$$\frac{\mathrm{d}}{\mathrm{d}\,x}\,(u \cdot v) = \frac{\mathrm{d}\,u}{\mathrm{d}\,v} \cdot v + u \cdot \frac{\mathrm{d}\,v}{\mathrm{d}\,x} \tag{1.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\,(f\,(x) \cdot g\,(x)) = f'\,(x) \cdot g\,(x) + f\,(x) \cdot g'\,(x) \tag{1.2}$$

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{\mathrm{d}\,y}{\mathrm{d}\,u} \cdot \frac{\mathrm{d}\,u}{\mathrm{d}\,x} \tag{1.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left[f\left(g\left(x\right)\right)\right] = f'\left(g\left(x\right)\right) \cdot g\left(x\right) \tag{1.4}$$

$$\int f(g(x)) \cdot g'(x) \, \mathrm{d}\, x = \int f(u) \, \mathrm{d}\, u \tag{1.5}$$

$$\int f(u) \cdot \frac{\mathrm{d}\, u}{\mathrm{d}\, x} \, \mathrm{d}\, x = \int f(u) \, \mathrm{d}\, u \tag{1.6}$$

$$u = f(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left[\sin\left(x\right)\right] = \cos\left(x\right)$$

$$\mathrm{d}\, v = g'\,(x)\,\mathrm{d}\, x$$

v

$$\int u \,\mathrm{d}\, v = u \cdot v - \int v \,\mathrm{d}\, u \tag{1.7}$$

$$\begin{aligned} u &= g(x) \qquad F(x): \quad F'(x) = f(x) = y \\ \frac{du}{dx} &= g'(x) \end{aligned} \tag{1.8}$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\,x}\left[F'\left(u\right)\right] &= F'\left(g\left(x\right)\right)\cdot g'\left(x\right) \\
&= f\left(g\left(x\right)\right)\cdot g'\left(x\right) \\
\implies f\left(g\left(x\right)\right)\cdot g'\left(x\right) &= \frac{\mathrm{d}}{\mathrm{d}\,x}\left[F\left(u\right)\right] \\
f\left(g\left(x\right)\right)\cdot g'\left(x\right) &= \frac{\mathrm{d}}{\mathrm{d}\,x}\left[F\left(u\right)+C\right]
\end{aligned}
\tag{1.9}
$$

$$\begin{aligned}\int f\left(g\left(x\right)\right)\cdot g'\left(x\right)\mathrm{d}\,x &= \int \frac{\mathrm{d}}{\mathrm{d}\,x}\left[F\left(u\right)+C\right]\mathrm{d}\,x \\ &= F\left(u\right)+C \\ &= \int f\left(u\right)\mathrm{d}\,u\end{aligned}$$

$$\int f\left(g\left(x\right)\right) \cdot g'\left(x\right) \mathrm{d}\, x = \int f\left(u\right) \mathrm{d}\, u$$

$$\int f\left(u\right) \cdot \frac{\mathrm{d}\, u}{\mathrm{d}\, x} \mathrm{d}\, x = \int f\left(u\right) \mathrm{d}\, u$$

(1.2)

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left[f\left(x\right)\cdot g\left(x\right)\right]=f'\left(x\right)\cdot g\left(x\right)+f\left(x\right)\cdot g'\left(x\right) \tag{1.10}$$

$$
\begin{array}{ll}
u = f\left(x\right) & v = g\left(x\right) \\
\frac{\mathrm{d}\,u}{\mathrm{d}\,x} = f'\left(x\right) & \frac{\mathrm{d}\,v}{\mathrm{d}\,x} = g'\left(x\right)
\end{array}
\tag{1.11}
$$

$$\int \left(\frac{\mathrm{d}\,u}{\mathrm{d}\,x} \cdot v + u \cdot \frac{\mathrm{d}\,v}{\mathrm{d}\,x}\right) \mathrm{d}\,x = u \cdot v$$

$$\int \left(v \cdot \frac{\mathrm{d}\,u}{\mathrm{d}\,x}\right) \mathrm{d}\,x + \int \left(u \cdot \frac{\mathrm{d}\,v}{\mathrm{d}\,x}\right) \mathrm{d}\,x = u \cdot v$$

(1.6)

$$\int v \,\mathrm{d}\, u + \int u \,\mathrm{d}\, v = u \cdot v$$
$$\int u \,\mathrm{d}\, v = u \cdot v - \int v \,\mathrm{d}\, u$$

$$\int [\ ]\,\mathrm{d}\,x$$

$$\left[f\left(u\right)\cdot\frac{\mathrm{d}\,u}{\mathrm{d}\,x}\right]=\left[f\left(g\left(x\right)\right)\cdot g'\left(x\right)\right]$$

$$\left[f\left(x\right)\cdot\frac{\mathrm{d}\,u}{\mathrm{d}\,x}\right]=\left[f\left(x\right)\cdot g'\left(x\right)\right]$$

$\mathrm{d}\,y$

$$\mathrm{d}\,x$$

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x}$$

$$\frac{\partial u}{\partial x}$$

$$\sum_{0}^{n}\left[a_0\left(x\right)\cdot\left(\frac{\mathrm{d}^n y}{\mathrm{d}\, x^n}\right)\right] \tag{1.12}$$

y

x

A differential equation of the form:

$$g\left(y\right) \cdot \frac{\mathrm{d}\, y}{\mathrm{d}\, x} = f\left(x\right) \tag{1.13}$$

Is a seperable Ordinary Differential Equation and has a solution:

$$\int g\left(y\right) \mathrm{d}\, y = \int f\left(x\right) \mathrm{d}\, x \tag{1.14}$$

$$
\begin{aligned}
g\left(y\right) \cdot \frac{\mathrm{d}\,y}{\mathrm{d}\,x} &= f\left(x\right) \\
\implies \int g\left(y\right) \frac{\mathrm{d}\,y}{\mathrm{d}\,x}\,\mathrm{d}\,x &= \int f\left(x\right)\mathrm{d}\,x
\end{aligned}
\tag{1.15}
$$

1.6

$$\int g\left(y\right)\mathrm{d}\,y = \int f\left(x\right)\mathrm{d}\,x \tag{1.16}$$

u

$$\mathrm{d}\,y\frac{}{\mathrm{d}\,x=f\left(\frac{x}{y}\right)}$$

$$
\begin{aligned}
u &= \frac{y}{x} \\
\implies y &= u \cdot x \\
\implies \frac{\mathrm{d}\, y}{\mathrm{d}\, x} &= \frac{\mathrm{d}\, u}{\mathrm{d}\, x} \cdot x + (1) \cdot u
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\,y}{\mathrm{d}\,x} &= f\left(\frac{y}{x}\right) \\
\frac{\mathrm{d}\,u}{\mathrm{d}\,x} \cdot x + u &= f\left(u\right) \\
\frac{\mathrm{d}\,u}{\mathrm{d}\,x} \cdot x &= f\left(u\right) - u \\
\frac{1}{f\left(u\right) - u} \cdot \frac{\mathrm{d}\,u}{\mathrm{d}\,x} \cdot x &= 1 \\
\frac{1}{f\left(u\right) - u} \cdot \frac{\mathrm{d}\,u}{\mathrm{d}\,x} &= \int \frac{1}{x}\,\mathrm{d}\,x \\
\int \frac{1}{f\left(u\right) - u} \cdot \frac{\mathrm{d}\,u}{\mathrm{d}\,x}\,\mathrm{d}\,x &= \int \frac{1}{x}\,\mathrm{d}\,x \\
\int \frac{1}{f\left(u\right) - u}\,\mathrm{d}\,u &= \ln\left|x\right| + c
\end{aligned}
\tag{1.17}
$$

$$\exists G\left(u\right):G\left(u\right)=\int\frac{1}{f(u)-u}\,\mathrm{d}\,u$$

$$
\begin{aligned}
G(u) &= \ln|x| + c \\
G\left(\frac{y}{x}\right) &= \ln|x| + c
\end{aligned}
$$

$$
G\left(\frac{y}{x}\right) + \ln|x| + c = 0 \tag{1.18}
$$

$$\sum_{0}^{n}\left[a_n\left(x\right)\cdot f^{(n)}\left(x\right)\right]=g\left(x\right)$$

If $g\left(x\right)=0$ it is said to be homogenous

$$a_1(x) \cdot \frac{\mathrm{d}\,y}{\mathrm{d}\,x} + a_0(x) \cdot y = g(x)$$

$$\text{Where } a(x) \text{ is a function} \tag{1.19}$$

Linear First Order ODE:

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} + p\,(x) \cdot y = f\,(x) \tag{1.20}$$

if $f\,(x) = 0$ the equation is said to be homogenous

y_h

y_p

$$y_h: \qquad \frac{\mathrm{d}\, y_h}{\mathrm{d}\, x} + p\,(x) \cdot y_h = 0$$

$$y_p: \qquad \frac{\mathrm{d}\, y_p}{\mathrm{d}\, x} + p\,(x) \cdot y_p = f\,(x)$$

- Rewrite the Equation in the standard form:
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} + p\left(x\right) \cdot y = f\left(x\right)$$
- Identify $p\left(x\right)$ and find the integrating factor:
$$e^{\int p(x)\,\mathrm{d}\,x}$$
- Multiply through by the integrating factor:
$$e^{\int p(x)\,\mathrm{d}\,x}\left(\frac{\mathrm{d}\,y}{\mathrm{d}\,x} + p\left(x\right) \cdot y\right) = e^{\int p(x)\,\mathrm{d}\,x} f\left(x\right)$$
  It may be concluded:
$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left[e^{\int p(x)\,\mathrm{d}\,x \cdot} \cdot y\right] = e^{\int p(x)\,\mathrm{d}\,x} \cdot f\left(x\right)$$
- Integrate both sides in order to solve:

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} + p\,(x) \cdot y = f\,(x) \tag{1.21}$$

$$\frac{\mathrm{d}\, y}{\mathrm{d}\, x} + p\,(x) \cdot y = 0 \qquad \Longrightarrow \;\; y = y_h \tag{1.22}$$

$$\frac{\mathrm{d}\, y}{\mathrm{d}\, x} + p\,(x) \cdot y = f(x) \Longrightarrow \;\; y = y_p \tag{1.23}$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\,x}\left(y_h+y_p\right)+p\left(x\right)\cdot\left(y_h+y_p\right)&=f\left(x\right)\\
\frac{\mathrm{d}\,y_h}{\mathrm{d}\,x}+\frac{\mathrm{d}\,y_p}{\mathrm{d}\,x}+p\left(x\right)\cdot y_h+p\left(x\right)\cdot y_p&=f\left(x\right)\\
\frac{\mathrm{d}\,y_h}{\mathrm{d}\,x}+p\left(x\right)\cdot y_h+\frac{\mathrm{d}\,y_p}{\mathrm{d}\,x}+p\left(x\right)\cdot y_p&=f\left(x\right)\\
0+f\left(x\right)&=f\left(x\right)
\end{aligned}
\tag{1.24}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\,y}{\mathrm{d}\,x} + p\left(x\right) \cdot y &= 0 \\
\frac{1}{y} \cdot \frac{\mathrm{d}\,y}{\mathrm{d}\,x} &= -p\left(x\right) \\
\ln\left|y\right| &= \int -p\left(x\right)\mathrm{d} + c \\
\left|y\right| &= e^{\int -p\left(x\right)x \,\mathrm{d}\,x} \cdot e^{c}
\end{aligned}
\tag{1.25}
$$

$y > 0$

$$\Longrightarrow \quad y_h = e^{-\int p(x)\,\mathrm{d}\,x} \cdot c$$

$$y_1 = e^{-\int p(x)\,\mathrm{d}\,x}$$

$$y_h = y_1(x) \cdot c \tag{1.26}$$

$$p\left(x\right)$$

c

$$f\left(x\right)$$

$$c = u\left(x\right)$$

$$u\left(x\right)$$

$$\begin{aligned} y_p &= u(x) \times y_h(x) \\ &= e^{-\int p(x)\,\mathrm{d}\,x} \cdot u(x) \end{aligned} \tag{1.27}$$

$$
\begin{aligned}
y_p &= e^{-\int p(x)\,\mathrm{d}\,x} \cdot u\left(x\right) \\
\frac{\mathrm{d}\,y_p}{\mathrm{d}\,x} + p\left(x\right) \cdot &= f\left(x\right) \\
\frac{\mathrm{d}}{\mathrm{d}\,x}\left(u\left(x\right) \cdot y_1\left(x\right)\right) + p\left(x\right) u\left(x\right) y_1\left(x\right) &= f\left(x\right) \\
\frac{\mathrm{d}\,u}{\mathrm{d}\,x} \cdot y_1\left(x\right) + \frac{\mathrm{d}\,y_1}{\mathrm{d}\,x} \cdot u\left(x\right) + p\left(x\right) \cdot u\left(x\right) \cdot y_1\left(x\right) &= f\left(x\right) \\
u\left(x\right)\left(\frac{\mathrm{d}\,y_1}{\mathrm{d}\,x} + p\left(x\right) y_1\right) + \frac{\mathrm{d}\,y}{\mathrm{d}\,x} \cdot y_1\left(x\right) &= f\left(x\right) \\
0 + \frac{\mathrm{d}\,y}{\mathrm{d}\,x} - y_1\left(x\right) &= f\left(x\right) \\
\frac{\mathrm{d}\,y}{\mathrm{d}\,x} &= f\left(x\right)/y_1\left(x\right) \\
\int \frac{\mathrm{d}\,u}{\mathrm{d}\,x}\,\mathrm{d}\,x &= \int f\left(x\right)/y_1\left(x\right)\mathrm{d}\,x \\
\int \mathrm{d}\,u &= \int f\left(x\right)/y_1\left(x\right)\mathrm{d}\,x \\
u &= \int f\left(x\right)/y_1\left(x\right)\mathrm{d}\,x \qquad (1.28)
\end{aligned}
$$

$$u = \int f(x) \cdot e^{\int p(x) \, \mathrm{d}\, x} \, \mathrm{d}\, x \tag{1.29}$$

$$y_p = u \cdot y_1$$

$$y_p = \frac{1}{y_1} \cdot \int f\left(x\right) \cdot e^{\int p(x)\,\mathrm{d}\,x}$$

$$y_p = e^{-\int p(x)\,\mathrm{d}\,x} \int f\left(x\right) \cdot e^{\int p(x)\,\mathrm{d}\,x} \tag{1.30}$$

$$e^{\int p(x)\,\mathrm{d}\,x}$$

$$\mathrm{e}^{\int p(x)\,\mathrm{d}\,x} \cdot y_p = e^{\int p(x)\,\mathrm{d}\,x} \cdot e^{-\int p(x)\,\mathrm{d}\,x} \int f\left(x\right) \cdot e^{\int p(x)\,\mathrm{d}\,x}$$
$$e^{\int p(x)\,\mathrm{d}\,x} \cdot y_p = \int f\left(x\right) \cdot e^{\int p(x)\,\mathrm{d}\,x}$$
$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left(e^{\int p(x)\,\mathrm{d}\,x} \cdot y_p\right) = \frac{\mathrm{d}}{\mathrm{d}\,x}\left(\int f\left(x\right) \cdot e^{\int p(x)\,\mathrm{d}\,x}\right)$$
$$= f\left(x\right) \cdot e^{\int p(x)\,\mathrm{d}\,x}$$
$$e^{\int p(x)\,\mathrm{d}\,x}\frac{\mathrm{d}\,y}{\mathrm{d}\,x} + p\left(x\right) \cdot e^{\int p(x)\,\mathrm{d}\,x} \cdot y = e^{\int p(x)\,\mathrm{d}\,x} \cdot f\left(x\right)$$
$$\implies \frac{\mathrm{d}\,y}{\mathrm{d}\,x} + p\left(x\right) \cdot y = f\left(x\right)$$

$$(x+1) \cdot \frac{\mathrm{d}\,y}{\mathrm{d}\,x} + y = \ln(x) \quad ; \qquad y(1) = 10 \tag{1.31}$$

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} + \frac{y}{x+1} = \frac{\ln(x)}{x+1} \quad : \qquad (x \in \mathbb{R} \setminus \{-1, 0\}) \tag{1.32}$$

$$
\begin{aligned}
u &= e^{\int \frac{1}{x+1}\,\mathrm{d}\,x} \\
&= e^{\int \ln|x+1|\,\mathrm{d}\,x} \\
&= |x+1|
\end{aligned} \tag{1.33}
$$

$$x > 0$$

$$
\begin{aligned}
(x+1) \cdot \frac{\mathrm{d}\, y}{\mathrm{d}\, x} + y &= \ln(x) \\
\implies \frac{\mathrm{d}}{\mathrm{d}\, x}\left((x+1) \cdot y\right) &= \ln(x)
\end{aligned} \tag{1.34}
$$

$$\int \frac{\mathrm{d}}{\mathrm{d}\,x}\left[(x+1)\cdot y\right]\mathrm{d}\,x = \int \ln{(x)}\,\mathrm{d}\,x$$

$$(x+1)\cdot y = \int \ln(x)\, \mathrm{d}\, x \tag{1.35}$$

$$\begin{aligned} u &= \ln(x) & \mathrm{d}\,v &= \mathrm{d}\,x \\ \mathrm{d}\,u &= \tfrac{1}{x}\,\mathrm{d}\,x & v &= x \end{aligned}$$

$$\implies \int u \,\mathrm{d}\,v = u \cdot v + \int v \,\mathrm{d}\,u$$

$$
\begin{aligned}
(x+1)\cdot y &= \ln(x)\cdot x - \int \mathrm{d}\,x \\
&= x\cdot(\ln(x)-1)+c \\
\implies y &= \frac{x\cdot(\ln(x)-1+c)}{x+1}
\end{aligned}
$$

$$y(1) = 10$$

$$
\begin{aligned}
10 &= \frac{1\left(\ln\left(1\right) - 1 + c\right)}{2} \\
20 &= 1\left(0 - 1\right) + c \\
c &= 19
\end{aligned} \tag{1.36}
$$

$$y = x(\ln(x) - 1 + 19)\frac{}{x+1 \quad ; \qquad \forall x \in \mathbb{C}\setminus\{-1{,}0\}}$$