(7) Complex Variables

Contents

Complex Values

A complex number is of the form $z = x + iy$ where x and y are real numbers:

$$\Re(z) = \mathrm{Re}\,(z) = x$$
$$\Im(z) = \mathrm{Im}\,(z) = y$$

Complex numbers are equal if and only if their components are equal:

$$(z_1 = z_2) \iff \Re z_1 = \Re z_2 \quad \wedge \quad \Im z_1 = \Im z_2$$

Arithmetic Operations

It's worth memorising these patterns to make everything quicker:

1. Addition

$$(x + iy) \pm (\alpha + i\beta) = (x + \alpha) + i\,(y + \beta)$$

2. Multiplication

   $(a+ib)\cdot(c+id) = (ac-bd) + i\cdot(bc+ad)$

3. Division

   $\frac{(a+ib)}{(c+id)} = \frac{(ac+bd)}{c^2+d^2} + i\frac{(bc-ad)}{c^2+d^2}$

4. Rotation

   $i^{4k} = 1 \quad ; \quad i^{4k+1} = -1 \quad ; \quad i^{4k+2} \quad ; \quad i^{4k+3}$

Modulus

The modulus is the distance from the origin, to a point on the complex plane:

$$|z| = \sqrt{x^+ y^2} \tag{1}$$

This definition of modulus is consistent with the definition of modulus/absolute-value used in real analysis:

$$|a| = \sqrt{a^2} = \begin{cases} a, a \geq 0 \\ -a, a < 0 \end{cases} \tag{2}$$

be careful though because even though this is consistent with the real definition the properties of real modulus values are not necessarily true for complex modulus values, e.g. $|z|^2 \neq |z^2|$

Properties of the Modulus of a Complex Number

- Distributive over Multiplication

  $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

- Modulus of conjugate is equivalent

  $|\bar{z}| = |z|$

- Sum of the the value and its conjugate corresponds to the real component

  $z + \bar{z} = \Re(z) \leq 2|z|$

- Difference of the value and its conjugate corresponds to the imaginary component

  $z - \bar{z} = 2\Im\{z\} \leq 2|z|$

- The triangle inequality holds for complex values as well

- $\Big|z_1 + z_2\Big| \leq |z_1| + |z_2|$

Conjugates

if $z = x + iy$ the conjugate $\bar{z} = x - iy$, on the complex plane it corresponds to a reflection about the x-axis:

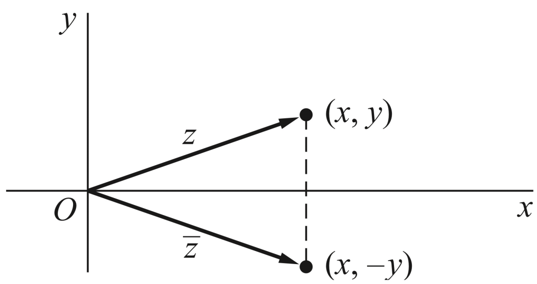

Figure .1:

This is useful because conjugates have many useful properties:
Memorise these

- $\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$
- $\overline{z_1 \cdot z_2} = \bar{z_1} \cdot \bar{z_2}$
- $\bar{\bar{z}} = z$
- $z \cdot \bar{z} = |z|^2$

This is useful for division, so for example:

$$\frac{(6+2i)}{(3+4i)} = \frac{(6+2i)\,(3-4i)}{\mid 3+4i \mid^2} = \frac{(26-6i)}{5^2}$$

Also it is useful to memorise:

$$i^{4k} \quad ; \quad i^{4k+1} = i \quad ; \quad i^{4k+1} \quad ; \quad i^{4k+2} \quad ; \quad i^{4k+3}$$

Geometry the Complex Plane

Circle In real analysis a circle is provided by the equation

$$x^2 + y^2 = r^2$$
$$\sqrt{x^2 + y^2} = r$$

Hence on the complex plane we can use represent a circle centred at $z_0 = \alpha + i \cdot \beta$ with a radius r by:

$$|z - z_0| = |z - \alpha - \beta i| = \sqrt{(x - \alpha)^2 + (x - \beta)^2} = r$$

Parabola

$$
\begin{aligned}
|z+3i| &= \Im(z)+4 \\
|x+(y+3)\,i| &= \Im(z)+4 \\
\sqrt{x^2+(y+3)^2} &= \Im(y)+4 \\
x^2+y^2+y^2+6y+9 &= 9y^2+8y+16 \\
y^2-y^2+6y &= -x^2-9+16 \\
6y &= -x^2+7 \\
y &= -\frac{1}{6}\cdot x^2+\frac{7}{6}
\end{aligned}
$$

Ellipse If p_1 and p_2 are points in the complex plane:

$$|p_1|+|p_2|=r$$

is an ellipse, because an elipse is a curve that is equal distance from two other points, there are helpful gifs on *Wikipedia* et cetera.

Complex Variables

Neighbourhoods

An open δ-neighbourhood (also known as an open δ disc) is a set of points inside a circle on the complex plane, so we could take some neighbourhood $N(\alpha,\delta)$ where:

- N is the name of the neighbourhood
- α is the centre of the neighbourhood or circle
- δ is the radius of the disc

This is basically interval notation on the complex plane.

Closed Neighbourhoods use $\leq$:

$$N_{Closed}(\alpha,\delta)=\{z:|z-\alpha|\leq\delta\}$$

Open Neighbourhoods use $<$:

$$N_{Open}(\alpha,\delta)=\{z:|z-\alpha|<\delta\}$$

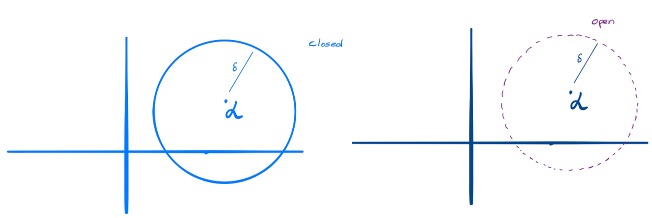

Figure .2: Example of Open and Closed Complex Neighbourhoods

Delteted Neighbourhoods There is also the concept of deleted Neighborhoods (also known as a punctured disk):

$$N \subset (\alpha, \delta) = \{z : 0 < |z - \alpha| < \delta\}$$
$$N \subset (\alpha, \delta) = \{z : 0 < |z - \alpha| \leq \delta\}$$

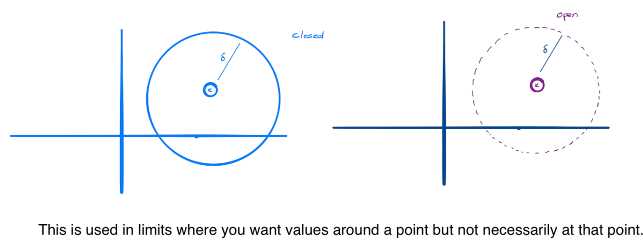

Figure .3: Diagram of a Deleted Neighbourhood on the Complex Plane

Interior and Boundary Point

Take some subset of the complex numbers $S \subset \mathbb{C}$ and some complex number $\alpha \in \mathbb{C}$.

The neighbourhood $N_o(\alpha, \delta)$ contains points inside and outside S no matter how small δ is made.

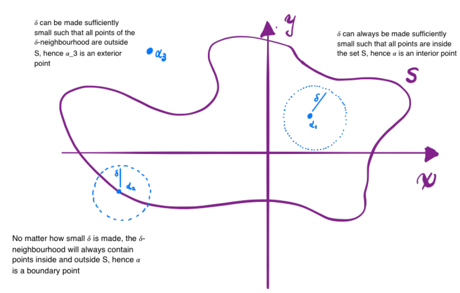

Figure .4: Example of Interior and Boundary Points

Limit Points

α is a limit point of S if for every δ value, $N_o(\alpha, \delta)$ contains at least ojne point in S.

So in the diagram above α_1 and α_2 are limit points, all interior and boundary points are limit points, generally however a limit point is a point that is arbitrarliy close but not contained by a neighbourhood, e.g.

> if the set S above was open, then $\alpha_2 \notin S$ and so α_2 is not a boundary point, however α_2 would still be a limit point of S.

Closed and Open Subsets

A subset $S \subseteq \mathbb{C}$ is:

- **Open** if every point of the set is an interior point

  an open neighbourhood has no boundary and is hence an open set

- **Closed** if every point of S is contained by S.

  a closed neighbourhood contains all of it's limit points (on the boundary), so it considered a closed set

The textbook mentions that it can be deduced from the definition, that S is closed if the following set is open:

$$S \setminus \mathbb{C} = \{z \in \mathbb{C} : z \notin S\}$$

Special Cases $\emptyset$ and $\mathbb{C}$ are simultaneously open and closed

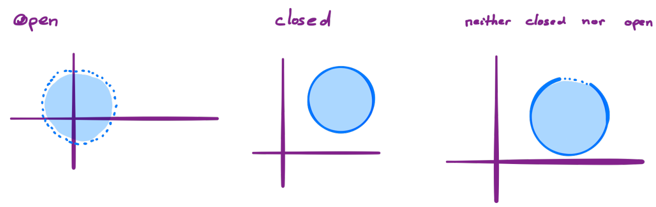

Figure .5: Examples of an open set, closed set and a set that is neither open nor closed

Closure of a Set The closure of a set (denoted $\overline{S}$ is the union of the set with its limit points, it has the effect of making any set closed, e.g.

- if S is open $\overline{S}$ is closed.
- if P is closed $\overline{P}$ is closed.
- if Q is neither open nor closed $\overline{Q}$ is closed.

Connected Sets

A connected set is a subset of the complex + ane where all points are touching, e.g.:

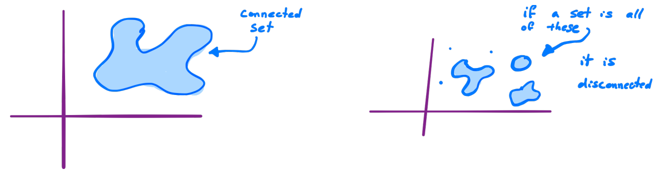

Figure .6:

Regions A *Region* is a connected set which is open and non-empty.

Simply Connected Sets A simply connected set is basically a connected set with no holes in it.

> Formally a Simply connected set is a connected Set set with a connected compliment

So in order to visualise this, consider the following connected sets:

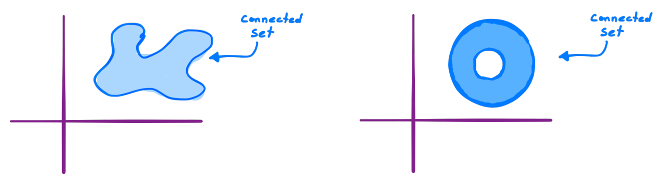

Figure .7: Two Connected Sets

Now consider the Compliments of these sets:

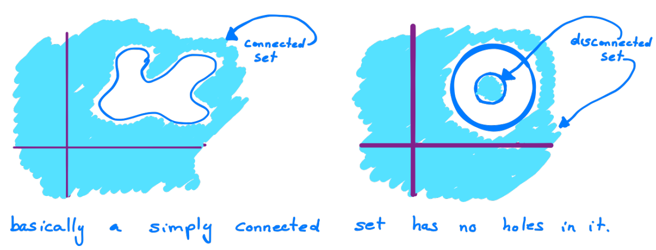

Figure .8: Corresponding Compliments

Observe that the right set has a disconnected compliment, hence only the left set is said to be 'simply connected'.

Bounded Sets

A set $S \in \mathbb{C}$ is bounded if every element of that set can be contained by a circle.

Definitions S is said to be bounded if and only if:

$$\forall z \in S, \exists M : |z| \leq M$$

- If a set is not bounded it is unbounded.
- Any δ-neighbourhood (i.e. a δ-disc) is bounded.

Compact Set A set which is bounded and closed is said to be compact.

So any δ-disc is compact

Polar Form

A Complex number z is of the form:

$$z = x + iy$$

if we took the angle from the positive-side of the x-axis we would have:

$$x = |z| \cdot \cos\theta \qquad y = |z| \cdot \sin\theta$$

This is more or less by definition of the trigonometric functions, now if we let $|z| = r$:

$$\begin{aligned} z = x + i \cdot y = x + i \cdot y &= r \cdot (\cos(\theta) + i \cdot \sin(\theta)) \\ &= r \cdot \mathrm{cis}(\theta) \\ &= r \cdot e^{i \cdot \theta} && \text{(by \textit{Euler's Formula}} \\ &= r \cdot \underline{/\theta} && \text{This is engineering notation} \end{aligned}$$

This notation is useful for simplifying the arithmetic of complex numbers This also means $\tan\theta = \frac{y}{x}$

Terms

- The angle θ is called the *argument*
- the radius r is called the *modulus*

In the context of Complex analysis we will define the **Principal Argument** as $\theta \in (-\pi, \pi]$, this range of values is not a universal convention, but it is a convenient:wq kjdefinition for use later.

Multiplication in Polar Form

If:

$$z_1 = r_1 \cdot \mathrm{cis}(\theta_1)$$
$$z_2 = r_2 \cdot \mathrm{cis}(\theta_2)$$

Then:

$$z_1 \cdot z_2 = r_1 \cdot r_2 \times [\cos\theta_1 + i\sin\theta_1] \times [\cos\theta_2 + i\sin\theta_2]$$

by the *double angle formula*

$$\begin{aligned}
&= r_1 r_2 \cdot \cos\theta + \theta_2 + i\sin\theta_1 + \theta_2 \\
&= r_1 r_2 \cdot \operatorname{cis}(\theta_1 + \theta_2)
\end{aligned}$$

This is best visualised by imagining addition and multiplication as geometric transformations, I highly recommend watching the video '*Euler's formula with introductory group theory*' by *3Blue1Brown* on *Youtube*.

Polar Multiplication

$$z_1 \cdot z_2 = r_1 r_2 \cdot \operatorname{cis}(\theta_1 + \theta_2) \qquad \text{Polar Multiplication} \qquad (3)$$

Indicies in Polar Form Indicies must be consistent with the motivating example of repeated multiplication, hence consider that with the benefit of polar notation:

$$\begin{aligned}
z^1 &= z \\
z^2 &= z \cdot z \\
&= r^2 \cdot \cos\theta + \theta \\
&= r^2 \cdot \cos 2\theta \\
z^3 &= z \cdot z \cdot z \\
&= r^3 \cdot \cos 3\theta \\
&\ldots
\end{aligned}$$

So observe that :

$$z^k = r^k \cdot \cos k\theta \implies r^{k+1} = r^{k+1} \cdot \cos((k+1)\,\theta)$$

and hence by induction:

Index Theorem

$$z^n = r^n \cdot \operatorname{cis}(n\theta) \qquad \text{Polar Indicies} \qquad (4)$$

Example Solve $z = \left(\frac{1+i}{1-i}\right)^{10}$
The first step here is to rewrite this as:

$$z = \left(\frac{z_1}{z_2}\right)^{10}$$

where:

$$\begin{aligned}
& z_1 = 1+i \ , \ |z| = \sqrt{2} \ , \ \operatorname{Arg}(z_1) = \frac{\pi}{4} \\
\Longrightarrow \ & z_1 = \sqrt{2} \cdot e^{i\frac{\pi}{4}} \\
& z_2 = 1-i \ , \ |z| = \sqrt{2} \ , \ \operatorname{Arg}(z_1) = \frac{-\pi}{4} \\
\Longrightarrow \ & z_2 = \sqrt{2} \cdot e^{i\frac{\pi}{4}}
\end{aligned}$$

Such that now we have:

$$\begin{aligned}
z = \frac{\sqrt{2} \cdot e^{\frac{\pi i}{4}}}{\sqrt{2} \cdot e^{\frac{-\pi i}{4}}} \\
&= \left(e^{\frac{\pi i}{4}} \cdot e^{-\frac{-\pi i}{4}}\right)^{10} \\
&= \left(e^{\frac{\pi i}{4}} \cdot e^{\frac{\pi i}{4}}\right)^{10} \\
&= \left(e^{\frac{2\cdot\pi i}{4}}\right)^{10} \\
&= \left(e^{\frac{\pi i}{2}}\right)^{10} \\
&= (i)^{10} \\
&= (i)^4 \cdot (i)^4 \cdot (i)^2 \\
&= 1 \times 1 \times (i)^2 \\
&= -1
\end{aligned}$$

and so we have shown that $\left(\frac{1+i}{1-i}\right)^{10} = -1$

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Roots in Polar Form This is similar to incicies with the difference that there will be multiple roots. An n^{th}-degree polynomial will have n roots, this is the *Fundamental Theorem of Algebra*.

> **Roots Theorem**
>
> $$z^{\frac{1}{n}} = r^{\frac{1}{n}} \cdot \text{cis}\left(\frac{\theta + 2\pi k}{n}\right) \qquad \text{Multiple Complex Roots}$$
>
> Remember to use the principal argument $\theta \in (-\pi, \pi]$

Proof let:

$$\begin{aligned} s \cdot \text{cis}(\phi) &= w = z^{\frac{1}{n}} \\ \implies r \cdot \text{cis}(\theta) &= z = w^n \end{aligned}$$

Now add the the terms like so:

$$\begin{aligned} \left(r \cdot \cos\theta^2\right) &= (s^n \cdot \cos n\phi)^2 \\ &+ \\ (r \cdot \sin\theta) &= (s^n \cdot \sin\theta)^2 \end{aligned}$$

$$\begin{aligned} r\left(\sin^2\theta + \cos^2\theta\right) &= s^n\left(\sin^2 n\phi + \cos^2 n\phi\right) \\ r &= s^n \\ s &= r^{\frac{1}{n}} \end{aligned}$$

Now we can use this result and some division to solve a value for ϕ and we will have the polar forrm of $w = z^{\frac{1}{n}}$:

$$\begin{aligned} r \cdot \sin\theta &= s^n \cdot \sin\phi \\ &\div \\ r \cdot \cos\theta &= s^n \cdot \cos(n\phi) \end{aligned}$$

$$\begin{aligned} \tan\theta &= \tan n\phi \\ n\phi &\equiv \theta \pmod{2\pi} \\ \phi &\equiv \frac{\theta}{n} \pmod{2\pi} \\ \phi &= \frac{\theta + 2\pi k}{n}, \quad \forall k \in \mathbb{Z} \end{aligned}$$

If you play with it for a little while, you will notice that the values of ϕ are uniqe for values of $k = 0, 1, 2, \ldots (n-1)$.

Hence an n^{th} degree polynomial will have n roots.

Roots of Unity The roots of unity are the roots of $z^n = 1$, they are distributed around the unit circle and form the vertices of a regular polygon.

Example

Find the roots of $z = 1^{\frac{1}{3}}$:

$$
\begin{aligned}
& z = 1^{\frac{1}{3}} \\
\implies & z = \left(1^{\frac{1}{3}} \cdot \operatorname{cis}(\theta)\right)^{\frac{1}{3}} \\
& z = e^{\left(\frac{2\pi k i}{3}\right)} = \exp\left(\frac{2\pi k i}{3}\right) = \operatorname{cis}\left(\frac{2\pi k}{3}\right) \quad : \quad k = 0, 1, 2
\end{aligned}
$$

So now we just need to consider the 3 unique roots provided by the varying values of k, it is unnessary to consider k greater than 2, because it's periodic and the roots will simply repeat.

k* = *0;

$$
\begin{aligned}
w_0 &= e^{\frac{e^{2k\pi i}}{3}} \\
&= e^0 \\
&= 1
\end{aligned}
$$

k* = *1;

$$
\begin{aligned}
w_1 &= e^{\frac{2\pi k i}{3}} \\
&= e^{\frac{2\pi i}{3}} \\
&= \operatorname{cis}\left(\frac{2\pi}{3}\right) \\
&= \operatorname{cis}(120°) \\
&= \cos 120° + i \cdot \sin 120° \\
&= -\cos 60° + i \cdot \sin 60° \\
&= -\frac{1}{2} + i\frac{\sqrt{3}}{2}
\end{aligned}
$$

k* = *2;

$$
\begin{aligned}
w_1 &= e^{\frac{2\pi k i}{3}} \\
&= e^{\frac{4\pi i}{3}} \\
&= \operatorname{cis}\left(\frac{4\pi}{3}\right) \\
&= \operatorname{cis}(210°) \\
&= \cos 240° + i \cdot \sin 240° \\
&= -\cos 60° - i \cdot \sin 60° \\
&= -\frac{1}{2} - i\frac{\sqrt{3}}{2}
\end{aligned}
$$

So we have the roots such that:

$$
w^3 = z = 1 \implies w = (1, 0), \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)
$$

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