12. Show that the hypothesis that the sequence $X := (x_n)$ is decreasing in Dirichlet's Test 9.3.4 can be replaced by the hypothesis that $\sum_{n=1}^{\infty} |x_n - x_{n+1}|$ is convergent.
13. If (a_n) is a bounded decreasing sequence and (b_n) is a bounded increasing sequence and if $x_n := a_n + b_n$ for $n \in \mathbb{N}$, show that $\sum_{n=1}^{\infty} |x_n - x_{n+1}|$ is convergent.
14. Show that if the partial sums s_n of the series $\sum_{k=1}^{\infty} a_k$ satisfy $|s_n| \leq Mn^r$ for some $r < 1$, then the series $\sum_{n=1}^{\infty} a_n/n$ converges.
15. Suppose that $\sum a_n$ is a convergent series of real numbers. Either prove that $\sum b_n$ converges or give a counter-example, when we define b_n by
    (a) a_n/n, (b) $\sqrt{a_n}/n \quad (a_n \geq 0)$,
    (c) $a_n \sin n$, (d) $\sqrt{a_n/n} \quad (a_n \geq 0)$,
    (e) $n^{1/n} a_n$, (f) $a_n/(1 + |a_n|)$.

Section 9.4 Series of Functions

Because of their frequent appearance and importance, we now present a discussion of infinite series of functions. Since the convergence of an infinite series is handled by examining the sequence of partial sums, questions concerning series of functions are answered by examining corresponding questions for sequences of functions. For this reason, a portion of the present section is merely a translation of facts already established for sequences of functions into series terminology. However, in the second part of the section, where we discuss power series, some new features arise because of the special character of the functions involved.

9.4.1 Definition If (f_n) is a sequence of functions defined on a subset D of $\mathbb{R}$ with values in $\mathbb{R}$, the sequence of **partial sums** (s_n) of the infinite series $\sum f_n$ is defined for x in D by

$$\begin{aligned} s_1(x) &:= f_1(x), \\ s_2(x) &:= s_1(x) + f_2(x) \\ &\cdots\cdots\cdots\cdots \\ s_{n+1}(x) &:= s_n(x) + f_{n+1}(x) \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

In case the sequence (s_n) of functions converges on D to a function f, we say that the infinite series of functions $\sum f_n$ **converges** to f on D. We will often write

$$\sum f_n \quad \text{or} \quad \sum_{n=1}^{\infty} f_n$$

to denote either the series or the limit function, when it exists.

If the series $\sum |f_n(x)|$ converges for each x in D, we say that $\sum f_n$ is **absolutely convergent** on D. If the sequence (s_n) of partial sums is uniformly convergent on D to f, we say that $\sum f_n$ is **uniformly convergent** on D, or that it **converges to** f **uniformly on** D.

One of the main reasons for the interest in uniformly convergent series of functions is the validity of the following results, which give conditions justifying the change of order of the summation and other limiting operations.

9.4.2 Theorem *If f_n is continuous on $D \subseteq \mathbb{R}$ to $\mathbb{R}$ for each $n \in \mathbb{N}$ and if $\sum f_n$ converges to f uniformly on D, then f is continuous on D.*

This is a direct translation of Theorem 8.2.2 for series. The next result is a translation of Theorem 8.2.4.

9.4.3 Theorem *Suppose that the real-valued functions f_n, $n \in \mathbb{N}$, are Riemann integrable on the interval $J := [a, b]$. If the series $\sum f_n$ converges to f uniformly on J, then f is Riemann integrable and*

$$\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n. \tag{1}$$

Next we turn to the corresponding theorem pertaining to differentiation. Here we assume the uniform convergence of the series obtained after term-by-term differentiation of the given series. This result is an immediate consequence of Theorem 8.2.3.

9.4.4 Theorem *For each $n \in \mathbb{N}$, let f_n be a real-valued junction on $J := [a, b]$ that has a derivative f_n' on J. Suppose that the series $\sum f_n$ converges for at least one point of J and that the series of derivatives $\sum f_n'$ converges uniformly on J.*

Then there exists a real-valued function f on J such that $\sum f_n$ converges uniformly on J to f. In addition, f has a derivative on J and $f' = \sum f_n'$.

Tests for Uniform Convergence

Since we have stated some consequences of uniform convergence of series, we shall now present a few tests that can be used to establish uniform convergence.

9.4.5 Cauchy Criterion *Let (f_n) be a sequence of functions on $D \subseteq \mathbb{R}$ to $\mathbb{R}$. The series $\sum f_n$ is uniformly convergent on D if and only if for every $\varepsilon > 0$ there exists an $M(\varepsilon)$ such that if $m > n \geq M(\varepsilon)$, then*

$$|f_{n+1}(x) + \cdots + f_m(x)| < \varepsilon \quad \textit{for all} \quad x \in D.$$

9.4.6 Weierstrass M-Test *Let (M_n) be a sequence of positive real numbers such that $|f_n(x)| \leq M_n$ for $x \in D$, $n \in \mathbb{N}$. If the series $\sum M_n$ is convergent, then $\sum f_n$ is uniformly convergent on D.*

Proof. If $m > n$, we have the relation

$$|f_{n+1}(x) + \cdots + f_m(x)| \leq M_{n+1} + \cdots + M_m \quad \text{for} \quad x \in D.$$

Now apply 3.7.4, 9.4.5, and the convergence of $\sum M_n$. Q.E.D.

In Appendix E we will use the Weierstrass M-Test to construct two interesting examples.

Power Series

We shall now turn to a discussion of power series. This is an important class of series of functions and enjoys properties that are *not* valid for general series of functions.

9.4.7 Definition A series of real functions $\sum f_n$ is said to be a **power series around** $x = c$ if the function f_n has the form

$$f_n(x) = a_n(x - c)^n,$$

where a_n and c belong to $\mathbb{R}$ and where $n = 0, 1, 2, \ldots$.

For the sake of simplicity of our notation, we shall treat only the case where $c = 0$. This is no loss of generality, however, since the translation $x' = x - c$ reduces a power series around c to a power series around 0. Thus, whenever we refer to a power series, we shall mean a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \cdots + a_n x^n + \cdots. \tag{2}$$

Even though the functions appearing in (2) are defined over all of $\mathbb{R}$, it is not to be expected that the series (2) will converge for all x in $\mathbb{R}$. For example, by using the Ratio Test 9.2.4, we can show that the series

$$\sum_{n=0}^{\infty} n! x^n, \quad \sum_{n=0}^{\infty} x^n, \quad \sum_{n=0}^{\infty} x^n/n!,$$

converge for x in the sets

$$\{0\}, \quad \{x \in \mathbb{R} : |x| < 1\}, \quad \mathbb{R},$$

respectively. Thus, the set on which a power series converges may be small, medium, or large. However, an arbitrary subset of $\mathbb{R}$ cannot be the precise set on which a power series converges, as we shall show.

If (b_n) is a bounded sequence of nonnegative real numbers, then we define the **limit superior** of (b_n) to be the infimum of those numbers v such that $b_n \le v$ for all sufficiently large $n \in \mathbb{N}$. This infimum is uniquely determined and is denoted by $\limsup(b_n)$. The only facts we need to know are (i) that if $v > \limsup(b_n)$, then $b_n \le v$ for all sufficiently large $n \in \mathbb{N}$, and (ii) that if $w < \limsup(b_n)$, then $w \le b_n$ for infinitely many $n \in \mathbb{N}$. (See 3.4.10 and 3.4.11.)

9.4.8 Definition Let $\sum a_n x^n$ be a power series. If the sequence $(|a_n|^{1/n})$ is bounded, we set $\rho := \limsup(|a_n|^{1/n})$; if this sequence is not bounded we set $\rho = +\infty$. We define the **radius of convergence** of $\sum a_n x^n$ to be given by

$$R := \begin{cases} 0 & \text{if} \quad \rho = +\infty, \\ 1/\rho & \text{if} \quad 0 < \rho < +\infty, \\ +\infty & \text{if} \quad \rho = 0. \end{cases}$$

The **interval of convergence** is the open interval $(-R, R)$.

We shall now justify the term "radius of convergence."

9.4.9 Cauchy-Hadamard Theorem *If R is the radius of convergence of the power series $\sum a_n x^n$, then the series is absolutely convergent if $|x| < R$ and is divergent if $|x| > R$.*

Proof. We shall treat only the case where $0 < R < +\infty$, leaving the cases $R = 0$ and $R = +\infty$ as exercises. If $0 < |x| < R$, then there exists a positive number $c < 1$ such that $|x| < cR$. Therefore $\rho < c/|x|$ and so it follows that if n is sufficiently large, then $|a_n|^{1/n} \le c/|x|$. This is equivalent to the statement that

$$|a_n x^n| \le c^n \tag{3}$$

for all sufficiently large n. Since $c < 1$, the absolute convergence of $\sum a_n x^n$ follows from the Comparison Test 3.7.7.

If $|x| > R = 1/\rho$, then there are infinitely many $n \in \mathbb{N}$ for which $|a_n|^{1/n} > 1/|x|$. Therefore, $|a_n x^n| > 1$ for infinitely many n, so that the sequence $(a_n x^n)$ does not converge to zero. Q.E.D.

Remark It will be noted that the Cauchy-Hadamard Theorem makes no statement as to whether the power series converges when $|x| = R$. Indeed, anything can happen, as the examples

$$\sum x^n, \quad \sum \frac{1}{n} x^n, \quad \sum \frac{1}{n^2} x^n,$$

show. Since $\lim(n^{1/n}) = 1$, each of these power series has radius of convergence equal to 1. The first power series converges at neither of the points $x = -1$ and $x = +1$; the second series converges at $x = -1$ but diverges at $x = +1$; and the third power series converges at both $x = -1$ and $x = +1$. (Find a power series with $R = 1$ that converges at $x = +1$ but diverges $x = -1$.)

It is an exercise to show that the radius of convergence of the series $\sum a_n x^n$ is also given by

$$\lim \left| \frac{a_n}{a_{n+1}} \right|, \tag{4}$$

provided this limit exists. Frequently, it is more convenient to use (4) than Definition 9.4.8.

The argument used in the proof of the Cauchy-Hadamard Theorem yields the uniform convergence of the power series on any fixed closed and bounded interval in the interval of convergence $(-R, R)$.

9.4.10 Theorem *Let R be the radius of convergence of $\sum a_n x^n$ and let K be a closed and bounded interval contained in the interval of convergence $(-R, R)$. Then the power series converges uniformly on K.*

Proof. The hypothesis on $K \subseteq (-R, R)$ implies that there exists a positive constant $c < 1$ such that $|x| < cR$ for all $x \in K$. (Why?) By the argument in 9.4.9, we infer that for sufficiently large n, the estimate (3) holds for all $x \in K$. Since $c < 1$, the uniform convergence of $\sum a_n x^n$ on K is a direct consequence of the Weierstrass M-test with $M_n := c^n$. Q.E.D.

9.4.11 Theorem *The limit of a power series is continuous on the interval of convergence. A power series can be integrated term-by-term over any closed and bounded interval contained in the interval of convergence.*

Proof. If $|x_0| < R$, then the preceding result asserts that $\sum a_n x^n$ converges uniformly on any closed and bounded neighborhood of x_0 contained in $(-R, R)$. The continuity at x_0 then follows from Theorem 9.4.2, and the term-by-term integration is justified by Theorem 9.4.3. Q.E.D.

We now show that a power series can be differentiated term-by-term. Unlike the situation for general series, we do not need to assume that the differentiated series is uniformly convergent. Hence this result is stronger than Theorem 9.4.4.

9.4.12 Differentiation Theorem *A power series can be differentiated term-by-term within the interval of convergence. In fact, if*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad \text{then} \quad f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad \text{for} \quad |x| < R.$$

Both series have the same radius of convergence.

Proof. Since $\lim(n^{1/n}) = 1$, the sequence $(|na_n|^{1/n})$ is bounded if and only if the sequence $(|a_n|^{1/n})$ is bounded. Moreover, it is easily seen that

$$\lim\sup\left(|na_n|^{1/n}\right) = \lim\sup\left(|a_n|^{1/n}\right).$$

Therefore, the radius of convergence of the two series is the same, so the formally differentiated series is uniformly convergent on each closed and bounded interval contained in the interval of convergence. We can then apply Theorem 9.4.4 to conclude that the formally differentiated series converges to the derivative of the given series. Q.E.D.

Remark It is to be observed that the theorem makes no assertion about the endpoints of the interval of convergence. If a series is convergent at an endpoint, then the differentiated series may or may not be convergent at this point. For example, the series $\sum_{n=1}^{\infty} x^n/n^2$ converges at both endpoints $x = -1$ and $x = +1$. However, the differentiated series given by $\sum_{n=1}^{\infty} x^{n-1}/n$ converges at $x = -1$ but diverges at $x = +1$.

By repeated application of the preceding result, we conclude that if $k \in \mathbb{N}$ then $\sum_{n=0}^{\infty} a_n x^n$ can be differentiated term-by-term k times to obtain

$$\sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k}. \tag{5}$$

Moreover, this series converges absolutely to $f^{(k)}(x)$ for $|x| < R$ and uniformly over any closed and bounded interval in the interval of convergence. If we substitute $x = 0$ in (5), we obtain the important formula

$$f^{(k)}(0) = k! a_k.$$

9.4.13 Uniqueness Theorem *If $\sum a_n x^n$ and $\sum b_n x^n$ converge on some interval $(-r, r)$, $r > 0$, to the same function f, then*

$$a_n = b_n \quad \text{for all} \quad n \in \mathbb{N}.$$

Proof. Our preceding remarks show that $n! a_n = f^{(n)}(0) = n! b_n$ for all $n \in \mathbb{N}$. Q.E.D.

Taylor Series

If a function f has derivatives of all orders at a point c in $\mathbb{R}$, then we can calculate the Taylor coefficients by $a_0 := f(c)$, $a_n := f^{(n)}(c)/n!$ for $n \in \mathbb{N}$ and in this way obtain a power series with these coefficients. However, it is not necessarily true that the resulting power series converges to the function f in an interval about c. (See Exercise 12 for an example.) The issue of convergence is resolved by the remainder term R_n in Taylor's Theorem 6.4.1. We will write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \tag{6}$$

for $|x - c| < R$ if and only if the sequence $(R_n(x))$ of remainders converges to 0 for each x in some interval $\{x : |x - c| < R\}$. In this case we say that the power series (6) is the **Taylor expansion** of f at c. We observe that the Taylor polynomials for f discussed in Section 6.4 are just the partial sums of the Taylor expansion (6) of f. (Recall that $0! := 1$.)

9.4.14 Examples **(a)** If $f(x) := \sin x$, $x \in \mathbb{R}$, we have $f^{(2n)}(x) = (-1)^n \sin x$ and $f^{(2n+1)}(x) = (-1)^n \cos x$ for $n \in \mathbb{N}$, $x \in \mathbb{R}$. Evaluating at $c = 0$, we get the Taylor coefficients $a_{2n} = 0$ and $a_{2n+1} = (-1)^n/(2n+1)!$ for $n \in \mathbb{N}$. Since $|\sin x| \leq 1$ and $|\cos x| \leq 1$ for all x, then $|R_n(x)| \leq |x|^n/n!$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Since $\lim(R_n(x)) = 0$ for each $x \in \mathbb{R}$, we obtain the Taylor expansion

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad \text{for all} \quad x \in \mathbb{R}.$$

An application of Theorem 9.4.12 gives us the Taylor expansion

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \qquad \text{for all} \quad x \in \mathbb{R}.$$

(b) If $g(x) := e^x$, $x \in \mathbb{R}$, then $g^{(n)}(x) = e^x$ for all $n \in \mathbb{N}$, and hence the Taylor coefficients are given by $a_n = 1/n!$ for $n \in \mathbb{N}$. For a given $x \in \mathbb{R}$, we have $|R_n(x)| \leq e^{|x|}|x|^n/n!$ and therefore $(R_n(x))$ tends to 0 as $n \to \infty$. Therefore, we obtain the Taylor expansion

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \qquad \text{for all} \quad x \in \mathbb{R}. \tag{7}$$

We can obtain the Taylor expansion at an arbitrary $c \in \mathbb{R}$ by the device of replacing x by $x - c$ in (7) and noting that

$$e^x = e^c \cdot e^{x-c} = e^c \sum_{n=0}^{\infty} \frac{1}{n!}(x-c)^n = \sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n \qquad \text{for} \quad x \in \mathbb{R}.$$ □

Exercises for Section 9.4

1. Discuss the convergence and the uniform convergence of the series $\sum f_n$, where $f_n(x)$ is given by:
   (a) $(x^2 + n^2)^{-1}$,
   (b) $(nx)^{-2}$ $(x \neq 0)$,
   (c) $\sin(x/n^2)$,
   (d) $(x^n + 1)^{-1}$ $(x \neq 0)$,
   (e) $x^n/(x^n + 1)$ $(x \geq 0)$,
   (f) $(-1)^n(n + x)^{-1}$ $(x \geq 0)$.
2. If $\sum a_n$ is an absolutely convergent series, then the series $\sum a_n \sin nx$ is absolutely and uniformly convergent.
3. Let (c_n) be a decreasing sequence of positive numbers. If $\sum c_n \sin nx$ is uniformly convergent, then $\lim(nc_n) = 0$.
4. Discuss the cases $R = 0$, $R = +\infty$ in the Cauchy-Hadamard Theorem 9.4.9.
5. Show that the radius of convergence R of the power series $\sum a_n x^n$ is given by $\lim(|a_n/a_{n+1}|)$ whenever this limit exists. Give an example of a power series where this limit does not exist.
6. Determine the radius of convergence of the series $\sum a_n x^n$, where a_n is given by:
   (a) $1/n^n$,
   (b) $n^\alpha/n!$,
   (c) $n^n/n!$,
   (d) $(\ln n)^{-1}$, $n \geq 2$,
   (e) $(n!)^2/(2n)!$,
   (f) $n^{-\sqrt{n}}$.

7. If $a_n := 1$ when n is the square of a natural number and $a_n := 0$ otherwise, find the radius of convergence of $\sum a_n x^n$. If $b_n := 1$ when $n = m!$ for $m \in \mathbb{N}$ and $b_n := 0$ otherwise, find the radius of convergence of the series $\sum b_n x^n$.
8. Prove in detail that $\limsup\left(|na_n|^{1/n}\right) = \limsup\left(|a_n|^{1/n}\right)$.
9. If $0 < p \le |a_n| \le q$ for all $n \in \mathbb{N}$, find the radius of convergence of $\sum a_n x^n$.
10. Let $f(x) = \sum a_n x^n$ for $|x| < R$. If $f(x) = f(-x)$ for all $|x| < R$, show that $a_n = 0$ for all odd n.
11. Prove that if f is defined for $|x| < r$ and if there exists a constant B such that $|f^{(n)}(x)| \le B$ for all $|x| < r$ and $n \in \mathbb{N}$, then the Taylor series expansion

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

converges to $f(x)$ for $|x| < r$.

12. Prove by Induction that the function given by $f(x) := e^{-1/x^2}$ for $x \ne 0$, $f(0) := 0$, has derivatives of all orders at every point and that all of these derivatives vanish at $x = 0$. Hence this function is not given by its Taylor expansion about $x = 0$.
13. Give an example of a function that is equal to its Taylor series expansion about $x = 0$ for $x \ge 0$, but is not equal to this expansion for $x < 0$.
14. Use the Lagrange form of the remainder to justify the general Binomial Expansion

$$(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n \qquad \text{for} \quad 0 \le x < 1\,.$$

15. (Geometric series) Show directly that if $|x| < 1$, then $1/(1-x) = \sum_{n=0}^{\infty} x^n$.
16. Show by integrating the series for $1/(1+x)$ that if $|x| < 1$, then

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n.$$

17. Show that if $|x| < 1$, then $\operatorname{Arctan} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$.
18. Show that if $|x| < 1$, then $\operatorname{Arcsin} x = \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} \cdot \frac{x^{2n+1}}{2n+1}$.
19. Find a series expansion for $\int_0^x e^{-t^2}\, dt$ for $x \in \mathbb{R}$.
20. If $\alpha \in \mathbb{R}$ and $|k| < 1$, the integral $F(\alpha, k) := \int_0^{\alpha} \left(1 - k^2 (\sin x)^2\right)^{-1/2} dx$ is called an **elliptic integral of the first kind**. Show that

$$F\left(\frac{\pi}{2}, k\right) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left(\frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n}\right)^2 k^{2n} \quad \text{for} \quad |k| < 1\,.$$