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COMPLEX VARIABLES AND THEIR APPLICATIONS

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Preface

In writing this volume it has been my intention to produce a book which is fairly versatile. My hope is that not only is this book suitable for both a first and second course in complex variables for mathematicians, but it contains enough applications to be of use and interest to engineering and other science students. It is written specifically with today's undergraduates in mind and only assumes a knowledge of basic real analysis and calculus. Some of the material has evolved from a lecture course that I have given to second-year undergraduates over a number of years. The text includes the standard techniques and applications of complex variables, with plentiful examples. Generally, I have tried to give generic applications in the form of residue theory, differential equations and integral transforms, but specific applications are given in Chapter 9. The book also presents the important analytical concepts and techniques used in deriving the standard results in complex analysis. Readers who are more interested in applications may wish to leave these derivations and go straight to the calculations and examples.

My hope is that this book can be used at different levels, depending on the sections that are chosen. Any material which is normally covered in a first course is included within the first six chapters. The next three chapters deal with applications other than residue theory. Although some of the applications concerning differential equations are not often given in books on complex variables, it is my belief that this is the natural setting for the material I have included. Some of the material in Chapters 5, 6, 9, 10, and possibly 11, is suitable for inclusion in a second course. A short bibliography, which suggests further reading, but which is by no means intended to be exhaustive, is given at the end of the book.

Notation which is possibly non-standard is defined when used. When no ambiguity can occur, Theorem 2.5 is sometimes abbreviated to 2.5 and equation (2.5) is sometimes abbreviated to (2.5).

I have endeavoured to provide a large number of exercises at the end of each chapter, ranging from routine, through those that test understanding of the main concepts, to hopefully quite challenging questions. The most challenging exercises are marked with a star. A large number of answers have been included. A solutions manual, which provides the full solutions to all the questions, is available to lecturers from the publishers.

I am indebted to Dr David Bedford, who read some of the first draft of my manuscript and made helpful suggestions. I also thank Professor Jeffrey and the anonymous reviewers for their helpful advice and constructive criticisms. Finally, I am grateful to the editorial staff at Addison Wesley Longman for their support and advice, and all the staff in the production department who made this book possible.

Anthony D. Osborne
Keele, April 1998

1 Functions of a Complex Variable

This book is concerned with functions which map complex numbers to complex numbers. Although anyone wishing to study complex variables will have some knowledge of complex numbers, we begin with a review of such numbers and their properties. The chapter continues by introducing functions of a complex variable and the standard elementary functions.

Complex Numbers

It first becomes clear that we need to consider numbers other than real numbers and, in particular, roots of negative real numbers, when we are seeking solutions of quadratic equations. For example, consider the quadratic equation,

$$x(10-x) = 40 \tag{1.1}$$

Then $x^2 - 10x + 40 = 0 \quad \Rightarrow \quad (x-5)^2 + 15 = 0 \quad \Rightarrow \quad x = 5 \pm \sqrt{-15}$ and the square of any real number is clearly non-negative. Hence x is not a real number and the two solutions are examples of complex numbers. Geometrically, the fact that there are no real solutions to (1.1) corresponds to the fact that the parabola with equation $y = x^2 - 10x + 40$ does not intersect the x-axis.

Definition

A **complex number** is a number of the form $x + iy$, where x and y are any real numbers and i represents $\sqrt{-1}$. Such a number is often written as the ordered pair (x, y).

Notation

The set of all complex numbers will be denoted by $\mathbb{C}$ and the set of all real numbers by $\mathbb{R}$.

The notation $x + iy$ for the ordered pair (x, y) can be formally justified by defining the sum and product of any two complex numbers as below. Note that $\mathbb{R}$ is a proper subset of $\mathbb{C}$.

Note

Engineers and scientists tend to use the symbol j for $\sqrt{-1}$ since i is reserved for current.

Historical Note

Roots of negative real numbers were first introduced by Cardano in his *Ars Magna*, on the solution of algebraic equations, in 1545. He gave a single example, the quadratic equation (1.1). Bombelli was the first mathematician to manipulate complex numbers, in his *Algebra*, in 1572. However, full acceptance of complex numbers only came in the nineteenth century. Gauss gave complex numbers their present name and used them in his proof of the fact that any polynomial equation of degree n has exactly n roots in $\mathbb{C}$. It was Euler who introduced the symbol i in a memoir in 1777.

Definitions

Let $z = x + iy = (x, y) \in \mathbb{C}$. Then x and y are called the **real part** and **imaginary part** of z respectively. We write $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$.

Note that it is y, not iy, that is the imaginary part of z. The word *imaginary* is unfortunate, but for historical reasons this misnomer has stuck. Equality of complex numbers corresponds to the definition of equality of ordered pairs.

Definition

Two complex numbers, z_1 and z_2, are **equal**, written $z_1 = z_2$ in the usual way, if their real parts are equal and their imaginary parts are equal. This is often termed comparing real and imaginary parts.

Turning now to the arithmetic of complex numbers, the most fruitful way to define the usual operations is in such a way as to ensure that $i^2 = -1$ and that $\mathbb{C}$ has the same algebraic properties as $\mathbb{R}$.

Definitions

Consider complex numbers as ordered pairs of real numbers. Given $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2) \in \mathbb{C}$, their **sum** and **product** are denoted and defined respectively by

$$z_1 + z_2 = (x_1 + x_2,\ y_1 + y_2)$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + y_1 x_2)$$

Notice that these definitions include the sum and product of real numbers as special cases and give the usual componentwise addition and multiplication by a scalar when $\mathbb{C}$ is treated as the vector space $\mathbb{R}^2$. It is easily checked that the second definition gives $(0, 1)(0, 1) = (-1, 0)$; that is, $i^2 = -1$. The first definition gives the formal justification for the notation $x + iy$ for (x, y). Written in a different way, these definitions read

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

$$(x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 + i(y_1 x_2 + x_1 y_2) + i^2 y_1 y_2$$

Hence complex numbers may be added and multiplied by manipulating them as real numbers and replacing i^2 by -1 whenever it occurs. It can be checked, using the definitions, that $\mathbb{C}$ has essentially the same algebraic properties as $\mathbb{R}$; that is, $\mathbb{C}$ is a **field**. This is left as an exercise.

Note The Field Properties of $\mathbb{C}$

Let z_1, z_2 and $z_3 \in \mathbb{C}$. Then

1. $z_1 + z_2 \in \mathbb{C}$ and $z_1 z_2 \in \mathbb{C}$
2. $z_1 + z_2 = z_2 + z_1$ and $z_1 z_2 = z_2 z_1$ (commutative laws)
3. $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ and $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ (associative laws)
4. $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ (distributive law)
5. There is a unique number $0 = (0, 0) \in \mathbb{C}$ such that $z + 0 = z$ for all $z \in \mathbb{C}$
6. Given $z \in \mathbb{C}$ there is unique number $-z \in \mathbb{C}$ such that $z + (-z) = 0$. Then $z_2 - z_1 = z_2 + (-z_1)$ (subtraction)
7. There is a unique number $1 = (1, 0) \in \mathbb{C}$ such that $1z = z$ for all $z \in \mathbb{C}$
8. Given $z \in \mathbb{C}$, $z \neq 0$ there is a unique number $z^{-1} \in \mathbb{C}$ such that $zz^{-1} = 1$. Then $\dfrac{z_2}{z_1} = z_2 z_1^{-1}$, $z_1 \neq 0$ (division)

Important Notes

(i) Using these eight rules, complex numbers may be added, subtracted, multiplied and divided by formally manipulating them as real numbers and replacing i^2 by -1 whenever it occurs.

(ii) The usual ordering in $\mathbb{R}$, which compares positions of real numbers on the real line, cannot be extended to $\mathbb{C}$ which, geometrically speaking, consists of points in the plane. Thus, a statement such as $z_1 < z_2$ has no meaning unless z_1, $z_2 \in \mathbb{R}$.

Example 1.1

(i) $(6 + 3i) - (4 - 2i) = 2 + 5i$

(ii) $(7 + 4i)(3 - 2i) = 21 + 12i - 14i - 8i^2 = 29 - 2i$

(iii) Suppose that a solution is sought to the equation

$$5x + i(3 - y) + (x + 1) + i(2y - 1) = (3 + 2i)(1 - i)$$

By using the usual algebraic rules and replacing i^2 when it occurs by -1, the equation simplifies to

$$(6x + 1) + i(y + 2) = 5 - i$$

It then follows by the definition of equality that

$$6x + 1 = 5 \quad \text{and} \quad y + 2 = -1 \text{ so that } x = 2/3 \quad \text{and} \quad y = -3$$

(iv) Let $(2+3i)/(1-2i) = x+iy$. Then

$$\begin{aligned} & 2+3i = (1-2i)(x+iy) = (x+2y) + i(y-2x) \\ \Rightarrow\quad & x+2y = 2 \quad\text{and}\quad y-2x = 3 \\ \Rightarrow\quad & x = -4/5 \quad\text{and}\quad y = 7/5 \end{aligned}$$

In practice, the division process is straightforward, once the ideas of the modulus and complex conjugate of a complex number have been introduced.

Definitions

Let $z = x+iy \in \mathbb{C}$. The **modulus** of z is denoted and defined by $|z| = \sqrt{x^2+y^2}$. The **complex conjugate** of z is denoted and defined by $\bar{z} = x - iy$.

Notes

(i) We follow the usual convention that if a is any positive real number, then $\sqrt{a}$ denotes the positive root of a.

(ii) Unlike in the real case, $|z|^2 \neq z^2$ in general. Although $z \in \mathbb{C}$, $|z| \in \mathbb{R}$. You have been warned!

Note that $|z|$ gives a measure of the magnitude of z. In particular, $z = 0$ if and only if $|z| = 0$. Note also that z is real if and only if $\bar{z} = z$. The following two results give the important properties of $\bar{z}$ and $|z|$. The modulus has some of the properties of its real counterpart.

Lemma 1.1. Properties of the Complex Conjugate

$$\left.\begin{array}{ll}\text{(i)} & \overline{z_1+z_2} = \overline{z_1}+\overline{z_2} \\ \text{(ii)} & \overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}\end{array}\right\}\text{for all } z_1, z_2 \in \mathbb{C}$$

$$\left.\begin{array}{ll}\text{(iii)} & \bar{\bar{z}} = z \\ \text{(iv)} & z\bar{z} = |z|^2\end{array}\right\}\text{for all } z \in \mathbb{C}$$

□

Proof

The results follow directly from the definitions.

(i) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

$$\begin{aligned} & \overline{z_1+z_2} = \overline{(x_1+x_2)+i(y_1+y_2)} = (x_1+x_2) - i(y_1+y_2) \\ \Rightarrow\quad & \overline{z_1+z_2} = (x_1 - iy_1) + (x_2 - iy_2) = \overline{z_1} + \overline{z_2} \end{aligned}$$

(ii) This is similar and is left as an exercise.

(iii) This is trivial.

(iv) Let $z = x + iy$. Then

$$z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$$

■

Note that (iv) gives the factorisation of the sum of two squares. And from (ii) it follows that $\overline{-z_2} = -(\overline{z_2})$, hence (i) implies that $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$.

Lemma 1.2. Properties of the Modulus of a Complex Number

(i) $|z_1 z_2| = |z_1||z_2|$ for all $z_1, z_2 \in \mathbb{C}$

$$\left.\begin{array}{ll}\text{(ii)} & |\overline{z}| = |z| \\ \text{(iii)} & z + \overline{z} = 2\,\text{Re}\, z \leqslant 2|z| \\ \text{(iv)} & z - \overline{z} = 2\,\text{Im}\, z \leqslant 2|z|\end{array}\right\}\text{for all } z \in \mathbb{C}$$

(v) $|z_1 + z_2| \leqslant |z_1| + |z_2|$ for all $z_1, z_2 \in \mathbb{C}$ (triangle inequality) □

Proof

(i)
$$\begin{aligned} |z_1 z_2|^2 &= (z_1 z_2)(\overline{z_1 z_2}) \quad \text{by } 1.1\text{(iv)} \\ &= (z_1 \overline{z_1})(z_2 \overline{z_2}) \quad \text{by } 1.1\text{(ii)} \\ &= |z_1|^2 |z_2|^2 \quad \text{by } 1.1\text{(iv)} \end{aligned}$$

The result follows on taking positive square roots since $|z| \geqslant 0$ for all $z \in \mathbb{C}$.

(ii) This is trivial.

(iii) Let $z = x + iy$. Then

$$z + \overline{z} = (x + iy) + (x - iy) = 2x \leqslant 2\sqrt{x^2 + y^2}$$

and similarly for (iv).

(v)
$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(\overline{z_1 + z_2}) \quad \text{by } 1.1\text{(iv)} \\ &= z_1 \overline{z_1} + z_2 \overline{z_1} + z_1 \overline{z_2} + z_2 \overline{z_2} \quad \text{by } 1.1\text{(i)} \\ &= |z_1|^2 + |z_2|^2 + (z_1 \overline{z_2} + \overline{z_1 \overline{z_2}}) \quad \text{by } 1.1\text{(ii)–(iv)} \\ &\leqslant |z_1|^2 + |z_2|^2 + 2|z_1 \overline{z_2}| \quad \text{by (iii)} \\ &= (|z_1| + |z_2|)^2 \quad \text{by (i), (ii)} \end{aligned}$$

Then taking the positive square root gives the result. ■

Important Notes

(i) Let $z_1, z_2 \in \mathbb{C}$ with $z_2 \neq 0$. Then, by 1.1(iv),

$$\frac{z_1}{z_2} = \frac{z_1 \overline{z_2}}{z_2 \overline{z_2}} = \frac{z_1 \overline{z_2}}{|z_2|^2}$$

Since $|z_2|$ is real, this provides a practical method for division.

(ii) As in the real case, it follows by 1.2(v) that

$$|z_i| = |z_i \pm z_j \mp z_j| \leqslant |z_i \pm z_j| + |z_j| \quad \text{for } i, j = 1, 2 \ (i \neq j)$$

Hence

$$||z_1| - |z_2|| \leqslant |z_1 \pm z_2|$$

Example 1.2

Using the above procedure for division,

$$\frac{2+3i}{1-2i} = \frac{(2+3i)(1+2i)}{(1-2i)(1+2i)} = \frac{-4+7i}{1+4} = -\frac{4}{5} + \frac{7i}{5}$$

(Compare this with Example 1.1(iv).)

The power of the triangle inequality is illustrated in the following example.

Example 1.3 The Enestrom–Kakeya Theorem

Let $a_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$ with $a_0 > a_1 > a_2 \ldots > a_n > 0$. Prove that all the roots of the polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

satisfy $|z| > 1$.

Solution

Note that

$$a_0 = (1-z)P(z) + (a_0 - a_1)z + \ldots + (a_{n-1} - a_n)z^n + a_n z^{n+1}$$

Hence by the triangle inequality, which is clearly a strict inequality in this case, for $|z| \leqslant 1$ with $z \neq 1$,

$$|a_0| < |(1-z)P(z)| + (a_0 - a_1) + \ldots + (a_{n-1} - a_n) + a_n$$

$$\Rightarrow \quad a_0 < |(1-z)P(z)| + a_0 \quad \Rightarrow \quad |P(z)| > 0$$

and $P(1) \neq 0$, as required.

Note

For any $z \in \mathbb{C}$, z^n for $n \in \mathbb{Z}$ is defined exactly as for powers of real numbers, so the usual rules of indices apply.

Exercise 1.1.1 Let $z_1 = 2 + i$, $z_2 = 3 - 4i$ and $z_3 = 7 + 5i$. Find

(i) $z_1 - 2z_2$

(ii) $z_1 z_3 + z_2$

(iii) z_2^3

(iv) $\dfrac{z_1}{z_3}$

(v) $\dfrac{z_1 z_2}{z_1 + \bar{z}_3}$

Exercise 1.1.2 Give a simple example to show that if $z \in \mathbb{C}$, $|z|^2 \neq z^2$ in general.

Exercise 1.1.3 Find the values of

(i) $\displaystyle\sum_{n=0}^{11} i^n$

(ii) $\left|\dfrac{1}{1+3i} - \dfrac{1}{1-3i}\right|$

Exercise 1.1.4 Let

$$z = \frac{1 - \cos\theta + i\sin\theta}{1 + \cos\theta - i\sin\theta}$$

where θ is real. Show that $\operatorname{Re} z = 0$ and $\operatorname{Im} z = \tan(\theta/2)$.

Exercise 1.1.5 Find the real numbers a and b such that $z = 1 + 2i$ is a solution to the cubic equation

$$z^3 + az + b = 0$$

and find all other solutions in this case.

Exercise 1.1.6 Prove that $z_1 z_2 = 0$ if and only if $z_1 = 0$ or $z_2 = 0$.

Exercise 1.1.7 Find the roots of the equation $z^3 = 1$. If $z^3 = 1$ with $z \neq 1$, show that $z^2 + z^4 = -1$.

Exercise 1.1.8 Find the values of $z \in \mathbb{C}$ which satisfy $|z - i| \leqslant |z - 2|$.

Exercise 1.1.9 Show that for α, $\beta \in \mathbb{R}$ with $0 \leqslant \alpha \leqslant 1$ and $0 \leqslant \beta \leqslant 1$, $\alpha^2 + \beta^2 \leqslant 1 + \alpha^2\beta^2$. Hence prove that for a, $z \in \mathbb{C}$, with $|a| \leqslant 1$ and $|z| \leqslant 1$,

$$\left|\frac{z - a}{1 - \bar{a}z}\right| \leqslant 1$$

Exercise 1.1.10

(i) Find all 2×2 matrices $\boldsymbol{A}$, with real entries, which satisfy $\boldsymbol{A}^2 = -\boldsymbol{I}_2$.

(ii) Prove that if n is odd, then there is no $n \times n$ matrix $\boldsymbol{A}$ with real entries such that $\boldsymbol{A}^2 = -\boldsymbol{I}_n$.

Exercise **1.1.11** Let

$$S = \left\{ \begin{bmatrix} x & y \\ -y & x \end{bmatrix}, \ x, y \in \mathbb{R} \right\}$$

and define $f : \mathbb{C} \to S$ by

$$f(z) = f(x + iy) = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$$

Prove that $f(z_1 + z_2) = f(z_1) + f(z_2)$ and $f(z_1 z_2) = f(z_1)f(z_2) = f(z_2)f(z_1)$ for all $z_1, z_2 \in \mathbb{C}$. (S is then algebraically identical to $\mathbb{C}$.)

Exercise **1.1.12** Prove that $\mathbb{C}$ satisfies the field axioms.

The Complex Plane

Recall that real numbers may be represented as points on a line and that a complex number z is an ordered pair of real numbers. Hence such a number is represented uniquely by the point P with Cartesian coordinates (x, y), as shown in Fig. 1.1. In this way, $\mathbb{C}$ is identified with $\mathbb{R}^2$. The x-axis is the **real axis** and the y-axis is the **imaginary axis**.

Note that the length of OP gives $|z|$ and the reflection of P in the x-axis represents $\bar{z}$. More generally, if a point P_1 represents z_1 and a point P_2 represents z_2, then $|z_1 - z_2|$ is the length of the line segment P_1P_2, by Pythagoras' theorem. The complex number z can be thought of as the position vector of the point $P(x, y)$. In this way, geometrically speaking, addition and subtraction of complex numbers is equivalent to addition and subtraction of the corresponding vectors in the complex plane. The triangle inequality for complex numbers, Lemma 1.2(v), corresponds to the triangle inequality for (two-dimensional) vectors. Lastly, notice that P can be located using the usual plane polar coordinates r and θ, (Fig. 1.1), so $x + iy = r(\cos\theta + i\sin\theta)$.

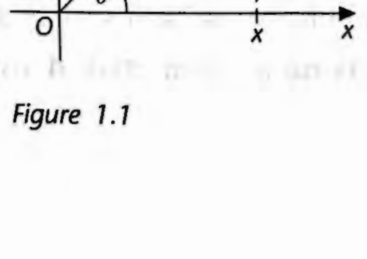


Figure 1.1

Historical Note

Complex numbers were first represented geometrically in John Wallis's *Algebra* of 1673. And in 1797 Casper Wessel published a paper describing the representation of a complex number as a point in the plane. This approach was investigated independently by Argand in 1806. In 1837 Hamilton gave the definition of a complex number as an ordered pair of real numbers. Together with some rules for their manipulation, Hamilton's definition placed complex numbers on a firm algebraic footing.

Since any point in the plane with Cartesian coordinates (x, y) can be identified with a complex number z, it is possible to translate Cartesian equations of curves in the plane to equations involving the single complex variable z, and vice versa. Perhaps the simplest and most important case is that of a circle. The following result is a consequence of the comments above.

Important Note

The equation of a circle, centred at $\alpha \in \mathbb{C}$, with radius r, is $|z - \alpha| = r$. This result can be verified algebraically by letting $z = x + iy$ and $\alpha = a + ib$.

Example 1.4

(i) Consider the equation

$$|z + 3i| = \operatorname{Im} z + 4$$

Letting $z = x + iy$ gives

$$x^2 + (y + 3)^2 = (y + 4)^2 \quad \Rightarrow \quad 2y = x^2 - 7$$

Hence the equation represents a parabola in the complex plane.

(ii) The equation $|z - 1| = 2$ is the equation of a circle centre $(1, 0)$ and radius 2. Hence the set of points satisfying $|z - 1| < 2$ is the set of points 'inside' this circle, excluding the circle itself. Thus the set of points satisfying $|z - 1| < 2$ and $|z + 1| < |z - 3|$ consists of those points (x, y) inside the given circle such that $x < 1$. This is shown as the shaded area in Fig. 1.2.

Certain curves in the complex plane are best described parametrically. This is particularly true of line segments. It follows easily using vector algebra that any point on the line passing through the endpoints of two fixed, non-parallel vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ has position vector $\boldsymbol{r} = (\boldsymbol{b} - \boldsymbol{a})t + \boldsymbol{a}$, where t is a real parameter. Hence we have the following result.

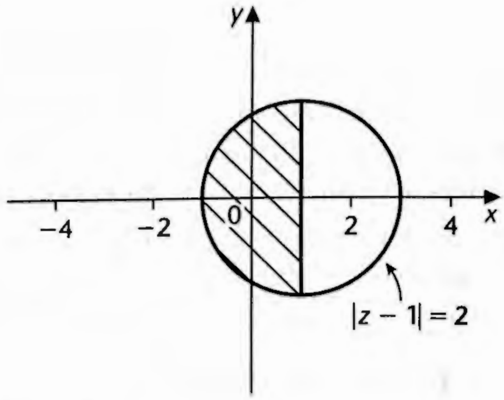


Figure 1.2

Important Note

The line segment with endpoints α and $\beta \in \mathbb{C}$ is the set of points $\{z = (1-t)\alpha + \beta t : 0 \leqslant t \leqslant 1\}$. For example, the line segment joining 1 to i is the set of points $\{z = (1-t) + it : 0 \leqslant t \leqslant 1\}$. This is equivalent to saying that the line segment has Cartesian equation $y = 1 - x$, $0 \leqslant x \leqslant 1$. In this case, if x is chosen as parameter, x runs between 1 and 0. More generally, a curve with Cartesian equation $y = f(x)$, $a \leqslant x \leqslant b$, is the set of points $\{z = t + if(t) : a \leqslant t \leqslant b\}$.

We require certain fundamental topological concepts in the complex plane for later work. Since $\mathbb{C}$ can be identified with $\mathbb{R}^2$, these are precisely the usual topological definitions in $\mathbb{R}^2$, which are natural generalisations of those in $\mathbb{R}$. Of particular importance are the ideas of open and closed neighbourhoods, which are the two-dimensional analogues of open and closed intervals of the real line. An open δ -neighbourhood of a point $\alpha \in \mathbb{C}$ is the set of all points whose distance is less than δ from α and so is the 'inside' of the circle, centre α and radius δ. A closed δ-neighbourhood includes all points on this circle as well. In a deleted δ-neighbourhood of α, the point α itself is excluded.

Definitions

Let $\alpha \in \mathbb{C}$ and $\delta \in \mathbb{R}^+$, the set of positive real numbers. An **open δ-neighbourhood (or open δ-disc) of α** is the set of points $\mathcal{N}(\alpha, \delta) = \{z : |z - \alpha| < \delta\}$. A **closed δ-neighbourhood (or closed δ-disc) of α** is the set of points $\overline{\mathcal{N}}(\alpha, \delta) = \{z : |z - \alpha| \leqslant \delta\}$. An **open deleted δ-neighbourhood (or punctured δ-disc) of α** is the set of points $\mathcal{N}'(\alpha, \delta) = \{z : 0 < |z - \alpha| < \delta\}$.

Two open neighbourhoods are pictured in Fig. 1.3. The ideas of interior, exterior and boundary points essentially correspond to our intuitive ideas.

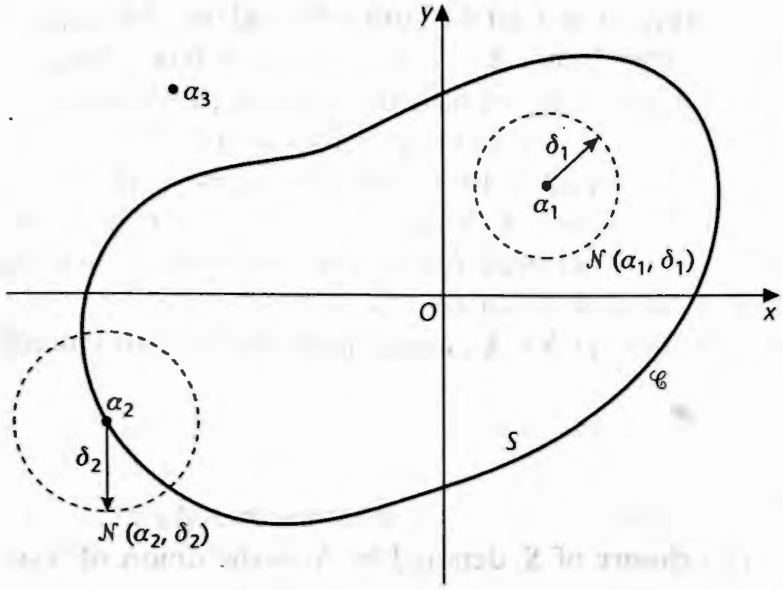


Figure 1.3

Definitions

Let $S \subseteq \mathbb{C}$. The complex number α is an **interior point** of S if there is some open δ-neighbourhood, $\mathcal{N}(\alpha, \delta)$, such that $\mathcal{N}(\alpha, \delta) \subset S$. The complex number α is a **boundary point** of S if every open neighbourhood of α contains points belonging to S and points not in S. If a point is not an interior or boundary point of S, then it is an **exterior** point of S.

For example, in Fig. 1.3, suppose that the set S consists of all points 'inside' or on the closed curve $\mathscr{C}$ shown. The point α_1 is an interior point of S, α_2 is a boundary point of S and α_3 is an exterior point of S. This is also true if S just consists of all points 'inside' $\mathscr{C}$.

In general, $S \subseteq \mathbb{C}$ need not have all three types of points. For example, a finite set has no interior points, and $\mathbb{C}$ has no exterior points.

Definition

The point $\alpha \in \mathbb{C}$ is a **limit point** of $S \subseteq \mathbb{C}$ if every deleted neighbourhood of α contains at least one point of S.

Thus, any limit point of S must have other points of S 'arbitrarily close' to it, but need not belong to S. For example, in Fig 1.3, α_1 and α_2 are limit points of S. If S excludes $\mathscr{C}$ itself, then $\alpha_2 \notin S$. It is clear that any limit point is an interior or boundary point.

Definitions

Let $S \subseteq \mathbb{C}$. Then S is **open** if every point of S is an interior point. The set S is **closed** if S contains all its limit points.

Intuitively, an open set is a set without a boundary. Any open neighbourhood is open. On the other hand, $S \subset \mathbb{C}$ is closed if it has a boundary. Any closed neighbourhood is closed. It can be deduced from the definitions that S is closed if the set $\mathbb{C}\backslash S = \{z \in \mathbb{C} : z \notin S\}$ is open. In Fig. 1.3, if S consists of the 'inside' of the closed curve $\mathscr{C}$ and $\mathscr{C}$ itself, then S is closed. If S just consists of the 'inside' of $\mathscr{C}$, then S is open. If S consists of the 'inside' of $\mathscr{C}$ and some, but not all of $\mathscr{C}$, then S is neither open nor closed. It can be shown that the only sets which are both open and closed are $\mathbb{C}$ and the empty set, $\varnothing$.

Given an arbitrary set $S \subset \mathbb{C}$, a closed set can be constructed by 'adding its boundary'.

Definition

Let $S \subseteq \mathbb{C}$. The **closure** of S, denoted by $\overline{S}$, is the union of S and the set of its limit points.

Clearly, the closure of any set is closed, and if S is closed then $\overline{S} = S$.

Definitions

A **(polygonally) connected set** $S \subseteq \mathbb{C}$ is a set for which any two points of S can be joined by a path consisting of line segments, all points of which lie in S. A **region** is a non-empty, open connected set.

Intuitively, a connected set consists of a single piece of the complex plane. The set S in Fig. 1.3 is connected. As we shall see later, it is often convenient to take the domain of a function of a complex variable to be a region of $\mathbb{C}$.

Definition

A connected set S is **simply connected** if $\mathbb{C}\backslash S$ is connected.

Basically speaking, $S \subseteq \mathbb{C}$ is simply connected if it has no 'holes'. For instance, the open neighbourhood $\{z : |z| < 2\}$ is a simply connected region, since $\{z : |z| \geqslant 2\}$ is connected, whereas the annular region $\{z : 1 < |z| < 2\}$ is not simply connected, since $A = \{z : |z| \leqslant 1 \text{ or } |z| \geqslant 2\}$ is not connected. This is demonstrated in Fig. 1.4, where there is clearly no path of line segments connecting α to β lying within A.

Definitions

A set $S \subseteq \mathbb{C}$ is **bounded** if there is a positive real constant M such that $|z| \leqslant M$ for all $z \in S$. An **unbounded** set is one which is not bounded. A set is **compact** if it is closed and bounded. Notice that any closed δ-neighbourhood is compact.

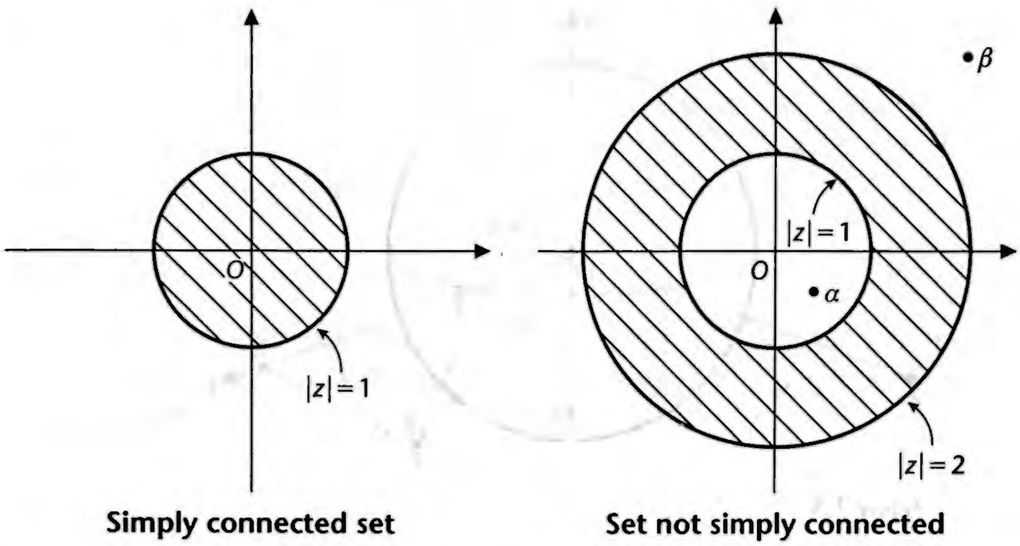


Figure 1.4

Important Note

Any closed curve in the plane which is not self-intersecting except at its end points divides the plane into two regions, one bounded, called the **inside**, and the other unbounded, called the **outside**, having the curve as a common boundary. This is the **Jordan curve theorem**.

The Riemann Sphere

Often it is useful to extend $\mathbb{C}$ by including a **point at infinity**. For example, this can be useful when considering the behaviour of complex functions as $|z| \to \infty$. This extended set, denoted by $\tilde{\mathbb{C}}$, is the **extended complex plane**. $\mathbb{C}$ can be regarded as being embedded in $\mathbb{R}^3$ by identifying $x + iy$ with $(x, y, 0)$. A geometric model for $\tilde{\mathbb{C}}$ is then the unit sphere with Cartesian equation $x^2 + y^2 + u^2 = 1$ in $\mathbb{R}^3$. A given point z in the plane may be associated with that point Z on the sphere such that the line passing through the north pole $N(0, 0, 1)$ and z intersects the sphere at Z, as shown in Fig. 1.5. Thus if $|z| < 1$, then Z lies in the southern hemisphere with the south pole $S(0, 0, -1)$ corresponding to 0; if $|z| = 1$, then Z coincides with z; if $|z| > 1$, then Z lies in the northern hemisphere, with N corresponding to the point at infinity. This geometric model for $\tilde{\mathbb{C}}$ is the **Riemann sphere** and the mapping $z \to Z$ is **stereographic projection.**

Important Note

There is only *one* point at infinity in $\tilde{\mathbb{C}}$, represented geometrically by the north pole on the Riemann sphere.

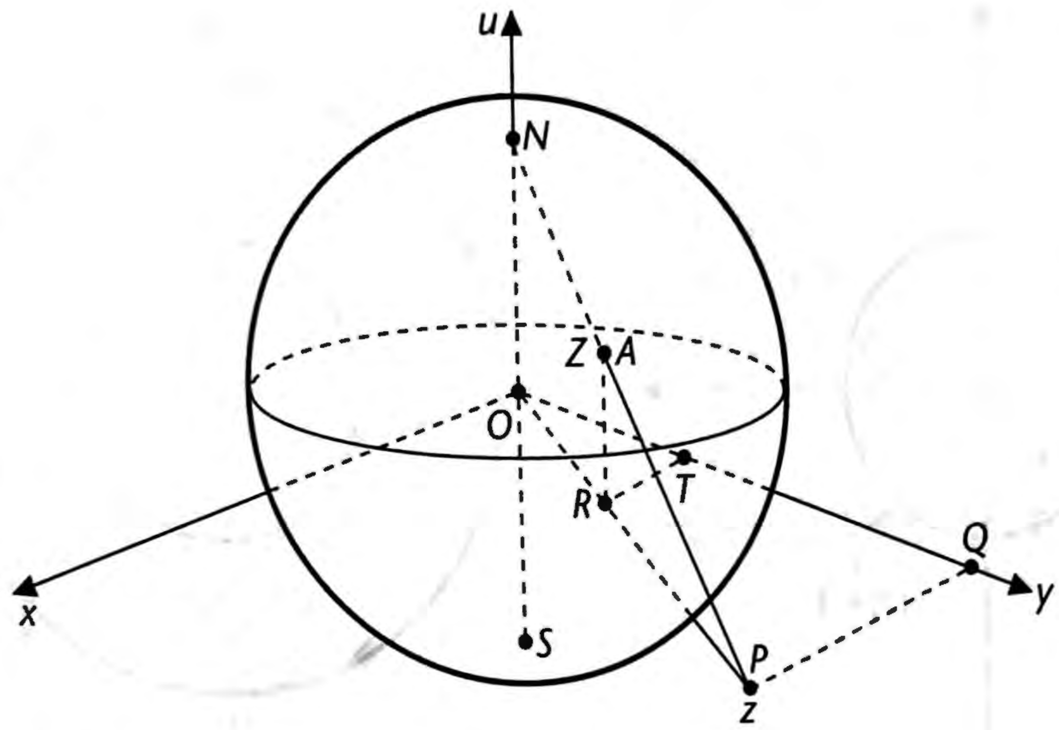


Figure 1.5

We can further investigate the correspondence between $z \in \mathbb{C}$ and its image Z on the Riemann sphere by using elementary geometry, as in the proof of the following lemma.

Lemma 1.3. Stereographic Projection

Let $Z(X, Y, U)$ be the image on the Riemann sphere of the point $z(x, y, 0)$. Then

$$\frac{x}{X} = \frac{y}{Y} = \frac{1}{1-U} \tag{1.2}$$

$$X = \frac{2x}{|z|^2+1} \qquad Y = \frac{2y}{|z|^2+1} \qquad U = \frac{|z|^2-1}{|z|^2+1} \tag{1.3}$$

□

Proof

Referring to Fig. 1.5, since triangles OPQ and ORT are similar,

$$\frac{x}{X} = \frac{y}{Y} = \frac{|z|}{\sqrt{X^2+Y^2}} \tag{1.4}$$

Since triangles ONP and RAP are also similar,

$$\frac{1}{|z|} = \frac{U}{|z| - \sqrt{X^2+Y^2}} \quad \Rightarrow \quad 1 - U = \frac{\sqrt{X^2+Y^2}}{|z|} \tag{1.5}$$

and (1.2) follows from (1.4) and (1.5). Finally, since $X^2 + Y^2 + U^2 = 1$, it follows from (1.2) that

$$|z|^2 + 1 = \frac{X^2}{(1-U)^2} + \frac{Y^2}{(1-U)^2} + 1 = \frac{2-2U}{(1-U)^2} = \frac{2}{1-U} \tag{1.6}$$

Then (1.3) follows from (1.2) and (1.6). ■

It can be shown, using Lemma 1.3, that under stereographic projection, the image of any circle in the complex plane is a circle on the Riemann sphere which does not pass through N, whereas the image of any line in the plane is a circle on the sphere which does pass through N. For example, the unit circle, centre 0, is clearly mapped to the equator on the sphere.

Example 1.5

(i) Consider the line with equation $x+y=1$ in the complex plane. It follows from (1.2) that the image of this line on the Riemann sphere is given by

$$\frac{X}{1-U}+\frac{Y}{1-U}=1 \quad \Rightarrow \quad X+Y+U=1$$

where $X^2+Y^2+U^2=1$, i.e. the intersection of a plane and the sphere, which is a circle passing through (0, 0, 1).

(ii) The image of the circle with equation $x^2+y^2=3$ in the complex plane, i.e. $|z|^2=3$, is given by $X=x/2$, $Y=y/2$ and $U=1/2$ by (1.3). Hence the image of this circle has equations $4X^2+4Y^2=3$, $U=1/2$, which is a circle on the sphere that does not pass through (0, 0, 1).

Exercise **1.2.1** Sketch the set of values of z in the complex plane for which

(i) $|z-2|<2$ and $|z-i|>2$

(ii) $\operatorname{Re} z - \operatorname{Im} z < 1$

(iii) $\left|\dfrac{z+2i}{z-2i}\right| \leqslant 3$

Exercise **1.2.2** Prove that

$$|z_1+z_2|^2+|z_1-z_2|^2=2\left(|z_1|^2+|z_2|^2\right)$$

for all $z_1, z_2 \in \mathbb{C}$. What is the geometrical interpretation of this result?

Exercise **1.2.3** Use a geometrical argument to support the fact that

$$|z-3i|+|z+3i|=12$$

is the equation of an ellipse in the complex plane. Prove this fact using an algebraic approach.

Exercise **1.2.4** Show that the equation

$$(\bar{z}+z)^2=2(\alpha\bar{z}+\bar{\alpha}z) \qquad (\alpha \in \mathbb{C})$$

represents a parabola in the complex plane if α is not real. What curve(s) does it represent if α is real?

Exercise **1.2.5** Which of the following sets of complex numbers are (a) open, (b) closed, (c) connected, (d) simply connected, (e) bounded, (f) compact? Give brief reasons for your answers.

(i) $\{z : \text{Im } z < 0\}$

(ii) $\{z : |z - \alpha| \geqslant 4\}$

(iii) $\{1, i, -1, -i\}$

(iv) $\{z : |z| < 1\} \cup \{z : |z - 2i| < 1\}$

(v) $\{z : 1 \leqslant |\text{Re } z| + |\text{Im } z| < 4\}$

(vi) $\mathbb{C}$

Exercise **1.2.6** Prove that $S \subseteq \mathbb{C}$ is closed if $\mathbb{C} \backslash S$ is open.

Exercise **1.2.7** Find the image of the circle with equation

$$(x - a)^2 + (y - b)^2 = r^2 \qquad (a, b, r \in \mathbb{R})$$

under stereographic projection.

Exercise ***1.2.8** Let A be the image of $z_1 = x_1 + iy_1$ and B the image of $z_2 = x_2 + iy_2$ on the Riemann sphere. Let $d(z_1, z_2)$ be the distance from A to B. Show that

$$d(z_1, z_2) = \begin{cases} \dfrac{2|z_1 - z_2|}{\sqrt{\left(1 + |z_1|^2\right)\left(1 + |z_2|^2\right)}} & (z_2 \neq \infty) \\[2ex] \dfrac{2}{\sqrt{1 + |z_1|^2}} & (z_2 = \infty) \end{cases}$$

The distance function d is the **chordal metric** for the Riemann sphere and can be shown to be a metric in the usual sense. Note that $d(z_1, z_2) \leqslant 2$ and so $\tilde{\mathbb{C}}$ is bounded relative to this metric. It then follows that $\mathbb{C}$ is compact with respect to d.

The Polar Form of a Complex Number

Consider any non-zero $z \in \mathbb{C}$ represented by a point $P(x, y)$ in the plane, as in Fig. 1.1. It is usual to denote $|z|$ by $r \geqslant 0$. If OP makes an angle θ, measured anticlockwise in radians, with the (positive) x-axis, then $x = r\cos\theta$ and $y = r\sin\theta$. Hence $z = x + iy$ can be expressed as

$$z = r(\cos\theta + i\sin\theta) \tag{1.7}$$

where $r = |z|$ and $\tan\theta = y/x$. This is the **polar representation** of z. This form of any complex number is particularly useful when calculating powers and roots and also gives a practical application of complex numbers to trigonometry. This form is sometimes abbreviated to $z = r\,\text{cis}\,\theta$, and in engineering texts it often appears as $r \angle \theta$.

Definitions

Any angle θ defining z in (1.7) is an **argument** of z and is written $\theta = \arg z$. We shall use the convention that the unique value of θ such that $-\pi < \theta \leqslant \pi$ is the **principal argument** of z and is denoted by Arg z.

Note

The given range of values for Arg z is not a universal convention. The reason for choosing this particular convention will become clear later.

Lemma 1.4. Multiplication and Division in Polar Form

Let $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$. Then

(i) $z_1 z_2 = r_1 r_2 \operatorname{cis} (\theta_1 + \theta_2)$

(ii) $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} \operatorname{cis} (\theta_1 - \theta_2),\ z_2 \neq 0$ □

Proof

(i)
$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2 (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)) \\ &= r_2 r_2 (\cos\ (\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \end{aligned}$$

(ii)
$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)} \\ &= \frac{r_1}{r_2}\,\frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2} \\ &= \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)) \end{aligned}$$
■

The following result is an important and elegant use of polar representation.

Theorem 1.5. De Moivre's Theorem

Let $z = r \operatorname{cis} \theta$. Then $z^n = r^n \operatorname{cis} n\theta$, for all $n \in \mathbb{N}$. □

Proof

We use the principle of mathematical induction. The result is trivially true for $n = 1$, so suppose that the result is true for $n = k$. Then

$$z^{k+1} = zz^k = (r \operatorname{cis} \theta)(r^k \operatorname{cis} k\theta) \quad \text{by assumption}$$

$$\Rightarrow \quad z^{k+1} = r^{k+1} \operatorname{cis} (k+1)\theta \qquad \text{by 1.4(i)}$$

Thus, the result is true for $n = 1$ and if true for $n = k$, it is true for $n = k + 1$. Hence the result is true for all $n \in \mathbb{N}$ by the principle of induction.

Example 1.6

Simplify $\dfrac{\left(\sqrt{3} - i\right)^{30}}{(1 + i)^{20}}$

Solution

$|\sqrt{3} - i| = 2$, $\operatorname{Arg}\left(\sqrt{3} - i\right) = -\pi/6$, $|1 + i| = \sqrt{2}$ and $\operatorname{Arg}(1 + i) = \pi/4$. Then by De Moivre's theorem and Lemma 1.4,

$$\frac{(\sqrt{3} - i)^{30}}{(1 + i)^{20}} = \frac{2^{30}(\operatorname{cis}(-\pi/6))^{30}}{2^{10}(\operatorname{cis}(\pi/4))^{20}} = \frac{2^{20}\operatorname{cis}(-5\pi)}{\operatorname{cis}(5\pi)} = 2^{20}\operatorname{cis}(-10\pi) = 2^{20}$$

The advantage of using polar form in this example is clear!

De Moivre's theorem can be applied to derive certain trigonometric identities. The method has the advantage of proving two identities at the same time. Note, however, that the proof of De Moivre's theorem itself depends explicitly on the addition formulae for sines and cosines.

Example 1.7

Use De Moivre's theorem to prove

$$\sin 4\theta = 4\sin\theta\cos\theta - 8\sin^3\theta\cos\theta \qquad (\text{for all } \theta \in \mathbb{R})$$

Solution

Let $s = \sin\theta$ and $c = \cos\theta$. By Theorem 1.5,

$$\operatorname{cis} 4\theta = \cos 4\theta + i\sin 4\theta = (\operatorname{cis}\theta)^4 = (c + is)^4$$

$$\begin{aligned} \Rightarrow \quad \cos 4\theta + i\sin 4\theta &= c^4 + 4c^3(is) + 6c^2(is)^2 + 4c(is)^3 + (is)^4 \\ &= (c^4 - 6c^2s^2 + s^4) + i(4c^3s - 4cs^3) \end{aligned}$$

since the binomial theorem clearly holds for complex numbers. Then comparing imaginary parts gives

$$\begin{aligned} \sin 4\theta &= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta \\ &= 4\cos\theta\sin\theta\,(1 - \sin^2\theta) - 4\cos\theta\sin^3\theta \\ &= 4\cos\theta\sin\theta - 8\cos\theta\sin^3\theta \end{aligned}$$

(Comparing real parts gives $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$, etc.)

Example 1.8

Prove, using De Moivre's theorem, that

$$\cos^5\theta = \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta \qquad (\text{for all } \theta \in \mathbb{R})$$

Solution

The given identity may be proved by employing the technique of the previous example to the right-hand side. However, a better way to proceed is as follows. Let $z = \text{cis}\ \theta$. Then by 1.5 and 1.4(ii),

$$z^n + z^{-n} = \text{cis}\ n\theta + \text{cis}(-n\theta) = 2\cos n\theta \qquad (\text{for all } n \in \mathbb{N})$$

Hence

$$\begin{aligned}(2\cos\theta)^5 &= (z + z^{-1})^5 \\ &= z^5 + 5z^4z^{-1} + 10z^3z^{-2} + 10z^2z^{-3} + 5zz^{-4} + z^{-5} \\ &= (z^5 + z^{-5}) + 5(z^3 + z^{-3}) + 10(z + z^{-1}) \\ &= 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta\end{aligned}$$

using the binomial theorem. The result then follows easily.

De Moivre's theorem also provides a practical way of determining the nth roots of a complex number. Let $w = z^{1/n}$ for $n \in \mathbb{N}$ so that, as in the real case, $z = w^n$. Let $z = r\,\text{cis}\ \theta$ and $w = s\,\text{cis}\ \phi$. Then by 1.5,

$$r(\cos\theta + i\sin\theta) = s^n(\cos n\phi + i\sin n\phi)$$

Comparing real and imaginary parts then gives

$$r\cos\theta = s^n\cos n\phi \qquad r\sin\theta = s^n\sin n\phi$$

Squaring and adding these two equations then gives $s = \sqrt[n]{r}$ so that $\tan\theta = \tan n\phi$, hence $\phi = (\theta + 2k\pi)/n$ for any $k \in \mathbb{Z}$. Now

$$\text{cis}\left(\frac{\theta + 2(k+n)\pi}{n}\right) = \text{cis}\left(\frac{\theta + 2k\pi}{n}\right) \qquad (\text{for any } k \in \mathbb{Z})$$

so that only n distinct values of $\text{cis}\ \phi$ exist. For simplicity, we take $k = 0, 1, \dots, n-1$ to produce these values. Hence, any complex number has exactly n nth roots.

Corollary 1.6 Roots of Complex Numbers

Let $z = r\,\text{cis}\ \theta$ where $\theta = \text{Arg}\ z$ without loss of generality. Then

$$z^{1/n} = \sqrt[n]{r}\,\text{cis}\left(\frac{\theta + 2k\pi}{n}\right) \qquad k = 0, 1, \dots n-1, \text{ for all } n \in \mathbb{N} \qquad \square$$

Example 1.9

Find the three exact cube roots of $1 - i$ in Cartesian form.

Solution

$|1 - i| = \sqrt{2}$ and $\operatorname{Arg}(1 - i) = -\pi/4$. Then by 1.6,

$$(1 - i)^{1/3} = \sqrt[6]{2}\operatorname{cis}\left(\frac{-\pi/4 + 2k\pi}{3}\right) \qquad (k = 0, 1, 2)$$

Let the three cube roots of $1 - i$ be z_0, z_1 and z_2. Using the fact tha $\pi/12 = \pi/3 - \pi/4$ and the standard trigonometric addition formulae, w obtain $\cos(\pi/12) = (\sqrt{6} + \sqrt{2})/4$ and $\sin(\pi/12) = (\sqrt{6} - \sqrt{2})/4$. Hence

$$z_0 = \sqrt[6]{2}(\cos(\pi/12) - i\sin(\pi/12)) = \frac{\sqrt[6]{2}}{4}\left((\sqrt{6} + \sqrt{2}) - i(\sqrt{6} - \sqrt{2})\right)$$

$$z_1 = \sqrt[6]{2}(\cos(7\pi/12) + i\sin(7\pi/12)) = \sqrt[6]{2}(-\sin(\pi/12) + i\cos(\pi/12))$$
$$= \frac{\sqrt[6]{2}}{4}\left((\sqrt{2} - \sqrt{6}) + i(\sqrt{2} + \sqrt{6})\right)$$

$$z_2 = \sqrt[6]{2}(\cos(5\pi/4) + i\sin(5\pi/4)) = -\sqrt[6]{2}(\cos(\pi/4) + i\sin(\pi/4))$$
$$= \frac{-(1 + i)}{\sqrt[3]{2}}$$

Example 1.10 The nth Roots of Unity

The nth roots of unity are the n roots of the equation $z^n = 1 = \operatorname{cis} 0$. Hence by 1.6, the n nth roots of unity are

$$\operatorname{cis}(2k\pi/n),\ k = 0, 1, \ldots, n - 1; \text{ that is, } 1, \omega, \omega^2, \ldots, \omega^{n-1}$$

where $\omega = \operatorname{cis}(2\pi/n)$. In the complex plane, these all lie on the unit circle $|z| = 1$, and form the vertices of a regular n-gon. Note that if $\omega \neq 1$ then

$$\omega^n = 1 \quad \Rightarrow \quad (\omega - 1)(1 + \omega + \omega^2 + \ldots + \omega^{n-1}) = 0$$
$$\Rightarrow \quad 1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0$$

Taking real parts, for instance, then gives

$$\cos(2\pi/n) + \cos(4\pi/n) + \ldots + \cos(2(n - 1)\pi/n) = -1 \qquad (\text{for all } n \in \mathbb{N})$$

Using polar representation, curves in the complex plane which are naturally defined in terms of polar coordinates can be rewritten in terms of a single complex variable. For example, the equation

$$|z|^3 = a\operatorname{Re}(z^2) \qquad (a \in \mathbb{R})$$

is equivalent to the polar equation

$$r = a\cos 2\theta$$

that is, the equation of a four-leaved rose.

Exercise **1.3.1** Find the polar representation of $\sin\theta - i\cos\theta, \theta \in \mathbb{R}$.

Exercise **1.3.2** Express the following in the form $a + ib$ by first finding the polar representation of each of the complex numbers involved:

(i) $(1+i)^{1000}$

(ii) $(1-i)^8(1+i\sqrt{3})^3$

(iii) $\dfrac{(\sqrt{3}-i)^3}{(-1+i\sqrt{3})^5}$

(iv) $27^{1/3}\, i^{-1/2}$

(v) $(-1)^{1/8}$

(vi) $(\sqrt{3}+i)^{1/4}$

Exercise **1.3.3** Find the two square roots of $3+4i$ in the form $a+ib$. Hence solve the equation

$$3z^2 + (2+7i)z + (2i-4) = 0$$

Exercise ***1.3.4** Use polar representation to prove that every complex number $z \neq -1$ of unit modulus can be expressed as

$$z = \frac{1+it}{1-it} \qquad \text{(for some } t \in \mathbb{R}\text{)}$$

Exercise **1.3.5** Use De Moivre's theorem to prove the following identities in real numbers. (You may assume that $\sin^2\theta + \cos^2\theta = 1$.)

(i) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(ii) $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

(iii) $\sin^4\theta = \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$

(iv) $\cos^8\theta + \sin^8\theta = \frac{1}{64}\cos 8\theta + \frac{7}{16}\cos 4\theta + \frac{35}{64}$

Exercise **1.3.6**

(i) Find the sixth roots of 1 and prove that they are the vertices of a regular hexagon, centre 0, in the complex plane.

*(ii) Prove that

$$\sin(\pi/n)\sin(2\pi/n)\ \ldots\ \sin((n-1)\pi/n) = n2^{1-n} \qquad (n \in \mathbb{N}, n \neq 1)$$

(*Hint*. Find the product of the non-zero roots of $(1-z)^n = 1$.)

Exercise **1.3.7** Let $k \in \mathbb{N}$. Prove that the non-negative integer powers of $\operatorname{cis}(3\pi/k)$ form a multiplicative cyclic group. Find the order of this group when (i) 3 divides k, (ii) 3 does not divide k.

Exercise **1.3.8** Show that the equation $|z+1||z-1| = 1$ is equivalent to the polar equation $r^2 = 2\cos 2\theta$, a lemniscate, if $z = r\operatorname{cis}\theta$. Sketch the curve in the complex plane.

Functions of a Complex Variable

This book is primarily concerned with functions which map A to B, where A and B are subsets of $\mathbb{C}$. The usual definitions associated with functions mapping sets to sets, which should be familiar to the reader, are given below.

Definitions

Let A and B be non-empty subsets of $\mathbb{C}$. A **function f mapping A to B** is a rule which associates with each $z \in A$, a unique $w \in B$. We write $f: A \to B$. We say w is the **image of z under f** or the **value of f at z** and write $w = f(z)$. A is the **domain** of f and B is the **codomain** of f. The **range** of f, denoted by $f(A)$, is the set of values of f. We say f is a **surjection** if $f(A) = B$; that is, the range of f is the whole of its codomain. The function f is an **injection** if $f(z_1) = f(z_2) \quad \Rightarrow \quad z_1 = z_2$, for all $z_1, z_2 \in A$; that is, distinct values of z give distinct values of w. Lastly, f is a **bijection** if it is an injection and a surjection.

Example 1.11

(i) $f: \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z^2$ is a function with domain $\mathbb{C}$. It is not an injection since $f(-1) = f(1) = 1$. The range of f is $\mathbb{C}$, so it is a surjection. On the other hand, g defined by $g(z) = z^{1/2}$ is not a function, since for every value of $z \neq 0$ there are two values of g.

(ii) $f: \mathbb{C} \to \mathbb{C}$ given by $f(z) = \bar{z}$ is a function which is a bijection. Geometrically speaking, f reflects any point in the real axis.

(iii) $f: \mathbb{C}\backslash\{0\} \to (-\pi, \pi]$ given by $f(z) = \operatorname{Arg} z$ is a function which is clearly a surjection but not an injection. For example, $f(i) = f(2i) = \pi/2$.

(iv) $f: \mathbb{C} \to \mathbb{R}$ given by $f(z) = \operatorname{Im} z$ is clearly a function which is a surjection but not an injection.

Important Conventions

(i) For simplicity, we usually take $B = \mathbb{C}$.

(ii) Most of the definitions and results concerning functions of real variables will be taken for granted and quoted without proof.

(iii) The letters x, y, u and v will always denote real variables and $z = x + iy$ and $w = u + iv$ will always denote complex variables. Thus, if $f: A \to \mathbb{C}$ and $w = f(z)$, then $u + iv = f(x + iy)$.

(iv) In later work, it is sometimes convenient to identify a function f with its values $f(z)$. With this convention, we can talk about 'the function $f(z^2)$', for example, instead of the composition of two functions, $f \circ g$, given in this case by $(f \circ g)(z) = f(g(z)) = f(z^2)$.

From a geometric point of view, functions which map $A \subseteq \mathbb{C}$ to $\mathbb{C}$ map points in the plane to points in the plane, so they map curves and regions in the z-plane to curves and regions in the w-plane. Hence such functions can be used to distort complicated curves and regions into simple curves and regions. This fact is extremely useful in applications to physical problems.

Example 1.12

Show that $f: \mathbb{C}\backslash\{0\} \to \mathbb{C}$, given by $w = f(z) = 1/z$, maps the inside of the circle $x^2 + y^2 = a^2$ in the z-plane to the outside of the circle $u^2 + v^2 = 1/a^2$ in the w-plane.

Solution

The equation of the given circle in the z-plane is $|z| = a$. Thus, along this circle, $|w| = 1/|z| = 1/a$; that is, $u^2 + v^2 = 1/a^2$. Also, $|z| < a \Rightarrow |w| > 1/a$, so that any non-zero point, A say, inside the circle $x^2 + y^2 = a^2$, gets mapped to a point A', outside the circle $u^2 + v^2 = 1/a^2$. This is shown in Fig. 1.6, which illustrates the case $a > 1$.

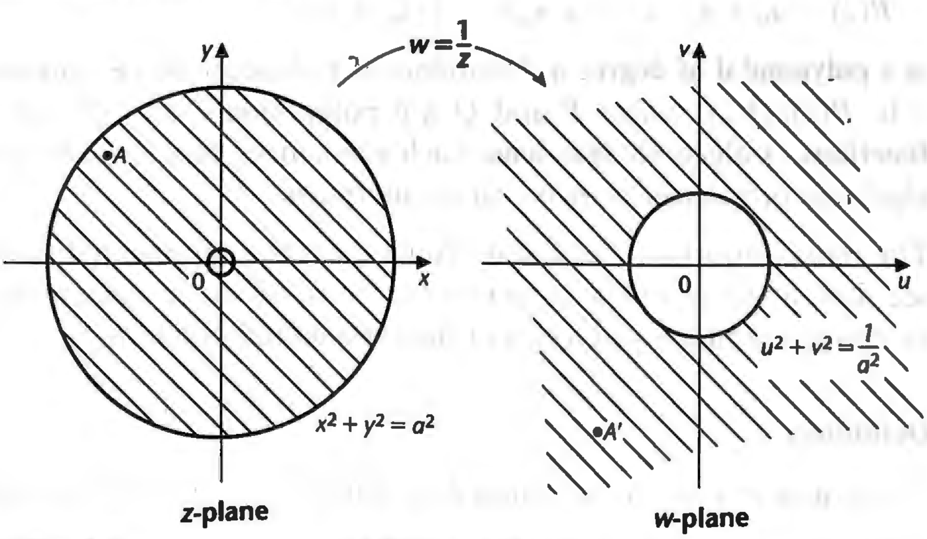


Figure 1.6

Convention

We shall adopt the usual convention that if a complex number is represented b a point A in the plane, then the image of A under some function f is denoted b A', etc.

Note that $f : A \subseteq \mathbb{C} \to \mathbb{C}$ is equivalent to a pair of real-valued function $u, v : \mathbb{R}^2 \to \mathbb{R}$; that is, if $w = u + iv = f(z) = f(x + iy)$, it is alway (theoretically) possible to find $u(x, y)$ and $v(x, y)$.

Example 1.13

Let $w = f(z) = z^3$ for all $z \in \mathbb{C}$. Then

$$w = u + iv = (x + iy)^3 = (x^3 - 3xy^2) + i(3yx^2 - y^3)$$

by the binomial theorem. Equating real and imaginary parts,

$$u(x, y) = x^3 - 3xy^2 \qquad v(x, y) = 3yx^2 - y^3$$

The Elementary Functions

We now investigate the elmentary functions that are the analogues of the usual elementary functions of real analysis. These functions have many properties in common with their real counterparts, although there are some fundamental differences.

Definitions

If $n \in \mathbb{Z}_{\geqslant 0}$, the set of non-negative integers, and $\alpha_i \in \mathbb{C}, i = 0, 1, \ldots, n$, then the function $P : \mathbb{C} \to \mathbb{C}$ defined by

$$P(z) = \alpha_0 + \alpha_1 z + \ldots + \alpha_n z^n \qquad (\alpha_n \neq 0)$$

is a **polynomial of degree *n*.** Quotients of polynomials, i.e. functions which map z to $P(z)/Q(z)$, where P and Q are polynomials with $Q(z) \not\equiv 0$, are **rational functions** or **algebraic fractions.** Such elementary functions have the same basic algebraic properties as their real counterparts.

The most important elementary function is the exponential function. We shall see that, apart from the algebraic fractions, all other elementary functions can be defined ultimately in terms of the exponential function.

Definition

The **exponential function**, denoted by $\exp : \mathbb{C} \to \mathbb{C}$, is defined for all $z \in \mathbb{C}$ by

$$\exp z = e^z = e^{x+iy} = e^x(\cos y + i \sin y)$$

Note

Although we take this as a definition, it is sometimes justified by examining the Maclaurin series expansions of the real functions involved. As a definition, it can be justified by noting that exp is the unique function that clearly includes the real exponential function as a special case, and as in the real case, is its own derivative. This will be demonstrated in the next chapter.

Historical Note

Euler stated that $e^{\sqrt{-1}x} = \cos x + \sqrt{-1}\sin x$ for real x in his *Introductio in Analysin Infinitorum* in 1748. The same statement, in a different form, was published by Cotes as early as 1714.

Important Note

Using the definition of the exponential function, the polar representation of any complex number z can be written simply as $z = re^{i\theta}$, where $r = |z|$ and $\theta = \operatorname{Arg} z$.

The exponential function obeys most of the usual rules for indices. However, unlike its real counterpart, it is periodic, which affects the rules for roots.

Lemma 1.7. Elementary Properties of the Exponential Function

(i) $e^{z+2\pi i} = e^z$ for all $z \in \mathbb{C}$

$$\left.\begin{aligned}&\text{(ii)}\quad e^{z_1+z_2} = e^{z_1}e^{z_2}\\ &\text{(iii)}\quad e^{z_1-z_2} = \frac{e^{z_1}}{e^{z_2}}\end{aligned}\right\}\text{for all } z_1, z_2 \in \mathbb{C}$$

$$\left.\begin{aligned}&\text{(iv)}\quad (e^z)^n = e^{nz}\\ &\text{(v)}\quad (e^z)^{1/n} = e^{(z+2k\pi i)/n}\\ &\qquad (k = 0, 1, \ldots, n-1)\end{aligned}\right\}\text{for all } z \in \mathbb{C}, n \in \mathbb{N}$$

□

Proof

Let $z = x + iy$. Then

$$e^{z+2\pi i} = e^{x+i(y+2\pi)} = e^x(\cos(y+2\pi) + i\sin(y+2\pi))$$

$$\Rightarrow \quad e^{z+2\pi i} = e^x(\cos y + i\sin y) = e^z$$

This proves (i). Since $e^{iy} = \operatorname{cis} y$, (ii) and (iii) are essentially a restatement of Lemma 1.4, and (iv) and (v) are essentially restatements of 1.5 and 1.6 respectively. ■

Important Note

By definition, $|e^z| = e^x$, so the range of exp is $\mathbb{C}\backslash\{0\}$. Also,

$$|e^{iy}| = 1 \qquad (\text{for any } y \in \mathbb{R})$$

Hence the circle with equation $|z - \alpha| = r$ has the parametric polar equation

$$z(\theta) = \alpha + re^{i\theta} \qquad (-\pi < \theta \leqslant \pi)$$

The exponential function has a practical application in finding certain real trigonometric sums, as the following example demonstrates.

Example 1.14

Prove that

$$1 + z + z^2 + \ldots + z^n = \frac{1 - z^{n+1}}{1 - z} \qquad (\text{for all } z \in \mathbb{C}\backslash\{1\}, n \in \mathbb{N})$$

Hence simplify

$$1 + \cos\theta\cos\phi + \cos^2\theta\cos 2\phi + \ldots + \cos^n\theta\cos n\phi \qquad (\theta, \phi \in \mathbb{R})$$

Solution

Let $S = 1 + z + z^2 + \ldots + z^n$. Then, as in the real case,

$$S - zS = (1 - z)S = 1 - z^{n+1}$$

Now let $z = \cos\theta e^{i\phi}$, so that $z^k = \cos^k\theta e^{ki\phi}$ for all $k \in \mathbb{N}$ by 1.7(iv), hence the given sum consists of powers of z when real parts are taken. Then, by the above,

$$1 + \cos\theta e^{i\phi} + \cos^2\theta e^{2i\phi} + \ldots + \cos^n\theta e^{ni\phi} = \frac{1 - \cos^{n+1}\theta e^{(n+1)i\phi}}{1 - \cos\theta e^{i\phi}}$$

Using the definition of $e^{i\phi}$ and the standard technique for complex division then gives

$$\begin{aligned}
&1 + \cos\theta\cos\phi + \cos^2\theta\cos 2\phi + \ldots + \cos^n\theta\cos n\phi \\
&= \mathrm{Re}(1 + \cos\theta e^{i\phi} + \cos^2\theta e^{2i\phi} + \ldots + \cos^n\theta e^{ni\phi}) \\
&= \mathrm{Re}\left(\frac{(1 - \cos^{n+1}\theta e^{(n+1)i\phi})(1 - \cos\theta e^{-i\phi})}{(1 - \cos\theta e^{i\phi})(1 - \cos\theta e^{-i\phi})}\right) \\
&= \frac{1 - \cos\theta\cos\phi - \cos^{n+1}\theta\cos(n+1)\phi + \cos^{n+2}\theta\cos n\phi}{1 - 2\cos\theta\cos\phi + \cos^2\theta}, \text{ etc.}
\end{aligned}$$

The hyperbolic functions are defined as in the case of real variables.

Definitions

For all $z \in \mathbb{C}$,

$$\sinh z = \tfrac{1}{2}(e^z - e^{-z}) \qquad \cosh z = \tfrac{1}{2}(e^z + e^{-z})$$

Then

$$\tanh z = \frac{\sinh z}{\cosh z} \quad (\cosh z \neq 0) \qquad \coth z = \frac{1}{\tanh z} \quad (\sinh z \neq 0)$$

$$\operatorname{sech} z = \frac{1}{\cosh z} \quad (\cosh z \neq 0) \qquad \operatorname{csch} z = \frac{1}{\sinh z} \quad (\sinh z \neq 0)$$

Since these definitions are algebraically identical to their real counterparts, the identities involving the hyperbolic functions are the same as those involving real variables. In particular, we have the following results.

Lemma 1.8. Elementary Properties of sinh and cosh

Let z, z_1 and z_2 denote any complex numbers. Then

(i) $\cosh^2 z - \sinh^2 z = 1$

(ii) $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$

(iii) $\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

(iv) $\sinh(-z) = -\sinh z$ and $\cosh(-z) = \cosh z$ □

Proof

The results follow from the definitions and the properties of e^z. We prove (i) as an example and the rest are left as an exercise.

$$\begin{aligned}\cosh^2 z - \sinh^2 z &= \tfrac{1}{4}(e^z + e^{-z})^2 - \tfrac{1}{4}(e^z - e^{-z})^2 \\ &= \tfrac{1}{4}(e^{2z} + e^{-2z} + 2 - e^{2z} - e^{-2z} + 2) \qquad \text{by 1.7} \\ &= 1\end{aligned}$$

■

It follows from the definition of $\exp : \mathbb{C} \to \mathbb{C}$ that

$$e^{iy} = \cos y + i \sin y \qquad e^{-iy} = \cos y - i \sin y \qquad (\text{for all } y \in \mathbb{R})$$

$$\Rightarrow \quad \sin y = \frac{1}{2i}(e^{iy} - e^{-iy}) \qquad \cos y = \frac{1}{2}(e^{iy} + e^{-iy}) \qquad (\text{for all } y \in \mathbb{R})$$

It is therefore natural to make the following definitions.

Definitions

For all $z \in \mathbb{C}$,

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) \qquad \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

Then

$$\tan z = \frac{\sin z}{\cos z} \quad (\cos z \neq 0) \qquad \cot z = \frac{1}{\tan z} \quad (\sin z \neq 0)$$

$$\sec z = \frac{1}{\cos z} \quad (\cos z \neq 0) \qquad \csc z = \frac{1}{\sin z} \quad (\sin z \neq 0)$$

These definitions include the corresponding trigonometric functions of real variables as special cases. All the usual algebraic identities hold, but note that it does not make sense to talk about 'the angle z' when finding $\sin z$, etc., if z is not real.

Lemma 1.9. Elementary Properties of sin and cos

Let z, z_1 and z_2 denote any complex numbers. Then

(i) $\sin^2 z + \cos^2 z = 1$

(ii) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

(iii) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

(iv) $\sin(-z) = -\sin z$ and $\cos(-z) = \cos z$ □

Proof

Once again, these identities follow directly from the definitions and the properties of e^z. We prove (ii) as an example.

$$\begin{aligned}
&\sin z_1 \cos z_2 + \cos z_1 \sin z_2 \\
&= \frac{1}{4i}(e^{iz_1} - e^{-iz_1})(e^{iz_2} + e^{-iz_2}) + \frac{1}{4i}(e^{iz_1} + e^{-iz_1})(e^{iz_2} - e^{-iz_2}) \\
&= \frac{1}{4i}(e^{i(z_1+z_2)} + e^{i(z_1-z_2)} - e^{i(z_2-z_1)} - e^{-i(z_1+z_2)} \\
&\quad + e^{i(z_1+z_2)} - e^{i(z_1-z_2)} + e^{i(z_2-z_1)} - e^{-i(z_1+z_2)}) \qquad \text{by 1.7} \\
&= \frac{1}{2i}(e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}) = \sin(z_1 + z_2)
\end{aligned}$$

■

The following correspondence follows directly from the definitions. In complex analysis, the trigonometric functions and the hyperbolic functions are not independent concepts; they are intimately related.

Lemma 1.10. Correspondence Between Trigonometric and Hyperbolic Functions

$$\sinh(iz) = i\sin z \qquad \cosh(iz) = \cos z$$
$$\sin(iz) = i\sinh z \qquad \cos(iz) = \cosh z$$

□

Hence Lemma 1.9 follows immediately from Lemma 1.8, or vice versa, using Lemma 1.10.

Word of Warning

Although functions of complex variables share many properties with their counterparts for real variables, this is not always the case. It is all too tempting to generalise familiar properties of functions of a real variable to the complex case, where they may not be true. For example, there is no real number x such that $\cosh x = 0$. On the other hand, for $z \in \mathbb{C}$, $\cosh z = 0$ if and only if $\cos(iz) = 0$ by Lemma 1.10, and it is easily shown that $\cos : \mathbb{C} \to \mathbb{C}$ has the same zeros as its real counterpart. Hence $\cosh z = 0$ if and only if $iz = (2n+1)\pi/2, n \in \mathbb{N}$.

The following result is a special case of 1.9(ii) and (iii), with $z_1 = x$ and $z_2 = iy$, and follows by 1.10. Corresponding results for sinh and cosh follow from 1.8 and 1.10.

Lemma 1.11. Real and Imaginary Parts of $\sin z$ and $\cos z$

Let $z = x + iy \in \mathbb{C}$. Then

(i) $\sin z = \sin x \cosh y + i\cos x \sinh y$

(ii) $\cos z = \cos x \cosh y - i\sin x \sinh y$ □

It follows immediately from 1.11 that the trigonometric functions are periodic with the same periods as their real counterparts. It also shows that, unlike in the real case, there exists $z \in \mathbb{C}$ such that $|\sin z| > 1$ and similarly for cos. For example, $\sin i = i\sinh 1 \Rightarrow |\sin i| = \sinh 1 > 1$. Lemma 1.11 can also be used to show that the trigonometric functions have the same zeros as their real counterparts.

Example 1.15

Show that $\cos z = 0$ if and only if $z = (n + 1/2)\pi$, for any $n \in \mathbb{Z}$, as in the case of a real variable.

Solution

Let $z = x + iy$. Then by 1.11,

$$\begin{aligned} |\cos z|^2 &= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y \\ &= \cos^2 x(1 + \sinh^2 y) + \sin^2 x \sinh^2 y \qquad \text{by 1.8} \\ &= \cos^2 x + \sinh^2 y \qquad \text{by 1.9} \end{aligned}$$

Then $\cos z = 0 \Leftrightarrow |\cos z| = 0 \Leftrightarrow \cos^2 x = -\sinh^2 y, x, y \in \mathbb{R} \Leftrightarrow$ $\cos x = \sinh y = 0 \Leftrightarrow x = (n + 1/2)\pi$ and $y = 0$, as required.

Example 1.16

Show that $\sin : \mathbb{C} \to \mathbb{C}$ maps the given points, boundary and shaded region in the z-plane, to the given points, boundary and the upper half w-plane, as shown in Fig. 1.7.

Solution

Let $w = \sin z$ with $z = x + iy$ and $w = u + iv$. By 1.11,

$$u(x, y) = \sin x \cosh y \qquad v(x, y) = \cos x \sinh y$$

Hence along the boundary line given by $x = -\pi/2$ and $y \geqslant 0$, $u = -\cosh y \leqslant -1$ and $v = 0$, with $A(-\pi/2, 0)$ mapped to $A'(-1, 0)$.

Along the boundary line given by $-\pi/2 \leqslant x \leqslant \pi/2$ and $y = 0$, $u = \sin x$ and $v = 0$, so that $-1 \leqslant u \leqslant 1$ and $v = 0$, with $O(0, 0)$ mapped to $O'(0, 0)$ and $B(\pi/2, 0)$ mapped to $B'(1, 0)$. Along the boundary line given by $x = \pi/2$ and $y \geqslant 0$, $u = \cosh y \geqslant 1$ and $v = 0$.

Finally, let $C(x, y)$ be any point in the shaded region in the z-plane, so that $-\pi/2 < x < \pi/2$ and $y > 0$. Then $v > 0$ but there is no restriction on u. Hence the shaded region in the z-plane is mapped to the upper half w-plane.

To complete our list of elementary functions, we need the complex analogue of the logarithmic function for real variables. Recall that the logarithmic function $\text{Log} : \mathbb{R}^+ \to \mathbb{R}$, is defined as the inverse of the exponential function $\exp : \mathbb{R} \to \mathbb{R}^+$, i.e.

$$\text{if} \quad x = e^y \quad \text{then} \quad y = \text{Log}\, x \qquad (\text{for all } x \in \mathbb{R}^+)$$

Clearly, $\text{Log} : \mathbb{R}^+ \to \mathbb{R}$ is a function since $\exp : \mathbb{R} \to \mathbb{R}^+$ is a bijection.

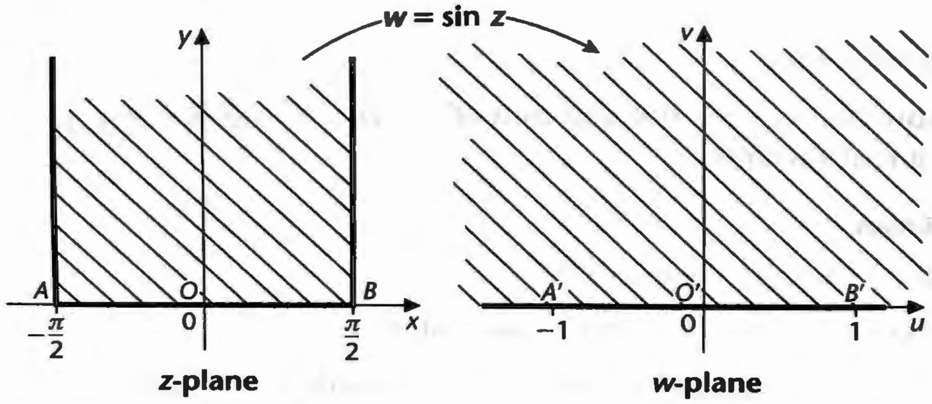


Figure 1.7

Notation

We use Log : $\mathbb{R}^+ \to \mathbb{R}$ to denote ln : $\mathbb{R}^+ \to \mathbb{R}$ in this setting. The reason for this notation will become clear.

This situation is altered in the complex case since exp : $\mathbb{C} \to \mathbb{C}\backslash\{0\}$ is not an injection by Lemma 1.7. Nevertheless, we try to mirror the real definition as far as possible.

Definition

The relation log : $\mathbb{C}\backslash\{0\} \to \mathbb{C}$ is defined by

$$w = \log z \quad \text{if} \quad z = e^w \qquad (\text{for all } z \in \mathbb{C}\backslash\{0\})$$

Note the use of lower case 'l' in this definition. Then 'log' is not a function since $\log z$ takes on an infinite number of values.

Lemma 1.12. Values of log

$$\log z = \mathrm{Log}|z| + i(\mathrm{Arg}\, z + 2k\pi) \text{ where } k \text{ is any integer} \qquad \square$$

Proof

Let $r = |z|$ and $\theta = \mathrm{Arg}\, z$, so that $z = r\,\mathrm{cis}\,\theta$. Let $z = e^w$ with $w = u + iv$. Then

$$z = e^w \quad \Rightarrow \quad r(\cos\theta + i\sin\theta) = e^u(\cos v + i\sin v)$$

Comapring real and imaginary parts and simplifying gives

$$r = e^u \text{ and } \sin\theta = \sin v \quad \Rightarrow \quad u = \mathrm{Log}\, r \text{ and } v = \theta + 2k\pi \qquad (k \in \mathbb{Z}) \qquad \blacksquare$$

Note that by 1.12, $\log(re^{i\theta}) = \mathrm{Log}\, r + i(\theta + 2k\pi)$.

Since $\log z$ has an infinite number of values, when we write $\log z$, we mean the set of values taken by $\log z$. In order to obtain a function from log, we need to restrict k to a particular value. Conventionally, this is $k = 0$ but any other fixed value of k will provide a (different) function.

Definitions

As z varies throughout $\mathbb{C}\backslash\{0\}$, each set of values of $\log z$ for a particular value of k is called a **branch** of log. That branch corresponding to $k = 0$ is called the **principal branch** of log. The **logarithmic function**, denoted by Log : $\mathbb{C}\backslash\{0\} \to C$ is defined by

$$\mathrm{Log}\, z = \mathrm{Log}|z| + i\,\mathrm{Arg}\, z \qquad (\text{for all } z \in \mathbb{C}\backslash\{0\})$$

Thus, Log is the principal branch of log and includes Log : $\mathbb{R}^+ \to \mathbb{R}$ as a special case. Since the principal branch of log is always taken when finding $\mathrm{Log}\, z$,

$\text{Log} : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ does not generally obey the same properties as its real counterpart. In particular, note that $\text{Log}(e^z) \neq z$ in general. For example $\text{Log}(e^{-i\pi}) = \text{Log}(-1) = i\pi$.

Lemma 1.13. Properties of Log

Let z, z_1 and z_2 denote any complex numbers. Then

(i) $\text{Log}(z_1 z_2) = \text{Log}\, z_1 + \text{Log}\, z_2 + 2k(z_1, z_2)\pi i$

(ii) $\text{Log}(z_1/z_2) = \text{Log}\, z_1 - \text{Log}\, z_2 + 2k(z_1, z_2)\pi i$

(iii) $\text{Log}(z^n) = n\,\text{Log}\, z + 2k(z)\pi i$ for all $n \in \mathbb{N}$

where in each case, k is a particular integer depending on z_1 and z_2 or z. □

Proof

(i) Let $|z_i| = r_i$ and $\text{Arg}\, z_i = \theta_i$, $i = 1, 2$. Then

$$z_1 z_2 = r_1 r_2 \,\text{cis}\,(\theta_1 + \theta_2) \qquad \text{by } 1.4$$

$$\Rightarrow \quad \text{Log}(z_1 z_2) = \text{Log}(r_1 r_2) + i(\theta_1 + \theta_2) + 2k(z_1, z_2)\pi i$$

$$\Rightarrow \quad \text{Log}(z_1 z_2) = (\text{Log}\, r_1 + i\theta_1) + (\text{Log}\, r_2 + i\theta_2) + 2k(z_1, z_2)\pi i$$

where $k(z_1, z_2)$ is an integer chosen so that $\theta_1 + \theta_2 + 2k\pi$ is the principal argument of $z_1 z_2$, as required.

(ii) This is similar and is left as an exercise.

(iii) Let $|z| = r$ and $\theta = \text{Arg}\, z$. Then $z^n = r^n \,\text{cis}\, n\theta$ by 1.5. Hence

$$\text{Log}(z^n) = \text{Log}(r^n) + in\theta + 2k(z)\pi i = n(\text{Log}\, r + i\theta) + 2k(z)\pi i$$

where $k(z)$ is an integer chosen so that $n\theta + 2k\pi$ is the principal argument of z^n. ■

Note

In Lemma 1.13(i) and (ii), $k(z_1, z_2)$ must take one of the values -1, 0 or 1.

Example 1.17

(i) $\text{Log}(i(i-1)) = \text{Log}(-1-i) = \text{Log}\sqrt{2} - 3\pi i/4$. Also, $\text{Log}\, i = i\pi/2$ and $\text{Log}(i-1) = \text{Log}\sqrt{2} + 3\pi i/4$. Hence

$$\text{Log}(i(i-1)) = \text{Log}\, i + \text{Log}(i-1) - 2\pi i$$

(ii) $\text{Log}(-1+i)^{10} = \text{Log}(2^5 \,\text{cis}\,(15\pi/2)) = 5\,\text{Log}\, 2 - i\pi/2$. Hence

$$\text{Log}(-1+i)^{10} = 10\,\text{Log}(-1+i) - 8\pi i$$

Having defined the complex exponential and logarithmic functions, it is easy to define complex exponents. From 1.12,

$$e^{p\log z} = \exp\,(p\,\mathrm{Log}\,r + ip\theta + 2k\pi ip) \qquad (\text{for all } p \in \mathbb{Q})$$

where $r = |z|$ and $\theta = \mathrm{Arg}\,z$. Hence

$$e^{p\log z} = r^p(\cos(p\theta + 2k\pi p) + i\sin(p\theta + 2k\pi p)) = z^p$$

by De Moivre's theorem and Corollary 1.6. Also, for any $x, a \in \mathbb{R}$, $x^a = e^{a\,\mathrm{Log}\,x}$. In order to be consistent with these results, we define z^α, where z and $\alpha \in \mathbb{C}$ as follows.

Definitions

$z^\alpha = e^{\alpha\log z}$ for all z, $\alpha \in \mathbb{C}, z \neq 0$. Such powers are clearly multivalued in general. The **principal branch** of z^α is defined by

$$z^\alpha = e^{\alpha\,\mathrm{Log}\,z} \qquad (\text{for all } z, \alpha \in \mathbb{C}, z \neq 0)$$

Convention

Unless otherwise stated, z^α will always be taken to mean the principal branch of z^α. For example,

$$i^{-i} = e^{-i\,\mathrm{Log}\,i} = e^{-i(\mathrm{Log}\,1 + i\pi/2)} = e^{\pi/2}$$

and so is real!

Historical Note

After explaining that $i^{-i} = e^{\pi/2}$ to one of his classes, Benjamin Peirce, a professor at Harvard in the nineteenth century, stated, 'Gentlemen, this is surely true, it is absolutely paradoxical, we can't understand it, and we haven't the slightest idea what the equation means, but we may be sure that it means something very important.'

Notice that if $n \in \mathbb{N}$, then the principal branch of $z^{1/n}$ is given by $\exp((\mathrm{Log}\,r + i\theta)/n) = \sqrt[n]{r}\,\mathrm{cis}\,(\theta/n)$, i.e., by $k = 0$ in 1.6. In particular, the principal value of $(-1)^{1/2} = \mathrm{cis}\,(\pi/2) = i$, with our convention of $-\pi < \mathrm{Arg}\,z \leqslant \pi$.

Note

Complex exponents obey some of the usual rules of indices, such as

$$z^{\alpha_1+\alpha_2} = z^{\alpha_1}z^{\alpha_2} \qquad z^{\alpha_1-\alpha_2} = z^{\alpha_1}/z^{\alpha_2}$$

but not all, since the exponential function does not (see Lemma 1.7 and the exercises).

The inverses of the trigonometric and hyperbolic functions are defined as f real variables. In general, if $f: A \to B$ where $A, B \subseteq \mathbb{C}$ and B is chosen to be t range of f, so that f is a surjection, then $f^{-1}: B \to A$ is defined by $f^{-1}(z) = w$ $z = f(w)$. Since f may not be an injection, f^{-1} need not be a function. Since t trigonometric and hyperbolic functions are defined in terms of the exponenti function, their inverses are expressible in terms of the relation log.

Example 1.18

The inverse of $\sin: \mathbb{C} \to \mathbb{C}$ is denoted by $\sin^{-1}: \mathbb{C} \to \mathbb{C}$ and is defined by

$$w = \sin^{-1} z \quad \text{if} \quad z = \sin w \qquad (\text{for all } z \in \mathbb{C})$$

Now if $z = \sin w$,

$$z = \frac{1}{2i}(e^{iw} - e^{-iw})$$

by definition and multiplying by $2ie^{iw}$ then gives

$$e^{2iw} - 2ize^{iw} - 1 = 0 \quad \Rightarrow \quad e^{iw} = iz + (1 - z^2)^{1/2}$$

solving the quadratic equation in e^{iw}. Hence

$$w = \sin^{-1} z = -i \log(iz + (1 - z^2)^{1/2})$$

To obtain the inverse **function**, the principal branch of log and the root ar taken. This ensures that $\sin^{-1} 0 = 0$ as in the real case. That is, the **inverse sin function** is given by

$$\sin^{-1} z = -i \operatorname{Log}(iz + (1 - z^2)^{1/2})$$

Exercise **1.4.1** For each of the following functions, express $f(z)$ in the form $u(x, y) + iv(x, y)$. Decide which of the functions are (a) an injection and (b) a surjection. Give reasons for your answers.

(i) $f: \mathbb{C} \to \mathbb{C}$ given by $f(z) = 3z + 1$

(ii) $f: \mathbb{C} \to \mathbb{C}$ given by $f(z) = z(2z - 1)$

(iii) $f: \mathbb{C}\backslash\{2/3\} \to \mathbb{C}$ given by $f(z) = \dfrac{\bar{z}}{3z + 2}$

(iv) $f: \mathbb{C}\backslash\mathbb{R}_{\geqslant 0} \to \mathbb{C}$ given by $f(z) = 1/\operatorname{Arg} \bar{z}$

(v) $f: \mathbb{C} \to \mathbb{R}$ given by $f(z) = \operatorname{Re} z$

Exercise **1.4.2** Let $w = u + iv$ and $z = x + iy = r \operatorname{cis} \theta$. Show that the transformation $w = z + 1/z$ maps the circle $x^2 + y^2 = a^2$, $a \neq 1$ in the z-plane, onto the ellipse with parametric equations $u = (a + 1/a)\cos\theta$, $v = (a - 1/a)\sin\theta$ in the w-plane.

Exercise **1.4.3** Let $w = z^{1/2}$, where $w = u + iv$ and $z = x + iy$. Find u and v as explicit functions of x and y.

Exercise **1.4.4** Let $z = x + iy$. Show that

$$\left|e^{ie^{\bar{z}}}\right| = e^{e^x \sin y}$$

Exercise **1.4.5** Use the elementary properties of the exponential function to evaluate the following sums:

(i) $\displaystyle\sum_{k=0}^{n} \sin k\theta$

(ii) $\displaystyle\sum_{k=0}^{n} \cos(\theta + k\phi)$

where $\theta, \phi \in \mathbb{R}$.

Exercise **1.4.6** Using only the definitions of the hyperbolic sine and cosine functions and properties of the exponential function, prove that

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2 \qquad (\text{for all } z_1, z_2 \in \mathbb{C})$$

(i) Use this result to prove that if $z = x + iy$, then

$$|\cosh z|^2 = \sinh^2 x + \cos^2 y$$

(ii) Find all the roots of the equation $\cosh z = 1/2$.

Exercise **1.4.7** Let $z = x + iy \in \mathbb{C}$. Find the real and imaginary parts of $\tan z$ in terms of x and y. Hence find all the roots of the equation $\tan z = 2i$.

Exercise **1.4.8** Find the images of the points $A(0, a)$, $B(\pi/2, a)$, $C(\pi, a)$, $D(3\pi/2, a)$ and $E(2\pi, a)$ in the z-plane, where $a \in \mathbb{R}^+$, under $w = \sin z$. Show that sin maps the line segment AE to an ellipse, with centre 0, in the w-plane.

Exercise **1.4.9** Find the images of the points, boundary and shaded region in the z-plane (Fig. 1.8) under $w = \operatorname{Log} z$.

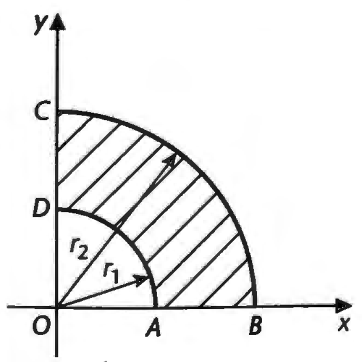


Figure 1.8

Exercise **1.4.10** Prove Lemma 1.13(ii).

Exercise **1.4.11**

(i) Prove that $z^\alpha z^\beta = z^{\alpha+\beta}$, for all z, α, $\beta \in \mathbb{C}$, $z \neq 0$. Take princi[pal] branches.

(ii) Prove that $(z_1 z_2)^\alpha = z_1{}^\alpha z_2{}^\alpha e^{2\pi i k\alpha}$, for all z_1, z_2, $\alpha \in \mathbb{C}$, $z \neq 0$, for so[me] integer k depending on z_1 and z_2. Take principal branches.

(iii) Give a simple example to show that $(z^\alpha)^\beta \neq z^{\alpha\beta}$ in general.

Exercise **1.4.12** Find all the values of $\tanh^{-1} 0$.

Exercise **1.4.13** Prove that the principal branch of $\operatorname{csch}^{-1} : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ is given by

$$\operatorname{csch}^{-1} z = \operatorname{Log}(1/z + (1/z^2 + 1)^{1/2})$$

Exercise **1.4.14** Functions of time occur in the analysis of many types of physi[cal] system. These functions may describe exponential growth or deca[y,] oscillatory behaviour, or behaviour which oscillates with an amplitude th[at] grows or decays exponentially with time. Many such functions f can b[e] defined by

$$f(t) = \operatorname{Re}(Fe^{st})$$

where $s = \sigma + i\omega \in \mathbb{C}$, is called the **complex frequency** of oscillation an[d] $F \in \mathbb{C}$. The number F is the **phasor** associated with f and is independent of [t.]

(i) Show that if $|F| = F_0$, $\operatorname{Arg} F = \theta$ and $\phi = \theta + \pi/2$, then

$$f(t) = F_0 e^{\sigma t}\cos(\theta + \omega t) = F_0 e^{\sigma t}\sin(\phi + \omega t)$$

For example, if s and $F \in \mathbb{R}$, so that $\omega = \theta = 0$, f represents exponent[ial] growth or decay. If θ, $\omega \neq 0$ and $\sigma < 0$, then f represents sinusoid[al] motion with decaying amplitude, and so on.

(ii) If f is given by $f(t) = \operatorname{Re}(Fe^{st})$, as above, prove that the phasor F, of f is unique provided that $\omega \neq 0$. What can we say when $\omega = 0$?

For an account of phasors, with applications, see the appendix to Chapte[r] in A. D. Wunsch, *Complex Variables with Applications*, 2nd edn, Addiso[n-]Wesley, 1994.

2 Differentiation and the Cauchy–Riemann Equations

This chapter introduces the limit of a function. It is a straightforward generalisation of the corresponding concept for a function of a real variable. This leads on to the ideas of continuity and derivatives, as for functions of a real variable. It is assumed that the reader is already familiar with these ideas from real analysis. The consideration of continuity also leads to the idea of branch points in the complex plane. We also give a famous characterisation of differentiable functions of a complex variable, in the form of the Cauchy–Riemann equations, which has some unexpected applications. The Cauchy–Riemann equations are of fundamental importance in complex analysis. Finally, we introduce the ideas of singular points and zeros of functions.

Limits of Functions

As in the case of real variables, to say $f(z)$ has limit ℓ as z tends to α essentially means that $f(z)$ can be made 'arbitrarily close' to ℓ by making z 'close enough' to α but distinct from it. This idea is formalised in the following definition.

Definition

Let $A \subseteq \mathbb{C}$ be an open set, $f: A \to \mathbb{C}$ and $\alpha \in \overline{A}$. Then **$f(z)$ has limit ℓ as z tends to α** if and only if, given any real $\varepsilon > 0$, there exists a real $\delta > 0$ (depending on ε) such that

$$0 < |z - \alpha| < \delta \quad \Rightarrow \quad |f(z) - \ell| < \varepsilon$$

If this definition is satisfied, we write

$$\lim_{z \to \alpha} f(z) = \ell$$

Geometrically speaking, the definition states that any open ε-neighbourhood of ℓ contains all the values of f, for z in some open δ-neighbourhood of α, except possibly for the value $f(\alpha)$. This is indicated in Fig. 2.1. In general, the smaller the given value of ε, the smaller δ will have to be.

Note

$\lim_{z\to\alpha} f(z)$ does not depend on $f(\alpha)$, since $f(\alpha)$ does not appear in the definition. Notice that α need not belong to the domain of f.

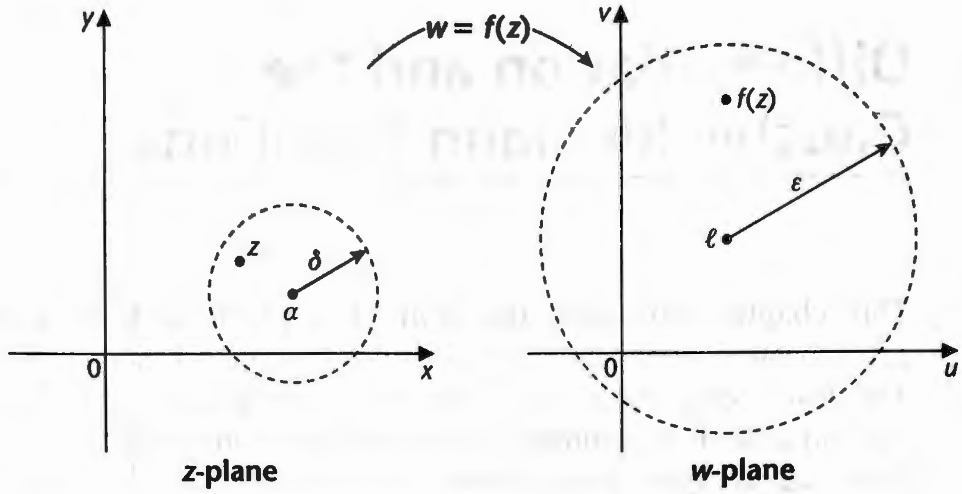


Figure 2.1

Example 2.1

(i) It follows immediately from the definition that $\lim_{z\to\alpha} z = \alpha$. Also $|\bar{z} - \bar{\alpha}| = |\overline{z - \alpha}| = |z - \alpha|$ so that it follows from the definition that $\lim_{z\to\alpha} \bar{z} = \bar{\alpha}$.

(ii) Suppose we wish to use the definition to prove that $\lim_{z\to i} (z^2 + 1) = 0$. It follows by the triangle inequality that

$$|(z^2 + 1) - 0| = |(z - i)(z - i + 2i)| \leqslant |z - i|(|z - i| + 2)$$

then $\quad 0 < |z - i| < \delta \quad \Rightarrow \quad |(z^2 + 1) - 0| < \delta(\delta + 2) < 3\delta$

as long as $\delta \leqslant 1$, so that $\delta^2 \leqslant \delta$. Then given $\varepsilon > 0$, we choose $\delta = \min(1, \varepsilon/3)$, so that for this choice of δ

$$0 < |z - i| < \delta \quad \Rightarrow \quad |(z^2 + 1) - 0| < \varepsilon$$

as required.

Important Notes

(i) With real analyis, when taking limits, there are only two directions in which a real variable can approach a real number a. But with complex analysis, in the definition above, z can approach α from any direction, along any curve.

(ii) Most of the results concerning limits of functions of a complex variable are essentially the same as the corresponding results for functions of a real variable. Since the definition is formally the same as the definition for a real variable and the triangle inequality holds for complex numbers (see Lemma 1.2(v)), the proofs of the following standard results are essentially the same as for a real variable. For this reason, these proofs are omitted.

Theorem 2.1. Properties of Limits of Functions

(i) If $\lim_{z\to\alpha} f(z)$ exists then it is unique.

Providing the following limits exist:

(ii) $\lim_{z\to\alpha}(kf(z)) = k\lim_{z\to\alpha} f(z)$ for any $k \in \mathbb{C}$

(iii) $\lim_{z\to\alpha}(f(z)+g(z)) = \lim_{z\to\alpha} f(z) + \lim_{z\to\alpha} g(z)$

(iv) $\lim_{z\to\alpha}(f(z)g(z)) = (\lim_{z\to\alpha} f(z))(\lim_{z\to\alpha} g(z))$

(v) $\lim_{z\to\alpha}\left(\dfrac{f(z)}{g(z)}\right) = \dfrac{\lim_{z\to\alpha} f(z)}{\lim_{z\to\alpha} g(z)}$, as long as $\lim_{z\to\alpha} g(z) \neq 0$ □

Note that in (v), it follows by definition that $\lim_{z\to\alpha} g(z) \neq 0$ ensures that $g(z) \neq 0$ in some deleted δ-neighbourhood of α.

Example 2.2

(i) It can be shown that $\lim_{z\to 0}(\sin z)/z$ exists. Recall that $\lim_{x\to 0}(\sin x)/x = 1$ for $x \in \mathbb{R}$. Hence letting $z \to 0$ along the positive x-axis and using 2.1(i) gives $\lim_{z\to 0}(\sin z)/z = \lim(\sin x)/x = 1$.

(ii) Using the same technique, $\lim_{z\to 0}|z|/z$ does not exist, since if $z \to 0$ along the positive x-axis, $\lim_{z\to 0}|z|/z = \lim_{x\to 0^+} x/x = 1$, whereas if $z \to 0$ along the positive y-axis, $\lim_{z\to 0}|z|/z = \lim_{y\to 0^+} y/iy = -i$, contradicting 2.1(i).

(iii) $\displaystyle \lim_{z\to i}\frac{z^2+iz+2}{z^2-3iz-2} = \lim_{z\to i}\frac{(z-i)(z+2i)}{(z-i)(z-2i)} = \lim_{z\to i}\frac{z+2i}{z-2i} = -3$

using 2.1(iii), (v).

Continuity

The concept of continuity at a point in complex analysis is again a straightforward generalisation of the corresponding concept in real analysis.

Definitions

Let $A \subseteq \mathbb{C}$ be open. A function $f: A \to \mathbb{C}$ is **continuous at** $\alpha \in A$ if and only if $\lim_{z\to\alpha} f(z) = f(\alpha)$. Function f is **continuous on a region** $\mathscr{R}$ if and only if it is continuous at every point of $\mathscr{R}$. Naively speaking, continuous functions map continuous curves in the z-plane to continuous curves in the w-plane. Once again, results concerning continuity at a point are essentially the same as those for real variables.

Theorem 2.2. Elementary Properties of Continuous Functions

(i) If $\lim_{z\to\alpha} f(z) = \ell$ and g is continuous in some open neighbourhood of ℓ, then

$$\lim_{z\to\alpha} g(f(z)) = g(\ell) = g(\lim_{z\to\alpha} f(z)) \qquad \text{(composite rule)}$$

(ii) Let $k \in \mathbb{C}$ and suppose that f and g are continuous at $\alpha \in \mathbb{C}$. Then $kf, f+g$ and fg are continuous at α, and f/g is continuous at α as long as $g(\alpha) \neq 0$.

(iii) If f and g are continuous at α and $f(\alpha)$ respectively, then the composition $g \circ f$ is continuous at α. □

It follows by Theorem 2.2 that every polynomial is continuous everywhere and, more generally, every algebraic fraction is continuous at those points at which the denominator is non-zero.

Note

As in real analysis, if a function f is continuous on a compact set, then f is bounded. This result is proved in Chapter 4 and appears as Theorem 4.5.

Example 2.3

(i) Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \bar{z}$. From Example 2.1(i), it follows that $\lim_{z\to\alpha} f(z) = \bar{\alpha} = f(\alpha)$ for any $\alpha \in \mathbb{C}$, so that f is continuous everywhere.

(ii) Let $f : \mathbb{C} \to \mathbb{C}$ be defined by

$$f(z) = \begin{cases} 1 \text{ if } |z| \text{ is rational} \\ 0 \text{ if } |z| \text{ is irrational} \end{cases}$$

It follows by definition that f is continuous at $\alpha \in \mathbb{C}$ if and only if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|z - \alpha| < \delta \quad \Rightarrow \quad |f(z) - f(\alpha)| < \varepsilon$$

Suppose that $|\alpha|$ is rational. No matter how small $\delta > 0$ is chosen, there exists z_1 with $|z_1 - \alpha| < \delta$, such that $|z_1|$ is irrational. Then

$$|f(z_1) - f(\alpha)| = |0 - 1| = 1 \not< \varepsilon \qquad (\varepsilon \leqslant 1)$$

Hence f is not continuous at α. Similarly, f is not continuous at any α for which $|\alpha|$ is irrational. Thus, f is discontinuous everywhere.

(iii) $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = \sec z = 1/\cos z$ (with f defined arbitrarily at values of z for which $\cos z = 0$, i.e. at $z = \alpha_n = (n + 1/2)\pi$, $n \in \mathbb{Z}$)) is continuous everywhere except at α_n, by 2.2(ii) and the fact that $\lim_{z\to\alpha_n} f(z)$ does not exist in $\mathbb{C}$.

(iv) The exponential function is easily shown to be continuous everywhere, so that by 2.2(i),

$$\lim_{z\to i} e^{\sin z} = e^{\lim_{z\to i} \sin z} = e^{i \sinh 1} = \cos(\sinh 1) + i \sin(\sinh 1)$$

Branch Points and Riemann Surfaces

As seen in Chapter 1, many elementary functions of a complex variable occur as specific branches of many-valued relations defined in terms of the relation arg. Such functions will be discontinuous at points along any line acting as a 'boundary' between different branches, corresponding to a discontinuity of the associated principal arguments. These somewhat arbitrary and artificial lines of discontinuity are known as branch cuts.

For example, because of our convention for Arg z, each branch of arg z is discontinuous along the line segment given by $z = x$, $x \leqslant 0$. Choosing a different convention for Arg z will produce a different line of discontinuity of each branch of arg z, emanating from the origin. Notice that 0 is the only point common to all branch cuts. Any complete continuous circuit of a closed curve, passing through $\alpha \in \mathbb{C}$ and not enclosing 0, i.e. one for which $|z|$ and arg z vary continuously, does not alter the initial value of arg α. On the other hand, upon completion of a continuous circuit of any closed curve passing through α and enclosing 0, any argument of α increases or decreases by 2π, so that arg α does not take the same value; the value of arg α now lies in a different branch since a branch cut has been crossed. This is indicated in Fig. 2.2.

This type of behaviour can be undesirable and needs to be identified when it occurs. These ideas are formalised in the following definitions.

Definitions

A many-valued relation which associates a subset of $\mathbb{C}$ with each point in some subset of $\mathbb{C}$ will be called a **multifunction**. A point $\alpha \in \mathbb{C}$ is a **branch point** of a multifunction g, defined on some open subset A of $\mathbb{C}$, if there exists at least one

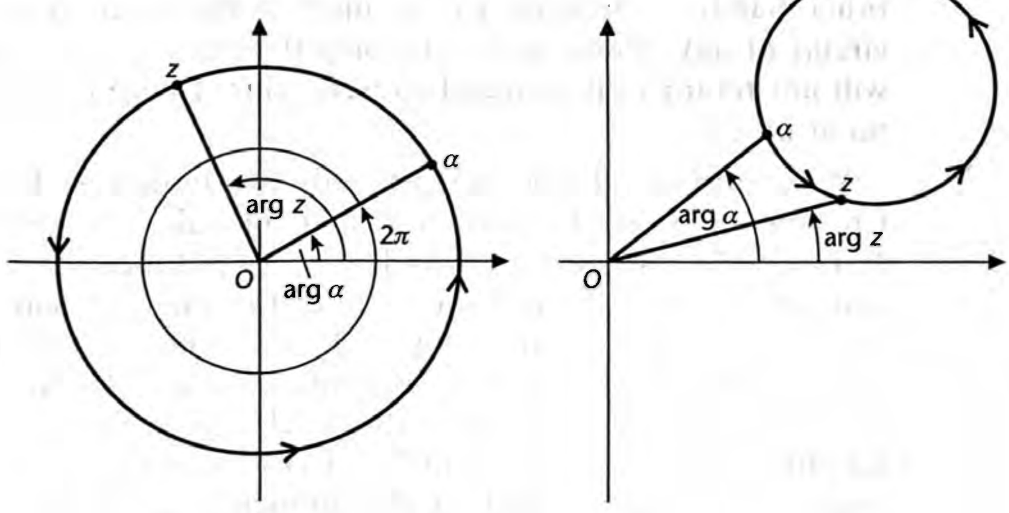


Figure 2.2

closed curve $\mathscr{C}$ enclosing α and lying in A, such that upon completion of a circuit of $\mathscr{C}$ for which any argument of $z-\alpha$ increases continuously by 2π, $g(z)$ does not return to its orginal chosen value. A **branch cut** of g is then a chosen line segment with initial point at a branch point α of g, such that there is one and only one branch cut in any neighbourhood of α containing no other branch point of g. A **branch** of g is any function with the same domain as g and obtained from g by making it single-valued, which is continuous at all points of its domain except along any chosen branch cuts of g.

Notes

When no ambiguity can occur, we shall use for multifunctions the same notation as we used for functions. The choice of branch cuts for a multifunction with branch points, set up to separate distinct branches, depends very much on a chosen convention. Crossing a branch cut has the effect of changing to a different branch of the multifunction. Notice that the image of any closed curve enclosing just one branch point of a multifunction g, under any branch of g, will not be closed, since any branch will be discontinuous along a branch cut. On the other hand, given a closed curve $\mathscr{C}$ not enclosing a branch point of g, there is a choice of convention which ensures that $\mathscr{C}$ does not cross any branch cut of g, so that the image of $\mathscr{C}$ under any branch of g is closed.

Example 2.4

(i) Consider the multifunction g defined on $\mathbb{C}$ by $g(z)=z^{1/2}$, i.e. $g(z)=\sqrt{r}e^{i\theta/2}$, where $r=|z|$ and $\theta=\arg z$. Any value of θ is not altered after a complete continuous circuit of any closed curve not enclosing 0, such as $\mathscr{C}_1$ or $\mathscr{C}_2$ shown in Fig. 2.3, so that if z is any point on the curve, $g(z)$ will return to its original chosen value. On the other hand, θ increases by 2π upon completion of any continuous circuit of any closed curve enclosing 0, such as $\mathscr{C}_3$ in Fig. 2.3, so $g(z)$ will not return to its original chosen value. Hence 0 is the only branch point of g.

With our convention, there are only two branches of g, as defined in Chapter 1. Consider the principal branch $f:\mathbb{C}\to\mathbb{C}$ defined by $f(z)=\sqrt{r}e^{i\theta/2}$, where $r=|z|$ and $\theta=\operatorname{Arg} z$. Since $-\pi<\operatorname{Arg} z\leqslant\pi$ by convention, f is discontinuous along the line segment OA given by $x\leqslant 0$, $y=0$, as shown in Fig. 2.3. This line segment is our chosen branch cut of g. Note that f is continuous at all points of $\mathscr{C}_1$ and a different convention for Arg z can be chosen so that f is continuous at all points of $\mathscr{C}_2$, i.e. the branch cut OA is moved. In these cases, the images of $\mathscr{C}_1$ and $\mathscr{C}_2$ under either branch of g are closed curves. The image of $\mathscr{C}_1$ under f is shown in Fig. 2.3. Starting at the point P, the

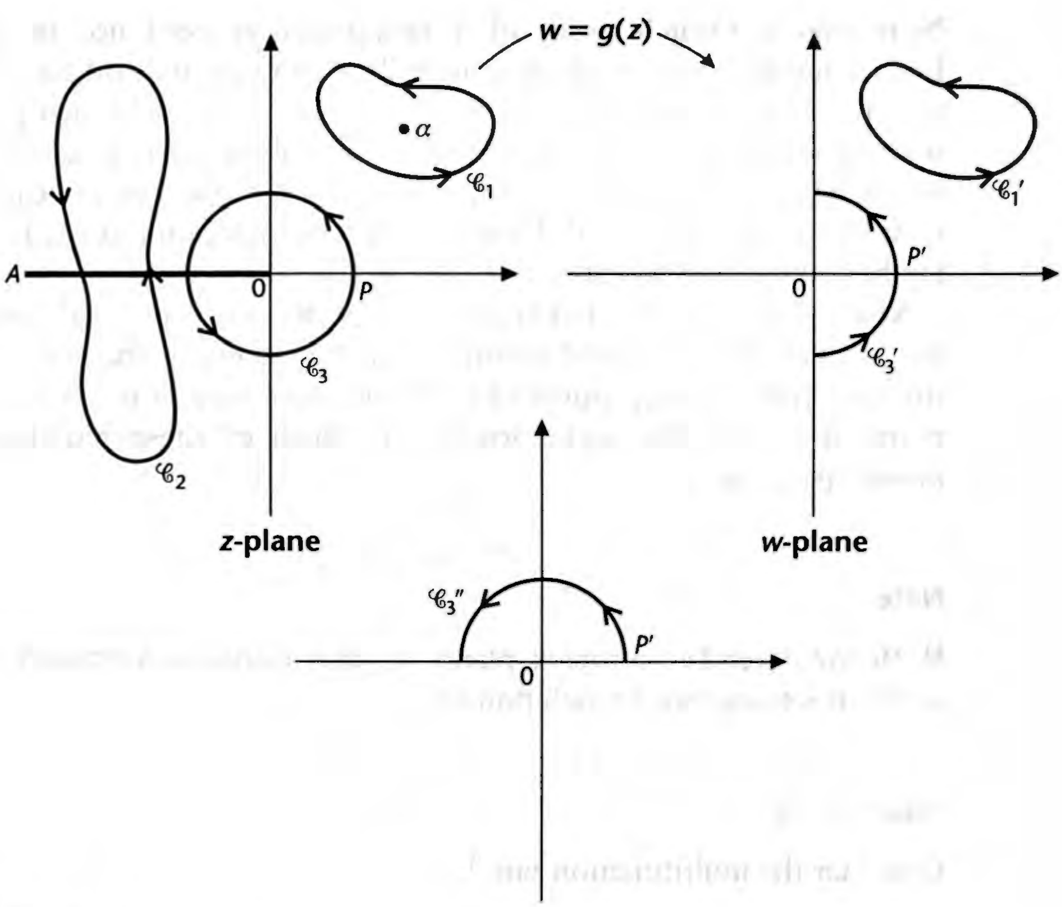


Figure 2.3

image of $\mathscr{C}_3$ under f, shown as $\mathscr{C}_3'$, is not a closed curve. A continuous image of $\mathscr{C}_3$ can be obtained by swopping branches of g when the branch cut OA is crossed; this is shown as $\mathscr{C}_3''$ in Fig. 2.3. Note that such a curve will still not be closed. Continuously traversing $\mathscr{C}_3$ twice gets us back to the orginal branch of g, so the image in this case, using both branches, will be closed.

(ii) Consider the multifunction log defined by

$$\log z = \mathrm{Log}\,|z| + i \arg z \qquad (z \neq 0)$$

(see Lemma 1.12). Once again 0 is the one and only branch point of log, and because of our convention chosen for Arg z, the line segment $x \leqslant 0$, $y = 0$ is the branch cut for log. Each branch of log, as defined in Chapter 1, is discontinous along the branch cut, so the image of any closed curve enclosing 0, under any branch of log, is not closed. In this case, since log has an infinite number of branches, no matter how many times a closed curve enclosing 0 is continuously traversed, its image can never be a closed curve, even if a different branch is employed when the branch cut is crossed.

Note that a branch point of a multifunction need not be in its domain. Extending the complex plane to include ∞, the point at infinity, as described in Chapter 1, we can investigate whether or not ∞ is a branch point of a given multifunction. For example, consider the multifunction g, given by $g(z) = z^{1/2}$, which has a branch point at 0. Letting $Z = 1/z$ gives $g(z) = G(Z) = Z^{-1/2}$ and G has a branch point at 0. Hence g has a branch point at ∞. In the same way, log has a branch point at ∞.

More generally, the multifunction given by $g(z) = (z - \alpha)^{1/n}$ where $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$ has n branches, and a simple transformation of the origin shows that α is the only finite branch point of g. In the same way, α is the only finite branch point of g given by $g(z) = \log(z - \alpha)$. Both of these multifunctions have a branch point at ∞.

Note

With the extended complex plane, we can think of a branch cut as any line segment joining two branch points.

Example 2.5

Consider the multifunction $\tan^{-1}$.

$$w = \tan^{-1} z \quad \Rightarrow \quad z = \frac{e^{2iw} - 1}{i(e^{2iw} + 1)} \quad \Rightarrow \quad e^{2iw} = \frac{1 + iz}{1 - iz}$$

$$\Rightarrow \quad w = \frac{1}{2i}\log\left(\frac{1 + iz}{1 - iz}\right)$$

$$\Rightarrow \quad 2i \tan^{-1} z = \log(i(z - i)) - \log(-i(z + i)) \qquad (z \neq \pm i)$$

(see Lemma 1.13). Hence $\tan^{-1}$ has branch points at $\pm i$ (but not at ∞ in $\tilde{\mathbb{C}}$).

Example 2.6

Define the multifunction g by $g(z) = (z^2 + 4)^{1/2}$. Show that $\pm 2i$ are branch points of g. Show also that a complete continuous circuit around any simple closed curve enclosing both points produces no change in an original chosen value of $g(z)$, on a particular branch of g, where z is any point on the closed curve. (A curve is **simple** if it has no self-intersections.) Indicate three possible branch cuts for g.

Solution

Note that $g(z) = (z + 2i)^{1/2}(z - 2i)^{1/2}$. Let $\mathscr{C}_1$ be a simple closed curve enclosing $2i$, and $\mathscr{C}_2$ be a simple closed curve enclosing $-2i$, as shown in Fig. 2.4.

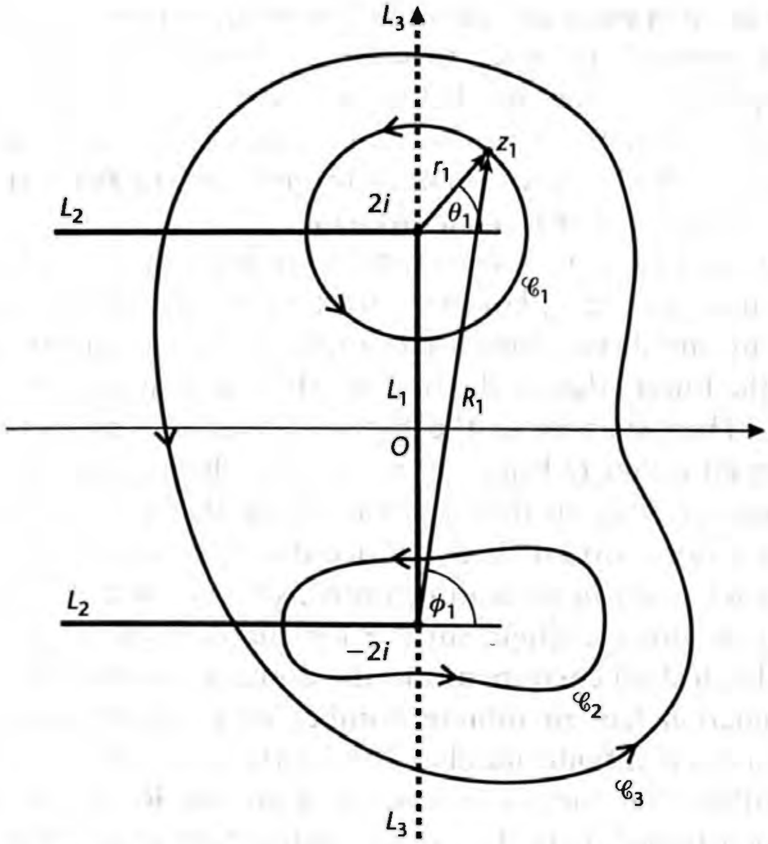


Figure 2.4

Let $z - 2i = re^{i\theta}$ and $z + 2i = Re^{i\phi}$. Choose a particular point z_1 on $\mathscr{C}_1$ and let $z_1 = r_1e^{i\theta_1} + 2i = R_1e^{i\phi_1} - 2i$. ($\theta_1$ can be any chosen value of arg $(z_1 - 2i)$ and ϕ_1 can be any chosen value of arg $(z_1 + 2i)$.) Let

$$w_1 = g(z_1) = \sqrt{r_1R_1}e^{i(\theta_1/2+\phi_1/2)}$$

As $\mathscr{C}_1$ is traversed anticlockwise once, so that θ and ϕ vary continuously, θ_1 increases by 2π whereas ϕ_1 is unchanged, so upon returning to z_1,

$$g(z_1) = \sqrt{r_1R_1}e^{i(\theta_1/2+\pi+\phi_1/2)} = -\sqrt{r_1R_1}e^{i(\theta_1/2+\phi_1/2)} = -w_1$$

Hence g does not return to its original chosen value, so $2i$ is a branch point of g. Similarly, $-2i$ is a branch point of g. Now suppose that $\mathscr{C}_3$ is a simple closed curve enclosing $2i$ and $-2i$ (Fig. 2.4). Then, using the same technique as above, if z is any point on $\mathscr{C}_3$, after one continuous circuit of $\mathscr{C}_3$, the change in arg $(z - 2i) = 2\pi$ and the change in $\arg(z + 2i) = 2\pi$, so that the change in arg $(g(z)) = 2\pi$, and $g(z)$ returns to its original chosen value, as required.

Clearly, $\pm 2i$ are the only finite branch points of g. Letting $Z = 1/z$ gives $g(z) = G(Z) = (4Z^2 + 1)^{1/2}/Z$, so that $Z = 0$ is not a branch point of G and ∞ is not branch point of g. Three possible branch cuts for g are labelled L_1, L_2 and L_3 in Fig. 2.4. In the figure, L_1 joins $-2i$ and $2i$ and passes through 0, whereas L_3 is the rest of the imaginary axis.

As we have seen, it is necessary to introduce branch cuts, which are somewhat arbitrary and artificial, in order to define a branch of a multifunction. An alternative approach is to ensure that the original multifunction is a continuous function directly, not defined just on the complex plane, but on many complex planes (sheets) attached to each other at branch cuts to form its domain. Such many-sheeted domains are **Riemann surfaces**.

For example, consider the multifunction given by $g(z) = z^{1/2}$ investigated in Example 2.4. Imagine the z-plane as consisting of two sheets superimposed on each other. 'Cut' the sheets along the branch cut OA, as shown in Fig. 2.3, and imagine that the lower edge of the bottom sheet is joined to the upper edge of the top sheet. Then starting at the bottom sheet and making one complete continuous circuit about O brings us to the top sheet. Now imagine the other cut edges joined together so that, by continuing the circuit, we pass from the top sheet back to the bottom sheet. Notice that this requires the two sheets to pass through each other in three dimensions, so it requires a little imagination! The joined sheets gives a single surface for the domain of g – the Riemann surface for g. Each sheet corresponds to the domain of a branch of g.

If a multifunction has an infinite number of branches, then the Riemann surface will have an infinite number of sheets, as in the case of log. For a particular multifunction, the construction of a suitable Riemann surface depends upon the chosen branch cuts. For the multifunction g, given in Example 2.6, choosing the branch cut L_1 shown in Fig 2.4, a possible Riemann surface for g consists of two sheets, with opposite edges joined along L_1, as described above.

Note

It is possible to give a formal topological definition of a Riemann surface but this is beyond the scope of this book. An extensive topological theory of Riemann surfaces exists and this more sophisticated approach gives a much deeper insight into multifunctions.

Historical Note

Riemann introduced the notion of a Riemann surface during the 1850s. Riemann surfaces were basic to the progress of analysis and topology in the twentieth century. For a comprehensive account of Riemann surfaces, see for example, G. Springer, *Introduction to Riemann Surfaces,* Addison-Wesley, 1957.

Exercise

2.1.1 Use the definition of a limit to prove the following results:

(i) $\lim_{z\to 0} (2z^2 + iz + i) = i$

(ii) $\lim_{z\to 1+i} z + \bar{z} = 2$

(iii) $\lim_{z\to -i} \dfrac{z+4}{z-3i} = i + 1/4$

Exercise **2.1.2** Use the definition of a limit to prove the following theorems:

(i) If $\lim_{z \to \alpha} f(z)$ exists then it is unique.

(ii) Providing the limits exist,

$$\lim_{z \to \alpha} (f(z) + g(z)) = \lim_{z \to \alpha} f(z) + \lim_{z \to \alpha} g(z)$$

Exercise **2.1.3** Find the following limits, quoting any results that you use:

(i) $\lim_{z \to 0} \frac{(z+i)^2 + 1}{z}$

(ii) $\lim_{z \to 2i} \frac{z^2 - 4iz - 4}{z^2 - 3iz - 2}$

Exercise **2.1.4** Show that the following limits do not exist, by letting $z \to 0$ along distinct paths:

(i) $\lim_{z \to 0} \frac{z}{\bar{z}}$

(ii) $\lim_{z \to 0} \left(\frac{z}{\bar{z}}\right)^2$

Exercise **2.1.5** Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \operatorname{Re} z$. Prove, using the definition, that $\lim_{z \to \alpha} f(z) = \operatorname{Re} \alpha$, so that f is continuous everywhere.

Exercise **2.1.6** Use the $\varepsilon - \delta$ definition to prove that $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = |z|^3/z, z \neq 0, f(0) = 0$, is continuous at 0.

Exercise **2.1.7** Use the properties of limits of functions to prove that if f and g are continuous at α with $g(\alpha) \neq 0$, then f/g is continuous at α.

Exercise **2.1.8** Find the image of the circle $|z| = a$ under $w = \operatorname{Log} z$. Starting at $z = a$, with the principal branch of log, find the image of $|z| = a$, traversed three times anticlockwise so that arg z increases continuously, under $w = \log z$.

Exercise **2.1.9** Locate the finite branch points of the following multifunctions:

(i) $g(z) = \frac{1}{(z^4 - 16)^{1/2}}$

(ii) $g(z) = \frac{\log(z^2 - 1)}{\cos(z^{1/2})}$

(iii) $g(z) = \coth^{-1} z$

Exercise **2.1.10**

(a) Show that $\pm i$ are branch points of the multifunction g, given by $g(z) = (z^2 + 1)^{1/n}$, $n \in \mathbb{N}$, $n > 1$, and find four different branch cuts for g.

(b) Show that 1, -2 and i are branch points of the multifunction g, given by $g(z) = (z^3 + z^2 - iz^2 - 2z - iz + 2i)^{1/2}$.

Exercise **2.1.11** Show that 0 is a branch point of g given by $g(z) = z^{1/3} - z^{1/4}$. After how many complete continuous circuits of a closed curve enclosing 0 will $g(z)$ resume its original value?

Exercise **2.1.12** Indicate three closed curves in the z-plane which cannot be deformed into each other without crossing a branch point of the multifunction g, given by $g(z) = (z^2 - 1)^{1/2}$, and which are such that when traversed continuously once, $g(z)$ resumes its initial value.

Derivatives

The definition of the derivative of a function of a complex variable is formally the same as the derivative of a function of a real variable, although in this case it does not have the same simple geometrical interpretation.

Definition

Let $A \subseteq \mathbb{C}$ be open, $f : A \to \mathbb{C}$ and $\alpha \in A$. Then the **derivative** of f at α (if it exists) is denoted and defined by

$$f'(\alpha) = \lim_{h \to 0} \frac{f(\alpha + h) - f(\alpha)}{h}$$

Function f is **differentiable** at α if and only if $f'(\alpha)$ exists.

Notes

(i) Remember that $h \in \mathbb{C}$ and can approach 0 in any direction along any path in the complex plane.

(ii) Letting $h = z - \alpha$ gives an alternative form of the definition as

$$f'(\alpha) = \lim_{z \to \alpha} \frac{f(z) - f(\alpha)}{z - \alpha}$$

(iii) As in the real case, if $w = f(z)$ we sometimes write $dw/dz = f'(z)$. Higher derivatives are defined in the usual way.

(iv) As in the case of real variables, if f is differentiable at α then it is continuous there (see Exercises 2.2).

(v) As the definition is identical in form to the definition for real variables, the following standard results can be derived from the definition by essentially the same steps used in real analysis, bearing in mind the results of Chapter 1 and the results of this chapter on limits of functions.

Theorem 2.3. Elementary Properties of Derivatives

(a) $f(z) = z^n$, $n \in \mathbb{N}$ $\Rightarrow$ $f'(z) = nz^{n-1}$ for all $z \in \mathbb{C}$.
(b) At each point where the derivatives exist:
 (i) If $h(z) = f(z) + g(z)$ then $h'(z) = f'(z) + g'(z)$.
 (ii) If $h(z) = f(z)g(z)$ then $h'(z) = f'(z)g(z) + f(z)g'(z)$.
 (iii) If $h(z) = (g \circ f)(z)$ and $w = f(z)$ then $h'(z) = g'(w)f'(z)$.
 (iv) If $h(z) = 1/f(z)$ then $h'(z) = -f'(z)/(f(z))^2$. □

Proof

The proof of each statement is identical in form to the proof of the corresponding result for a real variable. As examples, we prove (a) and (b)(ii).

(a) Using the binomial theorem gives

$$f'(z) = \lim_{h \to 0} \frac{(z+h)^n - z^n}{h}$$

$$= \lim_{h \to 0} \frac{(z^n + nz^{n-1}h + n(n-1)z^{n-2}h^2/2 + \ldots + h^n) - z^n}{h} = nz^{n-1}$$

(b) (ii) It follows from the definition that

$$h'(z) = \lim_{h \to 0} \frac{f(z+h)(g(z+h) - g(z)) + g(z)(f(z+h) - f(z))}{h}$$

$$= \lim_{h \to 0} (f(z+h)g'(z) + g(z)f'(z)) = f(z)g'(z) + g(z)f'(z)$$

by Theorem 2.1. ■

Notes

(i) It follows by 2.3(b)(ii) that if $h(z) = kf(z)$, then $h'(z) = kf'(z)$ for any $k \in \mathbb{C}$.

(ii) Letting $W = g(w)$ in 2.3(b)(iii), the result can be written as

$$\frac{dW}{dz} = \frac{dW}{dw} \cdot \frac{dw}{dz}$$

and is known as the **chain rule**, as in real calculus.

(iii) It follows by 2.3(b)(ii) and (iv) that

$$h(z) = \frac{f(z)}{g(z)} \quad \Rightarrow \quad h'(z) = \frac{g(z)f'(z) - f(z)g'(z)}{(g(z))^2}$$

at points where the derivatives exist, as in the real case. Hence by Theorem 2.3 polynomials and, more generally, algebraic fractions can be differentiated as in the real case.

Example 2.7

Use the definition of the derivative of a function to show that $f : \mathbb{C} \to \mathbb{R}$ defined by $f(z) = \operatorname{Re} z$ is nowhere differentiable.

Solution

Take any $z \in \mathbb{C}$. Since $f'(z)$ is a limit, if it exists it is unique, no matter how $h \to 0$, by 2.1(i). Let $z = x + iy$ and $h = \alpha + i\beta$, where x, y, α and β are real. Let $h \to 0$ along the real axis (in either direction), so that $\beta = 0$. Then

$$f'(z) = \lim_{h \to 0} \frac{\operatorname{Re}(z+h) - \operatorname{Re} z}{h} = \lim_{\alpha \to 0} \frac{(x+\alpha) - x}{\alpha} = 1$$

Now let $h \to 0$ along the imaginary axis, so that $\alpha = 0$. Then

$$f'(z) = \lim_{i\beta \to 0} \frac{x - x}{i\beta} = \lim_{\beta \to 0} \frac{0}{i\beta} = 0$$

This contradicts 2.1(i), so that f is nowhere differentiable.

It is easily shown that f is continuous everywhere (see Exercises 2.1). In contrast to functions mapping $\mathbb{R}$ to $\mathbb{R}$, functions mapping $\mathbb{C}$ to $\mathbb{C}$ which are continuous everywhere and differentiable nowhere are easy to find.

Analytic Functions and the Cauchy–Riemann Equations

Of great importance in complex analysis is the following result, which gives a necessary condition for differentiability. The proof uses the technique of Example 2.7.

Definitions

A function f is **analytic**, or **regular** or **holomorphic**, on a region $\mathcal{R}$ of $\mathbb{C}$ if and only if it is differentiable at each point of $\mathcal{R}$. A function f is **entire** if and only if it is analytic on $\mathbb{C}$.

Notation

Let $u : \mathbb{R}^2 \to \mathbb{R}$ and $w = u(x, y)$. We will usually write the first partial derivative of u with respect to x at (x, y) as $u_x(x, y)$ rather than as $\partial w/\partial x$. When no ambiguity can occur, we will write $u_x(x, y)$ as u_x, and similarly for y. We shall usually denote

$$\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)$$

by $u_{yx}(x, y)$ or simply by u_{yx}, and so on.

Theorem 2.4. The Cauchy-Riemann Equations

Let $f: A \subseteq \mathbb{C} \to \mathbb{C}$ be analytic on a region $\mathscr{R}$. Suppose that f is given by

$$f(z) = f(x + iy) = u(x, y) + iv(x, y) \qquad (\text{for all } z \in \mathscr{R}) \tag{2.1}$$

where u and v are real-valued functions on $\mathscr{R}$. Then the first partial derivatives of u and v, with respect to x and y, exist and satisfy the **Cauchy–Riemann equations**

$$u_x(x, y) = v_y(x, y) \qquad u_y(x, y) = -v_x(x, y) \tag{2.2}$$

at each point of $\mathscr{R}$. □

Proof

Since f is differentiable at a general point $z \in \mathscr{R}$,

$$f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$$

exists and is unique no matter how $h \to 0$, by 2.1(i). Let $h = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$ and, first of all, let $h \to 0$ along the real axis, so that $\beta = 0$. Then

$$f'(z) = \lim_{\alpha \to 0} \frac{(u(x + \alpha, y) - u(x, y)) + i(v(x + \alpha, y) - v(x, y))}{\alpha}$$

using (2.1). Hence by definition

$$f'(z) = u_x(x, y) + iv_x(x, y) \tag{2.3}$$

Now let $h \to 0$ along the imaginary axis, so that $\alpha = 0$. Note that, in this case, $h \to 0$ if and only if $\beta \to 0$. Then

$$f'(z) = \lim_{\beta \to 0} \frac{(u(x, y + \beta) - u(x, y)) + i(v(x, y + \beta) - v(x, y))}{i\beta}$$

$$\Rightarrow \quad f'(z) = u_y(x, y)/i + v_y(x, y) = v_y(x, y) - iu_y(x, y) \tag{2.4}$$

Then the result follows by equating real and imaginary parts of (2.3) and (2.4). ■

The following corollary follows directly from (2.2), (2.3) and (2.4).

Corollary 2.5. Calculation of Derivatives

If f is analytic on a region $\mathscr{R}$ and f is given by (2.1) then

$$f'(z) = u_x + iv_x = v_y - iu_y = u_x - iu_y = v_y + iv_x \qquad \text{for all } z \in \mathscr{R}$$ □

Historical Note

It is generally agreed that Cauchy was the originator of the study of functions of a complex variable. The first publication in this area appeared in 1825. Cauchy was the first mathematician to study analytic functions of a complex variable in the above sense, which he called **monogenic.** Weierstrass introduced the term 'analytic' through the study of power series expansions of such functions. Whereas Cauchy was interested in the analytical side of complex analysis, Riemann was more interested in geometrical aspects, as we have seen.

Example 2.8

(i) The function $f: \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2$ is entire by 2.3(a), with derivative $f'(z) = 2z$. Letting $z = x + iy$ and $f(z) = u + iv$,

$$\begin{aligned} &u(x,y) + iv(x,y) = (x+iy)^2 = (x^2 - y^2) + 2ixy \\ &\Rightarrow \quad u(x,y) = x^2 - y^2 \qquad v(x,y) = 2xy \\ &\Rightarrow u_x(x,y) = 2x = v_y(x,y) \qquad u_y(x,y) = -2y = -v_x(x,y) \end{aligned}$$

verifying 2.4 in this case. Also,

$$u_x + iv_x = 2(x+iy) = 2z = f'(z) \qquad (\text{for all } z \in \mathbb{C})$$

verifying 2.5.

(ii) Let $f: \mathbb{C} \to \mathbb{R}$, defined by $f(z) = \operatorname{Re} z$ be given by (2.1). Then

$$\begin{aligned} &u(x,y) = x \qquad v(x,y) = 0 \\ &\Rightarrow \quad u_x = 1, u_y = v_x = v_y = 0 \end{aligned}$$

so that (2.2) is satisfied nowhere. Hence by 2.4, f is nowhere differentiable. (Compare with Example 2.7.)

(iii) Let f be analytic in a region $\mathcal{R}$ with $f'(z) = 0$ for all $z \in \mathcal{R}$. Then, as for functions of a real variable, f is constant on $\mathcal{R}$. This is so because by hypothesis and 2.5,

$$f(x+iy) = u + iv \quad \Rightarrow \quad u_x = u_y = v_x = v_y = 0$$

so that $u(x,y)$ and $v(x,y)$ are constants. ■

Example 2.9

Show that there is no entire function f such that $f'(z) = xy^2$ for all $z \in \mathbb{C}$.

Solution

Suppose that f is entire with $f'(z) = xy^2$ at all points. Then by 2.5, $u_x = xy^2$ and $v_x = 0$, where f is given by (2.1). Then $u(x,y) = \frac{1}{2}x^2y^2 + F(y)$ and $v(x,y) = G(y)$, where F and G are arbitrary functions of y. Also, since f is entire, the Cauchy–Riemann equations are satisfied at all points, so that $v_y = xy^2$ and $u_y = 0$, and this is a contradiction.

Example 2.10 The Cauchy-Riemann Equations in Polar Form

If $w = f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ (2.5)

show that the Cauchy–Riemann equations reduce to

$$u_r(r,\theta) = \frac{1}{r} v_\theta(r,\theta),\ v_r(r,\theta) = \frac{-1}{r} u_\theta(r,\theta) \qquad (r \neq 0) \tag{2.6}$$

Hence find $f(z)$ if f is entire and $u(r,\theta) = r^2 \cos^2\theta$.

Solution

Note that $r = \sqrt{x^2+y^2}$ and $\tan\theta = y/x$. Hence by the chain rule for functions of two real variables

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{x}{r}\frac{\partial u}{\partial r} - \frac{y}{r^2}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{y}{r}\frac{\partial u}{\partial r} + \frac{x}{r^2}\frac{\partial u}{\partial \theta}$$

Similarly $$\frac{\partial v}{\partial x} = \frac{x}{r}\frac{\partial v}{\partial r} - \frac{y}{r^2}\frac{\partial v}{\partial \theta},\ \frac{\partial v}{\partial y} = \frac{y}{r}\frac{\partial v}{\partial r} + \frac{x}{r^2}\frac{\partial v}{\partial \theta}$$

Hence the Cauchy–Riemann equations give

$$\cos\theta u_r - \frac{\sin\theta}{r} u_\theta = \sin\theta v_r + \frac{\cos\theta}{r} v_\theta \tag{2.7}$$

$$\sin\theta u_r + \frac{\cos\theta}{r} u_\theta = -\cos\theta v_r + \frac{\sin\theta}{r} v_\theta \tag{2.8}$$

Mutiplying (2.7) by $\cos\theta$ and (2.8) by $\sin\theta$, and adding gives the first equation of (2.6). Multiplying (2.7) by $\sin\theta$ and (2.8) by $\cos\theta$ and subtracting gives the second equation of (2.6) as required.

If f is entire then the Cauchy–Riemann equations (2.6) are satisfied everywhere ($r \neq 0$), so $u(r,\theta) = r^2\cos 2\theta$ gives

$$\begin{aligned} & u_r = 2r\cos 2\theta \qquad u_\theta = -2r^2 \sin 2\theta \\ \Rightarrow\ & v_\theta = 2r^2\cos 2\theta \qquad v_r = 2r\sin 2\theta \\ \Rightarrow\ & v(r,\theta) = r^2\sin 2\theta + F(r) = r^2 \sin 2\theta + G(\theta) \end{aligned}$$

where F and G are arbitrary functions. Hence

$$v(r,\theta) = r^2\sin 2\theta + k \quad \Rightarrow \quad f(z) = f(re^{i\theta}) = r^2(\cos 2\theta + i\sin 2\theta) + ik$$

where k is an arbitrary real constant. The form of $f(z)$ is suggested by letting $\theta = 0$ in $f(re^{i\theta})$. This gives $f(r) = r^2 + ik$. In general,

$$f(z) = r^2(\cos\theta + i\sin\theta)^2 + ik = z^2 + ik$$

by De Moivre's theorem.

The following result is the partial converse of Theorem 2.4.

Theorem 2.6. Sufficient Conditions for Differentiability

Let a function f be given by (2.1) and suppose that the Cauchy–Riemann equations (2.2) are satisfied on a region $\mathscr{R}$, with the first-order partial derivatives of $u(x, y)$ and $v(x, y)$, with respect to x and y, continuous on $\mathscr{R}$. Then f is analytic on $\mathscr{R}$ and its derivative at any point $z \in \mathscr{R}$ is given by Corollary 2.5. □

Proof

Let $z = x + iy \in \mathscr{R}$ and $h = \alpha + i\beta$, where x, y, α and β are real. Then

$$f(z+h) - f(z) = [u(x+\alpha, y+\beta) - u(x,y)] + i[v(x+\alpha, y+\beta) - v(x,y)] \tag{2.9}$$

using (2.1). It follows by the mean value theorem for functions of one real variable that

$$\begin{aligned} u(x+\alpha, y+\beta) - u(x,y) &= [u(x+\alpha, y+\beta) - u(x, y+\beta)] \\ &\quad + [u(x, y+\beta) - u(x,y)] \\ &= \alpha u_x(x+\theta\alpha, y+\beta) + \beta u_y(x, y+\beta\psi) \end{aligned} \tag{2.10}$$

where $0 < \theta < 1$ and $0 < \psi < 1$. Now let

$$u_x(x+\theta\alpha, y+\beta) = u_x(x,y) + \varepsilon_1 \qquad u_y(x, y+\beta\psi) = u_y(x,y) + \varepsilon_2 \tag{2.11}$$

Then since u_x and u_y are continuous in $\mathscr{R}$,

$$\varepsilon_1, \varepsilon_2 \to 0 \quad \text{as} \quad (\alpha, \beta) \to (0,0)$$

independent of the chosen path. Substituting (2.11) into (2.10),

$$u(x+\alpha, y+\beta) - u(x,y) = \alpha u_x(x,y) + \beta u_y(x,y) + \alpha\varepsilon_1 + \beta\varepsilon_2 \tag{2.12}$$

Similarly,

$$v(x+\alpha, y+\beta) - v(x,y) = \alpha v_x(x,y) + \beta v_y(x,y) + \alpha\eta_1 + \beta\eta_2 \tag{2.13}$$

where $\eta_1, \eta_2 \to 0$ as $(\alpha, \beta) \to (0,0)$, independent of the chosen path. Hence from (2.9), (2.12) and (2.13), since the Cauchy–Riemann equations are satisfied in $\mathscr{R}$, it follows that

$$\begin{aligned} f(z+h) - f(z) &= (\alpha u_x - \beta v_x + \alpha\varepsilon_1 + \beta\varepsilon_2) + i(\alpha v_x + \beta u_x + \alpha\eta_1 + \beta\eta_2) \\ &= h(u_x + iv_x) + \alpha\sigma_1 + \beta\sigma_2 \text{ say} \end{aligned} \tag{2.14}$$

where $\lim_{h\to 0}\sigma_1 \lim_{h\to 0}\sigma_2 = 0$, independent of the chosen path in the complex plane. Hence, since

$$\left|\frac{\alpha\sigma_1 + \beta\sigma_2}{h}\right| \leqslant \frac{\max(|\alpha|, |\beta|)}{\sqrt{\alpha^2+\beta^2}}|\sigma_1+\sigma_2| \leqslant |\sigma_1+\sigma_2| \leqslant |\sigma_1| + |\sigma_2|$$

it follows from (2.14) that

$$f'(z) = \lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = u_x + iv_x \qquad (\text{for all } z \in \mathscr{R})$$

as required. The rest follows by 2.5. ■

Example 2.11

(i) Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = e^z$. It follows by definition that if f is given by (2.1) then

$$u(x,y) = e^x \cos y, \qquad v(x,y) = e^x \sin y$$

$$\Rightarrow u_x = e^x \cos y = v_y \qquad u_y = -e^x \sin y = -v_x$$

so that the Cauchy–Riemann equations are satisfied and the first-order partial derivatives are continuous everywhere. Hence f is entire by Theorem 2.6, and by Corollary 2.5

$$f'(z) = e^x(\cos y + i\sin y) = e^z = f(z)$$

as in the real case.

It is easy to show that exp is the unique entire function f with this property, such that $f(0) = 1$. Suppose a function f is entire and is given by (2.1) with $f(z) = f'(z)$. Then by 2.4, $u(x,y) = u_x(x,y)$ and $v(x,y) = v_x(x,y)$, so that

$$u(x,y) = e^x F(y) \quad \text{and} \quad v(x,y) = e^x G(y)$$

where $F(y) = G'(y)$ and $F'(y) = -G(y)$ by the Cauchy–Riemann equations. Hence $G''(y) = -G(y) \quad \Rightarrow \quad G(y) = \alpha\sin y + \beta\cos y$ where α and β are real constants. Then $f(z) = (\alpha + i\beta)e^z$. Using the initial condition $f(0) = 1$ gives the result.

(ii) Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \sin z$. It follows by Lemma 1.11 that if f is given by (2.1) then

$$u(x,y) = \sin x \cosh y \qquad v(x,y) = \cos x \sinh y$$

$$\Rightarrow \quad u_x = \cos x \cosh y = v_y \qquad u_y = \sin x \sinh y = -v_x$$

so that the Cauchy–Riemann equations are satisfied and the first-order partial derivatives are continuous at all points of $\mathbb{C}$. Hence by 2.6, f is differentiable for all $z \in \mathbb{C}$ and by 2.5 and 1.11,

$$f'(z) = \cos x \cosh y - i \sin x \sinh y = \cos z$$

as expected.

(iii) When investigating certain functions using Theorem 2.6, it is sometimes convenient to use alternative forms of the Cauchy–Riemann equations, such as the polar form given in (2.6). This form is particularly useful when investigating Log and associated functions.

For example, consider $f: \mathbb{C}\setminus\{0\} \to \mathbb{C}$ given by $f(z) = \operatorname{Log} z$. Let $z = re^{i\theta}$ where $r = |z|$ and $\theta = \operatorname{Arg} z$, and let f be given by (2.5). Then

$$u(r,\theta) = \operatorname{Log} r \qquad v(r,\theta) = \theta$$
$$\Rightarrow \quad u_r = \frac{1}{r} = \frac{1}{r}v_\theta \qquad v_r = 0 = \frac{-1}{r}u_\theta$$

Hence (2.6) is satisfied and the first-order partial derivatives are continuous at all points, except those given by $\operatorname{Arg} z = \pi$ and $z = 0$. Hence, by 2.6 and 2.4, f is differentiable for all $z \in \mathbb{C}$ except $z = 0$ and those points given by $\operatorname{Arg} z = \pi$, i.e. except at the branch point and along the branch cut of the associated multifunction. By 2.5, at points where $f'(z)$ exists,

$$f'(z) = u_x + iv_x = \left(\frac{x}{r}u_r - \frac{y}{r^2}u_\theta\right) + i\left(\frac{x}{r}v_r - \frac{y}{r^2}v_\theta\right)$$

using the results from Example 2.10. Then

$$f'(z) = \frac{x}{r^2} - \frac{iy}{r^2} = \frac{\cos\theta - i\sin\theta}{r} = \frac{1}{r(\cos\theta + i\sin\theta)} = \frac{1}{z}$$

Note

Theorem 2.6 is not true in general without the extra condition that the first-order partial derivatives are continuous, as the following example shows.

Example 2.12

Consider the function $f: \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \frac{x^3(1-i) + y^3(1+i)}{x^2 + y^2} \qquad (z \neq 0 \;\text{ and }\; f(0) = 0)$$

If f is given by (2.1), then

$$u(x,0) = x \qquad u(0,y) = y \qquad v(x,0) = -x \qquad v(0,y) = y$$
$$\Rightarrow \quad u_x(0,0) = v_y(0,0) = 1 \qquad u_y(0,0) = -v_x(0,0) = 1$$

so that the Cauchy–Riemann equations are satisfied at $(0,0)$. However, letting $h \to 0$ along the real axis in the definition of $f'(0)$, where $h = \alpha + i\beta$, gives

$$f'(0) = \lim_{h \to 0} \frac{f(h)}{h} = \lim_{\alpha \to 0} \frac{\alpha(1-i)}{\alpha} = 1 - i$$

Letting $h \to 0$ along the line $\alpha = \beta$ in the definition gives

$$f'(0) = \lim_{\alpha \to 0} \frac{\alpha((1-i)+(1+i))}{2\alpha(1+i)} = \frac{(1-i)}{2}$$

Hence f is not differentiable at 0. Notice that, in general,

$$u(x,y) = \frac{x^3+y^3}{x^2+y^2} \quad \text{and} \quad v(x,y) = \frac{y^3-x^3}{x^2+y^2}$$

and the first-order partial derivatives of u and v are clearly not continuous at $(0,0)$.

Important Notes

(i) It follows from Examples 2.11 and some of the previous comments, together with Theorem 2.3 and the definitions of Chapter 1, that the derivatives of the elementary functions and their inverses are formally the same as the corresponding results for functions of a real variable, at points where the functions are differentiable.

(ii) Since $z = x + iy \Rightarrow x = (z+\bar{z})/2,\ y = (z-\bar{z})/2i$, $u(x,y)$ and $v(x,y)$ in (2.1) may be formally regarded as functions of two independent variables, z and $\bar{z}$. Let $w = f(z)$. If u and v have continuous first-order partial derivatives with respect to x and y, then a necessary and sufficient condition that w is independent of $\bar{z}$ is $w_{\bar{z}} = 0$. This condition reduces to

$$w_{\bar{z}} = u_{\bar{z}} + iv_{\bar{z}} = 0 \quad \Leftrightarrow \quad u_x x_{\bar{z}} + u_y y_{\bar{z}} + i(v_x x_{\bar{z}} + v_y y_{\bar{z}}) = 0$$

$$\Leftrightarrow \quad \frac{1}{2}(u_x - v_y) + \frac{i}{2}(u_y + v_x) = 0$$

Hence this condition is satisfied if and only if the Cauchy–Riemann equations are satisfied. In other words, in any rule of association defining an analytic function, x and y can only occur in the combination $x + iy = z$.

Example 2.13

(i) Since $f(z) = e^z \Rightarrow f'(z) = e^z$ for all $z \in \mathbb{C}$, it follows by 2.3(b)(iii) that $f(z) = e^{kz} \Rightarrow f'(z) = ke^{kz}$ for all $z \in \mathbb{C}$ and any complex constant k. Now let

$$g(z) = \cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}) \qquad (\text{for all } z \in \mathbb{C})$$

Then using the above result and 2.3(b)(i), it follows that

$$g'(z) = \frac{1}{2}(ie^{iz} - ie^{-iz}) = \frac{-1}{2i}(e^{iz} - e^{-iz}) = -\sin z \qquad (\text{for all } z \in \mathbb{C})$$

Similarly $\dfrac{d}{dz}(\sin z) = \cos z$

Then if $h(z) = \tan z = \dfrac{\sin z}{\cos z}$

it follows by 2.3(b)(ii) and (iv) that

$$h'(z) = \frac{\cos^2 z + \sin^2 z}{\cos^2 z} = \sec^2 z \qquad (z \neq (2n+1)\pi/2, n \in \mathbb{Z})$$

and so on.

(ii) Let $f: \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \sinh^{-1} z$. Using the process demonstrated in Chapter 1, $f(z) = \operatorname{Log}(z + (z^2+1)^{1/2})$. Then

$$f'(z) = \frac{1 + z(z^2+1)^{-1/2}}{z + (z^2+1)^{1/2}} = \frac{((z^2+1)^{1/2} + z)((z^2+1)^{1/2} - z)}{(z(z^2+1)^{1/2} + z^2 + 1)((z^2+1)^{1/2} - z)}$$

$$\Rightarrow f'(z) = \frac{1}{(z^2+1)^{3/2} - z^2(z^2+1)^{1/2}} = \frac{1}{(z^2+1)^{1/2}} \qquad (z \neq \pm i)$$

using the result of Example 2.11(iii) and Theorem 2.3.

Alternatively, let $w = f(z) = \sinh^{-1} z$ so that $z = g(w) = \sinh w$. Then by (i) above, Theorem 2.3 and the results of Chapter 1,

$$g(w) = \frac{1}{2}(e^w - e^{-w}) \quad \Rightarrow \quad g'(w) = \frac{dz}{dw} = \frac{1}{2}(e^w + e^{-w}) = \cosh w$$

$$\Rightarrow \quad f'(z) = \frac{dw}{dz} = \left(\frac{dz}{dw}\right)^{-1} = \frac{1}{\cosh w} = \frac{1}{(\sinh^2 w + 1)^{1/2}} = \frac{1}{(z^2+1)^{1/2}}$$

Harmonic Functions

Let a function $f: A \subseteq \mathbb{C} \to \mathbb{C}$ be given by (2.1) and suppose that f is analytic in a region $\mathcal{R} \subseteq \mathbb{C}$, with the first- and second-order partial derivatives of $u(x,y)$ and $v(x,y)$ continuous in $\mathcal{R}$. Then, by Theorem 2.4,

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0 \qquad v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0$$

that is, $u(x,y)$ and $v(x,y)$ are both solutions of **Laplace's equation,**

$$\nabla^2\phi = \phi_{xx} + \phi_{yy} = 0 \tag{2.15}$$

in the region $\mathcal{R}$. (In fact, it will be shown in the next chapter that if f is analytic in $\mathcal{R}$, then the partial derivatives of $u(x,y)$ and $v(x,y)$, of all orders, exist and are continuous in $\mathcal{R}$.)

For example, $f: \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^3$ is entire and

$$z = x + iy \quad \Rightarrow \quad f(z) = (x^3 - 3xy^2) + i(3x^2y - y^3)$$
$$\Rightarrow \quad u(x,y) = x^3 - 3xy^2 \qquad v(x,y) = 3x^2y - y^3$$

It is easily checked that u and v satisfy Laplace's equation throughout the plane.

Laplace's equation is of the greatest importance in practical applications in many different areas of physics, such as in finding steady temperatures, electrostatic potentials, gravitational potentials in Newtonian theory, and in steady fluid flow.

Definitions

A function $\phi(x,y)$ is **harmonic** in a region $\mathcal{R}$ if it has continuous second-order partial derivatives and satisfies Laplace's equation (2.15) in $\mathcal{R}$. A function $v(x,y)$ is a **harmonic conjugate** of a harmonic function $u(x,y)$ if u and v satisfy the Cauchy–Riemann equations (2.2).

It follows that any number of harmonic functions can be generated simply by writing down any number of analytic functions and finding their real and imaginary parts. Note that any linear combination of harmonic functions is harmonic and that a constant is harmonic.

Example 2.14

Verify that $u(x,y) = \exp(x^2 - y^2)\cos 2xy$ is harmonic and find its harmonic conjugate. If $f: \mathbb{C} \to \mathbb{C}$ is entire and is given by (2.1), with $f(0) = 1$, find $f(z)$.

Solution

$$u_x = 2e^{(x^2-y^2)}(x\cos 2xy - y\sin 2xy)$$
$$\Rightarrow \quad u_{xx} = 2e^{(x^2-y^2)}((2x^2 - 2y^2 + 1)\cos 2xy - 4xy\sin 2xy)$$
$$u_y = -2e^{(x^2-y^2)}(y\cos 2xy + x\sin 2xy) \quad \Rightarrow \quad u_{yy} = -u_{xx}$$

as required. Let $v(x,y)$ be a harmonic conjugate of $u(x,y)$. Then

$$v_x = -u_y \quad \text{and} \quad v_y = u_x$$
$$\Rightarrow \quad v_x = 2e^{(x^2-y^2)}(y\cos 2xy + x\sin 2xy)$$
$$v_y = 2e^{(x^2-y^2)}(x\cos 2xy - y\sin 2xy)$$
$$\Rightarrow \quad v_x = \left(e^{(x^2-y^2)}\right)_x \sin 2xy + e^{(x^2-y^2)}(\sin 2xy)_x$$
$$\Rightarrow \quad v(x,y) = e^{(x^2-y^2)}\sin 2xy + F(y)$$

where $F(y)$ is an arbitrary function of y. Note that this solution for $v(x,y)$ is suggested by the form of $u(x,y)$. Then from the expression for v_y, it follows that $F'(y) = 0$ and so

$$f(x+iy) = u + iv = e^{(x^2-y^2)}(\cos 2xy + i(\sin 2xy + k))$$

where k is a real constant. Since $f(0) = 1, k = 0$. The form of $f(z)$ is suggested by taking $y = 0$, say, in the above expression. When $y = 0$, $f(x) = \exp x^2$ and this suggests that $f(z) = \exp z^2$ in general, as is indeed the case. From the expression above,

$$f(z) = e^{(x^2-y^2)}e^{2ixy} = e^{(x+iy)^2} = e^{z^2}$$

Example 2.15

From Example 2.11(iii), recall that $f\colon \mathbb{C}\backslash\{0\} \to \mathbb{C}$ given by $f(z) = \mathrm{Log}\, z$ is analytic for all z such that $\mathrm{Arg}\, z \neq \pi$ and $z \neq 0$. Hence $u(x,y) = \mathrm{Re}(f(z)) = \mathrm{Log}|z| = \mathrm{Log}\, r$, where $r^2 = x^2 + y^2$ is harmonic except at $r = 0$, as is easily checked. Then, more generally, $u(x,y) = a\mathrm{Log}\, r + b$, where a and b are real constants, is harmonic, except at $r = 0$.

Suppose we wish to find the steady temperature of water bounded by two concentric cylinders of radii 1 and 2. The inner cylinder is packed in ice and kept at 0 °C, while the outer cylinder is heated to K °C. The temperature does not depend on the length of the cylinders, so we can take a cross-section of the problem (Fig. 2.5). The steady temperature T is unchanging with time, so $T = T(x,y)$. It can be shown that T satisfies Laplace's equation (2.15) and that there is a unique solution to the problem. We thus require a harmonic function, $T(x,y)$, which satisfies the boundary conditions, $T = 0$ when $r = 1$ and $T = K$ when $r = 2$, as shown in Fig. 2.5.

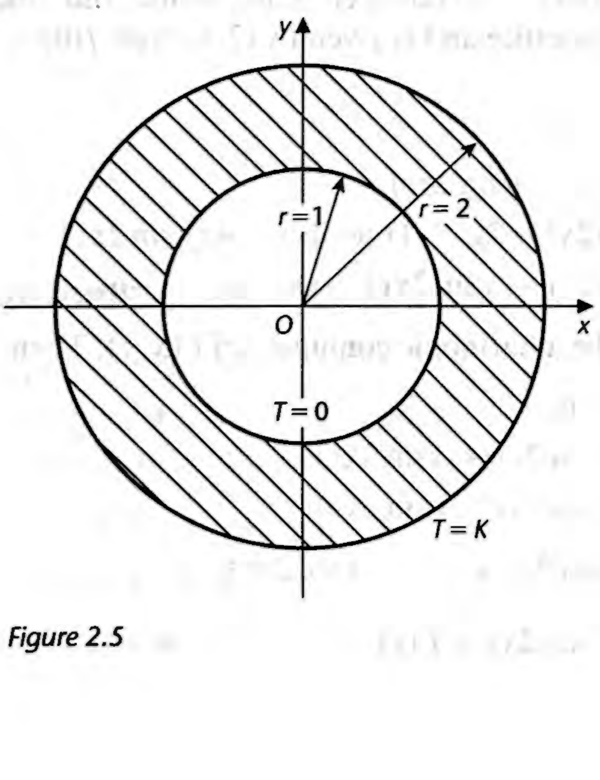


Figure 2.5

Since the boundaries are circles and the region excludes $r = 0$, the obvious choice for the solution is $T(x,y) = T(r) = a\,\mathrm{Log}\,r + b$ where a and b are constants. $T(r=1) = 0 \quad \Rightarrow \quad b = 0$ and $T(r=2) = K \quad \Rightarrow \quad a = K/\mathrm{Log}\,2$. Hence, the required solution is

$$T(x,y) = T(r) = \frac{K\,\mathrm{Log}\,r}{\mathrm{Log}\,2}$$

Harmonic functions are extremely important in applications and a large proportion of Chapter 9 is devoted to their properties and applications.

Exercise 2.2.1 Use the definition of the derivative of a function to prove that, at each point where the derivatives exist,

(i) if $h(z) = f(z) + g(z)$ then $h'(z) = f'(z) + g'(z)$

(ii) if $h(z) = 1/f(z)$ then $h'(z) = -f'(z)/(f(z))^2$

Exercise 2.2.2 Prove, using the definitions, that the existence of the derivative of a function of a complex variable at a point implies the continuity of the function there (as in real analysis). Show also, by considering $f: \mathbb{C} \to \mathbb{C}$ defined by $f(z) = |z|^2$, the continuity of a function of a complex variable at a point does not imply the existence of a derivative there.

Exercise 2.2.3 Show, directly from the definition, that the function $f: \mathbb{C} \to \mathbb{C}$ given by $f(z) = \bar{z}$ is nowhere differentiable.

Exercise 2.2.4 Use the Cauchy–Riemann equations and Theorems 2.4 to 2.6 to determine at which points, if any, the following functions are differentiable and find their derivatives at such points.

(i) $f: \mathbb{C} \to \mathbb{C}, f(z) = z^3$

(ii) $f: \mathbb{C} \to \mathbb{C}, f(z) = e^{\bar{z}}$

(iii) $f: \mathbb{C} \to \mathbb{C}, f(z) = \cosh z$

(iv) $f: \mathbb{C} \to \mathbb{C}, f(z) = z\,\mathrm{Im}\,z$

(v) $f: \mathbb{C}\setminus\{0\} \to \mathbb{C},\ f(z) = 1/z$

(vi) $f: \mathbb{C} \to \mathbb{C},\ f(z) = \sin\bar{z}$

Exercise 2.2.5 Use the Cauchy–Riemann equations to find the most general entire function f such that $\mathrm{Re}(f'(z)) = 0$.

Exercise 2.2.6 Use the Cauchy–Riemann equations to prove the following:

(i) If f is analytic on a region $\mathscr{R}$ and $|f(z)|$ is constant on $\mathscr{R}$, then $f(z)$ is constant on $\mathscr{R}$.

(ii) There is no analytic function f such that $\mathrm{Im}(f(z)) = x^2 - 2y$.

Exercise **2.2.7** Show that the family of curves given by $u(x,y) = k_1$, where k_1 is a constant, have gradient $-u_x/u_y$ at each point. Deduce that if f is entire and $f(z) = u(x,y) + iv(x,y)$, then the curves of this family intersect the curves of the family $v(x,y) = k_2$ (k_2 constant) orthogonally.

Exercise **2.2.8** If f is entire and $z = x + iy$, prove that, for all $z \in \mathbb{C}$,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

Exercise **2.2.9** Let $f : \mathbb{C} \to \mathbb{C}$ and let $w = \mathrm{R}e^{i\phi} = f(z) = f(x + iy)$. Use the chain rule for partial derivatives of functions of two real variables to show that the Cauchy–Riemann equations become

$$\frac{1}{R}R_x(x,y) = \phi_y(x,y) \qquad \frac{1}{R}R_y(x,y) = -\phi_x(x,y)$$

Suppose that f is entire and $\phi(x,y) = 2xy$. Find $R(x,y)$ and hence $w = f(z)$.

Exercise **2.2.10** Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = z^5/|z|^4, z \neq 0, f(0) = 0$. Prove that f is not differentiable at 0, but that the Cauchy–Riemann equations are satisfied there.

Exercise **2.2.11** Using the polar form of the Cauchy–Riemann equations (2.6) and Theorems 2.4 to 2.6, determine at which points $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^{1/2}$ is differentiable and find its derivative there. Use the fact that $f(z) = \exp(\frac{1}{2}\mathrm{Log}\, z)$ to verify your answers; assume the derivatives of exp and Log and use Theorem 2.3.

Exercise **2.2.12** Find the derivative of $\tanh^{-1} z$ at points where it exists

(i) by expressing $\tanh^{-1}$ in terms of Log

(ii) by letting $z = \tanh w$

Exercise **2.2.13** Verify that the following functions of two real variables are harmonic and find their harmonic conjugates. If f given by $f(z) = u(x,y) + iv(x,y)$ is entire, find $f(z)$ in each case.

(i) $u(x,y) = y^3 - 3x^2y$

(ii) $u(x,y) = \sinh x \sin y$

Exercise **2.2.14** By considering the entire function f given by $f(z) = z^2$, show that if $v(x,y)$ is a harmonic conjugate of $u(x,y)$ in some region, it is not in general true that u is a harmonic conjugate of v in that region.

Exercise **2.2.15** Show that $v(x,y) = \theta = \tan^{-1}(y/x), 0 \leqslant \theta < \pi$, is harmonic. Hence find the steady temperature $T(x,y)$, in the upper half-plane, given that $T(x,0) = 0, x > 0$ and $T(x,0) = K, x < 0$.

Singular Points and Zeros

We end this chapter by introducing the idea of singular points of functions, which are of fundamental importance when it comes to integration. Closely related is the idea of zeros of functions.

Definitions

(i) A function f is **analytic at a point** $\alpha \in \mathbb{C}$ if it is differentiable at α and at each point in some open neighbourhood of α.

(ii) A point $\alpha \in \mathbb{C}$ is a **singular point** of a function f if f fails to be analytic at α but is analytic at some point in every open neighbourhood of α.

(iii) A singular point of a function f is **isolated** if there some open neighbourhood of the point throughout which f is analytic except at the point itself.

Notes

(i) Definition (i) ensures that α is contained in a region on which f is analytic. Definition (ii) excludes the possibility of a function having a domain consisting entirely of singular points.

(ii) Singular points of a function need not belong to the domain of the function.

(iii) If f is not continuous at $\alpha \in \mathbb{C}$, then f is not analytic at α, so that α may be a singular point of f. It is easiest to look for this type of singular point first.

Example 2.16

(i) Let $f: \mathbb{C}\backslash\{0\} \to \mathbb{C}$ be defined by $f(z) = 1/z$. Then 0 is the only singular point of f (f is not continuous there), hence it is isolated.

(ii) It is easy to check using Theorem 2.4 that $f: \mathbb{C}\backslash\{0\} \to \mathbb{C}$ defined by $f(z) = 1/\bar{z}$ is differentiable nowhere and so is analytic nowhere. Hence, in contrast to (i), 0 is not a singular point of f.

(iii) Since $\sin(1/z) = 0$ for $z = 1/n\pi$ for all $n \in \mathbb{Z}$, it follows that 0 is a non-isolated singular point of f given by $f(z) = 1/\sin(1/z)$.

Note

It follows from the earlier section on branch points that points on a branch cut of a multifunction and the branch point itself are non-isolated singular points of any branch of that multifunction. However, as Example 2.16(iii) shows, not all non-isolated singular points fall into this category.

Apart from singular points arising from branch points and branch cuts of multifunctions, singular points can be classified in the following way.

Definitions

Let $\alpha \in \mathbb{C}$ be a singular point of a function f.

(i) The point α is a **removable singular point** of f if and only if $\lim_{z\to\alpha} f(z)$ exists.

(ii) The point α is a **pole of order n** ($n \in \mathbb{N}$) if and only if $\lim_{z\to\alpha}(z-\alpha)^n f(z) = k \neq 0$. A pole of order 1 is a **simple pole.**

(iii) A singular point of a function f which is not a removable singularity, a pole, or associated with a branch point or branch cut of a multifunction, is an **essential singular point**.

Notes

(i) Suppose that α is a removable singular point of f, so that $f'(\alpha)$ does not exist for some reason but $\lim_{z\to\alpha} f(z) = k$ say. Then redefining $f(\alpha) = k$ gives $f'(\alpha) = \lim_{z\to\alpha} f'(z)$. Hence the term 'removable' singular point.

(ii) If α is a pole of order n of f, then f can be expressed as $f(z) = g(z)/(z-\alpha)^n$, where $\lim_{z\to\alpha} g(z)$ exists and is non-zero. Hence either g is analytic at α or has a removable singular point at α and so can be made analytic at α by redefining $g(\alpha)$, by (i).

Hence if α is a pole of order n of f, then f can be expressed in the form $f(z) = g(z)/(z-\alpha)^n$, where g is analytic at α, with $g(\alpha) \neq 0$, without loss of generality. Notice that $\lim_{z\to\alpha} 1/f(z) = 0$.

Example 2.17

(i) Let $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ be defined by $f(z) = (\sin z)/z$, $z \neq 0$. Then 0 is a removable singular point of f since $\lim_{z\to 0} f(z) = 1$, and the domain of f can be extended so that $f(0) = 1$.

(ii) Function f defined by

$$f(z) = \frac{z+2}{(z^2+1)(z-4)^2(z+2i)^7}$$

has simple poles at $\pm i$, a pole of order 2 at 4 and a pole of order 7 at $-2i$.

(iii) Let f be defined by $f(z) = \csc z = 1/\sin z$. Then f has isolated singular points at $z = n\pi$, $n \in \mathbb{Z}$, since f is not continuous there. Note that $\lim_{z\to 0} zf(z) = 1$, so that 0 is a simple pole of f. Similarly, $n\pi$ is a simple pole of f for any $n \in \mathbb{Z}$.

(iv) Function f defined by $f(z) = 1/\sin(1/z)$ has a non-isolated singular point at 0. Clearly, this singular point is not removable and is not a pole. Also, f is not a branch of a multifunction, so that 0 is an essential singular point of f.

The function g defined by $g(z) = e^{-1/z}$ also has an essential singularity at 0, but in this case, the singular point is clearly isolated.

The following result is often useful when investigating singular points and is formally the same as its real counterpart.

Theorem 2.7. L'Hôpital's Rule

Let f and g be analytic in a region containing the point $\alpha \in \mathbb{C}$ and suppose that $f(\alpha) = g(\alpha) = 0$ with $g'(\alpha) \neq 0$. Then

$$\lim_{z \to \alpha} \frac{f(z)}{g(z)} = \frac{f'(\alpha)}{g'(\alpha)} \qquad \square$$

Proof

By hypothesis, since $g'(\alpha) \neq 0$,

$$\lim_{z \to \alpha} \frac{f(z)}{g(z)} = \lim_{z \to \alpha} \left(\frac{f(z) - f(\alpha)}{z - \alpha} \cdot \frac{z - \alpha}{g(z) - g(\alpha)} \right) = \frac{f'(\alpha)}{g'(\alpha)}$$

using Theorem 2.1 as required. ■

Notes

(i) As in the case of real variables, limits of other so-called indeterminate forms can often be evaluated by making the appropriate modifications to L'Hôpital's rule.

(ii) The rule can clearly be applied again if $f'(\alpha)$ and $g'(\alpha)$ are also zero, but $g''(\alpha) \neq 0$, etc.

Example 2.18

(i) $$\lim_{z \to i} \frac{z^6 + 1}{z^2 + 1} = \lim_{z \to i} \frac{6z^5}{2z} = \lim_{z \to i} 3z^4 = 3$$

by L'Hôpital's rule

(ii) $$\lim_{z \to 0} \frac{\sin z - \tan z}{z^2} = \lim_{z \to 0} \frac{\cos z - \sec^2 z}{2z} = \lim_{z \to 0} \frac{-\sin z - 2\sec^2 z \tan z}{2} = 0$$

by two applications of L'Hôpital's rule

(iii) $$\lim_{z\to 0}\frac{\operatorname{Log}(\cos z)}{z^2} = \lim_{z\to 0}\frac{(-\sin z)/\cos z}{2z}$$

$$= \left(\lim_{z\to 0}\frac{\sin z}{z}\right)\left(\lim_{z\to 0}\frac{-1}{2\cos z}\right) = -\frac{1}{2}$$

by one application of L'Hôpital's rule and Theorem 2.1. Now let $w = (\cos z)^{1/z^2}$. Then $\operatorname{Log} w = (\operatorname{Log}(\cos z))/z^2 \Rightarrow \lim_{z\to 0}\operatorname{Log} w = -1/2$ by above. Then by Theorem 2.2,

$$\operatorname{Log}(\lim_{z\to 0} w) = -1/2 \quad\Rightarrow\quad \lim_{z\to 0} w = \lim_{z\to 0}(\cos z)^{1/z^2} = e^{-1/2}$$

Example 2.19

(i) Returning to Example 2.17(i), $\lim_{z\to 0}(\sin z)/z = \lim_{z\to 0}\cos z = 1$, by 2.7. Redefining $f(0) = 1$ gives

$$f'(0) = \lim_{h\to 0}\frac{f(h)-1}{h} = \lim_{h\to 0}\frac{\sin h - h}{h^2} = \lim_{h\to 0} f'(h)$$

by 2.7 again. Then by a further two applications of 2.7,

$$f'(0) = \lim_{h\to 0}\frac{\cos h - 1}{2h} = \lim_{h\to 0}\frac{-\sin h}{2} = 0$$

(ii) Returning to Example 2.17(iii), by 2.7 it follows that

$$\lim_{z\to n\pi}(z - n\pi)f(z) = \lim_{z\to n\pi}\frac{z - n\pi}{\sin z} = \lim_{z\to n\pi}\frac{1}{\cos z} = (-1)^n$$

so that $n\pi$, $n \in \mathbb{Z}$, is a simple pole of f.

We turn now to the concept of the order of a zero of a given function.

Definition

A point $\alpha \in \mathbb{C}$ is a **zero of order n** ($n \in \mathbb{N}$) of a function f if and only if f can be expressed in the form $f(z) = (z-\alpha)^n g(z)$ for some function g with $\lim_{z\to\alpha} g(z) \neq 0$.

Note

If α is a zero of f, then $f(\alpha) = 0$. Once again, we can assume that g is analytic at α without loss of generality, with $g(\alpha) \neq 0$. Then α is a zero of order n of f if and only if it is a pole of order n of $1/f$.

Example 2.20

(i) Function f given by $f(z) = (z-2)^2(z^2+1)$ for all $z \in \mathbb{C}$ has a zero of order 2 at 2 and two zeros of order 1 (**simple zeros**) at $\pm i$.

(ii) Function f given by $f(z) = \tan z$ $(z \neq n\pi + \pi/2, n \in \mathbb{Z})$ clearly has zeros given by $\sin z = 0$ and so has zeros at $z = n\pi, n \in \mathbb{Z}$. Now

$$\lim_{z \to n\pi} \frac{f(z)}{z - n\pi} = \lim_{z \to n\pi} \frac{f'(z)}{1} = f'(n\pi) = \sec^2(n\pi) = 1 \neq 0$$

by L'Hôpital's rule, so that f has simple zeros at $z = n\pi$.

It is straightforward to show that the zeros of a function are isolated from each other in the plane.

Theorem 2.8. Zeros are Isolated

Let α be a zero of order n of f. Then this zero is **isolated**; that is, there exists an open neighbourhood of α which contains no other zero of f. □

Proof

By hypothesis, it follows that $f(z) = (z-\alpha)^n g(z)$, where g is analytic at α without loss of generality, and $g(\alpha) \neq 0$. Let $g(\alpha) = 2\beta$ say. Since g is continuous at α, there exists a real $\delta > 0$ such that

$$|z - \alpha| < \delta \quad \Rightarrow \quad |g(z) - g(\alpha)| < |\beta|$$

Hence, by the triangle inequality,

$$|z - \alpha| < \delta \quad \Rightarrow \quad |g(z)| \geqslant \Big| |g(\alpha)| - |g(z) - g(\alpha)| \Big| > |\beta|$$

so that $g(z)$ is not zero in the neighbourhood $|z - \alpha| < \delta$. ■

Note

Since f has a pole of order n at α if and only if $1/f$ has a zero of order n at α, it follows that poles of functions are isolated.

Exercise

2.3.1 Find the following limits by using L'Hôpital's rule:

(i) $\displaystyle\lim_{z \to -i} \frac{z^{11} - i}{z^7 - i}$

(ii) $\displaystyle\lim_{z \to 1} \frac{1 + \cos \pi z}{z^2 - 2z + 1}$

(iii) $\displaystyle\lim_{z \to i} (z + 1 - i)^{1/(z-i)}$

Exercise 2.3.2 Locate and classify the singular points of the following fu
Give reasons for your answers.

(i) $f(z) = \dfrac{z+1}{z(z^2+1)^3}$

(ii) $f(z) = \dfrac{\mathrm{Log}(z^2+9)}{(z+1)^2}$

(iii) $f(z) = \tan z$

(iv) $f(z) = \operatorname{sech} z$

(v) $f(z) = \dfrac{e^z - 1}{z}$

(vi) $f(z) = \sin(z + 1/z)$

Exercise 2.3.3 Locate the zeros of the following functions and find their orders:

(i) $f(z) = (z^2-1)^2(z^4+1)(z-i)^7$

(ii) $f(z) = \sinh^2 z$

(iii) $f(z) = \dfrac{e^z - 1}{z-1}$

(iv) $f(z) = \dfrac{\mathrm{Log}\, z}{z^2}$

Exercise 2.3.4 Prove that α is a pole of order n of f if and only if it is a zero
order n of $1/f$.

3 Integration, Cauchy's Theorems and Related Results

We introduce the idea of a complex definite integral in this chapter and give the famous results of Cauchy, together with related results. Some of the rich applications of these results will be demonstrated. For example, these techniques may be used to evaluate certain real definite integrals very simply, which would otherwise be difficult to evaluate. We shall also see how these results help us to locate the position of zeros of polynomials in the complex plane.

Definite Integrals

Initially, it seems logical to suppose that a definite integral of a complex function, evaluated between two points in the complex plane, will depend on the curve joining the two points. Hence the definition of a complex definite integral is very similar to that of a real line integral in the plane. We shall assume that the notion of a real definite integral (at least in the Riemannian sense) is already familiar to the reader. We first need to consider the case in which the complex-valued integrand is a function of one real variable only.

Preliminary Definition

Let $[a, b]$ be a closed interval of $\mathbb{R}$ and let $p, q : [a, b] \to \mathbb{R}$. Suppose that p and q are piecewise continuous on $[a, b]$. (In other words, p and q are continuous on $[a, b]$ except possibly for a finite number of jump discontinuities.) Let $g : [a, b] \to \mathbb{C}$ be defined by $g(t) = p(t) + iq(t)$ for $t \in [a, b]$. Then $\int_a^b g(t)\,dt$ is defined by

$$\int_a^b g(t)\,dt = \int_a^b p(t)\,dt + i\int_a^b q(t)\,dt \tag{3.1}$$

Notes

The conditions given are sufficient to ensure that the real integrals on the right of (3.1) exist (in the Riemannian sense). This is the obvious definition to make – the integral is evaluated by integrating the real and imaginary parts of the integrand. Clearly, all the usual properties of real (Riemann) integrals hold for $\int_a^b g(t)\,dt$.

We now define a sufficiently 'well-behaved' curve $\mathscr{C}$ joining two points in the complex plane, so that $\int_{\mathscr{C}} f(z)\,dz$ can be defined.

Definitions

(i) An **arc** $\mathscr{C}$ is a set of points $\{(x(t), y(t)) : t \in [a, b]\}$ in the complex plane where x and y are continuous functions of the real parameter t. It is convenient to describe the points on $\mathscr{C}$ by

$$z = z(t) = x(t) + iy(t) \qquad (a \leqslant t \leqslant b) \tag{3.2}$$

$\mathscr{C}$ is a **simple** or **Jordan** arc if it does not cross itself; that is, $z(t_1) = z(t_2) \quad \Rightarrow \quad t_1 = t_2$ for all $t_1, t_2 \in [a, b]$.

(ii) An arc $\mathscr{C}$ is **smooth** if $z'(t)$ exists and is non-zero for $t \in [a, b]$. (This implies that $\mathscr{C}$ has a continuously turning tangent.)

(iii) A (simple) **contour** is an arc consisting of a finite number of (simple) smooth arcs joined end to end. When only the initial and final values of $z(t)$ are the same, the contour is a (simple) **closed contour**.

Notice that, in the sense we have defined them, any arc and hence any contour joins one point to another point.

Recall the Jordan curve theorem (see Chapter 1). This result implies that any simple closed contour divides the complex plane into two regions, the bounded **inside** and the unbounded **outside**, having the curve as a common boundary. We shall not prove this topological result, which is intuitively obvious but very difficult to prove.

Example 3.1

In Fig. 3.1, $\mathscr{C}_1$ is an arc, $\mathscr{C}_2$ is a simple arc, $\mathscr{C}_3$ is a smooth arc, $\mathscr{C}_4$ is a simple smooth arc, $\mathscr{C}_5$ is a contour and $\mathscr{C}_6$ is a simple closed contour.

The easiest contours to describe by means of a real parameter, as in (3.2), consist of line segments and arcs of circles (see Chapter 1). It turns out that such contours are the most useful in applications.

Important Convention

With all closed contours, the direction of **increasing** parameter will be taken as **anticlockwise**. (Think of using the angular polar coordinate as a parameter for a circle.)

Main Definition

Let $\mathscr{C}$ be any contour in the complex plane represented by (3.2), extending from $\alpha = z(a)$ to $\beta = z(b)$. Let the domain of $f : A \subseteq \mathbb{C} \to \mathbb{C}$ contain the contour $\mathscr{C}$ and let f be piecewise continuous on $\mathscr{C}$; that is, $f(z(t))$ is piecewise continuous on

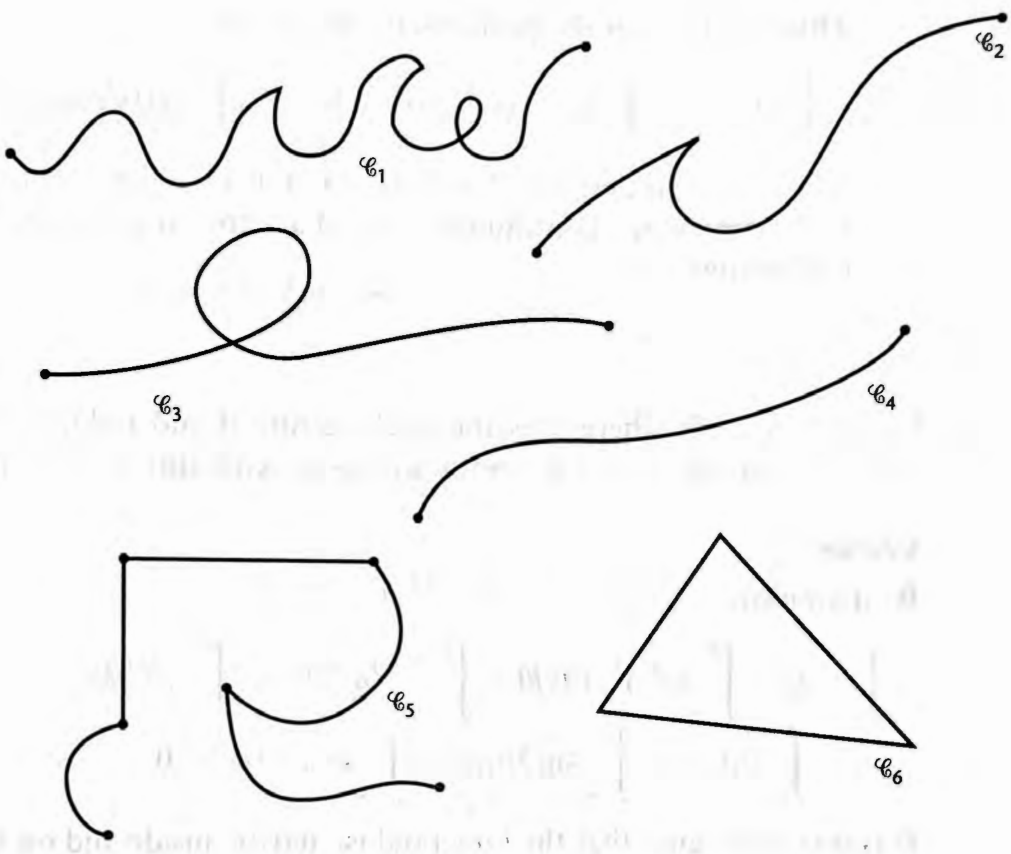


Figure 3.1

$[a, b]$. Then the **complex definite line integral of f along $\mathscr{C}$** is denoted and defined by

$$\int_{\mathscr{C}} f(z)\, dz = \int_a^b f(z(t))z'(t)\, dt \tag{3.3}$$

Notes

(i) The definition is reminiscent of integration by substitution for real integrals. It is independent of the choice of the parametrisation of $\mathscr{C}$. To show this, let $\mathscr{C}$ be a smooth simple arc without loss of generality and let $\mathscr{C}$ be also represented by $z = z(u)$ where $u \in \mathbb{R}$ with $u \in [c, d]$ and $\alpha = z(c)$ and $\beta = z(d)$, say. Then

$$\int_c^d f(z(u))z'(u)du = \int_a^b f(z(t))\left(\frac{dz}{dt}\frac{dt}{du}\right)\frac{du}{dt}dt = \int_a^b f(z(t))z'(t)\, dt$$

by the chain rule (Theorem 2.3(b) (iii)).

(ii) Let f be given by $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$ for all points in its domain. Then, from the definition of a derivative, it follows that

$$f(z(t))z'(t) = (u(t) + iv(t))(x'(t) + iy'(t))$$
$$\Rightarrow \quad f(z(t))z'(t) = [u(t)x'(t) - v(t)y'(t)] + i[v(t)x'(t) + u(t)y'(t)]$$

Thus, by (3.1) of the preliminary definition,

$$\int_{\mathscr{C}} f(z)\,dz = \int_a^b (u(t)x'(t) - v(t)y'(t))\,dt + i\int_a^b (v(t)x'(t) + u(t)y'(t))\,dt \quad (3.4)$$

and so is well defined. The real integrals exist since $\mathscr{C}$ is a contour and f is piecewise continuous, so that the integrands are piecewise continuous.

Example 3.2

Evaluate $\int_{\mathscr{C}} z^2\,dz$ where $\mathscr{C}$ is the circle, centre 0 and radius 1, represented by $z(\theta) = e^{i\theta}$ where $-\pi \leqslant \theta \leqslant \pi$ without loss of generality (see Chapter 1).

Solution

By definition

$$\int_{\mathscr{C}} z^2 dz = \int_{-\pi}^{\pi} (e^{i\theta})^2 z'(\theta) d\theta = \int_{-\pi}^{\pi} e^{2i\theta} i e^{i\theta} d\theta = i\int_{-\pi}^{\pi} e^{3i\theta} d\theta$$
$$\Rightarrow \quad \int_{\mathscr{C}} z^2 dz = -\int_{-\pi}^{\pi} \sin 3\theta\, d\theta + i\int_{-\pi}^{\pi} \cos 3\theta\, d\theta = 0$$

It is no coincidence that the integrand is analytic inside and on $\mathscr{C}$ and the result is zero. The connection will be revealed later.

Note

As the following examples show, in general given two points α and β in the complex plane, $\int_\alpha^\beta f(z)\,dz$ will depend on the choice of contour joining α to β.

Example 3.3

(i) Evaluate $\int_{\mathscr{C}_i} \bar{z}\,dz, i = 1, 2$, where $\mathscr{C}_1$ and $\mathscr{C}_2$ are the contours shown in Fig. 3.2.

(ii) Let $f: \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = x + iky$ for some $k \in \mathbb{Z}$. Let $\mathscr{C}_n$ be the contour joining 0 to $1 + i$ given by $y = x^n$, $n \in \mathbb{N}$. Show that $\int_{\mathscr{C}_n} f(z)\,dz$ is independent of n if and only if $k = 1$.

Solution

(i) It is advisable to try to find the simplest representation of a given contour, since the integral does not depend on the choice of parametrisation. For example, a suitable parametrisation of $\mathscr{C}_1$ is $z(t) = t$ for $0 \leqslant t \leqslant 1$ and $z(t) = 1 + i(t - 1)$ for $1 \leqslant t \leqslant 2$, but this is not the simplest choice. This is equivalent to splitting up $\mathscr{C}_1$ into two line

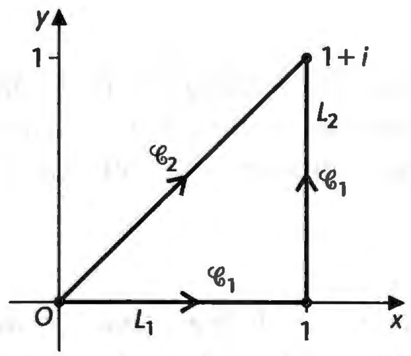


Figure 3.2

segments L_1 and L_2 as shown, where along L_1, $z(t) = t, 0 \leqslant t \leqslant 1$, and along L_2, $z(t) = 1 + it$, $0 \leqslant t \leqslant 1$. Then

$$\int_{\mathscr{C}_1} \bar{z}\, dz = \underset{(L_1)}{\int_0^1 t\, dt} + \underset{(L_2)}{\int_0^1 (1 - it)i\, dt} = 2\int_0^1 t\, dt + i\int_0^1 dt = 1 + i$$

The obvious representation of $\mathscr{C}_2$ is $z(t) = t + it, 0 \leqslant t \leqslant 1$, since $\mathscr{C}$ is part of the line $y = x$. Then by definition

$$\int_{\mathscr{C}_2} \bar{z}\, dz = \int_0^1 (t - it)(1 + i)\, dt = 2\int_0^1 t\, dt = 1$$

This shows that $\int_0^{1+i} \bar{z}\, dz$ depends on the choice of contour joining 0 to $1 + i$. Note that the integral taken round the whole triangle $\mathscr{C}$, in an anticlockwise sense, is given by

$$\int_{\mathscr{C}} \bar{z}\, dz = \int_{\mathscr{C}_1} \bar{z}\, dz - \int_{\mathscr{C}_2} \bar{z}\, dz = i$$

Note also that, in this case, the integrand is not analytic inside or on the closed contour $\mathscr{C}$.

(ii) The simplest parametrisation of each $\mathscr{C}_n$ is $z(t) = t + it^n, 0 \leqslant t \leqslant 1$. Then by definition,

$$\int_{\mathscr{C}_n} f(z)\, dz = \int_0^1 (t + ikt^n)(1 + int^{n-1})\, dt$$

$$= \int_0^1 (t - nkt^{2n-1})\, dt + i\int_0^1 (k + n)t^n dt = \frac{1}{2}(1 - k) + i\left(\frac{n + k}{n + 1}\right)$$

as required. It follows by the Cauchy–Riemann equations and 2.6 that f is analytic if and only if $k = 1$. In this special case,

$$\int_{\mathscr{C}_n} z\, dz = i = \left[\frac{z^2}{2}\right]_0^{1+i}$$

as in the real case. Once again, this is no accident.

Note

Some care has to be taken when evaluating integrals involving branches of multifunctions, as the following example illustrates. Very often they depend on the chosen convention for the principal argument of a complex number.

Example 3.4

Consider $\int_{\mathscr{C}} z^{-1/2}dz$, where $\mathscr{C}$ is the circle $|z| = 1$. Note that 0 is the branch point of the integrand, treated as a multifunction (see Example 2.4(i)). Now according to our convention, $z^{1/2} = \sqrt{r}e^{i\theta/2}$, where $r = |z|$, $\theta = \operatorname{Arg} z$, $-\pi < \theta \leqslant \pi$. Hence the simplest parametrisation of $\mathscr{C}$ to choose is $z(\theta) = e^{i\theta}$, $-\pi \leqslant \theta \leqslant \pi$, as in Example 3.2. Then by definition,

$$\int_{\mathscr{C}} \frac{dz}{z^{1/2}} = \int_{-\pi}^{\pi} ie^{i\theta}e^{-i\theta/2}d\theta = i\int_{-\pi}^{\pi} \cos(\theta/2)d\theta - \int_{-\pi}^{\pi} \sin(\theta/2)d\theta = 4i$$

Another parametrisation of $\mathscr{C}$ is $z(\phi) = e^{i\phi}$, $0 \leqslant \phi \leqslant 2\pi$. Then

$$\int_{\mathscr{C}} \frac{dz}{z^{1/2}} = \int_{0}^{\pi} ie^{i\phi}e^{-i\phi/2}d\phi + \int_{\pi}^{2\pi} ie^{i\phi}e^{-i(\phi-2\pi)/2}d\phi = 4i$$

as above. However, if the chosen convention is $0 \leqslant \operatorname{Arg} z < 2\pi$,

$$\int_{\mathscr{C}} \frac{dz}{z^{1/2}} = \int_{0}^{2\pi} ie^{i\phi}e^{-i\phi/2}d\phi = -4$$

The following elementary properties of definite integrals follow directly from the definition.

Lemma 3.1. Properties of Definite Integrals

Let f and g be piecewise continuous on $\mathscr{C}$.

(i) $$\int_{\mathscr{C}} (\alpha f(z) + \beta g(z))\, dz = \alpha\int_{\mathscr{C}} f(z)\, dz + \beta\int_{\mathscr{C}} g(z)\, dz$$

for any constants $\alpha, \beta \in \mathbb{C}$.

(ii) If $\mathscr{C}$ consists of a contour $\mathscr{C}_1$ joining α to β and a contour $\mathscr{C}_2$ joining β to γ, then

$$\int_{\mathscr{C}} f(z)\, dz = \int_{\mathscr{C}_1} f(z)\, dz + \int_{\mathscr{C}_2} f(z)\, dz$$

(iii) Let a contour $\mathscr{C}$ be described by $z = z(t)$, $a \leqslant t \leqslant b$ and let $-\mathscr{C}$ be the same set of points, taken in reverse order, so that $-\mathscr{C}$ is described by $z = z(-t)$, $-b \leqslant t \leqslant -a$. Then

$$\int_{-\mathscr{C}} f(z)\, dz = -\int_{\mathscr{C}} f(z)\, dz$$

□

Proof

All three results follow directly from the definition. We shall prove (i) as an example and leave the other two as exercises. Let $f(z) = u(x,y) + iv(x,y)$ and $g(z) = U(x,y) + iV(x,y)$ at all points of their domains, and let a parametrisation of $\mathscr{C}$ be given by (3.2). It follows by (3.4) that

$$\int_{\mathscr{C}} (f(z) + g(z))\,dz = \int_a^b ((u(t) + U(t))x'(t) - (v(t) + V(t))y'(t))\,dt$$

$$+i\int_a^b ((v(t) + V(t))x'(t) + (u(t) + U(t))y'(t))\,dt$$

$$\Rightarrow \quad \int_{\mathscr{C}} (f(z) + g(z))\,dz = \int_{\mathscr{C}} f(z)\,dz + \int_{\mathscr{C}} g(z)\,dz \tag{3.5}$$

using the linear property of real integrals and (3.4) again. Now let $\gamma = r + is, r, s \in \mathbb{R}$, be any complex constant. Then by (3.4) again,

$$\int_{\mathscr{C}} \gamma f(z)\,dz = \int_a^b ((ru(t) - sv(t))x'(t) - (rv(t) + su(t))y'(t))\,dt$$

$$+ i\int_a^b ((rv(t) + su(t))x'(t) + (ru(t) - sv(t))y'(t))\,dt$$

$$= (r + is)\left(\int_a^b (u(t)x'(t) - v(t)y'(t))\,dt + i\int_a^b (v(t)x'(t) + u(t)y'(t))\,dt\right)$$

$$\Rightarrow \quad \int_{\mathscr{C}} \gamma f(z)\,dz = \gamma\int_{\mathscr{C}} f(z)\,dz \tag{3.6}$$

Then the result follows by (3.5) and (3.6). ■

Notes

It follows from 3.1(ii) that if $\mathscr{C}$ is a closed contour then $\int_{\mathscr{C}} f(z)\,dz$ is independent of the choice of initial point of $\mathscr{C}$. Lemma 3.1(ii) is analogous to the result

$$\int_a^c f(t)\,dt = \int_a^b f(t)\,dt + \int_b^c f(t)\,dt$$

for real definite integrals, and 3.1(iii) is analogous to

$$\int_b^a f(t)\,dt = -\int_a^b f(t)\,dt$$

The following result, which gives an upper bound for the modulus of any complex definite integral, is of fundamental importance for future results and techniques.

Lemma 3.2. The 'ML Lemma'

Let $\mathscr{C}$ be any contour and let the function f be piecewise continuous on $\mathscr{C}$. Let L denote the length of $\mathscr{C}$; that is, if $\mathscr{C}$ is represented by $z = z(t)$, $t \in [a, b]$, then $L = \int_a^b |z'(t)|\, dt$. Suppose that $|f(z)| \leqslant M$ on $\mathscr{C}$ for some positive real constant M. Then

$$\left|\int_{\mathscr{C}} f(z)\, dz\right| \leqslant ML \qquad \square$$

Proof

We begin by considering $|\int_a^b g(t)\, dt|$ where $g : [a, b] \to \mathbb{C}$ is piecewise continuous, as in the preliminary definition. Suppose that $\int_a^b g(t)\, dt \neq 0$ and that the modulus of the integral is r and its principal argument is θ . Then $r = \int_a^b e^{-i\theta} g(t)\, dt$ since $e^{i\theta}$ is a constant. Then since r is real, $e^{-i\theta} g(t)$ is also real, so that

$$r = \int_a^b e^{-i\theta} g(t) \leqslant \int_a^b |e^{-i\theta} g(t)|\, dt = \int_a^b |g(t)|\, dt$$

$$\Rightarrow \quad r = \left|\int_a^b g(t)\, dt\right| \leqslant \int_a^b |g(t)|\, dt \tag{3.7}$$

as in the case of real definite integrals.

Now let the contour $\mathscr{C}$ have parametric equation (3.2). Then by definition and (3.7),

$$\left|\int_{\mathscr{C}} f(z)\, dz\right| = \left|\int_a^b f(z(t)) z'(t)\, dt\right| \leqslant \int_a^b |f(z(t))||z'(t)|\, dt$$

$$\Rightarrow \quad \left|\int_{\mathscr{C}} f(z)\, dz\right| \leqslant M \int_a^b |z'(t)| dt = ML \qquad \blacksquare$$

Notes

(i) Since f is piecewise continuous on the closed set $\mathscr{C}$, f is bounded on $\mathscr{C}$, so that the constant M exists.

(ii) If the contour $\mathscr{C}$ is a segment of the real axis joining a to b, then Lemma 3.2 and the definition give

$$\left|\int_a^b f(t)\, dt\right| \leqslant M(b - a)$$

a standard result from real calculus.

(iii) In contrast to the real case, it is not in general true that $|\int_{\mathscr{C}} f(z)\, dz| \leqslant \int_{\mathscr{C}} |f(z)| dz$ for complex integrals. Try $f(z) = 1/z$ with $\mathscr{C}$ the circle $|z| = 1$, for instance.

Example 3.5

Let $\mathscr{C}$ be the circle $|z| = R$ with $R > 1$ and consider

$$\left|\int_{\mathscr{C}} \frac{\operatorname{Log} z}{z^n}\, dz\right|$$

where $n \in \mathbb{N}$. The length of $\mathscr{C}$ is $2\pi R$ and on $\mathscr{C}$,

$$\left|\frac{\operatorname{Log} z}{z^n}\right| = \frac{|\operatorname{Log} z|}{R^n}$$

Also, by definition,

$$\operatorname{Log} z = \operatorname{Log}|z| + i \operatorname{Arg} z \qquad (-\pi < \operatorname{Arg} z \leqslant \pi)$$

Hence on $\mathscr{C}$,

$$|\operatorname{Log} z| \leqslant ((\operatorname{Log} R)^2 + \pi^2)^{1/2} < \operatorname{Log} R + \pi \,(\text{since } \operatorname{Log} R, \pi > 0)$$

Then by lemma 3.2 it follows that

$$\left|\int_{\mathscr{C}} \frac{\operatorname{Log} z}{z^n}\, dz\right| \leqslant 2\pi R\left(\frac{\operatorname{Log} R + \pi}{R^n}\right) = 2\pi R^{1-n}(\operatorname{Log} R + \pi)$$

As in real variable calculus, if a function f has a known antiderivative, it is relatively simple to evaluate $\int_{\mathscr{C}} f(z)\, dz$.

Theorem 3.3. Fundamental Theorem of Calculus

Let $F : A \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function with continuous derivative f in a region $\mathscr{R}$ containing the contour $\mathscr{C}$ parametrised by $z = z(t)$, $t \in [a, b]$. Then

$$\int_{\mathscr{C}} f(z)\, dz = F(\beta) - F(\alpha) = [F(z)]_\alpha^\beta$$

where $\alpha = z(a)$ and $\beta = z(b)$. □

Proof

Suppose first of all that $\mathscr{C}$ is a smooth arc. Then by hypothesis and the definition,

$$\int_{\mathscr{C}} f(z)\, dz = \int_{\mathscr{C}} F'(z)\, dz = \int_a^b F'(z(t))z'(t)\, dt = \int_a^b \frac{dF}{dz}\frac{dz}{dt}\, dt = \int_a^b \frac{dF}{dt}\, dt$$

by the chain rule. Let $F(z(t)) = U(t) + iV(t)$. Then, using the preliminary definition and the above result,

$$\int_{\mathscr{C}} f(z)\, dz = \int_a^b U'(t)\, dt + i\int_a^b V'(t)\, dt = [F(z(t))]_a^b = F(\beta) - F(\alpha),$$

as required. More generally, if $\mathscr{C}$ is a contour consisting of a finite number of smooth arcs $\mathscr{C}_k$, connecting α_k to β_k, then $\int_{\mathscr{C}_k} f(z)\, dz = F(\alpha_k) - F(\beta_k)$ for each k, by above. The result then follows by 3.1(ii). ■

Notes

(i) If f is the derivative of F, we write $F(z) = \int f(z)\,dz$ and call F an **indefinite integral** of f, as in real calculus. Since all the standard derivatives of the elementary functions are the same as in real variable calculus, the same applies to indefinite integrals. So, for example, $\int e^{iz}dz = e^{iz}/i$, and so on.

(ii) In contrast to the real case, the fundamental theorem of calculus is not the ultimate result concerning evaluation of definite integrals, as will soon be shown.

Example 3.6

Since sinh is the continuous derivative of the entire function cosh, it follows by Theorem 3.3 that

$$\int_{\operatorname{Log}2-i\pi}^{i\pi/2} \sinh z\,dz = \left[\cosh z\right]_{\operatorname{Log}2-i\pi}^{i\pi/2}$$

independent of the contour joining $\operatorname{Log}2 - i\pi$ to $i\pi/2$. Hence

$$\int_{\operatorname{Log}2-i\pi}^{i\pi/2} \sinh z\,dz$$

$$= \cos(\pi/2) - \cosh(\operatorname{Log}2)\cosh(i\pi) + \sinh(\operatorname{Log}2)\sinh(i\pi)$$

$$= \cosh(\operatorname{Log}2) = \tfrac{1}{2}\left(e^{\operatorname{Log}2} + e^{\operatorname{Log}1/2}\right) = \tfrac{5}{4}$$

using the results of Chapter 1.

The next result follows immediately from Theorem 3.3.

Corollary 3.4. Definite Integrals Independent of the Contour

If f is the continuous derivative of an analytic function in a region $\mathscr{R}$ containing α and β, then $\int_\alpha^\beta f(z)\,dz$ is independent of the contour joining α to β lying within $\mathscr{R}$, so that if $\mathscr{C}$ is any simple closed contour in $\mathscr{R}$, then $\int_{\mathscr{C}} f(z)\,dz = 0$. □

Functions f for which $\int_\alpha^\beta f(z)\,dz$ is independent of the choice of contour are sometimes called **integrable**.

Example 3.7

Consider $\int_{\mathscr{C}} z^{-2}dz$ where $\mathscr{C}$ is the circle $|z| = 1$. The integrand is the continuous derivative of the analytic function F given by $F(z) = -1/z, z \neq 0$, throughout any annular region containing $|z| = 1$ but excluding the origin. (Note that the region $\mathscr{R}$ in 3.4 does not have to be simply connected.) Hence by 3.4, $\int_{\mathscr{C}} z^{-2}dz = 0$, as can be checked using the definition.

Now consider $\int_{\mathscr{C}} z^{-2}dz$ where $\mathscr{C}$ is the circle $|z| = 1$ again. It is easy to check using the definition that $\int_{\mathscr{C}} z^{-2}dz = 2\pi i$. The fact that $\int_{\mathscr{C}} z^{-1}dz \neq 0$ is due to the fact that there is no region containing $\mathscr{C}$ throughout which the integrand is the continuous derivative of an analytic function. This is so because the derivative of Log z is $1/z$ and Log fails to be analytic along the branch cut given by $z = x$, $x \leqslant 0$, according to our convention. Under any convention, any region containing $\mathscr{C}$ will contain points on a branch cut of log, since any branch cut of log will pass through $\mathscr{C}$. Hence 3.4 does not apply to this integral. However, we can still use 3.3 to evaluate the integral, as long as we take some care. With our convention,

$$\int_{\mathscr{C}} \frac{dz}{z} = \lim_{\theta \to \pi -} [\operatorname{Log} z]_{z=e^{-i\theta}}^{z=e^{i\theta}} = \lim_{\theta \to \pi -} [\operatorname{Log}|z| + i \operatorname{Arg} z]_{z=e^{-i\theta}}^{z=e^{i\theta}}$$

(where $0 \leqslant \theta < \pi$), and so

$$\int_{\mathscr{C}} \frac{dz}{z} = \lim_{\theta \to \pi -} (2i\theta) = 2\pi i$$

The converse of Corollary 3.4 is also true.

Theorem 3.5. Integrable Functions are Derivatives

Let f be continuous in a region $\mathscr{R}$ and suppose that $\int_\alpha^\beta f(z)\, dz$ is independent of any contour joining α and β, contained wholly within $\mathscr{R}$. Then f is the derivative of an analytic function in $\mathscr{R}$. □

Proof

Let γ be any point in $\mathscr{R}$ and define F by $F(z) = \int_\gamma^z f(w)\, dw$, $z \in \mathscr{R}$. Then by hypothesis, F is well defined since the integral is independent of any contour joining γ to z, contained wholly within $\mathscr{R}$. Now,

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\left(\int_\gamma^{z+h} f(w)\, dw - \int_\gamma^z f(w)\, dw\right) = \frac{1}{h}\int_z^{z+h} f(w)\, dw$$

$$\Rightarrow \quad \left|\frac{F(z+h) - F(z)}{h} - f(z)\right| = \left|\frac{1}{h}\int_z^{z+h} (f(w) - f(z))\, dw\right| \tag{3.8}$$

where $z + h \in \mathscr{R}$ and $h \neq 0$, using 3.1 and 3.3. By hypothesis, the integration can be taken over the line segment joining z to $z + h$.

Since f is continuous in $\mathscr{R}$, given any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|h| < \delta \quad \Rightarrow \quad |f(z+h) - f(z)| < \varepsilon$$

It then follows by equation (3.8) and Lemma 3.2 that

$$0 < |h| < \delta \quad \Rightarrow \quad \left| \frac{F(z+h) - F(z)}{h} - f(z) \right| < \frac{1}{|h|}(\varepsilon|h|) = \varepsilon$$

$$\Rightarrow \quad f(z) = \lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = F'(z) \quad \text{as required.}$$ ■

Exercise **3.1.1** Evaluate $\int_{\mathscr{C}} z|z|dz$ where $\mathscr{C}$ is the contour given by $z(t) = 1 - t$ for $0 \leqslant t \leqslant 1, z(t) = i(t-1)$ for $1 \leqslant t \leqslant 2$.

Exercise **3.1.2** Evaluate

(i) $\int_{\mathscr{C}_k} z\,dz$

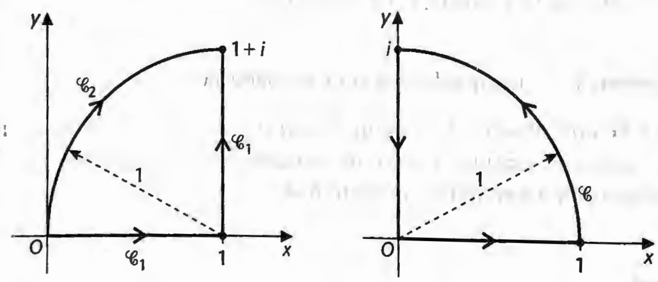
(ii) $\int_{\mathscr{C}_k} \bar{z}\,dz$

$k = 1, 2$, where $\mathscr{C}_1$ and $\mathscr{C}_2$ are the contours shown in Fig. 3.3(a). ($\mathscr{C}_2$ is an arc of a circle, centre 1 and radius 1.)

Verify that

$$\int_{\mathscr{C}_k} z\,dz = \left[\frac{z^2}{2}\right]_0^{1+i} \qquad (k = 1 \text{ or } 2)$$

and that in (ii) the result depends on the choice of contour.



(a) (b)

Figure 3.3

Exercise **3.1.3** Let $\mathscr{C}$ be the closed contour shown in Fig. 3.3(b), where the arc is part of a circle of unit radius, centre the origin. Show, by direct evaluation, that $\int_{\mathscr{C}} z^2 dz = 0$, but $\int_{\mathscr{C}} |z|^2 dz \neq 0$.

Exercise **3.1.4** Let $\mathscr{C}$ denote the circle $|z| = R$. Show, by direct evaluation, that $\int_{\mathscr{C}} \operatorname{Log} z\,dz$ and $\int_{\mathscr{C}} ((\operatorname{Log} z)/z)dz$ depend on the radius of $\mathscr{C}$, whereas $\int_{\mathscr{C}} z^n dz$ for any $n \in \mathbb{Z}$ does not.

Exercise **3.1.5** Evaluate

(i) $\int_{\mathscr{C}} z \operatorname{Im} z\,dz$

(ii) $\int_{\mathscr{C}} |z|^2 dz$

where $\mathscr{C}$ is the triangle with vertices at 0, 3 and $3 + 4i$.

Exercise **3.1.6** Evaluate $\int_{\mathscr{C}_k} \bar{z}\, dz$, $k = 1, 2$, where $\mathscr{C}_1$ is the arc of the ellipse with Cartesian equation $x^2/a^2 + y^2/b^2 = 1$, $x \geqslant 0$, and $\mathscr{C}_2$ has Cartesian equation $y = e^x$, $0 \leqslant x \leqslant 1$.

Exercise **3.1.7** Evaluate $\int_{\mathscr{C}} (1 - \operatorname{Im} z)\, dz$ where $\mathscr{C}$ is one arch of the cycloid with parametric equations $x = \theta - \sin\theta$, $y = 1 - \cos\theta$, $0 \leqslant \theta \leqslant 2\pi$.

Exercise **3.1.8** Evaluate $\int_{\mathscr{C}} z^{-1} dz$ where $\mathscr{C}$ is the rectangle with vertices at $\pm i \pm 2$.

Exercise **3.1.9** (a) Provided that the integrals exist, prove that if a contour $\mathscr{C}$ consists of a contour $\mathscr{C}_1$ joining α to β and a contour $\mathscr{C}_2$ joining β to γ, then $\int_{\mathscr{C}} f(z)\, dz = \int_{\mathscr{C}_1} f(z)\, dz + \int_{\mathscr{C}_2} f(z)\, dz$.

(b) Let a contour $\mathscr{C}$ be described by $z = z(t)$, $a \leqslant t \leqslant b$ and let $-\mathscr{C}$ be described by $z = z(-t)$, $-b \leqslant t \leqslant -a$. Prove that $\int_{-\mathscr{C}} f(z)\, dz = -\int_{\mathscr{C}} f(z)\, dz$, provided that the integrals exist.

Exercise **3.1.10** Without actually evaluating the integrals, use the ML lemma to show that

(i) $\left|\int_{\mathscr{C}} \dfrac{dz}{1+z^2}\right| \leqslant \dfrac{3\pi}{16}$, where $\mathscr{C}$ is the arc of the circle $|z| = 3$, joining 3 to $3i$ in the first quadrant

(ii) $|\int_{\mathscr{C}} e^{\bar{z} \operatorname{Im} z} dz| \leqslant 2\pi\sqrt{e}$, where $\mathscr{C}$ is the circle $|z| = 1$

(iii) $|\int_{\mathscr{C}} e^{\bar{z} \operatorname{Im} z} dz| \leqslant (2 + \sqrt{2})e$, where $\mathscr{C}$ is the triangle with vertices 0, 1 and $1 + i$

Exercise **3.1.11** Use the fundamental theorem of calculus to evaluate $\int_0^{\pi + i} \sin 2z\, dz$.

Exercise **3.1.12** Use the fundamental theorem of calculus to evaluate

(i) $\int_{\mathscr{C}_k} z^{1/2} dz$

(ii) $\int_{\mathscr{C}_k} \operatorname{Log} z\, dz$

$k = 1, 2$, where $\mathscr{C}_1$ is a simple closed contour not enclosing the origin and $\mathscr{C}_2$ is the circle $|z| = R$.

Cauchy's Theorem

It is often the case that a given complex definite integral can be evaluated without knowing the antiderivative of the integrand. In particular, the following famous result enables us to evaluate $\int_{\mathscr{C}} f(z)\, dz$, where $\mathscr{C}$ is a simple closed contour, without having to resort to the definition or the fundamental theorem of calculus, in a large number of cases. It is one of the foundations of the theory of analytic functions.

Theorem 3.6. Cauchy's Theorem

Let $f : A \subseteq \mathbb{C} \to \mathbb{C}$ be analytic inside and on a simple closed contour $\mathscr{C}$, with f' continuous inside and on $\mathscr{C}$. Then $\int_{\mathscr{C}} f(z)\,dz = 0$. □

Notes

(i) The result states that $\int_{\mathscr{C}} f(z)\,dz$ is independent of the contour and the integrand as long as f is analytic and f' is continuous inside and on $\mathscr{C}$. The result is easily extended to the case where $\mathscr{C}$ is expressible as a finite number of simple closed contours. For example,

$$\int_{\mathscr{C}} \frac{e^z}{z^2+9}\,dz = 0 \quad \text{where } \mathscr{C} \text{ is the circle } |z| = 1$$

$$\int_{\mathscr{C}} e^{z^2}\,dz = \int_{\mathscr{C}} \sin(z^3)\,dz = 0 \quad \text{where } \mathscr{C} \text{ is any closed contour}$$

(ii) It may be that for a particular contour $\mathscr{C}$, the conditions of 3.6 are not necessary for $\int_{\mathscr{C}} f(z)\,dz = 0$. For example, using the definition, $\int_{|z|=1} ((\operatorname{Log} z)/z)\,dz = 0$, although the conditions of 3.6 are not met.

The following revised form of 3.6 shows that the condition that f' is continuous inside and on $\mathscr{C}$ can be dropped.

Theorem 3.7. The Cauchy–Goursat Theorem

Let f be analytic in a simply connected region $\mathscr{R}$. Then if $\mathscr{C}$ is any simple closed contour lying totally within $\mathscr{R}$, $\int_{\mathscr{C}} f(z)\,dz = 0$. □

Historical Note

Cauchy first proved his theorem in 1814. Goursat proved his surprising amended version at the turn of the twentieth century. Almost all of the applications of the result were found by Cauchy.

The proof of Cauchy's theorem is staightforward and just uses Green's theorem from real calculus. It is assumed that the reader is familiar with this result. Without the additional assumption of f' continuous inside and on $\mathscr{C}$, the proof is much harder and we defer the proof of 3.7 until the end of the chapter.

Theorem 3.8. Green's Theorem

Let S be a closed set consisting of a simple closed contour $\mathscr{C}$ and its inside. If P, $Q : \mathbb{R}^2 \to \mathbb{R}$ are continuous with continuous first-order partial derivatives on S, then

$$\int_{\mathscr{C}} (P(x,y)\,dx + Q(x,y)dy) = \iint_S (Q_x(x,y) - P_y(x,y))\,dx\,dy$$ □

Proof of 3.6

Recall that if $z(t) = x(t) + iy(t)$, $a \leqslant t \leqslant b$, is any parametrisation of $\mathscr{C}$ and $f(z) = u(x,y) + iv(x,y)$, then

$$\int_{\mathscr{C}} f(z)\,dz = \int_a^b (u(t)x'(t) - v(t)y'(t))\,dt + i\int_a^b (v(t)x'(t) + u(t)y'(t))\,dt \tag{3.4}$$

This can be expressed in terms of real line integrals as

$$\int_{\mathscr{C}} f(z)\,dz = \int_{\mathscr{C}} (u(x,y)\,dx - v(x,y)dy) + i\int_{\mathscr{C}} (v(x,y)\,dx + u(x,y)dy) \tag{3.9}$$

Since f' is continuous on the closed set S, u_x, u_y, v_x and v_y are continuous in S by Theorem 2.5, and so by Green's theorem and (3.9) it follows that

$$\int_{\mathscr{C}} f(z)\,dz = -\iint_S (v_x + u_y)\,dx\,dy + i\iint_S (u_x - v_y)\,dx\,dy$$

Since f is analytic on S, the Cauchy–Riemann equations are satisfied on S by Theorem 2.4; that is, $u_y = -v_x$ and $u_x = v_y$ on S. Hence $\int_{\mathscr{C}} f(z)\,dz = 0$ as required. ■

The following example demonstrates how Cauchy's theorem may be used to evaluate certain real definite integrals.

Example 3.8

Let $\mathscr{C}$ be any simple closed contour not enclosing the origin. Then by Cauchy's theorem, $\int_{\mathscr{C}} (e^{iz}/z)dz = 0$. In particular, let $\mathscr{C}$ be the contour shown in Fig. 3.4. The semicircular arc Γ is parametrised by $z(\theta) = Re^{i\theta}$, $0 \leqslant \theta \leqslant \pi$ so that by definition and (3.7),

$$\left|\int_\Gamma \frac{e^{iz}}{z}dz\right| = \left|\int_0^\pi ie^{iR\cos\theta - R\sin\theta}d\theta\right| \leqslant \int_0^\pi |ie^{iR\cos\theta}||e^{-R\sin\theta}|d\theta$$

$$\Rightarrow \quad \left|\int_\Gamma \frac{e^{iz}}{z}dz\right| \leqslant \int_0^\pi e^{-R\sin\theta}d\theta = 2\int_0^{\pi/2} e^{-R\sin\theta}d\theta \leqslant 2\int_0^{\pi/2} e^{-2R\theta/\pi}d\theta$$

since $2\theta/\pi \leqslant \sin\theta$ for $0 \leqslant \theta \leqslant \pi/2$. Then

$$\left|\int_\Gamma \frac{e^{iz}}{z}dz\right| \leqslant \frac{\pi}{R}(1 - e^{-R}) < \frac{\pi}{R} \quad \text{so that} \quad \int_\Gamma \frac{e^{iz}}{z}dz \to 0 \quad \text{as} \quad R \to \infty$$

Also, $z(\theta) = \varepsilon e^{i\theta}$, $\pi \leqslant \theta \leqslant 0$ on the semicircular arc γ. Then

$$\int_\gamma \frac{e^{iz}}{z}dz = -\int_0^\pi ie^{i\varepsilon e^{i\theta}}d\theta \quad \Rightarrow \quad \lim_{\varepsilon\to 0}\int_\gamma \frac{e^{iz}}{z}dz = -\int_0^\pi i\,d\theta = -i\pi$$

since the integral is a continuous function of ε.

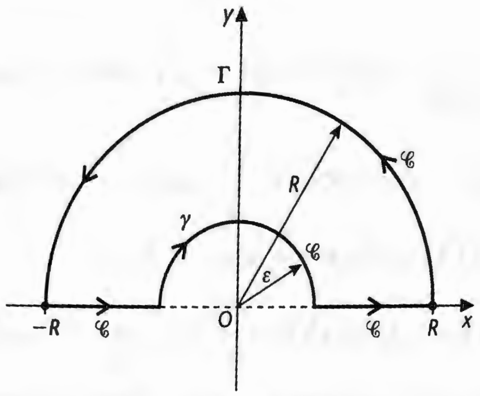


Figure 3.4

Hence, letting $R \to \infty$ and $\varepsilon \to 0$ gives

$$0 = \int_{\mathscr{C}} \frac{e^{iz}}{z} dz = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ix}}{x} dx - i\pi \quad \Rightarrow \quad \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

$$\Rightarrow \quad \int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

by comparing imaginary parts and noting that the last integrand is even. This real definite integral cannot be obtained directly by using the fundamental theorem of real variable calculus.

Corollary 3.9. **Definite Integrals Independent of the Contour**

Let α and β be any two points in a simply connected region $\mathscr{R}$. If $f : A \subseteq \mathbb{C} \to \mathbb{C}$ is analytic throughout $\mathscr{R}$, then $\int_\alpha^\beta f(z)\, dz$ is independent of the choice of contour contained wholly within $\mathscr{R}$ and joining α to β. □

Proof

Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be any two non-intersecting contours (without loss of generality) lying entirely within $\mathscr{R}$ and joining α to β, as shown in Fig 3.5. (Arrows denote the direction of increasing parameter.) $\mathscr{C}_1$ and $\mathscr{C}_2$ form a simple closed contour $\mathscr{C}$, and since $\mathscr{R}$ is simply connected, it follows by 3.7 and 3.1 that

$$\int_{\mathscr{C}} f(z)\, dz = \int_{\mathscr{C}_2} f(z)\, dz - \int_{\mathscr{C}_1} f(z)\, dz = 0$$

as required. ■

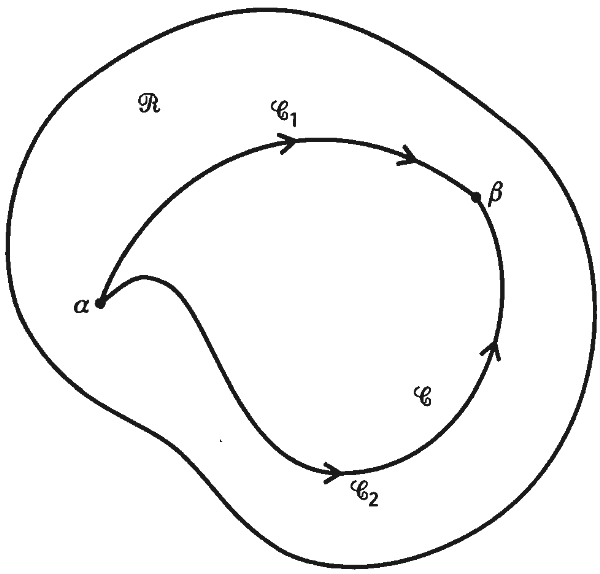


Figure 3.5

Note

Compare Corollaries 3.4 and 3.9. It follows by 3.9 and Theorem 3.5 that if f is analytic in a simply connected region $\mathcal{R}$ then there exists an analytic function F in $\mathcal{R}$ such that $F' = f$.

Cauchy's Integral Formula

Cauchy's integral formula concerns definite integrals around closed contours, in which the integrand has just one singularity, a simple pole, inside the closed contour. We require, first of all, the following result, which is important in its own right.

Lemma 3.10. A Deformation Result

Let $f : A \subseteq \mathbb{C} \to \mathbb{C}$ be analytic in a simply connected region containing a simple closed contour $\mathcal{C}$, except at a point α lying inside $\mathcal{C}$. If $\mathcal{C}'$ is a circle lying totally inside $\mathcal{C}$, centred at α, then

$$\int_{\mathcal{C}} f(z)\,dz = \int_{\mathcal{C}'} f(z)\,dz \qquad \square$$

Note how this result says that any simple closed contour enclosing just one singular point of the integrand can be conveniently replaced by a circle. The resulting integral is then easily evaluated using the definition.

Proof

Let β and γ be any two points on $\mathcal{C}$ and β' and γ' be any two points on $\mathcal{C}'$. Create contours Γ_1 and Γ_2 as shown in Fig. 3.6. (Note the direction of

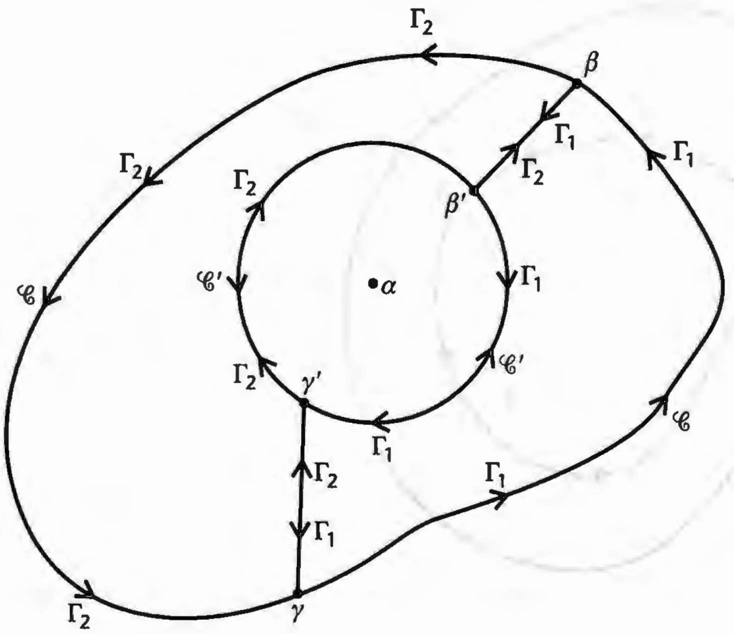


Figure 3.6

increasing parameters.) By 3.1, $\int_{\beta\,\Gamma_1}^{\beta'} f(z)\,dz = -\int_{\beta\,\Gamma_2}^{\beta'} f(z)\,dz$ and similarly for the other line segment,

$$\int_{\mathscr{C}} f(z)\,dz - \int_{\mathscr{C}'} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz$$

by 3.1 again. But f is analytic in a simply connected region containing both Γ_1 and Γ_2, so that by 3.7,

$$\int_{\mathscr{C}} f(z)\,dz - \int_{\mathscr{C}'} f(z)\,dz = 0$$ ∎

Notes

(i) Lemma 3.10 still holds if $\mathscr{C}'$ is replaced by any simple closed contour lying totally inside $\mathscr{C}$ and enclosing α.

(ii) Lemma 3.10 does not hold for integrands with a single branch point since branch points are not isolated singular points. For example, from Exercise 3.1.12(i), if $\mathscr{C}$ is the circle $|z| = R$ then $\int_{\mathscr{C}} z^{1/2} dz = -4iR^{3/2}/3$, so the result depends on the radius of the circle.

Example 3.9

Evaluate $\int_{\mathscr{C}} z^n dz$ for any $n \in \mathbb{Z}$ where $\mathscr{C}$ is any simple closed contour (i) enclosing 0, (ii) not enclosing 0.

Solution

Note that if $n \geqslant 0$ then the integrand is entire, so in either case $\int_{\mathscr{C}} z^n dz = 0$, by 3.6 or 3.7. Also note that in case (ii), if $n < 0$ then the integrand is analytic inside and on $\mathscr{C}$, so $\int_{\mathscr{C}} z^n dz = 0$, also by 3.6 or 3.7. Now consider (i) with $n < 0$, so Cauchy's theorem is not applicable. It follows by 3.10 that $\mathscr{C}$ may be replaced by the unit circle, centred at the origin. Hence let $z(\theta) = e^{i\theta}$ where $-\pi \leqslant \theta \leqslant \pi$ on $\mathscr{C}$ without loss of generality. Then by definition,

$$\int_{\mathscr{C}} z^n dz = \int_{-\pi}^{\pi} e^{ni\theta} ie^{i\theta} d\theta = 0 \qquad (n \neq -1) \quad \text{and} \quad \int_{\mathscr{C}} z^{-1} dz = 2\pi i$$

Alternatively, Theorem 3.3 may be used as in Example 3.7.

An extension of this result with $n = -1$ is the following lemma.

Lemma 3.11. An Integral Around a Circle

Let $\mathscr{C}'$ be the circle of radius ρ, centred at α. Then

$$\int_{\mathscr{C}'} \frac{dz}{z - \alpha} = 2\pi i \qquad \square$$

Proof

If z is any point on $\mathscr{C}'$, then $|z - \alpha| = \rho$, so that a suitable parametrisation of $\mathscr{C}'$ is $z(\theta) = \rho e^{i\theta} + \alpha$, $-\pi \leqslant \theta \leqslant \pi$ say. Then by definition,

$$\int_{\mathscr{C}'} \frac{dz}{z - \alpha} = \int_{-\pi}^{\pi} \frac{i\rho e^{i\theta}}{\rho e^{i\theta}} d\theta = 2\pi i \qquad \blacksquare$$

The last two results lead to the following remarkable result.

Theorem 3.12. Cauchy's Integral Formula

Let $f: A \subseteq \mathbb{C} \to \mathbb{C}$ be analytic in a simply connected region $\mathscr{R}$. If $\mathscr{C}$ is any simple closed contour in $\mathscr{R}$ and α is any point inside $\mathscr{C}$, then

$$f(\alpha) = \frac{1}{2\pi i} \int_{\mathscr{C}} \frac{f(z)}{z - \alpha} dz \qquad \square$$

Note

The result says that the given integral is independent of $\mathscr{C}$ and only depends on the value of f at the simple pole α of the integrand.

Proof

The proof can be conveniently split into two parts.

Step 1

$$I = \int_{\mathscr{C}} \frac{f(z)}{z-\alpha}\,dz = \int_{\mathscr{C}'} \frac{f(\alpha)}{z-\alpha}\,dz + \int_{\mathscr{C}'} \frac{f(z)-f(\alpha)}{z-\alpha}\,dz$$

where $\mathscr{C}'$ is any circle, centred at α, lying within $\mathscr{C}$ by 3.10.

$$\text{Hence} \quad I = 2\pi i f(\alpha) + I_1 \quad \text{where} \quad I_1 = \int_{\mathscr{C}'} \frac{f(z)-f(\alpha)}{z-\alpha}\,dz$$

by 3.11. Notice by 3.2, taking the radius of $\mathscr{C}'$ to be ρ, $|I_1| \leqslant 2\pi\rho M$, where

$$\left|\frac{f(z)-f(\alpha)}{z-\alpha}\right| = \frac{|f(z)-f(\alpha)|}{\rho} \leqslant M \quad \text{on } \mathscr{C}'$$

It remains to show that $I_1 = 0$.

Step 2

Since f is analytic at α, it is continuous at α, so that by definition, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|z-\alpha| < \delta \quad \Rightarrow \quad |f(z)-f(\alpha)| < \varepsilon$$

Then from step 1,

$$|I_1| < 2\pi\varepsilon \quad \text{whenever} \quad |z-\alpha| < \delta$$

But on $\mathscr{C}'$, $|z-\alpha| = \rho$, so that choosing $\rho < \delta$ gives $|I_1| < 2\pi\varepsilon$. Hence $\lim_{\rho\to 0} I_1 = 0$, as required. ∎

For example, it follows directly from 3.12 that if $\mathscr{C}$ is the rectangle with vertices at $\pm i$ and ± 2, then $\int_{\mathscr{C}} z^{-1}dz = 2\pi i$. Compare this method to using the definition – see the solution to Exercise 3.1.8!

Example 3.10

Let $\mathscr{C}$ be the circle $|z| = 3$. Using partial fractions,

$$\int_{\mathscr{C}} \frac{\cosh(z^2)}{z(z^2+4)}\,dz = \int_{\mathscr{C}} \frac{\cosh(z^2)}{4z}\,dz - \int_{\mathscr{C}} \frac{\cosh(z^2)}{8(z+2i)}\,dz - \int_{\mathscr{C}} \frac{\cosh(z^2)}{8(z-2i)}\,dz$$

The numerator in the integrand is entire and the points 0, $\pm 2i$ lie inside $\mathscr{C}$. Hence by Cauchy's integral formula,

$$\int_{\mathscr{C}} \frac{\cosh(z^2)}{z(z^2+4)}\,dz = \frac{2\pi i}{8}(2\cosh 0 - 2\cosh(-4)) = \frac{\pi i}{2}(1-\cosh 4)$$

Example 3.11

Let $\mathscr{C}$ be the unit circle, centred at the origin, parametrised by $z(\theta) = e^{i\theta}$, $-\pi \leqslant \theta \leqslant \pi$. Evaluate $\int_{\mathscr{C}} (e^{az}/z)\,dz$ where a is a non-zero real constant, hence evaluate the real integral $\int_0^\pi e^{a\cos\theta}\cos(a\sin\theta)d\theta$.

Solution

Note that the given integrands have no indefinite integrals expressible in terms of elementary functions. However, $f(z) = e^{az}$ is entire and 0 is inside $\mathscr{C}$. Then by Cauchy's integral formula,

$$\int_{\mathscr{C}} \frac{e^{az}}{z}\,dz = 2\pi i e^0 = 2\pi i$$

Then by definition, it follows that

$$\int_{\mathscr{C}} \frac{e^{az}}{z}\,dz = \int_{-\pi}^{\pi} \frac{(\exp(ae^{i\theta}))(ie^{i\theta})}{e^{i\theta}}\,d\theta = i\int_{-\pi}^{\pi} e^{a(\cos\theta + i\sin\theta)}\,d\theta = 2\pi i$$

$$\Rightarrow \quad \int_{-\pi}^{\pi} e^{a\cos\theta}(\cos(a\sin\theta) + i\sin(a\sin\theta))\,d\theta = 2\pi$$

Equating real parts gives

$$\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = 2\pi \quad \Rightarrow \quad \int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \pi$$

since the integrand is even.

A very important corollary of Cauchy's integral formula is the following result.

Theorem 3.13. Cauchy's Integral Formula for Derivatives

Let f be analytic in a simply connected region $\mathscr{R}$ and let $\mathscr{C}$ be a simple closed contour in $\mathscr{R}$. If α is any point inside $\mathscr{C}$ then

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i}\int_{\mathscr{C}} \frac{f(z)}{(z-\alpha)^{n+1}}\,dz \qquad n \in \mathbb{N}$$

□

Proof

Once again, we split the proof up into two parts.

Step 1

To illustrate the technique of the proof, we first show that the result is true for $n = 1$. By hypothesis and 3.12, if $\alpha + h$ lies inside $\mathscr{C}$ then

$$\frac{f(\alpha+h) - f(\alpha)}{h} = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{f(z)}{h}\left(\frac{1}{z-\alpha-h} - \frac{1}{z-\alpha}\right)dz = \frac{1}{2\pi i}I_1$$

where $$I_1 = \int_{\mathscr{C}} \frac{f(z)}{(z-\alpha-h)(z-\alpha)}\,dz$$

Now let $$I_2 = \int_{\mathscr{C}} \frac{f(z)}{(z-\alpha)^2}\,dz$$

It certainly seems reasonable that $I_1 \to I_2$ as $h \to 0$. We now show rigorously that this is indeed the case.

Let $|f(z)| \leqslant M$ on $\mathscr{C}$, L be the length of $\mathscr{C}$ and d the shortest distance from α to any point z on $\mathscr{C}$. Then $|z-\alpha| \geqslant d$ and it follows by the triangle inequality that

$$|z-\alpha| \leqslant |z-\alpha-h| + |h| \quad \Rightarrow \quad |z-\alpha-h| \geqslant d - |h|$$

Hence, altogether,

$$|z-\alpha| \geqslant d \ \text{ and } \ 2|h| \leqslant d \quad \Rightarrow \quad |z-\alpha-h| \geqslant d/2 \tag{3.10}$$

and it follows by 3.2 that for $2|h| \leqslant d$,

$$|I_1 - I_2| = \left| h \int_{\mathscr{C}} \frac{f(z)}{(z-\alpha)^2(z-\alpha-h)}\,dz \right| \leqslant \frac{2|h|ML}{d^3}$$

so that $I_1 - I_2 \to 0$ as $h \to 0$. Then

$$f'(\alpha) = \lim_{h\to 0} \frac{f(\alpha+h) - f(\alpha)}{h} = \frac{1}{2\pi i} I_2 = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{f(z)}{(z-\alpha)^2}\,dz$$

Step 2

The given result then follows by induction. In practice the inductive step is a little tedious, but uses the same technique as step 1. By above, the result is true for $n = 1$. Suppose that

$$f^{(k)}(\alpha) = \frac{k!}{2\pi i}\int_{\mathscr{C}} \frac{f(z)}{(z-\alpha)^{k+1}}\,dz$$

for some $k \in \mathbb{N}$.

Then $$\frac{f^{(k)}(\alpha+h) - f^{(k)}(\alpha)}{h} = \frac{k!}{2\pi i} I_1$$

where $$I_1 = \int_{\mathscr{C}} \frac{f(z)(w^{k+1} - (w-h)^{k+1})}{hw^{k+1}(w-h)^{k+1}}\,dz$$

and where $w = z - \alpha$.

Let $I_2 = (k+1)\displaystyle\int_{\mathscr{C}} \frac{f(z)}{w^{k+2}}\,dz$ then $|I_1 - I_2| = \left| \displaystyle\int_{\mathscr{C}} \frac{f(z)P(w,h)}{hw^{k+2}(w-h)^{k+1}}\,dz \right|$

where,

$$P(w,h) = w(w^{k+1} - (w-h)^{k+1}) - (k+1)h(w-h)^{k+1}$$
$$\Rightarrow \quad P(w,h) = w^{k+2} - (w+(k+1)h)(w-h)^{k+1}$$
$$\Rightarrow \quad P(w,h) = w^{k+2} - (w+(k+1)h)(w^{k+1} - (k+1)w^k h + \frac{k(k+1)}{2} w^{k-1}h^2$$
$$+ \text{ terms in higher powers of } h$$
$$\Rightarrow \quad P(w,h) = \tfrac{1}{2}(k+1)(k+2)h^2 w^k + \text{terms in higher powers of } h$$

using the binomial theorem. Then using Lemma 3.2 and (3.10),

$$|I_1 - I_2| \leqslant \frac{ML|P(w,h)|}{|h|d^{k+2}(d/2)^{k+1}}$$
$$\leqslant \frac{ML2^{k+1}}{d^{2k+3}}\left(\tfrac{1}{2}(k+1)(k+2)D^k + \text{terms in powers of } |h|\right)|h|$$

where $|z-\alpha| \leqslant D$ on $\mathscr{C}$. Then $I_1 \to I_2$ as $h \to 0$ and

$$f^{(k+1)}(\alpha) = \lim_{h\to 0} \frac{f^{(k)}(\alpha+h) - f^{(k)}(\alpha)}{h} = \frac{(k+1)k!}{2\pi i}\int_{\mathscr{C}} \frac{f(z)}{(z-\alpha)^{k+2}}dz$$

as required. ■

Note

What the bulk of this proof is really doing is showing that differentiation under the integral sign with respect to α is valid. Once this has been shown, the inductive proof of 3.13 is very straightforward (see Exercises 3.2).

Example 3.12

Let $\mathscr{C}$ be any simple closed contour enclosing 0. Then, since $f(z) = \sinh z$ is entire, it follows by 3.13 that

$$\int_{\mathscr{C}} \frac{\sinh z}{z^3}dz = \frac{2\pi i}{2!}f''(0) = \pi i \sinh 0 = 0$$

If a function f is analytic at a point α, there is an open neighbourhood of α in which f is analytic, so 3.13 holds in this neighbourhood. This gives the following immediate corollary.

Corollary 3.14. Derivatives of Analytic Functions

If a function f is analytic at a point, then its derivatives of all orders are also analytic at that point. □

Note

This result was used in Chapter 2 when discussing harmonic functions. Bu[t it] does not hold for functions mapping $\mathbb{R}$ to $\mathbb{R}$. It is easy to construct examples [of] such functions which are differentiable only once at a point. For instan[ce] consider $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^{3/2}$, $x \geqslant 0$ and $f(x) = 0$, $x < 0$. T[his] function is differentiable only once at 0.

The following result, which is the converse of the Cauchy–Goursat theore[m] can be deduced easily from 3.5 and 3.14. This result is often useful [in] determining whether or not a given function is analytic in some region. F[or] instance, it is a useful tool in the study of differentiability of functions defin[ed] by infinite series.

Theorem 3.15. Morera's Theorem

Let f be piecewise continuous in a simply connected region $\mathscr{R}$ and suppose tha[t] $\int_{\mathscr{C}} f(z)\,dz = 0$ for all closed contours contained within $\mathscr{R}$. Then f is analytic i[n] $\mathscr{R}$. □

Proof

It follows by hypothesis that if α and β are any two points in $\mathscr{R}$ then, using th[e] argument of the proof of 3.9, $\int_{\alpha}^{\beta} f(z)\,dz$ is independent of any contour joining [α] to β and contained within $\mathscr{R}$. Then by 3.5, f is the derivative of an analyti[c] function in $\mathscr{R}$. Hence f is also analytic in $\mathscr{R}$ by 3.14. ■

Exercise **3.2.1** Show, using the definition, that if $\mathscr{C}$ is the circle $|z| = 1$, then

$$\int_{\mathscr{C}} \frac{\operatorname{Log}(z^{1/2})}{z}\,dz = 0$$

even though the integrand does not meet the conditions of Cauchy's theorem.

Exercise **3.2.2** Use Cauchy's theorem or Cauchy's integral formula to evaluate

$$\int_{\mathscr{C}} \frac{z^2}{z^2+9}\,dz$$

(i) where $\mathscr{C}$ is the circle $|z| = 2$

(ii) where $\mathscr{C}$ is the circle $|z - 2i| = 2$

(iii) where $\mathscr{C}$ is the triangle with vertices at 0, $1 - 4i$ and $-2 - 5i$

Exercise **3.2.3** Using Cauchy's theorem, evaluate $\int_{\mathscr{C}} e^z dz$, where $\mathscr{C}$ is the semicircle comprising the semicircular arc $|z| = a$, $-\pi/2 \leqslant \operatorname{Arg} z \leqslant \pi/2$ and the line segment $|\operatorname{Im} z| \leqslant a, \operatorname{Re} z = 0$. Verify the result by using the fundamental theorem of calculus. Use the result to evaluate the real integral

$$\int_0^{\pi/2} e^{a\cos\theta} \cos(a\sin\theta + \theta)\,d\theta$$

Exercise **3.2.4**

(i) Evaluate $\int_{\mathscr{C}} e^{-z^2} dz$ where $\mathscr{C}$ is any simple closed contour.

(ii) Given that $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$, use the result of (i) with $\mathscr{C}$ as the rectangle with vertices $0, a, a+ib$ and ib, to evaluate $\int_0^\infty e^{-x^2} \cos 2bx\, dx$, and to show that

$$\int_0^\infty e^{-x^2} \sin 2bx\, dx = e^{-b^2} \int_0^b e^{x^2} dx$$

(iii) Use the result of (i) with $\mathscr{C}$ as the boundary of the sector given by $0 \leqslant |z| \leqslant R$, $0 \leqslant \operatorname{Arg} z \leqslant \pi/4$, to deduce that

$$\int_0^\infty \cos(x^2)\, dx = \int_0^\infty \sin(x^2)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} \qquad \text{(Fresnel's integrals)}$$

You may assume that the given integrals converge and that

$$\lim_{R\to\infty} \int_0^{\pi/4} Re^{-R^2 \cos 2\theta} d\theta = 0$$

Exercise **3.2.5** Evaluate $\int_{\mathscr{C}} z^{-1/2} dz$, where $\mathscr{C}$ is the circle $|z| = R$, using the fundamental theorem of calculus. Hence show that the deformation result, Lemma 3.10, does not hold in this case.

Exercise **3.2.6** Let a simple closed contour $\mathscr{C}$ enclose just two singular points of a function f at α and β. Let $\mathscr{C}'$ and $\mathscr{C}''$ be disjoint circles lying totally inside $\mathscr{C}$, centred at α and β respectively. Prove that $\int_{\mathscr{C}} f(z)\, dz = \int_{\mathscr{C}'} f(z)\, dz + \int_{\mathscr{C}''} f(z)\, dz$.

Exercise **3.2.7** Let $\mathscr{C}$ be a simple closed contour enclosing 0 and 1. Use the result of Exercise 3.2.6, Cauchy's theorem and the definition only, to evaluate $\int_{\mathscr{C}} ((2z-1)/z^2 - z))dz$.

Exercise **3.2.8** Use Cauchy's integral formula to prove that

$$\int_{\mathscr{C}} \frac{e^{zt}}{z^2+4} dz = \pi i \sin 2t \qquad (t > 0)$$

where $\mathscr{C}$ is the circle $|z| = 3$.

Exercise **3.2.9** Evaluate $\int_{\mathscr{C}} ((\cosh nz)/z) dz$ where $n \in \mathbb{N}$ and $\mathscr{C}$ is the circle $|z| = 1$, using Cauchy's integral formula. Hence evaluate $\int_0^{\pi/2} \cosh(n \cos\theta) \cos(n \sin\theta)\, d\theta$.

Exercise **3.2.10** Evaluate $\int_{\mathscr{C}} (e^{iz^2}/z) dz$ where $\mathscr{C}$ is the circle $|z| = 1$. Hence evaluate $\int_0^{2\pi} e^{-\sin 2\theta} \cos(\cos 2\theta)\, d\theta$.

Exercise **3.2.11** Use Cauchy's integral formula for derivatives to evaluate $\int_{\mathscr{C}} (e^{2z}/z^2)\, dz$, where $\mathscr{C}$ is the circle $|z| = 1$. Hence evaluate $\int_0^\pi e^{2\cos\theta} \cos(2\sin\theta - \theta)\, d\theta$.

Exercise **3.2.12** Let f be analytic in a simply connected region containing a simple closed contour $\mathscr{C}$. Assuming Cauchy's integral formula and that differentiation under the integral sign with respect to α is valid, prove Cauchy's integral formula for derivatives by induction.

Consequences of Cauchy's Integral Formulae

We now prove several important results which can be derived from Cauchy's integral formulae. The first result states that the only bounded entire functions are constant.

Theorem 3.16. Liouville's Theorem

If f is entire and $|f(z)| \leqslant M$ for some positive real constant M, for all $z \in \mathbb{C}$, then f is constant. □

Proof

Let $\mathscr{C}$ be a circle of radius r, centred at α. Since f is entire and bounded by M, it follows by 3.13 and 3.2 that

$$|f'(\alpha)| = \frac{1}{2\pi}\left|\int_{\mathscr{C}} \frac{f(z)}{(z-\alpha)^2}\, dz\right| \leqslant \frac{1}{2\pi}\left(\frac{M}{r^2}\right)2\pi r = \frac{M}{r}$$

for any chosen r. Hence $|f'(\alpha)| = 0$, so $f'(\alpha) = 0$ for any $\alpha \in \mathbb{C}$. It follows by the results of Chapter 2 that f is a constant. ■

Note

Liouville's theorem does not hold for functions mapping $\mathbb{R}$ to $\mathbb{R}$. For instance, $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin x$ is entire and $|\sin x| \leqslant 1$ for all $x \in \mathbb{R}$. On the other hand, $|\sin z|$ is not bounded for all $z \in \mathbb{C}$.

Liouville's theorem can be used to give an elegant proof of the following famous result.

Theorem 3.17. The Fundamental Theorem of Algebra

Every polynomial equation

$$P(z) = a_0 + a_1 z + \ldots + a_n z^n = 0$$

with degree n, $n \in \mathbb{N}$, has at least one root. It follows that $P(z) = 0$ has exactly n roots. □

Proof

Suppose that $P(z) = 0$ has no roots. Then $1/P(z)$ is entire and $|1/P(z)| \to 0$ as $|z| \to \infty$ so that $1/P(z)$ is bounded for all z. Hence $1/P(z)$ is a constant by 3.16, and this is a contradiction. Thus $P(z) = 0$ has at least one root. Suppose this root is $z = \alpha$. If $n > 1$ then $P(z) = P(z) - P(\alpha) = (z - \alpha)Q(z)$ say, where Q is a polynomial of degree $n - 1$, so $Q(z) = 0$ has at least one root by the above. Continuing this process shows that $P(z) = 0$ has exactly n roots, as required. ■

Another two very useful consequences of Cauchy's integral formula are the so-called maximum and minimum modulus theorems.

Theorem 3.18. The Maximum Modulus Theorem

If f is analytic inside a simple closed contour $\mathscr{C}$ and continuous on $\mathscr{C}$, and is not identically equal to a constant, then the maximum value of $|f(z)|$, inside and on $\mathscr{C}$, occurs on $\mathscr{C}$. □

Proof

Step 1

By hypothesis, f and hence $|f|$ is continuous inside and on $\mathscr{C}$. Suppose that the maximum value of $|f(z)|$ is attained at a point α inside $\mathscr{C}$ and $|f(\alpha)| = M$. Let $\mathscr{C}'$ be a circle lying inside $\mathscr{C}$, centred at α. Then, by hypothesis, there exists a point β inside $\mathscr{C}'$ such that $|f(\beta)| < M$, so that $|f(\beta)| = M - \varepsilon$ for some $\varepsilon > 0$, with $|f|$ continuous at β. Then given $\varepsilon/2 > 0$ there exists $\delta > 0$ such that

$$
\begin{aligned}
&|z - \beta| < \delta \quad \Rightarrow \quad \big||f(z)| - |f(\beta)|\big| < \varepsilon/2 \\
&\Rightarrow \quad |f(z)| < \varepsilon/2 + |f(\beta)| = M - \varepsilon/2
\end{aligned}
$$

Step 2

Suppose that $\mathscr{C}_1$ is a circle centred at β with radius $\rho < \delta$, chosen so that $\mathscr{C}_1$ is contained within $\mathscr{C}'$ as shown in Fig. 3.7. It then follows by step 1 that

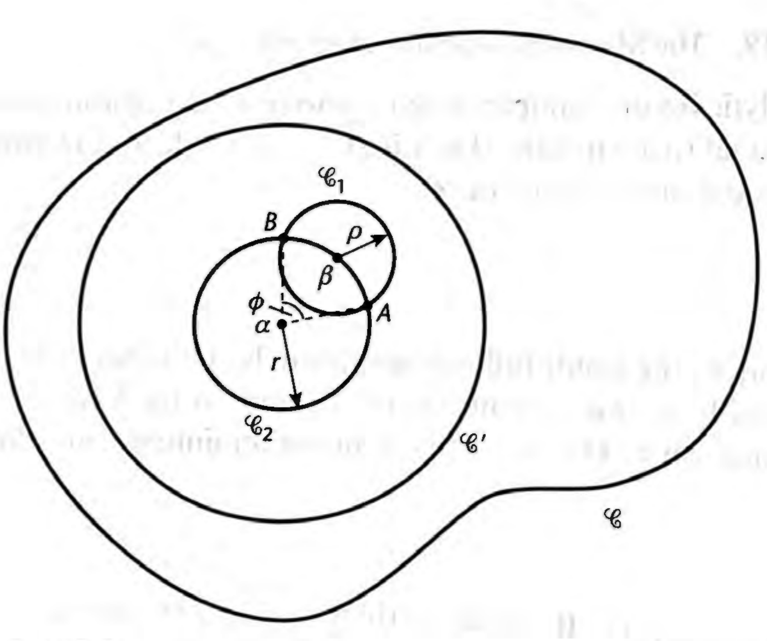


Figure 3.7

$|f(z)| < M - \varepsilon/2$ for all z inside $\mathscr{C}_1$. Now consider the circle $\mathscr{C}_2$ of radius r, centred at α, and passing through β as shown in Fig. 3.7. On the arc AB of $\mathscr{C}_2$, which is inside $\mathscr{C}_1$, $|f(z)| < M - \varepsilon/2$, and on the arc BA of $\mathscr{C}_2$, $|f(z)| \leqslant M$. Let ϕ be the angle subtended at α by the arc AB, as shown in Fig. 3.7.

Step 3

The circle $\mathscr{C}_2$ is parametrised by $z = \alpha + re^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$ say, where the point A is given by $\theta = 0$, without loss of generality. Then, since f is analytic at α, it follows by Cauchy's integral formula and the definition that

$$f(\alpha) = \frac{1}{2\pi i}\int_{\mathscr{C}_2} \frac{f(z)}{z-\alpha}\,dz = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha + re^{i\theta})\,d\theta$$

$$\Rightarrow \quad f(\alpha) = \frac{1}{2\pi}\int_0^{\phi} f(\alpha + re^{i\theta})\,d\theta + \frac{1}{2\pi}\int_{\phi}^{2\pi} f(\alpha + re^{i\theta})\,d\theta$$

$$\Rightarrow \quad |f(\alpha)| \leqslant \frac{1}{2\pi}\int_0^{\phi} (M - \varepsilon/2)\,d\theta + \frac{1}{2\pi}\int_{\phi}^{2\pi} M\,d\theta = M - \frac{\phi\varepsilon}{4\pi}$$

Hence $|f(\alpha)| = M \leqslant M - \phi\varepsilon/4\pi$, which is a contradiction, so $|f(z)|$ must attain its maximum value on $\mathscr{C}$. ∎

Note

Since f is continuous on the closed set consisting of $\mathscr{C}$ and its inside, it does have a maximum value on this domain, from the comments in Chapter 2 (see Theorem 4.5).

Theorem 3.19. The Minimum Modulus Theorem

Let f be analytic inside a simple closed contour $\mathscr{C}$ and continuous on $\mathscr{C}$, and not identically equal to a constant. Then if $f(z) \neq 0$ inside $\mathscr{C}$, the minimum value of $|f(z)|$, inside and on $\mathscr{C}$, occurs on $\mathscr{C}$. □

Proof

If $f(z) = 0$ on $\mathscr{C}$, the result follows immediately. Otherwise, by hypothesis, $1/f$ is analytic inside $\mathscr{C}$ and continuous on $\mathscr{C}$, and so by 3.18, $|1/f(z)|$ attains its maximum value on $\mathscr{C}$. Hence $|f(z)|$ attains its minimum value on $\mathscr{C}$. ∎

Note

It is clear that if $f(z) = 0$ inside $\mathscr{C}$, then $|f(z)|$ need not assume its minimum value, i.e. 0, on $\mathscr{C}$. Consider $f(z) = z$ and let $\mathscr{C}$ be the circle $|z| = 1$ for instance.

The Location of Roots of Equations

Another important application of the preceding results is a result concerning the number of poles and zeros of a particular type of function inside a simple closed contour, and the related Rouché's theorem, both very useful in locating the position of roots of equations in the complex plane. We first need the following generalisation of Lemma 3.10.

Lemma 3.20. A Generalised Deformation Result

Let $f: A \subseteq \mathbb{C} \to \mathbb{C}$ be analytic inside and on a simple closed contour $\mathscr{C}$, except at a finite number of singular points $z_1, z_2, \ldots, z_m$, lying inside $\mathscr{C}$. Then if $\mathscr{C}_k$ is a circle centred at z_k lying totally inside $\mathscr{C}$, for each $k = 1, \ldots, m$, such that none of the circles intersect, then

$$\int_{\mathscr{C}} f(z)\, dz = \sum_{k=1}^{m} \int_{\mathscr{C}_k} f(z)\, dz \qquad \square$$

Proof

Note that, since there are only a finite number of singular points of f inside $\mathscr{C}$, it is possible to construct the circles $\mathscr{C}_k$ with radii chosen so that none of them intersect. Connect each circle to $\mathscr{C}$ by two lines and create simple closed contours Γ and Γ_k for each k, which do not enclose $z_1, \ldots, z_m$, as shown in Fig. 3.8.

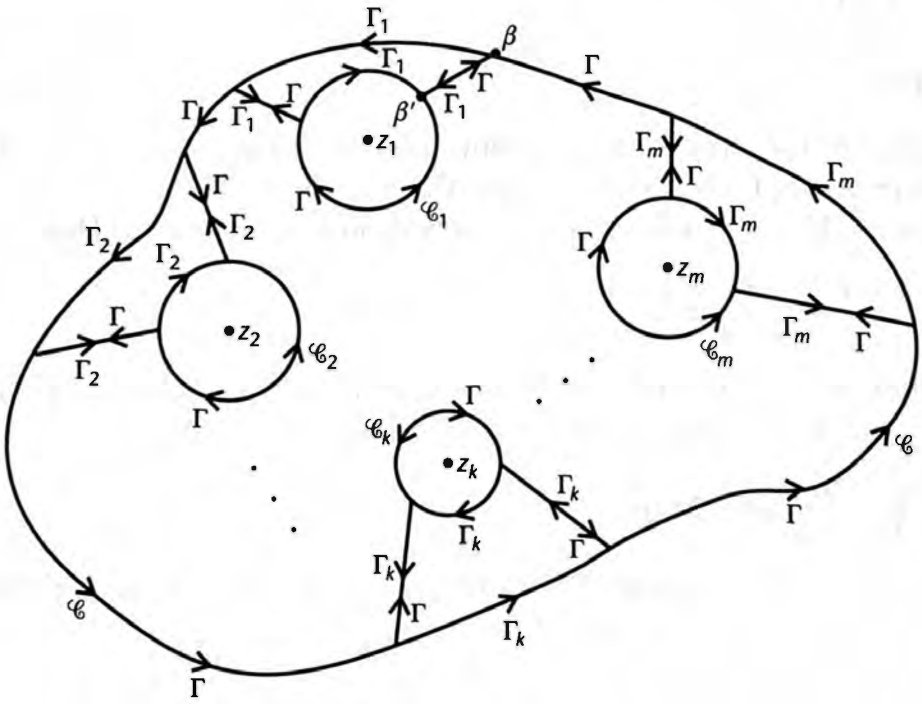


Figure 3.8

It follows by hypothesis and Cauchy's theorem that

$$\int_{\Gamma} f(z)\,dz = \int_{\Gamma_k} f(z)\,dz = 0 \qquad (k = 1, \ldots, m)$$

Also, by 3.1, $\int_{\beta\,\Gamma_1}^{\beta'} f(z)\,dz = -\int_{\beta\,\Gamma}^{\beta'} f(z)\,dz$ and similarly for the other line segments, so that, by 3.1 again,

$$\int_{\mathscr{C}} f(z)\,dz - \sum_{k=1}^{m} \int_{\mathscr{C}_k} f(z)\,dz = \int_{\Gamma} f(z)\,dz + \sum_{k=1}^{m} \int_{\Gamma_k} f(z)\,dz = 0$$ ■

Definition

A function which is analytic on a region $\mathscr{R}$, except possibly for poles, is **meromorphic** on $\mathscr{R}$.

Any function which has no branch points or essential singular points is meromorphic on $\mathbb{C}$. For example, algebraic fractions, the hyperbolic and trigonometric functions are all meromorphic on $\mathbb{C}$. An important result which concerns meromorphic functions is the following theorem.

Theorem 3.21. Poles and Zeros of Meromorphic Functions

Let f be analytic inside and on a simple closed contour $\mathscr{C}$, except possibly for P poles inside $\mathscr{C}$. Let $f(z) \neq 0$ on $\mathscr{C}$ and let f have Z zeros inside $\mathscr{C}$. (A pole or zero of order n is counted n times.) Then

$$\frac{1}{2\pi i}\int_{\mathscr{C}} \frac{f'(z)}{f(z)}\,dz = Z - P$$ □

Proof

Clearly, the only singular points of f'/f occur at the zeros and poles of f. Suppose that α is a zero of f of order n inside $\mathscr{C}$. Then $f(z) = (z-\alpha)^n g(z)$ where g is analytic and non-zero at α, without loss of generality. Hence

$$\frac{f'(z)}{f(z)} = \frac{n}{z-\alpha} + \frac{g'(z)}{g(z)}$$

where g'/g is analytic at α. It then follows by Cauchy's theorem and 3.11 that if $\mathscr{C}_\alpha$ is a circle, centred at α and not enclosing any other zeros or poles of f,

$$\int_{\mathscr{C}_\alpha} \frac{f'(z)}{f(z)}\,dz = 2n\pi i \tag{3.11}$$

Similarly, if β is a pole of f of order m inside $\mathscr{C}$ then $f(z) = h(z)/(z-\beta)^m$, where h is analytic and non-zero at β, so that

$$\frac{f'(z)}{f(z)} = \frac{h'(z)}{h(z)} - \frac{m}{z-\beta}$$

where h'/h is analytic at β. Then if $\mathscr{C}_\beta$ is a circle, centred at β and not enclosing any other zeros or poles of f,

$$\int_{\mathscr{C}_\beta} \frac{f'(z)}{f(z)}\,dz = -2m\pi i \tag{3.12}$$

The result then follows directly from (3.11), (3.12) and Lemma 3.20. ■

Example 3.13

Let $f(z) = (z^2+1)/z^3$. Then f has simple zeros at $\pm i$ and a pole of order 3 at 0. The number of zeros Z of f inside the circle $|z| = 2$, say, is 2 and the number of poles P of f inside the same circle is 3. Hence, by 3.21,

$$\int_{\mathscr{C}} \frac{f'(z)}{f(z)}\,dz = 2\pi i(Z-P) \quad \Rightarrow \quad \int_{\mathscr{C}} \frac{z^2+3}{z(z^2+1)}\,dz = 2\pi i$$

where $\mathscr{C}$ is the circle $|z| = 2$. This result can be checked by using partial fractions and Cauchy's integral formula:

$$\int_{\mathscr{C}} \frac{z^2+3}{z(z^2+1)}\,dz = \int_{\mathscr{C}} \frac{3}{z}\,dz - \int_{\mathscr{C}} \frac{dz}{z+i} - \int_{\mathscr{C}} \frac{dz}{z-i} = 6\pi i - 4\pi i = 2\pi i$$

Theorem 3.21 can be used to locate roots of equations in each of the four quadrants of the complex plane, as the following example demonstrates.

Example 3.14 The Principle of the Argument

It follows by 3.21 that if f is analytic inside and on a simple closed contour $\mathscr{C}$ and $f(z) \neq 0$ on $\mathscr{C}$, then the number of zeros of f inside $\mathscr{C}$ is

$$Z = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{f'(z)}{f(z)}\,dz$$

Also, for any particular branch of log, that is for a specific $k \in \mathbb{Z}$,

$$\frac{d}{dz}(\log(f(z))) = \frac{d}{dz}(\mathrm{Log}|f(z)| + i(\mathrm{Arg}(f(z)) + 2k\pi)) = \frac{f'(z)}{f(z)}$$

by Theorem 2.3(b)(iii), except along the associated branch cut. It follows by Theorem 3.3 that

$$Z = \frac{1}{2\pi}\Delta_{\mathscr{C}} \arg(f(z)) \tag{3.13}$$

where $\Delta_{\mathscr{C}} \arg(f(z))$ is the total change in any continuously varying argument of f around the contour $\mathscr{C}$. The formal proof of (3.13), usually known as the **principle of the argument**, is omitted.

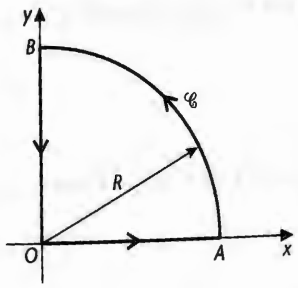


Figure 3.9

Now consider the equation $f(z) = z^5 - z + 16 = 0$. It is easily checked that this equation has no purely imaginary roots. Also for $x \in \mathbb{R}, f(x) = x^5 - x + 16 > 0$ for $x \geqslant 0$, (since $x < x^5 + 16$ for $x < 16$ and $x(x^4 - 1) + 16 > 0$ for $x \geqslant 1$). Function f has a local maximum when $x < 0$ with $5x^4 = 1$ and since $F(-1) > 0$ and $F(-2) < 0$, it follows (by the intermediate value theorem) that the given equation has exactly one root on the negative real axis.

We now use the principle of the argument to show that the given equation has exactly one root in the first quadrant. Note that f is entire. Consider the closed contour OAB, consisting of two line segments and the arc of a circle of radius R, shown in Fig. 3.9. Choose $\arg(f(z)) = 0$ on OA. Then on the arc AB, $z = Re^{i\theta}$, $0 \leqslant \theta \leqslant \pi/2$, so that $f(z) = R^5 e^{5i\theta} - Re^{i\theta} + 16$. Then as $R \to \infty$, $\Delta_{AB}\arg(f(z)) \to 5\pi/2$. On the line segment BO, $f(z) = i(y^5 - y) + 16$ $\Rightarrow \arg(f(z)) = \tan^{-1}(y^4(y-1)/16)$. Note that $g(y) = y^4(y-1)/16 = 0$ when $y = 0$ or 1 and that $g(y) \to \infty$ as $y \to \infty$. Hence as $R \to \infty$, $\Delta_{BO}\arg(f(z)) \to -\pi/2$. Then $\Delta_{\mathscr{C}}\arg(f(z)) \to 2\pi$ as $R \to \infty$ and it follows by (3.13) that f has exactly one zero in the first quadrant. Since f must have five zeros altogether (by the fundamental theorem of algebra) and its complex zeros occur in conjugate pairs, f has exactly one zero in each quadrant.

Theorem 3.21 also leads to the following theorem, which gives another method of locating roots of equations.

Theorem 3.22. Rouché's Theorem

If f and g are analytic inside and on a simple closed contour $\mathscr{C}$ and $|g(z)| < |f(z)|$ on $\mathscr{C}$, then f and $f + g$ have the same number of zeros inside $\mathscr{C}$. □

Proof

Step 1

Let the function F be defined inside and on $\mathscr{C}$ by $F(z, t) = f(z) + tg(z)$ where t is real with $t \in [0, 1]$. Since f and g are analytic inside and on $\mathscr{C}$, F has no poles inside or on $\mathscr{C}$, for fixed t. Also, since $|g(z)| < |f(z)|$ on $\mathscr{C}$, F does not have a zero at any point on $\mathscr{C}$. (Note that at a possible zero of f on $\mathscr{C}$, $f(z) = -tg(z) \Rightarrow |f(z)| \leqslant |g(z)|$ and this is a contradiction.) It follows by Theorem 3.21 that if

$$Z(t) = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{f'(z) + tg'(z)}{f(z) + tg(z)}\,dz$$

then $Z(t)$ is the number of zeros of F inside $\mathscr{C}$.

Step 2

We now show that Z is a continuous function of t. For $s \in [0, 1]$,

$$|Z(t) - Z(s)| = \frac{|t-s|}{2\pi}\left|\int_{\mathscr{C}} \frac{(f(z)g'(z) - f'(z)g(z))}{F(z,t)F(z,s)}\,dz\right| \tag{3.14}$$

It follows by the triangle inequality that

$$|F(z,t)| \geqslant |f(z)| - t|g(z)| \geqslant |f(z)| - |g(z)| > 0 \quad \text{on } \mathscr{C}$$

$$\text{and} \quad |F(z,s)| \geqslant |f(z)| - |g(z)| > 0 \quad \text{on } \mathscr{C}$$

Since f and g and their derivatives are continuous on $\mathscr{C}$, and $\mathscr{C}$ is a compact subset of $\mathbb{C}$, there exists a constant M such that

$$\left|\frac{f(z)g'(z) - f'(z)g(z)}{(|f(z)| - |g(z)|)^2}\right| \leqslant M \quad \text{on } \mathscr{C}$$

It then follows by the above, Lemma 3.2 and (3.14) that

$$|Z(t) - Z(s)| \leqslant \frac{ML}{2\pi}|t-s|$$

where L is the length of $\mathscr{C}$, so that Z is a continuous function of t, as required. Since Z is integer-valued it follows (by the intermediate value theorem) that $Z(t)$ is constant for $t \in [0, 1]$. Hence $Z(0) = Z(1)$, where $Z(0)$ is the number of zeros of f and $Z(1)$ is the number of zeros of $f + g$. ■

Rouché's theorem is particularly useful for locating roots of equations in circular and annular regions, as the following example demonstrates.

Example 3.15

Use Rouché's theorem to show that the four roots of $z^4 + 4(1+i)z + 1 = 0$ lie inside the circle $|z| = 2$, three of which lie in the annular region $1 \leqslant |z| < 2$.

Solution

Take $f(z) = z^4$ and $g(z) = 4(1+i)z + 1$, so that f and g are entire. On the circle $\mathscr{C}$ with equation $|z| = 2$,

$$|f(z)| = |z^4| = 16 \text{ and } |g(z)| \leqslant 4|1+i||z| + 1 = 4\sqrt{2}\cdot 2 + 1 < 16$$

so that $|g(z)| < |f(z)|$ on $\mathscr{C}$. Also f has one zero, a zero of order 4 at 0, and this point lies inside $\mathscr{C}$. Hence $f+g$ has exactly four zeros inside the same curve.

Now let $F(z) = 4(1+i)z$ and $G(z) = z^4 + 1$, so that, once again, F and G are entire. On the circle $\mathscr{C}'$ with equation $|z| = 1$

$$|F(z)| = 4\sqrt{2}|z| = 4\sqrt{2} \quad \text{and} \quad |G(z)| \leqslant |z|^4 + 1 = 2 < 4\sqrt{2}$$

so that $|G(z)| < |F(z)|$ on $\mathscr{C}'$. Clearly F has a simple zero at 0, which lies inside $\mathscr{C}'$, so $F+G$ has exactly one zero inside $\mathscr{C}'$, as required.

Exercise **3.3.1** Use Cauchy's integral formula for derivatives to prove the following results:

(i) If f is analytic in a simply connected region containing a circle with centre α and radius r, then

$$f^{(n)}(\alpha) = \frac{n!}{2\pi r^n}\int_0^{2\pi} e^{-ni\theta} f(\alpha + re^{i\theta})\,d\theta$$

(ii) If f is analytic with $|f(z)| \leqslant M$ in a simply connected region containing the circle $|z| = r$, then

$$|\alpha| < r \quad \Rightarrow \quad |f^{(n)}(\alpha)| \leqslant \frac{Mrn!}{(r - |\alpha|)^{n+1}}$$

Exercise **3.3.2** Use Cauchy's theorem and Cauchy's integral formula to prove that if f is analytic in a simply connected region containing the circle $\mathscr{C}$ with equation $|z| = r$, and α is any non-zero point inside $\mathscr{C}$, then

$$f(\alpha) = \frac{1}{2\pi i}\int_{\mathscr{C}} \left(\frac{1}{z-\alpha} - \frac{1}{z - r^2/\bar{\alpha}}\right) f(z)\,dz$$

Deduce that if $0 < \rho < r$, then

$$f(\rho e^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(r^2 - \rho^2)}{r^2 - 2r\rho\cos(\theta - \phi) + \rho^2} f(re^{i\theta})\,d\theta$$

(This result is known as Poisson's integral formula for a circle.)

Exercise **3.3.3** Let f be entire and $|f(z)| \geqslant m > 0$ for all $z \in \mathbb{C}$, where m is a constant. Use Liouville's theorem to prove that f is a constant.

Exercise **3.3.4** Use Liouville's theorem to prove that if

$$f(z + 2\pi) = f(z + 2\pi i) = f(z) \qquad \text{(for all } z) \qquad (*)$$

and f is entire, then f is a constant. (A function satisfying condition (*) is a **doubly periodic function**.)

Exercise **3.3.5*** Let f be an entire function which is not identically zero. Suppose there exist two positive real numbers r and M and a non-negative integer n such that

$$|f(z)| < M|z|^n \qquad (|z| > r)$$

Use Liouville's theorem and induction to prove that f is a polynomial of at most degree n.

Exercise **3.3.6*** Let f be a function that is analytic inside the unit circle $|z| = 1$ and which satisfies

$$f(0) = 0 \quad \text{and} \quad |f(z)| \leqslant 1 \qquad (0 < |z| < 1)$$

Use the maximum modulus theorem to prove **Schwarz's lemma**:

$$|f'(0)| \leqslant 1 \quad \text{and} \quad |f(z)| \leqslant |z| \qquad (0 < |z| < 1)$$

(*Hint*: Consider the function g defined by $g(z) = f(z)/z$ for $0 < |z| < 1$, with $g(0) = f'(0)$. Then g is analytic inside the unit circle.)

Exercise **3.3.7** Use Theorem 3.21 with $f(z) = (z-4)^3/(1+z^2)$ to evaluate

$$\int_{\mathscr{C}} \frac{z^2 + 8z + 3}{(z-4)(1+z^2)}\, dz$$

where $\mathscr{C}$ is the circle $|z| = 5$.

Exercise **3.3.8** Use the principle of the argument to show that $z^8 + 4z^3 + 5z + 3 = 0$ has exactly two roots in the first quadrant.

Exercise **3.3.9** Use the principle of the argument to investigate the position of the roots of the equation $z^7 + 3z + 1 = 0$, determining how many roots lie on each axis and in each quadrant of the complex plane.

Exercise **3.3.10** Use Rouché's theorem to prove that the equation $e^{2iz} = 12z^n$, $n \in \mathbb{N}$, has n roots inside the circle $|z| = 1$.

Exercise **3.3.11** Use Rouché's theorem to show that the equation $z^6 + 7z + 1 = 0$ has six roots inside the circle $|z| = 2$, five of which lie in the annular region $1 \leqslant |z| < 2$.

Exercise **3.3.12** Determine the number of roots of the equation $8z^3 + 4z^2 + 2z - 3 = 0$ inside the circle $|z| = 1$. (*Hint*: Multiply the given equation by $2z - 1$ and apply Rouché's theorem.)

Exercise **3.3.13** Determine all the roots of $z^4 + 4 = 0$. Prove that the equation

$$(z^2 - 1)^2 + 2(z-1)^2 + 1 = 0$$

has exactly one root in each quadrant, by simplifying the equation and using Rouché's theorem.

Exercise **3.3.14** Use Rouché's theorem to show that $z^5 - z + 16 = 0$ has exactly one root in the first quadrant, which lies inside the circle $|z| = 2$. (Compare with Example 3.14.)

The Cauchy–Goursat Theorem

We end this chapter with the promised proof of Theorem 3.7. The easiest way to proceed with the proof is to prove that the result holds for a closed polygon first of all. The simplest closed polygon is a triangle, so that will be our starting point.

Theorem 3.23. Cauchy's Theorem for a Triangle

Let f be analytic in a simply connected region containing a triangle Δ. Then

$$\int_{\Delta} f(z)\, dz = 0 \qquad \square$$

Proof

The basic idea is to approximate the integrand in a small enough region by an integrand which has an indefinite integral, so that the result follows by the fundamental theorem of calculus.

Step 1

Given the triangle Δ, form four new triangles T_1, T_2, T_3 and T_4 by connecting the midpoints of the sides of Δ as shown in Fig. 3.10. Let $I = \int_{\Delta} f(z)\, dz$ and $J_k = \int_{T_k} f(z)\, dz$, $k = 1, \ldots, 4$. Then

$$I = J_1 + J_2 + J_3 + J_4 \quad \Rightarrow \quad |I| \leqslant |J_1| + |J_2| + |J_3| + |J_4|$$

by 3.1 (since there are equal and opposite contributions to the integrals J_k along the line segments M_1M_2, M_1M_3 and M_2M_3). Let I_1 be the the integral among the J_k with maximum modulus. It then follows by the above inequality that $|I| \leqslant 4|I_1|$. Relabel the associated triangle as 'new' triangle Δ_1. By the same process, Δ_1 can be divided into four congruent triangles such that the contour integral I_2 of f along one of these triangles, Δ_2 say, satisfies $|I_1| \leqslant 4|I_2|$.

Continuing this process, after the nth stage, we obtain a sequence of triangles Δ, Δ_1, $\Delta_2, \ldots, \Delta_n$ for which the corresponding contour integrals I, I_1, $I_2, \ldots, I_n$ of f along these triangles satisfy

$$|I| \leqslant 4|I_1| \leqslant 4^2|I_2| \leqslant \ldots \leqslant 4^n|I_n| \tag{3.15}$$

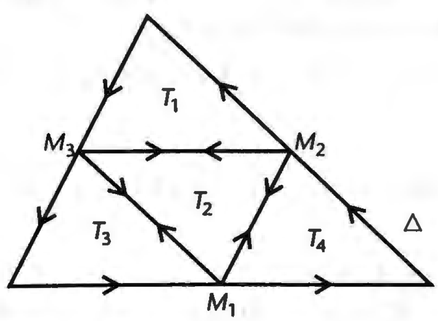


Figure 3.10

for any natural number n. The process also yields a sequence of nested triangular closed sets

$$\triangle^* \supseteq \triangle_1^* \supseteq \triangle_2^* \supseteq \ldots \supseteq \triangle_n^*$$

where $\triangle^*$ consists of $\triangle$ and its inside and similarly for $\triangle_k^*$.

Let L be the length of $\triangle$ and L_n be the length of $\triangle_n$ for each n. It follows by the way in which the triangles are constructed that $L_n = L/2^n$, so that $L_n \to 0$ as $n \to \infty$. Hence there is exactly one point, α say, which belongs to every $\triangle_n^*$.

Step 2

By hypothesis, f is differentiable at this point α, so that

$$f(z) = f(\alpha) + (z-\alpha)f'(\alpha) + \varepsilon(z,\alpha)(z-\alpha)$$

$$\text{where} \quad \varepsilon(z,\alpha) = \frac{f(z)-f(\alpha)}{z-\alpha} - f'(\alpha) \quad \Rightarrow \quad \lim_{z\to\alpha} \varepsilon(z,\alpha) = 0$$

It then follows that

$$I_n = \int_{\triangle_n} (f(\alpha) + (z-\alpha)f'(\alpha))\,dz + \int_{\triangle_n} \varepsilon(z,\alpha)(z-\alpha)\,dz$$

The integrand of the first integral is a linear function, so it has an indefinite integral. Hence, by Corollary 3.4, the integral is zero, so

$$I_n = \int_{\triangle_n} \varepsilon(z,\alpha)(z-\alpha)\,dz$$

Since $\lim_{z\to\alpha} \varepsilon(z,\alpha) = 0$, it follows that, given any $\varepsilon_1 > 0$, there is a $\delta > 0$ such that $|z-\alpha| < \delta \Rightarrow |\varepsilon(z,\alpha)| < \varepsilon_1$. Also, $\triangle_n$ lies within the circular region $|z-\alpha| < L_n$ and since $\lim_{n\to\infty} L_n = 0$, there is an $N \in \mathbb{N}$ such that $L_n < \delta$ for all $n > N$. Then by Lemma 3.2,

$$|I_n| \leqslant (\varepsilon_1 L_n)L_n \quad \Rightarrow \quad |I_n| \leqslant \varepsilon_1 \frac{L^2}{4^n} \qquad (n > N)$$

Hence from the chain of inequalities, (3.15), it follows that $|I| \leqslant L^2\varepsilon_1$ for any $\varepsilon_1 > 0$. Hence $I = 0$ as required. ■

Note

Cauchy's theorem for a rectangle follows in almost the same way; only very minor modifications are needed in the proof.

The next step in proving the Cauchy–Goursat theorem is to show that it holds for any closed polygon, not just a triangle. Essentially this is true because any polygon can be split up into a finite number of triangles.

Theorem 3.24. Cauchy's Theorem for a Polygon

Let f be analytic in a simply connected region containing a simple closed polygon $\mathscr{P}$. Then $\int_{\mathscr{P}} f(z)\,dz = 0$. □

Part Proof

It can be shown that any simple closed polygon can be **triangulated**; that is the boundary and inside of the polygon is the union of triangles $\Delta_1, \ldots, \Delta_m$ say, and their insides, any two of which have either a vertex, a whole side or no point in common, such that each side of a triangle is either a side of one other triangle or a side of the polygon. Such a triangulation is shown in Fig. 3.11.

As in the case of the Jordan curve theorem, we shall not prove this intuitive result; the formal proof is quite lengthy. (For the details, see for example, J. W. Dettman, *Applied Complex Variables*, Macmillan, 1965.)

Since the net contribution to the integral along the line segments forming the triangles in the triangulation of $\mathscr{P}$ is zero by 3.1,

$$\int_{\mathscr{P}} f(z)\,dz = \sum_{k=1}^{m} \int_{\Delta_k} f(z)\,dz = 0 \qquad \text{(by 3.23)}$$

as required. ■

Note that the above result is easily extended to any closed polygon. The last step in the proof of the Cauchy–Goursat theorem is to show that the integral of a continuous function along any simple smooth arc can be approximated by an integral along a simple polygon.

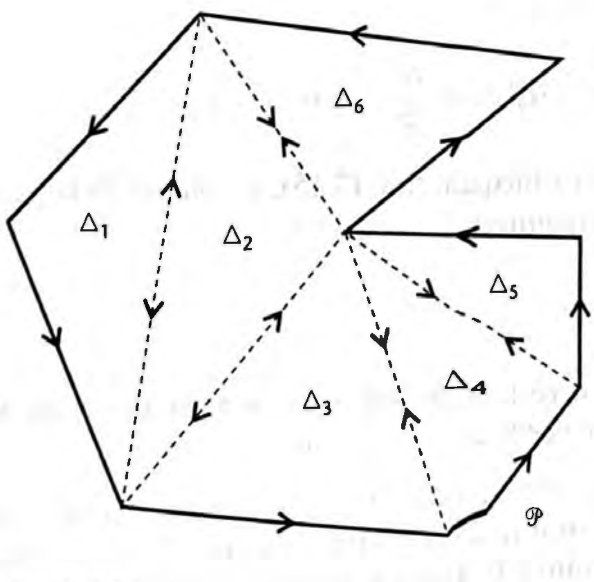


Figure 3.11

Theorem 3.25. Approximation by Polygons

Let $\mathscr{C}$ be a simple smooth arc and f a continuous function in some simply connected region $\mathscr{R}$ containing $\mathscr{C}$. Then given any $\varepsilon > 0$, there exists a simple polygon $\mathscr{P}$ contained in $\mathscr{R}$ such that

$$\left| \int_{\mathscr{C}} f(z)\, dz - \int_{\mathscr{P}} f(z)\, dz \right| < \varepsilon \qquad \square$$

Proof

Step 1

Note that $\mathscr{C}$ may be covered by a finite number of circles, centred at points on $\mathscr{C}$, which lie totally within $\mathscr{R}$, as indicated in Fig. 3.12. Let L denote the length of $\mathscr{C}$. Since f is continuous in $\mathscr{R}$, by choosing the largest radius of all the circles small enough, it is possible to construct such a system of circles which, together with their insides, form a closed set S with the property that given any $\varepsilon > 0$ there exists $\delta > 0$ such that for any points z and w in S,

$$|z - w| < \delta \quad \Rightarrow \quad |f(z) - f(w)| < \varepsilon/2L \tag{3.16}$$

Step 2

Let $\mathscr{C}$ be parametrised by $z = z(t)$, $t \in [a, b]$. Recall that $\int_{\mathscr{C}} f(z)\, dz$ can be expressed in terms of real line integrals as in (3.9). For any particular subdivision $a = t_0, t_1, \ldots, t_n = b$ of $[a, b]$, let $S_n = \sum_{k=1}^{n} f(z_{k-1})(z_k - z_{k-1})$ where $z_k = z(t_k)$. It then follows from the definition of a Riemann line integral

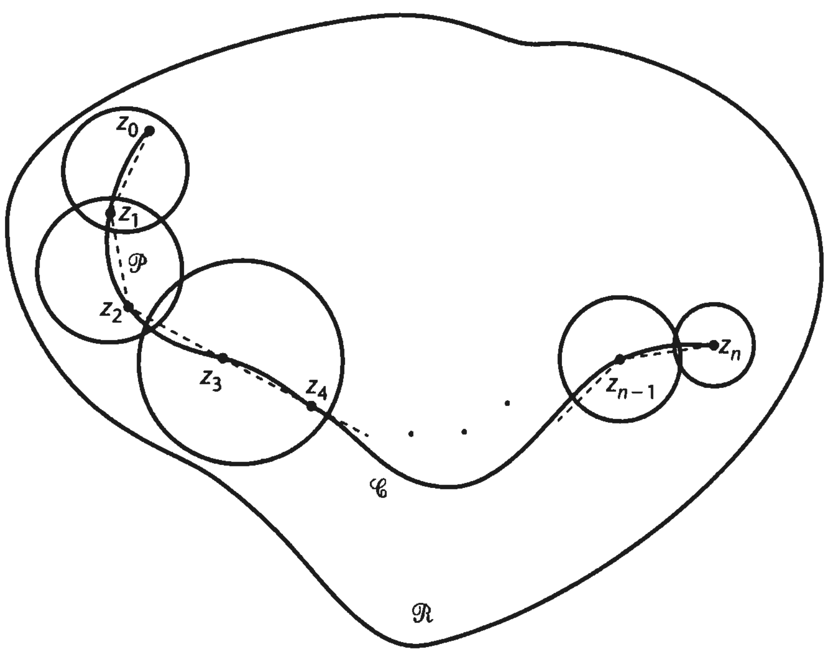


Figure 3.12

that $\int_{\mathscr{C}} f(z)\,dz = \lim_{\eta\to 0} S_n$ where $|z_k - z_{k-1}| \leqslant \eta$ for all k (so that n depends on η). Then, by (3.16), given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$\left| \int_{\mathscr{C}} f(z)\,dz - S_n \right| < \varepsilon/2 \tag{3.17}$$

and the polygon $\mathscr{P}$ with vertices $z_0, \ldots, z_n$ lies within the set S, with the property that $|f(z) - f(z_k)| < \varepsilon/2L$ along each line segment of $\mathscr{P}$ (Fig. 3.12).

Step 3

For the polygon $\mathscr{P}$ constructed in step 2,

$$\begin{aligned}\int_{\mathscr{P}} f(z)\,dz &= \int_{z_0}^{z_1} (f(z) - f(z_1) + f(z_1))\,dz + \ldots + \int_{z_{n-1}}^{z_n} (f(z) - f(z_n) + f(z_n))\,dz \\ &= \int_{z_0}^{z_1} (f(z) - f(z_1))\,dz + \ldots + \int_{z_{n-1}}^{z_n} (f(z) - f(z_n))\,dz + S_n\end{aligned}$$

It follows by the construction of step 2, (3.17) and 3.2 that

$$\left| S_n - \int_{\mathscr{P}} f(z)\,dz \right| < \frac{\varepsilon}{2L}(|z_1 - z_0| + \ldots + |z_n - z_{n-1}|) < \frac{\varepsilon}{2} \tag{3.18}$$

since $\sum_{k=1}^{n} |z_k - z_{k-1}|$, the length of $\mathscr{P}$, is necessarily less than L, the length of $\mathscr{C}$. Finally, by (3.17) and (3.18) we have

$$\begin{aligned}\left| \int_{\mathscr{C}} f(z)\,dz - \int_{\mathscr{P}} f(z)\,dz \right| &\leqslant \left| \int_{\mathscr{C}} f(z)\,dz - S_n \right| + \left| S_n - \int_{\mathscr{P}} f(z)\,dz \right| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon\end{aligned}$$

as required. ■

When $\mathscr{P}$ is a simple closed polygon, it is immediately true that the above result holds if $\mathscr{C}$ is replaced by any simple closed contour. It then follows by 3.24 that if $\mathscr{C}$ is a simple closed contour lying totally within a simply connected set in which f is analytic, $|\int_{\mathscr{C}} f(z)\,dz| < \varepsilon$ for any given ε and so the Cauchy–Goursat theorem follows.

Notes

(i) It is straightforward to show that if f is analytic inside and on a simple closed contour $\mathscr{C}$, then f is analytic in a simply connected region containing $\mathscr{C}$, so we only really need the former condition in order for the Cauchy–Goursat theorem to hold.

(ii) It can be shown that f only needs to be continuous on $\mathscr{C}$ and analytic inside $\mathscr{C}$ for the Cauchy–Goursat theorem to hold, but the proof of this more general result is quite involved. See, for example, R. L. Goodstein, *Complex Functions*, McGraw-Hill, 1965.

4 Infinite Series, Taylor Series and Laurent Series

The main aim of this chapter is the study of complex power series, leading to Taylor series and their generalisation to Laurent series. Such series have many important applications. For instance, we need results concerning Laurent series in order to proceed further with complex integration. It turns out that the treatment of Taylor series in the complex case is actually easier and more satisfactory that in the real case. Power series expansions are also of fundamental use for solving linear differential equations, as described in Chapter 7. In order to study these special infinite series we begin, as in the real case, with sequences.

Sequences

Analogous to the idea of a sequence of real numbers, a sequence of complex numbers is a list of complex numbers derived from a definite rule.

Definition

A **sequence** of complex numbers, denoted by (z_n), is determined by a function $f : \mathbb{N} \to \mathbb{C}$ defined by $f(n) = z_n$, where z_n is the ***n*th term** in the sequence.

Example 4.1

Certain sequences of complex numbers defined by recurrence relations have attracted a lot of interest in recent years because of their connection with fractals and chaos.

Consider the sequence (z_n) defined by

$$z_{n+1} = z_n^2 + \alpha \quad \text{with} \quad z_1, \alpha \in \mathbb{C} \text{ chosen}$$

For a particular choice of z_1, the set of values of α for which this sequence is bounded is a **Mandlebrot set**, $M_2(z_1)$. Mandlebrot sets are **fractals**; that is, they are geometrical figures in the complex plane that consist of an identical pattern repeating itself on an ever reduced scale. The set $M_2(z_1)$, pictured as a geometrical figure, dramatically changes shape for small variations in the initial value z_1. Such Mandlebrot sets have surprisingly complicated structures. See, for example, B. B. Mandlebrot, *Fractals: Form, Chance and Dimension*, W. H. Freeman, 1977.

In many cases, definitions and results concerning sequences of complex numbers are analogous to definitions and results for sequences of real numbers. Also, in many cases the proof of a result is essentially the same as the proof of the corresponding result for sequences of real numbers, and usually we shall omit it.

If (z_n) is a sequence of complex numbers and z_n becomes 'arbitrarily close' to $\alpha \in \mathbb{C}$ as n becomes 'arbitrarily large', then α is the limit of the sequence. This intuitive idea is formalised in the following definition.

Definition

A sequence (z_n) **converges to limit α**, where $\alpha \in \mathbb{C}$, if given any real $\varepsilon > 0$, a natural number N (depending on ε) can be found such that

$$|z_n - \alpha| < \varepsilon \quad \text{whenever} \quad n > N$$

If (z_n) has limit α, we write $z_n \to \alpha$ as $n \to \infty$, or $\lim_{n\to\infty} z_n = \alpha$.

Example 4.2

Intuitively, $n/(1+in) = 1/(1/n + i) \to 1/i = -i$ as $n \to \infty$. Prove that $n/(1+in) \to -i$ as $n \to \infty$, using the definition.

Solution

$$\left|\frac{n}{1+in} + i\right| = \left|\frac{i}{1+in}\right| = \frac{1}{(1+n^2)^{1/2}} < \frac{1}{n} \qquad (n \in \mathbb{N})$$

$$\text{Hence} \quad \left|\frac{n}{1+in} + i\right| < \frac{1}{n} < \frac{1}{N} \quad \text{whenever} \quad n > N$$

Then given any $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $N > 1/\varepsilon$. Then for this choice of N,

$$\left|\frac{n}{1+in} + i\right| < \varepsilon \quad \text{whenever} \quad n > N$$

as required.

The following theorem is a list of results which are analogous to results for real sequences and can be proved in exactly the same way, since $|z|$ for $z \in \mathbb{C}$ obeys the same properties as $|x|$ for $x \in \mathbb{R}$ and, in particular, the triangle inequality.

Theorem 4.1. Elementary Properties of Limits

(a) If the limit of a sequence exists, then it is unique.

(b) Any convergent sequence (z_n) is **bounded**; that is, there exists a positive real constant M such that $|z_n| \leqslant M$ for all $n \in \mathbb{N}$.

(c) Let $z_n \to \alpha$ and $w_n \to \beta$ as $n \to \infty$. Then

(i) $\lambda z_n \to \lambda\alpha$ as $n \to \infty$ for any $\lambda \in \mathbb{C}$

(ii) $z_n + w_n \to \alpha + \beta$ as $n \to \infty$

(iii) $z_n w_n \to \alpha\beta$ as $n \to \infty$

(iv) $z_n/w_n \to \alpha/\beta$ as $n \to \infty$ provided that $\beta \neq 0$

(d) Let $z_n \to \alpha$ as $n \to \infty$ and let $f: A \subseteq \mathbb{C} \to \mathbb{C}$ be continuous at each point of (z_n). Then $f(z_n) \to f(\alpha)$ as $n \to \infty$. □

Example 4.3

$n/(1+in) = 1/(1/n+i)$ and $1/n \to 0$ as $n \to \infty$. Hence by 4.1(c), $1/(1/n+i) \to 1/i$ as $n \to \infty$, so that $n/(1+in) \to -i$ as $n \to \infty$. (Compare with Example 4.2.)

Note

Results concerning real sequences involving inequalities of real numbers will not in general carry over to the complex case. For example, it makes no sense to talk of monotonic sequences of complex numbers, so there is no monotonic-bounded principle.

The following result implies that sequences of complex numbers can be investigated by examining their real and imaginary parts and using results for sequences of real numbers.

Theorem 4.2. Real and Imaginary Parts of Sequences

$z_n \to \alpha$ as $n \to \infty$ if and only if $\operatorname{Re} z_n \to \operatorname{Re}\alpha$ and $\operatorname{Im} z_n \to \operatorname{Im}\alpha$. □

Proof

Let $z_n = x_n + iy_n$ for all $n \in \mathbb{N}$ and $\alpha = a + ib$ where a and b are real. Suppose that $z_n \to \alpha$ as $n \to \infty$, so that given any $\varepsilon > 0$, there is a natural number N such that

$$|z_n - \alpha| = |(x_n - a) + i(y_n - b)| < \varepsilon \quad \text{whenever} \quad n > N$$

Now

$$|x_n - a| \leqslant |(x_n - a) + i(y_n - b)|$$
$$|y_n - b| \leqslant |(x_n - a) + i(y_n - b)|$$

for all $n \in \mathbb{N}$. Hence, given any $\varepsilon > 0$, there is a natural number N such that

$$|x_n - a| < \varepsilon \quad \text{and} \quad |y_n - b| < \varepsilon \quad \text{whenever} \quad n > N$$

Hence, by definition, $x_n \to a$ and $y_n \to b$ as $n \to \infty$.

Now suppose that these conditions hold. Then given any $\varepsilon > 0$, there exist natural numbers N_1 and N_2 such that $|x_n - a| < \varepsilon/2$ whenever $n > N_1$ and $|y_n - b| < \varepsilon/2$ whenever $n > N_2$. Then by the triangle inequality,

$$|z_n - \alpha| \leqslant |x_n - a| + |y_n - b| < \varepsilon \quad \text{whenever} \quad n > \max(N_1, N_2)$$

and so $z_n \to \alpha$ as $n \to \infty$, as required. ■

The standard results from real analysis concerning subsequences carry over to the complex case.

Definition

The sequence (w_n) is a **subsequence** of the sequence (z_n) if there exist natural numbers $n_1 < n_2 < \ldots$ such that $w_k = z_{n_k}, k \in \mathbb{N}$.

Theorem 4.3. Subsequences of Bounded Sequences

Any bounded sequence in $\mathbb{C}$ has a convergent subsequence. □

Proof

Let (z_n) be bounded so that there exists a positive real number M such that $|z_n| \leqslant M$ for all $n \in \mathbb{N}$. Then $|\operatorname{Re} z_n| \leqslant M$, so that $(\operatorname{Re} z_n)$ is a bounded sequence in $\mathbb{R}$ and hence has a convergent subsequence, $(\operatorname{Re} z_{n_k}) = (\operatorname{Re} w_k)$ say, in $\mathbb{R}$. By hypothesis, $(\operatorname{Im} w_k)$ is a bounded real sequence and so has a convergent subsequence, $(\operatorname{Im} w_{k_m})$ say, in $\mathbb{R}$. As a subsequence of $(\operatorname{Re} w_k)$, the sequence $(\operatorname{Re} w_{k_m})$ must converge too. Hence (w_{k_m}) is a convergent subsequence of (z_n), as required. ■

Recall the definition of a limit point given in Chapter 1. The following is a generalisation of a famous result from real analysis.

Theorem 4.4. The Bolzano–Weierstrass Theorem

Any bounded infinite subset S of $\mathbb{C}$ has a limit point. □

Proof

Since S is infinite, we may select a sequence (z_n) with the points z_n distinct and belonging to S. Since S is bounded, (z_n) has a convergent subsequence by 4.3. If

α is the limit of such a subsequence, by definition, every deleted neighbourhood of α contains points of (z_n) and hence S. Then α is a limit point of S, as required. ■

Using Theorem 4.4, we can proceed as in the real case to prove the following result, which was stated and used in the previous two chapters.

Theorem 4.5. Continuous Functions on Compact Sets

Let S be a compact subset of $\mathbb{C}$ and let $f: A \to \mathbb{C}$ be continuous at each point of S. Then f is bounded on S; that is, there exists a real constant M such that $|f(z)| \leqslant M$ for all $z \in S$. □

Proof

Suppose that f is unbounded on S. Then there exists distinct $z_n \in S$ such that $|f(z_n)| > n$ for each $n \in \mathbb{N}$. This defines a sequence (z_n) of points in S, which has a limit point α by 4.4. Note that $\alpha \in S$ since S is closed. Hence the function f is continuous at α, so $f(\alpha) = \lim_{z_n \to \alpha} f(z_n)$ by 4.1(d). But this is contradicted by the fact that $|f(z_n)| > n$ for each n. Hence f must be bounded in S. ■

In defining limits of sequences, we have defined $\lim_{n\to\infty} f(n)$ where $f: N \to \mathbb{C}$. In general, if $f: A \subseteq \mathbb{C} \to \mathbb{C}$, then $\lim_{z\to\infty} f(z)$ can be defined in terms of limits already defined in Chapter 2.

Definition

Let $f: A \subseteq \mathbb{C} \to \mathbb{C}$. Then **$f(z)$ has limit ℓ as z tends to infinity**, written $\lim\limits_{z\to\infty} f(z) = \ell$, if given any $\varepsilon > 0$ there is a $\delta > 0$ (depending on ε) such that

$$\left|\frac{1}{z}\right| < \delta \quad \Rightarrow \quad |f(z) - \ell| < \varepsilon$$

This definition makes sense since there is only one point at infinity in the extended complex plane. The definition formalises the idea that, as $|z|$ becomes 'arbitrarily large', i.e. $1/z$ becomes 'arbitrarily close' to 0, $f(z)$ becomes 'arbitrarily close' to ℓ. This concept will prove useful in Chapter 5.

Notes

(i) Theorem 4.1 applies to $\lim_{z\to\infty} f(z)$; in particular, if the limit exists, it is unique, no matter how $z \to \infty$.

(ii) It is easy to show that $\lim_{z\to\infty} f(z) = \ell$ if and only if $\lim_{z\to 0} g(z) = \ell$, where $g(z) = f(1/z)$.

Example 4.4

(i) Intuitively, $\lim_{z\to\infty} 1/z^k = 0$, for any $k \in \mathbb{R}^+$. This can be proved using the definition as follows.

$$\left|\frac{1}{z}\right| < \delta \quad \Rightarrow \quad \left|\frac{1}{z^k} - 0\right| = \frac{1}{|z|^k} < \delta^k \quad \text{since} \quad k > 0$$

Then given any $\varepsilon > 0$, we choose $\delta = \varepsilon^{1/k}$. For this choice of δ,

$$\left|\frac{1}{z}\right| < \delta \quad \Rightarrow \quad \left|\frac{1}{z^k} - 0\right| < \varepsilon$$

(ii) Although $\lim_{x\to+\infty} e^{-x} = 0$, $\lim_{z\to\infty} e^{-z}$ does not exist. For example, letting $z = -x, x \in \mathbb{R}^+$, $\lim_{z\to\infty} e^{-z}$ does not exist. Alternatively, letting $z = iy, e^{-z} = e^{-iy} = \cos y - i \sin y$ and $\lim_{y\to\infty} e^{-iy}$ does not exist.

Sequences of Functions

Let $(f_n(z))$ denote a sequence of values of functions $f_n : A \subseteq \mathbb{C} \to \mathbb{C}$, $n \in \mathbb{N}$. Each value of z in the domain A gives a different sequence and if each such sequence converges, it is possible to construct a limit function with values $f(z)$, called the **pointwise limit function** of the sequence $(f_n(z))$. In general, such a limit function will not be continuous on A, even if each f_n is continuous.

Example 4.5

Let $f_n(z) = 1 - z^n$ for $|z| \leqslant 1$. Then $f_n(z) \to 1$ as $n \to \infty$ for $|z| < 1$ but $f_n(1) = 0 \to 0$ as $n \to \infty$.

Clearly, it is important to find conditions which ensure that $(f_n(z))$ converges to a continuous pointwise limit function, and more important, conditions under which termwise integration and differentiation of such a sequence of functions is valid. This leads to the concept of uniform convergence, the definition of which is essentially the same as in the real case.

Note

Readers who are primarily interested in applications of Taylor and Laurent series and who are willing to accept that power series may be differentiated and integrated termwise within their domain of convergence, may wish to omit references to uniform convergence.

Definition

Let $f_n : A \subseteq \mathbb{C} \to \mathbb{C}$ for each $n \in \mathbb{N}$. A sequence of functions $(f_n(z))$ **converges uniformly** to its pointwise limit function $f(z)$ in the domain A, if given any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ (depending on ε but **independent of** z), such that

$$|f_n(z) - f(z)| < \varepsilon \quad \text{whenever } n > N \quad \text{for all } z \in A$$

Note

The crucial point to notice about uniform convergence is that the choice of N does not depend on the value of z; that is, the same N will satisfy the definition for all $z \in A$.

Example 4.6

The sequence (z^n) converges uniformly to the zero function for $|z| \leqslant 1/2$. To show this, notice first of all that the pointwise limit function is given by $f(z) = 0$, for $|z| \leqslant 1/2$. Now consider any value of z in the given domain. Then

$$|z| \leqslant 1/2 \quad \Rightarrow \quad |z^n - 0| = |z|^n \leqslant \frac{1}{2^n}$$

It follows by the binomial theorem that $2^n = (1+1)^n > n$ for all $n \in \mathbb{N}$. Hence

$$|z| \leqslant 1/2 \quad \Rightarrow \quad |z^n - 0| \leqslant \frac{1}{2^n} < \frac{1}{n} < \frac{1}{N} \quad \text{whenever} \quad n > N$$

Given any $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $N > 1/\varepsilon$. Note that N is independent of z. Then for this choice of N,

$$|z| \leqslant 1/2 \quad \Rightarrow \quad |z^n - 0| < \varepsilon \quad \text{whenever} \quad n > N$$

as required.

Note

In general, it is only possible to establish uniform convergence of a sequence of functions on a set A if A is compact, as in the above example.

Theorem 4.6. Uniform Convergence of Continuous Functions

If the sequence $(f_n(z))$ converges uniformly to $f(z)$ in a set A and if each f_n is continuous at each point of A, then f is continuous at each point of A. □

Proof

The proof is essentially the same as in the real case. For those readers unfamiliar with uniform convergence, we give the proof here. Let $\alpha \in A$ and z be any other point in A. By the triangle inequality it follows that

$$|f(z) - f(\alpha)| \leqslant |f(z) - f_n(z)| + |f_n(z) - f_n(\alpha)| + |f_n(\alpha) - f(\alpha)| \tag{4.1}$$

for any $n \in \mathbb{N}$. Since $(f_n(z))$ converges uniformly to $f(z)$ in A, given any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f(z) - f_n(z)| < \varepsilon/3 \quad \text{and} \quad |f_n(\alpha) - f(\alpha)| < \varepsilon/3 \quad \text{whenever} \quad n > N$$

Given the same $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|z - \alpha| < \delta \quad \Rightarrow \quad |f_n(z) - f_n(\alpha)| < \varepsilon/3 \qquad (\text{for all } n \in \mathbb{N})$$

since each f_n is continuous at α. It follows from (4.1) that, given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|z - \alpha| < \delta \quad \Rightarrow \quad |f(z) - f(\alpha)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

Hence f is continuous at $\alpha \in A$ as required. ■

It follows by Example 4.5 that the sequence $(1 - z^n)$ for $|z| \leqslant 1$ has a discontinuous pointwise limit function, so that by 4.6 the convergence is not uniform on $|z| \leqslant 1$.

Notes

(i) It is not true in general that if $(f_n(z))$ converges pointwise to $f(z)$ in a set A and if each f_n and f are continuous in A, then the convergence is uniform on A.

(ii) To test whether or not a pointwise convergent sequence of functions, $(f_n(z))$, converges uniformly on A, first of all find the pointwise limit function $f(z)$. Then calculate $U_n = \sup_{z \in A} |f_n(z) - f(z)|$. It follows from the definition that if $U_n \to 0$ as $n \to \infty$, then the convergence is uniform. In practice, it may be difficult to find U_n for each n.

Theorem 4.7 Integration of Uniformly Convergent Sequences

If $(f_n(z))$ converges uniformly to $f(z)$ on a simple contour $\mathscr{C}$ and if each f_n is continuous at each point of $\mathscr{C}$, then

$$\lim_{n \to \infty} \int_{\mathscr{C}} f_n(z)dz = \int_{\mathscr{C}} \lim_{n \to \infty} f_n(z)dz = \int_{\mathscr{C}} f(z)dz \qquad \square$$

Proof

Since each f_n is continuous on $\mathscr{C}$, f is continuous on $\mathscr{C}$ by 4.6, so the integrals exist. Let L be the length of $\mathscr{C}$. Then, by hypothesis, given any $\varepsilon > 0$, there exists N such that

$$|f_n(z) - f(z)| < \varepsilon/L \quad \text{whenever} \quad n > N \quad \text{for all } z \in \mathscr{C}$$

Then

$$\left| \int_{\mathscr{C}} f_n(z)dz - \int_{\mathscr{C}} f(z)dz \right| < \varepsilon \qquad (n > N)$$

by Theorem 3.2, as required. ■

Note

Clearly, the continuity condition in 4.7 can be weakened.

Theorem 4.8. Differentiation of Uniformly Convergent Sequences

For each $n \in \mathbb{N}$, let f_n be analytic on a simply connected region $\mathscr{R}$ and let $(f_n(z))$ be uniformly convergent to $f(z)$ on each compact subset of $\mathscr{R}$. Then f is analytic on $\mathscr{R}$ and $(f'_n(z))$ converges uniformly to $f'(z)$ on each compact subset of $\mathscr{R}$. □

Proof

Step 1

We need to show first of all that f is analytic in $\mathscr{R}$. Since, by hypothesis, each f_n is continuous in $\mathscr{R}$, it follows that f is continuous in $\mathscr{R}$ by 4.6. Then if $\mathscr{C}$ is any simple closed contour of length L in $\mathscr{R}$,

$$\int_{\mathscr{C}} f(z)dz = \int_{\mathscr{C}} (f(z) - f_n(z))dz + \int_{\mathscr{C}} f_n(z)dz \qquad (\text{for any } n \in \mathbb{N})$$

But $\int_{\mathscr{C}} f_n(z)dz = 0$ for any $n \in \mathbb{N}$ by Cauchy's theorem since f_n is analytic inside and on $\mathscr{C}$. Since $(f_n(z))$ is uniformly convergent on any compact subset of $\mathscr{R}$, given any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $z \in \mathscr{C}$,

$$|f(z) - f_n(z)| < \varepsilon/L \quad \text{whenever} \quad n > N$$

Hence by Theorem 3.2,

$$\left| \int_{\mathscr{C}} f(z)dz \right| = \left| \int_{\mathscr{C}} f(z) - f_n(z)dz \right| < \varepsilon \quad \text{whenever} \quad n > N$$

so that $\int_{\mathscr{C}} f(z)dz = 0$. Hence f is analytic in $\mathscr{R}$ by Morera's theorem, 3.15.

Step 2

Now let z be any point in $\mathscr{R}$ and $\mathscr{C}$ be a circle with centre z and radius r, lying entirely within $\mathscr{R}$. By hypothesis, given any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $w \in \mathscr{C}$,

$$|f(w) - f_n(w)| < r\varepsilon \quad \text{whenever} \quad n > N$$

Then, by Cauchy's theorem for derivatives and Theorem 3.2,

$$|f'(z) - f'_n(z)| = \left|\frac{1}{2\pi i}\int_{\mathscr{C}} \frac{f(w) - f_n(w)}{(w-z)^2}\,dw\right| < \frac{2\pi r.r\varepsilon}{2\pi r^2} = \varepsilon$$

whenever $n > N$. Hence $(f'_n(z))$ converges uniformly to $f'(z)$ in any compact subset of $\mathscr{R}$. ■

Exercise **4.1.1** Use the definition of the limit of a sequence to prove that if α is any complex number satisfying $|\alpha| < 1$, then

$$1 + \frac{(i\alpha)^n}{n^2} \to 1 \quad \text{as } n \to \infty$$

Exercise **4.1.2** Use the definition to prove that if $z_n \to \alpha$ as $n \to \infty$, then

(i) $\overline{z}_n \to \overline{\alpha}$ as $n \to \infty$

(ii) $|z_n| \to |\alpha|$ as $n \to \infty$

(iii) $z_n^{1/2} \to \alpha^{1/2}$ as $n \to \infty$

Give a simple example to show that $|z_n| \to |\alpha|$ as $n \to \infty \not\Rightarrow z_n \to \alpha$ as $n \to \infty$.

Exercise **4.1.3** Let $z_n \to \alpha$ and $w_n \to \beta$ as $n \to \infty$. Use the definition to prove that (i) $z_n - w_n \to \alpha - \beta$ as $n \to \infty$, (ii) $z_n w_n \to \alpha\beta$ as $n \to \infty$. (Use the fact that any convergent sequence is bounded in (ii).)

Exercise **4.1.4** Use Theorem 4.1 and, in the second case, Exercise 4.1.2(iii) to find the following limits:

(i) $\displaystyle\lim_{n\to\infty} \frac{2n - in^2}{(1+i)n^2 - 1}$

(ii) $\displaystyle\lim_{n\to\infty} \sqrt{n}\left((n+i)^{1/2} - (n-i)^{1/2}\right)$

Exercise **4.1.5** Prove that $\lim_{z\to\infty} f(z) = \ell$ if and only if $\lim_{z\to 0} g(z) = \ell$, where $g(z) = f(1/z)$.

Exercise **4.1.6** Prove that the sequence (n^{-z}) converges uniformly for $\operatorname{Re} z \geqslant a > 0$. Prove also that (n^{-z}) converges for $z \in A$, where $A = \{z : \operatorname{Re} z > 0 \text{ or } z = 0\}$ but that the convergence is not uniform on A.

Exercise **4.1.7** Prove that $(1/(1 + 2n^2 z))$ converges uniformly to 0 for all z satisfying $|z| \geqslant 1$.

Infinite Series

The concept of an infinite series of complex numbers is identical to that of an infinite series of real numbers. We begin with the usual definitions.

Definitions

Let (z_n) be a sequence of complex numbers. Then the sequence (S_k), defined by $S_k = \sum_{n=1}^{k} z_n = z_1 + z_2 + \ldots + z_k$, is the sequence of **partial sums** of the **infinite series** $z_1 + z_2 + z_3 + \ldots$. The infinite series **converges to sum S** if $S_k \to S$ as $k \to \infty$, and in this case we write $\sum_{n=1}^{\infty} z_n = S$. The infinite series **diverges** if (S_k) diverges.

Important Note

Let $R_k = \sum_{n=k+1}^{\infty} z_n$ for each k. Then R_k is called the **remainder** of the series after k terms. Since $|S_k - S| = |R_k - 0|$, it follows by the definition of the limit of a sequence that the series $\sum_{n=1}^{\infty} z_n$ converges to sum S if and only if $R_k \to 0$ as $k \to \infty$.

Once again, a lot of the properties of real series carry over to the complex case. It follows by the definition and 4.1(i) that if $\sum_{n=1}^{\infty} z_n$ converges then its sum is unique. Also the geometric series result is easily generalised to the complex case. The following standard results also follow from Theorem 4.1.

Theorem 4.9. Elementary Properties of Series

Suppose that the series $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=1}^{\infty} w_n$ converge. Then

(i) $\displaystyle\sum_{n=1}^{\infty} (\lambda z_n) = \lambda \sum_{n=1}^{\infty} z_n$ for all $\lambda \in \mathbb{C}$

(ii) $\displaystyle\sum_{n=1}^{\infty} (z_n + w_n) = \sum_{n=1}^{\infty} z_n + \sum_{n=1}^{\infty} w_n$

(iii) $z_n \to 0$ as $n \to \infty$ □

Recall that condition (iii) is necessary for convergence but it is not sufficient; for example, $\sum_{n=1}^{\infty} 1/n$ satisfies condition (iii) but it still diverges. The following theorem shows that series of complex numbers can be investigated by examining their real and imaginary parts.

Theorem 4.10. Real and Imaginary Parts of Series

The series $\sum_{n=1}^{\infty} z_n$ converges to sum S if and only if $\sum_{n=1}^{\infty} \operatorname{Re} z_n$ converges to $\operatorname{Re} S$ and $\sum_{n=1}^{\infty} \operatorname{Im} z_n$ converges to $\operatorname{Im} S$. □

Proof

Let $z_n = x_n + iy_n$ for each $n \in \mathbb{N}$ and $S = A + iB$, where A and B are real. Then

$$S_k = \sum_{n=1}^{k} z_n = \sum_{n=1}^{k} x_n + i\sum_{n=1}^{k} y_n = A_k + iB_k \quad \text{say, for all } k$$

It follows by Theorem 4.2 that $S_k \to S$ as $k \to \infty$ if and only if $A_k \to A$ and $B_k \to B$ as $k \to \infty$, and the result follows by definition. ■

Instead of using the above result, just as for series of real numbers, if we want to test the convergence of $\sum_{n=1}^{\infty} z_n$, we can test for absolute convergence.

Definition

The series $\sum_{n=1}^{\infty} z_n$ **converges absolutely** if $\sum_{n=1}^{\infty} |z_n|$ converges. Note that $\sum_{n=1}^{\infty} |z_n|$ is a series of non-negative real numbers and all the standard convergence tests, such as the comparison, ratio and nth root test, apply.

Theorem 4.11. Absolute Convergence Implies Convergence

Any absolutely convergent series converges. □

Proof

Let $z_n = x_n + iy_n$. By hypothesis, $\sum_{n=1}^{\infty} (x_n^2 + y_n^2)^{1/2}$ converges, and since $|x_n| \leqslant (x_n^2 + y_n^2)^{1/2}$, $|y_n| \leqslant (x_n^2 + y_n^2)^{1/2}$ for all $n \in \mathbb{N}$, it follows by the comparison test that $\sum_{n=1}^{\infty} |x_n|$ and $\sum_{n=1}^{\infty} |y_n|$ converge. Hence $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converge since any real absolutely convergent series converges. The result then follows by Theorem 4.10. ■

Notes

(i) It follows from results concerning series of real numbers and Theorems 4.10 and 4.11 that two absolutely convergent series can be multiplied to give another absolutely convergent series which converges to the product of the separate sums.

(ii) Recall that if a series is absolutely divergent, it is not necessarily divergent.

We now turn to the concept of uniform convergence of series of functions, which is the condition required for termwise differentiation and integration of such series.

Let $(f_n(z))$ be a sequence of values of functions $f_n : A \subseteq \mathbb{C} \to \mathbb{C}$. Then for each $z \in A$, $\sum_{n=1}^{\infty} f_n(z)$ is a series of complex numbers, which may or may not converge.

Definitions

A **series of functions** $\sum_{n=1}^{\infty} f_n(z)$ **converges pointwise** to **sum function** $F(z)$ if the sequence of **partial sum functions** $\left(\sum_{n=1}^{k} f_n(z)\right)$ converges pointwise to $F(z)$; that is, $\sum_{n=1}^{k} f_n(z) \to F(z)$ as $k \to \infty$ for each $z \in A$. The series of functions $\sum_{n=1}^{\infty} f_n(z)$ **converges uniformly** to $F(z)$ on A if the sequence $\left(\sum_{n=1}^{k} f_n(z)\right)$ converges uniformly to $F(z)$ on A.

The following result follows directly from the definition and Theorems 4.6, 4.7 and 4.8.

Theorem 4.12. Continuity, Differentiation and Integration of Uniformly Convergent Series

(i) If the series $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly to $F(z)$ on a set A and each f_n is continuous at each point of A, then F is continuous at each point of A.

(ii) If the series $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly to $F(z)$ on a simple contour $\mathscr{C}$ and if each f_n is continuous at each point of $\mathscr{C}$, then

$$\int_{\mathscr{C}} F(z)dz = \int_{\mathscr{C}} \sum_{n=1}^{\infty} f_n(z)dz = \sum_{n=1}^{\infty} \int_{\mathscr{C}} f_n(z)dz$$

(iii) If the series $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly to $F(z)$ in any compact subset of a simply connected region $\mathscr{R}$ and each f_n is analytic in $\mathscr{R}$, then F is analytic in $\mathscr{R}$ with $F'(z) = \sum_{n=1}^{\infty} f'_n(z)$, the convergence being uniform on any compact subset of $\mathscr{R}$. □

Proof

(i) By definition, $\left(\sum_{n=1}^{k} f_n(z)\right)$ converges uniformly to $F(z)$ in A, and since each f_n is continuous in A, $\sum_{n=1}^{k} f_n$ is continuous in A. Then the result follows by 4.6.

(ii) By hypothesis, $\left(\sum_{n=1}^{k} f_n(z)\right)$ converges uniformly to $F(z)$ on $\mathscr{C}$ where each f_n is continuous on $\mathscr{C}$ and so by (i), F is continuous on $\mathscr{C}$ and the integrals exist. It follows by 4.7 that

$$\int_{\mathscr{C}} \sum_{n=1}^{k} f_n(z)dz = \sum_{n=1}^{k} \int_{\mathscr{C}} f_n(z)dz \to \int_{\mathscr{C}} F(z)dz \quad \text{as } k \to \infty$$

as required.

(iii) This is similar and uses 4.8. It is left as an exercise. ■

A very useful test for uniform convergence of series of functions, which carries over from the real case, is the following result.

Theorem 4.13. Weierstrass's *M* test

Suppose that for each $n \in \mathbb{N}$, there exists a positive real number M_n such that $|f_n(z)| \leqslant M_n$ for all $z \in A$ and that $\sum_{n=1}^{\infty} M_n$ converges. Then $\sum_{n=1}^{\infty} f_n(z)$ is absolutely and uniformly convergent on A. □

Proof

Note that $\sum_{n=1}^{\infty} f_n(z)$ is absolutely convergent on A by hypothesis and the comparison test. Let the remainder of $\sum_{n=1}^{\infty} f_n(z)$ after k terms be $R_k(z)$ for each $z \in A$, and let the remainder of $\sum_{n=1}^{\infty} M_n$ after k terms be R_k^*. Then by hypothesis, $R_k^* \to 0$ as $k \to \infty$, so that given any $\varepsilon > 0$, there exists $K^* \in \mathbb{N}$ such that $|R_k^*| < \varepsilon$ whenever $k > K^*$. Also, since $|f_n(z)| \leqslant M_n$ for each $n \in \mathbb{N}$, it follows by Exercise 4.2.4 and the triangle inequality that

$$|R_k(z)| = \left|\sum_{n=k+1}^{\infty} f_n(z)\right| \leqslant \sum_{n=k+1}^{\infty} |f_n(z)| \leqslant \sum_{n=k+1}^{\infty} M_n = R_k^*$$

for each k. Hence, given any $\varepsilon > 0$, there exists K^* such that $|R_k(z)| < \varepsilon$ whenever $k > K^*$ for all values of z in the domain of convergence. Hence $R_k(z) \to 0$ uniformly on A as $k \to \infty$, as required. ■

Example 4.7

Prove that $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ is an analytic function for $\operatorname{Re} z > 1$. This is the **Riemann zeta function**, of fundamental importance in number theory.

Solution

Let $\operatorname{Re} z = x \geqslant \rho > 1$. Then $|n^{-z}| = n^{-x} \leqslant n^{-\rho}$. Let $M_n = n^{-\rho}$ in 4.13. Now $\sum_{n=1}^{\infty} n^{-\rho}$ converges by hypothesis and the real hyperharmonic series result, so the given series converges uniformly for $\operatorname{Re} z \geqslant \rho > 1$ by 4.13. Hence the given series converges uniformly on any compact subset of $\operatorname{Re} z > 1$, and so by 4.12, ζ is analytic on $\operatorname{Re} z > 1$ and $\zeta'(z) = \sum_{n=1}^{\infty} (\operatorname{Log} n)n^{-z}$ in this region.

Note

As in the real case, absolute convergence alone does not guarantee uniform convergence.

Power Series

The most important series of functions of complex variables consist of integer powers of a complex variable. These series have very wide applications.

Definition

A **power series** is a series of the form $\sum_{n=0}^{\infty} a_n(z-\alpha)^n$ where α and $a_n \in \mathbb{C}$ for any $n \in \mathbb{Z}_{\geqslant 0}$, and z may be any complex number in a stated domain. A trivial change of variable, $w = z - \alpha$, reduces such a series to the form $\sum_{n=0}^{\infty} a_n w^n$ and so we take $\alpha = 0$ without loss of generality. Note the slight change in convention for the values taken by n, which is for convenience.

Power series generally converge for certain values of z and diverge for others. Standard convergence tests for series of real numbers can be used to find the set of values of $|z|$ for which the series converges absolutely. As indicated earlier, it is generally much easier to test for absolute convergence than for convergence by other means.

Definition

The **radius of convergence** of the power series $\sum_{n=0}^{\infty} a_n z^n$ is denoted and defined by $R = \sup\{|z| : \sum_{n=0}^{\infty} |a_n z^n| \text{ converges}\}$. Then the series is absolutely convergent for all z inside the circle $\mathscr{C}$ given by $|z| = R$ and divergent for all z outside $\mathscr{C}$. $\mathscr{C}$ is the **circle of convergence** of the series.

Note

If a power series converges pointwise for all $z \in \mathbb{C}$, then the radius of convergence is taken to be infinite.

For convenience, we give below two standard convergence tests for real series of non-negative terms, which are the most useful for calculating the radius of convergence of a given power series.

Theorem 4.14. The Ratio and nth Root Tests

Let $\sum_{n=0}^{\infty} \alpha_n$ be a given series of complex numbers.

(i) **The ratio test**. Let $\lim_{n\to\infty} |\alpha_{n+1}/\alpha_n| = \lambda$, if it exists. Then $\sum_{n=0}^{\infty} |\alpha_n|$ converges if $\lambda < 1$ and diverges if $\lambda > 1$.

(ii) **The *n*th root test**. Let $\lim_{n\to\infty} |\alpha_n|^{1/n} = \lambda$, if it exists. Then $\sum_{n=0}^{\infty} |\alpha_n|$ converges if $\lambda < 1$ and diverges if $\lambda > 1$. □

Important Note

More generally, in the nth root test, if $\lim_{n\to\infty} |\alpha_n|^{1/n}$ does not exist, it can be replaced by $\limsup_{n\to\infty} |\alpha_n|^{1/n}$, the largest number to which any subsequence of $(|\alpha_n|^{1/n})$ converges, which always exists (allowing infinity as a possible value).

Note

The ratio and nth root tests give no information about the behaviour of a power series on the circle of convergence $\mathscr{C}$. For $|z| = R$, $\sum_{n=0}^{\infty} |a_n z^n| = \sum_{n=0}^{\infty} |a_n| R^n$ and the series can be tested for absolute convergence by other standard tests for real series of non-negative terms. More generally, it can be tested for convergence using Theorem 4.10.

Example 4.8

Find the radius of convergence of the following power series:

(i) $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$

(ii) $\displaystyle\sum_{n=1}^{\infty} n^2 z^n$

(iii) $3 + z + 3z^2 + z^3 + 3z^4 + \ldots$

Solution

(i) Let $\alpha_n = z^n/n!$. Then

$$\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \left|\frac{z^{n+1} n!}{(n+1)! z^n}\right| = \frac{|z|}{n+1} \quad \Rightarrow \quad \lim_{n\to\infty}\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = 0 \qquad (\text{for all } z \in \mathbb{C})$$

Hence, by the ratio test, the given series is absolutely convergent for all z. It follows by this example that $\sum_{n=0}^{\infty} n! z^n$ converges only at $z = 0$.

(ii) Let $\alpha_n = n^2 z^n$. Then

$$\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \frac{(n+1)^2}{n^2}|z| = \left(1 + \frac{1}{n}\right)^2 |z| \quad \Rightarrow \quad \lim_{n\to\infty}\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = |z|$$

Hence, by the ratio test, the given series is absolutely convergent for $|z| < 1$ and absolutely divergent for $|z| > 1$. In other words, the radius of convergence is 1. Note that when $|z| = 1$, then $|\alpha_n| = n^2$ and the series is absolutely divergent; it is in fact divergent, since in this case $\alpha_n = n^2 e^{ni \operatorname{Arg} z} \not\to 0$ as $n \to \infty$.

(iii) Let $\alpha_{2m} = 3z^{2m}$ and $\alpha_{2m+1} = z^{2m+1}$. In this case the ratio test is not appropriate since the required limit does not exist. On the other hand, $\lim_{m\to\infty} |\alpha_{2m}|^{1/2m} = \lim_{m\to\infty} 3^{1/2m}|z| = |z|$ and $\lim_{m\to\infty} |\alpha_{2m+1}|^{1/(2m+1)} = |z|$, so that $\lim_{n\to\infty} |\alpha_n|^{1/n} = |z|$. Hence, by the nth root test, the radius of convergence of the given series is 1.

Power series are particularly simple to manipulate since any power series is uniformly convergent within its domain of convergence.

Theorem 4.15. Uniform Convergence of Power Series

If the power series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R, then the series is uniformly convergent to its pointwise sum function in the closed set $|z| \leqslant \rho$ where $0 < \rho < R$. □

Proof

$|a_n z^n| \leqslant |a_n|\rho^n$ for $|z| \leqslant \rho$ and for each n. Also, $\sum_{n=0}^{\infty} |a_n|\rho^n$ is convergent by hypothesis. Hence $\sum_{n=0}^{\infty} a_n z^n$ is uniformly convergent for $|z| \leqslant \rho$ by Weierstrass's M test. ■

Since a power series is uniformly convergent within its circle of convergence, termwise integration is valid within this domain by 4.12(ii), and the next theorem follows immediately.

Theorem 4.16. Integration of Power Series

Suppose that $\sum_{n=0}^{\infty} a_n z^n$ converges to $f(z)$, inside the circle $|z| = R$ and that $\mathscr{C}$ is any contour lying inside this circle. Then

$$\int_{\mathscr{C}} f(z)dz = \int_{\mathscr{C}} \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \int_{\mathscr{C}} a_n z^n dz$$ □

We also have the following special case of Theorem 4.12(iii), which says that a power series can be differentiated termwise within its domain of convergence.

Theorem 4.17. Differentiation of Power Series

Suppose that $\sum_{n=0}^{\infty} a_n z^n$ converges to $f(z)$ and that its radius of convergence is R. Then $\sum_{n=1}^{\infty} n a_n z^{n-1}$ converges to $f'(z)$, with radius of convergence R. □

Proof

The fact that the power series can be differentiated termwise is a consequence of 4.12(iii) and 4.15. It remains to show that $\sum_{n=1}^{\infty} n a_n z^{n-1}$ has radius of convergence R.

Step 1

We first show that $\lim_{n\to\infty} n^{1/n} = 1$, by using the binomial theorem. Let $n^{1/n} = 1 + \alpha_n$, so that

$$n = (1+\alpha_n)^n > 1 + \frac{n(n-1)}{2}\alpha_n^2 \quad \Rightarrow \quad \alpha_n^2 < \frac{2n-2}{n(n-1)} < \frac{2n}{n(n/2)} = \frac{4}{n}$$

for $n > 2$. Hence $\alpha_n < 2/\sqrt{n}, n > 2$, and so $\alpha_n \to 0$ as $n \to \infty$.

Step 2

We next show that $\limsup_{n\to\infty}|na_n|^{1/n} = \limsup_{n\to\infty}|a_n|^{1/n}$. From step 1, $\limsup_{n\to\infty}|a_n|^{1/n} \leqslant \limsup_{n\to\infty}|na_n|^{1/n} \leqslant \lim_{n\to\infty} n^{1/n}.\limsup_{n\to\infty}|a_n|^{1/n} = \limsup_{n\to\infty}|a_n|^{1/n}$, as required.

Step 3

Note that $\lim_{n\to\infty}|z|^{1/n} = 1$ for any finite $|z|$. Hence from step 2, $\limsup_{n\to\infty}|na_nz^{n-1}|^{1/n} = \limsup_{n\to\infty}|a_nz^n|^{1/n}$ and so $\sum_{n=1}^{\infty} na_nz^{n-1}$ and $\sum_{n=0}^{\infty} a_nz^n$ have the same radius of convergence by Theorem 4.14(ii). ■

Important Notes

(i) Theorem 4.17 states that a power series defines an analytic function within its domain of convergence. Letting $f(z) = \sum_{n=0}^{\infty} a_nz^n$ for $|z| < R$, where the given power series has radius of convergence R, it follows by 4.17 and 3.14 that the derivatives of f of all orders exist at any point satisfying $|z| < R$, and that $a_n = f^{(n)}(0)/n!$ for each $n \in \mathbb{N}$. More generally, if $f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n$ for $|z| < R$, then $a_n = f^{(n)}(\alpha)/n!$.

(ii) Because of (i), power series are sometimes used to define standard analytic functions. For instance, the power series $\sum_{n=0}^{\infty} z^n/n!$ is absolutely convergent for all z and can be used to define the exponential function, which is entire. We shall show shortly that the definition $e^z = \sum_{n=0}^{\infty} z^n/n!$ is equivalent to the definition given in Chapter 2.

(iii) It follows by Theorem 4.9 that two power series may be added or subtracted termwise within the intersection of their domains of convergence. It also follows from the note after Theorem 4.11 that two power series may also be multiplied within the intersection of their domains of absolute convergence. We give a proof of this fact in the next section.

Exercise **4.2.1** Use the definition of a convergent series to prove that $\sum_{n=1}^{\infty} z^{n-1} = 1/(1-z)$ if $|z| < 1$, and that the series diverges for $|z| \geqslant 1$. (You may assume that $\lim_{n\to\infty} z^n = 0$ for $|z| < 1$ and diverges otherwise.) Hence find $\sum_{n=1}^{\infty} (1/4i)^n$.

Exercise **4.2.2** Let $\sum_{n=1}^{\infty} z_n$ be a convergent series. Prove that

(i) if $\lambda \in \mathbb{C}$ then $\sum_{n=1}^{\infty} (\lambda z_n) = \lambda \sum_{n=1}^{\infty} z_n$

(ii) $z_n \to 0$ as $n \to \infty$

Exercise 4.2.3 Use standard results concerning real infinite series to prove the following:

(i) $\sum_{n=1}^{\infty} 1/n^{\alpha}$, where $\alpha \in \mathbb{C}$, converges for $\operatorname{Re}\alpha > 1$ and diverges for $\operatorname{Re}\alpha \leqslant 0$

(ii) $\sum_{n=1}^{\infty} e^{in}/n^2$ and $\sum_{n=1}^{\infty} (1/n^2)\sin(n+i)$ both converge

Exercise 4.2.4 Prove that if $\sum_{n=1}^{\infty} z_n$ is absolutely convergent, then

$$\left|\sum_{n=1}^{\infty} z_n\right| \leqslant \sum_{n=1}^{\infty} |z_n|$$

Exercise 4.2.5 Show that

$$\sum_{n=1}^{\infty} \frac{z^4}{(1+z^4)^{n-1}} = 1 + z^4 \quad \text{for} \quad |1+z^4| > 1$$

By considering the remainder of the series after k terms, prove that the series is uniformly convergent in the region given by $|1+z^4| > 1$, but not uniformly convergent for $|1+z^4| \geqslant 1$.

Exercise 4.2.6 Let $\sum_{n=1}^{\infty} f_n(z)$ be uniformly convergent to $F(z)$ in any compact subset of a simply connected region $\mathscr{R}$ and suppose that each f_n is analytic in $\mathscr{R}$. Use Theorem 4.8 to prove that F is analytic in $\mathscr{R}$ and that $F'(z) = \sum_{n=1}^{\infty} f'_n(z)$ for all z in $\mathscr{R}$.

Exercise 4.2.7 Use Weierstrass's M test to prove that

(i) $\sum_{n=1}^{\infty} z^n/n^2(n+1)$ is uniformly convergent for $|z| \leqslant 1$

(ii) $\sum_{n=1}^{\infty} 1/(z^2+n^2)$ is uniformly convergent in any annular region $m-1 < |z| < m$ where $m \in \mathbb{N}$

(iii) $\sum_{n=1}^{\infty} (\sin nz)/n^3$ is not uniformly convergent on $|z| \leqslant a$ but is uniformly convergent on $|x| \leqslant a$ where $x \in \mathbb{R}$.

Exercise 4.2.8 Find the radius of convergence of the following power series:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 z^{2n}}{(3n)!}$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{(2z)^n}{\sqrt{n}}$

(iii) $\displaystyle\sum_{n=1}^{\infty} (-1)^n 3^{n+1} n z^{2n+1}$

(iv) $\displaystyle\sum_{n=1}^{\infty} n^n z^n$

(v) $\displaystyle\sum_{n=0}^{\infty} 3^n z^{(2n)!}$

(vi) $1 + 2z + z^2 + (2z)^3 + z^4 + (2z)^5 + \ldots$

Exercise **4.2.9** Show that, although $\sum_{n=1}^{\infty} f_n(z)$ and $\sum_{n=1}^{\infty} f'_n(z)$ have the same radius of convergence, where $f_n(z) = z^n/n^2$, they do not converge for the same set of values of z.

Exercise **4.2.10** Suppose that $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for $|z| < R$. Let z be any fixed point satisfying $|z| < R$ and let ρ satisfy $|z| < \rho < R$. Show that the series $\sum_{n=1}^{\infty} n|z|^n/\rho^n$ converges by the ratio test. Hence, by comparing $\sum_{n=1}^{\infty} |na_n z^{n-1}|$ with $\sum_{n=0}^{\infty} |a_n \rho^n|$, show that $\sum_{n=1}^{\infty} na_n z^{n-1}$ also converges absolutely for $|z| < R$.

Exercise **4.2.11** The exponential function may be defined by $e^z = \sum_{n=0}^{\infty} z^n/n!$ for all $z \in \mathbb{C}$.

(i) Use this definition to find the derivative of $f: \mathbb{C} \to \mathbb{C}$, defined by $f(z) = e^{-z} \cdot e^z$, for any $z \in \mathbb{C}$. Deduce that $e^{-z} = 1/e^z$ for all $z \in \mathbb{C}$.

(ii) Use the power series definition to find the derivative of $g: \mathbb{C} \to \mathbb{C}$, defined by $g(z) = e^{\alpha} \cdot e^z/e^{\alpha+z}$, α constant, for any $z \in \mathbb{C}$. Deduce that $e^{\alpha+z} = e^{\alpha} \cdot e^z$ for any z and $\alpha \in \mathbb{C}$.

Taylor Series

Suppose that the power series $\sum_{n=0}^{\infty} a_n(z-\alpha)^n$ converges absolutely to $f(z)$ within some circle of convergence, centred at α. We have shown in the previous section how it then follows by Theorem 4.17 that f is an analytic function and $a_n = f^{(n)}(\alpha)/n!$. We now show that the coefficients a_n can also be expressed in integral form, unlike in the case of real variables.

Let $f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n$ within some circle of convergence, centred at α. Then multiplying by $(z-\alpha)^{-1-m}$, where $m \in \mathbb{N}$, and integrating termwise, which is valid by Theorem 4.16, gives

$$\int_{\mathscr{C}} \frac{f(z)}{(z-\alpha)^{m+1}}\,dz = \sum_{n=0}^{\infty} a_n \int_{\mathscr{C}} (z-\alpha)^{n-m-1}\,dz \tag{4.2}$$

where $\mathscr{C}$ is any simple closed contour lying within the circle of convergence. It follows by the fundamental theorem of calculus that $\int_{\mathscr{C}} (z-\alpha)^{n-m-1}dz = 0$ if $n \neq m$, and it follows by Cauchy's integral formula that $\int_{\mathscr{C}} (z-\alpha)^{-1}dz = 2\pi i$. Hence, from (4.2) and Theorem 3.13, it follows that

$$a_m = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{f(z)}{(z-\alpha)^{m+1}}\,dz = \frac{f^{(m)}(\alpha)}{m!} \tag{4.3}$$

Essentially the converse of Theorem 4.17 is also true. In other words, any function which is analytic at a point $\alpha \in \mathbb{C}$ can be expanded in a power series, $\sum_{n=0}^{\infty} a_n(z-\alpha)^n$, about α, where the coefficients are given by (4.3). This power series is the Taylor series expansion of the function about α.

Theorem 4.18. Taylor Series

If $f: A \subseteq \mathbb{C} \to \mathbb{C}$ is analytic inside a circle, centre α, of radius r, then at each point z inside this circle,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n \quad \text{where} \quad a_n = \frac{f^{(n)}(\alpha)}{n!} \quad \text{for all } n \in \mathbb{N}$$

That is, the given power series converges to $f(z)$ for all z satisfying $|z-\alpha| < r$. □

Proof

Suppose first of all that f is analytic inside the circle $\mathscr{C}$ given by $|z| = r$. Then by Cauchy's integral formula, if w is any point inside $\mathscr{C}$ so that $|w| < |z|$,

$$f(w) = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{f(z)}{z-w}\,dz = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{f(z)}{z}\frac{1}{1-w/z}\,dz$$

$$\Rightarrow \quad f(w) = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{f(z)}{z}\sum_{n=0}^{\infty}\left(\frac{w}{z}\right)^n dz = \sum_{n=0}^{\infty}\frac{w^n}{2\pi i}\int_{\mathscr{C}}\frac{f(z)}{z^{n+1}}\,dz$$

using the geometric series result (see Exercise 4.2.1). The fact that the series can be integrated termwise follows from Theorem 4.12 since the series is uniformly convergent within the given circle by Weierstrass's M test. (Let $|w/z| = \rho < 1$. Then $|w/z|^n = \rho^n$ and $\sum_{n=0}^{\infty} \rho^n$ converges.) It then follows by (4.3) that

$$f(w) = \sum_{n=0}^{\infty} f^{(n)}(0)w^n/n!$$

Now suppose that f is analytic inside the circle $|z-\alpha| = r$. Letting $z - \alpha = w$ in the above result shows that

$$f(z) = f(w+\alpha) = \sum_{n=0}^{\infty}\frac{f^{(n)}(\alpha)(z-\alpha)^n}{n!} \qquad (|z-\alpha| < r)$$

■

The power series expansion of f in 4.18 is the **Taylor series** expansion of f about α and is formally the same as in the real case. When $\alpha = 0$ the Taylor series is sometimes called the **Maclaurin series** expansion of f. Any Taylor series can be obtained from a Maclaurin series by a change of origin.

Important Notes

(i) It follows by Theorem 4.18 and the preceding analysis that any power series is the Taylor series expansion of its pointwise sum function.

(ii) Note that the Taylor series expansion of f about α converges to $f(z)$ within the circle whose radius is the distance from α to the nearest point, β, where f fails to be analytic. It can be shown that this is actually the largest circle centred at α such that the Taylor series coverges to $f(z)$ for all z inside it. It may happen that the Taylor series converges for $|z-\alpha| = \rho$, where $\rho > r$, but in this case the series will not converge to $f(z)$ (see Exercise 4.3.4).

We shall later prove Theorem 4.18 as a corollary of a more general result, Laurent's theorem (Theorem 4.21), the proof of which does not depend on Cauchy's integral results or on results concerning uniform convergence.

Example 4.9

Since it is formally the same as in the real case, the Taylor series expansion of a function $f: \mathbb{C} \to \mathbb{C}$ takes exactly the same form as the Taylor series expansion of the corresponding function $f: \mathbb{R} \to \mathbb{R}$.

(i) Let $f(z) = e^z$ for all $z \in \mathbb{C}$ so that f is entire. $f^{(n)}(z) = e^z$ for any $n \in \mathbb{Z}_{\geqslant 0}$, so that $f^{(n)}(0) = 1$ for any such n. Hence the Maclaurin series expansion of f is

$$e^z = 1 + z + \frac{z^2}{2!} + \ldots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

and the series converges absolutely for all z. Then letting $z = w - i$ say, gives

$$e^{w-i} = \sum_{n=0}^{\infty} \frac{(w-i)^n}{n!} \quad \Rightarrow \quad e^z = e^i \sum_{n=0}^{\infty} \frac{(z-i)^n}{n!}$$

and this is the Taylor series expansion of f about i, which converges for all z. This expansion may also be found directly by using Theorem 4.18.

Similarly, the standard Maclaurin series expansions for the other elementary functions of a real variable hold for the analogous functions of a complex variable in the corresponding circle of convergence. In particular,

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \qquad \cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \tag{4.4}$$

and both series converge for all $z \in \mathbb{C}$. The Maclaurin series expansions of $\sin z$ and $\cos z$ follow easily from (4.4) using $\sin z = -i \sinh(iz)$ and $\cos z = \cosh(iz)$. Also,

$$\operatorname{Log}(1+z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1}$$

and the series converges for $|z| < 1$, since $\operatorname{Log}(1+z)$ has a singular point at $z = -1$.

(ii) The derivatives of the function $f: \mathbb{C}\backslash\{0\} \to \mathbb{C}$ defined by $f(z) = 1/z$ are $f^{(n)}(z) = (-1)^n n! z^{-n-1}$ for all $n \in \mathbb{Z}_{\geqslant 0}$, so that $f^{(n)}(1) = (-1)^n n!$ for each $n \in \mathbb{N}$ and $f(1) = 1$. Hence the Taylor series expansion of f about 1, say, is

$$\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n$$

Since f has a singular point at 0, the series converges for all z satisfying $|z-1| < 1$. Note that it diverges for $|z-1| \geqslant 1$.

(iii) Let $f(z) = (1+z)^\alpha$ where $\alpha \in \mathbb{R}$. Then $f'(z) = \alpha(1+z)^{\alpha-1}$, $f''(z) = \alpha(\alpha-1)(1+z)^{\alpha-2}, \ldots, f^{(n)}(\alpha) = \alpha(\alpha-1)\ldots(\alpha-n+1)(1+z)^{\alpha-n}$ for all $n \in \mathbb{N}$. Hence $f^{(n)}(0) = \alpha(\alpha-1)\ldots(\alpha-n+1)$ for all $n \in \mathbb{N}$. Then by 4.18, we obtain the usual **binomial series**

$$(1+z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!}z^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}z^3 + \ldots$$

which, if $\alpha \notin \mathbb{N}$, converges for $|z| < 1$ since the given function has a branch point or pole at $z = -1$.

Theorem 4.18 can be used to give a simple proof of the fact that two absolutely convergent power series can be multiplied termwise to give another absolutely convergent power series, whose sum is the product of the previous two.

Theorem 4.19. Multiplication of Power Series

Suppose that $\sum_{n=0}^{\infty} a_n z^n = f(z)$, with radius of convergence R_1, and $\sum_{n=0}^{\infty} b_n z^n = g(z)$, with radius of convergence R_2. Then $f(z)g(z) = \sum_{n=0}^{\infty} c_n z^n$ where $c_n = \sum_{r=0}^{n} a_r b_{n-r}$ and the series has radius of convergence of at least $R = \min(R_1, R_2)$. □

Proof

Both f and g are analytic for $|z| < R$ and $a_n = f^{(n)}(0)/n!$ and $b_n = g^{(n)}(0)/n!$ by 4.17. The product fg is also analytic for $|z| < R$ and is represented in this region by a Taylor series $f(z)g(z) = \sum_{n=0}^{\infty} c_n z^n$, where

$$c_n = \frac{(fg)^{(n)}(0)}{n!} = \sum_{r=0}^{n} \frac{1}{n!}\frac{n!}{r!(n-r)!} f^{(r)}(0) g^{(n-r)}(0) = \sum_{r=0}^{n} a_r b_{n-r}$$

by Leibniz's formula for the nth derivative of a product. ■

Important Note

Since any power series is the Taylor series expansion of its pointwise sum function, the coefficients in a Taylor series expansion can be calculated by a number of different means, not just by 4.18. Note also that Taylor series can be differentiated and integrated termwise within their circle of convergence by 4.16 and 4.17.

Example 4.10

(i) The **error function**, erf : $\mathbb{C} \to \mathbb{C}$ is defined by

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-w^2} dw$$

(Note that the integral does not depend on the choice of contour joining 0 to z, since the integrand is entire.) It follows by the Maclaurin series expansion of e^z that

$$e^{-w^2} = \sum_{n=0}^{\infty} (-1)^n \frac{w^{2n}}{n!}$$

and the series converges for all $w \in \mathbb{C}$. Then integrating termwise by 4.16 gives

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^z w^{2n} dw = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)n!}$$

as the Maclaurin series expansion of $\operatorname{erf}(z)$, which converges for all $z \in \mathbb{C}$.

(ii) It follows from the Maclaurin expansions of e^z and $\cos z$ and Theorem 4.19 that

$$\begin{aligned} e^z \sec z &= \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\right)\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots\right)^{-1} \\ &= \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\right)\left(1 + \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \dots\right)\right. \\ &\qquad \left. + \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \dots\right)^2 \dots\right) \end{aligned}$$

using the binomial series result of Example 4.9. Hence

$$e^z \sec z = \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots\right)\left(1 + \frac{z^2}{2} + \frac{5z^4}{24} + \dots\right)$$

$$\Rightarrow \quad e^z \sec z = 1 + z + z^2 + \frac{2z^3}{3} + \frac{z^4}{2} + \dots$$

and this is the Maclaurin series expansion of $e^z \sec z$, which converges for $|z| < \pi/2$ by 4.18.

Theorem 4.18 can be used to give another characterisation of zeros of a function.

Lemma 4.20. Zeros of a Function

Let f be analytic inside the circle $|z-\alpha| = r$, so that $f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n$ for $|z-\alpha| < r$. Then f has a zero of order m at α if and only if $a_n = 0$ for $n < m$ and $a_m \neq 0$. □

Proof

Suppose that $a_n = 0$ for $n < m$ with $a_m \neq 0$. Then

$$f(z) = (z-\alpha)^m \sum_{n=0}^{\infty} a_{n+m}(z-\alpha)^n$$

so that $f(z) = (z-\alpha)^m g(z)$ where g is analytic at α and $g(\alpha) = a_m \neq 0$, as required.

Now suppose that f has a zero of order m at α so that $f(z) = (z-\alpha)^m g(z)$ where $g(\alpha) \neq 0$ and g is analytic at α. Then by 4.18 and hypothesis, g has a Taylor series expansion about α of the form $g(z) = \sum_{n=m}^{\infty} a_n(z-\alpha)^{n-m}$, where $a_m \neq 0$, so that the result follows. ■

Note

Theorems 4.18 and 4.19 show that if f is analytic at α, then α is a zero of order m of f if and only if

$$f(\alpha) = f'(\alpha) = \ldots = f^{(m-1)}(\alpha) = 0 \quad \text{and} \quad f^{(m)}(\alpha) \neq 0$$

Theorem 4.18 can be used to give an alternative proof of Liouville's theorem, as the following example demonstrates.

Example 4.11

Suppose that f is analytic inside the circle $|z-\alpha| = r$ and that $|f(z)| \leqslant M(\rho)$ on the circle $\mathscr{C}$, given by $|z-\alpha| = \rho < r$, where $M(\rho)$ is a positive constant which depends only on the radius of the circle. It follows by 4.18 that $f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n$ for $|z-\alpha| < r$. Then by (4.3),

$$|a_n| = \left| \frac{1}{2\pi i} \int_{\mathscr{C}} \frac{f(z)}{(z-\alpha)^{n+1}} dz \right| \leqslant \frac{1}{2\pi} \cdot \frac{M(\rho)}{\rho^{n+1}} \cdot 2\pi\rho = \frac{M(\rho)}{\rho^n}$$

by the ML lemma, 3.2. Hence if f is entire and $|f(z)| < M$ for all z, then $|a_n| \leqslant M/\rho^n$ for arbitrary ρ. Hence $a_n = 0$ for $n > 0$ and $|a_0| \leqslant M$, so that $f(z) = a_0$, a constant. This gives Liouville's theorem.

Exercise

4.3.1 Use Theorem 4.18 to find (i) the Maclaurin series expansion of $\sinh z$, and (ii) the Taylor series expansion of $1/z^2$ about $z = 2$. State the circle of convergence in each case.

Exercise **4.3.2** Find the Maclaurin series expansion of $f(z) = 1/(1+z)$ using 4.18 and state the circle of convergence. Use this series to find

(i) the Taylor series expansion of $f(z)$ about 3

(ii) the Maclaurin series expansion of $1/(1+z)^2$

(iii) the Maclaurin series expansion of $\operatorname{Log}(1+z)$

State the circle of convergence of the series in each case.

Exercise **4.3.3** Use the Maclaurin series expansion of $\cos z$ and the binomial series to find the first four non-zero terms in the Maclaurin series expansion of $\sec z$.

Exercise **4.3.4** Find the Taylor series expansion of $z^{1/2}$ about $-1-i$, and show that the series converges to $z^{1/2}$ for $|z+1+i| < 1$. Use the ratio test to show that the series converges for $|z+1+i| < \sqrt{2}$.

Exercise **4.3.5** Use the Maclaurin series expansion of $\sin z$ to find

(i) the Maclaurin series expansion of the **Fresnel sine integral** defined by

$$S(z) = \sqrt{\frac{2}{\pi}} \int_0^z \sin(w^2)dw$$

(ii) the Maclaurin series expansion of $f(z) = \sin z/(1-z)$

Hence find the nth derivative of $\sin z/(1-z)$ at $z = 0$.

Exercise **4.3.6** Let f be an entire function which satisfies $|f(z)| \leqslant k|z|^m$ for all $z \in \mathbb{C}$, where k and m are positive constants. Use the integral formula for the coefficients in the Maclaurin series expansion of f and the ML lemma to prove that

(i) if $m \in \mathbb{N}$ then $f(z) = a_m z^m$ for some a_m

(ii) if $m \notin \mathbb{N}$ then $f(z) = 0$ for all z

(Compare this result with the result of Exercise 3.3.5.)

Laurent Series

Very often it is necessary to expand a function about an isolated singular point, rather than a non-singular point. Such expansions, which are not Taylor series, are particularly useful in integration.

Example 4.12

It follows by Theorem 4.18 that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1)$$

$$\Rightarrow \quad \frac{1}{z(1-z)} = \frac{1}{z} + \sum_{n=0}^{\infty} z^n \qquad (0 < |z| < 1)$$

Although the second series converges for all z in the annular region $0 < |z| < 1$, it is an expansion of $f(z) = 1/z(1-z)$ about the isolated singular point 0, so is not a Taylor series.

Similarly, from Example 4.9(i), it follows that

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} \qquad \text{(for all } z \neq 0\text{)}$$

which again is the expansion of a function about an isolated singular point.

In general, if $\alpha \in \mathbb{C}$ is any isolated singular point of a function f, then f has a series expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-\alpha)^n} \tag{4.5}$$

about α, where a_n and b_n are constants for all n, which converges for all z in some annular region $0 < |z-\alpha| < r_1$. The series in (4.5) is the **Laurent series** expansion of f about the isolated singular point α. Note that the constants a_n and b_n are not expressible in terms of values of derivatives of f at α since α is a singular point of f. However, they are expressible in terms of integrals, as in the case of the coefficients in a Taylor series. This is the content of the following important result.

Theorem 4.21. Laurent's Theorem

Let α be a singular point of $f: A \subseteq \mathbb{C} \to \mathbb{C}$. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two concentric circles, centred at α, with radii r_1 and r_2 respectively, where $r_2 < r_1$. If f is analytic on $\mathscr{C}_1$ and $\mathscr{C}_2$ and throughout the annular region between the circles, then at each point in that region,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-\alpha)^n} \tag{4.5}$$

$$\text{where} \quad a_n = \frac{1}{2\pi i}\int_{\mathscr{C}_1} \frac{f(z)}{(z-\alpha)^{1+n}}\,dz \qquad (\text{for all } n \in \mathbb{Z}_{\geqslant 0})$$

$$\text{and} \quad b_n = \frac{1}{2\pi i}\int_{\mathscr{C}_2} \frac{f(z)}{(z-\alpha)^{1-n}}\,dz \qquad (\text{for all } n \in \mathbb{N}) \qquad \square$$

Important Note

It is convenient for the proof of 4.21 to have two concentric circles defining the domain of convergence. However, it follows by 4.21 and Lemma 3.10 that if f is analytic at every point inside and on the circle $\mathscr{C}_1$, except at α itself, so that α is an isolated singular point of f, then for $0 < |z-\alpha| < r_1$,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-\alpha)^n \quad \text{where} \quad c_n = \frac{1}{2\pi i}\int_{\mathscr{C}_1} \frac{f(z)}{(z-\alpha)^{n+1}}\,dz \tag{4.6}$$

for all $n \in \mathbb{Z}$. More generally, if f is analytic in the annular region $r_2 < |z-\alpha| < r_1$, then $\mathscr{C}_1$ in (4.6) is replaced by any simple closed contour lying inside the annular region.

Proof of 4.21

It is convenient to split the proof up into a number of parts.

Step 1

Let z be any point in the annular region and let $\mathscr{C}$ be a circle centred at z, which lies entirely within the annular region. Construct contours $\Gamma_k, k = 1, 2, 3$ as shown in Fig. 4.1. Note that by hypothesis and Cauchy's theorem, $\int_{\Gamma_k}(f(w)/(w-z))\,dw = 0$, for each k. Hence it follows that, since the net contribution to the integrals along the line segments is zero by Lemma 3.1,

$$\int_{\mathscr{C}_1} \frac{f(w)}{w-z}\,dw - \int_{\mathscr{C}_2} \frac{f(w)}{w-z}\,dw - \int_{\mathscr{C}} \frac{f(w)}{w-z}\,dw = \sum_{n=1}^{3}\int_{\Gamma_k} \frac{f(w)}{w-z}\,dw = 0 \tag{4.7}$$

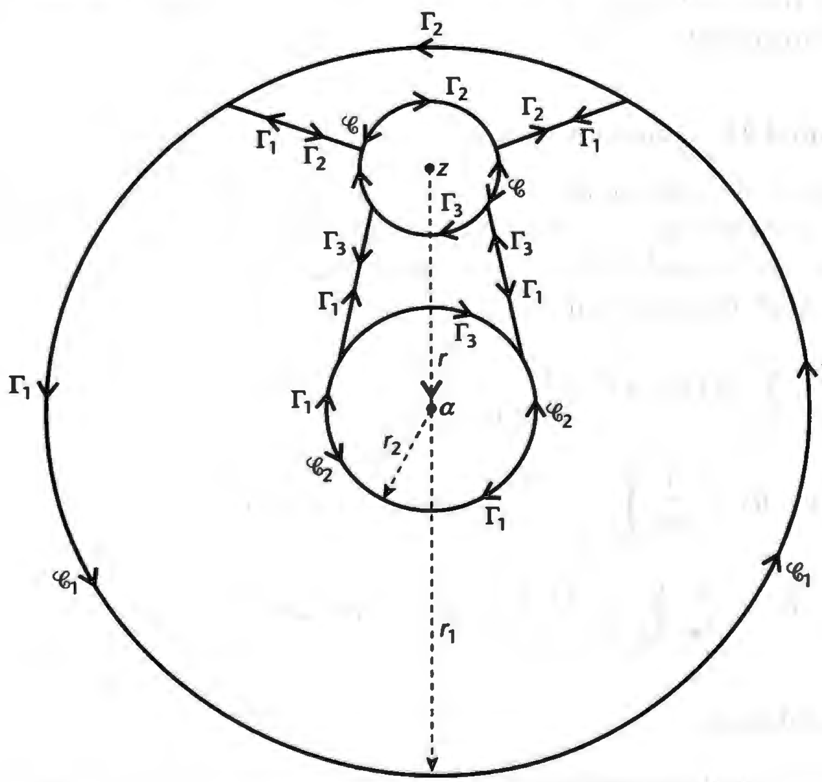


Figure 4.1

Also, by Cauchy's integral formula, $\int_{\mathscr{C}}(f(w)/(w-z))dw = 2\pi i f(z)$, so that by (4.7),

$$f(z) = \frac{1}{2\pi i}\int_{\mathscr{C}_1}\frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{\mathscr{C}_2}\frac{f(w)}{w-z}dw \tag{4.8}$$

In order to proceed further, it is necessary to expand the integrands in (4.8). It is possible to proceed using the technique used in the proof of 4.18 and uniform convergence results. Instead, we give an alternative approach which does not rely on uniform convergence results.

Step 2

Note that for any $\beta \in \mathbb{C}$, $\beta \neq 1$,

$$\frac{1}{1-\beta} = 1 + \beta + \beta^2 + \ldots + \beta^{k-1} + \frac{\beta^k}{1-\beta} \tag{4.9}$$

Letting $\beta = \dfrac{z-\alpha}{w-\alpha}$, $w \neq z$, in (4.9) gives

$$\frac{f(w)}{w-z} = \frac{f(w)}{w-\alpha}\frac{1}{1-(z-\alpha)/(w-\alpha)}$$

$$= \frac{f(w)}{w-\alpha} + \frac{f(w)}{(w-\alpha)^2}(z-\alpha) + \ldots + \frac{f(w)}{(w-\alpha)^k}(z-\alpha)^{k-1} + \frac{(z-\alpha)^k f(w)}{(w-\alpha)^k(w-z)}$$

$$\Rightarrow \quad \frac{1}{2\pi i}\int_{\mathscr{C}_1}\frac{f(w)}{w-z}dw = a_0 + a_1(z-\alpha) + \ldots + a_{k-1}(z-\alpha)^{k-1} + R_k(z) \tag{4.10}$$

where each a_n is given by the formula in the statement of the theorem and

$$R_k(z) = \frac{(z-\alpha)^k}{2\pi i}\int_{\mathscr{C}_1}\frac{f(w)}{(w-z)(w-\alpha)^k}dw \tag{4.11}$$

Similarly, letting $\beta = \dfrac{w-\alpha}{z-\alpha}$ in (4.9) gives

$$\frac{-f(w)}{w-z} = \frac{f(w)}{z-\alpha}\frac{1}{1-(w-\alpha)/(z-\alpha)}$$

$$= \frac{f(w)}{z-\alpha} + \frac{f(w)}{(z-\alpha)^2}(w-\alpha) + \ldots + \frac{f(w)}{(z-\alpha)^k}(w-\alpha)^{k-1} - \frac{f(w)(w-\alpha)^k}{(w-z)(z-\alpha)^k}$$

$$\Rightarrow \quad \frac{-1}{2\pi i}\int_{\mathscr{C}_2}\frac{f(w)}{w-z}dw = \frac{b_1}{z-\alpha} + \frac{b_2}{(z-\alpha)^2} + \ldots + \frac{b_k}{(z-\alpha)^k} + Q_k(z) \tag{4.12}$$

where the b_n are given by the formula in the theorem and

$$Q_k(z) = \frac{1}{2\pi i(z-\alpha)^k}\int_{\mathscr{C}_2} \frac{(w-\alpha)^k f(w)}{z-w}\,dw \tag{4.13}$$

Substituting (4.10) and (4.12) into (4.8) gives the required result as long as $\lim_{k\to\infty} R_k(z) = \lim_{k\to\infty} Q_k(z) = 0$.

Step 3

Let $r = |z-\alpha|$ as shown in Fig. 4.1. Then by the triangle inequality it follows that for any point w on $\mathscr{C}_1$,

$$|w-z| \geqslant |w-\alpha| - |z-\alpha| = r_1 - r$$

as can be seen in Fig. 4.1. Using the ML lemma, (4.11) then gives

$$|R_k(z)| = \frac{|z-\alpha|^k}{2\pi}\left|\int_{\mathscr{C}_1} \frac{f(w)}{(w-z)(w-\alpha)^k}\,dw\right| \leqslant \frac{r^k M}{2\pi(r_1-r){r_1}^k}(2\pi r_1)$$

where $|f(w)| \leqslant M$ on $\mathscr{C}_1$. (Notice that f is bounded on $\mathscr{C}_1$ by 4.5.) Hence

$$|R_k(z)| \leqslant \left(\frac{r}{r_1}\right)^k \frac{Mr_1}{r_1 - r} \quad \text{where} \quad \frac{r}{r_1} < 1$$

(see Fig. 4.1). Since $\lim_{k\to\infty} (r/r_1)^k = 0$, it follows by the sandwich theorem for real sequences that $\lim_{k\to\infty} R_k(z) = 0$.

Similarly, it follows by (4.13) and the ML lemma that

$$|Q_k(z)| = \frac{1}{2\pi r^k}\left|\int_{\mathscr{C}_2} \frac{(w-\alpha)^k f(w)}{z-w}\,dw\right| \leqslant \frac{r_2^k M^*}{2\pi r^k(r-r_2)}(2\pi r_2)$$

where $|f(w)| \leqslant M^*$ on $\mathscr{C}_2$. Hence

$$|Q_k(z)| \leqslant \left(\frac{r_2}{r}\right)^k \frac{M^* r_2}{r-r_2} \quad \text{where} \quad \frac{r_2}{r} < 1$$

so that $\lim_{k\to\infty} Q_k(z) = 0$, as required. ∎

Important Note

It may not be possible to expand a given function in any Laurent series about a non-isolated singular point. For example, Log z has no Laurent series expansion about the branch point at 0, since there is no annular region of the type described in Theorem 4.21, throughout which Log is analytic.

Taylor's theorem now follows easily from Laurent's theorem.

Proof of 4.18

Notice that Theorem 4.21 still holds if α is a non-singular point of f. If f is analytic at all points inside and on $\mathscr{C}_2$, including α, then

$$b_n = \frac{1}{2\pi i}\int_{\mathscr{C}_2} \frac{f(z)}{(z-\alpha)^{1-n}}\,dz = 0 \qquad (\text{for all } n \in \mathbb{N})$$

by Cauchy's theorem. Also,

$$a_n = \frac{1}{2\pi i}\int_{\mathscr{C}_1} \frac{f(z)}{(z-\alpha)^{n+1}}\,dz = \frac{f^{(n)}(\alpha)}{n!}$$

by Cauchy's integral theorem for derivatives, as required. ■

Important Note

Any Laurent series, given by (4.5), can be thought of as the sum of two power series, one in $z-\alpha$ and one in $(z-\alpha)^{-1}$, which converge absolutely with the domain of convergence of the Laurent series. It then follows from Theorem 4.15 that any Laurent series converges uniformly to its pointwise sum function within its domain of convergence. Hence by previous comments and Theorem 4.12:

(i) Any Laurent series can be multiplied by a constant, integrated or differentiated termwise within the domain of convergence.

(ii) Two Laurent series can be added or multiplied within the intersection of their domains of convergence.

In any case, the result is another convergent Laurent series.

In practice, it is not usually necessary to use Theorem 4.21 to calculate the Laurent series expansion of a given function about an isolated singular point. The following result states that the Laurent series expansion of a given function about a given point is unique, so the coefficients are not usually obtained by using their integral formulae.

Theorem 4.22. Uniqueness of Laurent Series

If the series

$$\sum_{n=-\infty}^{\infty} c_n(z-\alpha)^n = \sum_{n=0}^{\infty} a_n(z-\alpha)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-\alpha)^n}$$

(where a_k, b_k are constants for all k) converges to $f(z)$ at all points in some annular region about α, then it is the Laurent series expansion of $f(z)$ in that region. □

Proof

Let $\mathscr{C}$ be any circle lying in the given region, centred at α. Then by hypothesis and the above comments,

$$\frac{1}{2\pi i}\int_{\mathscr{C}} \frac{f(z)}{(z-\alpha)^{m+1}}\,dz = \sum_{n=-\infty}^{\infty} \frac{c_n}{2\pi i}\int_{\mathscr{C}} \frac{dz}{(z-\alpha)^{m-n+1}} \qquad (\text{for any } m \in \mathbb{Z})$$

Also, by definition

$$\int_{\mathscr{C}} \frac{dz}{(z-\alpha)^{m-n+1}} = \begin{cases} 2\pi i \text{ if } n = m \\ 0 \text{ if } n \neq m \end{cases}$$

Hence

$$c_m = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{f(z)}{(z-\alpha)^{m+1}}\,dz \quad (\text{for all } m \in \mathbb{Z})$$

as required. ∎

Example 4.13

(i) From the Maclaurin series for e^z it follows that

$$e^{1/z^2} = \sum_{n=0}^{\infty} \frac{1}{z^{2n} n!} \qquad (|z| > 0)$$

Also, for fixed $m \in \mathbb{N}$,

$$\frac{e^z}{z^m} = \frac{1}{z^m} + \ldots + \frac{1}{(m-1)!z} + \sum_{n=m}^{\infty} \frac{z^{n-m}}{n!} \qquad (|z| > 0)$$

These are the Laurent series expansions of the given functions about 0 by 4.22.

(ii) From the Maclaurin series for $\sin z$ it follows that

$$\sin z \sin(1/z) = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots\right)\left(\frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \frac{1}{7!z^7} + \ldots\right)$$

for $|z| > 0$.

Hence

$$\begin{aligned}\sin z \sin(1/z) = &\left(1 + \frac{1}{(3!)^2} + \frac{1}{(5!)^2} + \ldots\right) \\ &- \left(\frac{1}{3!} + \frac{1}{3!5!} + \frac{1}{5!7!} + \ldots\right)z^2 \\ &+ \left(\frac{1}{5!} + \frac{1}{3!7!} + \frac{1}{5!9!} + \ldots\right)z^4 - \ldots\end{aligned}$$

$$-\left(\frac{1}{3!}+\frac{1}{3!5!}+\frac{1}{5!7!}+\dots\right)\frac{1}{z^2}$$
$$+\left(\frac{1}{5!}+\frac{1}{3!7!}+\frac{1}{5!9!}+\dots\right)\frac{1}{z^4}-\dots \qquad (|z|>0)$$

Once again, this is a Laurent series expansion of the given function about 0.

(iii) Let $f(z) = 1/(1+z^2)$. Then f has singular points at $\pm i$. Let $w = z - i$, so that

$$f(z) = \frac{1}{w(w+2i)} = \frac{1}{2iw}(1 + w/2i)^{-1}$$

It then follows from the binomial series in Example 4.9(iii) and 4.22 that the Laurent series expansion of $f(z)$ about i is

$$f(z) = \frac{1}{2iw}\sum_{n=0}^{\infty}(-1)^n\left(\frac{w}{2i}\right)^n = \frac{1}{2i(z-i)}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{2i}\right)^n$$

which converges for $|w/2i| < 1$, i.e. for $|z - i| < 2$.

(iv) It follows from Example 4.12 that if $\mathscr{C}$ is any simple closed contour lying within the annular region $0 < |z| < 1$,

$$\int_{\mathscr{C}} \frac{dz}{z(1-z)} = \int_{\mathscr{C}} \frac{dz}{z} + \sum_{n=0}^{\infty}\int_{\mathscr{C}} z^n dz = 2\pi i$$

by Cauchy's theorem and Cauchy's integral formula.

Partial fractions are often useful when finding Laurent series expansions of algebraic fractions, as the following example demonstrates. This example also demonstrates the fact that a given function can clearly have different Laurent series expansions in different regions of the complex plane.

Example 4.14

The function

$$f(z) = \frac{1}{z(z+1)(z+2)}$$

has singular points, which are simple poles, at 0, -1 and -2. By Laurent's theorem, f has Laurent series expansions about 0 valid in the regions $0 < |z| < 1$, $1 < |z| < 2$ and $|z| > 2$. Note that

$$f(z) = \frac{1}{2z} + \frac{1}{2(z+2)} - \frac{1}{z+1}$$

Using the binomial series, it follows that

$$\frac{1}{z+2} = \frac{1}{2(1+z/2)} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n 2^{-n} z^n \qquad (|z/2| < 1)$$

$$\frac{1}{z+2} = \frac{1}{z(1+2/z)} = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n 2^n z^{-n} \qquad (|2/z| < 1)$$

$$\frac{1}{1+z} = \sum_{n=0}^{\infty}(-1)^n z^n \qquad (|z| < 1)$$

$$\frac{1}{1+z} = \frac{1}{z(1+1/z)} = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n z^{-n} \qquad (|1/z| < 1)$$

Hence it follows that

$$f(z) = \frac{1}{2z} + \sum_{n=0}^{\infty}(-1)^n(2^{-n-2} - 1)z^n \qquad (0 < |z| < 1)$$

$$f(z) = \frac{1}{2z} + \sum_{n=0}^{\infty}(-1)^n 2^{-n-2} z^n - \sum_{n=0}^{\infty}(-1)^n z^{-n-1} \qquad (1 < |z| < 2)$$

$$f(z) = \frac{1}{2z} + \sum_{n=0}^{\infty}(-1)^n(2^n - 1)z^{-n-1} \qquad (|z| > 2)$$

It follows by 4.22 that these are the Laurent series expansions of f about 0 in the given regions.

Laurent's theorem can be used to evaluate certain real integrals, by using the integral formulae for the coefficients in a known Laurent series, as the following example demonstrates.

Example 4.15

From the Maclaurin series expansion of e^z it follows that the Laurent series expansion of $e^{1/z}$ about 0 is

$$e^{1/z} = \sum_{n=0}^{\infty}\frac{1}{z^n n!} \qquad (|z| > 0)$$

(using Theorem 4.22). Comparing this series with (4.5) gives $a_0 = 1$, $a_n = 0$ for all $n > 1$ and $b_n = 1/n!$ for $n \in \mathbb{N}$. It follows by Theorem 4.21 that

$$b_n = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{e^{1/z}}{z^{1-n}}\,dz$$

where $\mathscr{C}$ is the circle $|z| = 1$, without loss of generality. Hence

$$\int_{\mathscr{C}} \frac{e^{1/z}}{z^{1-n}}\,dz = \frac{2\pi i}{n!} \qquad (n \in \mathbb{N})$$

On $|z| = 1$, $z = e^{i\theta}$, $-\pi \leqslant \theta \leqslant \pi$ say, so that by using the definition of a complex line integral,

$$\int_{-\pi}^{\pi} \frac{\exp(e^{-i\theta})}{e^{(1-n)i\theta}} ie^{i\theta} d\theta = \int_{-\pi}^{\pi} ie^{(\cos\theta - i\sin\theta + in\theta)} d\theta = \frac{2\pi i}{n!}$$

$$\Rightarrow \int_{-\pi}^{\pi} e^{\cos\theta}(\cos(n\theta - \sin\theta) + i\sin(n\theta - \sin\theta))d\theta = \frac{2\pi}{n!}$$

Comparing real parts gives

$$\int_{-\pi}^{\pi} e^{\cos\theta}\cos(n\theta - \sin\theta)d\theta = \frac{2\pi}{n!} \qquad \text{(for all } n \in \mathbb{N}\text{)}$$

Singular Points

Laurent series expansions can be used to give an easy classification of isolated singular points of functions. They can also be used to prove important theoretical results concerning certain types of isolated singular points. Recall that it follows from Theorem 2.8 that poles are isolated singular points. Clearly, removable singular points are also isolated.

Theorem 4.23. Classification of Isolated Singular Points

Let α be an isolated singular point of $f: A \subseteq \mathbb{C} \to \mathbb{C}$, so that f has a Laurent series expansion about α of the form (4.5), valid for $0 < |z - \alpha| < r$, for some r.

(i) The point α is a removable singular point of f if and only if $b_n = 0$ for all n in this Laurent series.

(ii) The point α is a pole of order m of $f (m \in \mathbb{N})$ if and only if $b_m \neq 0$ and $b_{m+1} = b_{m+2} = \ldots = 0$ in this Laurent series; that is, this Laurent series takes the form

$$f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n + \sum_{n=1}^{m} b_n(z-\alpha)^{-n}$$

(iii) The point α is an isolated essential singular point of f if and only if $b_n \neq 0$, except possibly for a finite number of values of n, in this Laurent series. □

Proof

(i) Let f have a Laurent series expansion about α of the form $f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n$, for $0 < |z - \alpha| < r$. Then $\lim_{z\to\alpha} f(z) = a_0$ since the series is uniformly convergent in the given region and

$\lim_{z\to\alpha}\sum_{n=0}^{\infty} a_n(z-\alpha)^n = a_0$. Hence α is a removable singular point of f. Now suppose α is a removable singular point of f, so that $\lim_{z\to\alpha} f(z)$ exists. Let $\lim_{z\to\alpha} f(z) = a_0$. Redefining $f(\alpha) = a_0$ makes f analytic at α (see Chapter 2) and so f has a Taylor series expansion about α, which converges for $|z-\alpha| < r$ say, as required.

(ii) Let f have a Laurent series expansion about α of the form

$$f(z) = \frac{b_m}{(z-\alpha)^m} + \frac{b_{m-1}}{(z-\alpha)^{m-1}} + \ldots + \frac{b_1}{(z-\alpha)} + \sum_{n=0}^{\infty} a_n(z-\alpha)^n$$

where $b_m \neq 0$, for $0 < |z-\alpha| < r$. Then $f(z) = (z-\alpha)^{-m}\sum_{n=0}^{\infty} d_n(z-\alpha)^n$, where $d_{m-n} = b_n$, $n = 1, \ldots, m$ and $d_{m+n} = a_n$, $n \in \mathbb{Z}_{\geqslant 0}$. Let $g(z) = \sum_{n=0}^{\infty} d_n(z-\alpha)^n$ for $0 < |z-\alpha| < r$. Then g is either analytic at α or has a removable singular point at α by (i). Then $\lim_{z\to\alpha} g(z) = d_0 = b_m \neq 0$ so that $\lim_{z\to\alpha}(z-\alpha)^m f(z) = b_m \neq 0$. Hence α is a pole of order m as required. Conversely, suppose α is a pole of order m of f, so that $(z-\alpha)^m f(z) = g(z)$ where g is analytic at α or has a removable singular point at α (see Chapter 2). Hence g either has a Taylor series expansion about α by 4.18, or a Laurent series expansion about α of the form (4.5) with $b_n = 0$ for all n by (i). Hence $g(z) = \sum_{n=0}^{\infty} d_n(z-\alpha)^n$ for $0 < |z-\alpha| < r$ say and f then has the required Laurent series expansion about α.

(iii) Let f have a Laurent series expansion of the form (4.5) for $0 < |z-\alpha| < r$, where $b_n \neq 0$, except possibly for a finite number of values of n. Then α is an isolated singular point of f which is not a removable singular point or a pole of f by (i) and (ii). Hence α is an isolated essential singular point of f. Conversely, if α is an isolated essential singular point of f, f has a Laurent series expansion about α of the form (4.5), valid in some region $0 < |z-\alpha| < r$. Since α is not a pole or removable singular point, this Laurent series takes the required form by (i) and (ii). ∎

Definitions

Let α be an isolated singular point of $f: A \subseteq \mathbb{C} \to \mathbb{C}$ so that f has a Laurent series expansion about α of the form (4.5), valid for $0 < |z-\alpha| < r$ for some r.

(i) The series $\sum_{n=1}^{\infty} b_n/(z-\alpha)^n$ is the **principal part** of this Laurent series.

(ii) The coefficient of $1/(z-\alpha)$ in this series, i.e. b_1, is the **residue** of f at α. We shall denote it by $\text{Res}_\alpha f(z)$.

Note that if f is analytic at α or has a removable singular point at α, so that the principal part of the Laurent series expansion of f about α is 0, then $\text{Res}_\alpha f(z) = 0$. Residues play a very important role when it comes to integrating functions with singular points, as will be shown in the next chapter.

Example 4.16

(i) From Theorem 4.23 and Example 4.13(i), it follows that 0 is an isolated essential singularity of e^{1/z^2} and 0 is a pole of order m of e^z/z^m.

(ii) Note that 0 is an isolated singular point of $(\sin z)/z$. It also follows from the Maclaurin series expansion of $\sin z$ that

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots \qquad (|z| > 0)$$

Hence 0 is a removable singular point of $(\sin z)/z$ by 4.23.

Example 4.17

Consider the function f given by $f(z) = 2/z(z-3)^2$. Note that f has isolated singular points at 0 and 3 (since f is not continuous at these points it is not differentiable there). By definition, 0 is a simple pole of f and 3 is a pole of order 2. It follows by the binomial series that

$$\frac{1}{(z-3)^2} = \frac{1}{(-3)^2(1-z/3)^2} = \frac{1}{9}\left(1 - \frac{z}{3}\right)^{-2}$$

$$\Rightarrow \quad \frac{1}{(z-3)^2} = \frac{1}{9}\left(1 + \frac{2z}{3} + \frac{(-2)(-3)}{2!}\left(\frac{z}{3}\right)^2 + \ldots\right) = \frac{1}{9} + \frac{2z}{27} + \frac{z^2}{27} + \ldots$$

for $|z/3| < 1$. Hence, by 4.22, the Laurent series expansion of f about 0, valid for $0 < |z| < 3$, is

$$f(z) = \frac{2}{9z} + \frac{4}{27} + \frac{2z}{27} + \ldots$$

This verifies 4.23 in the case of the simple pole of f at 0. Note also that $\mathrm{Res}_0 f(z) = 2/9$. To obtain the Laurent series expansion of f about 3, the simplest way to proceed is to let $w = z - 3$. Then

$$f(z) = \frac{2}{(w+3)w^2} = \frac{2}{3w^2}\left(1 + \frac{w}{3}\right)^{-1} = \frac{2}{3w^2}\left(1 - \frac{w}{3} + \frac{w^2}{9} - \frac{w^3}{27} + \ldots\right)$$

for $0 < |w| < 3$ by the binomial series again. Hence, by 4.22, the Laurent series expansion of f about 3, valid for $0 < |z-3| < 3$ is

$$f(z) = \frac{2}{3(z-3)^2} - \frac{2}{9(z-3)} + \frac{2}{27} - \frac{2(z-3)}{81} + \ldots$$

This verifies 4.23 in the case of the pole of order 2 of f at 3. Also, $\mathrm{Res}_3 f(z) = -2/9$.

We now present some results concerning isolated singular points, which depend either explicitly or implicitly on Laurent series. We begin with a preliminary result.

Theorem 4.24. Poles of Meromorphic Functions

A meromorphic function has only a finite number of poles in any bounded subset A of $\mathbb{C}$. □

Proof

Suppose that a meromorphic function f has an infinite number of poles in A. Then the set of these poles has a limit point α by Theorem 4.4. The function f is not analytic at α, since otherwise f would be analytic in some neighbourhood of α which excludes all the poles, and this contradicts the fact that α is a limit point of the poles. Hence α is a non-isolated singular point of f and so cannot be a pole. This contradicts the fact that f is meromorphic. ∎

Example 4.18

The function f defined by $f(z) = 1/\sin(1/z)$ has an infinite number of simple poles at the points $1/n\pi$, $n \in \mathbb{Z}$, $n \neq 0$. The set $\{1/n\pi : n \in \mathbb{Z}, n \neq 0\}$ is infinite and bounded and has limit point 0. This point is a non-isolated essential singular point of f, so f is not meromorphic.

Theorem 4.25. Weierstrass–Casorati Theorem

If $f : A \subseteq \mathbb{C} \to \mathbb{C}$ has an isolated essential singularity at α, then $f(z)$ approaches any given value arbitrarily closely in any neighbourhood of α. That is, given any $c \in \mathbb{C}$ and any $\varepsilon > 0$ and $\delta > 0$, there exists $z \in \mathbb{C}$ such that

$$|z - \alpha| < \delta \quad \text{and} \quad |f(z) - c| < \varepsilon$$ □

Proof

Suppose that the theorem is false. Then there exist δ, $\varepsilon > 0$ such that $|f(z) - c| \geqslant \varepsilon$ for all z satisfying $|z - \alpha| < \delta$. Let $g(z) = 1/(f(z) - c)$. Then by hypothesis there is some deleted open neighbourhood $\mathcal{N}$ say, of α throughout which g is analytic and $|g(z)| \leqslant 1/\varepsilon$; that is, g is analytic and bounded in $\mathcal{N}$. Then g has a Laurent series expansion about α, of the form (4.5), valid in $\mathcal{N}$, where

$$b_n = \frac{1}{2\pi i} \int_{\mathscr{C}} \frac{g(z)}{(z - \alpha)^{1-n}} \, dz$$

with $\mathscr{C}$ any circle of radius r, centred at α and contained in $\mathcal{N}$. Hence by the ML lemma, $|b_n| \leqslant (2\pi r/2\pi\varepsilon) r^{n-1} = r^n/\varepsilon$ and so $b_n \to 0$ as $r \to 0$, for fixed n.

Then $b_n = 0$ for all $n \in \mathbb{N}$, so g has a removable singular point at α by 4.23. If $\lim_{z\to\alpha} g(z) = a_0 \neq 0$, then since $f(z) = c + 1/g(z)$, f is analytic at α, which is a contradiction. Then $a_0 = 0$. Also, $g(z)$ cannot be identically zero in $\mathcal{N}$ since this contradicts the fact that f is analytic in a deleted neighbourhood of α. Hence g has an expansion of the form $g(z) = \sum_{n=k}^{\infty} a_n(z-\alpha)^n$, $a_k \neq 0$, for some k, in $\mathcal{N}$. Then $(z-\alpha)^{-k}g(z)$ is analytic and non-zero at α, so $(z-\alpha)^k/g(z)$ has a Taylor series expansion about α. Hence, by Theorem 4.23, f has a pole of order k at α, which is a contradiction. ■

A much stronger result than Theorem 4.25, although much harder to prove, is Picard's theorem.

Theorem 4.26. Picard's Theorem

Let f have an isolated essential singular point at α. Then $f(z)$ assumes every finite value, with one possible exception, an infinite number of times, in any neighbourhood of α. □

Example 4.19

The function f defined by $f(z) = e^{1/z^2}$ has an isolated essential singular point at 0. Now $e^z = 1$ when $z = 2n\pi i$ for $n \in \mathbb{Z}$, so that $e^{1/z^2} = 1$ when $z^2 = -i/2n\pi$. As an infinite number of these points lie in any neighbourhood of the origin, $f(z)$ assumes the value 1 an infinite number of times in any neighbourhood of 0. Note, however, that $f(z)$ never assumes the value 0.

The behaviour of a function f at the point at infinity in the extended complex plane $\tilde{\mathbb{C}}$, can be investigated by examining the behaviour of $f(1/z)$ in a neighbourhood of 0.

Definitions

Function $f(z)$ is respectively **analytic**, has a **removable singular point**, has a **pole of order** m or has an **essential singular point at** ∞ **in** $\tilde{\mathbb{C}}$ if and only if $f(1/z)$ is analytic, has a removable singular point, has a pole of order m or has an essential singular point at 0.

Example 4.20

(i) It is clear that any algebraic fraction is either analytic at ∞ or has a pole at ∞. For example, $f(z) = 1/z(1+z^2)$ has simple poles at 0 and $\pm i$. Also

$$f(1/z) = \frac{z}{1+1/z^2} = \frac{z^3}{1+z^2}$$

so $f(1/z)$ is analytic at 0. Hence f is analytic at ∞. Note that f has a zero of order 3 at ∞.

Now consider $g(z) = z^3/(z+1)$, which has a simple pole at -1. Then $g(1/z) = 1/z^2(1+z)$ has a pole of order 2 at 0. Hence g has a pole of order 2 at ∞.

(ii) Consider $f(z) = \sin z$, which is entire. Then $f(1/z) = \sin(1/z)$ has an isolated essential singularity at 0. Hence f has an isolated essential singularity at ∞.

It follows easily by Liouville's theorem that if a function is entire in the extended plane $\tilde{\mathbb{C}}$, then it must be constant.

Theorem 4.27. Entire Functions in the Extended Plane

Let f be entire in the extended plane $\tilde{\mathbb{C}}$. Then f is a constant. □

Proof

Since $f(z)$ is analytic at ∞, $f(1/z)$ is analytic at 0 and so there exists $r \in \mathbb{R}^+$ such that $f(1/z)$ is analytic, hence continuous, on the compact set $|z| \leqslant 1/r$. Then by 4.5, $f(1/z)$ is bounded for $|z| \leqslant 1/r$; that is, $f(z)$ is bounded for $|z| \geqslant r$. Since f is also continuous on the compact set $|z| \leqslant r$, it is also bounded on this set, by 4.5. Then f is bounded on $\mathbb{C}$ and the result follows by Liouville's theorem, 3.16. ■

Clearly, any algebraic fraction is meromorphic in $\tilde{\mathbb{C}}$. The converse of this result, which is an extension of 4.27, is also true.

Theorem 4.28. Meromorphic Functions in the Extended Plane

Let a function f be meromorphic in the extended complex plane. Then f is an algebraic fraction. □

Proof

Note that f is either analytic at ∞ or has a pole at ∞. Since poles are isolated, it is possible to construct a circle $\mathscr{C}$, centred at 0, such that ∞ is the only possible pole of f outside $\mathscr{C}$. There can only be a finite number of poles, $\alpha_1, \alpha_2, \ldots \alpha_n$ of f inside $\mathscr{C}$, by 4.24. If α_k is a pole of order m_k, then the principal part of the Laurent series expansion of f about α_k is of the form

$$g_k(z) = \frac{b_{1k}}{z-\alpha_k} + \frac{b_{2k}}{(z-\alpha_k)^2} + \ldots + \frac{b_{m_k k}}{(z-\alpha_k)^{m_k}}$$

where $b_{m_k k} \neq 0$. Since f is analytic or has a pole of order m say at ∞, $f(1/z)$ is analytic or has a pole of order m at 0, so that f has a Laurent series expansion of the form

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n} + b_1 z + b_2 z^2 + \ldots + b_m z^m \qquad (|z| > R \text{ say})$$

where each $b_k = 0$ or $b_m \neq 0$. In either case, the principal part of the Laurent series expansion of f about ∞ is of the form

$$h(z) = b_1 z + b_2 z^2 + \ldots + b_m z^m$$

for some non-zero b_m, or $b_k = 0$ for all k. Now consider the function ϕ defined by

$$\phi(z) = f(z) - \sum_{k=1}^{n} g_k(z) - h(z)$$

The function ϕ can clearly be made analytic at each α_k and is analytic at ∞ since $1/(z - \alpha_k)^N$ is analytic at ∞ for any non-zero α_k and $N \in \mathbb{N}$. Hence ϕ is entire in $\tilde{\mathbb{C}}$ and so is a constant by 4.27. Then $f(z)$ differs from $\sum_{k=1}^{n} g_k(z) + h(z)$ only by a constant and so is an algebraic fraction. ■

Exercise 4.4.1 Assuming the Maclaurin series expansions of $\sin z$ and $\cos z$:

(i) Find the first four terms in the Laurent series expansion of $\csc z$ in a region $0 < |z| < r$, stating the maximum value of r.

(ii) Find the Laurent series expansion of $(\cos z^2)/z^4$ about 0, valid for $z \neq 0$. Use this result to find the Laurent series expansion of $(\sin z^2)/z^3$ about 0, valid for $z \neq 0$.

Exercise 4.4.2 Assuming the Maclaurin series expansion for $\cos z$, find the Laurent series expansion of $z^2 \cos (z-1)^{-1}$ about 1, in the region $|z-1| > 0$.

Exercise 4.4.3 Assuming the Maclaurin series for e^z and the binomial series, find the Laurent series expansion of $e^z/(z^2-1)$

(i) about 0 in the region $|z| > 1$

(ii) about 1 in the region $0 < |z-1| < 2$

(iii) about 1 in the region $|z-1| > 2$

Exercise 4.4.4 Find the Laurent series expansions of $f(z) = 1/(z-2)(z-3)$ in the regions

(i) $2 < |z| < 3$

(ii) $|z| > 3$

(iii) $0 < |z-2| < 1$

(iv) $|z-2| > 1$

Exercise **4.4.5** Let f be given by $f(z) = 1/z^3 \cosh z$. Use the Maclaurin series expansion of $\cosh z$ and the binomial series to show that the Laurent series expansion of f about 0 is

$$f(z) = \frac{1}{z^3} - \frac{1}{2z} + \frac{5z}{24} + \dots \qquad (0 < |z| < \pi/2)$$

Integrate termwise and use Cauchy's theorem and the fundamental theorem of calculus to show that

$$\int_{|z|=1} f(z)dz = -\pi i = 2\pi i \mathrm{Res}_0 f(z)$$

Exercise **4.4.6** Use Laurent's theorem to show that $\exp(z + 1/z) = \sum_{n=-\infty}^{\infty} c_n z^n$, for $z \neq 0$, where $c_n = (1/\pi)\int_0^\pi \exp(2\cos\theta)\cos n\theta\, d\theta$ for all $n \in \mathbb{Z}$. Hence show that

$$\int_0^\pi \exp(2\cos\theta)d\theta = \pi \sum_{n=0}^{\infty} \frac{1}{(n!)^2}$$

Exercise **4.4.7** The **Bessel function** $J_n(z)$, $n \in \mathbb{Z}$, can be defined as the coefficient of w^n in the Laurent series of $e^{z(w-1/w)/2}$, valid in the region $|w| > 0$. Show that $J_n(z) = (1/\pi)\int_0^\pi \cos(n\theta - z\sin\theta)d\theta$.

Exercise **4.4.8** Locate the isolated singular points of the following functions. For each singular point of each function, find a Laurent series expansion of the function which converges in some deleted neighbourhood of the singular point. Hence classify the singular points and calculate the residue of each function at each singular point.

(i) $f(z) = ze^{1/z}$

(ii) $f(z) = \dfrac{e^z - 1}{z}$

(iii) $f(z) = \dfrac{1}{z^2(1+z^2)}$

(iv) $f(z) = \dfrac{1}{\sinh^3 z}$

(v) $f(z) = \sin(z + 1/z)$

(vi) $f(z) = \dfrac{e^z}{(4+z^2)^2}$

(vii) $f(z) = e^{\cosh(1/z)}$

(viii) $f(z) = \dfrac{1}{1+e^z}$

Exercise **4.4.9** Let g be a function which is analytic at α. Expand g as a Taylor series to show that α is a removable singular point of the function f given by $f(z) = g(z)/(z - \alpha)$ if $g(\alpha) = 0$. Show also that if $g(\alpha) \neq 0$, the point α is a simple pole of f with residue $g(\alpha)$.

Exercise **4.4.10** For each of the functions listed in Exercise 4.4.8, decide whether or not the point at infinity in the extended complex plane is a singular point. Classify the point at infinity in the cases where it is a singular point. (You need not find any Laurent series expansions.)

Exercise **4.4.11** Let f be a non-constant function, entire on $\mathbb{C}$, with $f(z) \neq 0$ for any $z \in \mathbb{C}$. Prove that f has an isolated essential singularity at ∞ in $\tilde{\mathbb{C}}$.

Exercise **4.4.12** Prove that a function entire on $\mathbb{C}$ having a non-essential singular point at ∞ in $\tilde{\mathbb{C}}$ must be a polynomial.

Exercise **4.4.13** Use Picard's theorem to prove that a function f entire on $\mathbb{C}$ with $f(z) \neq 0$ and $f(z) \neq 1$ for any $z \in \mathbb{C}$ must be a constant.

5 The Residue Theorem and its Applications

This chapter is concerned with Cauchy's residue theorem, which is a generalisation of Cauchy's integral formula, given in Chapter 3. The residue theorem is a very powerful result which can be used to evaluate definite integrals and hence a large class of real definite integrals. A large number of real definite integrals which can be evaluated by this technique are difficult to evaluate by other means. The residue theorem can also be used to sum certain convergent series of real numbers, which can be difficult to sum by other means.

Historical Note

Cauchy's original application of the residue theorem was to the evaluation of certain real definite integrals.

Cauchy's Residue Theorem and Calculation of Residues

The residue theorem implies that an integral evaluated around a closed contour only depends on the behaviour of the integrand at any singular points inside the contour. In fact, the only quantities which affect the integral are the residues of the integrand at its singular points.

Recall that if α is an isolated singular point of $f: A \subseteq \mathbb{C} \to \mathbb{C}$, then f has a Laurent series expansion about α of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-\alpha)^n} \tag{5.1}$$

where a_n and b_n are constants for all n, which converges for all z in some region $0 < |z-\alpha| < r$. The coefficient of $1/(z-\alpha)$ in (5.1), i.e. b_1, is the **residue** of f at α and will be denoted by $\operatorname{Res}_\alpha f(z)$.

Important Note

It is seldom necessary to calculate more than a small number of terms in this Laurent series expansion of f order to calculate $\operatorname{Res}_\alpha f(z)$.

Theorem 5.1. The Residue Theorem

Let $\mathscr{C}$ be a simple closed contour within and on which $f: A \subseteq \mathbb{C} \to \mathbb{C}$ is analytic except at a finite number of singular points, $z_1, z_2, \ldots, z_m$ inside $\mathscr{C}$. Then

$$\int_{\mathscr{C}} f(z)\,dz = 2\pi i \sum_{k=1}^{m} \mathrm{Res}_{z_k} f(z) \quad (= 2\pi i \times \text{sum of residues})$$ □

Proof

Step 1

Since there are a finite number of singular points, they are isolated, so it is possible to construct circles $\mathscr{C}_k$, $k = 1, 2, \ldots m$, centred at z_k for each k, which all lie inside $\mathscr{C}$, with radii small enough so that no two of them intersect, as in Fig. 3.8. It then follows by the generalised deformation result, Lemma 3.20, that

$$\int_{\mathscr{C}} f(z)\,dz = \sum_{k=1}^{m} \int_{\mathscr{C}_k} f(z)\,dz \tag{5.2}$$

(Recall that this result depends on Cauchy's theorem.)

Step 2

The radius of each $\mathscr{C}_k$ can be chosen small enough so that each circle is within the domain of convergence of the Laurent series expansion of f about z_k, of the form (5.1). It follows by Laurent's theorem, 4.21, that if b_{nk} is the coefficient of $(z - z_k)^{-n}$ in each Laurent series, then

$$b_{nk} = \frac{1}{2\pi i} \int_{\mathscr{C}_k} \frac{f(z)}{(z - z_k)^{1-n}}\,dz \qquad (n \in \mathbb{N})$$

$$\Rightarrow \qquad \int_{\mathscr{C}_k} f(z)\,dz = 2\pi i b_{1k} = 2\pi i\,\mathrm{Res}_{z_k} f(z) \qquad (\text{for each } k) \tag{5.3}$$

Substituting (5.3) into (5.2) gives the required result. ■

Notes

(i) The result (5.3) can also be obtained by noting that each Laurent series can be integrated termwise around each of the circles, by the results of Chapter 4. All terms give zero, by Cauchy's theorem and the fundamental theorem of calculus, except for $2\pi i b_{1k}$, which follows from the definition. This is the term left after integration, hence the term 'residue' for b_{1k}.

(ii) Theorem 5.1 reduces to Cauchy's theorem as a special case. If f has no singular points inside and on $\mathscr{C}$, then the sum of the residues is 0, so that $\int_{\mathscr{C}} f(z)\,dz = 0$. Theorem 5.1 also includes Cauchy's integral formula as a special case. If f has one singular point, α say, inside $\mathscr{C}$, which is a simple pole, then $\int_{\mathscr{C}} f(z)\,dz = 2\pi i \mathrm{Res}_\alpha f(z)$ and $f(z) = g(z)/(z - \alpha)$, where g is analytic at α without loss of generality. Hence $\mathrm{Res}_\alpha f(z) = g(\alpha)$ and so $\int_{\mathscr{C}} (g(z)/(z - \alpha))\,dz = 2\pi i g(\alpha)$.

(iii) Remember that it is only necessary to calculate the residues at the singular points **enclosed** by the contour $\mathscr{C}$ in order to apply 5.1 to obtain $\int_{\mathscr{C}} f(z)\,dz$.

Example 5.1

(i) From the Maclaurin series expansion of e^z it follows that $z^2e^{1/z} = \sum_{n=0}^{\infty} z^{2-n}/n!$ for all $z \neq 0$, so that 0 is an essential singular point of $z^2e^{1/z}$ and $\mathrm{Res}_0(z^2e^{1/z}) = 1/3!$. Clearly 0 is the only singular point of $z^2e^{1/z}$. Hence, if $\mathscr{C}$ is any simple closed contour enclosing the origin, then by 5.1, $\int_{\mathscr{C}} z^2e^{1/z}dz = \pi i/3$. Note however, that the real indefinite integral $\int x^2e^{1/x}dx$ cannot be evaluated in terms of elementary functions.

(ii) Similarly, it follows from the Maclaurin series expansion of $\sin z$ that

$$\sin(z^{-k}) = \sum_{n=0}^{\infty}(-1)^n z^{-(2n+1)k}/(2n+1)! \text{ for } k \in \mathbb{N}.$$

Hence 0 is an essential singular point of $\sin(z^{-k})$ and $\mathrm{Res}_0(\sin(z^{-k})) = 1$ if $k = 1$ and 0 otherwise. Clearly, 0 is the only singular point of $\sin(z^{-k})$. Hence if $\mathscr{C}$ is any simple closed contour enclosing the origin, then

$$\int_{\mathscr{C}} \sin(z^{-k})\,dz = 0 \quad \text{if} \quad k \neq 1 \quad \text{and} \quad \int_{\mathscr{C}} \sin(1/z)\,dz = 2\pi i$$

Let $\mathscr{C}$ be the circle $|z| = 1$. Then $\mathscr{C}$ is parametrised by $z = e^{i\theta}$, $-\pi \leqslant \theta \leqslant \pi$ say. It follows by the above result and by definition that

$$\int_{-\pi}^{\pi} \sin(e^{-i\theta})ie^{i\theta}d\theta = 2\pi i \Rightarrow \int_{-\pi}^{\pi} \sin(\cos\theta - i\sin\theta)(\cos\theta + i\sin\theta)d\theta = 2\pi$$

$$\Rightarrow \int_{-\pi}^{\pi} (\sin(\cos\theta)\cosh(\sin\theta) - i\cos(\cos\theta)\sinh(\sin\theta))(\cos\theta + i\sin\theta)\,d\theta$$

$$= 2\pi$$

$$\Rightarrow \int_{-\pi}^{\pi} (\sin(\cos\theta)\cosh(\sin\theta)\cos\theta + \cos(\cos\theta)\sinh(\sin\theta)\sin\theta))\,d\theta$$

$$= 2\pi.$$

Since the calculation of residues is of fundamental importance, it is advantageous to look for methods of calculating them, other than the definition. The following result enables us, at least in theory, to calculate the residue at a pole without having to resort to the definition.

Lemma 5.2. The Residue at a Pole

Let α be a pole of order m of $f: A \subseteq \mathbb{C} \to \mathbb{C}$, so that $f(z) = g(z)/(z-\alpha)^m$, $m \in \mathbb{N}$, for some g, analytic at α with $g(\alpha) \neq 0$, without loss of generality. Then $\operatorname{Res}_\alpha f(z) = g^{(m-1)}(\alpha)/(m-1)!$. □

Proof

Since g is analytic at α, it has, by 4.18, a Taylor series expansion in some neighbourhood of α of the form

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(\alpha)(z-\alpha)^n}{n!} \quad \Rightarrow \quad f(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(\alpha)(z-\alpha)^{n-m}}{n!} \tag{5.4}$$

Hence by definition,

$$\operatorname{Res}_\alpha f(z) = \text{coefficient of } \frac{1}{z-\alpha} \quad \text{in} \quad (5.4) = \frac{g^{(m-1)}(\alpha)}{(m-1)!}$$

■

Notes

(i) If α is a simple pole of f then by 5.2,

$$\operatorname{Res}_\alpha f(z) = g(\alpha) = \lim_{z \to \alpha} (z-\alpha) f(z)$$

This is the case in which 5.2 is most applicable.

(ii) If α is a pole of f of order greater than 2, then it is probably easier to use the definition to calculate $\operatorname{Res}_\alpha f(z)$, rather than 5.2.

(iii) Lemma 5.2 also follows by Cauchy's integral formula for derivatives and the residue theorem (see Exercises 5.1).

Example 5.2

(i) Let $f(z) = \dfrac{1}{(z-1)^2(z^2+1)} = \dfrac{1}{(z-1)^2(z+i)(z-i)}$

Then f has isolated singular points at 1 and at $\pm i$ (since f is not continuous there). Note that by definition, $\pm i$ are simple poles and 1 is a pole of f of order 2. Then by 5.2,

$$\operatorname{Res}_{\pm i} f(z) = \left.\frac{1}{(z-1)^2(z \pm i)}\right|_{z=\pm i} = \frac{1}{4}$$

$$\operatorname{Res}_1 f(z) = \left.\frac{d}{dz}\left(\frac{1}{z^2+1}\right)\right|_{z=1} = \left.\frac{-2z}{(z^2+1)^2}\right|_{z=1} = -\frac{1}{2}$$

Alternatively, we can use the definition to find $\operatorname{Res}_1 f(z)$. Let $w = z - 1$.

Then

$$f(z) = \frac{1}{w^2(w^2+2w+2)} = \frac{1}{2w^2}\left(1 + w + \frac{w^2}{2}\right)^{-1}$$

$$\Rightarrow \quad f(z) = \frac{1}{2w^2}\left(1 - w - \frac{w^2}{2} + \left(w + \frac{w^2}{2}\right)^2 + \dots\right)$$

$$= \frac{1}{2w^2}\left(1 - w + \frac{w^2}{2} + \dots\right)$$

in some neighbourhood of $w = 0$ using the binomial series. Then by definition, $\mathrm{Res}_1 f(z) = -\frac{1}{2}$.

(ii) Consider $\int_{\mathscr{C}}((\mathrm{Log}\, z)/(z^2+1)^2)\, dz$, where $\mathscr{C}$ is the contour shown in Fig. 5.1. Let $f(z) = (\mathrm{Log}\, z)/(z^2+1)^2$. Then f has singular points at $\pm i$, 0 and by convention at all points on the negative real axis. Only i lies within $\mathscr{C}$ and is a pole of f of order 2. By 5.2,

$$\mathrm{Res}_i f(z) = \frac{d}{dz}\left(\frac{\mathrm{Log}\, z}{(z+i)^2}\right)\bigg|_{z=i} = \left(\frac{1}{z(z+i)^2} - \frac{2\,\mathrm{Log}\, z}{(z+i)^3}\right)\bigg|_{z=i} = \frac{\frac{1}{2}i\pi - 1}{4i}$$

Alternatively, $\mathrm{Res}_i f(z)$ can be found using the definition. Then by the residue theorem,

$$\int_{\mathscr{C}} \frac{\mathrm{Log}\, z}{(z^2+1)^2}\, dz = 2\pi i \cdot \frac{1}{4i}\left(\frac{i\pi}{2} - 1\right) = \frac{\pi}{2}\left(\frac{i\pi}{2} - 1\right)$$

It is often the case that a given function cannot easily be expressed in 'factorised' form, so 5.2 may not be immediately applicable. In such cases, where the pole is simple, the following result is sometimes useful.

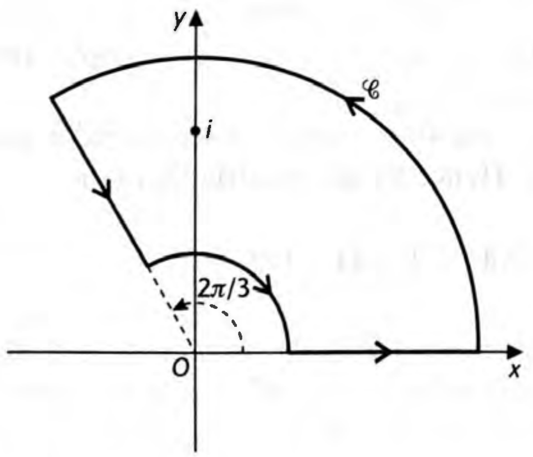


Figure 5.1

Lemma 5.3. The Residue at a Simple Pole

Suppose that $f(z)$ can be expressed in the form $f(z) = g(z)/h(z)$, where g and h are both analytic at α, with $g(\alpha) \neq 0$, $h(\alpha) = 0$ and $h'(\alpha) \neq 0$. Then α is a simple pole of f and $\text{Res}_\alpha f(z) = g(\alpha)/h'(\alpha)$. □

Note

It follows by the hypothesis of 5.3 and Lemma 4.20 that h has a simple zero at α.

Proof

It follows by 4.18 and hypothesis that $f(z) = g(z)/(z-\alpha)H(z)$ where g and H are analytic and non-zero at α. Hence g/H is analytic and non-zero at α and so α is a simple pole of f. Then by 5.2, $\text{Res}_\alpha f(z) = \lim_{z\to\alpha}(z-\alpha)g(z)/h(z)$ so that

$$\text{Res}_\alpha f(z) = \lim_{z\to\alpha} g(z) \cdot \lim_{z\to\alpha} \frac{(z-\alpha)}{h(z)-h(\alpha)} = \frac{g(\alpha)}{h'(\alpha)}$$

by hypothesis and the definition of $h'(\alpha)$. ■

Example 5.3

(i) Let $f(z) = \coth z = \cosh z/\sinh z$. Then f has an infinite number of isolated singular points at $n\pi i$, $n \in \mathbb{Z}$, and these are the only singular points of f since $\sinh z = 0$ iff $\sin iz = 0$ iff $z = n\pi i$. Clearly, 5.2 is not directly appropriate here, unless L'Hôpital's rule is used. Let $g(z) = \cosh z$ and $h(z) = \sinh z$. Then g and h are entire and certainly analytic at $n\pi i$. Also, $g(n\pi i) = h'(n\pi i) = \cos n\pi \neq 0$ and $h(n\pi i) = \sin n\pi = 0$. Hence by 5.3, $n\pi i$ are simple poles of f and $\text{Res}_{n\pi i} f(z) = \cosh z/\cosh z|_{z=n\pi i} = 1$ for all $n \in \mathbb{Z}$.

The circle $|z| = 3\pi/2$ encloses exactly three singular points of $\coth z$, those at $-i\pi$, 0 and $i\pi$. Hence by the residue theorem

$$\int_{|z|=3\pi/2} \coth z\, dz = 2\pi i(1+1+1) = 6\pi i$$

(ii) Consider $f(z) = z/\cos z(e^z - 1)$. Note that $\cos z = 0$ if and only if $z = (2n+1)\pi/2$, $n \in \mathbb{Z}$, and $e^z = 1$ if and only if $z = 2n\pi i$, $n \in \mathbb{Z}$. Then f has singular points at 0, $2n\pi i$, $n \neq 0$ and $(2n+1)\pi/2$. Note also that

$$f(z) = \frac{1}{(1 - z^2/2! + \ldots)(1 + z/2! + z^2/3! + \ldots)} = 1 - \frac{z}{2} + \frac{7z^2}{12} + \ldots$$

in some deleted neighbourhood of 0, so that 0 is a removable singular point of f and $\text{Res}_0 f(z) = 0$.

Let $g(z) = z/\cos z$ and $h(z) = e^z - 1$. Then g and h are analytic at $2n\pi i$, $g(2n\pi i) = 2n\pi i/\cosh 2n\pi \neq 0$ if $n \neq 0$, $h(2n\pi i) = 0$ and $h'(2n\pi i) = e^{2n\pi i} = 1 \neq 0$. Then by 5.3, $2n\pi i$, $n \neq 0$, are simple poles of f and $\text{Res}_{2n\pi i} f(z) = 2n\pi i/\cosh 2n\pi$.

Now let $g(z) = z/(e^z - 1)$ and $h(z) = \cos z$. Then the conditions of 5.3 are now satisfied for $\alpha = (2n+1)\pi/2$, so that $(2n+1)\pi/2$ are simple poles of f with

$$\text{Res}_{(2n+1)\pi/2} f(z) = \frac{(-1)^{n+1}(2n+1)\pi/2}{(e^{(2n+1)\pi/2} - 1)}$$

Notes

(i) It is usually clear which of 5.2 or 5.3 to use when calculating the residue at a simple pole. However, even in the case of simple poles, it is sometimes easier to resort to the definition to calculate the residue. Consider $f(z) = (z^2 \sinh z)/(e^z - 1)^4$. In this case, f has a simple pole at 0 by 5.4 below, but neither 5.2 or 5.3 is easy to apply.

(ii) There is no simple generalisation of 5.3 to the case of a pole of any order. The special case of a pole of order 2 is dealt with in Exercises 5.1. However, it is useful to bear in mind the following result, used in conjunction with 4.20.

Lemma 5.4. The Order of a Pole

Suppose that $f(z) = g(z)/h(z)$, where g and h are both analytic at α. Suppose also that α is a zero of order k of g and α is a zero of order m of h. Then α is a pole of f of order $m - k$ if $k < m$, α is a removable singular point of f if $k = m$ and α is a zero of order $k - m$ if $m < k$. □

Proof

By hypothesis $f(z) = (z - \alpha)^{k-m} G(z)/H(z)$ where G and H are analytic at α, so that G/H is also analytic at α, and $G(\alpha) \neq 0$ and $H(\alpha) \neq 0$. Then the results follow from the definitions. ■

For example, it follows by 4.20 that 0 is a zero of order 3 of $g(z) = z^2 \sinh z$ and a zero of order 4 of $h(z) = (e^z - 1)^4$. Then by 5.4, 0 is simple pole of $f(z) = (z^2 \sinh z)/(e^z - 1)^4$, as stated above.

Exercise **5.1.1** Use the residue theorem to evaluate the following integrals:

(i) $\displaystyle\int_{|z|=1} z\cos(1/z)\,dz$

(ii) $\displaystyle\int_{|z|=2} \frac{e^z}{z-1}\,dz$

Exercise **5.1.2** Evaluate $\int_{\mathscr{C}} z\sinh(1/z^2)\,dz$, where $\mathscr{C}$ is the circle $|z| = 1$, using the residue theorem. Let $z = e^{i\theta}$, $-\pi \leqslant \theta \leqslant \pi$ in this result to obtain the result of a particular real definite integral.

Exercise **5.1.3** Prove Lemma 5.2 using Cauchy's integral formula for derivatives and the residue theorem.

Exercise **5.1.4** Calculate the residue at each singular point of the following functions, using Lemma 5.2:

(i) $f(z) = \dfrac{1}{z^2 + a^2}$

(ii) $f(z) = \dfrac{z^2}{(z^2+4)^2}$

(iii) $f(z) = \dfrac{e^z}{z^2(z^2-1)}$

(iv) $f(z) = \dfrac{z^2+5z+1}{(z+1)^2(z^2+1)}$

Exercise **5.1.5** Calculate the residue at each singular point of the following functions, using Lemma 5.3 or the definition:

(i) $f(z) = \sec z$

(ii) $f(z) = z\coth z$

(iii) $f(z) = \dfrac{1}{z^n - 1}$, $n \in \mathbb{N}$

(iv) $f(z) = \dfrac{e^z}{\sin^2 z}$

(v) $f(z) = \dfrac{z^2\sinh z}{(e^z-1)^4}$

Exercise **5.1.6** Prove that if $f(z) = g(z)/h(z)$ where g and h are analytic at α, $g(\alpha) \neq 0$, $h(\alpha) = h'(\alpha) = 0$ and $h''(\alpha) \neq 0$, then α is a pole of f of order 2 and

$$\mathrm{Res}_\alpha f(z) = \frac{2g'(\alpha)}{h''(\alpha)} - \frac{2g(\alpha)h'''(\alpha)}{3(h''(\alpha))^2}$$

Exercise **5.1.7** Evaluate the following integrals using the residue theorem:

(i) $\displaystyle\int_{|z|=2} \frac{e^z}{z^3+z}\,dz$

(ii) $\displaystyle\int_{|z|=1} \frac{z+a}{z^n(z+b)}\,dz,\ n \in \mathbb{N}$

(iii) $\displaystyle\int_{\mathscr{C}} \frac{\sin z}{z^2 \sinh z}\,dz$ where $\mathscr{C}$ is the quadrilateral with vertices ± 1, $3i\pi/2$ and $-i\pi/2$

Exercise **5.1.8** Evaluate $\displaystyle\int_{\mathscr{C}} \frac{\operatorname{Log}(z+i)}{z^2+1}\,dz$

where $\mathscr{C}$ is the semicircle in the upper half-plane, of radius $R > 1$ and centre 0. Assuming that $\lim_{R\to\infty} \int_\Gamma f(z)\,dz = 0$, where Γ is the semicircular arc of $\mathscr{C}$, use this result to find

$$\int_0^\infty \frac{\operatorname{Log}(x^2+1)}{x^2+1}\,dx$$

Evaluation of Real Definite Integrals Using the Residue Theorem

The residue theorem can be used to great effect in the evaluation of certain real definite integrals. A suitable complex definite integral taken round a closed contour must be found which can be related to the given integral. It is useful to bear in mind that nearly every closed contour of practical use in this context consists of line segments and arcs of circles. We begin by investigating integrals of four commonly occuring types, which are the simplest to deal with.

Integrals of Type I

Relatively simple to deal with are real definite integrals which are related to the type

$$\boxed{\int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta \qquad (\mathrm{I})}$$

The idea here is essentially to use the definition of a complex definite integral in reverse. By definition,

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \quad \text{and} \quad \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

Hence letting $z = e^{i\theta}$ where $0 \leqslant \theta \leqslant 2\pi$ and noting $z'(\theta) = iz$,

$$\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta = \int_{\mathscr{C}} f\left(\frac{1}{2i}\left(z - \frac{1}{z}\right), \frac{1}{2}\left(z + \frac{1}{z}\right)\right)\frac{1}{iz}\, dz$$

where $\mathscr{C}$ is the circle $|z| = 1$. The transformed integral can be evaluated by the residue theorem in the normal way.

Example 5.4

(i) Consider $I = \int_0^{2\pi}(1/(5 + 4\sin\theta))\, d\theta$, which is of type I. Letting $z = e^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$, gives

$$I = \int_{\mathscr{C}} \frac{dz}{iz(5 - 2i(z - 1/z))} = \int_{\mathscr{C}} \frac{dz}{2z^2 + 5iz - 2} = \int_{\mathscr{C}} \frac{dz}{(2z + i)(z + 2i)}$$

where $\mathscr{C}$ is the circle $|z| = 1$. The integrand has simple poles at $-i/2$ and $-2i$. Only $-i/2$ lies within $\mathscr{C}$. It follows by 5.2 that

$$\mathrm{Res}_{-i/2}\left(\frac{1}{(2z + i)(z + 2i)}\right) = \left.\frac{1}{2(z + 2i)}\right|_{z=-i/2} = \frac{1}{3i}$$

Hence by the residue theorem, $\int_0^{2\pi}(1/(5 + 4\sin\theta))\, d\theta = 2\pi i/3i = 2\pi/3$.

(ii) Consider $\int_0^{2\pi}((\cos 2\theta)/(3\cos\theta + 5)^2)\, d\theta$, which is implicitly of type I. It is possible to substitute $z = e^{i\theta}$ directly but the evaluation is made a lot simpler by noting that

$$\int_0^{2\pi} \frac{\cos 2\theta}{(3\cos\theta + 5)^2}\, d\theta = \mathrm{Re}\left(\int_0^{2\pi} \frac{e^{2i\theta}}{(3\cos\theta + 5)^2}\, d\theta\right) = \mathrm{Re}\, I \quad \text{say}$$

Then letting $z = e^{i\theta}$ in I in the usual way gives

$$I = \int_{\mathscr{C}} \frac{z^2 dz}{iz(3(z + 1/z)/2 + 5)^2} = \frac{4}{i}\int_{\mathscr{C}} \frac{z^3 dz}{(3z + 1)^2(z + 3)^2}$$

where $\mathscr{C}$ is the circle $|z| = 1$. The integrand $f(z) = z^3/(3z + 1)^2(z + 3)^2$ has poles of order 2 at $-1/3$ and -3. Only $-1/3$ lies within $\mathscr{C}$. To find $\mathrm{Res}_{-1/3} f(z)$, we resort to the definition. Let $w = z + 1/3$.

$$f(z) = \frac{(w - 1/3)^3}{9w^2(w + 8/3)^2} = \frac{(-1/3)^3(1 - 3w)^3(1 + 3/8w)^{-2}}{9w^2(8/3)^2}$$

$$\Rightarrow \quad f(z) = \frac{-1}{27 \cdot 64w^2}(1 - 9w + \ldots)\left(1 - \frac{3w}{4} + \ldots\right)$$

in some deleted neighbourhood of $w = 0$. Hence

$$\mathrm{Res}_{-1/3} f(z) = \frac{-1}{27.64}\left(-9 - \frac{3}{4}\right) = \frac{13}{36.64}$$

Then by the residue theorem

$$I = 8\pi \mathrm{Res}_{-1/3} f(z) = \frac{13\pi}{288} \Rightarrow \int_0^{2\pi} \frac{\cos 2\theta}{(3\cos\theta + 5)^2}\, d\theta = \mathrm{Re}\, I = \frac{13\pi}{288}$$

Integrals of Type II

The next case we consider is a wide class of real improper integrals, which can all be evaluated using the same basic technique. We consider integrals related to the form

$$\boxed{\int_{-\infty}^{\infty} f(x)\, dx \quad f \text{ an algebraic fraction with no real singular points} \qquad \text{(II)}}$$

Important Note

It will be supposed in (II) that the degree of the denominator of f is at least 2 greater than the degree of the numerator of f. Then $\lim_{x\to\pm\infty} x^2 f(x)$ exists, so that by standard results from real analysis, the given real improper integral exists and is equal to its principal value, i.e.

$$\int_{-\infty}^{\infty} f(x)\, dx = \lim_{R,S\to\infty} \int_{-S}^{R} f(x)\, dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx \tag{5.5}$$

The idea is to choose a closed contour in the complex plane which encloses all relevant singular points of f and which includes an interval of the real axis. Since, in general, circles are the easiest contours to deal with, the method proceeds as follows.

Step 1

Consider $\int_{\mathscr{C}} f(z)\, dz$ where $\mathscr{C}$ is the semicircle in the upper half-plane of radius R, as shown in Fig. 5.2. The radius R is chosen large enough so that $\mathscr{C}$ encloses any singularities of f in the upper half-plane. This is possible since f has only a finite number of singular points. Let the semicircular arc of $\mathscr{C}$ be denoted by Γ, as shown.

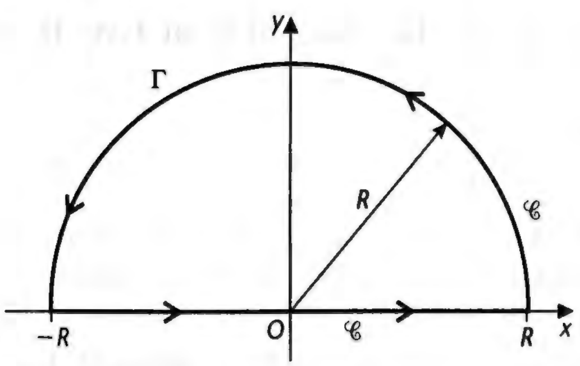


Figure 5.2

Since there are no singular points of f on $\mathscr{C}$, evaluate $\int_{\mathscr{C}} f(z)\,dz$ using the residue theorem. There is nothing special about the position of $\mathscr{C}$. We could equally well take a semicircle in the lower half-plane and ultimately get the same result.

Step 2

The next step is to make sure that $\lim_{R\to\infty}\int_\Gamma f(z)\,dz = 0$. Then $\lim_{R\to\infty}\int_{\mathscr{C}} f(z)\,dz = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx$ and the desired integral is obtained, because of (5.5). The following result is used at this stage.

Lemma 5.5. Convergence of $\int_C f(z)\,dz$ as $R \to \infty$

Let C be an arc of a circle with radius R and centre 0. Let f be continuous on C, with $\lim_{z\to\infty} zf(z) = 0$. Then $\lim_{R\to\infty}\int_C f(z)\,dz = 0$. □

Proof

Note that $|z| = R$ on C. By hypothesis, given any real $\varepsilon > 0$, there exists a real $\delta > 0$ such that

$$|1/z| < \delta \Rightarrow |zf(z)| < \varepsilon \quad \text{so that} \quad 1/R < \delta \Rightarrow |f(z)| < \varepsilon/R \quad \text{on } C.$$

Hence by Lemma 3.2, given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$1/R < \delta \quad \Rightarrow \quad \left|\int_C f(z)\,dz\right| < \alpha\pi R(\varepsilon/R) = \alpha\pi\varepsilon$$

where $0 < \alpha \leqslant 2$. Hence, by definition, $\lim_{R\to\infty}\int_C f(z)\,dz = 0$. ■

It follows by 5.5 that for integrals of type II, $\lim_{R\to\infty}\int_\Gamma f(z)\,dz = 0$, as required.

Example 5.5

(i) Suppose we wish to evaluate

$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

where $a, b > 0$ with $a \neq b$. The integral is of type II, so we consider $\int_{\mathscr{C}} f(z)\,dz$ where

$$f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$$

and $\mathscr{C}$ is the semicircle in Fig. 5.2; f has simple poles at $\pm ai$ and $\pm bi$. Only ai and bi lie in the upper half-plane, so we choose $R > \max(a, b)$.

$$\mathrm{Res}_{ai} f(z) = \frac{1}{2ai(b^2-a^2)} \quad \text{by 5.2 so that} \quad \mathrm{Res}_{bi} f(z) = \frac{1}{2bi(a^2-b^2)}$$

Then by the residue theorem,

$$\int_{\mathscr{C}} f(z)\,dz = 2\pi i(\mathrm{Res}_{ai} f(z) + \mathrm{Res}_{bi} f(z)) = \frac{\pi}{ab(a+b)}$$

Note that

$$\lim_{z\to\infty} zf(z) = \lim_{z\to\infty} \frac{1}{(z+a^2/z)(z^2+b^2)} = 0$$

so that by 5.5, $\lim_{R\to\infty}\int_\Gamma f(z)\,dz = 0$. Hence, letting $R\to\infty$ gives

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{ab(a+b)} \quad\Rightarrow\quad \int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

since the integrand is even.

(ii) Consider $\int_0^\infty (1/(1+x^4))\,dx$, which is once again of type II. Let $f(z) = 1/(1+z^4)$ and consider $\int_{\mathscr{C}} f(z)\,dz$ where $\mathscr{C}$ is the semicircle in Fig. 5.2. The integrand f has isolated singular points when $z^4 = -1 = e^{-i\pi + 2ki\pi}$, $k \in \mathbb{Z}$. Hence f has singular points $z_k = e^{-i\pi/4 + ik\pi/2}$, $k = 0, 1, 2, 3$. It is easily checked that only z_1 and z_2 lie in the upper half plane, so let $R > 1$. It follows by 5.3 that all singular points are simple poles and

$$\mathrm{Res}_{z_1} f(z) = \frac{1}{4z_1^3} = \frac{1}{4}e^{-3i\pi/4} \qquad \mathrm{Res}_{z_2} f(z) = \frac{1}{4z_2^3} = \frac{1}{4}e^{-i\pi/4}$$

Hence by 5.1,

$$\int_{\mathscr{C}} f(z)\,dz = \frac{\pi i}{2}(\cos(\pi/4) - i\sin(\pi/4) - \sin(\pi/4) - i\cos(\pi/4)) = \frac{\sqrt{2}\pi}{2}$$

Clearly $\lim_{z\to\infty} zf(z) = 0$ so that by 5.5, $\lim_{R\to\infty}\int_\Gamma f(z)\,dz = 0$. Then letting $R\to\infty$ gives

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\sqrt{2}\pi}{2} \quad\Rightarrow\quad \int_0^\infty \frac{dx}{1+x^4} = \frac{\sqrt{2}\pi}{4}$$

since the integrand is even.

Integrals of Type III

We next consider improper real integrals related to the form

$$\boxed{\int_{-\infty}^{\infty} f(x)g(x)\,dx \quad f \text{ an algebraic fraction with no real singular points and } g(x) = \sin mx \text{ or } \cos mx \ (m \in \mathbb{R}^+) \qquad \text{(III)}}$$

Important Note

It is supposed in (III) that the degree of the denominator of f is at least one greater than the degree of the numerator of f. The given real improper integral is absolutely convergent if the degree of the denominator is at least two greater than the numerator, by previous comments. If the degree of the denominator of f is only one greater than the numerator, then f' is an algebraic fraction with the degree of the denominator two greater than the numerator. Hence integration by parts shows that the given integral is (conditionally) convergent. Then in either case,

$$\int_{-\infty}^{\infty} f(x)g(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)g(x)\,dx$$

The same basic method employed for integrals of type II can clearly be employed in this case too. The only problem is with the convergence to 0 of the integral along the semicircular arc Γ. Generally $\int_\Gamma f(z)g(z)\,dz$ diverges as $R\to\infty$, but $\lim_{R\to\infty}\int_\Gamma e^{miz}f(z)\,dz = 0$. Hence the standard method is as follows.

Step 1

Consider $\int_{\mathscr{C}} e^{miz}f(z)\,dz$ where $\mathscr{C}$ is the standard semicircle of radius R shown in Fig. 5.2, with R chosen large enough so that $\mathscr{C}$ encloses any singular points of f in the upper half-plane. Since there are no singular points on $\mathscr{C}$, evaluate $\int_{\mathscr{C}} e^{miz}f(z)\,dz$ using the residue theorem.

Step 2

Show that $\lim_{R\to\infty}\int_\Gamma e^{miz}f(z)\,dz = 0$, so that letting $R\to\infty$ gives $\lim_{R\to\infty}\int_{\mathscr{C}} e^{miz}f(z)\,dz = \int_{-\infty}^{\infty} e^{mix}f(x)\,dx$. Then the required integral follows by comparing real or imaginary parts. In this case, the following result is used.

Theorem 5.6. Jordan's Lemma

Let Γ be the semicircular arc parametrised by $z = Re^{i\theta}$, $0\leqslant\theta\leqslant\pi$. Let f be continuous on Γ, with $\lim_{z\to\infty} f(z) = 0$. Then $\lim_{R\to\infty}\int_\Gamma e^{miz}f(z)\,dz = 0$ for $m\in\mathbb{R}^+$. □

Proof

Part 1

Using the given parametrisation of Γ gives

$$\left|\int_\Gamma e^{miz}\,dz\right| = \left|\int_0^\pi e^{miR(\cos\theta + i\sin\theta)}\,iRe^{i\theta}\,d\theta\right|$$

$$\leqslant \int_0^{\pi} |e^{miR\cos\theta}||e^{-mR\sin\theta}||iRe^{i\theta}|d\theta$$

$$\Rightarrow \quad \left|\int_\Gamma e^{miz}dz\right| \leqslant R\int_0^{\pi} e^{-mR\sin\theta}d\theta = 2R\int_0^{\pi/2} e^{-mR\sin\theta}d\theta \tag{5.6}$$

using (3.7) in the proof of 3.2 and noting that $e^{-mR\sin\theta}$ is symmetric about $\theta = \pi/2$ in the interval $[0, \pi]$.

Part 2

Note that $2\theta/\pi \leqslant \sin\theta$ for $\theta \in [0, \pi/2]$ since $2\theta/\pi = \sin\theta$ when $\theta = 0$ or $\pi/2$ in this interval and if $f(\theta) = 2\theta/\pi$ and $g(\theta) = \sin\theta$ then $f'(0) < g'(0)$. Hence inequality (5.6) becomes

$$\left|\int_\Gamma e^{miz}dz\right| \leqslant 2R\int_0^{\pi/2} e^{-2mR\theta/\pi}d\theta = \frac{\pi}{m}(1 - e^{-mR}) < \frac{\pi}{m} \tag{5.7}$$

Part 3

By hypothesis, $\lim_{z\to\infty} f(z) = 0$ so that given any real $\varepsilon > 0$, there exists a real $\delta > 0$ such that

$$|1/z| < \delta \quad \Rightarrow \quad |f(z)| < \varepsilon \quad \text{so that} \quad 1/R < \delta \quad \Rightarrow \quad |f(z)| < \varepsilon \quad \text{on } \Gamma$$

Then by (5.7), given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$1/R < \delta \quad \Rightarrow \quad \left|\int_\Gamma e^{miz} f(z)\, dz\right| < \varepsilon\left|\int_\Gamma e^{miz}dz\right| < \frac{\pi\varepsilon}{m}$$

Hence $\lim_{R\to\infty} \int_\Gamma e^{miz} f(z)\, dz = 0$ as required. ■

Example 5.6

(i) To evaluate

$$\int_{-\infty}^{\infty} \frac{x\sin mx}{x^2 + a^2}dx \qquad (m, a > 0)$$

which is of type III, we consider $\int_{\mathscr{C}} e^{miz} f(z)\, dz$ where $f(z) = z/(z^2 + a^2)$ and where $\mathscr{C}$ is the semicircle shown in Fig. 5.2; f has simple poles at $\pm ai$ and only ai lies in the upper half-plane, so let $R > a$. By 5.2, $\operatorname{Res}_{ai}(e^{miz} f(z)) = e^{-ma}/2$, and so by 5.1, $\int_{\mathscr{C}} e^{miz} f(z)\, dz = \pi i e^{-ma}$. Also, $\lim_{z\to\infty} f(z) = \lim_{z\to\infty} 1/(z + a^2/z) = 0$, so that by Jordan's lemma, $\lim_{R\to\infty} \int_\Gamma e^{miz} f(z)\, dz = 0$. Hence

$$\lim_{R\to\infty} \int_{\mathscr{C}} e^{miz} f(z) = \pi i e^{-ma} = \int_{-\infty}^{\infty} e^{mix} f(x)\, dx$$

Equating imaginary parts gives

$$\int_{-\infty}^{\infty} \frac{x \sin mx}{x^2 + a^2} dx = \pi e^{-ma} \quad \Rightarrow \quad \int_0^{\infty} \frac{x \sin mx}{x^2 + a^2} dx = \frac{\pi}{2e^{ma}}$$

since the integrand is even.

(ii) To evaluate

$$\int_0^{\infty} \frac{\cos mx}{(x^2 + 1)^2} dx$$

we consider $\int_{\mathscr{C}} e^{miz} f(z)\, dz$ where $f(z) = 1/(z^2 + 1)^2$ and $\mathscr{C}$ is the semicircle shown in Fig. 5.2, where $R > 1$ since f has poles of order 2 at $\pm i$. Then only i lies inside $\mathscr{C}$. Using Lemma 5.2, $\mathrm{Res}_i(e^{miz} f(z)) = g'(i)$ where $g(z) = e^{miz}/(z + i)^2$, so that $\mathrm{Res}_i(e^{miz} f(z)) = (m + 1)e^{-m}/4i$ and then by 5.1, $\int_{\mathscr{C}} e^{miz} f(z)\, dz = (m + 1)e^{-m}\pi/2$. Letting $R \to \infty$, it follows by Jordan's lemma that

$$\int_{-\infty}^{\infty} e^{mix} f(x)\, dx = \frac{(m + 1)e^{-m}\pi}{2} \quad \Rightarrow \quad \int_0^{\infty} \frac{\cos mx}{(x^2 + 1)^2} dx = \frac{(m + 1)e^{-m}\pi}{4}$$

by comparing real parts and noting that the integrand is even.

Integrals of Type IV

It is also possible to evaluate convergent integrals of type III, except that f has real singular points, by amending the standard semicircle suitably. A closed contour is created which does not contain any real singular points, by constructing a small semicircle centred at each real singular point.

Important Note

Recall that it may happen that a given real improper integral, $\int_{-\infty}^{\infty} f(x)\, dx$ does not exist, whereas its Cauchy principal value $\lim_{R \to \infty} \int_{-R}^{R} f(x)\, dx$ does exist. Similarly, if f has a singular point at $x = c$, where $a < c < b$, it may happen that the improper integral

$$\int_a^b f(x)\, dx = \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)\, dx + \lim_{\eta \to 0} \int_{c+\eta}^{b} f(x)\, dx$$

does not exist, whereas its Cauchy principal value

$$\lim_{\varepsilon \to 0} \left(\int_a^{c-\varepsilon} f(x)\, dx + \int_{c+\varepsilon}^{b} f(x)\, dx \right)$$

does exist. In any case, we shall always take $\int_{-\infty}^{\infty} f(x)\, dx$ to mean the Cauchy principal value of the given improper integral. The notation reduces to its usual meaning when the integral converges in the usual sense.

Consider real convergent improper integrals, or more generally, integrals whose Cauchy principal value exists, which are related to the form

$\int_{-\infty}^{\infty} f(x)g(x)\,dx$	f an algebraic fraction with singular points on the real axis and $g(x) = \sin mx$ or $\cos mx$ $(m \in \mathbb{R}^+)$	(IV)

Note

A necessary condition for convergence is that the degree of the denominator of f is at least one greater than the numerator of f.

The standard method of evaluating integrals of type IV is as follows.

Step 1

Suppose that f has real singular points at $a_1, \ldots, a_k$ where $a_1 < a_2 < \ldots < a_k$. Consider $\int_{\mathscr{C}} e^{miz} f(z)\,dz$ where $m \in R^+$ and $\mathscr{C}$ is the contour shown in Fig. 5.3, consisting of the large semicircular arc Γ of radius R, line segments of the real axis, and small semicircular arcs $\gamma_1, \ldots, \gamma_k$ of radii $r_1, \ldots, r_k$. The radius R is chosen large enough and the radii $r_1, \ldots, r_k$ are chosen small enough so that $\mathscr{C}$ encloses any singularities of f in the upper half-plane. Since there are no singular points of f on $\mathscr{C}$, evaluate $\int_{\mathscr{C}} e^{miz} f(z)\,dz$ using the residue theorem.

Step 2

Show that $\lim_{R\to\infty} \int_{\Gamma} e^{miz} f(z)\,dz = 0$ using Jordan's lemma as in type III.

Step 3

Find $\lim_{r_k\to 0} \int_{\gamma_k} e^{miz} f(z)\,dz$ for each k using Lemma 5.7 or Corollary 5.8.

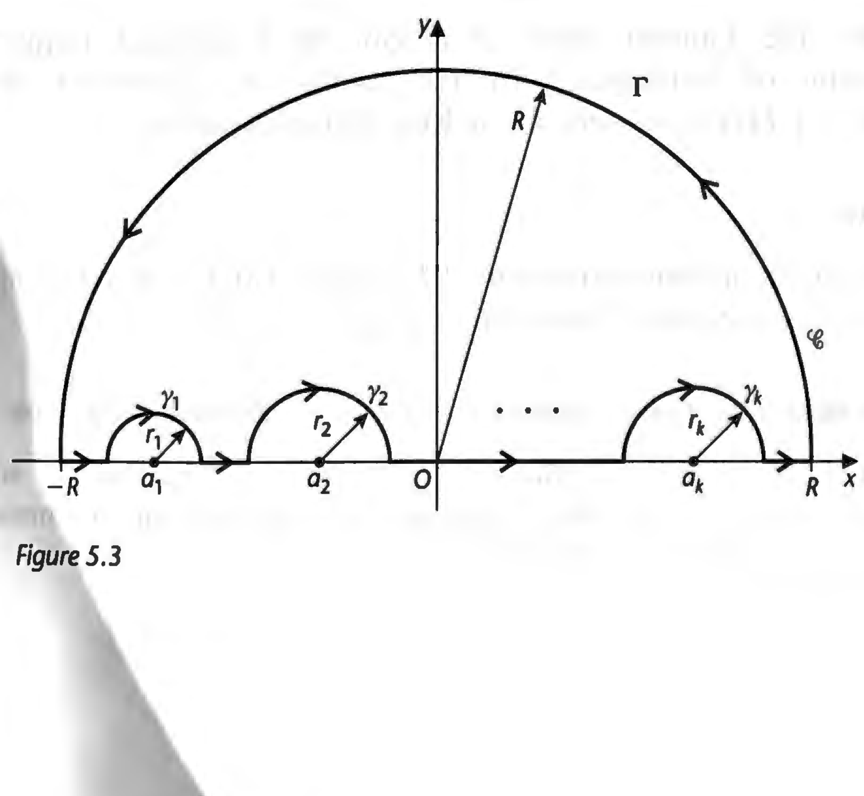


Figure 5.3

Step 4

Finally, letting $R \to \infty$ and $r_k \to 0$ for each k in $\mathscr{C}$ gives, provided the integrals converge,

$$\int_{\mathscr{C}} e^{miz} f(z)\, dz = \lim_{R\to\infty} \int_{-R}^{R} e^{mix} f(x)\, dx - i\pi \sum_{n=1}^{k} \mathrm{Res}_{a_n}(e^{miz} f(z))$$

The desired real integral is then obtained by equating real and imaginary parts.

Lemma 5.7. Convergence of $\int_\gamma F(z)\, dz$ as $r \to 0$

Let F be analytic in the region $0 < |z-a| < r_1$, with a singular point at $a \in \mathbb{R}$. Let γ be a semicircular arc of radius $r < r_1$, centred at a, parametrised by $z - a = re^{i\theta}$, where θ *decreases* from π to 0. If the Laurent series expansion of F about a, valid for $0 < |z-a| < r_1$, contains no even negative powers in its principal part, then $\lim_{r\to 0} \int_\gamma F(z)\, dz = -i\pi \mathrm{Res}_a F(z)$. □

Proof

By hypothesis, F has a Laurent series expansion about a of the form

$$F(z) = \sum_{n=0}^{\infty} \frac{b_{2m+1}}{(z-a)^{2m+1}} + \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{for} \quad 0 < |z-a| < r_1 \tag{5.8}$$

By definition,

$$\int_\gamma (z-a)^n dz = -ir^{n+1} \int_0^\pi e^{(n+1)i\theta} d\theta = \frac{r^{n+1}}{n+1}(1 + \cos n\pi)$$

where $n \in \mathbb{Z}$ with $n \neq -1$. Hence for $n \geqslant 0$ and $n < -1$ with n odd, $\lim_{r\to 0} \int_\gamma (z-a)^n dz = 0$. Also by definition,

$$\lim_{r\to 0} \int_\gamma (z-a)^{-1} dz = \lim_{r\to 0} \int_0^\pi -i d\theta = -i\pi$$

Since the Laurent series (5.8) can be integrated termwise within its domain of convergence by the results of Chapter 4, it follows that $\lim_{r\to 0} \int_\gamma F(z)\, dz = -i\pi b_1 = -i\pi\, \mathrm{Res}_a F(z)$ as required. ■

Note

If the Laurent series expansion of F about a, valid for $0 < |z-a| < r_1$, contains even negative powers, then $\lim_{r\to 0} \int_\gamma F(z)\, dz$ diverges.

Corollary 5.8. Convergence of $\int_\gamma F(z)\, dz$ as $r \to 0$ with a Simple Pole

Let F be analytic in the region $0 < |z-a| < r_1$, with a simple pole at $a \in \mathbb{R}$. Let γ be the semicircular arc as in Lemma 5.7. Then $\lim_{r\to 0} \int_\gamma F(z)\, dz = -i\pi\, \mathrm{Res}_a F(z)$. □

Example 5.7

(i) One of the easiest type IV integrals to deal with is the famous improper integral $\int_0^\infty ((\sin x)/x)\,dx$. Consider $\int_{\mathscr{C}} (e^{iz}/z)\,dz$ where $\mathscr{C}$ is the contour as shown in Fig. 5.4, since the integrand has only one singular point, a simple pole at 0. It follows by the residue theorem that $\int_{\mathscr{C}} (e^{iz}/z)\,dz = 0$. Also, by Jordan's lemma, $\lim_{R\to\infty} \int_\Gamma (e^{iz}/z)\,dz = 0$. $\mathrm{Res}_0(e^{iz}/z) = 1$ by 5.2, so by 5.8, $\lim_{r\to 0} \int_\gamma (e^{iz}/z)\,dz = -i\pi$. Then letting $R \to \infty$ and $r \to 0$ in $\mathscr{C}$ gives

$$\int_{\mathscr{C}} \frac{e^{iz}}{z}\,dz = 0 = \lim_{R\to\infty, r\to 0}\left(\int_{-R}^{-r} \frac{e^{ix}}{x}\,dx + \int_r^R \frac{e^{ix}}{x}\,dx\right) - i\pi$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = i\pi \Rightarrow \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi \Rightarrow \int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

by equating imaginary parts and noting that the resulting integrand is even. Compare this with Example 3.8.

(ii) Another convergent real improper integral of type IV is

$$\int_{-\infty}^{\infty} \frac{\cos \pi x}{1-4x^2}\,dx$$

(since the singular points at $x = \pm 1/2$ are removable). Following the standard technique, consider $\int_{\mathscr{C}} F(z)\,dz$ where $F(z) = e^{i\pi z}/(1-4z^2)$. Since F has only two singular points, which are simple poles at $\pm 1/2$, $\mathscr{C}$ is chosen to be the contour shown in Fig 5.5, where $r < 1/2$ and $R > 1$. It follows by the residue theorem that $\int_{\mathscr{C}} F(z)\,dz = 0$. And by Jordan's lemma, $\lim_{R\to\infty} \int_\Gamma F(z)\,dz = 0$. Also by Lemma 5.2, $\mathrm{Res}_{\pm 1/2} F(z) = 1/4i$. Hence letting $R \to \infty$ and $r \to 0$ in $\mathscr{C}$ and using 5.8 gives

$$\int_{\mathscr{C}} F(z)\,dz = 0 = \int_{-\infty}^{\infty} F(x)\,dx - i\pi\left(\frac{2}{4i}\right) \Rightarrow \int_{-\infty}^{\infty} \frac{\cos \pi x}{1-4x^2}\,dx = \frac{\pi}{2}$$

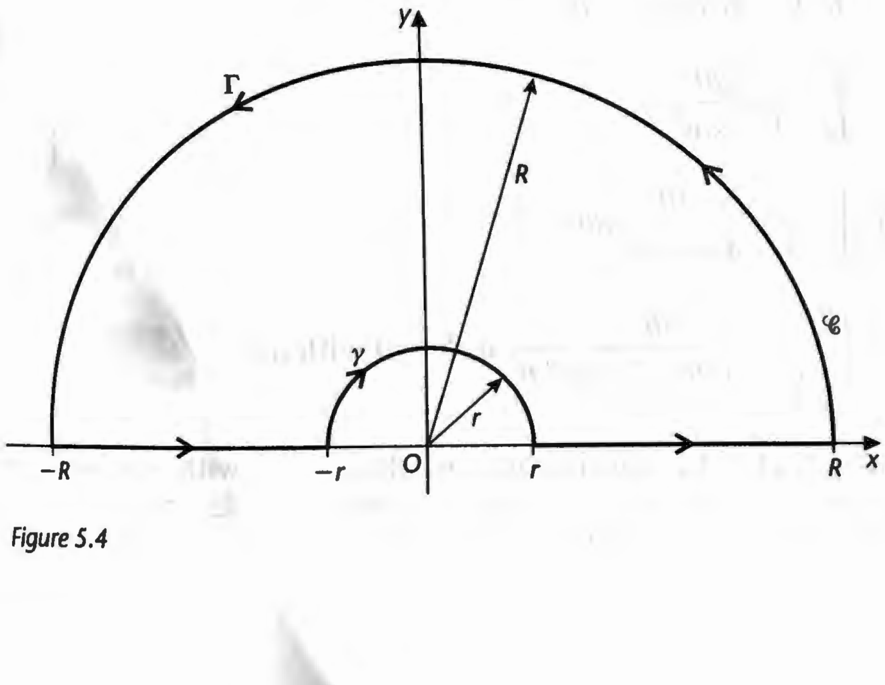


Figure 5.4

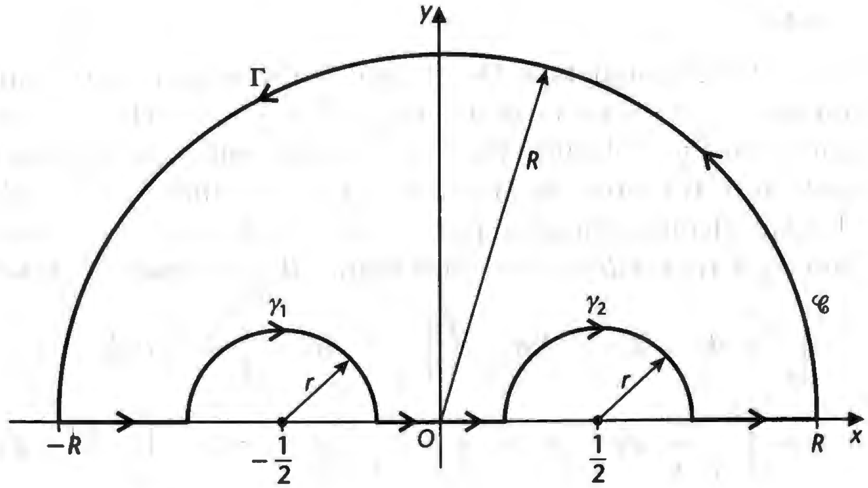


Figure 5.5

Exercise **5.2.1** Evaluate the following real integrals using the residue theorem:

(i) $\displaystyle\int_0^{\pi} \frac{d\theta}{a + b\cos\theta}$, $a > b > 0$

(ii) $\displaystyle\int_0^{2\pi} e^{2\cos\theta}\, d\theta$

(iii) $\displaystyle\int_0^{2\pi} \frac{d\theta}{(3\cos\theta + 5)^2}$

(iv) $\displaystyle\int_0^{\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2}\, d\theta$, $|a| < 1$

(v) $\displaystyle\int_0^{\pi/2} \frac{d\theta}{1 + \sin^2\theta}$

(vi) $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta}\, d\theta$

(vii) $\displaystyle\int_0^{2\pi} \frac{d\theta}{a^2\sin^2\theta + b^2\cos^2\theta}$, $a, b > 0$ with $a \neq b$

Exercise **5.2.2** Let f be continuous on the arc $\mathscr{C}$ with parametric representation $z = Re^{i\theta}$, $\alpha \leqslant \theta \leqslant \beta$, and let $\lim_{z\to\infty} zf(z) = \ell$. Prove that $\lim_{R\to\infty} \int_{\mathscr{C}} f(z)\, dz = (\beta - \alpha)\ell i$.

Exercise 5.2.3 Evaluate the following real convergent improper integrals where a, $b > 0$, $a \neq b$ and $n \in \mathbb{N}$, using the residue theorem:

(i) $$\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}\, dx$$

(ii) $$\int_0^\infty \frac{x^2}{(x^2+a^2)^2}\, dx$$

(iii) $$\int_0^\infty \frac{x^2}{x^6+a^6}\, dx$$

(iv) $$\int_0^\infty \frac{x^4}{(x^2+a^2)^3}\, dx$$

(v) $$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)^2}$$

(vi) $$\int_0^\infty \frac{x^2}{x^4+x^2+1}\, dx$$

(vii) $$\int_{-\infty}^\infty \frac{dx}{(x^2+x+1)^2}$$

(viii) $$\int_0^\infty \frac{dx}{(x^2+1)^n}$$

Exercise 5.2.4 Evaluate the following real improper integrals, where a, $m > 0$ and $n \in \mathbb{N}$, using the residue theorem:

(i) $$\int_0^\infty \frac{\cos^2 x}{x^2+a^2}\, dx$$

(ii) $$\int_0^\infty \frac{x^3 \sin mx}{(x^2+1)^2}\, dx$$

(iii) $$\int_{-\infty}^\infty \frac{x \sin x}{x^2+2x+2}\, dx$$

(iv) $$\int_0^\infty \frac{\cos mx}{(x^2+1)^3}\, dx$$

(v) $$\int_0^\infty \frac{x \sin x}{x^4+a^4}\, dx$$

Exercise 5.2.5 Let F be analytic in the region $0 < |z-a| < r_1$, with a simple pole at a. Let $\mathscr{C}$ be the arc of a circle parametrised by $z - a = re^{i\theta}$ where $\alpha\pi \leqslant \theta \leqslant \beta\pi$ and $r < r_1$. Prove that $\lim_{r\to 0} \int_{\mathscr{C}} F(z)\, dz = \pi(\beta-\alpha) i \operatorname{Res}_a f(z)$.

Exercise 5.2.6 Evaluate the following real improper integrals using the residue theorem.

(i) $\displaystyle\int_0^\infty \frac{\sin x}{x(x^2+1)}\,dx$

(ii) $\displaystyle\int_0^\infty \frac{\cos(3\pi x/2)}{1-9x^2}\,dx$

(iii) $\displaystyle\int_0^\infty \frac{\sin x}{x(1+x^2)^2}\,dx$

(iv) $\displaystyle\int_0^\infty \frac{\sin^3 x}{x^3}\,dx$

(v) $\displaystyle\int_0^\infty \frac{\sin \pi x}{x(1-x^2)}\,dx$

Exercise 5.2.7 Use the residue theorem to evaluate $\int_{-\infty}^{\infty}(1/(1-x^3))\,dx$. (In this case, the Cauchy principal value exists, although the improper integral does not converge.)

Evaluation of Other Real Definite Integrals

The techniques used to evaluate integrals of types II to IV can be adapted to evaluate various other real convergent improper integrals. In every case the basic idea is to find a suitable contour, consisting of arcs of circles and line segments, that includes an interval of $\mathbb{R}$. Singularities on the real axis are avoided as in type IV. The residue theorem is used to evaluate the associated contour integral and then lemmas similar to 5.5, 5.6, 5.7 and 5.8 are used to reduce the contour integral to the desired real integral.

Example 5.8

Consider $\int_0^\infty(1/(1+x^3))\,dx$. This improper integral cannot be directly related to an integral of type II since the integrand is not even. Hence the techniques given so far fail. Instead, we adapt the method given for integrals of type II and use a sector of the semicircle instead. Hence consider $\int_{\mathscr{C}}(1/(1+z^3))\,dz$ where $\mathscr{C}$ is the contour as shown in Fig. 5.6; angle θ is fixed. (Note that the only real singularity of the integrand is -1.) Let $f(z)=1/(1+z^3)$. The idea is to choose θ suitably so that $\int_L f(z)\,dz$ can be related to the given integral. The integrand f has isolated singular points given by $z^3=-1=e^{-i\pi+2ki\pi}$, $k=0,1,2$. Hence the singular points of f are $z_k=e^{-i\pi/3+2ki\pi/3}$, $k=0,1,2$. Only $z_1=e^{i\pi/3}$ lies in the

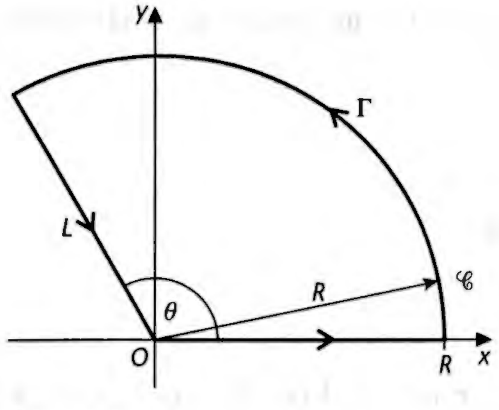


Figure 5.6

upper half-plane, so take $\pi/3 < \theta < \pi$ and $R > 1$ for $\mathscr{C}$. Note that by 5.3, $e^{i\pi/3}$ is a simple pole of f and

$$\mathrm{Res}_{e^{i\pi/3}} f(z) = \left.\frac{1}{3z^2}\right|_{z=e^{i\pi/3}} = \frac{e^{-2i\pi/3}}{3}$$

It follows by 5.1 that $\int_{\mathscr{C}} f(z)\,dz = \frac{2}{3}\pi i e^{-2i\pi/3}$. It also follows by 5.5 that $\lim_{R\to\infty} \int_\Gamma f(z)\,dz = 0$. Hence letting $R \to \infty$ gives

$$\int_{\mathscr{C}} \frac{dz}{1+z^3} = \frac{2\pi i e^{-2i\pi/3}}{3} = \lim_{R\to\infty} \int_0^R \frac{dx}{1+x^3} + \lim_{R\to\infty} \int_R^0 \frac{e^{i\theta}\,d\rho}{1+\rho^3 e^{3i\theta}}$$

since along the line segment L, $z = \rho e^{i\theta}$, with θ fixed and where ρ decreases from R to 0. Hence in order to obtain the desired integral, we choose θ such that $e^{3i\theta} = 1$, i.e., $\theta = 2\pi/3$. Then by the above it follows that

$$\int_0^\infty \frac{dx}{1+x^3} = \frac{2\pi i e^{-2i\pi/3}}{3(1-e^{2i\pi/3})} = \frac{2\pi i(-1-i\sqrt{3})}{3(3-i\sqrt{3})} = \frac{2\sqrt{3}\pi}{9}$$

This technique can be used to evaluate any integral of the form

$$\int_0^\infty \frac{dx}{(x^n+a^n)^m} \qquad (m, n \in \mathbb{N})$$

Integrals Involving Branch Points

Very often we wish to evaluate a real convergent improper integral using the residue theorem where the associated complex integral involves a particular branch of a multifunction. The techniques discussed so far can usually be adapted in such cases as long as any chosen contour does not enclose a branch point and so cross a branch cut. Hence, in certain cases, the previous techniques can be applied directly. For example,

$$\int_0^\infty \frac{\operatorname{Log}(x^2+1)}{x^2+1}\,dx$$

can be evaluated by considering

$$\int_{\mathscr{C}} \frac{\operatorname{Log}(z+i)}{z^2+1}\,dz$$

where $\mathscr{C}$ is the semicircular contour in Fig. 5.2, and using Lemma 5.5 (see Exercise 5.1.8. More generally, if a contour of a previous type crosses a branch cut, it can easily be amended so it does not. Specifically, convergent integrals related to the form $\int_0^\infty f(x)g(x)\,dx$ where f is an algebraic fraction and $g(x) = \operatorname{Log} x$ or x^λ where λ is real but not an integer, can usually be evaluated by using one of the four contours shown in Fig. 5.7. (The choice of contour depends on the behaviour of $f(x)$ and is sometimes a matter of personal preference.) The idea is to let $\theta \to \pi$ or $\theta \to 0$, 2π, etc. on L, $L\pi$, or L_2, as well as $r \to 0$ and $R \to \infty$ to obtain the desired real integral.

The following result is useful when dealing with this type of integral.

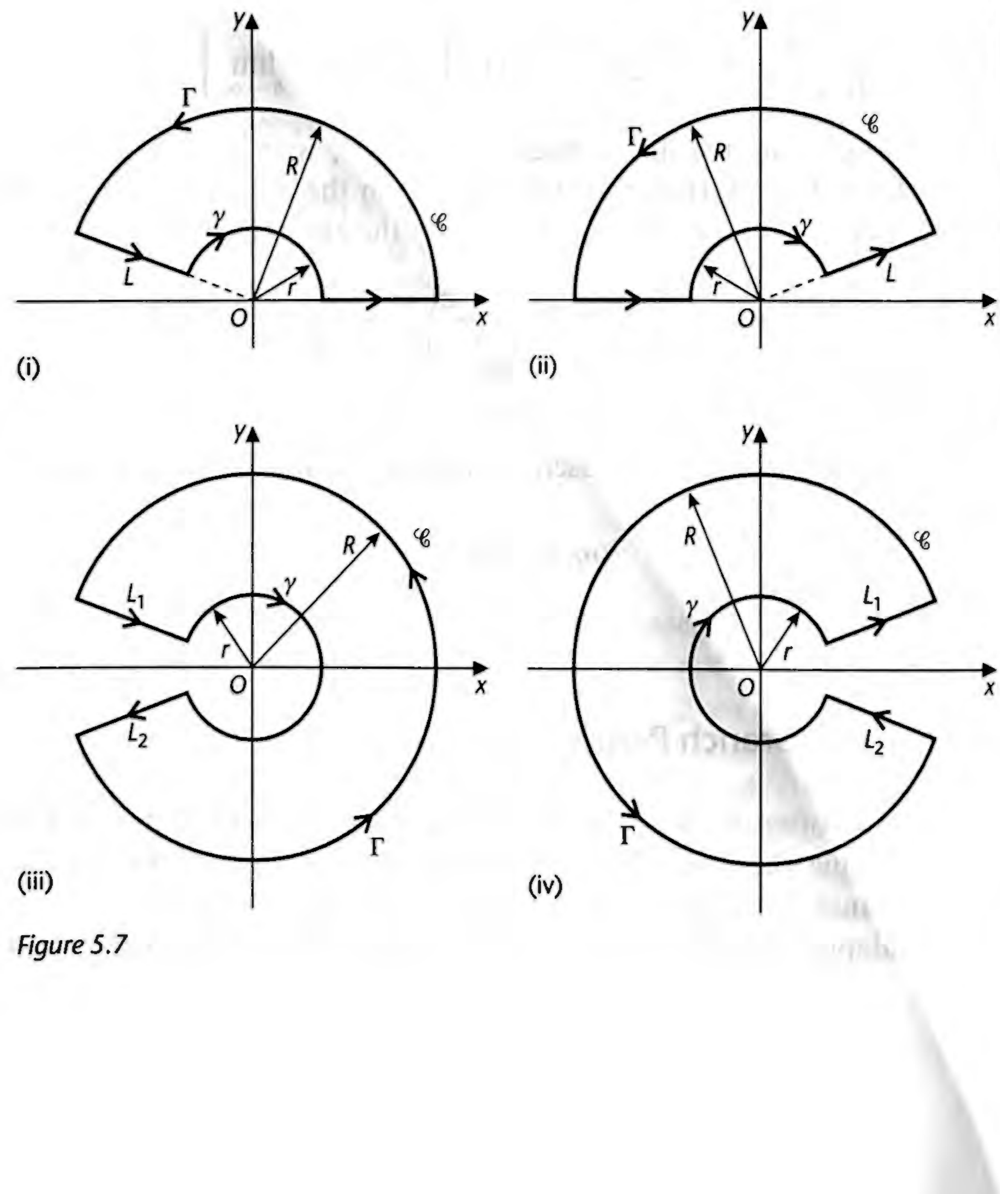


Figure 5.7

Lemma 5.9. Convergence of $\int_\gamma f(z)\,dz$ as $r \to 0$.

Let γ be an arc of a circle of radius r, centred at 0. Let f be continuous on γ with $\lim_{z\to 0} zf(z) = 0$. Then $\lim_{r\to 0}\int_\gamma f(z)\,dz = 0$. □

(Compare with Corollary 5.8. If 0 is a simple pole of f then $\lim_{z\to 0} zf(z) \neq 0$.)

Proof

Let the length of γ be $\alpha\pi r$ for some α, $0 < \alpha \leqslant 2$. By hypothesis, given any real $\varepsilon > 0$, there is a real $\delta > 0$ such that

$$0 < |z| < \delta \quad \Rightarrow \quad |zf(z)| < \varepsilon$$

On γ, $|z| = r$, so that as long as $r < \delta$ then $|f(z)| < \varepsilon/r$. It then follows by 3.2 that $r < \delta \Rightarrow |\int_\gamma f(z)\,dz| < (\alpha\pi r)\varepsilon/r = \alpha\pi\varepsilon$. Hence $\lim_{r\to 0}\int_\gamma f(z)\,dz = 0$, as required. ■

Example 5.9

Consider the convergent integral

$$\int_0^\infty \frac{\operatorname{Log} x}{x^2 + a^2}\,dx \qquad (a \neq 0)$$

In order to evaluate this integral we consider $\int_{\mathscr{C}} f(z)\,dz$ where $f(z) = (\operatorname{Log} z)/(z^2 + a^2)$ and $\mathscr{C}$ is the contour in Fig. 5.7(i), where $R > a$. This contour is chosen since 0 is a branch point of the multifunction log and, by our convention, the non-positive real axis is the branch cut for log, so that Log is singular along this line segment. Note, however, that if $z = \rho e^{i\pi}$ then $z^2 = \rho^2$.

The integrand f has simple poles at $\pm ai$ and only ai lies within $\mathscr{C}$. By 5.2, $\operatorname{Res}_{ai} f(z) = (\operatorname{Log} ai)/2ai = (\operatorname{Log} a + i\pi/2)/2ai$. Then by the residue theorem, $\int_{\mathscr{C}} f(z)\,dz = (\pi/a)(\operatorname{Log} a + \frac{1}{2}i\pi)$. It follows by L'Hôpital's rule that

$$\lim_{z\to\infty} zf(z) = \lim_{z\to 0} \frac{f(1/z)}{z} = \lim_{z\to 0} \frac{-\operatorname{Log} z}{a^2 z + 1/z} = \lim_{z\to 0} \frac{-z}{a^2 z^2 - 1} = 0$$

Hence by 5.5, $\lim_{R\to\infty}\int_\Gamma f(z)\,dz = 0$. Also by L'Hôpital's rule,

$$\lim_{z\to 0} zf(z) = \lim_{z\to 0} \frac{\operatorname{Log} z}{z + a^2/z} = \lim_{z\to 0} \frac{1/z}{1 - a^2/z^2} = \lim_{z\to 0} \frac{z}{z^2 - a^2} = 0$$

Hence by 5.9, $\lim_{r\to 0}\int_\gamma f(z)\,dz = 0$. Note that along the line segment L, $z = \rho e^{i\theta}$ with θ fixed and ρ decreasing from R. Hence letting $R \to \infty$, $r \to 0$ and $\theta \to \pi$ on $\mathscr{C}$ gives

$$\int_{\mathscr{C}} f(z)\,dz = \int_0^\infty \frac{\operatorname{Log} x}{x^2 + a^2}\,dx + \int_\infty^0 \frac{(\operatorname{Log}\rho + i\pi)e^{i\pi}}{\rho^2 e^{2i\pi} + a^2}\,d\rho = \frac{\pi}{a}\left(\operatorname{Log} a + \frac{i\pi}{2}\right)$$

$$\Rightarrow 2\int_0^\infty \frac{\operatorname{Log} x}{x^2 + a^2}\,dx + i\pi\int_0^\infty \frac{dx}{x^2 + a^2} = \frac{\pi}{a}\left(\operatorname{Log} a + \frac{i\pi}{2}\right) \Rightarrow \int_0^\infty \frac{\operatorname{Log} x}{x^2 + a^2}\,dx = \frac{\pi \operatorname{Log} a}{2a}$$

comparing real parts. Note that comparing imaginary parts gives an elementary result!

Example 5.10

To evaluate $\int_0^\infty (x^{\lambda-1}/(1+x))\,dx$, where λ is real but not an integer, which converges for $0<\lambda<1$, we consider $\int_{\mathscr{C}} f(z)\,dz$ where $f(z)=z^{\lambda-1}/(1+z)$ and $\mathscr{C}$ is the contour in Fig. 5.7(iv), where $R>1$. This contour is chosen since f has a branch point at 0 and if $z=\rho e^{i\theta}$, then $z=\rho$ if $\theta=2n\pi$ but $z=-\rho$ for $\theta=(2n+1)\pi$, $n\in\mathbb{Z}$. Note also that f has a simple pole at -1 and this lies on the contour in Fig. 5.7(ii). Hence, in this case, $z^{\lambda-1}$ is chosen as a branch of a multifunction whose branch cut is the non-negative real axis.

There is only one singular point of f inside $\mathscr{C}$ and that is the simple pole at -1. By 5.2, $\operatorname{Res}_{-1} f(z)=e^{(\lambda-1)\log(-1)}=-e^{i\pi\lambda}$ and so by the residue theorem, $\int_{\mathscr{C}} f(z)\,dz=-2\pi i e^{i\pi\lambda}$. Note that $\lim_{z\to\infty} zf(z)=\lim_{z\to\infty} z^\lambda/(1+z)=0$ if and only if $\lambda<1$. Then by 5.5, $\lim_{R\to\infty}\int_\Gamma f(z)\,dz=0$ if and only if $\lambda<1$. Also $\lim_{z\to 0} zf(z)=0$ if and only if $\lambda>0$. Then by 5.9, $\lim_{r\to 0}\int_\gamma f(z)\,dz=0$ if and only if $\lambda>0$. Note that $z=\rho e^{i\theta}$ on the line segments L_1 and L_2, where θ is fixed and ρ increases to R on L_1 and decreases from R on L_2. Then letting $R\to\infty$, $r\to 0$, $\theta\to 0$ on L_1, and $\theta\to 2\pi$ on L_2,

$$\int_0^\infty \frac{x^{\lambda-1}}{1+x}\,dx+\int_\infty^0 \frac{\rho^{\lambda-1}e^{2i\pi(\lambda-1)}e^{2i\pi}}{1+\rho e^{2i\pi}}\,d\rho=-2\pi i e^{\lambda i\pi}$$

$$\Rightarrow\quad \int_0^\infty \frac{x^{\lambda-1}}{1+x}\,dx=\frac{-2\pi i e^{\lambda i\pi}}{1-e^{2i\lambda\pi}}=\frac{2\pi i}{e^{i\lambda\pi}-e^{-i\lambda\pi}}=\frac{\pi}{\sin\lambda\pi}\qquad(0<\lambda<1)$$

Integrals with an Infinite Number of Singular Points

Techniques described so far will fail on real improper integrals whose integrands have an infinite number of non-real singular points, since any semicircle will never enclose all the singular points, no matter how large the radius. Instead a rectangle is chosen. Possibly indented by small semicircles to avoid singular points, the rectangle should include an interval of the real axis and enclose a finite number of singular points of the integrand.

Example 5.11

Consider the integral $\int_{-\infty}^\infty (e^{\lambda x}/(1+e^x))\,dx$, where λ is to be determined for convergence. To evaluate this integral, we consider $\int_{\mathscr{C}} f(z)\,dz$ where $f(z)=e^{\lambda z}/(1+e^z)$ and $\mathscr{C}$ is the rectangle as shown in Fig. 5.8. This rectangle is chosen since $e^{z+2\pi i}=e^z$ and it is chosen to be non-symmetric to avoid the need to take the principal value of the given real integral. The only singular point of f

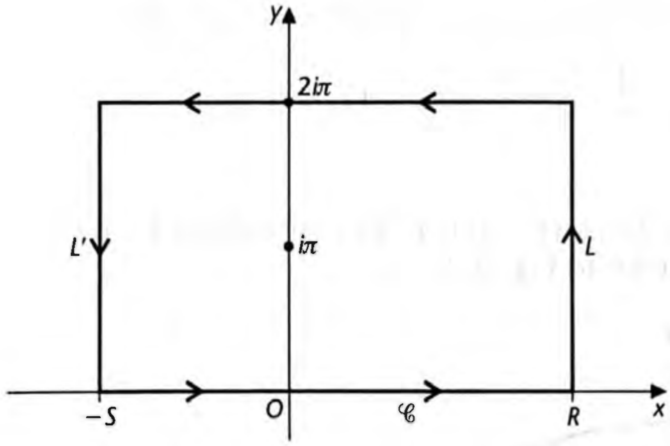


Figure 5.8

inside $\mathscr{C}$ is πi and this is a simple pole with $\text{Res}_{i\pi}f(z) = -e^{\lambda i\pi}$, by Lemma 5.3. Hence by the residue theorem, $\int_{\mathscr{C}} f(z)\,dz = -2\pi i e^{\lambda i\pi}$. Now on the line segment L, $z = R + iy$, $0 \leqslant y \leqslant 2\pi$. Hence,

$$\int_L f(z)\,dz = \int_0^{2\pi} \frac{ie^{\lambda(R+iy)}}{1+e^{R+iy}}\,dy$$

It follows by the triangle inequality that

$$|1 + e^R e^{iy}| \geqslant |1 - |e^R e^{iy}|| = e^R - 1$$

Hence by 3.2, $|\int_L f(z)\,dz| \leqslant 2\pi e^{\lambda R}/(e^R - 1) \to 0$ as $R \to \infty$ if and only if $\lambda < 1$. Similarly, $|\int_{L'} f(z)\,dz| \leqslant 2\pi e^{-\lambda S}/(1 - e^{-S}) \to 0$ as $S \to \infty$ if and only if $\lambda > 0$. Then letting $R \to \infty$ and $S \to \infty$ in $\mathscr{C}$ gives

$$\int_{\mathscr{C}} f(z)\,dz = \int_{-\infty}^{\infty} \frac{e^{\lambda x}}{1+e^x}\,dx + \int_{\infty}^{-\infty} \frac{e^{\lambda(x+2i\pi)}}{1+e^{x+2i\pi}}\,dx = -2\pi i e^{\lambda i\pi}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \frac{e^{\lambda x}}{1+e^x}\,dx = \frac{2\pi i e^{\lambda i\pi}}{e^{2\lambda i\pi} - 1} = \frac{\pi}{\sin \lambda\pi} \qquad (0 < \lambda < 1)$$

Note that letting $y = e^x$ in this result gives $\int_0^{\infty} (y^{\lambda-1}/(1+y))\,dy = \pi/\sin\lambda\pi$ for $0 < \lambda < 1$, which is the result of Example 5.10.

Exercise

5.3.1 Use the technique employed in Example 5.8 to evaluate the following real convergent improper integrals ($a \neq 0$):

(i) $\displaystyle\int_0^{\infty} \frac{dx}{x^4 + a^4}$ (ii) $\displaystyle\int_0^{\infty} \frac{dx}{1 + x^n}$, $n \in \mathbb{N}$, $n > 1$ (iii) $\displaystyle\int_0^{\infty} \frac{dx}{(1 + x^3)^2}$

Exercise

5.3.2 Use the technique of Example 5.8 to show that

$$\frac{1}{a}\int_0^\infty \frac{\sin x}{x+a}\,dx = \int_0^\infty \frac{e^{-x}}{x^2+a^2}\,dx \qquad (a > 0)$$

Exercise

5.3.3 Evaluate $\int_0^\infty (1/(1-x^3))\,dx$ by considering $\int_{\mathscr{C}} (1/(1-z^3))\,dz$ where $\mathscr{C}$ is the contour shown in Fig. 5.9.

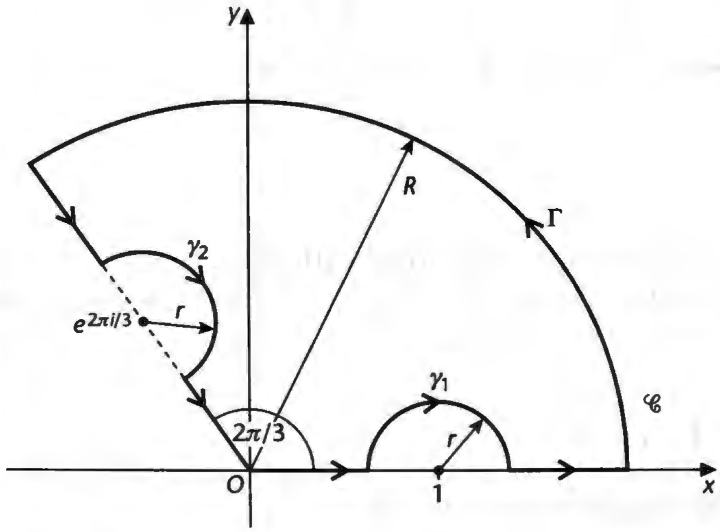


Figure 5.9

Exercise

5.3.4 Evaluate the following real convergent improper integrals, where λ is real but not an integer. State the range of values of λ necessary and sufficient for convergence where appropriate.

(i) $\displaystyle\int_0^\infty \frac{x^\lambda}{x^2+1}\,dx$

(ii) $\displaystyle\int_0^\infty \frac{\operatorname{Log} x}{(1+x^2)^2}\,dx$

(iii) $\displaystyle\int_0^\infty \frac{x^\lambda}{(x^2+a^2)^2}\,dx$

(iv) $\displaystyle\int_0^\infty \frac{(\operatorname{Log} x)^2}{1+x^2}\,dx$

(v) $\displaystyle\int_0^\infty \frac{x^\lambda \operatorname{Log} x}{1+x}\,dx$

(vi) $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(x^3+1)}$

Exercise

5.3.5 Evaluate $\int_{\mathscr{C}} (1/z^\lambda(1-z))\,dz$, $0<\lambda<1$, where $\mathscr{C}$ is the contour as shown in Fig. 5.10. Hence evaluate $\int_0^\infty (1/x^\lambda(1+x))\,dx$ and $\int_0^\infty (1/x^\lambda(1-x))\,dx$.

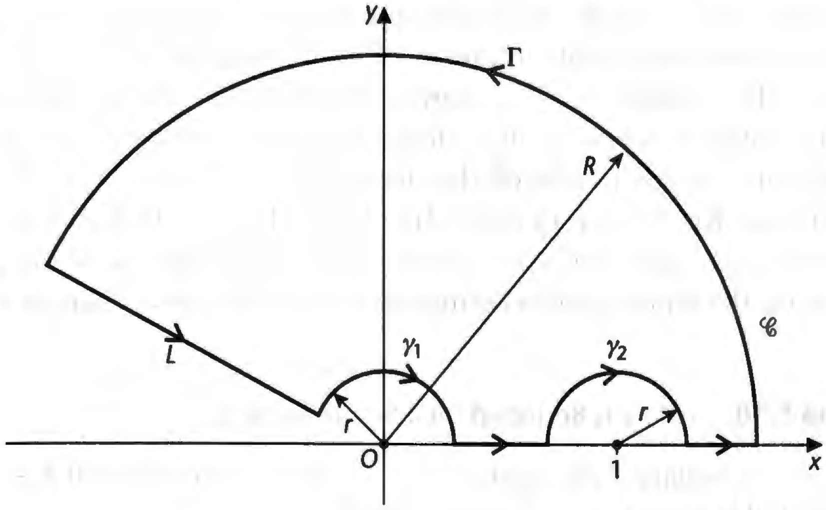


Figure 5.10

Exercise

5.3.6 Use the technique of Example 5.9 to show that

$$\int_0^\infty x^{\lambda-1}\sin x\,dx = \Gamma(\lambda)\sin(\pi\lambda/2) \qquad (0<\lambda<1)$$

where the **gamma function** is defined by $\Gamma(\lambda)=\int_0^\infty e^{-x}x^{\lambda-1}dx$.

Exercise

5.3.7 Evaluate the following convergent integrals.

(i) $\displaystyle\int_{-\infty}^\infty \frac{e^{\lambda x}}{\cosh x}\,dx$ and hence $\displaystyle\int_0^\infty \frac{\cosh \lambda x}{\cosh x}\,dx$, $-1<\lambda<1$

(ii) $\displaystyle\int_0^\infty \frac{\sin \lambda x}{\sinh x}\,dx$, $\lambda>0$

(iii) $\displaystyle\int_0^\infty \frac{x}{\sinh x}\,dx$

Exercise

***5.3.8** (Cauchy). Evaluate $\int_{\mathscr{C}} (z/(\lambda-e^{-iz}))\,dz$, $0<\lambda<1$, where $\mathscr{C}$ is the rectangle with vertices $\pm\pi$ and $\pm\pi+iR$, $R>1$. Hence show that

$$\int_0^\pi \frac{x\sin x}{\lambda^2-2\lambda\cos x+1}\,dx=\frac{\pi}{\lambda}\mathrm{Log}(1+\lambda)$$

Exercise

***5.3.9** Adapt the technique used in Example 5.11 to evaluate

$$\int_0^\infty \frac{\sin x}{e^x+1}\,dx$$

Summation of Series Using the Residue Theorem

The residue theorem can also be used to sum certain convergent infinite series of real numbers. Essentially the technique consists of evaluating an integral, whose integrand has an infinite number of real singular points, using the residue theorem. To evaluate $\sum_{n=-\infty}^{\infty} f(n)$ where f is an algebraic fraction, we consider $\int_{\mathscr{C}_k} \pi f(z) \cot \pi z \, dz$ where $\mathscr{C}_k$ is a closed contour for each $k \in \mathbb{N}$ enclosing a finite number of singular points of the integrand, and let $k \to \infty$. This integrand is chosen since $\text{Res}_n(\pi f(z) \cot \pi z) = f(n)$ for each $n \in \mathbb{Z}$ by Lemma 5.3, as long as f is non-singular and non-zero when $z = n$. It is easier to show convergence as $k \to \infty$ on the set of squares defined in the next Lemma, than on a set of circles.

Lemma 5.10. $\cot \pi z$ **is Bounded on a Set of Squares**

Let $\mathscr{C}_k$ be a square with vertices $(k + 1/2)(\pm 1 \pm i)$ for each $k \in \mathbb{N}$ as shown in Fig. 5.11. Then $|\cot \pi z| < 2$ on $\mathscr{C}_k$ for all k. □

Proof

Let $\alpha = k + 1/2$. On the line segments L_1 and L_3, $z = \pm\alpha + iy$ respectively, where $|y| \leqslant \alpha$. Note that $\cos \alpha\pi = 0$ and $\sin \alpha\pi = (-1)^k$. Hence on L_1 and L_3

$$|\cot \pi z| = \left| \frac{\cos \alpha\pi \cosh \pi y \mp i \sin \alpha\pi \sinh \pi y}{\pm \sin \alpha\pi \cosh \pi y + i \cos \alpha\pi \sinh \pi y} \right| = |\tanh \pi y| < 1$$

for all y, independent of k. On the line segments L_2 and L_4 $z = x \pm i\alpha$ respectively, where $|x| \leqslant \alpha$. Hence on these lines

$$|\cot \pi z| = \left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right| = \left| \frac{e^{i\pi x} e^{\mp\alpha\pi} + e^{-i\pi x} e^{\pm\alpha\pi}}{e^{i\pi x} e^{\mp\alpha\pi} - e^{-i\pi x} e^{\pm\alpha\pi}} \right|$$

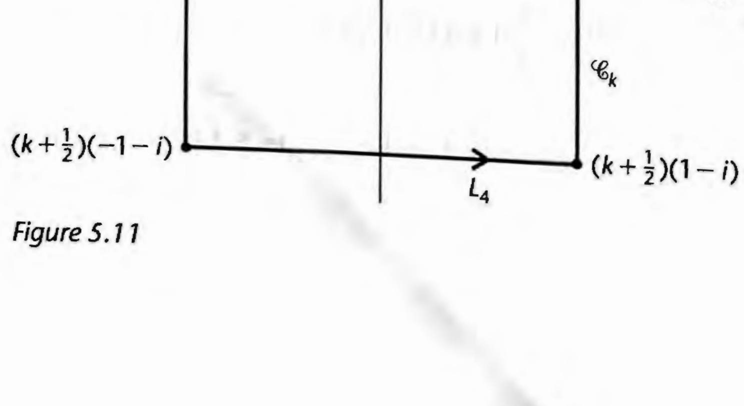


Figure 5.11

Also, it follows by the triangle inequality that

$$|e^{i\pi x}e^{\mp\alpha\pi} + e^{-i\pi x}e^{\pm\alpha\pi}| \leqslant |e^{\mp\alpha\pi}| + |e^{\pm\alpha\pi}| = e^{\alpha\pi} + e^{-\alpha\pi}$$

and

$$|e^{i\pi x}e^{\mp\alpha\pi} - e^{-i\pi x}e^{\pm\alpha\pi}| \geqslant |e^{\mp\alpha\pi} - e^{\pm\alpha\pi}| = e^{\alpha\pi} - e^{-\alpha\pi}$$

It then follows that

$$|\coth \pi z| \leqslant \frac{e^{\alpha\pi} + e^{-\alpha\pi}}{e^{\alpha\pi} - e^{-\alpha\pi}} = \coth \alpha\pi \leqslant \coth \frac{3}{2}\pi < 2$$

on these lines, independent of k, since $\coth x$ is monotonic decreasing for $x >$ ■

Theorem 5.11. Convergence of $\int_{\mathscr{C}_k} f(z) \cot \pi z \, dz$ as $k \to \infty$

Let $\mathscr{C}_k$ be a square with vertices $(k + 1/2)(\pm 1 \pm i)$ for each $k \in \mathbb{N}$. Let f be meromorphic, with f continuous on $\mathscr{C}_k$ for $k > k_1$ say, and $\lim_{z\to\infty} zf(z) = 0$. Then $\lim_{k\to\infty} \int_{\mathscr{C}_k} f(z) \cot \pi z \, dz = 0$. □

Proof

On each square $\mathscr{C}_k$, $|z| \geqslant k + 1/2 > k$ and by hypothesis, given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\frac{1}{|z|} < \delta \quad \Rightarrow \quad |zf(z)| < \varepsilon$$

Then choosing $k > 1/\delta$ shows that given any $\varepsilon > 0$ there exists k such that $|f(z)| < \varepsilon/(k + 1/2)$ on $\mathscr{C}_k$. The length of $\mathscr{C}_k$ is $8(k + 1/2)$, so that by 3.2 and 5.10,

$$\frac{1}{k} < \delta \quad \Rightarrow \quad \left| \int_{\mathscr{C}_k} f(z) \cot \pi z \, dz \right| \leqslant \frac{8(k + 1/2)2\varepsilon}{k + 1/2} = 16\varepsilon$$

Hence $\lim_{k\to\infty} \int_{\mathscr{C}_k} f(z) \cot \pi z \, dz = 0$, as required. ■

It is due to this convergence result that the residue theorem can be used to evaluate $\sum_{n=-\infty}^{\infty} f(n)$ for certain functions f. In particular, we have the following result.

Theorem 5.12. The Sum of an Infinite Series

Let f be an algebraic fraction such that $\lim_{z\to\infty} zf(z) = 0$. Let f be non-zero at $n \in \mathbb{Z}$ and have non-integer singular points $z_1, \ldots, z_m$. Then

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum_{n=1}^{m} \operatorname{Res}_{z_n}(\pi f(z) \cot \pi z)$$ □

Proof

Consider $\int_{\mathscr{C}_k} \pi f(z) \cot \pi z \, dz$ where $\mathscr{C}_k$ is a member of the set of squares in 5.11, with k chosen large enough so that all the singular points of f lie within $\mathscr{C}_k$. By hypothesis and Lemma 5.3, $F(z) = \pi f(z) \cot \pi z$ has simple poles at $n \in \mathbb{Z}$ and

$$\operatorname{Res}_n F(z) = \left. \frac{\pi f(z) \cos \pi z}{\pi \cos \pi z} \right|_{z=n} = f(n)$$

Also by hypothesis, F has non-integer singular points $z_1, z_2, \ldots, z_m$. Hence by 5.1,

$$\int_{\mathscr{C}_k} F(z) \, dz = 2\pi i \left(\sum_{n=-k}^{k} f(n) + \sum_{n=1}^{m} \operatorname{Res}_{z_n} F(z) \right)$$

The result then follows by letting $k \to \infty$ and using 5.11. ∎

Notes

(i) If $\lim_{z\to\infty} z f(z) \neq 0$ then the infinite series will not converge as in the case of the harmonic series $\sum_{n=1}^{\infty} 1/n$.

(ii) The above result is easily adapted in the case where at least one of the singular points of f is an integer, as demonstrated in the second of the following examples.

Example 5.12

(i) Let $f(z) = 1/(2z+1)^2$. Then f has a pole of order two at $-1/2$ and $\lim_{z\to\infty} z f(z) = 0$. Also by Lemma 5.2,

$$\operatorname{Res}_{-1/2}(\pi f(z) \cot \pi z) = \left. \frac{d}{dz} \left(\frac{\pi \cot \pi z}{4} \right) \right|_{z=-1/2} = -\frac{\pi^2 \csc^2(-\pi/2)}{4} = -\frac{\pi^2}{4}$$

Hence by Theorem 5.12,

$$\sum_{n=-\infty}^{\infty} f(n) = \frac{\pi^2}{4} \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{4}$$

Letting $n = m + 1$ gives

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \quad \text{hence} \quad \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

(ii) Let $f(z) = 1/z^2$ so that f has a pole of order 2 at 0 and $\lim_{z\to\infty} z f(z) = 0$. Theorem 5.12 is not directly applicable in this case but the technique is easily adapted. F defined by $F(z) = \pi f(z) \cot \pi z$ has a pole of order 3 at 0 and simple poles at $n \in \mathbb{Z}$, $n \neq 0$. As in the proof of 5.12, $\mathrm{Res}_n F(z) = f(n)$ for $n \neq 0$. Also, in some neighbourhood of 0,

$$F(z) = \frac{\pi}{z^2}\left(\frac{1}{\pi z} - \frac{\pi z}{3} - \frac{\pi^3 z^3}{45} + \dots\right)$$

so that $\mathrm{Res}_0 F(z) = -\pi^2/3$. Now let $\mathscr{C}_k$ be a square with vertices $(k+1/2)(\pm 1 \pm i)$. Then by 5.1,

$$\int_{\mathscr{C}_k} F(z)\,dz = 2\pi i\left(\sum_{\substack{n=-k \\ n\neq 0}}^{k} f(n) - \frac{\pi^2}{3}\right)$$

Letting $k \to \infty$ and using 5.11 gives

$$\sum_{n=-\infty}^{-1} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3} \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

It is also possible to sum certain convergent alternating sign series using the residue theorem. To sum $\sum_{n=-\infty}^{\infty} (-1)^n f(n)$ where f is an algebraic fraction, we consider $\int_{\mathscr{C}_k} \pi f(z) \csc \pi z\,dz$, where $\mathscr{C}_k$ is a square as in 5.10, since if f is non-singular and non-zero at $n \in \mathbb{Z}$, $\mathrm{Res}_n(\pi f(z) \csc \pi z) = (-1)^n f(n)$. The proofs of the following results are very similar to the proofs of 5.10, 5.11 and 5.12; they are left as an exercise.

Theorem 5.13. Convergence of $\int_{\mathscr{C}_k} f(z) \csc \pi z\,dz$ as $k \to \infty$

Let $\mathscr{C}_k$ be a square with vertices $(k+1/2)(\pm 1 \pm i)$ for each $k \in \mathbb{N}$. Let f be meromorphic, with f continuous on $\mathscr{C}_k$ for $k > k_1$ say, and $\lim_{z\to\infty} zf(z) = 0$. Then $\lim_{k\to\infty} \int_{\mathscr{C}_k} f(z) \csc \pi z\,dz = 0$. □

Theorem 5.14. The Sum of an Alternating Sign Series

Let f be an algebraic fraction such that $\lim_{z\to\infty} zf(z) = 0$. Let f be non-zero at $n \in \mathbb{Z}$ with non-integer singular points $z_1, \dots, z_m$. Then

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\sum_{n=1}^{m} \mathrm{Res}_{z_n}(\pi f(z) \csc \pi z) \qquad \square$$

The technique used to prove 5.12 and 5.14 can be adapted to sum other series, as demonstrated in the following example.

Example 5.13

Let $f(z) = (\operatorname{csch} \pi z)/z^3$ so that f is meromorphic, $\lim_{z\to\infty} zf(z) = 0$, f has a pole of order 4 at 0 and simple poles at in, $n \in \mathbb{Z}$, $n \neq 0$. Let $F(z) = \pi f(z) \csc \pi z$.

Then F has simple poles at n and in, $n \in \mathbb{Z}$, $n \neq 0$ and a pole of order 5 at 0. It follows by Lemma 5.3 that

$$\operatorname{Res}_n F(z) = (-1)^n f(n) \qquad \operatorname{Res}_{in} F(z) = \left.\frac{\pi}{\pi z^3 \sin \pi z \cosh \pi z}\right|_{z=in} = (-1)^n f(n)$$

Also, in some neighbourhood of 0,

$$F(z) = \frac{\pi}{z^3}\left(\frac{1}{\pi z} + \frac{\pi z}{6} + \frac{7\pi^3 z^3}{360} + \dots\right)\left(\frac{1}{\pi z} - \frac{\pi z}{6} + \frac{7\pi^3 z^3}{360} - \dots\right)$$

$$\text{Hence} \quad \operatorname{Res}_0 F(z) = \frac{7\pi^3}{360} - \frac{\pi^3}{36} + \frac{7\pi^3}{360} = \frac{\pi^3}{90}$$

It then follows by the residue theorem that

$$\int_{\mathscr{C}_k} F(z)\,dz = 2\pi i \left(\frac{\pi^3}{90} + 2\sum_{\substack{n=-k \\ n\neq 0}}^{k} (-1)^n f(n)\right)$$

where $\mathscr{C}_k$ is a square with vertices $(k + 1/2)(\pm 1 \pm i)$. Hence by 5.13 it follows that

$$4\sum_{n=1}^{\infty}(-1)^n f(n) = \frac{-\pi^3}{90} \quad \Rightarrow \quad \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\operatorname{csch} n\pi}{n^3} = \frac{\pi^3}{360}$$

Partial Fraction Expansions

Besides summing certain series of real numbers, the residue theorem can also be used to expand certain functions as a series of partial fractions involving their poles. Once again, this idea is originally due to Cauchy. The following result is the simplest of its type but can be generalised to include non-simple poles.

Theorem 5.15. Partial Fraction Expansion of a Meromorphic Function with Simple Poles

Let f be a meromorphic function with only simple poles $z_1, z_2, z_3, \dots$ with $|z_1| \leqslant |z_2| \leqslant |z_3| \leqslant \dots$. Let $\operatorname{Res}_{z_n} f(z) = b_n$ for each n. Let $(\mathscr{C}_k)$ be a sequence of nested squares with vertices $a_k(\pm 1 \pm i)$ such that $a_k \to \infty$ as $k \to \infty$, which do not pass through any pole of f. Suppose that f is analytic at 0 and that there exists $M \in \mathbb{R}^+$, independent of k, such that $|f(z)| \leqslant M$ on $\mathscr{C}_k$ for each k. Then except at the poles of f,

$$f(z) = f(0) + \sum_n b_n\left(\frac{1}{z - z_n} + \frac{1}{z_n}\right) \qquad \square$$

Proof

Step 1

Suppose that α is not a pole of f and define g by $g(z) = f(z)/(z-\alpha)$. Then g has simple poles at z_n for each n, and at α, with $\mathrm{Res}_{z_n} g(z) = b_n/(z_n - \alpha)$ and $\mathrm{Res}_\alpha g(z) = f(\alpha)$ by 5.2. Then

$$\frac{1}{2\pi i}\int_{\mathscr{C}_k} g(z)\,dz = f(\alpha) + \sum_n \frac{b_n}{z_n - \alpha} \tag{5.9}$$

by 5.1, where $\sum_n$ is taken over all poles of f inside $\mathscr{C}_k$ and k is chosen large enough so that α lies inside $\mathscr{C}_k$. Letting $\alpha = 0$, which lies inside all the $\mathscr{C}_k$, in (5.9) gives

$$\frac{1}{2\pi i}\int_{\mathscr{C}_k} \frac{f(z)}{z}\,dz = f(0) + \sum_n \frac{b_n}{z_n} \tag{5.10}$$

Subtracting (5.10) from (5.9) gives

$$\frac{\alpha}{2\pi i}\int_{\mathscr{C}_k} \frac{f(z)}{z(z-\alpha)}\,dz = f(\alpha) - f(0) + \sum_n b_n\left(\frac{1}{z_n - \alpha} - \frac{1}{z_n}\right) \tag{5.11}$$

Step 2

Note that the length of each square $\mathscr{C}_k$ is $8a_k$ and that $|z| \geqslant a_k$ on each $\mathscr{C}_k$, so that by the triangle inequality, $|z-\alpha| \geqslant a_k - |\alpha|$ on each $\mathscr{C}_k$. Then by hypothesis and Lemma 3.2,

$$\left|\int_{\mathscr{C}_k} \frac{f(z)}{z(z-\alpha)}\,dz\right| \leqslant \frac{8Ma_k}{a_k(a_k - |\alpha|)} = \frac{8M}{a_k - |\alpha|}$$

Then by hypothesis, $\int_{\mathscr{C}_k}(f(z)/z(z-\alpha))\,dz$ and the result follows from (5.11). ■

Example 5.14

Let $f(z) = \tan z$. Then f has simple poles at $(n + 1/2)\pi$, $n \in \mathbb{Z}$ and f is analytic at 0. Note that by Lemma 5.3, $\mathrm{Res}_{(n+1/2)\pi} f(z) = -1$ for all n. Also, using the technique of the proof of 5.10, it is easily seen that $|f(z)| < 2$ on any square $\mathscr{C}_k$ with vertices $k(\pm 1 \pm i)$, $k \in \mathbb{N}$ (see Exercise 5.4.5). Clearly, f is analytic on each $\mathscr{C}_k$. Then by 5.15,

$$\tan z = -\sum_{n=-\infty}^{\infty}\left(\frac{1}{z-(n+1/2)\pi} + \frac{1}{(n+1/2)\pi}\right)$$

$$\Rightarrow \quad \tan z = -\sum_{n=0}^{\infty}\left(\frac{1}{z-(n+1/2)\pi} + \frac{1}{(n+1/2)\pi}\right)$$

$$-\sum_{n=1}^{\infty}\left(\frac{1}{z+(n-1/2)\pi} + \frac{1}{(-n+1/2)\pi}\right)$$

$$\Rightarrow \quad \tan z = -\sum_{n=0}^{\infty}\left(\frac{1}{z-(n+1/2)\pi}+\frac{1}{(n+1/2)\pi}\right)$$
$$-\sum_{m=0}^{\infty}\left(\frac{1}{z+(m+1/2)\pi}-\frac{1}{(m+1/2)\pi}\right)$$

letting $n = m + 1$ in the second series. Then simplifying gives

$$\tan z = \sum_{n=0}^{\infty}\frac{2z}{(n+1/2)^2\pi^2 - z^2}$$

except at the simple poles of tan. Letting $z \to 0$ in this result,

$$\lim_{z\to 0}\frac{\tan z}{z} = 1 = \sum_{n=0}^{\infty}\frac{2}{(n+1/2)^2\pi^2} \quad \Rightarrow \quad \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

which is the result of Example 5.12(i). Letting $z = \pi/4$ gives

$$\sum_{n=0}^{\infty}\frac{1}{4(2n+1)^2 - 1} = \frac{\pi}{8}$$

Note

Suppose that the conditions of 5.15 are satisfied, except that 0 is also a simple pole of f. In this case, by definition, $\lim_{z\to 0} zf(z) = k$ say, so define g by $g(z) = f(z) - k/z$ with $g(0) = \lim_{z\to 0}(f(z) - k/z)$. Then g is analytic at 0 and is bounded on the same sequence of nested squares as f, so that 5.15 can be applied to g (see Exercises 5.4).

Exercise **5.4.1** Find the sum for each of the following convergent series:

(i) $\displaystyle\sum_{n=-\infty}^{\infty}\frac{1}{(n-a)^2},\ a \notin \mathbb{Z}$

(ii) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n^2+a^2},\ a > 0$

(iii) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4+a^4},\ a > 0$

(iv) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4}$

Exercise **5.4.2** Prove that $|\csc \pi z| \leqslant 1$ on the square $\mathscr{C}_k$ with vertices $(k+1/2)(\pm 1 \pm i)$ for each k. Use this result to prove Theorem 5.13 and hence prove Theorem 5.14.

Exercise **5.4.3** Find the sum of each of the following convergent series:

(i) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+a^2}, \ a > 0$

(ii) $\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(2n+1)^3}$

(iii) $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n^2+1)^2}$

Exercise **5.4.4** Use the technique of Example 5.13, with

$$f(z) = \frac{\operatorname{sech}(z+1/2)\pi}{2z+1}$$

to sum the infinite series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\cosh(n+1/2)\pi}$$

Exercise **5.4.5** Prove that $|\tan \pi z| < 2$ on the square $\mathscr{C}_k$ with vertices $k(\pm 1 \pm i)$ for each $k \in \mathbb{N}$. Now let f be an algebraic fraction with $\lim_{z\to\infty} zf(z) = 0$. Prove that $\lim_{k\to\infty} \int_{\mathscr{C}_k} f(z) \tan \pi z \, dz = 0$. Hence use the residue theorem to prove that if $f(n+1/2) \neq 0$ and f has singular points $z_1, z_2, \ldots, z_m \neq n + 1/2$, for any $n \in \mathbb{Z}$ then

$$\sum_{n=-\infty}^{\infty} f(n+1/2) = \sum_{n=1}^{m} \operatorname{Res}_{z_n}(\pi f(z) \tan \pi z)$$

Exercise **5.4.6** Use Theorem 5.15 to prove the following results:

(i) $\cot z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2\pi^2}$, except at the poles of $\cot z$

(ii) $\pi \operatorname{csch} \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(-1)^n 2z}{z^2+n^2}$, except at the poles of $\operatorname{csch} \pi z$

(Compare with Exercise 5.4.3(i).

Exercise **5.4.7** Let $f(z) = (\sin az)/(\sin \pi z)$ where $0 < a < \pi$ and let $\mathscr{C}_k$ be a square with vertices $(k+1/2)(\pm 1 \pm i)$ for each $k \in \mathbb{N}$. Show that $|f(z)| \leqslant M$ on each $\mathscr{C}_k$ for some $M \in \mathbb{R}^+$, independent of k. Hence apply Theorem 5.15 to prove that, except at the poles of f,

$$\frac{\sin az}{\sin \pi z} = \frac{a}{\pi} + \frac{2z^2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sin an}{n(z^2 - n^2)}$$

Exercise

5.4.8 Let $f(z) = g(z) + \sum_{n=1}^{m} a_n/z^n$ where g is analytic in the region $|z| < R$ for some R, so that f has a pole of order m at 0. Prove that if $\mathscr{C}$ is any simple closed contour inside $|z| < R$ and α is a point inside $\mathscr{C}$, then

$$\frac{1}{2\pi i}\int_{\mathscr{C}} \frac{f(z)}{z-\alpha}\, dz = f(\alpha) - \sum_{n=1}^{m} \frac{a_n}{\alpha^n}$$