Exam 1 Equation Sheet

You must know the Gauss Elimination and Gauss Jordan Methods - no equations are given.

Cramer's Theorem $$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \cdots, x_n = \frac{D_n}{D}$$

Determinant using Rule of Minors $$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad \text{for } j = 1, 2, \cdots, \text{ or } n$$

You are allowed to use the Alternative Method for the determinant of a 3x3 or 2x2, but that equation is not given on this sheet.

Eigenvalues are the roots of the Characteristic Equation: $|A - \lambda \cdot I| = 0$

Eigenvectors are the solutions (x's) for this homogeneous equation, using each of the eigenvalues (λ's). $(A - \lambda \cdot I) \cdot x = 0$

Doolittle's Method for L U Decomposition

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{a_{21}}{a_{11}} & 1 & 0 \\ \frac{a_{31}}{a_{11}} & \frac{a_{32} - \frac{a_{31}}{a_{11}} a_{12}}{a_{22} - \frac{a_{21}}{a_{11}} a_{12}} & 1 \end{bmatrix}$$

$$\mathbf{Ax} = \mathbf{b}$$

$$\mathbf{A} = \mathbf{LU}$$

$$\mathbf{LUx} = \mathbf{b}$$

$$\mathbf{Ux} = \mathbf{y}$$

$$\mathbf{Ly} = \mathbf{b}$$

$$\mathbf{U} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} - \frac{a_{21}}{a_{11}} a_{12} & a_{23} - \frac{a_{21}}{a_{11}} a_{13} \\ 0 & 0 & a_{33} - \frac{a_{31}}{a_{11}} a_{13} - \frac{a_{32} - \frac{a_{31}}{a_{11}} a_{12}}{a_{22} - \frac{a_{21}}{a_{11}} a_{12}} \left(a_{23} - \frac{a_{21}}{a_{11}} a_{13} \right) \end{bmatrix}$$

Example equations for 3x3 Gauss-Jacobi

$$xnew_1 = \frac{a_{1,4} - a_{1,3} \cdot x_3 - a_{1,2} \cdot x_2}{a_{1,1}}$$

$$xnew_2 = \frac{a_{2,4} - a_{2,3} \cdot x_3 - a_{2,1} \cdot x_1}{a_{2,2}}$$

$$xnew_3 = \frac{a_{3,4} - a_{3,2} \cdot x_2 - a_{3,1} \cdot x_1}{a_{3,3}}$$

Example equations for 3x3 Gauss-Seidel

$$x_1 = \frac{a_{1,4} - a_{1,3} \cdot x_3 - a_{1,2} \cdot x_2}{a_{1,1}}$$

$$x_2 = \frac{a_{2,4} - a_{2,3} \cdot x_3 - a_{2,1} \cdot x_1}{a_{2,2}}$$

$$x_3 = \frac{a_{3,4} - a_{3,2} \cdot x_2 - a_{3,1} \cdot x_1}{a_{3,3}}$$

Gauss-Jacobi Convergence Check: Write our system of equations as $\mathbf{A^* x} = \mathbf{b}$ where $\mathbf{A^*}$ is made to have $a^*_{jj} = 1$

sufficient condition for convergence is: $\|\mathbf{C}\| < 1 \qquad \mathbf{C} = \mathbf{I} - \mathbf{A^*} \qquad \|\mathbf{C}\| = \sqrt{\sum_{j=1}^{n} \sum_{k=1}^{n} c_{jk}^2}$

Gauss-Seidel Convergence Check: Will not be covered on this exam

Condtion number for a Matrix: Will not be covered on this exam

Least Squares Curve Fitting Method - for any polynomial (including a straight line)

$$y(x) = b_0 + b_1 x + b_2 x^2$$

$$b_0 n + b_1 \sum_{j=1}^{n} x_j + b_2 \sum_{j=1}^{n} x_j^2 = \sum_{j=1}^{n} y_j$$

$$b_0 \sum_{j=1}^{n} x_j + b_1 \sum_{j=1}^{n} x_j^2 + b_2 \sum_{j=1}^{n} x_j^3 = \sum_{j=1}^{n} x_j y_j$$

$$b_0 \sum_{j=1}^{n} x_j^2 + b_1 \sum_{j=1}^{n} x_j^3 + b_2 \sum_{j=1}^{n} x_j^4 = \sum_{j=1}^{n} x_j^2 y_j$$

Exam 2 Equation Sheet

Fixed Point Method

$$x = g(x)$$

Newton - Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Secant Method

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

Lagrange Interpolation

$$p_n(x) = \sum_{k=0}^{n} L_k(x) f(x_k) = \sum_{k=0}^{n} \frac{l_k(x)}{l_k(x_k)} f(x_k)$$

where

$$\begin{aligned} l_0(x) &= (x - x_1)(x - x_2)\cdots(x - x_n) \\ l_k(x) &= (x - x_0)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n) \\ l_n(x) &= (x - x_0)(x - x_1)\cdots(x - x_{n-1}) \end{aligned}$$

Newton Divided Difference

$$\begin{aligned} p_1(x) &= a_0 + a_1(x - x_0) \\ p_2(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \\ p_3(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \\ &\quad + a_3(x - x_0)(x - x_1)(x - x_2) \\ &\cdots \\ p_n(x) &= p_{n-1} + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}) \end{aligned}$$

where

$$\begin{aligned} a_0 &= f(x_0) \\ a_1 &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\ a_2 &= \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2 - x_0} \end{aligned}$$

Cubic Spline Interpolation

$$q_j(x) = a_{j0} + a_{j1}(x - x_j) + a_{j2}(x - x_j)^2 + a_{j3}(x - x_j)^3$$
$$j = 0, 1, \ldots, n-1$$

where

$$a_{j0} = f_j$$
$$a_{j1} = k_j$$
$$a_{j2} = \frac{3}{h_j^2}(f_{j+1} - f_j) - \frac{1}{h_j}(k_{j+1} + 2k_j)$$
$$a_{j3} = \frac{2}{h_j^3}(f_j - f_{j+1}) + \frac{1}{h_j^2}(k_{j+1} + k_j)$$

$$k_{j-1} + 4k_j + k_{j+1} = \frac{3}{h}(f_{j+1} - f_{j-1}) \quad j = 1, \ldots, n-1$$

Trapezoidal Rule

$$J = \int_a^b f(x)\,dx \approx \frac{h}{2}[f(a)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(b)]$$

Simpson's Rule

$$\int_a^b f(x)\,dx \approx \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 2f_{2m-2} + 4f_{2m-1} + f_{2m})$$

Gauss Integration

$$J \approx \frac{b-a}{2}\sum_{j=1}^{n} A_j f(x_j) \qquad x_j = \tfrac{b+a}{2} + \tfrac{b-a}{2}t_j$$

n	Nodes t_j	Coefficients A_j	Degree of Precision
2	−0.5773502692 0.5773502692	1 1	3
3	−0.7745966692 0 0.7745966692	0.5555555556 0.8888888889 0.5555555556	5
4	−0.8611363116 −0.3399810436 0.3399810436 0.8611363116	0.3478548451 0.6521451549 0.6521451549 0.3478548451	7

Laplace Transforms

$$F(s) = \mathscr{L}(f) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\mathscr{L}\{e^{at} f(t)\} = F(s-a)$$
$$e^{at} f(t) = \mathscr{L}^{-1}\{F(s-a)\}$$

$$\mathscr{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$$
$$f(t-a)u(t-a) = \mathscr{L}^{-1}\{e^{-as}F(s)\}$$

$$\mathscr{L}(f') = s\mathscr{L}(f) - f(0)$$
$$\mathscr{L}(f'') = s^2\mathscr{L}(f) - sf(0) - f'(0)$$

$$\mathscr{L}(f^{(n)}) = s^n\mathscr{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

You must know how to solve a differential equation using the Laplace Transform method

You must know how to do Partial Fraction Expansion if needed

	$f(t)$	$\mathscr{L}(f)$
1	1	$1/s$
2	t	$1/s^2$
3	t^2	$2!/s^3$
4	t^n $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$
5	t^a (a positive)	$\dfrac{\Gamma(a+1)}{s^{a+1}}$
6	e^{at}	$\dfrac{1}{s-a}$

	$f(t)$	$\mathscr{L}(f)$
7	$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
8	$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
9	$\cosh at$	$\dfrac{s}{s^2-a^2}$
10	$\sinh at$	$\dfrac{a}{s^2-a^2}$
11	$e^{at}\cos \omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
12	$e^{at}\sin \omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$

$$\mathscr{L}\{u(t-a)\} = \frac{e^{-as}}{s} \qquad \mathscr{L}\{\delta(t-a)\} = e^{-as}$$

For a function with period $2L$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx \qquad n = 1, 2, \cdots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx \qquad n = 1, 2, \cdots$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \sin ax\,dx = -\frac{\cos ax}{a}$$

$$\int x\sin ax\,dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a}$$

$$\int x^2\sin ax\,dx = \frac{2x}{a^2}\sin ax + \left(\frac{2}{a^3} - \frac{x^2}{a}\right)\cos ax$$

$$\int x^3\sin ax\,dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right)\sin ax + \left(\frac{6x}{a^3} - \frac{x^3}{a}\right)\cos ax$$

$$\int \cos ax\,dx = \frac{\sin ax}{a}$$

$$\int x\cos ax\,dx = \frac{\cos ax}{a^2} + \frac{x\sin ax}{a}$$

$$\int x^2\cos ax\,dx = \frac{2x}{a^2}\cos ax + \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\sin ax$$

$$\int x^3\cos ax\,dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right)\cos ax + \left(\frac{x^3}{a} - \frac{6x}{a^3}\right)\sin ax$$

Exam 3 Equation Sheet

Euler's Method $$y_{n+1} = y_n + (t_{n+1} - t_n) f(y_n, t_n)$$

Improved Euler $$y_{n+1} = y_n + \frac{\Delta t}{2}(f(y_n, t_n) + f(y^*_{n+1}, t_{n+1}))$$

Runge-Kutta $$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = \Delta t f(y_n, t_n)$$
$$k_2 = \Delta t f(y_n + 0.5k_1, t_n + 0.5\Delta t)$$
$$k_3 = \Delta t f(y_n + 0.5k_2, t_n + 0.5\Delta t)$$
$$k_4 = \Delta t f(y_n + k_3, t_n + \Delta t)$$

Mean: $\bar{x} = \frac{1}{n} \sum_{j=1}^{n} x_j$

Variance: $s^2 = \frac{1}{n-1} \sum_{j=1}^{n} (x_j - \bar{x})^2$

Standard deviation: $s = \sqrt{\frac{1}{n-1} \sum_{j=1}^{n} (x_j - \bar{x})^2}$

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx \qquad \text{Continuous distribution} \qquad \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

$$P(A) = \frac{\text{Number of points in } A}{\text{Number of points in } S}$$

The probability of two mutually exclusive ($A \cap B = \emptyset$) is:

$$P(A \cup B) = P(A) + P(B)$$

The probability of the complement of A is:

$$P(A^C) = 1 - P(A)$$

For two events that are not mutually exclusive:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

For independent events:

$$P(A \cap B) = P(A)P(B)$$

$$\mathrm{Perm}(n) = n!$$

$$\mathrm{PermNo}(n,k) = \frac{n!}{(n-k)!}$$ no repetition

$$\mathrm{PermWith}(n,k) = n^k$$ with repetition

$$\mathrm{CombNo}(n,k) = \frac{n!}{k!\cdot(n-k)!}$$ no repetition

$$\mathrm{CombWith}(n,k) = \frac{(n+k-1)!}{k!\cdot(n-1)!}$$ with repetition

$$\mathrm{Binomial}(n,x,p) = \mathrm{CombNo}(n,x)\cdot p^x (1-p)^{n-x}$$

$$\mu = n\cdot p$$
$$\sigma^2 = \mu$$

$$\mathrm{Poisson}(x,\mu) = \frac{\mu^x}{x!} e^{-\mu}$$

$$\mathrm{HyperGeometric}(x,n,N,M) = \frac{\mathrm{CombNo}(M,x)\cdot \mathrm{CombNo}(N-M,n-x)}{\mathrm{CombNo}(N,n)}$$

Normal Distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

Cumulative Normal

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left[-\frac{1}{2}\left(\frac{v-\mu}{\sigma}\right)^2\right] dv$$

Conversion to Standardized Normal

Standardized Cumulative Normal

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du$$

$$z = \frac{x-\mu}{\sigma}$$

Interval Probability

$$P(a < X \leq b) = F(b) - F(a) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

$$P(\mu - \sigma < X \leq \mu + \sigma) \approx 68\%$$
$$P(\mu - 2\sigma < X \leq \mu + 2\sigma) \approx 95.5\%$$
$$P(\mu - 3\sigma < X \leq \mu + 3\sigma) \approx 99.7\%$$

$$P(\mu - 1.96\sigma < X \leq \mu + 1.96\sigma) = 95\%$$
$$P(\mu - 2.58\sigma < X \leq \mu + 2.58\sigma) = 99\%$$
$$P(\mu - 3.29\sigma < X \leq \mu + 3.29\sigma) = 99.9\%$$

I will give segments of Tables A7 thru A11 as needed

Confidence Interval - known variance:

Choose your confidence level (γ) as 90%, 95%, ...

Find the corresponding confidence interval value (c)

γ	0.9	0.95	0.99	0.999
c	1.645	1.96	2.576	3.291

Values are obtained for Table A8 using $z(D)$ column

Find the mean of your sample

Calculate $k = \frac{c\sigma}{\sqrt{n}}$

Confidence interval is: $\text{Conf}_\gamma = \bar{x} - k \leq \mu \leq \bar{x} + k$

Hypothesis Testing for Population Mean

Create a null hypothesis
e.g., the mean yield strength is X GPa

Create an alternative hypothesis
e.g., the mean yield strength is less than, greater than or not X

Choose a significance level
you want a 5% chance that you make a mistake

Calculate the T value: $$T = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

Determine c (the critical value) using d.f. = (n-1)

If MORE is BAD: T > c (of right tail), reject the null hypothesis

If LESS is BAD T < c (of left tail), reject the null hypothesis

If BOTH are BAD T < c-left or T > c-right, reject the null hypothesis
but redistribute the significance (ex 5% becomes -2.5% and +2.5%)

Confidence Interval - unknown variance:

Choose your confidence level (γ) as 90%, 95%, ...

Determine the solution, c, of the equation:

$$F(c) = \frac{1}{2}(1+\gamma)$$

Use Table A-9 (t-distribution) with $n-1$ degrees of freedom

Find the sample mean and standard deviation

Calculate $k = \frac{cs}{\sqrt{n}}$ and subtract/add to mean

Confidence interval

$$k = \frac{cs}{\sqrt{n}} \qquad \text{Conf}_\gamma = \bar{x} - k \leq \mu \leq \bar{x} + k$$

I will not ask you to calculate r^2 on this exam (unless I give you r first).

I will not ask you to do ANOVA on this exam

Table A7 **Normal Distribution**

Values of the distribution function $\Phi(z)$ [see (3), Sec. 24.8]. $\Phi(-z) = 1 - \Phi(z)$

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
	0.		0.		0.		0.		0.		0.
0.01	5040	0.51	6950	1.01	8438	1.51	9345	2.01	9778	2.51	9940
0.02	5080	0.52	6985	1.02	8461	1.52	9357	2.02	9783	2.52	9941
0.03	5120	0.53	7019	1.03	8485	1.53	9370	2.03	9788	2.53	9943
0.04	5160	0.54	7054	1.04	8508	1.54	9382	2.04	9793	2.54	9945
0.05	5199	0.55	7088	1.05	8531	1.55	9394	2.05	9798	2.55	9946

Table A8 **Normal Distribution**

Values of z for given values of $\Phi(z)$ [see (3), Sec. 24.8] and $D(z) = \Phi(z) - \Phi(-$
Example: $z = 0.279$ if $\Phi(z) = 61\%$; $z = 0.860$ if $D(z) = 61\%$.

%	$z(\Phi)$	$z(D)$	%	$z(\Phi)$	$z(D)$	%	$z(\Phi)$	$z(D)$
1	−2.326	0.013	41	−0.228	0.539	81	0.878	1.311
2	−2.054	0.025	42	−0.202	0.553	82	0.915	1.341
3	−1.881	0.038	43	−0.176	0.568	83	0.954	1.372
4	−1.751	0.050	44	−0.151	0.583	84	0.994	1.405
5	−1.645	0.063	45	−0.126	0.598	85	1.036	1.440

Table A9 ***t*-Distribution**

Values of z for given values of the distribution function $F(z)$ (see (8) in Sec. 25.3). Example: For 9 degrees of freedom, $z = 1.83$ when $F(z) = 0.95$.

$F(z)$	Number of Degrees of Freedom									
	1	2	3	4	5	6	7	8	9	10
0.5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.6	0.32	0.29	0.28	0.27	0.27	0.26	0.26	0.26	0.26	0.26
0.7	0.73	0.62	0.58	0.57	0.56	0.55	0.55	0.55	0.54	0.54
0.8	1.38	1.06	0.98	0.94	0.92	0.91	0.90	0.89	0.88	0.88
0.9	3.08	1.89	1.64	1.53	1.48	1.44	1.41	1.40	1.38	1.37
0.95	6.31	2.92	2.35	2.13	2.02	1.94	1.89	1.86	1.83	1.81
0.975	12.7	4.30	3.18	2.78	2.57	2.45	2.36	2.31	2.26	2.23
0.99	31.8	6.96	4.54	3.75	3.36	3.14	3.00	2.90	2.82	2.76
0.995	63.7	9.92	5.84	4.60	4.03	3.71	3.50	3.36	3.25	3.17
0.999	318.3	22.3	10.2	7.17	5.89	5.21	4.79	4.50	4.30	4.14

Table A11 ***F*-Distribution with (*m*, *n*) Degrees of Freedom**

Values of z for which the distribution function $F(z)$ [see (13), Sec. 25.4] has the value **0.95**
Example: For (7, 4) d.f., $z = 6.09$ if $F(z) = 0.95$.

n	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$	$m = 8$	$m = 9$
1	161	200	216	225	230	234	237	239	241
2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4
3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77

Complex Numbers

Addition: $(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

Subtraction:
$(x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$

Multiplication:
$(x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

Division: $\dfrac{x_1 + iy_1}{x_2 + iy_2} = \dfrac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$

Complex Conjugates $\bar{z} = x - iy$

$$\overline{(z_1 + z_2)} = \overline{z_1} + \overline{z_2} \qquad \overline{(z_1 - z_2)} = \overline{z_1} - \overline{z_2}$$

$$\overline{(z_1 z_2)} = \overline{z_1 z_2} \qquad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

Polar Form $z = r(\cos\theta + i\sin\theta)$ $\quad r = \sqrt{x^2 + y^2} = |z|$ $\quad \tan\theta = \frac{y}{x}$

Multiplication: $z_1 z_2 = \ r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

$$|z_1 z_2| = r_1 r_2 = |z_1||z_2| \qquad \text{arg}(z_1 z_2) = \text{arg}(z_1) + \text{arg}(z_2)$$

Division: $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ $\quad \text{arg}\left(\dfrac{z_1}{z_2}\right) = \text{arg}(z_1) - \text{arg}(z_2)$

Power $z^n = r^n(\cos n\theta + i\sin n\theta)$

Root $\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\dfrac{\theta + 2k\pi}{n} + i\sin\dfrac{\theta + 2k\pi}{n}\right)$

Cartesian form of a complex function $w = f(z) = u(x, y) + iv(x, y)$

The derivative of a function $f'(z_0) = \lim\limits_{\Delta z \to 0} \dfrac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$