TOPIC #1 SUMMARY

ANALYSIS 200023 | TB: *INTRODUCTION TO REAL ANALYSIS* BARTLE AND SHERBERT

**The textbook does not cover Complex Analysis*

CONTENTS

1. SETS AND FUNCTIONS

Week 1 Material, Due Thur. 7 March,

Ch. [1.1], [1.2], [1.3],[1.4]

SETS [1.1]

P. 2 [1.1] OF TB

A set is a collection of elements, if an element is in that set it is written:

A set can contain anything, and

- Is **Unordered**
  - A set is unordered in the sense that two sets with the same elements in a different order are the same
- **Repetition** does not matter
  - Sets can only contain one copy of the same item, e.g.

$$
\begin{aligned}
A &= \{1, 3, 8, 9\} \\
&= \{1, 9, 3, 8\} \\
&= \{8, 3, 9, 8, 1\} \\
&\neq \{9, 1, \{3, 8\}, 1, 3\} \\
&\neq \{9, 1, \{3, 8\}, 1\}
\end{aligned}
$$

If an element is contained or not contained within a set it is expressed:

x* is an element of *A (\in)	***x* is not an element of *A*** (\notin)
$x \in A$	$x \notin A$

SET OPERATIONS

SUBSET NOTATION[1]

If A is a set for which some or all elements are contained by B it is written”

$$A \subseteq B \iff B \supseteq A$$

Observe the following definitions:

Subset Symbol	$A \subset B \iff B \supset A$	A contains some elements of B
Proper Subset Symbol	$P \subseteq Q \iff P \supseteq P$	P Contains some or all elements of Q
Subset, but not equal	$M \subsetneq N \iff N \supsetneq N$	M contains some elements of N, but not all elements

Consider also, the algebraic structure of these sets

Set	***Algebraic Structure***	***Justification***
$\mathbb{N}, \mathbb{Z}^+$	Not a ring Technically a semiring	∵ The set is not closed under subtraction Which is logically equivalent to saying there is no additive inverse
$\mathbb{Z}$	Integral Domain	∵ It is a ring that is commutative, with unity and no zero Divisors
$\mathbb{Q}, \mathbb{R}, \mathbb{C}$	Fields	∵ They are integral Domains in which every non-zero element is a unity

[1] Mathsisfun.com. (2018). *Set Symbols*. [online] Available at: https://www.mathsisfun.com/sets/symbols.html [Accessed 7 Mar. 2018].

An element in some ring is:

- A **Unit** if there exists a multiplicative inverse for that element within that ring
- A **Zero Divisor** if there exists another element within that ring that will muyltiply to give zero, e.g.:
    - $2 \cdot 3 = 6 = 0 \in [\mathbb{Z}]_6$

SET OPERATIONS

Operation	Definition	Description
Union	$A \cup B \coloneqq \{x \ : \ x \in A \ \vee \ x \in B\}$	Take the elements of both Sets (i.e. a set containing anything that's in either A or B)
Intersect	$A \ \cap B \coloneqq \{x \ : \ x \in A \ \wedge x \in B\}$	Take the elements that both sets have in common (i.e. a set containing anything that is in both A and B)
Compliment	$A \ \setminus \ B := \{x \ : \ x \in A \ \wedge x \notin B\}$	So basically everything that's in A but take-away anything in B (i.e. a set that contains everything in A, not in BJ)

DE MORGAN'S LAW

De Morgan's law is a law in set theory that exemplifies set Operations. The law states:

$$A \ \setminus (B \cup C) = (A \ \ B) \cap (A \ C)$$

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$$

There is a proof for this in *Goodnotes/Notability*

FUNCTIONS [1.1]

CARTESIAN PRODUCT

Before introducing functions, consider the cartesian product, the cartesian product combines elements of a set, so for example

if:

$A \subset \mathbb{Z} : A = \{1, 3, 4\}$

$B \subset \mathbb{Z}: B = \{1,0\}$

Then:

$A \times B = \{ (1,1), (1,3), (1,4), (0,1), (0,3), (0,4) \}$

CARTESIAN PLANE

So for example the cartesian plane would be the cartesian product of two real sets of numbers:

$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

And this starts to demonstrate the somewhat deeper connection between set theory, abstract algebra, discrete mathematics and continuous calculus.

DEFINITION OF A FUNCTION

A function f from a set A into some set B:

$$f : A \to B$$

Is a 'rule of correspondence',

That maps from, every element in A, to some other element in B:

$$(\forall x \in A)(!\exists x \in B)$$

The issue with this definition is that it doesn't clearly establish what is meant by a rule of correspondence, in order to give more meaning to that set theory is used.

PRECISE DEFINITION OF A FUNCTION WITH SETS

Define a function f as a set of ordered pairs.

In Discrete Mathematics a set of ordered pairs is typically known as a graph

This might seem arbitrary, but the whole point is to define very unambiguously what a function is before we start trying to play with the mathematics of functions and calculus.

So to be clear a function is really a set of all the possible combinations of input/output, formally this is expressed:

Some function f from A to B:

$f: A \to B$

Is defined by a set $f \subsetneq (A \times B)$, such that:

1. $\forall a \in A,\ \exists b \in B\ :\ (a, b) \in f$
    a. So basically every input value $a \in A$ is assigned to some output
2. $(a, b) \in f\ \wedge (\ a, q) \in f\ \implies\ b = q$
    a. So This says an input can only have one corresponding output value
        i. Careful though, multiple input values may share a common output
            1. E.g. $f(x) = 3$ is a function but there is no function such that $f(3) = x\ ,\ \forall x \in \mathbb{R}$
        ii. This is essentially the vertical Line test

This can be encompassed in a single statement:

$$\forall a \in A, !\, \exists b \in B\ :\ (a, b) \in f$$

Here the ! character means that there is a unique value, so this statement would be read as:

For any given value in $a \in A$ there is a single unique value $b \in B$ such that the ordered pair $(a, b) \in f$

DOMAIN AND RANGE

DOMAIN

The domain is the set of allowable input values,

The notation $D(f)$ is used to express the domain set

Generally a function is given and the domain restrited after the fact

e.g. $f(x) = \frac{1}{x}$ has a domain $D(f) = \{x \ : \ x \in \mathbb{R}, x \neq 0\}$

CODOMAIN

The Codomain is the output set , it is a broad set that is either a superset or equal to the range set

e.g. the codomain of $f(x)$ above would be all $\mathbb{R}$

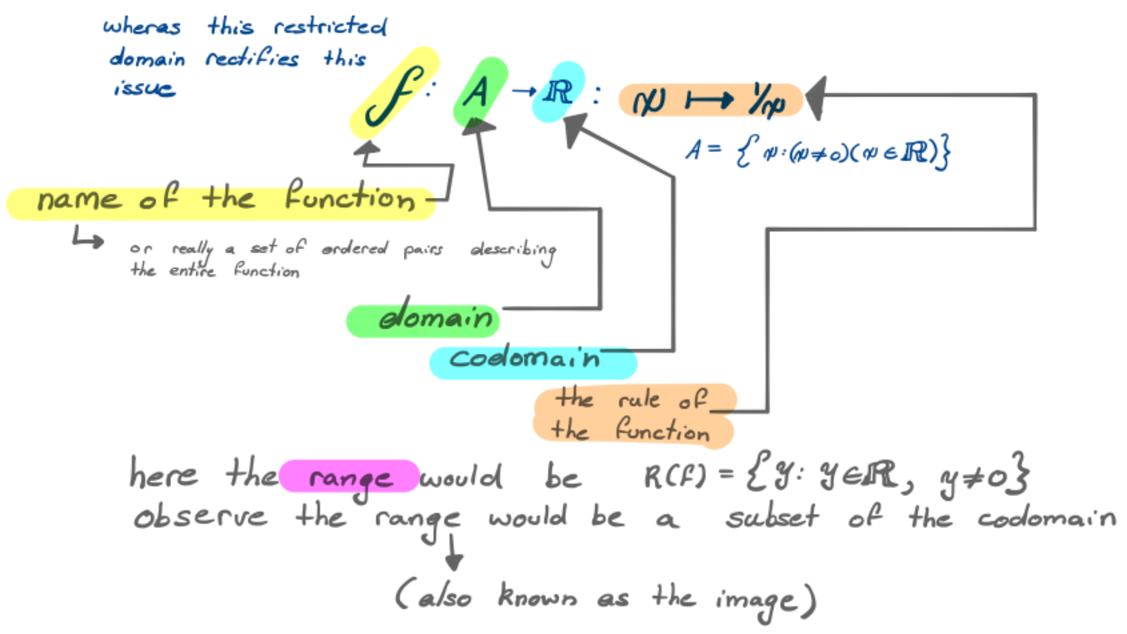
The codomain would represent the entire y-axis whereas the Range would represent the set of all y-values that the function crosses (or is defined for)

RANGE / IMAGE

The range (also known as the image of a set) is the set of all possible output values of a function, given the input,

The notation $R(f)$ is used to express the set of range values

e.g. $f(x) = \frac{1}{x}$ has a range $R(f) = \{y : y \in \mathbb{R}\,, y \neq 0\}$



DIRECT AND INVERSE IMAGES

Take some arbitrary function $f\colon A \to B$, where:

$E \subseteq A$ and $H \subseteq R(f) \subseteq B$

The '***Direct Image*** of E under f' is essentially the range of the function that corresponds only to input values from E

In other words, the direct image of E under f is the range of the function if it was such that E was the domain:

$$f(E) = \{f(x) : x \in E\}$$

A simple analogy is that it is the function applied to the set E

The '***Inverse Image'*** of E under f is the set of input values corresponding to the output values of H

In other words, the direct image of H under f is the domain of the function if it was such that H was the Range:

$$f^{-1}(E) = \{ x \in D(f) : f(x) \in H\}$$

A simple analogy is that it is the function applied to the set E

Observe that in the inverse image, only x values that are in the domain will be returned, hence the inverse image will be defined if E is a subset of the codomain rather than the more restrictive requirement that E be a subset of the range.

INJECTIONS, SURJECTIONS AND BIJECTIONS

Injections, surjections and bijections are best illustrated by comparing them to the two necessary properties of functions:

SURJECTIONS

A surjection or surjective function is also known as an onto function,

- A **function** must have a corresponding output for all elements within the domain,
  - $\forall x \in A, \exists y \in B \ : f(x) = y$
    - In terms of the function as a set
      - $\forall a \in A \,, \exists b \in B \ : \ (a, b) \in f$
- A **Surjection** must have a corresponding input for any given element within the codomain
  - $\forall y \in B, \exists x \in A \ : (a, b) \in f$
    - In terms of the function as a set:
      - $\forall b \in B, \exists a \in A \ : (a, b) \in f$

If a function is a surjection, the codomain must also be equal to the range because the range cannot contain any extra elements (i.e. $B = R(f)$)

INJECTIONS

An injection or injective function is also known as a on-to-one function.

- A **function** must have only one output value corresponding to an input value:
  - $f(x) = f(a) \ \Rightarrow \ x = a$
    - In terms of the function as a set:
      - $(a, b) \in f \land (a, q) \in f \Longrightarrow b = q$
- An **injection** must *also* have only one input value corresponding to any given output value.
  - $x = a \ \Longrightarrow f(x) = f(a)$
    - Hence an injection is such that:
      - $f(x) = f(a) \Leftrightarrow x = a$
    - In terms of the function as a set:
      - $(a, b) \in f \land (p, b) \in f \Longrightarrow a = p$

BIJECTIONS

A bijejction or bijective function is a function that is both injective and surjective, the whole point is that the inverse process of the function satisfies the requirements of a function,

$$\forall x \in A, !\exists y \in B: f(x) = y$$

In terms of the function as a set:

$$\forall a \in A, !\exists b \in B: (a, b) \in f$$

MATHEMATICAL INDUCTION [1.2]

WELL-ORDERING PRINCPILE (WOP)

The *Well-Ordering Principle* is an axiom that states:

If a set contains:

1. Only Natural Numbers, and
2. That sset is non-empty

Then one of those elements within the set must be the smallest.

WOP AND THE INTEGERS

The *WOP* doesn't necessarily apply to other sets such as the integers, for example:

$$\mathbb{Z} = \{-\infty, \dots, -2, -1, 0, 1, 2, 3, \dots \infty\}$$

Take any value $g \in \mathbb{Z}$, observe that:

$g \in \mathbb{Z} \Longrightarrow \mathbb{Z}$

This necessarily implies that the set of integers cannot have a least element.

MATHEMATICAL INDUCTION

Mathematical induction is a method for proving the truth of a statement or proposition, it makes these proofs a little simpler but it doesn't provide any help in finding statements/propositions to begin with.

IN PLAIN ENGLISH

Mathematical induction is a priunciple of proofs that provides:

A statement or proposition $P(n)$ is true if:

1. The first statement is true,
2. Given One Statement is true, then so is the next

If both of these conditions are satisfied, then mathematical induction provides that $P(n)$ is true for all positive whole values of n.

IN MATHEMATICAL TERMS

Let $P(n)$ be a statement about some $n \in \mathbb{N}$, this statement is true $\forall n \in \mathbb{N}$ if the following conditions are satisfied:

1. $p(k)$ is true, and;
2. $(\forall n \geq k)$, $\big(P(n) \Longrightarrow P(n+1)\big)$

Condition (1) is known typically as the *base* and (2) as the *bridge*.

> This can be formally proved by contradictions (e.g. suppose $S \neq \mathbb{N}$ hence there must be some $a \in \mathbb{N}$ such that $a \notin S$, but there isn't thus $\mathbb{N} = S$)

IF/THEN STATEMENTS

In order to prove the if/then statement of (2), first assume that is true and then demonstrate that as a consequence of that $(p(n)+1)$ is true $\forall n \geq k$.

> Also be aware that it is more that possible that (2) might be true for some or all values of even if (1) is false, there is no necessary connection between these two conditions or logical statements, the truth of one doesn't imply the truth of the other and only together do they establisht the overall truth of $P(n)$.

ESTABLISHING A RIGOUROUS FOUNDATION FOR INDUCTION

INDUCTION FOR $S \subseteq \mathbb{N}$

Let $S \subseteq \mathbb{N}$, if:

1. $1 \in S$
2. $(\forall k \in S),\ k \in S \Longrightarrow (k+1) \in S$

It must be such that S is actually the entire set $\mathbb{N}$

INDUCTION FOR ANY STATEMENT

Take the above form of induction but now change S for some arbitrary proposition $P(n), \forall n \ \in \mathbb{N}\ :$ $P(n)$ is true:

$$S = \{n \in \mathbb{N}\ :\ \ p(n) \text{ is true}\}$$

Now if S can be shown to be the set of $\mathbb{N}$ then $P(n)$ must be true $\forall n \in \mathbb{N}$

> (or that S contains all $\mathbb{N}$ for values greater than some base n_0)

Hence, $p(n)$ is true $\forall n \in \mathbb{N}\ :\ \ n \geq n_0$ if the following conditions are satisfied:

1. $n_0 \in S$
   a. $\because$ this implies $P(n_0)$ is true
2. $(\forall k \geq n_0)\ ,\ k \in S \Longrightarrow (k+1\ ) \in S$
   a. $\because$ this implies if $P(n)$ is true $P(n+1)$ must be true.

PROPERTIES OF $\mathbb{R}$ [2.1]

The set of real numbers is a field, in order to understand fields it is necessary to understand rings.

THE RING

Typically a ring is introduced by way of a set congruence classes:

$$\mathbb{Z}_n = \{[0]_n, [1]_n, [2]_n, \dots [n-1]_n\}$$

So, for example:

$$\mathbb{Z}_3 = \{[0]_3, [1]_3, [2]_3\}$$

> Where, for examples

$$\begin{aligned} [2]_3 &= \{\dots -1, 2, 5, 8, 1, \dots\} \\ &= \{x : x = 3n+2, \forall n \in \mathbb{Z}\} \end{aligned}$$

More formally however a ring is any structure that satisfies the ring axioms:

DEFINITION OF A RING

A ring is some set R for which two operations are defined:

- One that is associated with Addition
  - For which the following symbol is used (+)
- One that is associated with Multiplication (*)
  - Ocassionaly $(\cdot)$ is used as well

And the following axioms are satisfied:

And satisfies the axioms of a ring:

1. ***Associativity of Addition***
   a) $(\forall a, b, c \in R)(a+b)+c = a+(b+c)$
2. ***Commutativity of Addition***
   a) $(\forall a, b \in R) \;\; a+b = b+a$
3. ***Additive Elements Exists***
   a) $(\forall a, b \in R) \wedge (\exists 0 \in R) \;\; a+0 = 0+a = 0$
4. ***Associativity of Addition***
   a) $(\forall a, b, c \in R)(a+b)+c = a+(b+c)$
5. ***Commutativity of Addition***
   a) $(\forall a, b \in R) \;\; a+b = b+a$
6. ***Additive Elements Exists***
   a) $(\forall a, b \in R) \wedge (\exists 0 \in R) \;\; a+0 = 0+a = 0$

These further axioms are not necessary for a structure to be considered a ring, however if they are satisfied a ring can be further classified:

7. **Commutativity of Multiplication***
   1. $(\forall a, b \in R) a \cdot b = b \cdot a$
      1. A ring that satisfies this property is called a **commutative Ring**
8. **Existence of a Multiplicative identity Element (a ring with Unity)**
   1. $(\exists 1 \in R)(\forall a \in R) \;\; 1 \cdot a = a \cdot 1 = a$
      1. A ring that satisfies this property is called a **a ring with identity** or equivalently a **a ring with unity**

INTEGRAL DOMAIN AND FIELDS

An Integral domain is a ring that is:

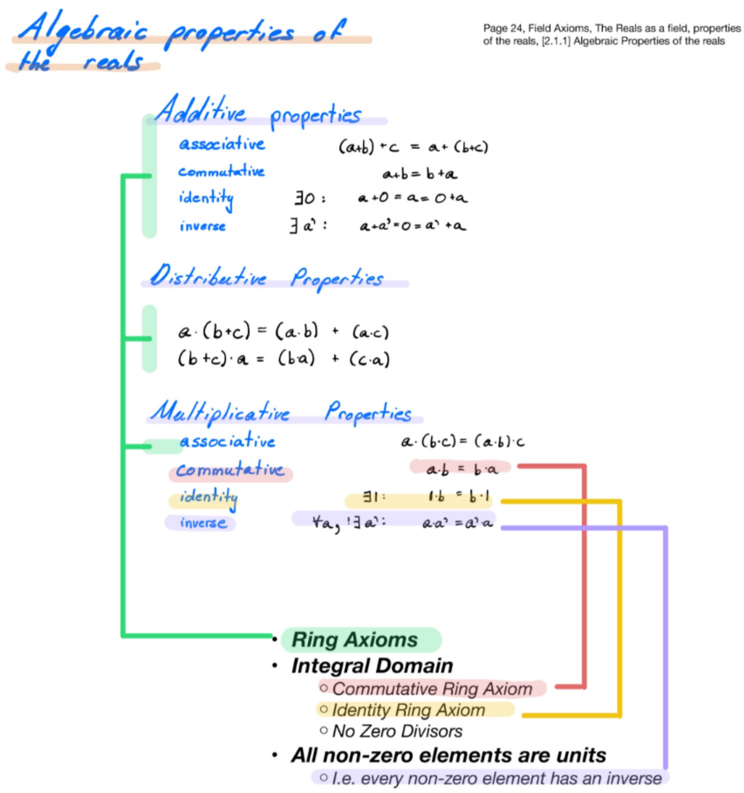
- Commutative (under multiplication)
- Is with unity/identity
- Has no Zero Divisors

A Field is an Integral Domain that is:

- Every non-zero element is a unit
  - i.e. has a multiplicative inverse.

ALGEBRAIC PROPERTIES

The textbook goes various properties of the real numbers that more or less repeat these axioms/properties:



THE ORDER PROPERTIES OF THE REALS

THE POSITIVE NUMBERS

The set of positive real numbers is a subset of the reals, denoted $\mathbb{P} \subset \mathbb{R}$.

The definition of a positive number is:

1. $a, b \in \mathbb{P} \Longrightarrow (a + b) \in \mathbb{P}$
2. $a, b \in \mathbb{P} \Longrightarrow (a \times b\ ) \in \mathbb{P}$
3. $a \in \mathbb{P} \Longrightarrow (a \in \mathbb{P}) \vee (\ a = 0\ ) \vee (-a \in \mathbb{P})$
    a. This is called the trichotomy principle, basically it says:
        i. By the definition of a positive number, a number must be negative, positive or 0
            1. Pay attention to semantics here, the positive numbers are defined such that they in turn define the negative numbers and that positive numbers only exist if the trichotomy principle is satisfied
                a. 0 is guaranteed by the 6th ring axiom above.

LESS THAN OR EQUAL TO

This definition of positive numbers can be used to define the relation of less than/greater than.

LESS THAN / GREATER THAN

$$\forall a, b \in \mathbb{R}, \;\; a - b \in \mathbb{P} \Longrightarrow (a > b) \;\wedge\; (b > a)$$

OR EQUAL TO

$$\forall a, b \in \mathbb{R}, \;\; (a - b) \in \mathbb{P} \cup \{0\} \Longrightarrow (a > b) \;\wedge\; (b > a)$$

So now we have a formal definition for greater than or less than, the trichotomy principle can be applied here to guarantee that:

$$\forall a, b \in \mathbb{R} \Longrightarrow (a > b) \vee (a = 0) \vee (a < b)$$

THE NATURAL NUMBERS

The set of natural numbers is typically:

$$\{1, 2, 3, 4, 5, \dots\}$$

However there is no clear definition on whether or not to include 0 (even though in this unit we elect not to).[2]

Hence use the following notation to avoid ambiguity:

$\mathbb{Z}$	Integers	$\{\dots, -2, -1, 0, 1, 2 \dots\}$
$\mathbb{Z}^+$	Positive Integers	$\{1, 2, 3, \dots\}$
$\mathbb{Z}^-$	Negative Integers	$\{\dots, -3, -2, -1\}$
$\mathbb{Z}^*$	Non-Negative Integers	$\{0, 1, 2, 3, 4, \dots\}$

NO SMALLEST REAL NUMBER

The real numbers cannot have a smallest value, because the fractions are infinitely divisible, this is fairly obvious

USING THIS AS A METHOD OF PROOF [2.1.9]

This is where we get a little bit tricky:

If $a \in \mathbb{R}$ is such that $0 \leq a < \varepsilon \;\;\forall\;\; \varepsilon > 0$

Then $a = 0$

This is a lot like the definition of limits and will be used with sets later

ABSOLUTE VALUES [2.2]

Now that we have a definition for positive numbers and hence a definition for < or > we can define the absolute value function:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

So what this says is that if x is a positive number, the absolute value just returns that value, if it's a negative number it gets multiplied by (-1) and then is hence positive, so it's not a complicated function.

GEOMETRIC INTERPRETATION

[2] http://mathworld.wolfram.com/NaturalNumber.html

The Interpretation generally of $|x|$ is as the distance from 0 on the real number line, more broadly the distance between any two points on the real number line is $|a-b|$.

USEFUL PROPERTIES

$$|ab| = |a| \cdot |b|$$

$$|a^2| = |a| \cdot |a|$$

$$c \geq 0, \qquad |a| \leq c \implies -c \leq a \leq c$$

$$-|a| \leq a \leq |a|, \qquad \forall a \in \mathbb{R}$$

THE TRIANGLE INEQUALITY

$$\forall a, by \in \mathbb{R}, |a+b| \leq |a| + |b|$$

$$\begin{aligned} &a \wedge b > 0 \implies ab > 0 \\ &\qquad \implies |a+b| = |a| + |b| \end{aligned}$$

SIMPLIFYING ABSOLUTE SET CONDITIONS BY CASE ANALYSIS

This is best exemplified by way of an example, but in essence you need to separate the absolute function into its constituent conditions

EXAMPLE FROM TEXTBOOK

Take some set $B = \{ x \in \mathbb{R} : |x-1| < |x| \}$

in order to solve this it is necessary to consider all cases, that is, all values of x that may have lead to a different output value.

recall the definition of $|x| = \begin{cases} x, & x > 0 \\ -x, & -x < 0 \\ 0, & x = 0 \end{cases}$

hence all cases that must be considered are

$(x-1) < 0$
$(x < 1)$

- $|x| < 0 \implies x < 1$: $|x-1| < |x| \implies -(x-1) < -x$ (1)
- $|x| > 0 \implies x \in (0,1)$: $|x-1| < |x| \implies -(x-1) < x$ (2)
- $|x| = 0 \implies$ False

$(x-1) > 0$
$(x > 1)$

- $|x| < 0 \implies$ False
- $|x| > 0 \implies x > 1$: $|x-1| < |x| \implies (x-1) < x$ (3)
- $|x| = 0 \implies$ False

$(x-1) = 0$
$(x = 1)$

- $|x| < 0 \implies$ False
- $|x| > 0 \implies x = 1$: $|x-1| < |x| \implies 0 < 1$ (4)
- $|x| = 0 \implies$ False

now putting (1), (2), (3), (4) together:

$x < 1 \Rightarrow -(x-1) < -x$ ______ (1)

$\Rightarrow x-1 > x$

$0 > 1$

False, $x \not< 1$ ______ (a)

$x \in (0,1) \Rightarrow -(x-1) < x$ ______ (2)

$\Rightarrow 1-x < x$

$-2x < -1$

$2x > 1$

$x > 1/2$ ______ (b)

$x > 1 \Rightarrow (x-1) < x$ ______ (3)

$\Rightarrow 0 < 1$

True, $x > 1$ ______ (c)

$x = 1 \Rightarrow 0 < 1$ ______ (4)

True, $x = 1$ ______ (d)

Now put (a), (b), (c), (d) together

This condition is satisfied for:

$$\underset{a}{x \not< 1} \vee \underset{b}{x > 1/2} \vee \underset{c}{x > 1} \vee \underset{d}{x = 1}$$

$\Rightarrow x > 1/2 \vee x > 1 \vee x = 1$

$\Rightarrow x > 1/2$

hence,

$$B = \{x \in \mathbb{R} : |x-1| < |x|\}$$
$$= \{x \in \mathbb{R} : x > 1/2\}$$

Solving Inequalities

Problem

Solve values of x that satisfy: $\mid x-5\mid + \mid x-3 \mid \geq 12$

Solution

Recall that the definition of the absolute value function for some value y is:

$$\mid y \mid = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Hence in order, to solve the problem it is necessary to consider every case of the absolute values:

1. $(x-5 \geq 0) \quad \wedge \quad (x-3 \geq 0)$
2. $(x-5 \geq 0) \quad \wedge \quad (x-3 < 0)$
3. $(x-5 < 0) \quad \wedge \quad (x-3 \geq 0)$
4. $(x-5 < 0) \quad \wedge \quad (x-3 < 0)$

(1) Case 1 : $(x-5 \geq 0)(x-3 \geq 0)$

Consider Intervals

First see whether these two conditions are consistent:

$$(x-5 \geq 0)(x-3 \geq 0) \implies (x \geq 5)(x \geq 3) \implies (x \geq 5)$$

Consider Inequality

Now consider the inequality:

$$(x-5 \geq 0) \implies |x-5| = x-5 \tag{1}$$
$$(x-3) \implies |x-3| = x-3 \tag{2}$$
$$\tag{3}$$

hence,

$$|x-5| + |x-3| \geq 12 \implies (x-5)+(x-3) \geq 12 \tag{4}$$
$$\implies 2x-8 \geq 12 \tag{5}$$
$$\implies x-4 \geq 6 \tag{6}$$
$$\implies x \geq 10 \tag{7}$$
$$\tag{8}$$

Put the Conditions together

It follows that a solution for x given the inequality is:

$$x \geq 10 \wedge x \geq 5 \implies x \geq 10$$

$\therefore x \geq 10$ is a solution for the inequality

1

(2) Case 2 : $(x - 5 \geq 0)(x - 3 < 0)$

Consider Intervals

First see whether these two conditions are consistent:

$$
\begin{align}
&(x - 5 \geq 0)(x - 3 < 0) \tag{9}\\
&\implies (x \geq 5)(x < 0) \tag{10}\\
&\implies (x \notin \mathbb{R}) \tag{11}
\end{align}
$$

(3) Case 1 : $(x - 5 < 0)(x - 3 \geq 0)$

Consider Intervals

First see whether these two conditions are consistent:

$$(x - 5 < 0)(x - 3 \geq 0) \implies (x < 5)(x \geq 3) \implies x \in [3, 5)]$$

Consider Inequality

Now consider the inequality:

$$
\begin{align}
(x - 5 \geq 0) \implies |x - 5| &= -(x - 5) \tag{12}\\
&= 5 - x \tag{13}\\
(x - 3) \implies |x - 3| &= x - 3 \tag{14}\\
\tag{15}
\end{align}
$$

hence,

$$
\begin{align}
|x - 5| + |x - 3| \geq 12 &\implies (x - 5) + 3 - x \geq 12 \tag{16}\\
&\implies -3 \geq 12 \tag{17}\\
&\implies x \notin \mathbb{R} \tag{18}
\end{align}
$$

This is a contradiction which means the premise must be false, the premise is that there is an x value that satisfies the inequality given that $x \in [3, 5)$, hence we conclude that there is no $x \in [3, 5)$ that satisfies the inequality.

(4) Case 4 : $(x - 5 < 0)(x - 3 < 0)$

Consider Intervals

First see whether these two conditions are consistent:

$$(x - 5 < 0)(x - 3 < 0) \implies (x < 5)(x < 3) \implies (x < 3)$$

Consider Inequality

Now consider the inequality:

$$
\begin{aligned}
(x-5<0) &\implies |x-5| = -(x-5) && (19)\\
&\qquad\qquad = 5-x && (20)\\
(x-3<0) &\implies |x-3| = -(x-3) && (21)\\
&\qquad\qquad = 3-x && (22)\\
& && (23)
\end{aligned}
$$

hence,

$$
\begin{aligned}
|x-5|+|x-3| \geq 12 &\implies (5-x)+(3-x) \geq 12 && (24)\\
&\implies -2x+8 \geq 12 && (25)\\
&\implies -x+4 \geq 6 && (26)\\
&\implies x \leq -2 && (27)\\
& && (28)
\end{aligned}
$$

Put the Conditions together

It follows that a solution for x given the inequality is:

$$x < 3 \wedge x < -2 \implies x \leq -2$$

Conclusion

The possible solutions for x given the inequality are:

$$x \geq 10 \vee x \leq -2 \implies x \in p : p \notin [-2, 10]$$

[2.3] Completeness Property of the Reals

Upper and Lower Bounds

Upper Bound An upper bound is any value greater than or equal to all elements of a set, e.g. u is an upper bound of A if:

$$\forall s \in S, \exists u \in \mathbb{R} : u \geq s \tag{1}$$

Lower Bound A lower bound is any value less than or equal to all elements of a set, e.g. w is a lower bound of A if:

$$\forall s \in S, \exists w \in \mathbb{R} : w \leq s \tag{2}$$

Supremea and Infima

Supremum The suprema of a set is the smallest upper bound value of some set.
This value would be the maximum value of the set if the set had a maximum value.
Let V be the set of all upper bound values, u is a suprema iff:

$$u \leq v, \forall v \in V \tag{3}$$

Infimum The infimum of a set is the largest lower bound value of some set.
This value would be the maximum value of the set if the set had a maximum value.
Let T be the set of all upper bound values, w is a suprema iff:

$$w \leq t, \forall t \in T \tag{4}$$