ECE 405 Assignment 2

Ryan Mah
20814200

Question 1

a.

$|\psi\rangle$ is already normalized:

$$\langle\psi|\psi\rangle = \left(\frac{1}{\sqrt{2}}\langle\phi_1| - \frac{i}{\sqrt{2}}\langle\phi_2|\right)\left(\frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{i}{\sqrt{2}}|\phi_2\rangle\right)$$

$$= \frac{1}{2}\langle\phi_1|\phi_1\rangle + \frac{i}{2}\langle\phi_2|\phi_1\rangle - \frac{i}{2}\langle\phi_1|\phi_2\rangle + \frac{1}{2}\langle\phi_2|\phi_2\rangle$$

Since $|\phi_1\rangle$ and $|\phi_2\rangle$ are eigenstates of a Hermitian operator, they are orthogonal to each other.

Thus, $\langle\phi_1|\phi_2\rangle = \langle\phi_2|\phi_1\rangle = 0$, and $\langle\phi_1|\phi_1\rangle = \langle\phi_2|\phi_2\rangle = 1$, meaning:

$$\langle\psi|\psi\rangle = 1$$

b.

A measurement made on $|\psi\rangle$ with respect to $\mathbf{\Phi}$ can only yield either ϕ_1 or ϕ_2, since they are the eigenvales of $\mathbf{\Phi}$.

The probability of measuring ϕ_1 is given by $P(\phi_1) = |\langle\phi_1|\psi\rangle|^2$.

$$\begin{aligned}\langle\phi_1|\psi\rangle &= \langle\phi_1|\left(\frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{i}{\sqrt{2}}|\phi_2\rangle\right)\\ &= \frac{1}{\sqrt{2}}\langle\phi_1|\phi_1\rangle + \frac{i}{\sqrt{2}}\langle\phi_1|\phi_2\rangle\\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

Thus,

$$P(\phi_1) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

Similarly, for ϕ_2, $P(\phi_2) = |\langle\phi_2|\psi\rangle|^2$.

$$\begin{aligned}\langle\phi_2|\psi\rangle &= \langle\phi_2|\left(\frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{i}{\sqrt{2}}|\phi_2\rangle\right)\\ &= \frac{1}{\sqrt{2}}\langle\phi_2|\phi_1\rangle + \frac{i}{\sqrt{2}}\langle\phi_2|\phi_2\rangle\\ &= \frac{i}{\sqrt{2}}\end{aligned}$$

Thus,

$$P(\phi_2) = \left|\frac{i}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

c.

To find another quantum state that is orthogonal to $|\Psi\rangle$, we can set the inner product of the two states to 0.

Let $|\Psi_2\rangle = \alpha|\phi_1\rangle + \beta|\phi_2\rangle,\ \alpha, \beta \in \mathbb{C}$,

$$\begin{aligned}\langle\Psi|\Psi_2\rangle = 0 &= \left(\frac{1}{\sqrt{2}}\langle\phi_1| - \frac{i}{\sqrt{2}}\langle\phi_2|\right)(\alpha|\phi_1\rangle + \beta|\phi_2\rangle)\\ &= \frac{\alpha}{\sqrt{2}}\langle\phi_1|\phi_1\rangle + \frac{\beta}{\sqrt{2}}\langle\phi_2|\phi_1\rangle - \frac{i\alpha}{\sqrt{2}}\langle\phi_1|\phi_2\rangle - \frac{i\beta}{\sqrt{2}}\langle\phi_2|\phi_2\rangle\\ &= \frac{\alpha}{\sqrt{2}} - \frac{i\beta}{\sqrt{2}}\\ &= \frac{1}{\sqrt{2}}(\alpha - i\beta)\end{aligned}$$

Thus, we get that $0 = \alpha - i\beta$. Let $\alpha = x_\alpha + iy_\alpha$ and $\beta = x_\beta + iy_\beta$, where $x_\alpha, y_\alpha, x_\beta, y_\beta \in \mathbb{R}$.

Then,

$$\begin{aligned}0 &= \alpha - i\beta\\ 0 &= (x_\alpha + iy_\alpha) - i(x_\beta + iy_\beta)\\ 0 &= x_\alpha + iy_\alpha - ix_\beta + y_\beta\\ 0 &= (x_\alpha + y_\beta) + i(y_\alpha - x_\beta)\end{aligned}$$

So, $x_\alpha = -y_\beta$ and $x_\beta = y_\alpha$. Let $\alpha = \frac{1}{\sqrt{2}}(1+i)$ and $\beta = \frac{1}{\sqrt{2}}(1-i)$.

Thus, another quantum state that is orthogonal to $|\Psi\rangle$ is:

$$|\Psi_2\rangle = \frac{1}{\sqrt{2}}(1+i)\,|\phi_1\rangle + \frac{1}{\sqrt{2}}(1-i)\,|\phi_2\rangle$$

d.

The probability of finding the system in the state $|\psi_2\rangle$ if a if a measurement is made is given by $P(\psi_2) = |\langle\psi_2|\psi\rangle|^2$.

Since $|\psi_2\rangle$ is orthogonal to $|\psi\rangle$, $\langle\psi_2|\psi\rangle = 0$, meaning that the probability of finding the system in the state $|\psi_2\rangle$ is 0.

Question 2

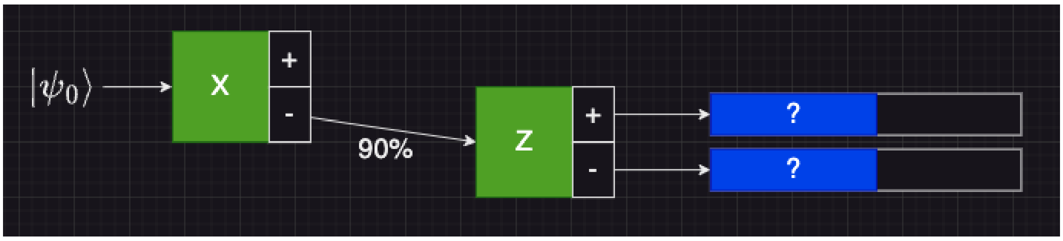
a.

The probability of measuring $|\psi_0\rangle$ in the $+x$ state is given by $P(+x) = |\langle +x|\psi_0\rangle|^2$.

$$\begin{aligned}
\langle +x \mid \psi_0\rangle &= \langle +x| \left(\frac{i}{\sqrt{10}}\,|+x\rangle - \frac{3}{\sqrt{10}}\,|-x\rangle\right) \\
&= \frac{i}{\sqrt{10}}\,\langle +x \mid +x\rangle - \frac{3}{\sqrt{10}}\,\langle +x \mid -x\rangle \\
&= \frac{i}{\sqrt{10}}
\end{aligned}$$

Thus, the probability of finding $|\psi_0\rangle$ in the $+x$ state is 10%, and the probability of finding $|\psi_0\rangle$ in the $-x$ state is 90%.

Knowing this, we can now draw the SG-experiment.



b.

The possble outcomes of the S_z measurement are $|+z\rangle$ and $|-z\rangle$. This is because the eigenvalues of S_z are $\pm\frac{\hbar}{2}$.

c.

The probability of finding the system in $|-z\rangle$ after the last analyzer is given by:

$$P(-z) = |\langle -z \mid -x\rangle|^2 \, |\langle -x \mid \psi_0\rangle|^2 \iff |\langle -x \mid \psi_0\rangle|^2 = 0.9$$

So,

$$\begin{aligned}
\langle -z \mid -x\rangle &= \langle -z| \left(\frac{1}{\sqrt{2}} |+z\rangle - \frac{1}{\sqrt{2}} |-z\rangle \right) \\
&= \frac{1}{\sqrt{2}} \langle -z \mid +z\rangle - \frac{1}{\sqrt{2}} \langle -z \mid -z\rangle \\
&= -\frac{1}{\sqrt{2}}
\end{aligned}$$

Thus, the probability of finding the system in $|-z\rangle$ after the last analyzer is:

$$\begin{aligned}
P(-z) &= \left| -\frac{1}{\sqrt{2}} \right|^2 |\langle -x \mid \psi_0\rangle|^2 \\
&= \frac{1}{2} \cdot 0.9 \\
&= 0.45
\end{aligned}$$

Similarly, for the $+z$ state, we have $P(+z) = |\langle +z \mid -x\rangle|^2 \, |\langle -x \mid \psi_0\rangle|^2$.

$$\begin{aligned}
\langle +z \mid -x\rangle &= \langle +z| \left(\frac{1}{\sqrt{2}} |+z\rangle - \frac{1}{\sqrt{2}} |-z\rangle \right) \\
&= \frac{1}{\sqrt{2}} \langle +z \mid +z\rangle - \frac{1}{\sqrt{2}} \langle +z \mid -z\rangle \\
&= \frac{1}{\sqrt{2}}
\end{aligned}$$

So,

$$\begin{aligned}
P(+z) &= \left| \frac{1}{\sqrt{2}} \right|^2 |\langle -x \mid \psi_0\rangle|^2 \\
&= \frac{1}{2} \cdot 0.9 \\
&= 0.45
\end{aligned}$$

Question 3

a.

$$\begin{aligned}
\langle\psi_1|\psi_1\rangle = 1 &= \left(a|\phi_1\rangle + b|\phi_2\rangle - \frac{1}{2}|\phi_3\rangle\right)^* \left(a|\phi_1\rangle + b|\phi_2\rangle - \frac{1}{2}|\phi_3\rangle\right) \\
&= \left(a^*\langle\phi_1| + b^*\langle\phi_2| - \frac{1}{2}\langle\phi_3|\right)\left(a|\phi_1\rangle + b|\phi_2\rangle - \frac{1}{2}|\phi_3\rangle\right) \\
&= |a|^2\langle\phi_1|\phi_1\rangle + |b|^2\langle\phi_2|\phi_2\rangle + \frac{1}{4}\langle\phi_3|\phi_3\rangle \\
&= |a|^2 + |b|^2 + \frac{1}{4}
\end{aligned}$$

Thus, we have a circle:

$$\frac{3}{4} = |a|^2 + |b|^2$$

Any a and b that satisfy the above equation will normalize $|\psi_1\rangle$.

However, given that a and b are not unique, $a = b$, meaning

$$\begin{aligned}
\frac{3}{4} &= 2|a|^2 \\
\frac{3}{8} &= |a|^2 \\
|a| = |b| &= \frac{\sqrt{3}}{2\sqrt{2}}
\end{aligned}$$

Picking a and b to have the zero phase (the most simple case), we get:

$$a = b = \frac{\sqrt{3}}{2\sqrt{2}}$$

b.

$$\begin{aligned}
\langle\psi_2|\psi_1\rangle = 0 &= \left(-i\langle\phi_2| + c^*\langle\phi_3|\right)\left(a|\phi_1\rangle + b|\phi_2\rangle + \frac{1}{2}|\phi_3\rangle\right) \\
&= -ib\langle\phi_2|\phi_2\rangle + \frac{c^*}{2}\langle\phi_3|\phi_3\rangle \\
&= -ib + \frac{c^*}{2}
\end{aligned}$$

Thus,

$$c = 2ib = \frac{\sqrt{3}i}{\sqrt{2}}$$

The above value of c will make psi_1 and psi_2 orthogonal to each other. However, psi_2 must be normalized. Skipping steps, since we have done this many times before, we get:

$$|\psi_2\rangle = \sqrt{\frac{2}{5}} \left(i\,|\phi_2\rangle + \sqrt{\frac{3}{2}} i\,|\phi_3\rangle \right) = \sqrt{\frac{2}{5}} i\,|\phi_2\rangle + \sqrt{\frac{3}{5}} i\,|\phi_3\rangle$$

c.

$$\langle\phi_1|\psi_1\rangle = \langle\phi_1| \left(a\,|\phi_1\rangle + b\,|\phi_2\rangle - \frac{1}{2}\,|\phi_3\rangle \right) = a = \frac{\sqrt{3}}{2\sqrt{2}}$$

$$\langle\phi_2|\psi_1\rangle = \langle\phi_2| \left(a\,|\phi_1\rangle + b\,|\phi_2\rangle - \frac{1}{2}\,|\phi_3\rangle \right) = b = \frac{\sqrt{3}}{2\sqrt{2}}$$

$$\langle\phi_3|\psi_1\rangle = \langle\phi_3| \left(a\,|\phi_1\rangle + b\,|\phi_2\rangle - \frac{1}{2}\,|\phi_3\rangle \right) = -\frac{1}{2}$$

Thus, the probabilities are as such:

$$P(\phi_1) = |\langle\phi_1|\psi_1\rangle|^2 = \frac{3}{8}$$

$$P(\phi_2) = |\langle\phi_2|\psi_1\rangle|^2 = \frac{3}{8}$$

$$P(\phi_3) = |\langle\phi_3|\psi_1\rangle|^2 = \frac{1}{4}$$

Hence, if a measurement on $|\psi_1\rangle$ is made, the most likely outcome is $|\phi_1\rangle$ or $|\phi_2\rangle$, with a probability of $\frac{3}{8}$ each.

d.

$$\langle\phi_1|\psi_2\rangle = \langle\phi_1|\left(\sqrt{\frac{2}{5}}i\,|\phi_2\rangle + \sqrt{\frac{3}{5}}i\,|\phi_3\rangle\right) = 0$$
$$\langle\phi_2|\psi_2\rangle = \langle\phi_2|\left(\sqrt{\frac{2}{5}}i\,|\phi_2\rangle + \sqrt{\frac{3}{5}}i\,|\phi_3\rangle\right) = \sqrt{\frac{2}{5}}i$$
$$\langle\phi_3|\psi_2\rangle = \langle\phi_3|\left(\sqrt{\frac{2}{5}}i\,|\phi_2\rangle + \sqrt{\frac{3}{5}}i\,|\phi_3\rangle\right) = \sqrt{\frac{3}{5}}i$$

Thus, the probabilities are as such:

$$P(\phi_1) = |\langle\phi_1|\psi_2\rangle|^2 = 0$$
$$P(\phi_2) = |\langle\phi_2|\psi_2\rangle|^2 = \frac{2}{5}$$
$$P(\phi_3) = |\langle\phi_3|\psi_2\rangle|^2 = \frac{3}{5}$$

Hence, if a measurement on $|\psi_2\rangle$ is made, the most likely outcome is $|\phi_3\rangle$.

e.

The probability of finding the system in state $|\psi_2\rangle$ after a measurement is made on $|\psi_1\rangle$ is given by:

$$P(\psi_2) = |\langle\psi_2|\psi_1\rangle|^2$$

Since $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal to each other, $\langle\psi_2|\psi_1\rangle = 0$, meaning that the probability is 0.

Question 4

First, we normalize $|\psi\rangle$. Skipping steps, we get:

$$|\psi\rangle = \sqrt{\frac{3}{5}}\left(\frac{i}{\sqrt{3}}\,|+z\rangle - \frac{2}{\sqrt{3}}\,|-z\rangle\right) = \frac{i}{\sqrt{5}}\,|+z\rangle - \frac{2}{\sqrt{5}}\,|-z\rangle$$

Now,

$$
\begin{aligned}
\langle\chi|\psi\rangle &= \left(\frac{1}{\sqrt{2}}\left|+z\right\rangle + \frac{1}{\sqrt{2}}\left|-z\right\rangle\right)^* \left(\frac{i}{\sqrt{5}}\left|+z\right\rangle - \frac{2}{\sqrt{5}}\left|-z\right\rangle\right) \\
&= \frac{i}{\sqrt{10}}\left\langle +z\right| + z\rangle - \frac{2}{\sqrt{10}}\left\langle -z\right| - z\rangle \\
&= \frac{i}{\sqrt{10}} - \frac{2}{\sqrt{10}}
\end{aligned}
$$

Thus,

$$
P(\chi) = \left|\langle\chi|\psi\rangle\right|^2 = \frac{1}{10} + \frac{4}{10} = \frac{1}{2}
$$