1 The Clohessy Wiltshire Model

The relative motion of a chase spacecraft with respect to a target spacecraft that is in a circular orbit about a central body, considered a point mass, can be described by the CW equations of motion

$$\begin{aligned}
\ddot{x} &= 3n^2 x + 2n\dot{y} \\
\ddot{y} &= -2n\dot{x} \\
\ddot{z} &= -n^2 z
\end{aligned}$$

where the x-axis is along the radius vector of the target spacecraft, the z-axis is along the angular momentum vector of the target spacecraft, and the y-axis completes the right handed system. With this definition, the central body is towards the negative x direction and the y-axis points along the velocity vector of the target spacecraft. Motion along y-axis is considered 'along-track', and motion along the positive and negative z-axis is considered 'out-of-plane' motion.

These equations have a closed form solution given by

$$\begin{aligned}
x(t) &= (4 - 3\cos nt)x_0 + \frac{\sin nt}{n}\dot{x}_0 + \frac{2}{n}(1 - \cos nt)\dot{y}_0 \\
y(t) &= 6(\sin nt - nt)x_0 + y_0 - \frac{2}{n}(1 - \cos nt)\dot{x}_0 + \frac{4\sin nt - 3nt}{n}\dot{y}_0 \\
z(t) &= z_0 \cos nt + \frac{\dot{z}_0}{n}\sin nt
\end{aligned}$$

where

$$n = \sqrt{\frac{\mu}{a_t^3}}$$

and a_t is the semi-major axis of the target vehicle's orbit.

The velocities are then can be computed by taking the first time derivative of the position components.

If a state vector is defined as

$$\delta\mathbf{s} = \begin{pmatrix} \delta\mathbf{r} \\ \delta\mathbf{v} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$

The solution of the system is given by

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix}
4 - 3\cos nt & 0 & 0 & \frac{1}{n}\sin nt & \frac{2}{n}(1 - \cos nt) & 0 \\
6(\sin nt - nt) & 1 & 0 & -\frac{2}{n}(1 - \cos nt) & \frac{1}{n}(4\sin nt - 3nt) & 0 \\
0 & 0 & \cos nt & 0 & 0 & \frac{1}{n}\sin nt \\
3n\sin nt & 0 & 0 & \cos nt & 2\sin nt & 0 \\
-6n(1 - \cos nt) & 0 & 0 & -2\sin nt & 4\cos nt - 3 & 0 \\
0 & 0 & -n\sin nt & 0 & 0 & \cos nt
\end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \end{pmatrix}$$

The propagation matrix premultiplying the initial condition vector is also referred to as the state transition matrix for the CW equations. This matrix is labeled as $\Phi(t)$, and is a time varying. The system can be partitined as

$$\begin{pmatrix} \delta\mathbf{r}(t) \\ \delta\mathbf{v}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{M}(t) & \mathbf{N}(t) \\ \mathbf{S}(t) & \mathbf{T}(t) \end{pmatrix} \begin{pmatrix} \delta\mathbf{r}_0 \\ \delta\mathbf{v}_0 \end{pmatrix}$$

So that the position and velocities are given as

$$\begin{aligned} \delta\mathbf{r}(t) &= \mathbf{M}(t)\delta\mathbf{r}_0 + \mathbf{N}(t)\delta\mathbf{v}_0 \\ \delta\mathbf{v}(t) &= \mathbf{S}(t)\delta\mathbf{r}_0 + \mathbf{T}(t)\delta\mathbf{v}_0 \end{aligned}$$

In the text (Wiesel, Spaceflight Dynamics), the variables are related to the ones shown as

$$\begin{aligned} \psi &= nt \\ \delta r &= x \\ r_0\delta\theta &= y \\ \delta z &= z \\ \Phi_{rr} &= \mathbf{M} \\ \Phi_{rv} &= \mathbf{N} \\ \Phi_{vr} &= \mathbf{S} \\ \Phi_{vv} &= \mathbf{T} \end{aligned}$$