50. CAUCHY INTEGRAL FORMULA

Another fundamental result will now be established.

Theorem. *Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then*

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z - z_0}. \tag{1}$$

Formula (1) is called the *Cauchy integral formula*. It tells us that if a function f is to be analytic within and on a simple closed contour C, then the values of f interior to C are completely determined by the values of f on C.

When the Cauchy integral formula is written as

$$\int_C \frac{f(z)\,dz}{z - z_0} = 2\pi i f(z_0), \tag{2}$$

it can be used to evaluate certain integrals along simple closed contours.

EXAMPLE. Let C be the positively oriented circle $|z| = 2$. Since the function

$$f(z) = \frac{z}{9 - z^2}$$

is analytic within and on C and since the point $z_0 = -i$ is interior to C, formula (2) tells us that

$$\int_C \frac{z\,dz}{(9 - z^2)(z + i)} = \int_C \frac{z/(9 - z^2)}{z - (-i)}\,dz = 2\pi i\left(\frac{-i}{10}\right) = \frac{\pi}{5}.$$

We begin the proof of the theorem by letting C_ρ denote a positively oriented circle $|z - z_0| = \rho$, where ρ is small enough that C_ρ is interior to C (see Fig. 66). Since the quotient $f(z)/(z - z_0)$ is analytic between and on the contours C_ρ and C, it follows from the principle of deformation of paths (Sec. 49) that

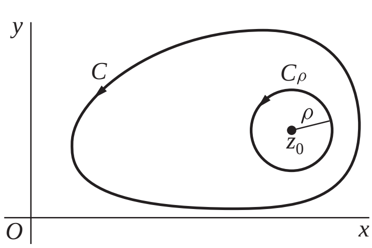

FIGURE 66

$$\int_C \frac{f(z)\,dz}{z-z_0} = \int_{C_\rho} \frac{f(z)\,dz}{z-z_0}.$$

This enables us to write

$$\int_C \frac{f(z)\,dz}{z-z_0} - f(z_0)\int_{C_\rho} \frac{dz}{z-z_0} = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz. \tag{3}$$

But [see Exercise 10*(b)*, Sec. 42]

$$\int_{C_\rho} \frac{dz}{z-z_0} = 2\pi i;$$

and so equation (3) becomes

$$\int_C \frac{f(z)\,dz}{z-z_0} - 2\pi i f(z_0) = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz. \tag{4}$$

Now the fact that f is analytic, and therefore continuous, at z_0 ensures that corresponding to each positive number ε, however small, there is a positive number δ such that

$$|f(z)-f(z_0)| < \varepsilon \quad \text{whenever} \quad |z-z_0| < \delta. \tag{5}$$

Let the radius ρ of the circle C_ρ be smaller than the number δ in the second of these inequalities. Since $|z-z_0| = \rho < \delta$ when z is on C_ρ, it follows that the *first* of inequalities (5) holds when z is such a point; and the theorem in Sec. 43, giving upper bounds for the moduli of contour integrals, tells us that

$$\left| \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz \right| < \frac{\varepsilon}{\rho}\,2\pi\rho = 2\pi\varepsilon.$$

In view of equation (4), then,

$$\left| \int_C \frac{f(z)\,dz}{z-z_0} - 2\pi i f(z_0) \right| < 2\pi\varepsilon.$$

Since the left-hand side of this inequality is a nonnegative constant that is less than an arbitrarily small positive number, it must be equal to zero. Hence equation (2) is valid, and the theorem is proved.

51. AN EXTENSION OF THE CAUCHY INTEGRAL FORMULA

The Cauchy integral formula in the theorem in Sec. 50 can be extended so as to provide an integral representation for derivatives of f at z_0. To obtain the extension, we consider a function f that is analytic everywhere inside and on a simple closed contour C, taken in the positive sense. We then write the Cauchy integral formula as

$$(1) \qquad f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{s-z},$$

where z is interior to C and where s denotes points on C. Differentiating *formally* with respect to z under the integral sign here, without rigorous justification, we find that

$$(2) \qquad f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2}.$$

To verify that $f'(z)$ exists and that expression (2) is in fact valid, we led d denote the smallest distance from z to points s on C and use expression (1) to write

$$\begin{aligned}\frac{f(z+\Delta z)-f(z)}{\Delta z} &= \frac{1}{2\pi i}\int_C \left(\frac{1}{s-z-\Delta z} - \frac{1}{s-z}\right)\frac{f(s)}{\Delta z}\,ds \\ &= \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z-\Delta z)(s-z)},\end{aligned}$$

where $0 < |\Delta z| < d$ (see Fig. 67). Evidently, then,

$$(3) \qquad \frac{f(z+\Delta z)-f(z)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2} = \frac{1}{2\pi i}\int_C \frac{\Delta z\, f(s)\,ds}{(s-z-\Delta z)(s-z)^2}.$$

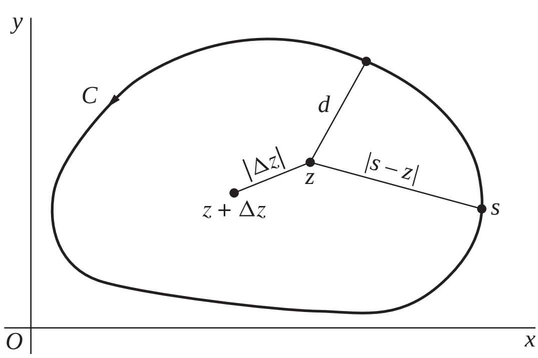

FIGURE 67

Next, we let M denote the maximum value of $|f(s)|$ on C and observe that since $|s-z| \ge d$ and $|\Delta z| < d$,

$$|s - z - \Delta z| = |(s - z) - \Delta z| \geq ||s - z| - |\Delta z|| \geq d - |\Delta z| > 0.$$

Thus

$$\left| \int_C \frac{\Delta z\, f(s)\, ds}{(s - z - \Delta z)(s - z)^2} \right| \leq \frac{|\Delta z| M}{(d - |\Delta z|) d^2} L,$$

where L is the length of C. Upon letting Δz tend to zero, we find from this inequality that the right-hand side of equation (3) also tends to zero. Consequently,

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi i} \int_C \frac{f(s)\, ds}{(s - z)^2} = 0\,;$$

and the desired expression for $f'(z)$ is established.

The same technique can be used to suggest and verify the expression

$$f''(z) = \frac{1}{\pi i} \int_C \frac{f(s)\, ds}{(s - z)^3}. \tag{4}$$

The details, which are outlined in Exercise 9, Sec. 52, are left to the reader. Mathematical induction can, moreover, be used to obtain the formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(s)\, ds}{(s - z)^{n+1}} \qquad (n = 1, 2, \ldots). \tag{5}$$

The verification is considerably more involved than for just $n = 1$ and $n = 2$, and we refer the interested reader to other texts for it.* Note that with the agreement that

$$f^{(0)}(z) = f(z) \quad \text{and} \quad 0! = 1,$$

expression (5) is also valid when $n = 0$, in which case it becomes the Cauchy integral formula (1).

When written in the form

$$\int_C \frac{f(z)\; dz}{(z - z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0) \qquad (n = 0, 1, 2, \ldots), \tag{6}$$

expressions (1) and (5) can be useful in evaluating certain integrals when f is analytic inside and on a simple closed contour C, taken in the positive sense, and z_0 is any point interior to C. It has already been illustrated in Sec. 50 when $n = 0$.

EXAMPLE 1. If C is the positively oriented unit circle $|z| = 1$ and

$$f(z) = \exp(2z),$$

*See, for example, pp. 299–301 in Vol. I of the book by Markushevich, cited in Appendix 1.

then

$$\int_C \frac{\exp(2z)\,dz}{z^4} = \int_C \frac{f(z)\,dz}{(z-0)^{3+1}} = \frac{2\pi i}{3!} f'''(0) = \frac{8\pi i}{3}.$$

EXAMPLE 2. Let z_0 be any point interior to a positively oriented simple closed contour C. When $f(z) = 1$, expression (6) shows that

$$\int_C \frac{dz}{z - z_0} = 2\pi i$$

and

$$\int_C \frac{dz}{(z - z_0)^{n+1}} = 0 \qquad (n = 1, 2, \ldots).$$

(Compare with Exercise 10*(b)*, Sec. 42.)

52. SOME CONSEQUENCES OF THE EXTENSION

We turn now to some important consequences of the extension of the Cauchy integral formula in the previous section.

Theorem 1. *If a function f is analytic at a given point, then its derivatives of all orders are analytic there too.*

To prove this remarkable theorem, we assume that a function f is analytic at a point z_0. There must, then, be a neighborhood $|z - z_0| < \varepsilon$ of z_0 throughout which f is analytic (see Sec. 24). Consequently, there is a positively oriented circle C_0, centered at z_0 and with radius $\varepsilon/2$, such that f is analytic inside and on C_0 (Fig. 68). From expression (4), Sec. 51, we know that

$$f''(z) = \frac{1}{\pi i} \int_{C_0} \frac{f(s)\,ds}{(s - z)^3}$$

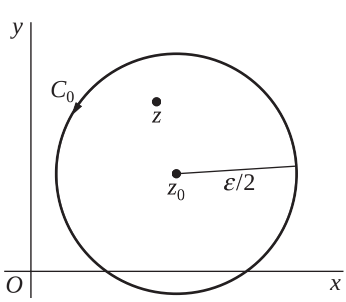

FIGURE 68

at each point z interior to C_0, and the existence of $f''(z)$ throughout the neighborhood $|z - z_0| < \varepsilon/2$ means that f' is analytic at z_0. One can apply the same argument to the analytic function f' to conclude that its derivative f'' is analytic, etc. Theorem 1 is now established.

As a consequence, when a function

$$f(z) = u(x, y) + iv(x, y)$$

is analytic at a point $z = (x, y)$, the differentiability of f' ensures the continuity of f' there (Sec. 19). Then, since (Sec. 21)

$$f'(z) = u_x + iv_x = v_y - iu_y,$$

we may conclude that the first-order partial derivatives of u and v are continuous at that point. Furthermore, since f'' is analytic and continuous at z and since

$$f''(z) = u_{xx} + iv_{xx} = v_{yx} - iu_{yx},$$

etc., we arrive at a corollary that was anticipated in Sec. 26, where harmonic functions were introduced.

Corollary. *If a function $f(z) = u(x, y) + iv(x, y)$ is analytic at a point $z = (x, y)$, then the component functions u and v have continuous partial derivatives of all orders at that point.*

The proof of the next theorem, due to E. Morera (1856–1909), depends on the fact that the derivative of an analytic function is itself analytic, as stated in Theorem 1.

Theorem 2. *Let f be continuous on a domain D. If*

$$\int_C f(z)\,dz = 0 \tag{1}$$

for every closed contour C in D, then f is analytic throughout D.

In particular, when D is *simply connected*, we have for the class of continuous functions defined on D the converse of the theorem in Sec. 48, which is the adaptation of the Cauchy–Goursat theorem to such domains.

To prove the theorem here, we observe that when its hypothesis is satisfied, the theorem in Sec. 44 ensures that f has an antiderivative in D; that is, there exists an analytic function F such that $F'(z) = f(z)$ at each point in D. Since f is the derivative of F, it then follows from Theorem 1 that f is analytic in D.

Our final theorem here will be essential in the next section.

Theorem 3. *Suppose that a function f is analytic inside and on a positively oriented circle C_R, centered at z_0 and with radius R* (Fig. 69). *If M_R denotes the maximum value of $|f(z)|$ on C_R, then*

$$|f^{(n)}(z_0)| \le \frac{n!M_R}{R^n} \quad (n = 1, 2, \ldots). \tag{2}$$

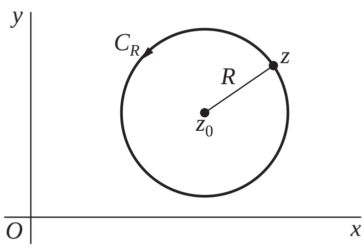

FIGURE 69

Inequality (2) is called *Cauchy's inequality* and is an immediate consequence of the expression

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_R} \frac{f(z)\,dz}{(z - z_0)^{n+1}} \quad (n = 1, 2, \ldots),$$

which is a slightly different form of equation (6), Sec. 51, when n is a positive integer. We need only apply the theorem in Sec. 43, which gives upper bounds for the moduli of the values of contour integrals, to see that

$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi}\cdot\frac{M_R}{R^{n+1}}2\pi R \quad (n = 1, 2, \ldots),$$

where M_R is as in the statement of Theorem 3. This inequality is, of course, the same as inequality (2).

EXERCISES

1. Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals:

(a) $\displaystyle\int_C \frac{e^{-z}\,dz}{z - (\pi i/2)}$; *(b)* $\displaystyle\int_C \frac{\cos z}{z(z^2 + 8)}\,dz$; *(c)* $\displaystyle\int_C \frac{z\,dz}{2z + 1}$;

(d) $\displaystyle\int_C \frac{\cosh z}{z^4}\,dz$; *(e)* $\displaystyle\int_C \frac{\tan(z/2)}{(z - x_0)^2}\,dz \quad (-2 < x_0 < 2)$.

Ans. *(a)* 2π; *(b)* $\pi i/4$; *(c)* $-\pi i/2$; *(d)* 0; *(e)* $i\pi\sec^2(x_0/2)$.

2. Find the value of the integral of $g(z)$ around the circle $|z - i| = 2$ in the positive sense when

(a) $g(z) = \dfrac{1}{z^2 + 4}$; *(b)* $g(z) = \dfrac{1}{(z^2 + 4)^2}$.

Ans. *(a)* $\pi/2$; *(b)* $\pi/16$.