

(03) SERIES

WEEK 4 MATERIAL; DUE 28 MARCH

THE CAUCHY CRITERION (3.5)

THE CAUCHY CONVERGENCE CRITERION (3.5.5)

A sequence is convergent if and only if it is a Cauchy sequence

Cauchy Sequence implies **Convergence**

Every Cauchy sequence of real numbers is bounded, hence by the Bolzano-Weierstrass theorem the sequence has a convergent subsequence, hence is itself convergent.

Convergence implies **Cauchy Sequence**

If two terms can be made arbitrarily close then any term can be made arbitrarily close to another term in the set (which will be the limit point).

PROPERLY DIVERGENT (3.6)

A series (x_n) is said to be properly divergent if $\lim_{n \rightarrow \infty} (x_n) \neq \pm \infty$

DEFINITION OF A SERIES [3.7.1]

If (x_n) is a sequence, then the **series** generated by the sequence is $S = (s_k)$:

The terms of the sequence are $(x_n) = (x_1, x_2, x_3, x_4, \dots, x_n)$,

The terms of the series are $(s_n) = (s_1, s_2, s_3, s_4, \dots, s_n)$

The terms of the series are called the **partial sums** and are defined as such:

$$\begin{aligned} S_1 &= x_1 &&= x_1 \\ S_2 &= S_1 + x_2 &&= x_1 + x_2 \\ S_3 &= S_2 + x_3 &&= x_1 + x_2 + x_3 \\ S_4 &= S_3 + x_4 &&= x_1 + x_2 + x_3 \\ &\dots \\ S_n &= S_{n-1} + x_n = x_1 + x_2 + x_3 + \dots + x_n &&= \sum_{i=1}^n [x_i] \end{aligned}$$

COMMON SERIES TYPES

These are series that we are expected to memorise because they so often appear in series problems (and moreover we will need them for the exam).

GEOMETRIC SERIES (3.7.6 (A))

The Geometric Series is Convergent for $|r| < 1$:

$$\sum_{n=1}^{\infty} [r^n] = 1 + r + r^2 + r^3 + \dots r^n$$

iff $|r| < 1$ then this is convergent

$$|r| < 1 \Rightarrow \sum_{n=1}^{\infty} [r^n] = \frac{1}{1-r}$$

$$\begin{aligned} r \geq 1 &\Rightarrow \lim(r^n) > 0 \\ &\Rightarrow \text{divergence} \end{aligned}$$

Thus it is convergent if and only if $|r| < 1$

$$\sum_{n=1}^{\infty} [r^n] = 1 + r + r^2 + r^3 + \dots r^n$$

iff $|r| < 1$ then this is convergent

$$|r| < 1 \Rightarrow \sum_{n=1}^{\infty} [r^n] = \frac{1}{1-r}$$

$$\begin{aligned} r \geq 1 &\Rightarrow \lim(r^n) > 0 \\ &\Rightarrow \text{divergence} \end{aligned}$$

HARMONIC SERIES (3.7.6 (B))

$$\sum_{n=1}^{\infty} [r^n] = 1 + r + r^2 + r^3 + \dots + r^n$$

iff $|r| < 1$ then this is convergent

$$|r| < 1 \Rightarrow \sum_{n=1}^{\infty} [r^n] = \frac{1}{1-r}$$

$$r \geq 1 \Rightarrow \lim(r^n) > 0 \\ \Rightarrow \text{divergence}$$

The Harmonic series $\sum_{n=1}^{\infty} [1/n]$ is divergent:

assume S converges to a number:

$$\begin{aligned} S &= (1 + 1/2) + (1/3 + 1/4) + (1/5 + 1/6) \dots + (1/(2n-1) + 1/2n) \\ &> (1/2 + 1/2) + (1/4 + 1/4) + (1/6 + 1/6) \dots + (1/2n + 1/2n) \\ &= (1) + (2/4) + (2/6) \dots + (1/n) \\ &= 1 + (1/2) + (1/3) \dots + (1/n) \\ &= S \end{aligned}$$

\therefore the assumption that $\sum_{n=1}^{\infty} [1/n] = S$ implies $S > S$
hence S DNE and the series diverges.

P-SERIES (3.7.6 (D))

PROPERTIES OF SERIES

THE n^{TH} TERM TEST (3.7.3)

if $\sum [x_n]$ is convergent, then $\lim(x_n) = 0$

be careful!

The p -Series is convergent for $p > 1$:

if $\lim(x_n) = 0$,
 $\sum_{n=1}^{\infty} [1/n^p]$ is convergent may or may not converge

for $0 < p \leq 1$ this is divergent.

for $p=1$ this is the harmonic sequence

for $p=-1$ this is the geometric sequence.

This condition is necessary but not sufficient in and of itself to establish the convergence of a series.

CAUCHY CRITERION FOR SERIES (3.7.4)

If a sequence is convergent it must be a Cauchy sequence, hence all convergent series are composed of *Cauchy* sequences (as a necessary but not sufficient condition).

A series converges if and only if it is a Cauchy Series:

$$\forall \varepsilon > 0, \exists M : \\ m, n \geq M \Rightarrow |s_m - s_n| = |u_{n+1} + u_{n+2} + \dots + u_m| < \varepsilon$$

CONVERGENCE CONDITION GENERALLY

$$\forall \varepsilon > 0, \exists K : \\ n \geq N \Rightarrow |s_n - s| = |u_1 + u_2 + u_3 + \dots + u_n| < \varepsilon$$

CONVERGENCE TESTS

TYPES OF CONVERGENCE

A series $\sum [x_n]$ is **absolutely convergent** if:

$$\sum [|x_n|] \text{ is convergent}$$

This is important because:

$$\text{Convergence of } \sum [|x_n|] \Rightarrow \text{Convergence of } \sum [x_n]$$

But be careful because this does not work the other way, a convergent series may not necessarily be absolutely convergent, this is discussed at (9.1) of TB.

If a series is not **absolutely convergent** it is said to be **conditionally convergent**.

So below I have split the tests into three categories, the comparison tests demonstrate nonabsolute convergence but are simple and useful and so are introduced first, the absolute convergence tests show absolute convergence and the latter tests are more obscure tests as well as tests for alternating series.

CHOOSING A TEST

Choosing the right Test can be tricky, so I have included an appendix with a flow chart that we should probably memorise for want of the exam.

MANIPULATING SERIES

Sometimes you'll be given a series in an odd way for example:

$$S_n = \sum_{n=1}^{\infty} \left[\frac{1}{n \cdot (n+1)} \right]$$

Now this could be shown to be convergent using the limit comparison test (which is below) but if you are asked to find the value to which the series converges to there is a bit more work involved.

Generally if you are asked to find what value a series converges to it will be either:

- A Geometric Series (3.7.6 (a) of TB)
- A Telescoping Series

Geometric Series have already been shown, but a telescoping series is new and not covered in the textbook, basically, it is a series where most of the terms cancel out by way of rearrangement and grouping to leave only one or two terms left.

Often it is necessary to manipulate the terms somewhat in order for them to exhibit the cancelling/telescoping property, often by way of partial fractions (remember from *Mathematics 1B*), for an example of this refer to Q3(c) of the corresponding tutorial (tutorial #4 of wk 4 material, due wk. 5, topic 3 from learning guide)

PARTIAL FRACTIONS

You may be given a rational expression of the form:

$$\frac{1}{(3n-2) \cdot (3n+1)}$$

In this case because it is not a geometric series, it must be a telescoping series (because otherwise we wouldn't be asked to find the value to which it converges to, we only know how to find the convergence values of those two series, so we know it's telescoping, in order to get it into a form that will work, use partial fractions¹:

$$\begin{aligned} \frac{1}{(3n-2) \cdot (3n+1)} &= \frac{A}{3n-2} + \frac{B}{3n+1} \\ &= \frac{\frac{1}{3}}{3n-2} + \frac{-\frac{1}{3}}{3n+1} \end{aligned}$$

From here we would manipulate the series using grouping and rearrangement

GROUPING OF SERIES (9.1.3)

Grouping terms in a series does not affect the value to which it converges,

this flows from the associativity of addition, a property exhibited by the reals which is the codomain of the sequence function

So in the above example the regrouping necessary to demonstrate the telescoping nature:

$$\begin{aligned} & \left(\frac{1}{3} - \frac{1}{12} \right) + \left(\frac{1}{12} - \frac{1}{21} \right) + \left(\frac{1}{21} - \frac{1}{33} \right) \cdots \left(\frac{1}{3n-2} - \frac{1}{3n+1} \right) \\ &= \frac{1}{3} - \frac{1}{12} + \frac{1}{12} - \frac{1}{21} + \frac{1}{21} - \frac{1}{33} \cdots \frac{1}{3n-2} - \frac{1}{3n+1} \\ &= \frac{1}{3} + \left(-\frac{1}{12} + \frac{1}{12} \right) + \left(-\frac{1}{21} + \frac{1}{21} \right) + \left(-\frac{1}{33} + \frac{1}{33} \right) \cdots \left(-\frac{1}{3n-2} + \frac{1}{3n-2} \right) - \frac{1}{3n+1} \end{aligned}$$

¹ <http://tutorial.math.lamar.edu/Classes/CalcII/PartialFractions.aspx>

REARRANGEMENTS (9.1.5)

If a series is absolutely convergent then you can rearrange the terms and the series will converge to the same value (otherwise you can't so be careful)

So say you have some series and you rearrange it, if this new series is absolutely convergent then it's fine.

However, if you rearrange some series and the new series is only conditionally convergent, then the rearrangement wasn't logically valid and this convergence value is erroneous.

So in our example the series is absolutely convergent so we could rearrange it:

$$= \frac{1}{3} + \cancel{\frac{-1/3}{3n+1}} + \left(\frac{1}{12} - \frac{1}{12} \right) + \left(\frac{1}{21} - \frac{1}{21} \right) + \left(\frac{1}{33} - \frac{1}{33} \right) \dots + \left(\frac{1/3}{3n-2} - \frac{1/3}{3n-1} \right)$$
$$= \frac{1}{3} - \frac{1}{9n+3} + 0$$

$$\frac{1}{3} > \frac{1}{3} \cdot \frac{1}{3n+1}$$
$$\Rightarrow 0 < \frac{1}{3} - \frac{1}{9n+3}$$
$$\Rightarrow \frac{1}{3} - \frac{1}{9n+3} = \left| \frac{1}{3} - \frac{1}{9n+3} \right|$$

Thus, the new series is absolutely convergent

And simply for want of completing the problem:

$$\sum \left[\frac{1}{(3n-2) \cdot (3n+1)} \right] = \lim_{n \rightarrow \infty} \left(\frac{1}{3} - \frac{1}{9n+3} \right)$$
$$= \frac{1}{3}$$

IDENTITIES TO REMEMBER

For the exam you're going to want to remember that:

LIMITS OF EXPONENTIALS

$$\begin{aligned}\lim_{n \rightarrow \infty} (n^{1/n}) &= \lim_{n \rightarrow \infty} (e^{\ln(n^{1/n})}) \\ &= e^{\lim_{n \rightarrow \infty} (\ln(n^{1/n}))} \\ &= e^{\lim_{n \rightarrow \infty} (1/n \cdot \ln(n))} \\ &= e^{\lim_{n \rightarrow \infty} \left(\frac{d/dn (\ln(n))}{d/dn (1/n)} \right)} \\ &= e^{\lim_{n \rightarrow \infty} (1/n^2)} \\ &= e^0 \\ &= 1\end{aligned}$$

Because the exponential function is continuous

By the exponentiation log law

By L'Hospital's Rule

$\therefore \lim_{n \rightarrow \infty} (n^{1/n}) = 1$

DEALING WITH INEQUALITIES

$$\begin{aligned}a < n < b &\iff \frac{1}{b} < \frac{1}{n} < \frac{1}{a} \\ \text{e.g. } 2 < 5 < 7 &\iff \frac{1}{7} < \frac{1}{5} < \frac{1}{2} \\ \text{if } 5 > 2 &\text{ then } 1/5 < 1/2 \\ \text{if } 7 > 5 &\text{ then } 1/7 < 1/5 \\ \text{thus } 1/7 < 1/5 < 1/2\end{aligned}$$
$$\begin{aligned}a < n &\Rightarrow n > a \Rightarrow \frac{1}{a} < \frac{1}{n} \\ n < b &\Rightarrow b > n \Rightarrow \frac{1}{n} < \frac{1}{b} \\ &\Rightarrow \frac{1}{a} < \frac{1}{n} < \frac{1}{b}\end{aligned}$$

COMPARISON TESTS (3.7.7)

COMPARISON TEST (3.7.7)

take positive real sequences and some $k \in \mathbb{N}$:

$$n \geq k \Rightarrow 0 < x_n \leq y_n$$

so this order only needs to hold for any tail of the sequence.

a) if $\sum [y_n]$ converges, then $\sum [x_n]$ converges

b) if $\sum [x_n]$ diverges, then $\sum [y_n]$ diverges

LIMIT COMPARISON TEST (3.7.8)

Sometimes it can be difficult to establish the inequalities of the first test and a ratio would be easier to use, in that case this test can be used:

if the following limit exists:

$$r = \lim \left(\frac{x_n}{y_n} \right)$$

then:

a) if $r \neq 0$
 $\sum [x_n]$ is convergent $\Leftrightarrow \sum [y_n]$ is convergent

b) if $r = 0$
 $\sum [y_n]$ is convergent $\Rightarrow \sum [x_n]$ is convergent

be careful, if $\sum [x_n]$ is convergent
 $\sum [y_n]$ may or may not
be convergent.

ABSOLUTE CONVERGENCE TESTS (9.2)

If these tests are satisfied they will establish that the series is absolutely convergent.

LIMIT COMPARISON TEST II (9.2.1) (FOR ABSOLUTE CONVERGENCE)

This version of the test is useful for establishing absolute convergence, it may be more difficult to establish however.

if the following limit exists:

$$r = \lim \left(\frac{x_n}{y_n} \right)$$

then:

a) if $r \neq 0$

$\sum [x_n]$ is convergent $\Leftrightarrow \sum [y_n]$ is convergent

b) if $r = 0$

$\sum [y_n]$ is convergent $\Rightarrow \sum [x_n]$ is convergent

be careful, if $\sum [x_n]$ is convergent
 $\sum [y_n]$ may or may not
be convergent.

if the following limit exists:

$$r = \lim \left(\frac{x_n}{y_n} \right)$$

then:

a) if $r \neq 0$

$\sum [x_n]$ is convergent $\Leftrightarrow \sum [y_n]$ is convergent

b) if $r = 0$

$\sum [y_n]$ is convergent $\Rightarrow \sum [x_n]$ is convergent

be careful, if $\sum [x_n]$ is convergent
 $\sum [y_n]$ may or may not
be convergent.

RATIO TEST (9.2.4)

Take a sequence of non-zero real numbers (n_n) and some $K \in \mathbb{N}$

consider:

$$L_1 = \left| \frac{n_{n+1}}{n_n} \right| \quad (n \geq K) \quad \left| \quad L_2 = \lim \left(\left| \frac{n_{n+1}}{n_n} \right| \right) \quad \right| \quad L_3 = \limsup_{n \rightarrow \infty} \left(\left| \frac{n_{n+1}}{n_n} \right| \right)$$

Any of these three tests is logically valid and will provide the following:

if $L = 1$ the test tells us nothing
if $L < 1$ then $\sum [n_n]$ is absolutely convergent
if $L > 1$ then $\sum [n_n]$ is divergent.

RATIO TEST (9.2.4) (GENERALISED D'ALAMBERT)

This can be useful where the ratio test fails for want of $(-1)^n$, the $\limsup()$ operator will strip that away for a $(+1)$.

It is worth remembering that a sequence (x_n) is convergent if and only if:

$$\liminf (x_n) = \limsup (x_n) = \lim (x_n)$$

In this test however, we simply need to show that the $\limsup()$ exists (which it will if the ratio-sequence has an upper bound), it isn't necessary to show that the ratio-sequence is convergent.

(However it is necessary that the sequence which generates the series converges to 0, otherwise the series will be divergent)

ROOT TEST (9.2.2)

Take a sequence of non-zero real numbers (x_n) and some $K \in \mathbb{N}$

consider:

$$q_1 = |x_n|^{\frac{1}{n}} \quad (n \geq K) \quad \bigg| \quad q_2 = \lim (|x_n|^{\frac{1}{n}}) \quad \bigg| \quad q_3 = \limsup_{n \rightarrow \infty} (|x_n|^{\frac{1}{n}})$$

Any of these three tests is logically valid and will provide the following:

if $q = 1$ the test tells us nothing
if $q < 1$ then $\sum [x_n]$ is absolutely convergent
if $q > 1$ then $\sum [x_n]$ is divergent.

ROOT TEST (GENERALISED CAUCHY TEST)

This can be useful where the root test fails for want of $(-1)^n$, the \limsup () operator will strip that away for a (+1).

INTEGRAL TEST

If the series is of a function that is positive and decreasing, then the series could converge if and only if the integral converges:

Let $f(k)$ be a positive decreasing function:

$$\sum_{n=1}^{\infty} [f(n)] \text{ converges} \iff \int_1^{\infty} f(x) dx \text{ converges}$$

Anything below here will not be in the class test so don't worry about it.

NONABSOLUTE CONVERGENCE TESTS (9.3)

An alternating Series is if the form

DEFINITION OF AN ALTERNATING SERIES

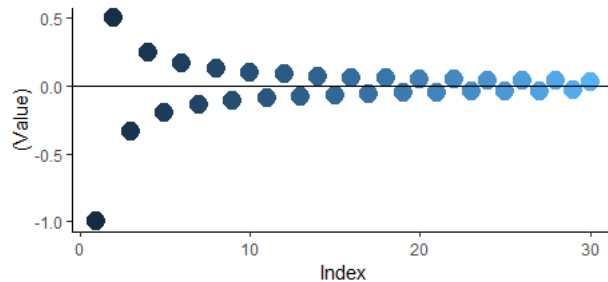
An Alternating sequence is a sequence where the terms alternate between positive and negative at each step, so for example:

This is alternating

$$x_n = \frac{(-1)^{n+1}}{\sqrt{n}}$$

Plot of the Sequence; $x_n = \frac{\sin(n)}{n}$

This sequence alternates between values, but, it is not an alternating sequence because it does not alternate between positive and negative at each iteration



This is an alternating sequence because at each succession the sequence changes sign.

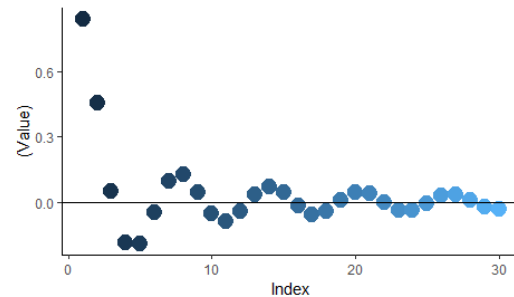
Microsoft Excel 2010
Worksheet: Sheet1
Cell: A1
Formula: =SIN(A2)/A2
Value: 0.5

This is not alternating

$$x_n = \frac{\sin(n)}{n}$$

Plot of the Sequence; $x_n = \frac{\sin(n)}{n}$

This sequence alternates between values, but, it is not an alternating sequence because it does not alternate between positive and negative at each iteration



Although this series is alternating between values it is not an alternating sequence because the terms doesn't alternate . This at each succession

Microsoft Excel 2010
Worksheet: Sheet1
Cell: A1
Formula: =SIN(A2)/A2
Value: 0.5

ALTERNATING SERIES

Take a decreasing sequence of positive numbers:

if (Z_n) :

$$Z_{n+1} < Z_n \text{ and } Z_n > 0 \quad \forall n \in \mathbb{Z}$$

then $\sum_{n=1}^{\infty} [(-1)^{n+1} \cdot Z_n]$ is convergent

So basically if the sequence is decreasing, then the series of the alternating sequence will hence converge.

So for example $(x_n) = \frac{1}{n}$ is a convergent sequence, the series are:

$$\sum \left[\frac{1}{n} \right] \text{ divergent} \quad \sum \left[(-1)^{n+1} \cdot \frac{1}{n} \right] \text{ convergent}$$

THE DIRICHLET AND ABEL TESTS

These tests are useful for establishing an absolute convergence, so for example they could be useful for series generated by sequences that alternate but are not, strictly speaking, alternating sequences.

They are derived from Abel's Lemma, which is also known as the partial summation formula (because it corresponds to integration by parts for functions from the reals onto the reals, the partial summation formula is the same process for functions from the naturals onto the reals, i.e. sequences and series)

PARTIAL SUMMATION FORMULA

let $X = (x_n)$ and $Y = (y_n)$ be sequences in \mathbb{R}

let the partial sums of $\sum [y_n]$ be denoted by S_n

where $S_0 := 0$

if $m > n$ then:

$$\sum [x_k y_k] = (x_m S_m - x_{n+1} S_n) + \sum (x_k - x_{k+1}) \cdot S_k$$

$$\lim_{n \rightarrow \infty} \left(\lim_{n \rightarrow \infty} \left(n^{\frac{1}{n}} \right) \right) = \lim_{n \rightarrow \infty} \left(e^{\ln(n^{\frac{1}{n}})} \right)$$

$$= e^{\lim_{n \rightarrow \infty} \left(\lim_{n \rightarrow \infty} \left(\frac{1}{n} \cdot \ln(n) \right) \right)}$$

$$= e^{\lim_{n \rightarrow \infty} \left(\frac{1}{n} \cdot \ln(n) \right)}$$

$$= e^{\lim_{n \rightarrow \infty} \left(\lim_{n \rightarrow \infty} \left(\frac{d/dn \left(\frac{1}{n} \right)}{d/dn \left(\ln(n) \right)} \right) \right)} = \lim_{n \rightarrow \infty} \left(e^{\ln(n^{\frac{1}{n}})} \right)$$

$$= e^{\lim_{n \rightarrow \infty} \left(\frac{1}{n^2} \right)}$$

$$= e^{\lim_{n \rightarrow \infty} \left(\ln(n^{\frac{1}{n}}) \right)}$$

$$= e^{\lim_{n \rightarrow \infty} \left(\frac{1}{n} \cdot \ln(n) \right)}$$

$$= e^{\lim_{n \rightarrow \infty} \left(\frac{d/dn \left(\ln(n) \right)}{d/dn \left(n \right)} \right)}$$

$$= e^{\lim_{n \rightarrow \infty} \left(\frac{1}{n^2} \right)}$$

$$= e^0$$

$$= 1$$

$$\therefore \lim_{n \rightarrow \infty} \left(n^{\frac{1}{n}} \right) = 1$$

$$\therefore \lim_{n \rightarrow \infty} \left(n^{\frac{1}{n}} \right) = 1$$

T

$$\begin{aligned}\lim_{n \rightarrow \infty} (n^{1/n}) &= \lim_{n \rightarrow \infty} (e^{\ln(n^{1/n})}) \\&= e^{\lim_{n \rightarrow \infty} (\ln(n^{1/n}))} \\&= e^{\lim_{n \rightarrow \infty} (1/n \cdot \ln(n))} \\&= e^{\lim_{n \rightarrow \infty} \left(\frac{d/dn (\ln(n))}{d/dn (1/n)} \right)} \\&= e^{\lim_{n \rightarrow \infty} (1/n^2)} \\&= e^0 \\&= 1\end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} (n^{1/n}) = 1$$



Section 4-12 : Strategy for Series

Now that we've got all of our tests out of the way it's time to think about organizing all of them into a general set of guidelines to help us determine the convergence of a series.

Note that these are a general set of guidelines and because some series can have more than one test applied to them we will get a different result depending on the path that we take through this set of guidelines. In fact, because more than one test may apply, you should always go completely through the guidelines and identify all possible tests that can be used on a given series. Once this has been done you can identify the test that you feel will be the easiest for you to use.

With that said here is the set of guidelines for determining the convergence of a series.

1. With a quick glance does it look like the series terms don't converge to zero in the limit, *i.e.* does $\lim_{n \rightarrow \infty} a_n \neq 0$? If so, use the Divergence Test. Note that you should only do the Divergence Test if a quick glance suggests that the series terms may not converge to zero in the limit.
2. Is the series a p -series ($\sum \frac{1}{n^p}$) or a geometric series ($\sum_{n=0}^{\infty} ar^n$ or $\sum_{n=1}^{\infty} ar^{n-1}$)? If so use the fact that p -series will only converge if $p > 1$ and a geometric series will only converge if $|r| < 1$. Remember as well that often some algebraic manipulation is required to get a geometric series into the correct form.
3. Is the series similar to a p -series or a geometric series? If so, try the Comparison Test.
4. Is the series a rational expression involving only polynomials or polynomials under radicals (*i.e.* a fraction involving only polynomials or polynomials under radicals)? If so, try the Comparison Test and/or the Limit Comparison Test. Remember however, that in order to use the Comparison Test and the Limit Comparison Test the series terms all need to be positive.
5. Does the series contain factorials or constants raised to powers involving n ? If so, then the Ratio Test may work. Note that if the series term contains a factorial then the only test that we've got that will work is the Ratio Test.
6. Can the series terms be written in the form $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$? If so, then the Alternating Series Test may work.
7. Can the series terms be written in the form $a_n = (b_n)^n$? If so, then the Root Test may work.
8. If $a_n = f(n)$ for some positive, decreasing function and $\int_a^{\infty} f(x) dx$ is easy to evaluate then the Integral Test may work.

Again, remember that these are only a set of guidelines and not a set of hard and fast rules to use when trying to determine the best test to use on a series. If more than one test can be used try to use the test that will be the easiest for you to use and remember that what is easy for someone else may not be easy for you!

Also, just so we can put all the tests into one place here is a quick listing of all the tests that we've got.

Divergence Test

If $\lim_{n \rightarrow \infty} a_n \neq 0$ then $\sum a_n$ will diverge

Integral Test

Suppose that $f(x)$ is a positive, decreasing function on the interval $[k, \infty)$ and that $f(n) = a_n$ then,

1. If $\int_k^{\infty} f(x) dx$ is convergent then so is $\sum_{n=k}^{\infty} a_n$.
2. If $\int_k^{\infty} f(x) dx$ is divergent then so is $\sum_{n=k}^{\infty} a_n$.

Comparison Test

Suppose that we have two series $\sum a_n$ and $\sum b_n$ with $a_n, b_n \geq 0$ for all n and $a_n \leq b_n$ for all n . Then,

1. If $\sum b_n$ is convergent then so is $\sum a_n$.
2. If $\sum a_n$ is divergent then so is $\sum b_n$.

Limit Comparison Test

Suppose that we have two series $\sum a_n$ and $\sum b_n$ with $a_n, b_n \geq 0$ for all n . Define,

$$c = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}$$

If c is positive (i.e. $c > 0$) and is finite (i.e. $c < \infty$) then either both series converge or both series diverge.

Alternating Series Test

Suppose that we have a series $\sum a_n$ and either $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ where $b_n \geq 0$ for all n . Then if,

1. $\lim_{n \rightarrow \infty} b_n = 0$ and,
2. $\{b_n\}$ is eventually a decreasing sequence

the series $\sum a_n$ is convergent

Ratio Test

Suppose we have the series $\sum a_n$. Define,

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Then,

1. if $L < 1$ the series is absolutely convergent (and hence convergent).
2. if $L > 1$ the series is divergent.
3. if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.

Root Test

Suppose that we have the series $\sum a_n$. Define,

$$L = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$$

Then,

1. if $L < 1$ the series is absolutely convergent (and hence convergent).
2. if $L > 1$ the series is divergent.
3. if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.