_j(z)\ dz. \tag{6}$$

But

$$\int_{C_j} dz = 0 \quad \text{and} \quad \int_{C_j} z\, dz = 0$$

since the functions 1 and z possess antiderivatives everywhere in the finite plane. So equation (6) reduces to

$$\int_{C_j} f(z)\, dz = \int_{C_j} (z - z_j)\delta_j(z)\, dz \qquad (j = 1, 2, \ldots, n). \tag{7}$$

The sum of all n integrals on the left in equations (7) can be written

$$\sum_{j=1}^{n} \int_{C_j} f(z)\, dz = \int_C f(z)\, dz$$

since the two integrals along the common boundary of every pair of adjacent subregions cancel each other, the integral being taken in one sense along that line segment in one subregion and in the opposite sense in the other (Fig. 57). Only the integrals along the arcs that are parts of C remain. Thus, in view of equations (7),

$$\int_C f(z)\, dz = \sum_{j=1}^{n} \int_{C_j} (z - z_j)\delta_j(z)\, dz\,;$$

and so

$$\left| \int_C f(z)\, dz \right| \le \sum_{j=1}^{n} \left| \int_{C_j} (z - z_j)\delta_j(z)\, dz \right|. \tag{8}$$

We now use the theorem in Sec. 43 to find an upper bound for each modulus on the right in inequality (8). To do this, we first recall that each C_j coincides either

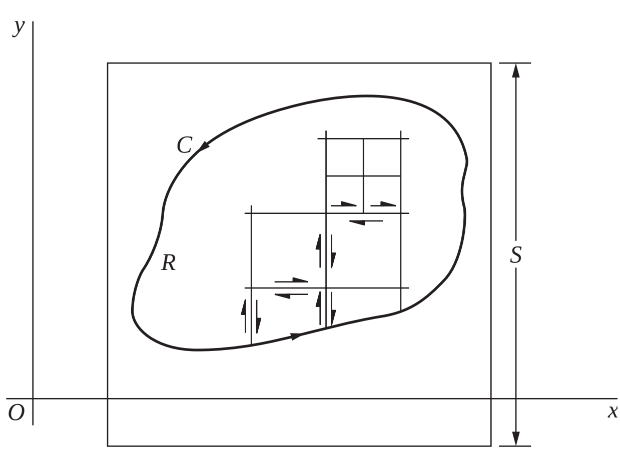

FIGURE 57

entirely or partially with the boundary of a square. In either case, we let s_j denote the length of a side of the square. Since, in the jth integral, both the variable z and the point z_j lie in that square,

$$|z - z_j| \le \sqrt{2} s_j .$$

In view of inequality (5), then, we know that each integrand on the right in inequality (8) satisfies the condition

$$|(z - z_j)\delta_j(z)| = |z - z_j|\,|\delta_j(z)| < \sqrt{2} s_j \varepsilon. \tag{9}$$

As for the length of the path C_j, it is $4s_j$ if C_j is the boundary of a square. In that case, we let A_j denote the area of the square and observe that

$$\left| \int_{C_j} (z - z_j)\delta_j(z)\, dz \right| < \sqrt{2} s_j \varepsilon 4 s_j = 4\sqrt{2} A_j \varepsilon. \tag{10}$$

If C_j is the boundary of a partial square, its length does not exceed $4s_j + L_j$, where L_j is the length of that part of C_j which is also a part of C. Again letting A_j denote the area of the full square, we find that

$$\left| \int_{C_j} (z - z_j)\delta_j(z)\, dz \right| < \sqrt{2} s_j \varepsilon (4s_j + L_j) < 4\sqrt{2} A_j \varepsilon + \sqrt{2} S L_j \varepsilon, \tag{11}$$

where S is the length of a side of some square that encloses the entire contour C as well as all of the squares originally used in covering R (Fig. 57). Note that the sum of all the A_j's does not exceed S^2.

If L denotes the length of C, it now follows from inequalities (8), (10), and (11) that

$$\left| \int_C f(z)\, dz \right| < (4\sqrt{2} S^2 + \sqrt{2} S L)\varepsilon.$$

Since the value of the positive number ε is arbitrary, we can choose it so that the right-hand side of this last inequality is as small as we please. The left-hand side, which is independent of ε, must therefore be equal to zero; and statement (3) follows. This completes the proof of the Cauchy–Goursat theorem.

48. SIMPLY CONNECTED DOMAINS

A *simply connected* domain D is a domain such that every simple closed contour within it encloses only points of D. The set of points interior to a simple closed contour is an example. The annular domain between two concentric circles is, however, not simply connected. Domains that are not simply connected are discussed in the next section.

The closed contour in the Cauchy–Goursat theorem (Sec. 46) need not be simple when the theorem is adapted to simply connected domains. More precisely, the contour can actually cross itself. The following theorem allows for this possibility.

Theorem. *If a function f is analytic throughout a simply connected domain D, then*

$$\int_C f(z)\,dz = 0 \tag{1}$$

for every closed contour C lying in D.

The proof is easy if C is a *simple* closed contour or if it is a closed contour that intersects itself a *finite* number of times. For if C is simple and lies in D, the function f is analytic at each point interior to and on C; and the Cauchy–Goursat theorem ensures that equation (1) holds. Furthermore, if C is closed but intersects itself a finite number of times, it consists of a finite number of simple closed contours. This is illustrated in Fig. 58, where the simple closed contours C_k $(k = 1, 2, 3, 4)$ make up C. Since the value of the integral around each C_k is zero, according to the Cauchy–Goursat theorem, it follows that

$$\int_C f(z)\,dz = \sum_{k=1}^{4} \int_{C_k} f(z)\,dz = 0.$$

Subtleties arise if the closed contour has an *infinite* number of self-intersection points. One method that can sometimes be used to show that the theorem still applies is illustrated in Exercise 5, Sec. 49.*

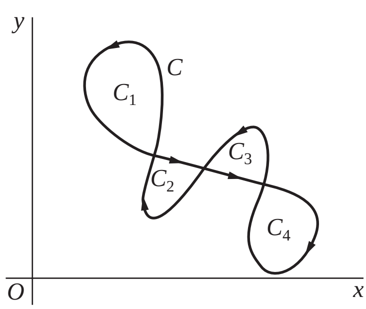

FIGURE 58

EXAMPLE. If C denotes any closed contour lying in the open disk $|z| < 2$ (Fig. 59), then

$$\int_C \frac{z\,e^z}{(z^2+9)^5}\,dz = 0.$$

*For a proof of the theorem involving more general paths of finite length, see, for example, Secs. 63–65 in Vol. I of the book by Markushevich that is cited in Appendix 1.

This is because the disk is a simply connected domain and the two singularities $z = \pm 3i$ of the integrand are exterior to the disk.

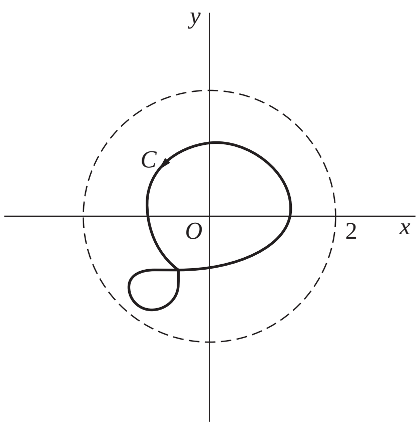

FIGURE 59

Corollary. *A function f that is analytic throughout a simply connected domain D must have an antiderivative everywhere in D.*

We begin the proof of this corollary with the observation that a function f is continuous on a domain D when it is analytic there. Consequently, since equation (1) holds for the function in the hypothesis of this corollary and for each closed contour C in D, f has an antiderivative throughout D, according to the theorem in Sec. 44. Note that since the finite plane is simply connected, the corollary tells us that *entire functions always possess antiderivatives*.

49. MULTIPLY CONNECTED DOMAINS

A domain that is not simply connected (Sec. 48) is said to be *multiply connected*. The following theorem is an adaptation of the Cauchy–Goursat theorem to multiply connected domains.

Theorem. *Suppose that*

(a) C is a simple closed contour, described in the counterclockwise direction;

(b) C_k $(k = 1, 2, \ldots, n)$ are simple closed contours interior to C, all described in the clockwise direction, that are disjoint and whose interiors have no points in common (Fig. 60).

If a function f is analytic on all of these contours and throughout the multiply connected domain consisting of the points inside C and exterior to each C_k, then

$$\int_C f(z)\,dz + \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0. \tag{1}$$

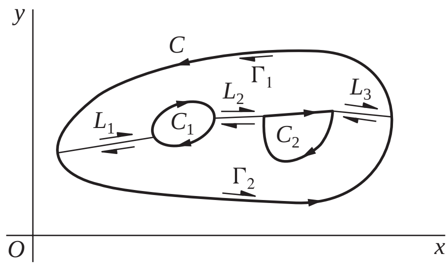

FIGURE 60

Note that in equation (1), the direction of each path of integration is such that the multiply connected domain lies to the *left* of that path.

To prove the theorem, we introduce a polygonal path L_1, consisting of a finite number of line segments joined end to end, to connect the outer contour C to the inner contour C_1. We introduce another polygonal path L_2 which connects C_1 to C_2; and we continue in this manner, with L_{n+1} connecting C_n to C. As indicated by the single-barbed arrows in Fig. 60, two simple closed contours Γ_1 and Γ_2 can be formed, each consisting of polygonal paths L_k or $-L_k$ and pieces of C and C_k and each described in such a direction that the points enclosed by them lie to the left. The Cauchy–Goursat theorem can now be applied to f on Γ_1 and Γ_2, and the sum of the values of the integrals over those contours is found to be zero. Since the integrals in opposite directions along each path L_k cancel, only the integrals along C and the C_k remain; and we arrive at statement (1).

Corollary. *Let C_1 and C_2 denote positively oriented simple closed contours, where C_1 is interior to C_2* (Fig. 61). *If a function f is analytic in the closed region consisting of those contours and all points between them, then*

$$\int_{C_2} f(z)\,dz = \int_{C_1} f(z)\,dz. \tag{2}$$

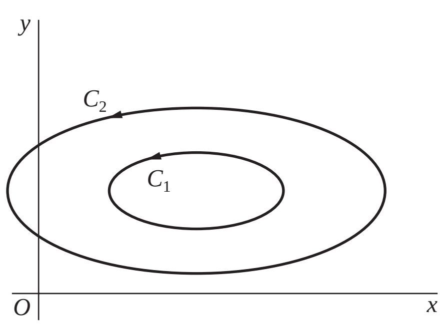

FIGURE 61

This corollary is known as the *principle of deformation of paths* since it tells us that if C_1 is continuously deformed into C_2, always passing through points at which f is analytic, then the value of the integral of f over C_1 never changes. To verify the corollary, we need only write equation (2) as

$$\int_{C_2} f(z)\,dz + \int_{-C_1} f(z)\,dz = 0$$

and apply the theroem.

EXAMPLE. When C is any positively oriented simple closed contour surrounding the origin, the corollary can be used to show that

$$\int_C \frac{dz}{z} = 2\pi i.$$

This is done by constructing a positively oriented circle C_0 with center at the origin and radius so small that C_0 lies entirely inside C (Fig. 62). Since (see Example 2, Sec. 42)

$$\int_{C_0} \frac{dz}{z} = 2\pi i$$

and since $1/z$ is analytic everywhere except at $z = 0$, the desired result follows.

Note that the radius of C_0 could equally well have been so large that C lies entirely inside C_0.

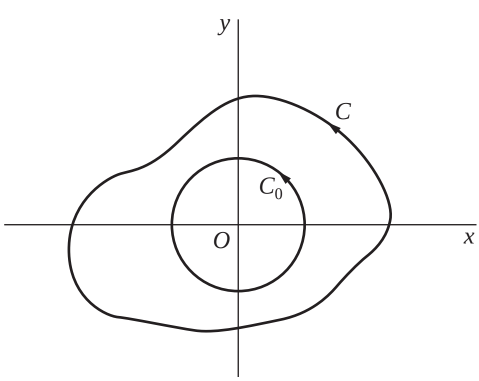

FIGURE 62

EXERCISES

1. Apply the Cauchy–Goursat theorem to show that

$$\int_C f(z)\,dz = 0$$

when the contour C is the unit circle $|z| = 1$, in either direction, and when

(a) $f(z) = \dfrac{z^2}{z-3}$; *(b)* $f(z) = z\,e^{-z}$; *(c)* $f(z) = \dfrac{1}{z^2+2z+2}$;

(d) $f(z) = \operatorname{sech} z$; *(e)* $f(z) = \tan z$; *(f)* $f(z) = \operatorname{Log}\,(z+2)$.

2. Let C_1 denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1$, $y = \pm 1$ and let C_2 be the positively oriented circle $|z| = 4$ (Fig. 63). With the aid of the corollary in Sec. 49, point out why

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

when

(a) $f(z) = \dfrac{1}{3z^2 + 1}$; *(b)* $f(z) = \dfrac{z + 2}{\sin(z/2)}$; *(c)* $f(z) = \dfrac{z}{1 - e^z}$.

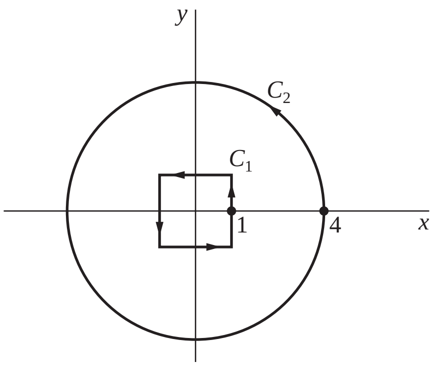

FIGURE 63

3. If C_0 denotes a positively oriented circle $|z - z_0| = R$, then

$$\int_{C_0} (z - z_0)^{n-1}\,dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \ldots, \\ 2\pi i & \text{when } n = 0, \end{cases}$$

according to Exercise 10*(b)*, Sec. 42. Use that result and the corollary in Sec. 49 to show that if C is the boundary of the rectangle $0 \le x \le 3, 0 \le y \le 2$, described in the positive sense, then

$$\int_{C} (z - 2 - i)^{n-1}\,dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \ldots, \\ 2\pi i & \text{when } n = 0. \end{cases}$$

4. Use the following method to derive the integration formula

$$\int_0^\infty e^{-x^2} \cos 2bx\,dx = \frac{\sqrt{\pi}}{2} e^{-b^2} \qquad (b > 0).$$

(a) Show that the sum of the integrals of e^{-z^2} along the lower and upper horizontal legs of the rectangular path in Fig. 64 can be written

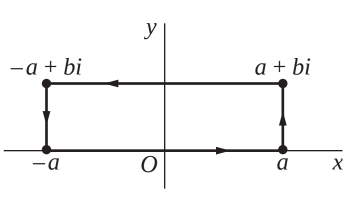

FIGURE 64

$$2\int_0^a e^{-x^2}dx - 2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\ dx$$

and that the sum of the integrals along the vertical legs on the right and left can be written

$$ie^{-a^2}\int_0^b e^{y^2}e^{-i2ay}dy - ie^{-a^2}\int_0^b e^{y^2}e^{i2ay}dy.$$

Thus, with the aid of the Cauchy–Goursat theorem, show that

$$\int_0^a e^{-x^2}\cos 2bx\,dx = e^{-b^2}\int_0^a e^{-x^2}dx + e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin 2ay\ dy.$$

(b) By accepting the fact that*

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

and observing that

$$\left|\int_0^b e^{y^2}\sin 2ay\,dy\right| \le \int_0^b e^{y^2}dy,$$

obtain the desired integration formula by letting a tend to infinity in the equation at the end of part *(a)*.

5. According to Exercise 6, Sec. 39, the path C_1 from the origin to the point $z = 1$ along the graph of the function defined by means of the equations

$$y(x) = \begin{cases} x^3\sin(\pi/x) & \text{when } 0 < x \le 1, \\ 0 & \text{when } x = 0 \end{cases}$$

is a smooth arc that intersects the real axis an infinite number of times. Let C_2 denote the line segment along the real axis from $z = 1$ back to the origin, and let C_3 denote any smooth arc from the origin to $z = 1$ that does not intersect itself and has only its end points in common with the arcs C_1 and C_2 (Fig. 65). Apply the Cauchy–Goursat theorem to show that if a function f is entire, then

$$\int_{C_1} f(z)\ dz = \int_{C_3} f(z)\ dz \quad \text{and} \quad \int_{C_2} f(z)\ dz = -\int_{C_3} f(z)\ dz.$$

Conclude that even though the closed contour $C = C_1 + C_2$ intersects itself an infinite number of times,

$$\int_C f(z)\ dz = 0.$$

*The usual way to evaluate this integral is by writing its square as

$$\int_0^\infty e^{-x^2}dx\int_0^\infty e^{-y^2}dy = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy$$

and then evaluating this iterated integral by changing to polar coordinates. Details are given in, for example, A. E. Taylor and W. R. Mann, "Advanced Calculus," 3d ed., pp. 680–681, 1983.

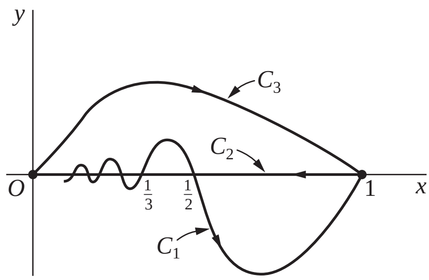

FIGURE 65

6. Let C denote the positively oriented boundary of the half disk $0 \le r \le 1, 0 \le \theta \le \pi$, and let $f(z)$ be a continuous function defined on that half disk by writing $f(0) = 0$ and using the branch

$$f(z) = \sqrt{r}e^{i\theta/2} \qquad \left(r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right)$$

of the multiple-valued function $z^{1/2}$. Show that

$$\int_C f(z)\, dz = 0$$

by evaluating separately the integrals of $f(z)$ over the semicircle and the two radii which make up C. Why does the Cauchy–Goursat theorem not apply here?

7. Show that if C is a positively oriented simple closed contour, then the area of the region enclosed by C can be written

$$\frac{1}{2i}\int_C \overline{z}\, dz.$$

Suggestion: Note that expression (4), Sec. 46, can be used here even though the function $f(z) = \overline{z}$ is not analytic anywhere [see Example 2, Sec. 19].

8. *Nested Intervals*. An infinite sequence of closed intervals $a_n \le x \le b_n$ $(n = 0, 1, 2, \ldots)$ is formed in the following way. The interval $a_1 \le x \le b_1$ is either the left-hand or right-hand half of the first interval $a_0 \le x \le b_0$, and the interval $a_2 \le x \le b_2$ is then one of the two halves of $a_1 \le x \le b_1$, etc. Prove that there is a point x_0 which belongs to every one of the closed intervals $a_n \le x \le b_n$.

Suggestion: Note that the left-hand end points a_n represent a bounded nondecreasing sequence of numbers, since $a_0 \le a_n \le a_{n+1} < b_0$; hence they have a limit A as n tends to infinity. Show that the end points b_n also have a limit B. Then show that $A = B$, and write $x_0 = A = B$.

9. *Nested Squares*. A square $\sigma_0 : a_0 \le x \le b_0, c_0 \le y \le d_0$ is divided into four equal squares by line segments parallel to the coordinate axes. One of those four smaller squares $\sigma_1 : a_1 \le x \le b_1, c_1 \le y \le d_1$ is selected according to some rule. It, in turn, is divided into four equal squares one of which, called σ_2, is selected, etc. (see Sec. 47). Prove that there is a point (x_0, y_0) which belongs to each of the closed regions of the infinite sequence $\sigma_0, \sigma_1, \sigma_2, \ldots$.

Suggestion: Apply the result in Exercise 8 to each of the sequences of closed intervals $a_n \le x \le b_n$ and $c_n \le y \le d_n$ $(n = 0, 1, 2, \ldots)$.

50. CAUCHY INTEGRAL FORMULA

Another fundamental result will now be established.

Theorem. *Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then*

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z - z_0}. \tag{1}$$

Formula (1) is called the *Cauchy integral formula*. It tells us that if a function f is to be analytic within and on a simple closed contour C, then the values of f interior to C are completely determined by the values of f on C.

When the Cauchy integral formula is written as

$$\int_C \frac{f(z)\,dz}{z - z_0} = 2\pi i f(z_0), \tag{2}$$

it can be used to evaluate certain integrals along simple closed contours.

EXAMPLE. Let C be the positively oriented circle $|z| = 2$. Since the function

$$f(z) = \frac{z}{9 - z^2}$$

is analytic within and on C and since the point $z_0 = -i$ is interior to C, formula (2) tells us that

$$\int_C \frac{z\,dz}{(9 - z^2)(z + i)} = \int_C \frac{z/(9 - z^2)}{z - (-i)}\,dz = 2\pi i\left(\frac{-i}{10}\right) = \frac{\pi}{5}.$$

We begin the proof of the theorem by letting C_ρ denote a positively oriented circle $|z - z_0| = \rho$, where ρ is small enough that C_ρ is interior to C (see Fig. 66). Since the quotient $f(z)/(z - z_0)$ is analytic between and on the contours C_ρ and C, it follows from the principle of deformation of paths (Sec. 49) that

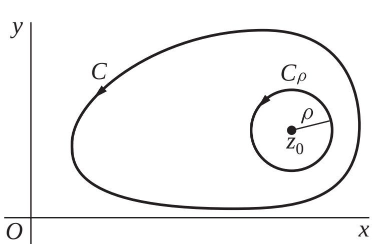

FIGURE 66

$$\int_C \frac{f(z)\,dz}{z-z_0} = \int_{C_\rho} \frac{f(z)\,dz}{z-z_0}.$$

This enables us to write

$$\int_C \frac{f(z)\,dz}{z-z_0} - f(z_0)\int_{C_\rho} \frac{dz}{z-z_0} = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz. \tag{3}$$

But [see Exercise 10*(b)*, Sec. 42]

$$\int_{C_\rho} \frac{dz}{z-z_0} = 2\pi i;$$

and so equation (3) becomes

$$\int_C \frac{f(z)\,dz}{z-z_0} - 2\pi i f(z_0) = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz. \tag{4}$$

Now the fact that f is analytic, and therefore continuous, at z_0 ensures that corresponding to each positive number ε, however small, there is a positive number δ such that

$$|f(z)-f(z_0)| < \varepsilon \quad \text{whenever} \quad |z-z_0| < \delta. \tag{5}$$

Let the radius ρ of the circle C_ρ be smaller than the number δ in the second of these inequalities. Since $|z-z_0| = \rho < \delta$ when z is on C_ρ, it follows that the *first* of inequalities (5) holds when z is such a point; and the theorem in Sec. 43, giving upper bounds for the moduli of contour integrals, tells us that

$$\left|\int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| < \frac{\varepsilon}{\rho}\,2\pi\rho = 2\pi\varepsilon.$$

In view of equation (4), then,

$$\left|\int_C \frac{f(z)\,dz}{z-z_0} - 2\pi i f(z_0)\right| < 2\pi\varepsilon.$$

Since the left-hand side of this inequality is a nonnegative constant that is less than an arbitrarily small positive number, it must be equal to zero. Hence equation (2) is valid, and the theorem is proved.

51. AN EXTENSION OF THE CAUCHY INTEGRAL FORMULA

The Cauchy integral formula in the theorem in Sec. 50 can be extended so as to provide an integral representation for derivatives of f at z_0. To obtain the extension, we consider a function f that is analytic everywhere inside and on a simple closed contour C, taken in the positive sense. We then write the Cauchy integral formula as

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{s-z}, \tag{1}$$

where z is interior to C and where s denotes points on C. Differentiating *formally* with respect to z under the integral sign here, without rigorous justification, we find that

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2}. \tag{2}$$

To verify that $f'(z)$ exists and that expression (2) is in fact valid, we led d denote the smallest distance from z to points s on C and use expression (1) to write

$$\begin{aligned}\frac{f(z+\Delta z)-f(z)}{\Delta z} &= \frac{1}{2\pi i}\int_C \left(\frac{1}{s-z-\Delta z} - \frac{1}{s-z}\right)\frac{f(s)}{\Delta z}\,ds \\ &= \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z-\Delta z)(s-z)},\end{aligned}$$

where $0 < |\Delta z| < d$ (see Fig. 67). Evidently, then,

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2} = \frac{1}{2\pi i}\int_C \frac{\Delta z\, f(s)\,ds}{(s-z-\Delta z)(s-z)^2}. \tag{3}$$

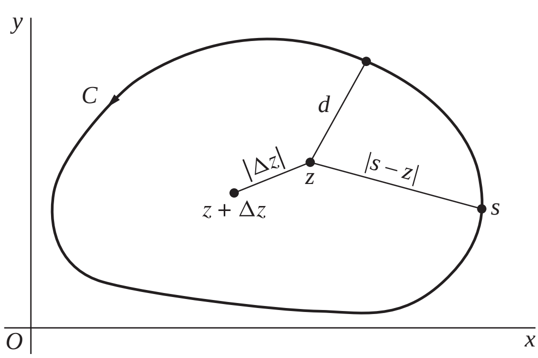

FIGURE 67

Next, we let M denote the maximum value of $|f(s)|$ on C and observe that since $|s-z| \ge d$ and $|\Delta z| < d$,

$$|s - z - \Delta z| = |(s - z) - \Delta z| \geq ||s - z| - |\Delta z|| \geq d - |\Delta z| > 0.$$

Thus

$$\left| \int_C \frac{\Delta z\, f(s)\, ds}{(s - z - \Delta z)(s - z)^2} \right| \leq \frac{|\Delta z| M}{(d - |\Delta z|)d^2} L,$$

where L is the length of C. Upon letting Δz tend to zero, we find from this inequality that the right-hand side of equation (3) also tends to zero. Consequently,

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi i} \int_C \frac{f(s)\, ds}{(s - z)^2} = 0\,;$$

and the desired expression for $f'(z)$ is established.

The same technique can be used to suggest and verify the expression

$$f''(z) = \frac{1}{\pi i} \int_C \frac{f(s)\, ds}{(s - z)^3}. \tag{4}$$

The details, which are outlined in Exercise 9, Sec. 52, are left to the reader. Mathematical induction can, moreover, be used to obtain the formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(s)\, ds}{(s - z)^{n+1}} \qquad (n = 1, 2, \ldots). \tag{5}$$

The verification is considerably more involved than for just $n = 1$ and $n = 2$, and we refer the interested reader to other texts for it.* Note that with the agreement that

$$f^{(0)}(z) = f(z) \quad \text{and} \quad 0! = 1,$$

expression (5) is also valid when $n = 0$, in which case it becomes the Cauchy integral formula (1).

When written in the form

$$\int_C \frac{f(z)\ dz}{(z - z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0) \qquad (n = 0, 1, 2, \ldots), \tag{6}$$

expressions (1) and (5) can be useful in evaluating certain integrals when f is analytic inside and on a simple closed contour C, taken in the positive sense, and z_0 is any point interior to C. It has already been illustrated in Sec. 50 when $n = 0$.

EXAMPLE 1. If C is the positively oriented unit circle $|z| = 1$ and

$$f(z) = \exp(2z),$$

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*See, for example, pp. 299–301 in Vol. I of the book by Markushevich, cited in Appendix 1.

then

$$\int_C \frac{\exp(2z)\,dz}{z^4} = \int_C \frac{f(z)\,dz}{(z-0)^{3+1}} = \frac{2\pi i}{3!} f'''(0) = \frac{8\pi i}{3}.$$

EXAMPLE 2. Let z_0 be any point interior to a positively oriented simple closed contour C. When $f(z) = 1$, expression (6) shows that

$$\int_C \frac{dz}{z - z_0} = 2\pi i$$

and

$$\int_C \frac{dz}{(z-z_0)^{n+1}} = 0 \qquad (n = 1, 2, \ldots).$$

(Compare with Exercise 10*(b)*, Sec. 42.)

52. SOME CONSEQUENCES OF THE EXTENSION

We turn now to some important consequences of the extension of the Cauchy integral formula in the previous section.

Theorem 1. *If a function f is analytic at a given point, then its derivatives of all orders are analytic there too.*

To prove this remarkable theorem, we assume that a function f is analytic at a point z_0. There must, then, be a neighborhood $|z - z_0| < \varepsilon$ of z_0 throughout which f is analytic (see Sec. 24). Consequently, there is a positively oriented circle C_0, centered at z_0 and with radius $\varepsilon/2$, such that f is analytic inside and on C_0 (Fig. 68). From expression (4), Sec. 51, we know that

$$f''(z) = \frac{1}{\pi i}\int_{C_0} \frac{f(s)\,ds}{(s-z)^3}$$

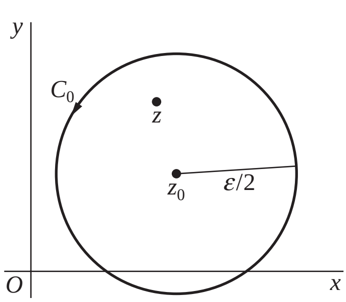

FIGURE 68

at each point z interior to C_0, and the existence of $f''(z)$ throughout the neighborhood $|z - z_0| < \varepsilon/2$ means that f' is analytic at z_0. One can apply the same argument to the analytic function f' to conclude that its derivative f'' is analytic, etc. Theorem 1 is now established.

As a consequence, when a function

$$f(z) = u(x, y) + iv(x, y)$$

is analytic at a point $z = (x, y)$, the differentiability of f' ensures the continuity of f' there (Sec. 19). Then, since (Sec. 21)

$$f'(z) = u_x + iv_x = v_y - iu_y,$$

we may conclude that the first-order partial derivatives of u and v are continuous at that point. Furthermore, since f'' is analytic and continuous at z and since

$$f''(z) = u_{xx} + iv_{xx} = v_{yx} - iu_{yx},$$

etc., we arrive at a corollary that was anticipated in Sec. 26, where harmonic functions were introduced.

Corollary. *If a function $f(z) = u(x, y) + iv(x, y)$ is analytic at a point $z = (x, y)$, then the component functions u and v have continuous partial derivatives of all orders at that point.*

The proof of the next theorem, due to E. Morera (1856–1909), depends on the fact that the derivative of an analytic function is itself analytic, as stated in Theorem 1.

Theorem 2. *Let f be continuous on a domain D. If*

$$\int_C f(z)\, dz = 0 \tag{1}$$

for every closed contour C in D, then f is analytic throughout D.

In particular, when D is *simply connected*, we have for the class of continuous functions defined on D the converse of the theorem in Sec. 48, which is the adaptation of the Cauchy–Goursat theorem to such domains.

To prove the theorem here, we observe that when its hypothesis is satisfied, the theorem in Sec. 44 ensures that f has an antiderivative in D; that is, there exists an analytic function F such that $F'(z) = f(z)$ at each point in D. Since f is the derivative of F, it then follows from Theorem 1 that f is analytic in D.

Our final theorem here will be essential in the next section.

Theorem 3. *Suppose that a function f is analytic inside and on a positively oriented circle C_R, centered at z_0 and with radius R* (Fig. 69). *If M_R denotes the maximum value of $|f(z)|$ on C_R, then*

$$|f^{(n)}(z_0)| \le \frac{n!M_R}{R^n} \quad (n = 1, 2, \ldots). \tag{2}$$

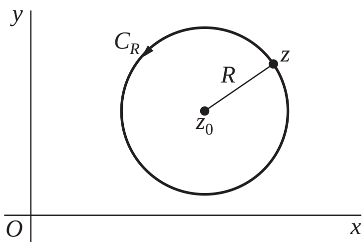

FIGURE 69

Inequality (2) is called *Cauchy's inequality* and is an immediate consequence of the expression

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_R} \frac{f(z)\,dz}{(z - z_0)^{n+1}} \quad (n = 1, 2, \ldots),$$

which is a slightly different form of equation (6), Sec. 51, when n is a positive integer. We need only apply the theorem in Sec. 43, which gives upper bounds for the moduli of the values of contour integrals, to see that

$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi} \cdot \frac{M_R}{R^{n+1}} 2\pi R \quad (n = 1, 2, \ldots),$$

where M_R is as in the statement of Theorem 3. This inequality is, of course, the same as inequality (2).

EXERCISES

1. Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals:

(a) $\displaystyle\int_C \frac{e^{-z}\,dz}{z - (\pi i/2)}$; *(b)* $\displaystyle\int_C \frac{\cos z}{z(z^2 + 8)}\,dz$; *(c)* $\displaystyle\int_C \frac{z\,dz}{2z + 1}$;

(d) $\displaystyle\int_C \frac{\cosh z}{z^4}\,dz$; *(e)* $\displaystyle\int_C \frac{\tan(z/2)}{(z - x_0)^2}\,dz \quad (-2 < x_0 < 2)$.

Ans. (a) 2π; *(b)* $\pi i/4$; *(c)* $-\pi i/2$; *(d)* 0; *(e)* $i\pi \sec^2(x_0/2)$.

2. Find the value of the integral of $g(z)$ around the circle $|z - i| = 2$ in the positive sense when

(a) $g(z) = \dfrac{1}{z^2 + 4}$; *(b)* $g(z) = \dfrac{1}{(z^2 + 4)^2}$.

Ans. (a) $\pi/2$; *(b)* $\pi/16$.

3. Let C be the circle $|z| = 3$, described in the positive sense. Show that if

$$g(z) = \int_C \frac{2s^2 - s - 2}{s - z}\, ds \qquad (|z| \neq 3),$$

then $g(2) = 8\pi i$. What is the value of $g(z)$ when $|z| > 3$?

4. Let C be any simple closed contour, described in the positive sense in the z plane, and write

$$g(z) = \int_C \frac{s^3 + 2s}{(s - z)^3}\, ds.$$

Show that $g(z) = 6\pi i z$ when z is inside C and that $g(z) = 0$ when z is outside.

5. Show that if f is analytic within and on a simple closed contour C and z_0 is not on C, then

$$\int_C \frac{f'(z)\, dz}{z - z_0} = \int_C \frac{f(z)\, dz}{(z - z_0)^2}.$$

6. Let f denote a function that is *continuous* on a simple closed contour C. Following a procedure used in Sec. 51, prove that the function

$$g(z) = \frac{1}{2\pi i} \int_C \frac{f(s)\, ds}{s - z}$$

is *analytic* at each point z interior to C and that

$$g'(z) = \frac{1}{2\pi i} \int_C \frac{f(s)\, ds}{(s - z)^2}$$

at such a point.

7. Let C be the unit circle $z = e^{i\theta}\,(-\pi \leq \theta \leq \pi)$. First show that for any real constant a,

$$\int_C \frac{e^{az}}{z}\, dz = 2\pi i.$$

Then write this integral in terms of θ to derive the integration formula

$$\int_0^{\pi} e^{a\cos\theta} \cos(a \sin\theta)\, d\theta = \pi.$$

8. *(a)* With the aid of the binomial formula (Sec. 3), show that for each value of n, the function

$$P_n(z) = \frac{1}{n!\, 2^n} \frac{d^n}{dz^n}(z^2 - 1)^n \qquad (n = 0, 1, 2, \ldots)$$

is a polynomial of degree n.*

*These are Legendre polynomials, which appear in Exercise 7, Sec. 43, when $z = x$. See the footnote to that exercise.

(b) Let C denote any positively oriented simple closed contour surrounding a fixed point z. With the aid of the integral representation (5), Sec. 51, for the nth derivative of a function, show that the polynomials in part *(a)* can be expressed in the form

$$P_n(z) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(s^2-1)^n}{(s-z)^{n+1}}\,ds \qquad (n = 0, 1, 2, \ldots).$$

(c) Point out how the integrand in the representation for $P_n(z)$ in part *(b)* can be written $(s+1)^n/(s-1)$ if $z = 1$. Then apply the Cauchy integral formula to show that

$$P_n(1) = 1 \qquad (n = 0, 1, 2, \ldots).$$

Similarly, show that

$$P_n(-1) = (-1)^n \qquad (n = 0, 1, 2, \ldots).$$

9. Follow these steps below to verify the expression

$$f''(z) = \frac{1}{\pi i}\int_C \frac{f(s)\,ds}{(s-z)^3}$$

in Sec. 51.

(a) Use expression (2) in Sec. 51 for $f'(z)$ to show that

$$\frac{f'(z+\Delta z) - f'(z)}{\Delta z} - \frac{1}{\pi i}\int_C \frac{f(s)\,ds}{(s-z)^3} = \frac{1}{2\pi i}\int_C \frac{3(s-z)\Delta z - 2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3} f(s)\,ds.$$

(b) Let D and d denote the largest and smallest distances, respectively, from z to points on C. Also, let M be the maximum value of $|f(s)|$ on C and L the length of C. With the aid of the triangle inequality and by referring to the derivation of expression (2) in Sec. 51 for $f'(z)$, show that when $0 < |\Delta z| < d$, the value of the integral on the right-hand side in part *(a)* is bounded from above by

$$\frac{(3D|\Delta z| + 2|\Delta z|^2)M}{(d - |\Delta z|)^2 d^3} L.$$

(c) Use the results in parts *(a)* and *(b)* to obtain the desired expression for $f''(z)$.

10. Let f be an entire function such that $|f(z)| \le A|z|$ for all z, where A is a fixed positive number. Show that $f(z) = a_1 z$, where a_1 is a complex constant.

Suggestion: Use Cauchy's inequality (Sec. 52) to show that the second derivative $f''(z)$ is zero everywhere in the plane. Note that the constant M_R in Cauchy's inequality is less than or equal to $A(|z_0| + R)$.

53. LIOUVILLE'S THEOREM AND THE FUNDAMENTAL THEOREM OF ALGEBRA

Cauchy's inequality in Theorem 3 of Sec. 52 can be used to show that no entire function except a constant is bounded in the complex plane. Our first theorem here,