Contents

Part I

Differential Equations

The references in this section are a combination of the zill textbook I got off ebay[1], that big red textbook [2] and my lecture notes.

Topic 1

First Order Differential Equations

1.1 Preliminary Calculus Rules

The Chain Rule and Product Rule can be established visually fairly easily [1] and the proofs are fairly straightforward. [2]

Product Rule

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left(u\cdot v\right)=\frac{\mathrm{d}\,u}{\mathrm{d}\,v}\cdot v+u\cdot\frac{\mathrm{d}\,v}{\mathrm{d}\,x} \tag{1.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left(f\left(x\right)\cdot g\left(x\right)\right)=f'\left(x\right)\cdot g\left(x\right)+f\left(x\right)\cdot g'\left(x\right) \tag{1.2}$$

Chain Rule

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x}=\frac{\mathrm{d}\,y}{\mathrm{d}\,u}\cdot\frac{\mathrm{d}\,u}{\mathrm{d}\,x} \tag{1.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left[f\left(g\left(x\right)\right)\right]=f'\left(g\left(x\right)\right)\cdot g\left(x\right) \tag{1.4}$$

Integration by Substitution
The chain rule can be used for integration with some clever substitution:

$$\int f\left(g\left(x\right)\right)\cdot g'\left(x\right)\mathrm{d}\,x=\int f\left(u\right)\mathrm{d}\,u \tag{1.5}$$

$$\int f\left(u\right)\cdot\frac{\mathrm{d}\,u}{\mathrm{d}\,x}\mathrm{d}\,x=\int f\left(u\right)\mathrm{d}\,u \tag{1.6}$$

The product rule can be used for integration, but it's only fruitful when:

1. You can choose some $u=f\left(x\right)$ that simplifies when differentiated
   Or atleast stays the same e.g. $\frac{\mathrm{d}}{\mathrm{d}\,x}\left[\sin\left(x\right)\right]=\cos\left(x\right)$
2. $\mathrm{d}\,v=g'\left(x\right)\mathrm{d}\,x$ can be chosen such that the differential can be readily integrated to give v.

$$\int u\,\mathrm{d}\,v=u\cdot v-\int v\,\mathrm{d}\,u \tag{1.7}$$

Let:

$$\begin{aligned} u &= g(x) \\ \tfrac{du}{dx} &= g'(x) \end{aligned}\qquad F(x):\quad F'(x)=f(x)=y \tag{1.8}$$

Now by direct substitution into the chain rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,x}\left[F'\left(u\right)\right]&=F'\left(g\left(x\right)\right)\cdot g'\left(x\right)\\&=f\left(g\left(x\right)\right)\cdot g'\left(x\right)\\\implies f\left(g\left(x\right)\right)\cdot g'\left(x\right)&=\frac{\mathrm{d}}{\mathrm{d}\,x}\left[F\left(u\right)\right]\\f\left(g\left(x\right)\right)\cdot g'\left(x\right)&=\frac{\mathrm{d}}{\mathrm{d}\,x}\left[F\left(u\right)+C\right]\end{aligned} \tag{1.9}$$

Now by integrating both sides:

$$\begin{aligned}\int f\left(g\left(x\right)\right)\cdot g'\left(x\right)\mathrm{d}\,x&=\int\frac{\mathrm{d}}{\mathrm{d}\,x}\left[F\left(u\right)+C\right]\mathrm{d}\,x\\&=F\left(u\right)+C\\&=\int f\left(u\right)\mathrm{d}\,u\end{aligned}$$

So what we have is integration by substitution:

$$\int f\left(g\left(x\right)\right)\cdot g'\left(x\right)\mathrm{d}\,x=\int f\left(u\right)\mathrm{d}\,u \tag{1.5 revisited}$$

$$\int f\left(u\right)\cdot\frac{\mathrm{d}\,u}{\mathrm{d}\,x}\mathrm{d}\,x=\int f\left(u\right)\mathrm{d}\,u \tag{1.6 revisited}$$

This basically means that if an integral looks like the differentials could cancel out, they do, making the *Leibniz* notation particularly useful.

Consider the Product Rule (1.2):

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left[f\left(x\right)\cdot g\left(x\right)\right]=f'\left(x\right)\cdot g\left(x\right)+f\left(x\right)\cdot g'\left(x\right) \tag{1.10}$$

Let,

$$\begin{aligned} u &= f\left(x\right) & v &= g\left(x\right)\\ \tfrac{\mathrm{d}\,u}{\mathrm{d}\,x} &= f'\left(x\right) & \tfrac{\mathrm{d}\,v}{\mathrm{d}\,x} &= g'\left(x\right)\end{aligned} \tag{1.11}$$

Now we have:

$$\int\left(\frac{\mathrm{d}\,u}{\mathrm{d}\,x}\cdot v+u\cdot\frac{\mathrm{d}\,v}{\mathrm{d}\,x}\right)\mathrm{d}\,x=u\cdot v$$

$$\int\left(v\cdot\frac{\mathrm{d}\,u}{\mathrm{d}\,x}\right)\mathrm{d}\,x+\int\left(u\cdot\frac{\mathrm{d}\,v}{\mathrm{d}\,x}\right)\mathrm{d}\,x=u\cdot v$$

By Rule (1.6) we have:

$$\int v\,\mathrm{d}\,u+\int u\,\mathrm{d}\,v=u\cdot v$$

$$\int u\,\mathrm{d}\,v=u\cdot v-\int v\,\mathrm{d}\,u \tag{1.7 revisited}$$

[1] Visualizing the chain rule and product rule
[2] Differentiation Rules Proof

Application These are really the only two rules we've got (other than manipulation with partial fractions if possible) so the only trick is choosing when to use which one:

- Look at the intergrand $\int [\ ] \mathrm{d}\, x$:
  - If it's of the form $\left[f(u) \cdot \frac{\mathrm{d}\, u}{\mathrm{d}\, x}\right] = \left[f(g(x)) \cdot g'(x)\right]$
    * Use Integration by Substitution
  - If it's of the form: $\left[f(x) \cdot \frac{\mathrm{d}\, u}{\mathrm{d}\, x}\right] = \left[f(x) \cdot g'(x)\right]$
    * Use Integration by Parts

1.2 Introduction to Ordinary Differential Equations

Equations involving differentials like $\mathrm{d}\, y$ or $\mathrm{d}\, x$ or $\frac{\mathrm{d}\, y}{\mathrm{d}\, x}$ are differential equations.

If the derivatives correspond to only a single independent variable and do not involve partials e.g. $\left(\frac{\partial u}{\partial x}\right)$ they are said to be *Ordinary Differential Equations* (***ODE***).

Classifying Differential Equations

Order and Degree

- The Order of a differential corresponds to the higest derivative taken
- The Degree is the highest power of the highest order derivative of the *ODE*

Linearity A Linear ODE is of the form:

$$\sum_{0}^{n}\left[a_0(x) \cdot \left(\frac{\mathrm{d}^n\, y}{\mathrm{d}\, x^n}\right)\right] \tag{1.12}$$

Solutions to Differential Equations In order of preference, to solve a differential equaiton:

1. Solve for y (explicit)
2. Solve for x (explicit)
3. Solve for 0 (implicit)

1.3 Seperable Differential Equations

A differential equation of the form:

$$g(y) \cdot \frac{\mathrm{d}\, y}{\mathrm{d}\, x} = f(x) \tag{1.13}$$

Is a seperable Ordinary Differential Equation and has a solution:

$$\int g(y)\, \mathrm{d}\, y = \int f(x)\, \mathrm{d}\, x \tag{1.14}$$

Proof

$$\begin{aligned} g(y) \cdot \frac{\mathrm{d}\, y}{\mathrm{d}\, x} &= f(x) \\ \implies \int g(y) \frac{\mathrm{d}\, y}{\mathrm{d}\, x}\, \mathrm{d}\, x &= \int f(x)\, \mathrm{d}\, x \end{aligned} \tag{1.15}$$

By the Substitution Rule at (1.6):

$$\int g(y)\, \mathrm{d}\, y = \int f(x)\, \mathrm{d}\, x \tag{1.16}$$

1.4 Equations Reducible to Separable Equations

Some equations can be tricky to deal with, there is a method of u-Substitution:

Take some equation of the form:

$$\frac{\mathrm{d}\, y}{\mathrm{d}\, x} = f\left(\frac{x}{y}\right)$$

We can perform the u-substitution:

$$\begin{aligned} u &= \frac{y}{x} \\ \implies y &= u \cdot x \\ \implies \frac{\mathrm{d}\, y}{\mathrm{d}\, x} &= \frac{\mathrm{d}\, u}{\mathrm{d}\, x} \cdot x + (1) \cdot u \end{aligned}$$

Substituting in the terms:

$$\begin{aligned} \frac{\mathrm{d}\, y}{\mathrm{d}\, x} &= f\left(\frac{y}{x}\right) \\ \frac{\mathrm{d}\, u}{\mathrm{d}\, x} \cdot x + u &= f(u) \\ \frac{\mathrm{d}\, u}{\mathrm{d}\, x} \cdot x &= f(u) - u \\ \frac{1}{f(u)-u} \cdot \frac{\mathrm{d}\, u}{\mathrm{d}\, x} \cdot x &= 1 \\ \frac{1}{f(u)-u} \cdot \frac{\mathrm{d}\, u}{\mathrm{d}\, x} &= \int \frac{1}{x}\, \mathrm{d}\, x \\ \int \frac{1}{f(u)-u} \cdot \frac{\mathrm{d}\, u}{\mathrm{d}\, x}\, \mathrm{d}\, x &= \int \frac{1}{x}\, \mathrm{d}\, x \\ \int \frac{1}{f(u)-u}\, \mathrm{d}\, u &= \ln|x| + c \end{aligned} \tag{1.17}$$

Now presume $\exists G(u) : G(u) = \int \frac{1}{f(u)-u}\, \mathrm{d}\, u$

$$\begin{aligned} G(u) &= \ln|x| + c \\ G\left(\frac{y}{x}\right) &= \ln|x| + c \\ G\left(\frac{y}{x}\right) + \ln|x| + c &= 0 \end{aligned} \tag{1.18}$$

Hence by by (1.18) there must atleast be an implicit solution to the equation assuming that the integral can be solved.

1.5 First Order Linear ODE

1.5.1 Summary

A Linear Odinary Differential Equation is of the form:

$$\sum_{0}^{n}\left[a_n(x) \cdot f^{(n)}(x)\right] = g(x)$$

If $g(x) = 0$ it is said to be homogenous

A first Order Linear ODE is of the form:

$$a_1(x) \cdot \frac{\mathrm{d}\, y}{\mathrm{d}\, x} + a_0(x) \cdot y = g(x) \tag{1.19}$$

Where $a(x)$ is a function

It is typical to rewrite this as:

Linear First Order ODE:

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} + p\,(x) \cdot y = f\,(x) \tag{1.20}$$

if $f\,(x) = 0$ the equation is said to be homogenous

Let the homogenous be y_h and the particular solution be y_p, i.e.:

- y_h : $\frac{\mathrm{d}\,y_h}{\mathrm{d}\,x} + p\,(x) \cdot y_h = 0$
- y_p : $\frac{\mathrm{d}\,y_p}{\mathrm{d}\,x} + p\,(x) \cdot y_p = f\,(x)$

In order to find a solution a solution for a First Order Linear ODE, don't remember an equation, remember the technique:

1. Rewrite the Equation in the standard form:
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} + p\,(x) \cdot y = f\,(x)$$
2. Identify $p\,(x)$ and find the integrating factor:
$$e^{\int p(x)\,\mathrm{d}\,x}$$
3. Multiply through by the integrating factor:
$$e^{\int p(x)\,\mathrm{d}\,x}\left(\frac{\mathrm{d}\,y}{\mathrm{d}\,x} + p\,(x) \cdot y\right) = e^{\int p(x)\,\mathrm{d}\,x} f\,(x)$$
It may be concluded:
$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left[e^{\int p(x)\,\mathrm{d}\,x \cdot} \cdot y\right] = e^{\int p(x)\,\mathrm{d}\,x} \cdot f\,(x)$$
4. Integrate both sides in order to solve:

1.5.2 Proof

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} + p\,(x) \cdot y = f\,(x) \tag{1.21}$$

Consider the homogenous and particular solution:

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} + p\,(x) \cdot y = 0 \quad \Longrightarrow \quad y = y_h \tag{1.22}$$
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} + p\,(x) \cdot y = f(x) \Longrightarrow \quad y = y_p \tag{1.23}$$

Observe that the sum of these solutions is a valid solution:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\,x}\,(y_h + y_p) + p\,(x) \cdot (y_h + y_p) &= f\,(x)\\
\frac{\mathrm{d}\,y_h}{\mathrm{d}\,x} + \frac{\mathrm{d}\,y_p}{\mathrm{d}\,x} + p\,(x) \cdot y_h + p\,(x) \cdot y_p &= f\,(x)\\
\frac{\mathrm{d}\,y_h}{\mathrm{d}\,x} + p\,(x) \cdot y_h + \frac{\mathrm{d}\,y_p}{\mathrm{d}\,x} + p\,(x) \cdot y_p &= f\,(x)\\
0 + f\,(x) &= f\,(x) \qquad (1.24)
\end{aligned}$$

The point of showing (1.24) is that we need y_h to find y_p anyway:

$$\begin{aligned}
\frac{\mathrm{d}\,y}{\mathrm{d}\,x} + p\,(x) \cdot y &= 0\\
\frac{1}{y} \cdot \frac{\mathrm{d}\,y}{\mathrm{d}\,x} &= -p\,(x)\\
\ln|y| &= \int -p\,(x)\,\mathrm{d} + c\\
|y| &= e^{\int -p(x)x\,\mathrm{d}\,x} \cdot e^{c} \qquad (1.25)
\end{aligned}$$

assume $y > 0$:

$$\Longrightarrow \; y_h = e^{-\int p(x)\,\mathrm{d}\,x} \cdot c$$

Let $y_1 = e^{-\int p(x)\,\mathrm{d}\,x}$:

$$y_h = y_1(x) \cdot c \tag{1.26}$$

The homogenous solution involves only the parameters $p\,(x)$ and c, if the 0 value was changed to $f\,(x)$, the solution would have to reflect this change as $c = u\,(x)$ and hence we assume that their is an integrating factor $u\,(x)$:

$$\begin{aligned}
y_p &= u\,(x) \times y_h\,(x)\\
&= e^{-\int p(x)\,\mathrm{d}\,x} \cdot u\,(x) \qquad (1.27)
\end{aligned}$$

Now in order to solve this $u\,(x)$ substitute (1.27) into (1.21):

$$\begin{aligned}
y_p &= e^{-\int p(x)\,\mathrm{d}\,x} \cdot u\,(x)\\
\frac{\mathrm{d}\,y_p}{\mathrm{d}\,x} + p\,(x) \cdot &= f\,(x)\\
\frac{\mathrm{d}}{\mathrm{d}\,x}\,(u\,(x) \cdot y_1\,(x)) + p\,(x)\,u\,(x)\,y_1\,(x) &= f\,(x)\\
\frac{\mathrm{d}\,u}{\mathrm{d}\,x} \cdot y_1\,(x) + \frac{\mathrm{d}\,y_1}{\mathrm{d}\,x} \cdot u\,(x) + p\,(x) \cdot u\,(x) \cdot y_1\,(x) &= f\,(x)\\
u\,(x)\left(\frac{\mathrm{d}\,y_1}{\mathrm{d}\,x} + p\,(x)\,y_1\right) + \frac{\mathrm{d}\,y}{\mathrm{d}\,x} \cdot y_1\,(x) &= f\,(x)\\
0 + \frac{\mathrm{d}\,y}{\mathrm{d}\,x} - y_1\,(x) &= f\,(x)\\
\frac{\mathrm{d}\,y}{\mathrm{d}\,x} &= f\,(x)\,/y_1\,(x)\\
\int \frac{\mathrm{d}\,u}{\mathrm{d}\,x}\,\mathrm{d}\,x &= \int f\,(x)\,/y_1\,(x)\,\mathrm{d}\,x\\
\int \mathrm{d}\,u &= \int f\,(x)\,/y_1\,(x)\,\mathrm{d}\,x\\
u &= \int f\,(x)\,/y_1\,(x)\,\mathrm{d}\,x
\end{aligned} \tag{1.28}$$

This is more or less where

If we susbstitute in the value of $y_1 = e^{-\int p(x)\,\mathrm{d}\,x}$ at (1.26) into (1.28):

$$u = \int f\,(x) \cdot e^{\int p(x)\,\mathrm{d}\,x}\,\mathrm{d}\,x \tag{1.29}$$

Now if we substitute in $y_p = u \cdot y_1$ from (1.27):

$$\begin{aligned}
y_p &= \frac{1}{y_1} \cdot \int f\,(x) \cdot e^{\int p(x)\,\mathrm{d}\,x}\\
y_p &= e^{-\int p(x)\,\mathrm{d}\,x} \int f\,(x) \cdot e^{\int p(x)\,\mathrm{d}\,x} \qquad (1.30)
\end{aligned}$$

So that gives a formula for the value of the particular solution to a linear first-order ODE, however that should not be memorised, instead observe if we use the factor $e^{\int p(x)\,\mathrm{d}\,x}$:

Proof.

$$
\begin{aligned}
e^{\int p(x)\,\mathrm{d}\,x} \cdot y_p &= e^{\int p(x)\,\mathrm{d}\,x} \cdot e^{-\int p(x)\,\mathrm{d}\,x} \int f(x) \cdot e^{\int p(x)\,\mathrm{d}\,x} \\
e^{\int p(x)\,\mathrm{d}\,x} \cdot y_p &= \int f(x) \cdot e^{\int p(x)\,\mathrm{d}\,x} \\
\frac{\mathrm{d}}{\mathrm{d}\,x}\left(e^{\int p(x)\,\mathrm{d}\,x} \cdot y_p\right) &= \frac{\mathrm{d}}{\mathrm{d}\,x}\left(\int f(x) \cdot e^{\int p(x)\,\mathrm{d}\,x}\right) \\
&= f(x) \cdot e^{\int p(x)\,\mathrm{d}\,x} \\
e^{\int p(x)\,\mathrm{d}\,x} \frac{\mathrm{d}\,y}{\mathrm{d}\,x} + p(x) \cdot e^{\int p(x)\,\mathrm{d}\,x} \cdot y &= e^{\int p(x)\,\mathrm{d}\,x} \cdot f(x) \\
\implies \frac{\mathrm{d}\,y}{\mathrm{d}\,x} + p(x) \cdot y &= f(x)
\end{aligned}
\tag{1.31}
$$

□

This gives us a technique to follow each time we see an equation of this form:

1. get it into the form in (1.21)
2. multiply through by $e^{\int p(x)\,\mathrm{d}\,x}$
3. The LHS of the resulting equation is automatically the derivative of y and $e^{\int p(x)\,\mathrm{d}\,x}$
4. Integrate both sides

1.5.3 Exemplars

$$
(x+1) \cdot \frac{\mathrm{d}\,y}{\mathrm{d}\,x} + y = \ln(x) \quad ; \qquad y(1) = 10 \tag{1.32}
$$

(1) Put the equation into the Standard Form:

$$
\frac{\mathrm{d}\,y}{\mathrm{d}\,x} + \frac{y}{x+1} = \frac{\ln(x)}{x+1} \quad : \qquad (x \in \mathbb{R} \setminus \{-1, 0\}) \tag{1.33}
$$

(2) Solve the integrating factor

$$
\begin{aligned}
u &= e^{\int \frac{1}{x+1}\,\mathrm{d}\,x} \\
&= e^{\int \ln|x+1|\,\mathrm{d}\,x} \\
&= |x+1|
\end{aligned}
\tag{1.34}
$$

Assume $x > 0$

(3) Multiply through by the Integrating Factor

$$
\begin{aligned}
(x+1) \cdot \frac{\mathrm{d}\,y}{\mathrm{d}\,x} + y &= \ln(x) \\
\implies \frac{\mathrm{d}}{\mathrm{d}\,x}\left((x+1) \cdot y\right) &= \ln(x)
\end{aligned}
\tag{1.35}
$$

(4) Integrate Both Sides

$$
\int \frac{\mathrm{d}}{\mathrm{d}\,x}\left[(x+1) \cdot y\right] \mathrm{d}\,x = \int \ln(x)\,\mathrm{d}\,x
$$

by the chain rule:

$$
(x+1) \cdot y = \int \ln(x)\,\mathrm{d}\,x \tag{1.36}
$$

Let:

$$
\begin{aligned}
u &= \ln(x) & \mathrm{d}\,v &= \mathrm{d}\,x \\
\mathrm{d}\,u &= \tfrac{1}{x}\,\mathrm{d}\,x & v &= x
\end{aligned}
$$

$$
\implies \int u\,\mathrm{d}\,v = u \cdot v + \int v\,\mathrm{d}\,u
$$

integration by substitution provides:

$$
\begin{aligned}
(x+1) \cdot y &= \ln(x) \cdot x - \int \mathrm{d}\,x \\
&= x \cdot (\ln(x) - 1) + c \\
\implies y &= \frac{x \cdot (\ln(x) - 1 + c)}{x+1}
\end{aligned}
$$

(5) Consider the Initial Condition
Substitute $y(1) = 10$:

$$
\begin{aligned}
10 &= \frac{1\,(\ln(1) - 1 + c)}{2} \\
20 &= 1\,(0 - 1) + c \\
c &= 19
\end{aligned}
\tag{1.37}
$$

A first order ODE will only have one family of solutions, hence the solution on an interval will be a solution for the entire domain, hence we have:

$$
y = \frac{x\,(\ln(x) - 1 + 19)}{x+1} \quad ; \qquad \forall x \in \mathbb{C} \setminus \{-1, 0\}
$$

1.6 First Order Exact Equations

Refer to Ch. 2.4 of the Textbook [1]

1.6.1 Summary

1.6.2 Proof

1.6.3 Exemplars

chapterSecond Order Differential Equations

1.7 Linear ODE Theory

Refer to Ch. 4.1 of the Textbook [1]

1.7.1 Summary

1.7.2 Proof

1.7.3 Exemplars

Part II

Mathematical Modelling

The Textbook refers to *A First Course in Mathematical Modelling 5th Ed*[3]

Topic 1

(01) Modelling Change, Wk. 1-3, TB Ch. 1

1.1 Difference Equations (Recurrence Relations)

1.1.1 Summary

Creating a Mathematical Model

In order to a mathematical model:

1. Identify relevant quantities

   Make Simplifying assumptions in order to limit the number of assumptions

2. Use assumptions to mathematically relate variables together

3. Solve the equations and interpret the results

4. Compare the Model results with Observations

Definitions

- A ***Sequence*** is a function from the naturals to the reals $f : \mathbb{N} \mapsto \mathbb{R}$
- A ***dynamical System*** is a relationship among terms in a sequence
- ***numerical solution*** is a table of values satisfying the dynamical system.

Difference Equations When creating difference equations, always remember that the notationis:

$$\Delta a_n = a_{n+1} - a_n$$

We will often need to find the change of sequence in terms of some function, if that function involves a preceeding term it is known as a ***Recurrence Relation***.

Proportionality if two rates are proportional the following notation is used:

$$\Delta \propto p \implies \Delta p_n = k \cdot p_n \qquad \exists k \in \mathbb{R} \tag{1.1}$$

Population Growth A population with p members and a carrying capacity of C will have a difference equation:[1]

$$\Delta S_n \propto (C - S) \cdot S \qquad \exists k \in \mathbb{R} \tag{1.2}$$

[1] Refer to P. 11 of TB

1.1.2 Notes and Proofs

Modelling Population Growth Imagine an influenza infection spreading throughout a school campus, if there are $C = 400$ students, S infected students the rate of disease spread will be proportional to the number of uninfected students $(C - S)$, which will in turn be proportional to the number of infected students that could spread the disease, giving a model:

$$\begin{aligned} \Delta S_n &\propto (C - S) \cdot S \\ \implies \Delta S_n &= (C - S) \cdot S \cdot k \quad \exists k \in \mathbb{R} \\ \implies S_{n+1} - S_n &= (C - S) \cdot S \cdot k \\ \implies S_{n+1} &= (C - S) \cdot S \cdot k + S_n \end{aligned}$$

Equilibrium Value A number a is an equilibrium value (i.e. a fixed point) if:

$$a_k = a \forall k \in \mathbb{Z} \tag{1.3}$$

Linear Dynamical Systems

A Linear Dynamical System of the form: [a]

$$a_{n+1} = r \tag{1.4}$$

Has a solution $r \neq 0$:

$$a_k = r^k a_0 \tag{1.5}$$

by mindful that this is in effect a solution to the homogenous recurrence relation

[a] refer to p. 21 of TB

1.1.3 Exemplars

1.2 Systems of Difference Equations

1.2.1 Worked Example

Consider the following Price and Supply model:

$$P_{n+1} = 0.7 \cdot P_n - 0.1 (Q_n - 500); \tag{1.6}$$
$$Q_{n+1} = Q_n + 0.2 \cdot (P_n - 100) \tag{1.7}$$

Expand the equations:

$$x =$$

This system has equilibrium values P, Q:

$$\begin{aligned} P &= 0.7 \cdot P - 0.1\left(Q - 500\right); \\ Q &= Q + 0.2 \cdot \left(P - 100\right) \end{aligned}$$

Observe that (1.7):

$$\begin{aligned} Q &= Q + 0.2P - 0.2 \times 100 \\ 0 &= 0.2P - 20 \\ \implies P &= 100‘ \end{aligned}$$

By substituting that value into (1.6) we have:

$$\begin{aligned} P &= 0.7 \cdot P - 0.1\left(Q - 500\right) \\ 100 &= 0.7 \times 100 - 0.1\left(Q - 500\right) \\ \implies Q &= 200 \end{aligned}$$

Hence the equilibrium values are :

$$\begin{aligned} P &= 100 \\ Q &= 200 \end{aligned}$$

Long-Term Behaviour The real trick is being able to assess the long term behaviour of the system using eigenvalues and eigenvectors:

First rewrite the system in matrix form:

$$\begin{aligned} \begin{bmatrix} P_{n+1} \\ Q_{n+1} \end{bmatrix} &= \begin{bmatrix} 0.7P - 0.1Q + 50 \\ Q + 0.2P - 20 \end{bmatrix} \\ &= \begin{bmatrix} 0.7P - 0.1Q \\ Q + 0.2P \end{bmatrix} + \begin{bmatrix} 50 \\ -20 \end{bmatrix} \\ &= \begin{bmatrix} 0.7 & -0.1 \\ 0.2 & 1 \end{bmatrix} \cdot \begin{bmatrix} P \\ Q \end{bmatrix} + \begin{bmatrix} 50 \\ -20 \end{bmatrix} \end{aligned}$$

Rewrite the matrix as a single expression:

$$M_{n+1} = A \times M_n + B \tag{1.8}$$

Where:

$$\begin{aligned} M_n &= \begin{bmatrix} P_n \\ q_n \end{bmatrix} \\ A &= \begin{bmatrix} 0.7 & -0.1 \\ 0.2 & 1 \end{bmatrix} \\ B &= -\begin{bmatrix} 50 \\ 20 \end{bmatrix} \end{aligned}$$

Now we've seen this before, the problem is that we cannot use the method of undetermined coefficients to solve the particular solution, hence we use the method of conjecture to relate this to a geometric series and hence solve the geometric series which may be solved via subtraction of terms:

Observe that:

$$\begin{aligned} M_1 &= AM_0 + B \\ M_2 &= AM_1 + B \\ &= A\left(AM_0 + B\right) \\ &= A^2 M_0 + AB + B \\ M_3 &= AM_2 + B \\ &= A\left(A^2 M_0 + AB + B\right) + B \\ &= A^3 M_0 + A^2 B + AB + B \end{aligned}$$

By the Method of Conjecture we have

$$\begin{aligned} &\dots \\ M_n &= A^n M_0 + A^{n-1}B + \cdots A^2 B + AB + A^0 B \\ &= A^n M_0 + \sum_{i=0}^{n-1} \left[A^i\right] \times B \\ &= A^n M_0 + S_n B \end{aligned} \tag{1.9}$$

$$\tag{1.10}$$

Recall that the Geometric Series is such that:[2]

Similar such reasoning may be applied in the context of Matrices:

$$S_n = A^0 + A^1 + A^2 + A^3 + \cdots + A^{n-1} \tag{1.11}$$

Multiply through by the coefficient A and we have:

$$S_n A = A^1 + A^2 + A^3 + A^4 \cdots + A^n \tag{1.12}$$

Now Subtract (1.12) from (1.11)

$$\begin{aligned} S_n - AS &= I - A^n \\ S_n\left(I - A\right) &= I - A^n \\ S_n &= \left(I - A^n\right)\left(I - A\right)^{-1} \end{aligned} \tag{1.13}$$

Now simply substitute (1.13) into (1.9):

$$\begin{aligned} &= A^n M_0 + S_n B \\ &= A^n M_0 + \left(I - A^n\right)\left(I - A\right)^{-1} B \end{aligned}$$

So if we can deal with A^n we have our solution, this is the whole trick, raising matrices to an index is computationally difficult, instead it is neessary to use eigenvalues and eigenvectors.

An eigenvalue (λ) is a number that behaves like the matrix when multiplied by the corresponding eigenvector (X): [3]

[2] Refer to p. 95 of Bartle & Sherbert's *Real Analysis*

[3] Refer to Ron Larsons *Linear Algebra*

Topic 2

(02) The Modelling Process, Wk. 4-5, TB Ch. 2

Topic 3

(03) Proportionality, Wk. 5, Notes Ch. 3

Topic 4

(04) Model Fitting, Wk. 6, Notes Ch. 4

Topic 5

(05) Graphs of Functions as Models , Wk. 7-8, TB Ch. 15

5.1 Arms Race Question

Every year the question in the exams has been the same, so just learn how to deal with this question and you're good to go, it will be in the final and in the second class test, for the past two years the second class test has been identical, but, the first test has been different each year and both times it has been very difficult.

5.1.1 Problem for this topic

Assume two countries X and Y are involved in a nuclear arms race.

Each country assumes the other is the aggressor that wishes to conduct a first strike. To deter this, each country is 'forced' to have enough nuclear missiles so that those that survive a first strike will have a sufficient number remaining in order to destroy the aggressor.

Defence experts in country Y believe they need y_0 surviving missiles to deter country X from attacking in the first place.

They also beilieve that the proportion of missiles targeted and surviving an attack would be s, where $0 < s < 1$. Let variables x and y respectively denote the number of missiles posessed by country X and Y. Assume that one missile can destroy at most one missile of the opposing country.

1. if $x < y$, determine the minimum number of missiles country Y feels is required to deter an attack from x.

2. if $y < x < 2y$, what is the least number of missiles country Y now requires to deter country X?

3. Suppose at some point $x = 2y$. At least how many missiles does country Y require to feel safe?

5.1.2 Working

1. The number of missiles struck by an initial atack will be x, of those missiles $s \cdot x$ will survive.

   The number of missiles not struck will be $y - x$.

   The number of missiles required will me y_0 :

$$\begin{aligned} y_0 &= y - x + s \cdot x \\ \implies y &= y_0 + x - s \cdot x \\ &= y_0 + x(1 - s) \end{aligned}$$

2. Some missiles will be struck twice while others will be struck once.

   The number of missilles struck once but not twice is N_1 :

$$\begin{aligned} N_1 &= x - (y - x) \\ &= x - y + x \\ &= 2x - y \end{aligned}$$

   The proportion that will survive is $N_1 \cdot s$.

   The number of missiles struck twice is N_2 :

$$N_2 = y - x$$

   of those a proportion will survive.

We assume that the missile destruction is probablistic and unknown by the other party, this way missiles are not redirected following an attack.

Following the first barrage of $s \cdot N_2$ missiles will survive, following a subsequent attack $N_2 \cdot s^2$ will survive.

y_0 missiles will be required to match the surviving number of missiles so:

$$\begin{aligned} y_0 &= s \cdot N_1 + s^2 \cdot N_2 \\ &= S(2x - y) + s^2 (y - x) \\ y_0 &= 2sx - sy + s^2 y - s^2 x \\ y_0 &= 2sx - sy + s^2 y - s^2 x \\ y_0 &= 2sx + y\left(-s + s^2\right) - s^2 x \\ y\left(s - s^2\right) &= -y_0 + 2sx - s^2 x \\ y &= \frac{-y_0 + 2sx - s^2 x}{(s - s^2)} \end{aligned}$$

Topic 6

(06) Modelling with Differential Equations, Wk. 8, TB. Ch. 11

6.1 Modelling with Differential Equations

6.1.1 Seperable Differential Equations

Consider the ***Fundamental Theorom of Calculus***:

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left(\int_a^x f(x)\,\mathrm{d}\,x\right) = f(x) \tag{6.1}$$

$$\int_a^b f(x)\,\mathrm{d}\,x = F(b) - F(a) \tag{6.2}$$

This gives a relationship between integration and differentiation and from this we hence define the antiderivative and the indefinite integral of a functoin:

$$\frac{\mathrm{d}}{\mathrm{d}\,x}(F(x)+C) = f(x) \iff \int f(x)\,\mathrm{d}\,x = F(x) + C \tag{6.3}$$

That's basically where the C comes from, you lose it in differentiation and $\int \mathrm{d}\,x$ means give me the value that we differentiated with respect to x to get here.

The magic of tying all that into differentiation is the fundamental theorem of calculus.

Establish the Substitution Rule From the definition of the indefinite integral (6.3):

$$\begin{aligned} F(x) &= \int f(x)\,\mathrm{d}\,x \\ F'(x) &= \frac{\mathrm{d}}{\mathrm{d}\,x}\left(\int f(x)\,\mathrm{d}\,x\right) \\ f(x) &= \frac{\mathrm{d}}{\mathrm{d}\,x}\left(\int f(x)\,\mathrm{d}\,x\right) \end{aligned} \tag{6.4}$$

This is basically a formulation of the definition of the indefinite integral backwards, we'll use this in a moment.

Let;

$$\begin{aligned} u &= g(x) \\ \frac{\mathrm{d}\,u}{\mathrm{d}\,x} &= g'(x) \\ F'(x) &= f(x) = y \end{aligned} \tag{6.5}$$

Now consider the derivative of u :

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\,x}(F(u)) &= f'(u)\cdot\frac{\mathrm{d}}{\mathrm{d}\,x}(u) \\ &= f(u)\cdot\frac{\mathrm{d}\,u}{\mathrm{d}\,x} \end{aligned} \tag{6.6}$$

So now we have:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\,x}(F(u)) &= f(u)\frac{\mathrm{d}\,u}{\mathrm{d}\,x} \\ \int \frac{\mathrm{d}}{\mathrm{d}\,x}(F(u))\,\mathrm{d}\,x &= \int f(u)\cdot\frac{\mathrm{d}\,u}{\mathrm{d}\,x}\,\mathrm{d}\,x \end{aligned} \tag{6.7}$$

Now we can take (6.7) and use the definition of the indefinite integral from (6.4) to basically cancel out the integral of the derivative [1]:

$$\begin{aligned} F(u) &= \int f(u)\cdot\frac{\mathrm{d}\,u}{\mathrm{d}\,x}\,\mathrm{d}\,x \\ \implies \int f(u)\cdot\frac{\mathrm{d}\,u}{\mathrm{d}\,x}\,\mathrm{d}\,x &= \int f(u)\,\mathrm{d}\,u \end{aligned} \tag{6.8}$$

Establish Seperable Equations Now that we have the substitution rule we can use that to establish seperable differentiable equaitons; say we have:

$$\frac{\mathrm{d}\,P}{\mathrm{d}\,t} \propto P$$

We can rewrite this as:

$$\begin{aligned} \frac{1}{P}\cdot\frac{\mathrm{d}\,P}{\mathrm{d}\,t} &= k \\ \int \frac{1}{P}\cdot\frac{\mathrm{d}\,P}{\mathrm{d}\,t}\,\mathrm{d} &= k\cdot\int \mathrm{d}\,t \\ \int \frac{1}{P}\cdot\frac{\mathrm{d}\,P}{\mathrm{d}\,t}\,\mathrm{d} &= kt + C \end{aligned}$$

Now by the substitution rule that was established at (6.8):

$$\begin{aligned} \int \frac{1}{P}\,\mathrm{d}\,P &= kt + C \\ \ln(P) &= kt + C \\ \implies P &= C_1 e^{kt} \end{aligned} \tag{6.9}$$

[1] Now be mindful that the integral of the derivative is the function because of the definition of the indefinite integral / anti-derivative (which we defined because of the FTC), where as the derivative of the integral $\left(\frac{\mathrm{d}}{\mathrm{d}\,x}\left(\int \mathrm{d}\,x\right)\right)$ is the original function because of the FTC

6.2 Modelling with Populatoin Differential Equations

6.2.1 Worked Example

Consider te spreading of a disease on an isolated island with population size N. A portion of the population travels abroad and returns to the island infected with the disease. you would like to predict the number of people X who will have been infected by some time t. Consider the following model where $k > 0$ is constant:

$$\frac{\mathrm{d}\,X}{\mathrm{d}\,t} = k \cdot X \cdot (N - X)$$

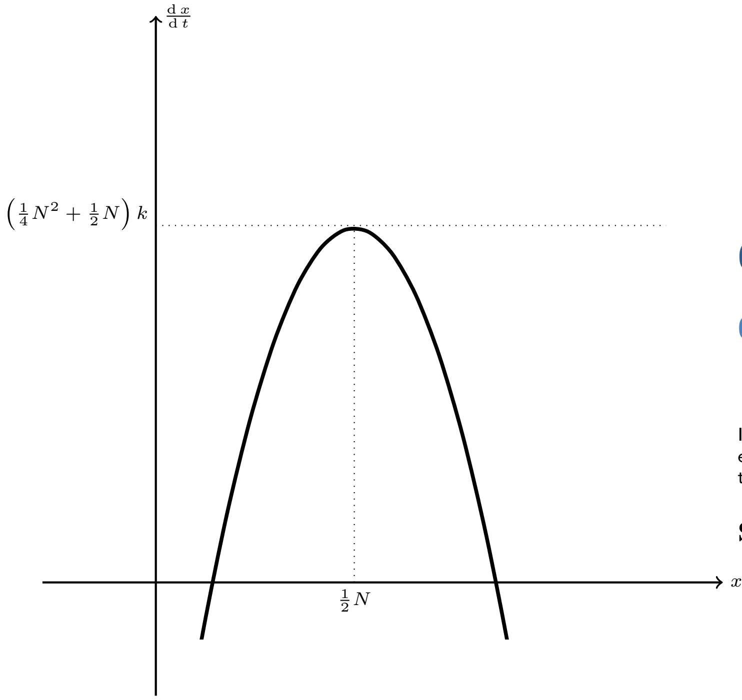
Assumptions in the Model There is an assumption that the rate of infection is directly proportional to the number of infected people, this relationship may be non linear and may depend on other things such as temperature, humidity et cetera. the assumption that the rate of change of the disease is proportional to the number of uninflected people may be such that the rate of change is a non linear function of the number of remaining people. as mentioned there will likely be more factors that affect the rate of disease communication, the biggest one being temperature

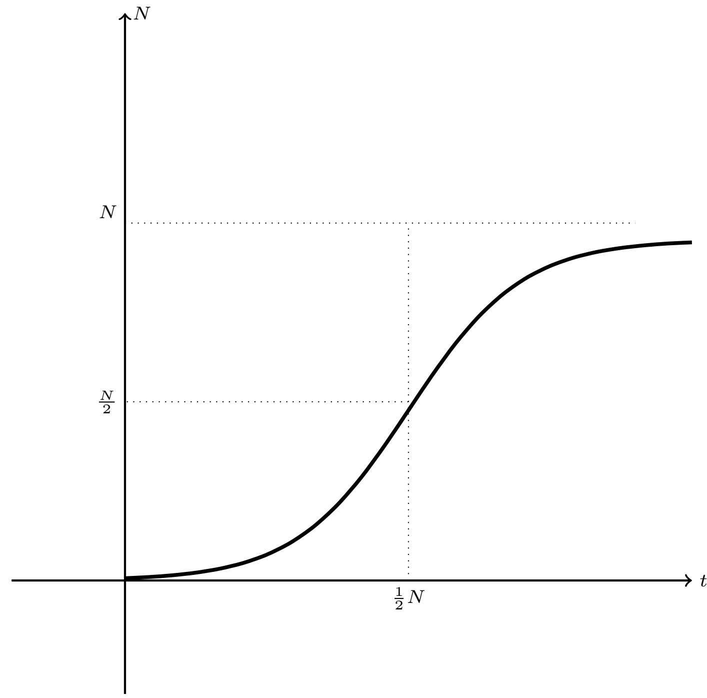
Graph $\frac{\mathrm{d}\,X}{\mathrm{d}\,t}$ verses X observe that:

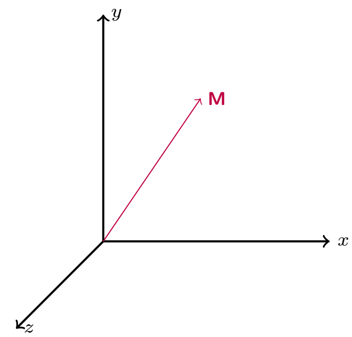
$$\begin{aligned}\frac{\mathrm{d}\,X}{\mathrm{d}\,t} &= k \cdot \left(XN - X^2\right) \\ &= \left(-X^2 - Xn\right) \cdot k \qquad (6.10)\end{aligned}$$

$$X_{\textsf{Vertex}} = -\frac{1}{2} \cdot N \qquad (6.11)$$

Which corresponds to the plots:







6.3 Modelling Drug Concentrations

6.3.1 Worked Examples, Tutorial of Week 11

If $k = 0.05$ hr^{-1} and the highest safe concentration is e times the lowest effective concentration, find the length of the time between repeated doses that will ensure safe but effective concentrations.

Solve the Decay Model let:

t	be the time (variable) since the last dose was administered
T	be the constant time between doses
$c_n(t)$	be the blood concentration, after a period of t, following the nth dose
c_0	be the initial dose administered, which will also be the constant dosage and initial blood concentration at $t = 0$
$H = r \cdot L,$	be the maximum safe dosage, $\exists! r \in \mathbb{R}^+$
L	be the minimum effective dosage
C_n	be the drug level immediately following administration
R_n	be the drug level remainining immediately preceeding administration.

Presume that the rate of drug metabolism is proportional to the drug levels $c(t)$:

$$\begin{gathered}\frac{\mathrm{d}\,c}{\mathrm{d}\,t} \propto c(t) \\ \ln|c(t)| = -kt + \lambda, \quad \exists \lambda \in \mathbb{R}\end{gathered}$$

blood levels will be positive and so the absolute value may be dispenced with:

$$\begin{aligned}\ln(c(t)) &= -kt+\kappa\\ \implies c(t) &= \lambda^*\cdot e^{-kt}, \quad \exists \lambda^*\in\mathbb{R} \end{aligned} \tag{6.12}$$

Applying the initial condition that $c(0)=c_0$:

$$\begin{aligned}c(0)=c_0 &= \lambda^*\cdot e^{-k0}\\ \implies \lambda^* &= c_0\end{aligned} \tag{6.13}$$

Hence the blood concentration levels, as a function of time will be:

$$c(t)=c_0\cdot e^{-kt} \tag{6.14}$$

Solve the Time between doses Presume when a dose is applied that the level instantaneously reaches the higher level as shown in the diagram at 7.1.

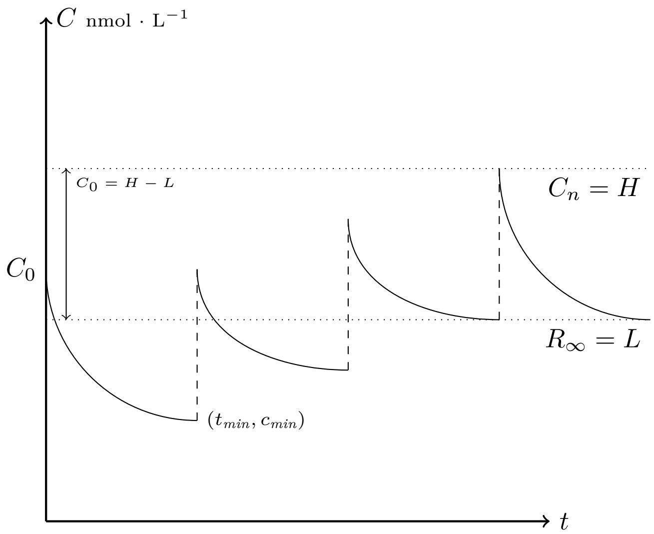


Figure 6.1: Diagram of Blood Levels over time

The blood concentration will not necessarily reach the overdose threshold H or the minimum effective threshold L following the first dose, as shown in the figure at 7.1, there will however be a maximum (t_{max}, c_{max}) and a minimum concentration (t_{min}, c_{min}):

$$\begin{aligned}c &= c_0 e^{-kt}\\ e^{-kt} &= \frac{c}{c_0}\\ -kt &= \ln\left(\frac{c}{c_0}\right)\\ t &= \frac{1}{-k}\cdot\ln\left(\frac{c}{c_0}\right)\end{aligned} \tag{6.15}$$

The time between dosages, if the dosages are constant, must be the difference between the maximum concentration and the minimum concentration, if the minimum concentration is limited by the effective minimum dosage L and the maximum effective dosage is limited by the safe threshod $H = r\cdot L$ then we have:

$$\begin{aligned}T &= t(c_{\max}) - t(c_{\min})\\ &= \frac{1}{-k}\left[\ln\left(\frac{c_{\max}}{c_0}\right)-\ln\left(\frac{c_{\min}}{c_0}\right)\right]\\ &= \frac{1}{-k}\left[\ln\left(\frac{L\cdot r}{c_0}\right)-\ln\left(\frac{L}{c_0}\right)\right]\\ &= \frac{1}{-k}\cdot\ln(r)\end{aligned} \tag{6.16}$$

Thus the time between repeated doses must be less than T, where T is defined as above at (6.16).

What is the size of the dose It is not possible to determine the size of each dose, we would need to know the upper/lower limits or, in this case, r.

Suppose $k = 0.05$ hr^{-1} and $T = 10$ hr; what is the smallest n such that $R_n > 0.5\cdot R$

Solve the Decay Function

$$\begin{aligned}\frac{\mathrm{d}\,C}{\mathrm{d}\,t} &\propto C\\ \implies \frac{1}{C}\cdot\frac{\mathrm{d}\,C}{\mathrm{d}\,t} &= -k, \quad \exists k\in\mathbb{R}^-\\ \implies \ln|C| &= -k\cdot t+\lambda, \quad \exists\lambda\in\mathbb{R}\\ \implies C(t) &= \lambda^*\cdot e^{-k\cdot t}, \quad \exists\lambda^*\in\mathbb{R}\end{aligned} \tag{6.17}$$

Now by using the initial condition that $C(0)=C_0$:

$$\begin{aligned}C(0) &= \lambda^*\cdot e^0\\ \implies \lambda^* &= C_0\\ \implies C(t) &= C_0\cdot e^{-k\cdot t}\end{aligned} \tag{6.18}$$

Solve the levles for repeated doses The function $C_n(t)$ that describes drug levels, given the simplifying assumption that drug levels immediately rise following administration of the drug, is described by a sequence of seperate functions, $(C_1(t), C_2(t), C_3(t)\dots C_n(t))$ corresponding to the domain $((n-1)\cdot T, T)$ respectively.

Solve the value of C_n Following the initial dose of C_0, a subsequent dose will need to be administered after a period of time T, which correspoonds to the constant dosing schedule, at this time the blood levels will be:

$$\begin{aligned}R_1 &= C(T)\\ &= C_0\cdot e^{-k\cdot t}\end{aligned}$$

At this time, the simplifying assumption is made that the levels rise immediately to reach C_2, which is given by the initial value pluse the dose C_0 (which is also assumed constant):

$$\begin{aligned}C_1 &= C_0+R_1\\ &= C_0+C_0\cdot e^{-k\cdot t}\end{aligned} \tag{6.19}$$

Following this the levles will again decrease up until the time of the next dose, after a period of $t=T$, but this time they will fall from an initial value of C_1:

$$\begin{aligned}R_2 &= C_1\cdot e^{-kT}\\ &= \left(C_0+C_0\cdot e^{-kT}\right)\cdot e^{-kT}\\ &= C_0e^{-kT}+C_0\cdot e^{-2kT}\end{aligned} \tag{6.20}$$

following the preceeding logic:

$$\begin{aligned}C_2 &= R_2+C_0\\ &= C_0+C_0\cdot e^{-kT}+C_0\cdot e^{-2kT}\end{aligned} \tag{6.21}$$

now by the geometric series we have $\sum_{i=0}^{n-1}[r^n] = \frac{1-r^n}{1-r}$ so:

$$\begin{aligned}R_3 &= C_2\cdot e^{-kT}\\ &= \left(C_0poC_0\cdot e^{-kT}+C_0\cdot e^{-2kT}\right)\cdot e^{-kT}\\ &= c)e^{-kT}+C_0e^{-2kT}+C_0e^{-3kT}\\ \dots\\ R_n &= C_0\cdot\sum_{i=1}^{n}\left[\left(e^{-kT}\right)^i\right]\\ &= \frac{C_0\cdot e^{-kT}\cdot\left(1-e^{-kTn}\right)}{(1-e^{-kT})}\end{aligned} \tag{6.22}$$

The long term behaviour of the concentration levels will be:

$$\begin{aligned}
R = R_\infty &= \lim_{n\to\infty} [R_n] \\
&= \lim_{n\to\infty} \left[\frac{C_0 - e^{-kTn}}{(e^{-kT}-1)}\right] \\
&= \frac{C_0 - \lim_{n\to\infty}\left[e^{-kTn}\right]}{(e^{-kT}-1)} \\
&= \frac{C_0 - 0}{(e^{-kT}-1)} \\
&= \frac{C_0}{(e^{-kT}-1)}
\end{aligned} \tag{6.23}$$

The concentration level is hence given by:

$$\begin{aligned}
C_n &= C_0 + R_n \\
&= c_0 + 1 + \frac{1 - e^{-kTn}}{(e^{-kT}-1)} \\
c_\infty &= \lim_{n\to\infty} [C_n(t)] \\
&= C_0 + R_\infty \\
&= c_0\left(1 + \frac{1}{(e^{-kT}-1)}\right)
\end{aligned} \tag{6.24}$$

Substitute the Dose Schedule

From above we have that the dose schedule is:

$$T = \ln(r) \cdot \frac{1}{k}$$

hence by substitution:

$$\begin{aligned}
10 &= 100 \cdot \ln(r) \\
r &= e^{\frac{1}{10}}
\end{aligned}$$

Hence we may conclude:

$$H = e^{\frac{1}{10}} \tag{6.25}$$

Now in order to solve n :

$$\begin{aligned}
r_n &> 0.5 \cdot R \\
\frac{C_0 \cdot e^{-kT}\left(1 - e^{-kTn}\right)}{1 - e^{-kt}} &> \frac{C_0}{2\left(e^{kt}-1\right)}
\end{aligned}$$

multiply the RHS by $\frac{e^{-kt}}{e^{-kt}}$

$$\frac{C_0 \cdot e^{-kT}\left(1 - e^{-kTn}\right)}{1 - e^{-kt}} > \frac{C_0 \cdot e^{-kt}}{2\left(1 - e^{-kt}\right)} \tag{6.26}$$

because $\left(1 - e^{-kt}\right) > 1$:

$$\begin{aligned}
C_0 \cdot e^{-kT}\left(1 - e^{-kTn}\right) &> C_0 \cdot e^{-kT} \cdot \frac{1}{2} \\
1 - e^{-kTn} &> \frac{1}{2} \\
\frac{1}{2} &> e^{-kTn} \\
-\ln(2) &> -kTn \\
\ln(2) &< kTn \\
\frac{\ln(2)}{kT} &< n \\
n &> \frac{\ln(2)}{kT} \\
n &> 10 \times \ln(2) \\
n &> 6.9
\end{aligned} \tag{6.27}$$

$\therefore n = 7$ is the minimum value of n that satisfies that condition.

Suppose that $k = 0.2$ hr^{-1} and that the smallest concetration is 0.03 mg/ml. A signle dose that produces a concentration of 0.1 mg/ml is administered. Approximately how many hours will the drug remain effective?

From the previous working we have:

$$\frac{\mathrm{d}\,C}{\mathrm{d}\,t} = C(t) \implies C(t) = C_0 \cdot e^{k\cdot t}$$

and the question provides:

$$\begin{aligned}
C_0 &= 0.1 \\
L &= 0.03
\end{aligned}$$

Now substitute the values and find t :

$$\begin{aligned}
L &= C(t) \\
&= C_0 \cdot e^{k\cdot t} \\
kt &= \ln\left(\frac{C_0}{L}\right) \\
t &= \frac{1}{k} \cdot \ln\left(\frac{C_0}{L}\right) \\
&= 5 \cdot \ln\left(\frac{10}{3}\right) \\
&= 6 \text{ hours, } 1 \text{ min}
\end{aligned}$$

$\therefore$ the dosage will be effective for only six hours

A patient is given a dosage of Q of a drug at regular intervals of time T. The concentration of the drug in the blood behaves differently in this scenario, and it has been found that the concentration level C is given by:

$$\frac{\mathrm{d}\,C}{\mathrm{d}\,t} = -ke^{C}, \qquad \exists k \in \mathbb{R}^+ \tag{6.28}$$

(a) Solve the First Residual

The first dose is administered at $t = 0$, after T hours the residual in the blood is:

$$R_1 = -\ln\left(kT + e^{-Q}\right) \tag{6.29}$$

In order to show this consider that The residual corresponds to $t = T$ and solve $C(t)$:

$$\begin{aligned}
e^{-C} \cdot \frac{\mathrm{d}\,C}{\mathrm{d}\,t} &= -k \\
\int e^{-C} dC \,\mathrm{d}\,C &= -kt + A_1, \qquad \exists A_i \in \mathbb{R}, \forall iin\mathbb{Z}^+ \\
-e^{-C} &= -kt + A_1 \\
e^{-C} &= kt + A_2 \\
-C &= \ln(kt + A_2) \\
C &= -\ln(kt + A_2)
\end{aligned} \tag{6.30}$$

Consider the initial condition:

$$\begin{aligned}
C(0) = Q &= -\ln(kt + A_2) \\
-Q &= \ln(A_2) \\
A_2 &= e^{-Q}
\end{aligned}$$

hence, given this model, we may conclude:

$$\begin{aligned}
C(t) &= -\ln(kt + C_2) \\
&= -\ln\left(kt + e^{-Q}\right)
\end{aligned}$$

Now that we have solved the exponential model corresponding to the time following the initial dose and before the subsequent dose, i.e. $t \in [0, T]$, in order to solve the first residual:

$$\begin{aligned}
R_1 &= C(T) \\
&= -\ln\left(kT + e^{-Q}\right)
\end{aligned}$$

(b) Solve the Second Residual Assume an instantanous rise in concentration whenever the drug is administrated, show that after the second dose and another T hours have elapsed that the residual concetration in the blood will be given by:

$$R_2 = -\ln\left[kT\left(1+e^{-Q}\right)+e^{-2Q}\right] \tag{6.31}$$

The first thing to be mindful of here is that, owing to the subsequent readministration of the drug and the simplifying assumption that the readministration will lead to an instantaneous rise in blood concentration, the blood concentration will be described by a sequence of function $(C_i(t))$ corresponding to the domain $(t \in [(i-1)\cdot T, i\cdot T])$ wherein $\frac{\mathrm{d}}{\mathrm{d}\,t}(C_i) = -k\cdot e^C$ and the constant dose is the initial dose C_0.

As described above, $(C_i(t))$ is a sequence of functoins describing the blood level over the applicable domain, while $(R_i(t))$ is a sequence of values describing the blood level at remaining in the blood immediately preceeding the upcoming dose.

Following a subsequent dose, the blood levels will rise to C_1 :

$$C_1 = R_1 + Q \tag{6.32}$$
$$= Q - \ln\left(kT + e^{-Q}\right) \tag{6.33}$$

$R_2 = C_1(T)$ will be such that $\frac{\mathrm{d}}{\mathrm{d}\,t}(C_i(t)) \propto C_i, \ \ \forall i \in \mathbb{Z}^+$ with the initial condition now being that $C_1(0) = R_1 + Q$, this change in initial condition being how the seqence of functions evolves over iteration.

$$\begin{aligned} C_1(0) = C_1 &= R_1 + Q \\ &= Q - \ln\left(kT + e^{-Q}\right) \end{aligned}$$

Now we have from earlier in (6.30) that:

$$\frac{\mathrm{d}}{\mathrm{d}\,t}(C_i(t)) \implies C_i(t) = -\ln(kt + A_2) \tag{6.34}$$
$$\exists A_i \in \mathbb{R}, \forall \ i \in \mathbb{Z}^+$$

so by applying this new initial condition:

$$\begin{aligned} C_1(0) = C_1 &= R_1 + Q \\ &\implies -\ln(k\times 0 + A_2) = R_1 + Q \\ &\implies A_2 = -e^{R_1}\cdot e^Q \end{aligned} \tag{6.35}$$

Now by substituting the value from above (6.35) into the solution from (6.30) we have:

$$C_1(t) = -\ln\left(kt + e^{R_1}\cdot e^Q\right) \tag{6.36}$$

Find R_2 The residual R_2 will correspond to $C_1(T)$:
I may have made a mistake with negatives here

$$\begin{aligned} R_2 = C_1(T) &= -\ln\left(kTe^{R_1}\cdot e^Q\right) \\ &= -\ln\left(kt + e^{-Q}\left(kT + e^{-Q}\right)\right) \\ &= -\ln\left(kt\left(1+e^{-Q}\right)+e^{-2Q}\right) \end{aligned} \tag{6.37}$$

(c) Solve the Residual Limit Given that the residual in the bloodstreamafter nT hours is given by

$$R_n = -\ln\left[kT\left(1+e^{-Q}+e^{-2Q}+\cdots+e^{-(n-1)Q}\right)+e^{-nQ}\right]$$

Show that the limiting value R of the residual concentration for doses of Q mg/ml repeated at intervals T hours is given by the formula

$$R = -\ln\left(\frac{kT}{1-e^{-Q}}\right)\text{mg/ml}$$

Apply the Geometric Series by continuing this pattern we would find that:

$$\begin{aligned} R_n &= -\ln\left(kT\cdot\sum_{i=0}^{n-1}\left[\left(e^{-Q}\right)^n\right]+e^{-nQ}\right) \\ &= -\ln\left(kT\cdot\frac{1-e^{-Qn}}{1-e^{-Q}}+e^{-nQ}\right) \\ \lim_{n\to\infty}[R_n] &= \lim_{n\to\infty}\left[-\ln\left(kT\cdot\frac{1-e^{-Qn}}{1-e^{-Q}}+e^{-nQ}\right)\right] \end{aligned} \tag{6.38}$$

by the definition of continuity

$$\begin{aligned} \lim_{n\to\infty}[R_n] &= -\ln\left(kT\cdot\lim_{n\to\infty}\left[\frac{1-e^{-Qn}}{1-e^{-Q}}+e^{-nQ}\right]\right) \\ &= -\ln\left(kT\cdot\frac{1-0}{1-e^{-Q}}+0\right) \\ &= -\ln\left(kT\cdot\frac{1}{1-e^{-Q}}\right) \end{aligned} \tag{6.39}$$

(d) Solve the Dose Period T Assuming the drugis ineffective below a concentration of L mg/ml and can be harmful above concentrations of H mg/ml. Show that the dose schedule T for a safe and effective concentration of the drug in the bloodsatisfies the formula

$$T = \frac{1}{k}\left(e^{-L}-e^{-H}\right) \tag{6.40}$$

If the drug is effective only above L and inneffective below ht we would want

$$T = C_n - R_n \quad \exists n \in \mathbb{Z}^+ : \qquad C_n \leq H \ \wedge \ R_n \geq L \tag{6.41}$$

Setting C_0 as H would not be appropriate, instead:

$$C_0 = H - L$$

That if the dose is taken when it is not effective at L way the blood levels will i'mediately reach the threshold avoiding an overdose and maximising the dosage schedule

$$C_0 = H - L \implies C_n :$$
$$c_n = -\ln\left(kt + e^{L-H}\right) \tag{6.42}$$

and we would hence have $R_1 = L$:

$$\begin{aligned} R_\infty &= -\ln\left(kT\times\frac{1}{1-e^{-Q}}\right) \\ R_\infty &= -\ln\left(kT\times\frac{1}{1-e^{L-H}}\right) \\ -L &= \ln\left(kT\times\frac{1}{1-e^{L-H}}\right) \\ e^{-L} &= e^{-L}\left(1-e^{L-H}\right) \\ &= e^{-L}-e^{-H} \\ T &= 1/k\left(e^{-L}-e^{-H}\right) \end{aligned}$$

$\therefore$ the dosage schedule must be exactly T.

Topic 7

(07) Modelling with Differential Equations, Wk. 11, TB. Ch. 12

7.1 Systems of Differential Equations

7.1.1 Higher Order Linear ODE

Consider the second order ODE:

$$0 = 2 \cdot \frac{\mathrm{d}^2}{\mathrm{d}\, x^2} (y(x)) - 5 \frac{\mathrm{d}}{\mathrm{d}\, x} (y(x)) - 3 \cdot y(x)$$

$$0 = 2y'' - 5y' - 3y \tag{7.1}$$

$$\tag{7.2}$$

The homogenous solution will be of the form $y = e^{mx}$ such that:

$$0 = 2m^2 - 5m - 3 \tag{7.3}$$

$$m = -\frac{1}{2},\ 3 \tag{7.4}$$

Hence the solution to the equation will be of the form:

$$y = c_1 \cdot e^{-\frac{1}{2}x} + c_2 \cdot e^{3x} \tag{7.5}$$

Now for the sake of the argument, let $C_1 = 5 + 7 \cdot i$ and $C_2 = 3 + 9 \cdot i$, in reality any value $C_i \in \mathbb{C},\ \forall i \in \mathbb{Z}^+$ will satisfy the equation.

Our conjectured solution is:

$$\begin{aligned} y &= c_1 \cdot e^{-\frac{1}{2}x} + c_2 \cdot e^{3x} \\ \implies \frac{\mathrm{d}\, y}{\mathrm{d}\, x} &= -\frac{c_1}{2} \cdot e^{-\frac{1}{2}x} + c_2 \cdot 3 \cdot e^{3x} \\ \implies \frac{\mathrm{d}^2}{\mathrm{d}\, x^2}(y) &= \frac{c_2}{4} \cdot e^{-\frac{1}{2}x} + c_2 \cdot 9 \cdot e^{3x} \end{aligned} \tag{7.6}$$

Our equation is:

$$0 = 2 \cdot \frac{\mathrm{d}^2}{\mathrm{d}\, x^2} (y(x)) - 5 \frac{\mathrm{d}}{\mathrm{d}\, x} (y(x)) - 3 \cdot y(x)$$

$$0 = 2y'' - 5y' - 3y \tag{7.7}$$

$$0 = 2\left(\frac{c_2}{4} \cdot e^{-\frac{1}{2}x} + c_2 \cdot 9 \cdot e^{3x}\right) - 5\left(-\frac{c_1}{2} \cdot e^{-\frac{1}{2}x} + c_2 \cdot 3 \cdot e^{3x}\right) - 3\left(c_1 \cdot e^{-\frac{1}{2}x} + c_2 \cdot e^{3x}\right) \tag{7.8}$$

$$0 = \frac{c_2}{2} \cdot e^{-\frac{1}{2}x} + c_2 \cdot 18 \cdot e^{3x} + \frac{5 \cdot c_1}{2} \cdot e^{-\frac{1}{2}x} - c_2 \cdot 15 \cdot e^{3x} - 3 \cdot c_1 \cdot e^{-\frac{1}{2}x} - 3 \cdot c_2 \cdot e^{3x} \tag{7.9}$$

...

$$\begin{aligned} 0 &= c_1 \times (0) \times e^{-\frac{1}{2} \cdot x} + c_2 \times (0) \times e^{3x} \\ 0 &= 0 \end{aligned} \tag{7.10}$$

Observe that there was no need to substitute in our c_1 and c_2, they were totally arbitrary, hence they must be complex values because there is no real restriction placed upon them (recall that $\mathbb{R} \subset \mathbb{C}$).

This is important because if we get complex roots to the characteristic equation, say $m = 1 \pm i = \sqrt{2}e^{\pm i \cdot \frac{\pi}{4}}$, then we need to make that assumption in order to find a real solution for y.

$$\begin{aligned} m = \alpha + \beta \cdot i \implies y &\propto e^{(\alpha \pm \beta \cdot i) \cdot x} \\ y &= e^{\alpha \cdot x}\left(c_1 e^{\beta \cdot x \cdot i} + c_2 e^{-\beta \cdot x \cdot i}\right) \\ y &= e^{\alpha \cdot x} \cdot (c_1 \cdot \cos(\beta \cdot x) + c_1 \cdot i \cdot \sin(\beta \cdot x) + c_2 \cdot \cos(-\beta \cdot \\ &= e^{\alpha \cdot x}\left[(c_1 + c_2) \cdot \cos(\beta \cdot x) + i \cdot (c_1 - c_2) \cdot \sin(\beta \cdot x)\right] \\ &= e^{\alpha \cdot x}\left[k_1 \cdot \cos(\beta \cdot x) + k_2 \cdot \sin(\beta \cdot x)\right] \quad \exists k_{1,2} \in \mathbb{C} \end{aligned} \tag{7.11}$$

See what makes this solution interesting is that if y is real then k_1 and k_2 must both be real, that however implies that c_1 and c_2 must both be purely imaginary.

At equilibrium the system will be:

$$x_n = x_{n+1} \iff \Delta x_n = 0 \tag{7.12}$$

$$y_n = y_{n+1} \iff \Delta y_n = 0 \tag{7.13}$$

so we have (for x, y at equilibrium):

$$\begin{aligned} 0 &= -\delta x_n + T \\ x_n &= \frac{T}{\delta} \\ &= \frac{25}{0.01} \\ &= 2500 \end{aligned}$$

$$\begin{aligned} 0 &= y(\beta - \delta) + \beta x - T \\ y(\delta - \beta) &= \beta x - T \\ y &= \frac{\beta x - T}{\delta - \beta} \\ &= \frac{0.07 \times 2500 - 25}{0.01 - 0.07} \\ &= -2500 \end{aligned}$$

$$\begin{aligned} x_{n+1} &= 0.99 x_n + T \\ y_{n+1} &= 0.07 x_n + 1.06 y_n - T \end{aligned}$$

Figure 7.1: Diagram of Blood Levels over time

Bibliography

[1] Dennis G Zill and Michael R Cullen. *Differential equations*. Brooks/Cole, 7 edition, 2009.

[2] Erwin Kreyszig. *Advanced engineering mathematics*. Wiley, 8 edition, 1999.

[3] Steven Horton Frank Giordano, William Fox. *A First Course in Mathematical Modelling*. Brooks/Cole, 5 edition, 2014.

Topic 8

Final Exam Revision

8.1 Discrete Systems Problem

Humpback whales were almost extinct along the east coast of Australia when whaling ceased in Australia in 1963, with only about 400 whales left. In an effort to monitor and understand humpback whales better, scientists began a tagging and tracking program. Let x_n, and y_n be, respectively denote the number of whales that have been tagged and not tagged, n years on from the beggining of 1963 (not $x_n + y_n$ is the whale population n years on from 1963).

- if β denotes the annual birth rate of humpback whales (that is, if there are x whales, the)
- Suppose for the rest of this question that the birth rate for humpback whales is $\beta = 0.07$, death rat is $\delta = 0.01$ and $T = 25$ whales are tagged each year. Determine x_0 and y_0 when whaling ceased, and determine x_1 and ky_1 one year after the tagging program began.
- Determine the equilibrium values for the system of equations, and interpret their meaning, if any.
- Reformulate the system into a matrix difference equation.
- Formulate a solution for the system of difference equations.
- The change in the tagged whales will be the number of tagged whales that perish less the new whales that are tagged:

  $$\Delta x_n = -\delta \cdot x_n + T$$

  The change in untagged whales will be the number of newborn whales (from either tagged or untagged whales) less those that are tagged less the untagged whales that die:

  $$\begin{aligned} \Delta y_n &= \beta \left(x_n + y_n\right) - \delta \cdot y_n - T \\ &= \beta \cdot x_n + \beta y_n - \delta y_n - T \\ &= \beta x_n + y_n \left(\beta - \delta\right) - T \end{aligned}$$

  Presuming that the rate of population change is not affected by tagging.
- Given these constant values the rate of change of the population will be:

  $$\Delta x_n = -0.01 \cdot \delta x_n + T \tag{8.1}$$
  $$\Delta y_n = 0.07 \cdot x_n + 0.06 y_n - T \tag{8.2}$$
  $$\implies \begin{bmatrix} \Delta x_n \\ \Delta y_n \end{bmatrix} = \begin{bmatrix} -0.01 & 0 \\ 0.07 & 0.06 \end{bmatrix} \cdot \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} 25 \\ -25 \end{bmatrix} \tag{8.3}$$

  let :

  - $A = \begin{bmatrix} -0.01 & 0 \\ 0.07 & 0.06 \end{bmatrix}$
  - $B = \begin{bmatrix} 25 \\ -25 \end{bmatrix}$
  - $X = \begin{bmatrix} \Delta x_n \\ \Delta y_n \end{bmatrix}$

  So we have:

  $$P_{n+1} = A \cdot P_n + B$$

  The initial number of whales, following cessation of whaling is 400:

  $$x_n + y_n = 400$$

  we would expect initially no whales to be tagged:

  $$x_0 = 0; y_0 = 400 \implies p_0 = \begin{bmatrix} 0 \\ 400 \end{bmatrix}$$

  So p_1 may be determined:

  $$P_1 = A \cdot p_0 + B = -0.01$$

  wait, no, hold on:

  $$\begin{aligned} a_{n+1} - x_n &= -0.01 \delta x_n + T \\ x_{n+1} &= 0.99 x_n + 0 \cdot y_n + T \\ y_{n+1} - y_n &= 0.07 \cdot x_n + 0.06 \cdot y_n - T \\ y_{n+1} &= 0.07 \cdot x_n + 1.06 \cdot y_n - T \end{aligned}$$

8.1.1 Solutions