Chapter 9: Discrete-time control design II: State-space

ECE 481 - Digital Control Systems

Yash Vardhan Pant

Based on course notes by Professor Chris Nielsen

Outline

[X] The stabilization problem

    [ ] Controllability
    [ ] State feedback control (linear)
    [ ] Deadbeat control
    [ ] Intersample ripple

[ ] Output-feedback stabilization

    [ ] Observers and observability
    [ ] Observer-based control

[ ] Tracking and regulation via state-space design

    [ ] Preliminaries
    [ ] Solution to the regulation problem

X = The upcoming topic

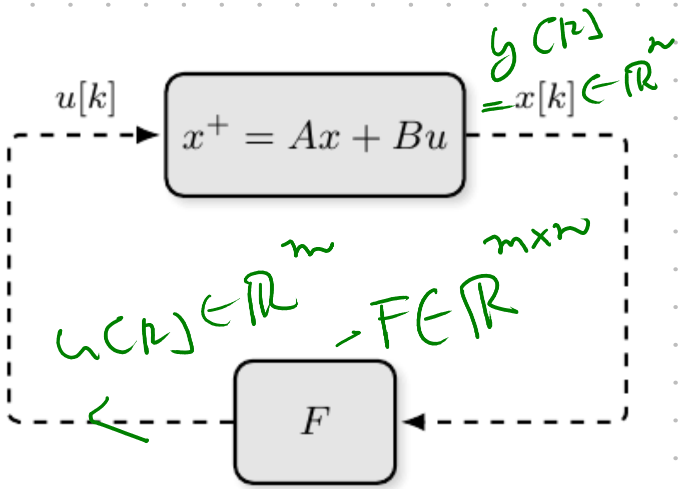
- = Topic that has been covered

Stabilization problem

Consider the system $x^+ = Ax + Bu$ and let 0 be the desired equilibrium point for the state x.
Assume x[k] is available as a measurement (or consider output equations y = Cx, where C = I).
Full state

Recall the discussion on closed-loop internal stability from Chapter 6. We aim to design a simple(r) state-feedback controller $\underbrace{u[k] = Fx[k]}$ such that the closed-loop system below is stable.
Linear feedback law.

What are the closed-loop dynamics for such a system?



$x^+ = Ax + Bu, \quad u = Fx$

$\Rightarrow x^+ = Ax + BFx = \underbrace{(A+BF)}_{A_{cl}} x$

With this, we can define the stabilization problem:

Stabilization Problem. *Given $\underbrace{(A, B)}$, find F so that $A + BF$ is stable.*

Note: the problem is pretty much the same in CT or DT, only the conditions for stability change.

What property are we looking for A+BF to have? Eigenvalues of $A+BF \subset$ Unit disk

Is it always possible to find such a real state-feedback matrix F? No!

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Controllability

Consider the DT system $x^+ = Ax + Bu, \qquad x[0] = 0.$ [Initial state = 0 w.l.o.g]

Definition 9.1.1. System (9.1) is **controllable** (or the pair (A, B) is **controllable**) if for every target vector x_f, there is a time $k_f \geq 0$ and a control signal $u[k]$, $k \in \{0, 1, \ldots, k_f - 1\}$, such that $x[k_f] = x_f$.

$k_f = n$ in next slides.

i.e., controllability says that every state is "reachable" in finite time, starting from the origin.

This leads to the notion of a controllability matrix for LTI systems:

$$W_c := \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$ which has n rows and nm columns

The span of the columns of Wc, or the range-space of the matrix, is the set of reachable states. How?

Def: $\text{Range}(A) = \{ v \mid \exists x .\ v = Ax \}$

$= \text{Span of columns of } A.$

e.g. $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad Ax = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} x_2$

$x[0] = 0,$

$x[1] = Bu[0] + Ax[0] = Bu[0]$

$x[2] = Ax[1] + Bu[1] = ABu[0] + Bu[1]$

$\vdots$

$x[N] = A^{N-1}Bu[0] + A^{N-2}Bu[1] + \dots + Bu[N-1]$

We want $x[N] = x_f$ (target state)

$$x[N] = \underbrace{[A^{N-1}B \quad A^{N-2}B \quad \dots \quad AB \quad B]}_{\tilde{C}} \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[N-1] \end{bmatrix}$$

Range($\tilde{C}$) is our set of reachable states in N time steps.

We want $x_f \in$ Range($\tilde{C}$), if yes x_f reached in finite time.

our reachable space seems dependent on N. We want to remove this dependency.

Cayley-Hamilton Theorem: Every square matrix satisfies its own characteristic equation.

For $A \in \mathbb{R}^{n \times n}$, Characteristic eq^n,

$$\det(sI - A) = 0 = s^n + c_{n-1}s^{n-1} + \dots + c_1 s + c_0$$

Thus, $A^n + c_{n-1}A^{n-1} + \dots + c_1 A + c_0 I = 0_{n \times n}$

$\Rightarrow$ A^n = linear combination of terms upto A^{n-1}.

For any $N > n$, A^N is also a linear combination of terms with matrix power upto A^{n-1}.

e.g. Consider $N = n+1$, $A^{n+1} = A \cdot A^n = A \cdot$ (linear combination $\dots A^n$)

Recurse on this, = lin fun$_1$(A^n)

= lin fun$_2$(of power A^{n-1})

$\therefore$ Span of columns of $[B \;\; AB \;\; \cdots \;\; A^{N-2}B \;\; A^{N-1}B]$ $(= \tilde{C})$

$= $ span of columns of $\underbrace{[B \;\; AB \;\; \cdots \;\; A^{n-1}B]}$ (W_c)

In other words, for N > n, the set of reachable states is the same as the set of reachable states after n time steps, where n is the number of states in the system (dim. of state vector).

Theorem: $x^+ = Ax + Bu$ is controllable

iff W_c has rank $= n$ (full row rank)

$x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $W_c \in \mathbb{R}^{n \times nm}$

(Result is same for CT systems).

PBH test for controllability

We say that an eigenvalue of A is controllable if $\operatorname{rank}\begin{bmatrix} A-\lambda I & B \end{bmatrix} = n.$

(mode)

If this holds for all eigenvalues of A, then the system is controllable (iff).

Summary: controllability

$$\begin{aligned}
\text{every state is reachable} &\Leftrightarrow (A,B) \text{ is controllable} \\
&\Leftrightarrow \operatorname{rank} W_c = n \\
&\Leftrightarrow \operatorname{rank}\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = n \\
&\Leftrightarrow \operatorname{rank}\begin{bmatrix} A-\lambda I & B \end{bmatrix} = n \text{ for every eigenvalue } \lambda \text{ of } A.
\end{aligned}$$

Examples

Example 9.1.1: $A = \begin{bmatrix} 2 & 0 & 2 \\ 3 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $W_c = \left[\begin{array}{ccc} B & AB & A^2B \end{array}\right] = \begin{bmatrix} 0 & 2 & 6 \\ 0 & 0 & 6 \\ 1 & 1 & 3 \end{bmatrix}$.

Eyeball Wc, is it full rank? If so, the system is controllable.

Matlab's ctrb command computes the controllability matrix for CT/DT LTI systems:

```
A = [2 0 2; 3 1 0; 1 4 1];
B = [0;0;1];
Wc = ctrb(A,B);
det(Wc)
```

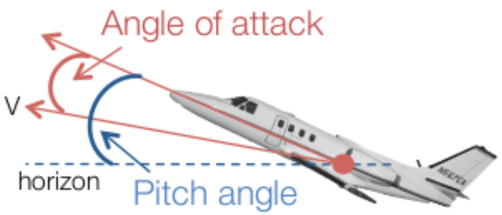
$\neq 0$ (iff controllable)

Single input system

Linearized continuous-time model:
(at altitude of 5000m and a speed of 128.2 m/sec)

$$\dot{x} = \begin{bmatrix} -1.2822 & 0 & 0.98 & 0 \\ 0 & 0 & 1 & 0 \\ -5.4293 & 0 & -1.8366 & 0 \\ -128.2 & 128.2 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} -0.3 \\ 0 \\ -17 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x$$



- Input: elevator angle
- States: x_1: angle of attack, x_2: pitch angle, x_3: pitch rate, x_4: altitude
- Outputs: pitch angle and altitude
- Constraints: elevator angle ± 0.262rad ($\pm 15°$), elevator rate ± 0.524rad ($\pm 60°$), pitch angle ± 0.349 ($\pm 39°$)

Open-loop response is unstable (open-loop poles: $0, 0, -1.5594 \pm 2.29i$)

Recall that in ch. 3., traditional pole placement failed. However, the system is controllable! See Matlab code.

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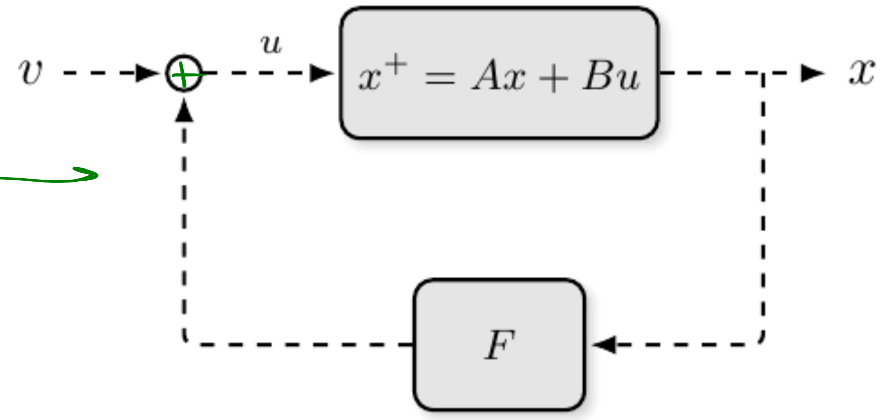
State feedback control: introduction

Why does controllability matter?
We can re-assign eigenvalues of controllable systems via state-feedback!

Consider $u = Fx + v, \qquad v$ is an external input



The new state-space model is $x^+ = (\underbrace{A + BF}_{A_{cl}})x + Bv$
where v is a new independent input.

This gives us the Pole/Eigenvalue assignment Theorem:

> **Theorem 9.1.3** (Pole-Assignment Theorem). *The pair (A, B) is controllable if, and only if, for every set of desired eigenvalues, there exists a matrix F such that $A + BF$ has exactly that set of eigenvalues.*

This is still called a "pole placement" theorem since the eigenvalues of A+BF are the poles of the closed-loop TF below:

$$\left[\begin{array}{c|c} A + BF & B \\ \hline C = I & 0 \end{array}\right](z) = (zI - A - BF)^{-1}B.$$

$$\left\{ G(z) = C(zI - A)^{-1}B + D \right\}$$

Computing state feedback matrices (Single-input case)

We know how to go from a DT TF to a state-space model (chapter 6). Recall the controllable canonical form below:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

COMPANION FORM

This corresponds to the characteristic polynomial

$$\det(zI - A) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n.$$

Note, a system with (A, B) in controllable canonical form is controllable.

The matrix A here is said to be in "companion form".

Example

Consider a system x[k+1] = Ax[k] + Bu[k] where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

You can verify that the controllability matrix is:

$$W_c = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_3 \\ 1 & -a_3 & -a_2 + a_3^2 \end{bmatrix}.$$

Is Wc full rank ? (yes, its columns span $\mathbb{R}^3$).

We'll see next that it is easy to design a state feedback controller u = Fx when (A, B) is in the form above.

Example 9.1.3

Consider a single-input system x[k+1] = Ax[k] + Bu[k], with A, B in controllable canonical form below:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$u = Fx = \sum_{i=1}^{3} f_i x_i$

Our task is to design a state feedback matrix $F = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}$ to place the closed-loop poles/eigenvalues at {-0.1, -0.2, -0.3}. [Stabilization]

What is A+BF ?

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1+f_1 & -1+f_2 & -1+f_3 \end{bmatrix}$$

STILL COMPANION FORM

Given A+BF is in companion form, we can write its characteristic polynomial just from its last row:

Closed loop ch.p. $\pi = z^3 + (1-f_3)z^2 + (+1-f_2)z - 1 - f_1$

For the given desired roots, the ch.p. is: $(z+0.1)(z+0.2)(z+0.3) = z^3 + 0.6z^2 + 0.11z + 0.006.$

Does this look familiar ? What do we do next ?

Equating coefficients in the two polynomials from the previous slide, we get:

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} -0.4 \\ -0.89 \\ 1.006 \end{bmatrix}$$

This has a unique solution, and thus we have the state feedback matrix

$$F = \begin{bmatrix} -1.006 & 0.89 & 0.4 \end{bmatrix}.$$

Verify that $\lambda(A+BF) = \{-0.1, -0.2, -0.3\}$

Transforming single input systems into controllable canonical form

There exists a coordinate change (recall similarity transforms from chapter 6) to transform any (A, B) pair into a controllable canonical form IFF (A, B) is controllable and B is n x 1 (n-states, 1 input).

Theorem 9.1.5. *Suppose (A, B) is controllable and B is $n \times 1$. Let the ch.p. of A be given by*

$$z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n.$$

Define

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$$\tilde{A} := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix}, \qquad \tilde{B} := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

$\tilde{A}$, $\tilde{B}$ in Controllable canonical form.

Then there exists an invertible matrix $W \in \mathbb{R}^{n \times n}$ such that

$$W^{-1} A W = \tilde{A}, \qquad W^{-1} B = \tilde{B}.$$

[See discussion on similarity transforms]

We will not prove this theorem. Note, the theorem points us to a procedure for designing state-feedback matrices that assign desired closed-loop eigenvalues!

$$\tilde{W}_c = [\tilde{B} \;\; \tilde{A}\tilde{B} \;\; \tilde{A}^2\tilde{B} \;\; \cdots \;\; \tilde{A}^{n-1}\tilde{B}] \in \mathbb{R}^{n \times n}$$

A recipe for pole assignment via state-feedback control (for single-input systems)

Copy-pasted from the course notes:

[Note: W_c is full-rank conditioned on our assumptions].

1. Compute the controllability matrix W_c of (A, B).
2. Find the ch.p. of A ($\det(zI-A)$)
$$z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n$$
and use it to define $\tilde{A}$ as in Theorem 9.1.5. $\tilde{A}$ in companion form.
3. Let $\tilde{W}_c$ be the controllability matrix of $(\tilde{A}, \tilde{B})$ where $\tilde{B}$ is as in Theorem 9.1.5. [$\tilde{W}_c$ full rank. Prove!]
4. Define $W := W_c \tilde{W}_c^{-1}$. Then $\tilde{A} = W^{-1} A W$, $\tilde{B} = W^{-1} B$. [W is full rank].
5. Compute the unique $\tilde{F} \in \mathbb{R}^{1\times n}$ so that $\tilde{A} + \tilde{B}\tilde{F}$ has eigenvalues in the desired locations

[See Ex. 9.1.3]
$$\tilde{F} = \begin{bmatrix} a_n - \alpha_0 & a_{n-1} - \alpha_1 & \cdots & a_1 - \alpha_{n-1} \end{bmatrix}.$$
where the α_i come from the desired ch.p. $z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_1 z + \alpha_0$.
6. Set $F = \tilde{F} W^{-1}$.

$$(A, B) \xleftarrow{W} (\tilde{A}, \tilde{B})$$

[$\tilde{A} + \tilde{B}\tilde{F}$ is stable $\iff$ $A + BF$ is stable]

Summary: the controllability matrices here define a state transform that makes it easy to compute a state feedback matrix for the transformed system. Since the transform is invertible, we can recover a the state-feedback matrix for the original system (A, B). This of course only works when (A, B) is controllable and B has one column.

Example: Double integrator (symbolic T)

The continuous time system $\ddot{x} = u$ (the double integrator) has a discretization below (see example 7.2.4):

$$x^+ = \overbrace{\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}}^{A} x + T \overbrace{\begin{bmatrix} \frac{T}{2} \\ 1 \end{bmatrix}}^{B} u.$$

[ZOH, Step-invariant discretization].

We will design a state feedback controller u = Fx to get a desired ch.p. of $z^2 + \alpha_1 z + \alpha_0$.

First, is the system controllable?

$$W_c = [B \ AB] = T \begin{bmatrix} \frac{T}{2} & \frac{3}{2}T \\ 1 & 1 \end{bmatrix}, \quad \det(W_c) \neq 0$$

As expected it is (otherwise, what's the point of this example?), so we can proceed.

The A matrix of the DT double integrator is upper triangular, so its ch.p. is $(z-1)(z-1) = z^2 - 2z + 1$

This gives us the 'transformed' companion matrix $\tilde{A} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$. [See the last row here.]

Next, we will compute the controllability matrix for the associated controllable canonical form, and then compute the state transform.

Note that $\tilde{B} = [0 \quad 1]'$ in the canonical form, thus

$$\tilde{W}_c = [\tilde{B} \quad \tilde{A}\tilde{B}] = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \qquad \Rightarrow \qquad W = W_c \tilde{W}_c^{-1} = T \begin{bmatrix} \frac{T}{2} & \frac{T}{2} \\ -1 & 1 \end{bmatrix}.$$

Now, we can compute the feedback matrix in this transformed space, i.e., $\tilde{F}$ to get the desired ch.p:

$$\tilde{A} + \tilde{B}\tilde{F} = \begin{bmatrix} 0 & 1 \\ -1 + \tilde{f}_1 & 2 + \tilde{f}_2 \end{bmatrix}.$$ Here $\tilde{F} = [\tilde{f}_1 \quad \tilde{f}_2]^{\cancel{\prime}}$.

No transpose, $\tilde{B}\tilde{F} \in \mathbb{R}^{2\times 2}$

This is, as expected, a companion matrix and has a ch.p. $z^2 - (2 + f_2)z + 1 - f_1$.

Compare this to the desired ch.p. $z^2 + \alpha_1 z + \alpha_0$. We get: $\tilde{F} = \begin{bmatrix} 1 - \alpha_0 & -\alpha_1 - 2 \end{bmatrix}$.

We can now get the feedback matrix F that places the eigenvalues of A + BF where we want them:

$$F = \tilde{F}W^{-1} = -\frac{1}{2T^2} \begin{bmatrix} 2(\alpha_0 + \alpha_1 + 1) & T(\alpha_1 - \alpha_0 + 3) \end{bmatrix}.$$

Example: Double integrator (numerical)

Say T = 0.1, and the desired closed-loop poles are {0.2, 0.5}, i.e., the desired ch.p. is

$$(z-0.2)(z-0.5) = z^2 - 0.7z + 0.1$$

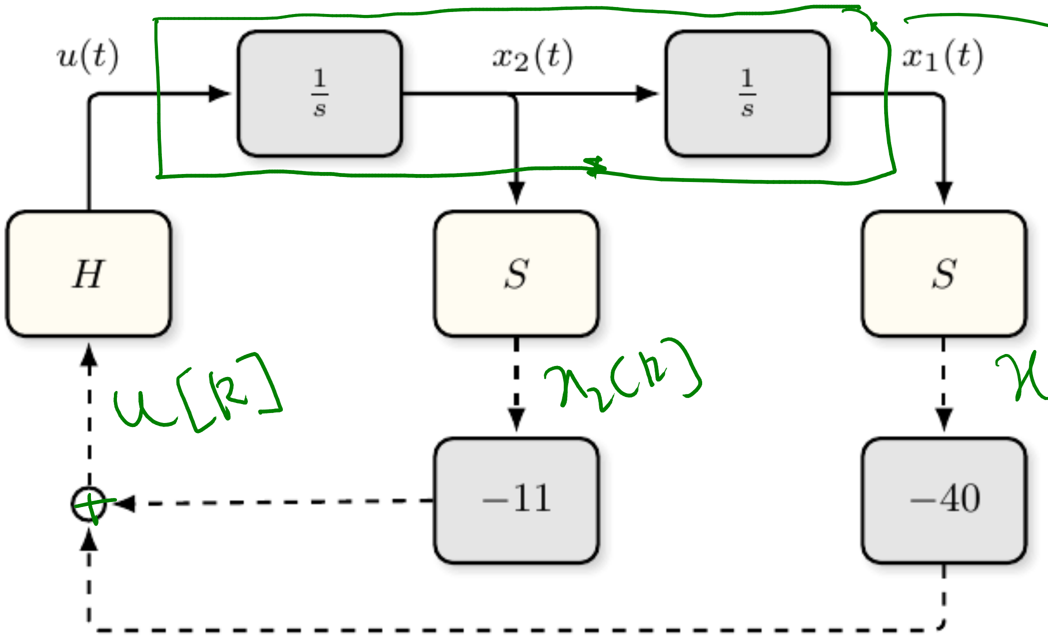
Compare this to the previous example's desired ch.p. of $z^2 + \alpha_1 z + \alpha_0$.

We have values for the alphas, and hence the feedback matrix, which we had computed as:

$$F = \tilde{F}W^{-1} = -\frac{1}{2T^2}\begin{bmatrix} 2(\alpha_0+\alpha_1+1) & T(\alpha_1-\alpha_0+3) \end{bmatrix}.$$

Substituting, we get $F = -\begin{bmatrix} 40 & 11 \end{bmatrix}$. Verify that $\lambda(A+BF) = \{0.2, 0.5\}$

The block-diagram for such a feedback controller implementation (step-invariant TF) is below:



$\ddot{x} = u$

← Full state feedback

$u = Fx$
$= f_1x_1 + f_2x_2$

The procedure we have described might not be the most convenient one to compute F by hand, but lends itself well to numerical implementation. Matlab's 'place' or 'acker' commands can be used to compute F.

See code for computing F for the DT double integrator in Matlab.

Stabilizability

Example: Consider a DT system with the following (A, B) pair: $A = \frac{1}{4}\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

Eigenvalues of A are {-0.5, 1}. We can consider designing a feedback controller u = Fx to stabilize the system in closed-loop. Is (A, B) controllable?

$$W_c = [B \quad AB] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

This is not full rank, so no. However we might still be able to stabilize the system via u = Fx.

Definition 9.1.7. System (9.1) is **stabilizable** (or the pair of matrices (A, B) is **stabilizable**) if there exists a matrix F so that the eigenvalues of $A + BF$ are all inside the open unit disk.

Turns out, such a F exists IFF all unstable eigenvalues of A (i.e., outside the unit disk) are controllable. We can use the PBH test on the unstable eigenvalues to test this.

By theorem 9.1.3, if (A, B) is controllable, then this implies than (A, B) is also stabilizable. But as we will see in this example, not the other way around.

Stabilizability is thus a weaker condition, but a useful one for many systems.

The idea is: the uncontrollable eigenvalue is stable, the unstable one is controllable and can be stabilized.

Example

Continuing with a DT system with (A, B): $A = \frac{1}{4}\begin{bmatrix}1 & 3\\ 3 & 1\end{bmatrix}, \qquad B = \begin{bmatrix}1\\1\end{bmatrix}.$

Let us check for stabilizability, or if the unstable eigenvalue (=1) is controllable. Apply the PBH test:

$$\text{rank}\begin{bmatrix}A-(1)I & B\end{bmatrix} = \text{rank}\begin{bmatrix}-3/4 & 3/4 & 1\\ 3/4 & -3/4 & 1\end{bmatrix} = 2.$$ = number of states (n), or full row rank!

Our design for F is not going to bother with the stable eigenvalue at -0.5 (because we can't, it's not contro -llable). We will instead 'stabilize' the system by moving the eigenvalue at 1 to 0.1 (interior of unit disk).

The desired ch.p. is thus: $\pi_{\text{des}}[z] = \underbrace{(z+0.5)}_{\text{fixed}}(z-0.1) = z^2 + 0.4z - 0.05.$ Let $F = \begin{bmatrix}f_1 & f_2\end{bmatrix}$, thus:

→ uncontrollable but stable eigenvalue

$$A + BF = \begin{bmatrix}0.25+f_1 & 0.75+f_2\\ 0.75+f_1 & 0.25+f_2\end{bmatrix}$$ and $\det(zI - A - BF) = z^2 + (-0.5 - f_1 - f_2)z - 0.5f_1 - 0.5f_2 - 0.5.$

Comparing coefficients, we get $\begin{bmatrix}-1 & -1\\ -0.5 & -0.5\end{bmatrix}\begin{bmatrix}f_1\\f_2\end{bmatrix} = \begin{bmatrix}0.9\\0.45\end{bmatrix}.$

This doesn't have a unique solution, and we can simply pick $F = -\begin{bmatrix}0.45 & 0.45\end{bmatrix}$.

Verify that A + BF has stable eigenvalues.

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Deadbeat control

Straightforward idea: place all closed-loop poles/eigenvalues at z = 0 (only works for DT systems).

Consider an autonomous DT system: $x^+ = Ax,\ x[0] = x_0$

We want the state to go to zero in finite time for any initial state, i.e., $x_k \to 0$ in finite time for any x_0.

This is the same as $A^k \to 0_{n\times n}$, where k is finite. $x[k] = A^k x_0$

$\iff$ $A^n = 0$ Using the Cayley Hamilton theorem.

$\iff$ all eigenvalues of A are zero

$\iff$ A is nilpotent.

$\to$ All CL λ's are $= 0$.

Thus, we have a recipe for state feedback control, choose $\pi_{\text{des}}[z] = z^n$ for the closed-loop ch.p.

The resulting closed-loop system is $x^+ = A_{cl}x = (A + BF)x$

A + BF is nilpotent (design F such that this happens), then x[k] -> 0 in finite time k<=n.

Note: We cannot get this behavior for CT systems $\dot{x} = (A + BF)x$. Why?

This is because the state response is $x(t) = e^{t(A+BF)}x(0)$, a decaying exponential that cannot go to zero in finite time!

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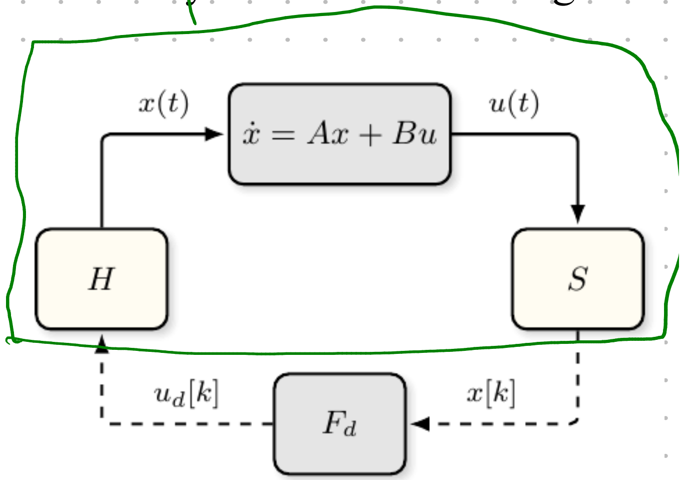
Intersample ripple

Step-inv. TF of CT

When a stable discretized (step-invariant) system's state goes to zero, does the CT system's state also go to zero?



Consider the CT system $\dot{x} = Ax + Bu.$

Its step-invariant discretized model (for some T) is: $x^+ = A_d x + B_d u_d.$

Say we have a feedback controller $u = F_d x$ so that all egivenvalues of $A_d + B_d F_d$ are inside the unit disk (i.e., closed-loop stable).

By definition, x[k] = x(kT) converges to zero as k goes to infinity, for all initial states x(0).
What about x(t)? Does it also converge to 0? (i.e., stay at 0, and not hit 0 only at sampling instants alone).

Yes!

Proof:

Course notes.

Intersample ripple: summary

a) Continuous time state converges to zero at all intersample times.

b) If the closed loop DT matrix $A_d + B_d F_d$ is nilpotent, then x(kT) converges to zero in finite time.
x(t) then also converges to zero in finite time. This follows a similar proof mechanism.
Note: This does not contradict our comment towards the end of the section on deadbeat control!

Outline

[-] The stabilization problem

    [-] Controllability

    [-] State feedback control (linear)

    [-] Deadbeat control

    [-] Intersample ripple

[ ] Output-feedback stabilization

    [ ] Observers and observability

    [ ] Observer-based control

[ ] Tracking and regulation via state-space design

    [ ] Preliminaries

    [ ] Solution to the regulation problem

X = The upcoming topic

- = Topic that has been covered

Information for the final exam

- Date, time, location:

Thursday, August 10, 2023. 9:00 AM-10:30 AM, MC 4063.

- Rules: Similar to midterm,
  <> Can only bring course notes/your own notes from class/lecture slides.
  No cheat sheets/tutorial material/assignments etc.
  <> Can bring a computer/tablet to access above described material. Download notes apriori.
  No internet/search/Learn access, no use of computer for anything other than described above.
  <> Can bring a calculator. No smartphones, or phones of any kind should be on you during the exam.

- Material covered: the final exam covers everything that we've studied so far, with an emphasis on chapters 4 and onwards. In general, we have studied (from the course notes):

  Chapters 2-4 in entirety.
  Chapter 5: 5.1-5.4 in the course notes.
  Chapter 6: Everything except 6.8.3 (Jury test).
  Chapter 7: 7.1-7.4, 7.6.
  Chapter 9: 9.1
  Extra Chapter on system identification.