3.5 - Cauchy Criterion Problems (Textbook)

Problem 2, (a)

Problem

Show that the follwoing sequence is a Cauchy Sequence

$$X = \frac{n_{+}1}{n} \tag{1}$$

Solution

Layout the Proof In order to show that the sequence X is a cauchy sequence it must be shown that:

$$\begin{aligned} &\forall \varepsilon > 0,\ \exists H \in \mathbb{N} : \\ &\qquad m, n > H \implies \mid x_n - x_m \mid < \varepsilon \end{aligned} \tag{2}$$

so first we will consider the restriction required by ε and work backwards to find a sufficient value for H.

Consider the ε Restriction

$$\begin{aligned} \mid \frac{n+1}{n} - \frac{m+1}{m} \mid &= \mid \frac{mn + m - mn + n}{mn} \\ &= \mid \frac{m+n}{mn} \mid \\ &= \frac{m+n}{mn} && \text{Because } m, n \in \mathbb{N} \\ &= (m+n) \cdot \frac{1}{mn} \end{aligned} \tag{3}$$

Hence we have:

$$\begin{aligned}
&\mid \frac{n+1}{n} - \frac{m+1}{m} \mid < \varepsilon \\
\implies &(m+n) \cdot \frac{1}{mn} < \varepsilon \qquad (4)
\end{aligned}$$

Assume a Value for H Now assume an arbitrary value for for H, we will use $H \geq 3$, this implies from (2):

$$\begin{aligned}
m, n &\geq H \\
m, n &\geq 3 && \text{sub } H \geq 3 \\
m \cdot n &\geq 9 \\
\frac{1}{mn} &\leq \frac{1}{9} \\
\frac{1}{mn} &\leq \frac{1}{9} \\
(m+n) \cdot \frac{1}{mn} &\leq \frac{1}{9} \cdot mn
\end{aligned}$$

and from (4) we have:

$$(m+n) \cdot \frac{1}{mn} \leq \varepsilon$$

Apply the restriction to H So we will choose H:

$$\frac{1}{9}(m+n) > \varepsilon \qquad (5)$$

So re arranging this to solve some value for m, n, H

$$\begin{aligned}
\frac{1}{9}(m+n) &> \varepsilon \\
(m+n) &> 9 \cdot \varepsilon
\end{aligned} \qquad (6)$$

So if we choose a H value such that $H > \frac{9\varepsilon}{2}$ then we will have $m > \frac{9\varepsilon}{2}$ and $n > \frac{9\varepsilon}{2}$ and so $(m+n) > \varepsilon$

Choose the Specific H Value Now there are two values for H, we need a value of $H \geq 3$ and $H > \frac{9\varepsilon}{2}$, this is satisfied by taking $H = \sup \left\{ 9 \cap \left(\frac{2\varepsilon}{2}, \infty \right) \right\}$

The actual proof

$$\forall \varepsilon, \ \ \exists H = \sup \left\{ 9 \cap \left(\frac{9\varepsilon}{2}, \infty \right) \right\}$$

Now assume that $m, n > H$, and consider $\mid x_n - x_m \mid$:

$$\mid x_n - x_m \mid = \mid \frac{n+1}{n} - \frac{m+1}{m} \mid \tag{7}$$
$$= (m+n) \cdot \frac{1}{mn} \tag{8}$$

Now because $H \geq 9$ and $m, n \geq H$

$$< \frac{1}{9}(m+n)$$

because $H > \frac{9\varepsilon}{2}$

$$< \frac{1}{9} \cdot \left(\frac{9\varepsilon}{2} + \frac{9\varepsilon}{2} \right)$$
$$< \varepsilon \tag{9}$$

Now because we have shown that $\forall \varepsilon, \ \ \exists H = \sup \left\{ 9 \cap \left(\frac{9\varepsilon}{2}, \infty \right) \right\}$ such that:

$m, n \geq H \implies \mid x_n - x_m < \varepsilon$
It is established that X must be a Cauchy Sequence.