Modal logic on $\mathbb{L}^4$

by Ryan J. Buchanan | Roseburg, OR
rjbuchanan2000@gmail.com

§0. Conventions

A *causal set*, $\underline{\mathscr{Caus}}$, is a locally path-connected space with an induced preorder

$$\tau_{<k}(\Pi_\infty(r)) \precsim \tau_k(\Pi_\infty(r)),$$

so that for distinct points [0, >0], the point corresponding to 0 is said to precede the others, or to take precedence.

We use the vocabulary:

$\Box / \Box\Box$: necessary/necessarily necessary

$\Box\Diamond = \Diamond$: possible

$\Diamond^2$: possibly possible[1]

$\Diamond^{-1}$: *was* possible

$\Box^{-1}$: *was* necessary

Notice, there is a distinct lack of a symbol for "will be necessary[2]." This is because the future cone of $\mathbb{L}^4$ is stratified by $\underline{\mathscr{Caus}}\Gamma$ and has the universal property that it consists only in purely possible moments with powers greater than zero.

The *worldline* of a particle is denoted by $\mathfrak{W}_Q$, with Q any particle. Mathematically, we consider particles to be atomic zero-vectors in the category **Vect**.

§1. Candidate Topologies

§1.1 $\mathbb{R}^n$

$\mathbb{R}^n$ is an excellent candidate topology, as it is used in definitions of algebraic topology quite a bit, specifically where "local resemblance" of some space is required. So, if there is a topologization of a causal set, then it automatically "locally resembles" this space.

§1.2 $\mathbb{L}^4$

We will be focusing here on the Minkowski lightcone, $\mathbb{L}^4$. The lightcone has a natural triadic stratification into upper, lower, and central parts, with the central part dominating the upper and lower parts at a much higher density.

Cuts of $\mathbb{L}^4$ *locally resemble* $\mathbb{R}^n$, or any Hermitian slice of a holomorphic space. We will write $\mathfrak{b}$ for the singularity of the lightcone, in place of the repetitive Sing($\mathbb{L}^4$) where it is clear that we are working with a Minkowski spacetime. There are some technicalities involving the invocation of this

---

[1] For the nth power case, we extend this to possibly possibly .... possible (**n times**)

[2] I.e., positive power exponents for squares

invariant. We will discuss these details later on. For now, we shall simply focus on the properties of the lightcone that make it a "natural" space to begin discourse about causal structures.

Categorically, there is a very good notion of a "push-out" and a "pull-back," and these are rooted in the timewise directions. Analytically, there is a very good notion of a barycenter, and that is $\mathfrak{b}$. Abstractly, there is a $\mathfrak{b}$-module for every projective slice of the strata of $\mathbb{L}^4$. In addition to satisfying all of these "niceness condition" for the various islands of mathematics, there are the facts that:

- **Theorem A** There is an evident map

  $$\mathfrak{b} \rightarrow \mathbb{E}_\infty^{\mathrm{CW}}$$

  taking a $\mathfrak{b}$-adic gauge to an open sector of a ringed space.

- **Theorem B** $\mathrm{Hom}(\Box,\Diamond) = \mathrm{LocSys}(1_{\mathfrak{b}})$

  This proposition relates the functorial (operational) *connection* between "nowness" (as described by necessity), and *possibility* to the constant $\mathfrak{b}$.

§1.3 Detour

We do not recollect all the facts about the lightcone here. Instead, we shall refer the reader to the various, and scattered about resources available in the literature. However, we do draw just enough information to make use of the following (crucial) lemma:

Lemma A (Kosz.) Let $\mathcal{M}$ be an open submanifold of a Nash manifold $\mathcal{N}$. Then, there is a shortest possible path $\mathcal{P}{:}\mathcal{M} \rightarrow \mathcal{N}$ of objects (x,y) in $\mathrm{Hom}(\mathcal{M},\mathcal{N})$.

This lemma is an existence statement for rank-one connections, which are in some classical sense the "strictest" kind of a connection (fibration) which is admitted by any two objects. We have used the terminology "Hurewicz isomorphism" in the past to describe a connection of this type, which is fully faithful and coextensive with both its adjoints.

§1.4 $\mathcal{T}_{\mathrm{GM}}$

The canonical Teichmüller space $\mathcal{T}_{\mathrm{GM}}$ is honorably mentioned here as a possibly exotic candidate structure for contemporary modal logic. This is because there is an associated *rich* syntax that comes with discussions of this well-established hyperbolic space. For instance, one immediately gains the language of *earthquakes*, *laminations*, *foliations*, etc.

Furthermore, it is of some interest to the author that the works on constructing Euclidean buildings in $\mathcal{T}_{\mathrm{GM}}$ by, e.g. L. Ji, do not go unnoticed. These are potential scaffolds for our project, and so they bear mentioning, although in this paper we do not explicitly provide recollection of the crucial aspects of these spaces, such as Borel-Serre compactification, etc.

§2. Worldlines in the Minkowski Cone

Definition 2.0.0. A *worldline* is a (locally) flat path-connected space

$$\square^{-n} \longrightarrow \square^{m}$$

such that each place bears witness to an ∞-topos corresponding to a one hundred percent probability.

Remark 1 This definition rules out unnecessary places, and so we obtain a restriction on the degrees of freedom of the system in question.

Definition 2.0.1 A *body*, $\mathcal{B}$ is a *witness* to a 3-space

$$\mathbb{R}^3 \sim \mathfrak{b} \in \mathbb{L}^4$$

In the following isomorphism, a sort function on particles is calibrated as follows: for every body $\mathcal{B}$, there is a sub-object identifier $\{\alpha, \beta, \ldots, \omega\}$, which corresponds to a unique type.

Definition 2.0.2 An *action*, **A**, is a step in a concurrent system approximating a change of one unit currency[3].

Finally, we have:

$$\mathcal{B}_\alpha{}^{A_1} \leftrightarrow \mathcal{B}_\beta{}^{A_2} \leftrightarrow \ldots \leftrightarrow \mathcal{B}_\omega{}^{A_n} \simeq Sort(\alpha, \beta, \ldots, \omega) \simeq cofib(\mathcal{B}_\infty)$$

This gives a *topological* relationship to every object by providing a minimal geodesic between each of the identities assigned to a coordinate in $\mathbb{R}^3$. The set of co-fibrations out of all the bodies is categorically equivalent to this "sort" function, as defined [here](). Borrowing terminology, the set of equivalences will be called the bond set..

We have

$$\underline{\mathscr{Caus}}\Gamma \rightarrow \mathcal{B}_\infty \simeq \lim_{\square \rightarrow \Diamond^\infty},$$

and also the monad

$$\text{Push}(\Gamma) = (\{\alpha, \beta, \ldots, \omega\}, \simeq, \text{Proj}(\mathcal{B}))$$

[3] By necessity, this action is to be *semi-continuous.*