(08) Complex Variables

Analysis (200023)

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Contents

Functions of a Complex Variable

A complex function is a function from a complex plane onto another complex + ane:

$$A, B \subseteq \mathbb{C},$$
$$f : A \to B$$

All the usual definitions of functions still apply, e.g.:

- Functions are rigourously defined using sets
- There is a do main, range, codomain, image etc.
- ...

The *Churchill's* Textbook mentions these conventions however

1. Usually the codomain is taken as the set of all complex values.
2. Most of the results concerning real functions are taken as already established without justification
3. x, y, u, v denote real variables where as z and w denote complex variables

   $z = x + iy$

   w = u + i v

   $f(z) = w \implies f(z) = u + iv$

4. Sometimes there won't be a clear distinction between the values of a function and the function itself, e.g.:

$$g(z) = z^2$$
$$f(g(z)) = f(z^2) \qquad \text{Here } z^2 \text{ is shorthand for the function}$$

Geometric Interpretation

Imagine the real function $y = 3$:

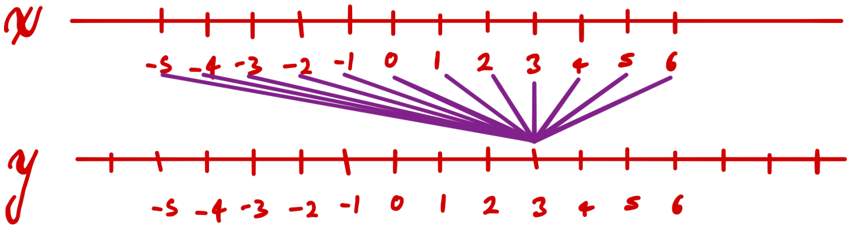

Figure 1:

A similar constant complex function would be $f(z) = 3 + 4i$:

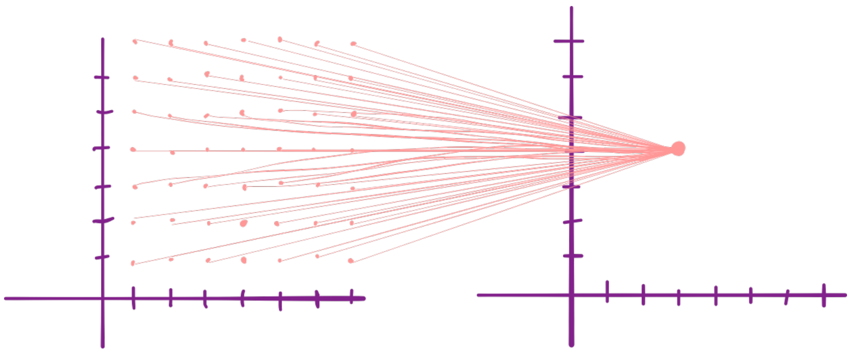

Figure 2:

It isn't possible to make a graph like it is with simple single variable real functions because that would require four spacial dimensions in order to plot it, this has the dissapointing consequence that geometric interpretations of derivatives as a slope and integrals as area beneath a curve are no longer helpful.

It isn't uncommon to use a 3D Cartesian plane to illustrate a function from the reals onto the complex, e.g. imagine $y = x^2$, if a complex domain is ullustrated as an x /y plane and a perpendicular z-axis represents the real codomain, the surface representing the values would always have two roots, even if the're not real, the *Welch Labs* video *Imaginary Numbers are real*[1] is really good for getting a visualisation of this but the general visualisation is:

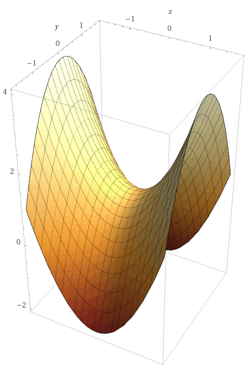

Figure 3: This can be generated in *Wolfram* or something else by using $f(x, y) = (x + iy)^2 + 1$

[1] https://www.youtube.com/watch?v=T647CGsuOVU

Complex functions Components

Complex functions can be illustrated as a pair of two-variable real functions. Take a function:

$$f(z) = w$$

The w variable can be expanded:

$$f(z) = u + iv$$

the z variable can also be expanded:

$$f(x + iy) = u + iv$$

Observe that the value u is in essence a function of the x and y input variables, the same is true also for v, hence, this can be rewritten:

$$f(x + iy) = u(x, y) + i \times v(x, y)$$

Example

$$\begin{aligned} f(z) &= z^2 \\ &= (x + iy)^2 \\ &= x^2 - y^2 + i \cdot 2xy \\ &= \left(x^2 - y^2\right) + i \cdot (2xy) \end{aligned}$$

So in this case the component functions would be:

$$\begin{aligned} u(x, y) &= \left(x^2 - y^2\right) && \text{and,} \\ v(x, y) &= (2xy) \end{aligned}$$

Essentially a complex-valued function is a pair of two variable real functions.

Limits

if f is defined on all poiints in a *deleted neighbourhood* of α it is written:

$$\lim_{z \to \alpha} f(z) = L, \qquad \text{equivalently,} \qquad f(z) \to w_0 \quad \text{as} \quad z \to z_0.$$

if and only if:

$f(z)$ can be made arbitrarily close to L by making z sufficiently close to α.

In formal notation this is expressed:

> **Formal Definition of a Complex Limit**
>
> If $f : A \to \mathbb{C}$ and $\alpha \in \overline{A}$
>
> $$\begin{aligned} &\forall \varepsilon > 0, \ \exists \delta : \\ &\qquad 0 < |z - \alpha| < \delta \implies |f(z) - L| < \varepsilon \end{aligned}$$

Limits in Terms of Sequences A sequence of complex numbers $\{z_n\}_1^\infty$ has a limit z (i.e. it converges to z) if:

Formal Definintion of a Limit to a Complex Sequence

$$\forall \varepsilon > 0, \ \exists N \in \mathbb{Z} :$$
$$in > N \implies |z_n - z| < \varepsilon$$

So this says, the limit of the sequence is z iif the terms of the sequence can be made arbitrarily close to the limit value by moving sufficiently far along the sequence.

Limit values are unique, a function can only have a single limite value at a point (or no limit value if the limit is undefined at that point).

Limits from multiple directions

A single variable real functions can only approach a variable from the left-hand side or the right-hand side, complex funcitons however can approach a varible along any curve in the complex plane.

So for example consider the limit of a function as $z \to 0$, z could approach zero along:

- the real-axis $(x, 0) : x \to 0 \implies z \to 0$
- the imaginary-axis $(0, y) : y \to 0 \implies z \to 0$
- any straight-line $y = mx : y \to 0 \implies z \to 0$
- along a parabola $y = x^2 : y \to 0 \implies z \to 0$
- any curve whatsoever at all...

What makes this more confusing is that a limit may approach a value along one curve but not another, maybe for example our function approaches $w = f(z) = L$ as the variable approaches 0 on both the x-axis and the y-axis, despite this it's entirely possible that our function approaches the value 33 along a parabola, the value 42 along a straight line and maybe $6\pi + 4i$ along a cubic curve.

So it's really worth noting that as a **necessary but not sufficient condition**, the limit taken along the axis must be equal in order for the limit to exist, if they are equal however, the limit is not guaranteed to exist, it may be another value along a different curve. It's worth reading *Pauls Online Notes* [2]

The reason for often taking limits along the axis (as opposed to some other arbitrary curve), is because the axis zeroes out a term which can be simpler and because the partial derivatives are also taken along the axis, which is used in developing the *Cauchy Riemann* equations later, but, really, there is no difference taking the limit along arbitrary curves or along the axis, the function doesn't necessarily care.

[2] http://tutorial.math.lamar.edu/Classes/CalcIII/Limits.aspx

Theorems on Limits The idea here is to establishh a connection between limits of complex functions and limits of real functions so we can use all the pre-established properties of real limits from calculus.

if:

$$z = x + iy$$
$$f(z) = u(x, y) + i \cdot v(x, y)$$

Then we have:

$$\lim_{z \to \alpha} (f(z)) = L$$

if and only if:

$$\lim_{(x,y) \to (a,b)} [u(x, y)] = \mathrm{Re}(L) \quad \text{and} \quad \lim_{(x,y) \to (a,b)} [v(x, y)] = \mathrm{Im}(L)$$

So now we can break the complex limits up into real components that we already know how to deal with, and all the familiar *Limit Laws* carry over from earlier calculus.

Limit Laws

Distribution over Addition

$$\lim_{z \to z_0} [f(z) g(z)] = \lim_{z \to z_0} [f(z)] + \lim_{z \to z_0} [g(z)]$$

Distribution over Multiplication

$$\lim_{z \to z_0} [f(z) \cdot g(z)] = \lim_{z \to z_0} [f(z)] \cdot \lim_{z \to z_0} [g(z)]$$

Distribution over Division
Assume that $\lim_{z \to z_0} [g(z)] \neq 0$:

$$\lim_{z \to z_0} \left[\frac{f(z)}{g(z)} \right] = \frac{\lim_{z \to z_0} [f(z)]}{\lim_{z \to z_0} [g(z)]}$$

Riemann Sphere Limits at infinity are given a theoretical foundation using an idea called the *Riemann Sphere*, it's interesting but a deep understanding of the theory isn't necessary in order to work with limits at infinity so dont worry about it.

Continuity

A function f is *continuous* at a point z_0 if for all points $\lim_{z \to z_0} [f(z)] = f(z_0)$.

This is generally broken up into three conditions for want of decomposing problems:

Conditions of Continuity
A function f is *Continuous* at z_0 if the following three conditions are all satisfied:

1. $\lim_{z \to z_0} [f(z)]$
2. $f(z_0)$ exists
3. $\lim_{z \to z_0} [f(z)] = f(z_0)$ (which implies the above 2)

If a function is continuous on some neighbourhood, it's limit value for any point in that neighbourhood is the function value, this means, if we did, for instance, take the limit at a point along both axis (or along any two arbitrary curves), and they were equal, then the limit would be defined at that point, because it would be the function value.

If a function can be differentiated at a point, the function is continuous at that point.

So if we could show that a derivative exists on all points of some neighbourhood, and that the derivative was continuous at some point, then that neighbourhood would be continuous and the limit at that point would certainly exist.

This might seem a little bit contrived, but these are the pieces that are used for the *Cauchy Riemann* equations

Function Composition
A composition of continuous functions is continuous, e.g.

if:

$$f(x) = x^2 \qquad \text{is continuous}$$
$$g(x) = e^x \qquad \text{is contninous}$$

Then:

$$f \circ g = f[g(x)] = e^{x^2} \qquad \text{is continuous}$$

Again, this might seem obvious, but it's useful for complex functions and is necessary in the *Cauchy Riemann* equations.

Continuity of Complex Functions A Complex function is only continuous if the real two-variable components $u(x, y)$ and $v(x, y)$ are continuous:

This is because a composition of continuous functions is continuous

$$\underbrace{f(z)} = \underbrace{u(x,y)} + i\,\underbrace{v(x,y)}$$

this is continuous
if and only if
these are continous

Again, remember this for later when we are doing the *Cauchy Riemann* equations.

Derivatives

Derivatives have the same definition in complex analysis as they do in real calculus, with the difference that the variable is now complex, the derivative of f at a is:

$$\begin{aligned} w &= f(z) \\ \implies \frac{\mathrm{d}\, w}{\mathrm{d}\, z} &= \lim_{\Delta Z \to 0} \left[\frac{\Delta w}{\Delta z}\right] \end{aligned}$$

Or in a more useful fashion:

$$f'(a) = \lim_{\Delta Z \to 0} \left[\frac{f(z) - f(a)}{z - a}\right] = \lim_{\Delta Z \to 0} \left[\frac{f(a + \Delta z) - f(a)}{\Delta z}\right]$$

A function may be differentiable at a point z but not necessarily at any other points in the neighbourhood of z.

The real and imaginary components of a function may have continuous partial derivatives (of all orders) yet this does not imply that the function is differentiable there,

> e.g. $f(z) = |z|$ has continuous partial derivatives away from $z = 0$, but is not differentiable anywhere, because the limits as $\Delta z \to 0$ are different depending on which path is taken.

Conditions of Continuity

- if a function is continuous it may or may not be differentiable

  continuity $\not\Longrightarrow$ differentiability

- if a function is differentiable it must be continuous

  differentiability $\Longrightarrow$ continuity

Derivatives to Memorise

The following complex functions are nowhere differentiable:

- $f(z) = \Re(z)$
- $f(z) = \Im(z)$
- $f(z) = \overline{z}$

The function $f(z) = |z|^2$ is differentiable only at $z = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}\, z} \left(|z|^2\right) \Big|_{z=0} = f'(0) = 0$$

bear in mind however that this function is nowhere analytic, because it is not differentiable on a neighbourhood of 0, this is covered further down.

How to Deal with Derivatives If you are given a function purely of z, then all the familiar differentiation rules carry over, their proofs are different, but the rules come out the same. [3]

If however you are given a function with terms of z, x and y, then you will need to first break the function up into it's components:

$$f(z) = u(x, y) + i \cdot v(x, y)$$

and then you will need to use the *Cauchy Riemann* equations, which we will get to further down.

[3] I mean, be careful with log functions and roots because they have multiple values

Example Find the derivative of $w = f(z) = \frac{1}{z}$:

$$\begin{aligned}
\lim_{\Delta Z \to 0}\left[\frac{\Delta w}{\Delta z}\right] &= \lim_{\Delta Z \to 0}\left[\frac{f(z+\Delta z) - f(z)}{\Delta z}\right] \\
&= \lim_{\Delta Z \to 0}\left[\frac{\frac{1}{z+\Delta z} - \frac{1}{z}}{\Delta z}\right] \\
&= \lim_{\Delta Z \to 0}\left[\frac{z + \Delta z - z}{z(z+\Delta z)\Delta z}\right] \\
&= \lim_{\Delta Z \to 0}\left[\frac{\cancel{\Delta z}}{z(z+\Delta z)\cancel{\Delta z}}\right] \\
&= \frac{-1}{z(z+0)} \\
&= -z^{-2}
\end{aligned}$$

$\therefore \frac{\mathrm{d}}{\mathrm{d}\,z}\left(z^{-1}\right) = -z^{-2} \quad \forall z \in \mathbb{C}$

Differentiation Rules

All the typical rules carry over, they are however derived with different proofs, refer to the *Churchill* textbook.

- $\frac{\mathrm{d}}{\mathrm{d}\,z}(z) = 1$
- $\frac{\mathrm{d}}{\mathrm{d}\,z}(c) = 0$
- Contants:
  $f(cz) = c \cdot f(z)$
- Power Rule:
  $\frac{\mathrm{d}}{\mathrm{d}\,z}(az^n) = a(n-1)a^{n-1}$
- Product Rule:
  $\frac{\mathrm{d}}{\mathrm{d}\,z}(f(z) \cdot g(z)) = \frac{\mathrm{d}}{\mathrm{d}\,z}(f(z)) \cdot g(z) + f(z) \cdot \frac{\mathrm{d}}{\mathrm{d}\,z}(g(z))$
- Quotient Rule:
  $\frac{\mathrm{d}}{\mathrm{d}\,z}\left(\frac{f(z)}{g(z)}\right) = \frac{\frac{\mathrm{d}}{\mathrm{d}\,z}(f(z)) \cdot g(z) - f(z) \cdot \frac{\mathrm{d}}{\mathrm{d}\,z}(g(z))}{\left(g(z)^2\right)}$
- Chain Rule:
  $\frac{\mathrm{d}}{\mathrm{d}\,z}[f(g(z))] = f'(g(z)) \cdot g(z)$
  $\frac{\mathrm{d}\,w}{\mathrm{d}\,z} = \frac{\mathrm{d}\,w}{\mathrm{d}\,u} \cdot \frac{\mathrm{d}\,u}{\mathrm{d}\,z}$

The *Cauchy Riemann* Equations

The idea of these is to provide a means to solve derivatives when the function is of the form:

$$f(z) = u(x, y) + i \cdot v(x, y)$$

because in this case we can't just use our typical differentiation formula.

The proof for this is in the textbook by *Churchill*, and it's well worth going through because it's pretty simple and establishes where this comes from, but the main idea of this is:

***Cauchy-Riemann* Necessary Condition** If a complex function:

$$f(z) = u(x, y) + i \cdot v(x, y)$$

Is differentiable at some value (z_0), then the derivative must be:

$$f'(z_0) = \frac{\partial u}{\partial x} + i \cdot \frac{\partial v}{\partial x} = \frac{\partial v}{\partial x} + i \cdot \frac{\partial u}{\partial y} \tag{1}$$

Which implies that:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Now this isn't very useful because it can't tell us if the derivative actually exists, only what the value would have to be if it did.

However, we also have the sufficient condition:

***Cauchy-Riemann* Sufficient Condition**
if all first order partial derivatives:

- exist on some neighbourhood of z_0
- are continuous at z_0

then $f'(z)$ must exist.

It is necessary to memorise these.

Technique So if we wanted to solve the derivative of some complex function, that isn't solely in terms of z, we would:

1. solve the partials,
2. Test that they exist on a neighbourhood around where we are interested
3. Test that they are continuous at the point we're interested in
4. make sure that $u_x = v_y$ and $u_y = -u_x$

Then we just write the derivative as $f'(z) = u_x + iv_x$

Polar Co-Ordinates
This also works for polar co-ordinates:

> ***Cauchy-Riemann* Necessary Condition**
> If a complex function:
>
> $$f(z) = u(r,\theta) + i \cdot v(r,\theta)$$
>
> is differentiable, then the derivative will be:
>
> $$f'(z_0) = e^{i\theta}(u_r + i \cdot v_r)$$
>
> And the partials will be such that:
>
> $$r \cdot \frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \qquad \frac{\partial u}{\partial \theta} = -r \cdot \frac{\partial v}{\partial r}$$

Example is $f(z) = z^2$ diffentiable everywhere?

Decompose the function

$$\begin{aligned} z^2 &= (x+iy)^2 = x^2 - y^2 + i(2xy) \\ &= u + iv : \end{aligned}$$

Solve the Partials

$$u = x^2 - y^2 \qquad\qquad v = 2xy$$

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\left(x^2 - c\right) \\ &= \frac{\partial}{\partial x}\left(x^2\right) \\ &= 2x \end{aligned} \qquad\qquad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -2y \qquad\qquad \frac{\partial v}{\partial x} = 2y$$

The partials are defined in the domain of the function and continuous throughout, hence $f'(z)$ must exist by the second test above.

Use the Cauchy Riemann equations
Because a derivative exist, it is a necessary condition that:

$$\begin{aligned} \frac{\partial u}{\partial x} = u_x = v_y = \frac{\partial v}{\partial x} \qquad\qquad \frac{\partial v}{\partial x} = v_x = -u_y = \frac{\partial u}{\partial y} \\ 2x = 2x \qquad\qquad\qquad\qquad 2y = -(-2y) \end{aligned}$$

Thus $f\left(z^2\right)$ is differentiable on the complex domain (i.e. f is analytic in $\mathbb{C}$, which means that it is an entire function).

Analytic Functions

Analytic Functions are also known as regular or holomorphic functions.

- A function is analytic at a point if:

  The function is analytic on some neighbourhood of that point.

- A function is analytic on an open set if:

  Thhe derivative exists on the entire set.

These neighbourhoods are open, there's a reason for this but I've forgotten, it's not important really because if a function is differentiable on an entire closed set like:

$$S_{Closed} = \{z : |z| \leq 3\}$$

it can be easily be redefined as equivalent open set (this is similar to the ideas of suprema and infima):

$$S_{Open} = \{z : |z| < r : r > 3\}$$

A function is said to be entire if it is differentiable over the entire complex plane, e.g. $w = z, w = z^2, w = e^z$ are all entire functions.

The textbook by Osborne (i.e. the prescribed textbook) talks about a function being analytic on a region, a region is an open connected set.

If a function is analytic on some domain:

- The function is continuous on that domain.

  because a differentiable function must be continuous

- The Cauchy Riemann equations are satisfied.

  because they are a necessary condition of differentiability

Distribution of Analyticity

- The sum and difference of analytic functions are still analytic

  because $\frac{\mathrm{d}}{\mathrm{d}\,x}\left[f(x) + g(x)\right] = f'(x) + g(x)$

- The composition of analytic functions are analytic

  because $\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{\mathrm{d}\,y}{\mathrm{d}\,u} \cdot \frac{\mathrm{d}\,u}{\mathrm{d}\,x}$

Derivatives of constants If f is an analytic function such that:

$$f'(z) = 0$$

then $f(z)$ is a constant functions:

$$f(z) = \alpha : \quad \exists\ \alpha \in \mathbb{C}$$

Special Cases of Analyticity

- $w = \frac{1}{z}$ is analytic on every non-zero point of $\mathbb{C}$, because every point **around** 0 has a neighbourhood of differential points.
- $w = |z|^2$ is analytic nowhere because it is only differentiable **at** 0, this means that there are no neighbourhoods throughout which the function is differentiable.

Singular Points If a function is not analytic at some point z_0,
but is analytic at some point in every neighbourhood of z_0,
that point is said to be a singularity of f.

So for example the point 0 of $w = \frac{1}{z}$ is a singular point of w.

Singular points are of fundamental importance whin it comes to integration and are closely related to the ideas of zeroes and roots.

Definitions Take some function $f(z)$ and some point in the complex plane $\alpha \in \mathbb{C}$.

- ***Analytic at a point***

  f is analytic at $\alpha \in \mathbb{C}$ if it is differentiable at α and in some nieghbourhood of α.

- ***Singular Point***

  α is a singular point of f if f is not analytic at α but is analytic at some point in every open neighbourhood of α. $\frac{1}{(z-1)(z+1)}$, $z=1$ and $z=-1$ are singular points

- ***Isolated Point***

  α is isolated if there is some open neighbourhood of α throughout which f is analytic everywhere but α.

  so 0 is an isolated point of $w = \frac{1}{z}$

The textbooks also talk about harmonic functions which are important for physics et cetera, however we don't seem to be covering it.