Chapter 3: Pole placement using polynomials

ECE 481 - Digital Control Systems

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Based on course notes by Professor Chris Nielsen.

Example: Cessna Citation Aircraft

[Slide courtesy of Prof. Manfred Morari, University of Pennsylvania.]

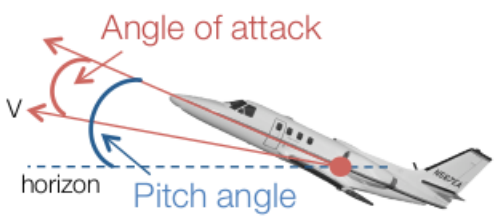
Linearized continuous-time model:
(at altitude of 5000m and a speed of 128.2 m/sec)

$$\dot{x} = \begin{bmatrix} -1.2822 & 0 & 0.98 & 0 \\ 0 & 0 & 1 & 0 \\ -5.4293 & 0 & -1.8366 & 0 \\ -128.2 & 128.2 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} -0.3 \\ 0 \\ -17 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

$x \in \mathbb{R}^n$,
$u \in \mathbb{R}^m$
$y \in \mathbb{R}^p$



- Input: elevator angle
- States: x_1: angle of attack, x_2: pitch angle, x_3: pitch rate, x_4: altitude
- Outputs: pitch angle and altitude
- Constraints: elevator angle ± 0.262rad ($\pm 15°$), elevator rate ± 0.524rad ($\pm 60°$), pitch angle ± 0.349 ($\pm 39°$)

Open-loop response is unstable (open-loop poles: $0, 0, -1.5594 \pm 2.29i$)

Outline

[X] Converting design specifications into desired pole locations

<> Assumptions, notation, good regions for poles

<> Step-response characteristics into constraints for closed-loop poles

[ ] Pole placement design

<> Plant and controller model

<> Pole Placement Problem (PPP) and existence of controllers

<> Placing poles where we need them to be: polynomial Diophantine equation

<> Limitations of pole placement

[ ] Pole placement for reference tracking control

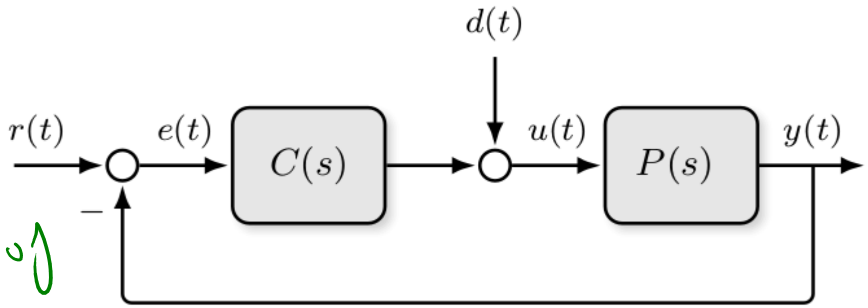
<> Asymptotic step tracking

<> Practical (relaxed) step tracking

X = The upcoming topic

\- = Topic that has been covered

Assumptions and notations



Assumption 3.1.1: The plant TF P(s) is rational and proper. The numerator and denominator of P(s) are co-prime and the denominator polynomial is monic.

Highest power of 's' has a co-efficient of 1, i.e.

$D(s) = s^n + a_{n-1}s^{n-1} + \dots + a_0$

Transfer functions and characteristic polynomial of the closed-loop system:

$$P(s) = \frac{N_p(s)}{D_p(s)}, \; C(s) = \frac{N_c(s)}{D_c(s)}$$

Characteristic polynomial of closed-loop system is thus:

$$\pi(s) = N_p N_c + D_p D_c$$

Objective of control design: Put the closed-loop poles in a good region

Given time-domain and/or frequency-domain specifications for the desired closed-loop response, put poles, or roots of the characteristic polynomial into a "good region" where these specs are met.

$$\text{Good region is a set } \mathbb{C}_g = \{s \in \mathbb{C}.\, \pi(s) = 0, \text{ other constraints on s}\} \subseteq \mathbb{C}^-$$

How do we obtain a "good region"?

We do so by approximating the closed-loop system as a prototypical 2nd order system, and use the relations of pole locations to time/frequency-domain characterstics that we have studied before.

Will justify this approximation in the final design (kind of).

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Step-response specification: Settling time

Given desired settling time $T_s^{\max}$, closed loop poles should be such that

$$\mathrm{Re}(s) \leq -\frac{4}{T_s^{\max}}$$

This gives us constraints to define a "good region":

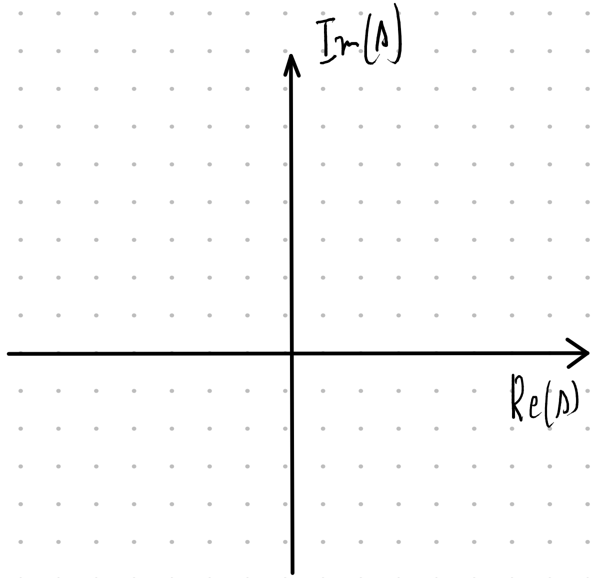


Figure: Good region for settling time to be below desired max.

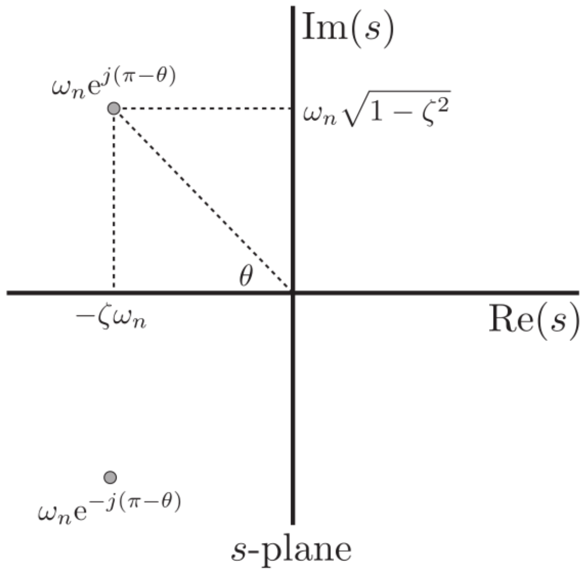
Step-response specification: Overshoot

As seen in the previous chapter, given max overshoot limit $\%OS^{max}$, the poles should be such that:

$$\mathbb{C}_g \subseteq \{s \in \mathbb{C}.\, |arg(s)| \geq \pi - \theta^{max}\}$$

where, $arg(s) \in (-\pi, \pi]$, and,

$$\theta^{max} = \arccos\left(-\frac{\ln(\%OS^{max})}{\sqrt{\pi^2 + (\ln(\%OS^{max}))^2}}\right)$$



Frequency-domain specification: Phase margin

Gain crossover frequency (GCF)

$|G| = 1$

$20 \log_{10} |G| = 0$

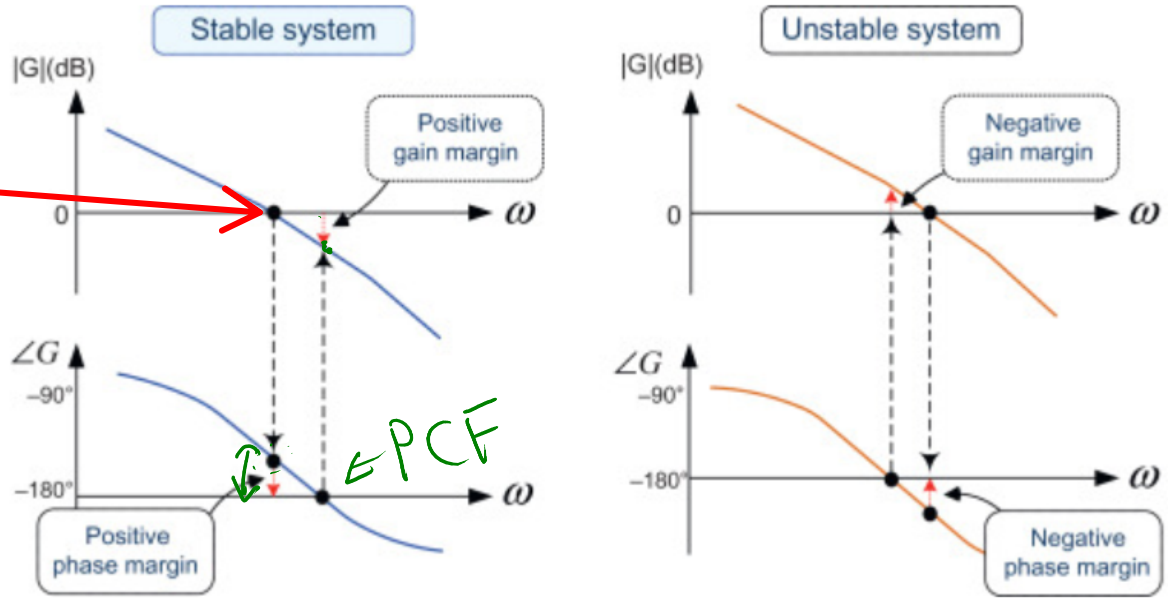


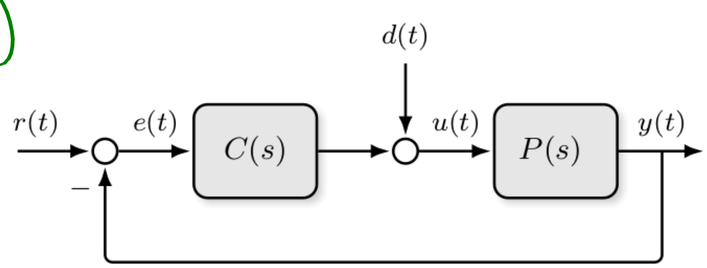
Figure from S-H Kim, Electric Motor Control, 2017.

Specification: Closed-loop system has a positive phase margin (for stability).

Computing phase margin for prototypical 2nd order systems

Loop gain

Consider a closed-loop system s.t. $C(s)P(s) = \dfrac{\omega_n^2}{s(s+2\zeta\omega_n)}$ $= L(s)$



The closed-loop transfer function from r to y is thus

$$\frac{Y(s)}{R(s)} = \frac{PC}{1+PC} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

← Prototypical 2nd order system.

We need to compute the GCF to compute the PM, set loop-gain to 1 and solve for w, i.e., $|C(j\omega)P(j\omega)| = 1$

This gives us:

$$\omega_{\mathrm{gc}} = \omega_n \sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}$$

$\angle L(j\omega_{gc})$

From this, we can get an expression for PM (via the dark arts):

$$\Rightarrow \quad \Phi_{\mathrm{pm}} = \arctan\left(\frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}}\right)$$

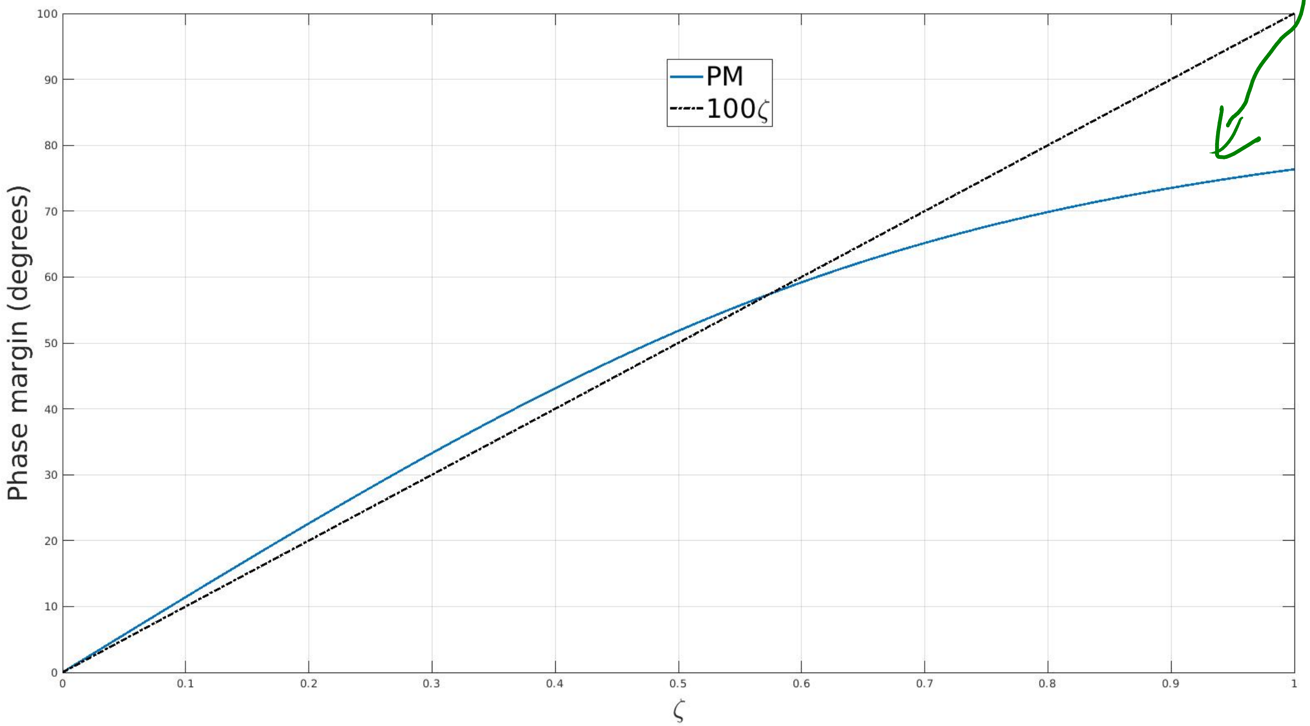
→ only depends on damping ratio (ζ)

Phase margin as a function of damping ratio (and an approximation)

Note: PM is only a function of the damping ratio and not of the natural frequency.

$$\Phi_{\text{pm}} = \arctan\left(\frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^4}-2\zeta^2}}\right)$$

We can get a linear approximation of the function as shown below:



The approximation (in degrees) $PM = 100\zeta$ works well when $0 \leq \zeta \leq 0.7$

Alternatively, we can also approximate PM (in degrees) in terms of overshoot
$\text{PM} = 60(1 - \%OS)$, where $\%OS \in [0.4, 1]$

Summary: Good region to place closed-loop poles from specifications

0. Assume a 2nd order closed-loop system.
1. We get constraints for the closed-loop poles from specifications.
2. The constraints define regions where the poles should lie in.
3. Intersection of these regions (a region for each specification) gives us the "good region" where the closed-loop poles should lie.

e.g., We want settling time below a threshold, and max overshoot below a given value.

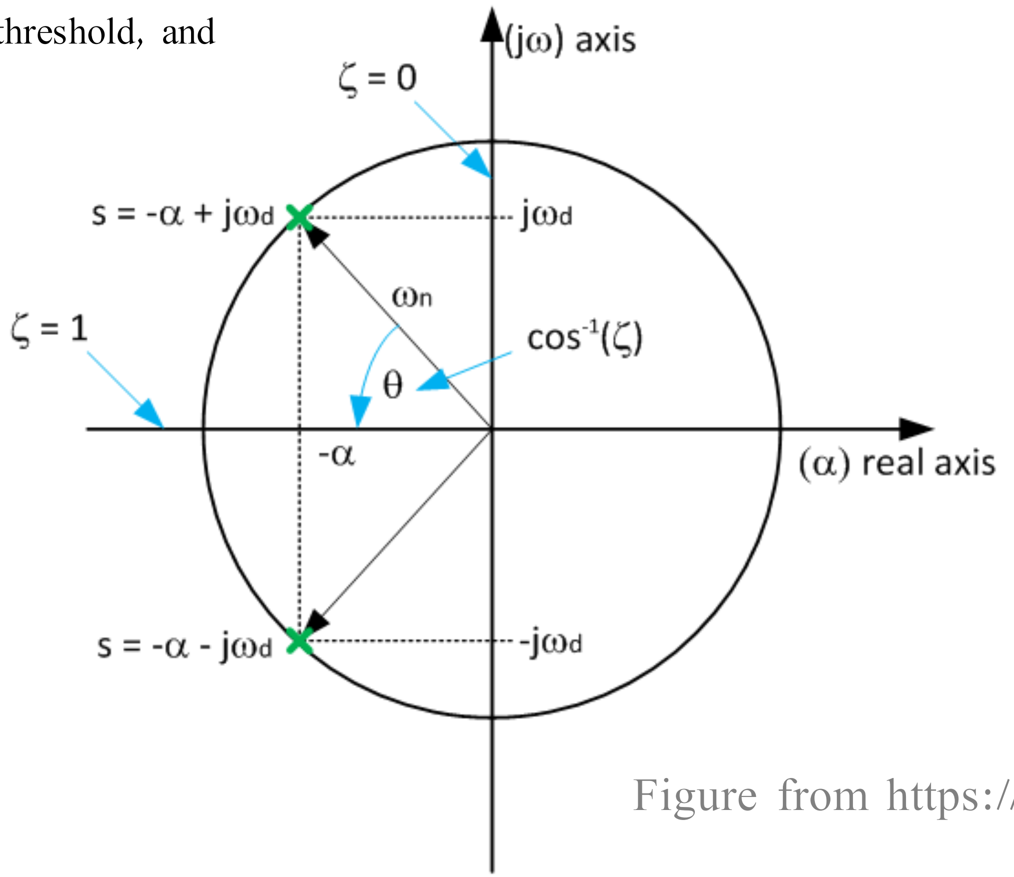


Figure from https://controlsystemsacademy.com

So how do we design a controller such that closed-loop poles are as selected (in the good region)?

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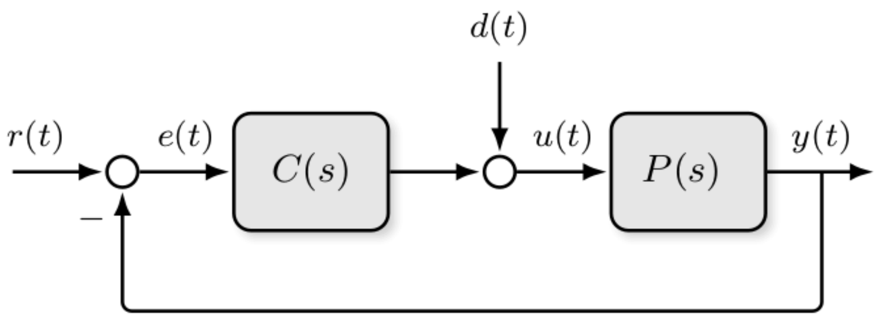
Pole Placement Design

We want to design a controller such that the closed-loop poles are placed at points in the "good region".



Consider a general problem with the following plant (with known coefficients):

$$P(s) = \frac{N_p(s)}{D_p(s)} = \frac{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

We want to find the coefficients (or parameters) for the controller of order 'm'

$$C(s) = \frac{N_c(s)}{D_c(s)} = \frac{g_m s^m + g_{m-1}s^{m-1} + \cdots + g_1 s + g_0}{f_m s^m + f_{m-1}s^{m-1} + \cdots + f_1 s + f_0}$$

How many design parameters?

$2(m+1) = 2m+2$

The closed-loop characteristic polynomial is thus:

$$\pi(s) = D_p(s)D_c(s) + N_p(s)N_c(s)$$

What is the (max) degree of the characteristic polynomial?

$n+m$

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Pole Placement Problem (PPP), if $\lambda \in \mathbb{C} \in \Lambda$ then $\bar{\lambda}$ is also $\in \Lambda$

Given plant $P(s)$ and symmetric set of n+m desired closed-loop pole locations $\Lambda = \{\lambda_1, \ldots, \lambda_{n+m}\} \subset \mathbb{C}_g$, find the parameters of a controller C(s) such that the closed-loop poles of the system all lie in Λ.

Assumption 3.1.1. The plant TF $P(s)$ is rational and proper. The numerator and denominator polynomials of the plant are coprime and the denominator polynomial is monic. ◀

Pole Placement Problem (PPP): existence of a controller

Given plant $P(s)$ and symmetric set of n+m desired closed-loop pole locations $\Lambda = \{\lambda_1, \ldots, \lambda_{n+m}\} \subset \mathbb{C}_g$, find the parameters of a controller C(s) such that the closed-loop poles of the system all lie in Λ.

Assumption 3.1.1. The plant TF $P(s)$ is rational and proper. The numerator and denominator polynomials of the plant are coprime and the denominator polynomial is monic. ◀

Theorem 3.2.1. *Suppose that Assumption 3.1.1 holds. There exists an mth order controller that solves the Pole Placement Problem for any symmetric set* $\underbrace{\{\lambda_1, \ldots, \lambda_{n+m}\}}_{\Lambda \text{ (desired pole locations)}}$ *if, and only if,* $m \geq n - 1$.

Note: The controller C(s) is unique IFF m = n -1. If m > n-1, then there could exist multiple controllers to get the same closed-loop poles. Finally, if m < n-1, a solution to the PPP may not exist.

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Deriving the controller design equations

Plant n
Controller $m = n-1$

Let $\{\lambda_1, \ldots, \lambda_{2n-1}\}$ be the 2n-1 desired pole locations of the closed-loop system picked to lie in the 'good region'.

The resulting (desired) closed-loop characteristic polynomical is thus:

$$\pi_{des}(s) = \prod_{i=1}^{2n-1} (s - \lambda_i)$$

$$:= s^{2n-1} + \alpha_{2n-2} s^{2n-2} + \ldots + \alpha_1 s + \alpha_0$$

$$= N_p N_c + D_p D_c$$

We know λ_i's $\Rightarrow$ we can compute α_i's.

What should the coefficients of the desired characteristic polynomial be？（Polynomial Diophantine Equation）

Recall:

$$P(s) = \frac{N_p(s)}{D_p(s)} = \frac{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

$$C(s) = \frac{N_c(s)}{D_c(s)} = \frac{g_m s^m + g_{m-1}s^{m-1} + \cdots + g_1 s + g_0}{f_m s^m + f_{m-1}s^{m-1} + \cdots + f_1 s + f_0}$$ Where, $m = n-1$

Can write equations as:

$Ax = b$

We want:

$$\pi = D_p D_c + N_p N_c = \pi_{des} = s^{2n-1} + \alpha_{2n-2}s^{2n-2} + \ldots + \alpha_1 s + \alpha_0$$

Let's compare coefficients of the ch.p. to those of the desired ch.p.:

$2n-1$: $(b_n g_{n-1} + 1 \cdot f_{n-1}) = 1$

$2n-2$: $\ldots$

Design variables.

$\vdots$

0 : $b_0 g_0 + a_0 f_0 = \alpha_0$

2n equations

Compare the coefficients: Sylvester matrix

Write the system of equations in a vector form:

Vector of unknowns (controller params)

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & b_n & \cdots & \cdots & 0 \\ a_{n-1} & 1 & & \vdots & b_{n-1} & b_n & & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & \vdots & & \ddots & b_n \\ a_0 & \cdots & \cdots & a_{n-1} & b_0 & \cdots & \cdots & b_{n-1} \\ 0 & \ddots & & \vdots & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & a_0 & 0 & \cdots & \cdots & b_0 \end{bmatrix} \begin{bmatrix} f_{n-1} \\ \vdots \\ \vdots \\ f_0 \\ g_{n-1} \\ \vdots \\ \vdots \\ g_0 \end{bmatrix} = \begin{bmatrix} 1 \\ \alpha_{2n-2} \\ \vdots \\ \alpha_n \\ \alpha_{n-1} \\ \vdots \\ \alpha_0 \end{bmatrix}$$

Desired ch. p. coeffs.

Plant num & den coeffs

Compare the coefficients: Sylvester matrix

Write the system of equations in a vector form:

$$
\begin{bmatrix}
1 & 0 & \cdots & 0 & b_n & \cdots & \cdots & 0 \\
a_{n-1} & 1 & & \vdots & b_{n-1} & b_n & & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & 1 & \vdots & & \ddots & b_n \\
a_0 & \cdots & \cdots & a_{n-1} & b_0 & \cdots & \cdots & b_{n-1} \\
0 & \ddots & & \vdots & 0 & \ddots & & \vdots \\
\vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\
0 & \cdots & \cdots & a_0 & 0 & \cdots & \cdots & b_0
\end{bmatrix}
\begin{bmatrix}
f_{n-1} \\ \vdots \\ \vdots \\ f_0 \\ g_{n-1} \\ \vdots \\ \vdots \\ g_0
\end{bmatrix}
=
\begin{bmatrix}
1 \\ \alpha_{2n-2} \\ \vdots \\ \alpha_n \\ \alpha_{n-1} \\ \vdots \\ \alpha_0
\end{bmatrix}
$$

Solve this system of linear equations to get controller parameters (or gains);

$$f_{n-1}, f_{n-2}, \ldots, f_0, g_{n-1}, g_{n-2}, \ldots, g_0$$

When is the Sylvester matrix full-rank (or invertible)?

Let $a(s)$, $b(s)$ be two polynomials of 's' such that: (monic)

$$a(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots a_1 s + a_0, \qquad a_n \neq 0, \quad a_n = 1$$

$$b(s) = b_m s^m + b_{m-1} s^{m-1} + \cdots b_1 s + b_0, \qquad b_m \neq 0$$

Denote the Sylvester matrix formed by these as:

$$\mathbf{S}(a(s), b(s)) := \begin{bmatrix} 1 & 0 & \cdots & 0 & b_n & \cdots & \cdots & 0 \\ a_{n-1} & 1 & & \vdots & b_{n-1} & b_n & & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & \vdots & & \ddots & b_n \\ a_0 & \cdots & \cdots & a_{n-1} & b_0 & \cdots & \cdots & b_{n-1} \\ 0 & \ddots & & \vdots & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & a_0 & 0 & \cdots & \cdots & b_0 \end{bmatrix}$$

Note: Laziness made n=m here, but they do not have to be equal. a(s) and b(s) should denote the denominator and numerator (respectively) of a rational, proper transfer function.

When is the Sylvester matrix full-rank (or invertible)?

Let $a(s)$, $b(s)$ be two polynomials of 's' as seen earlier,
then we have the following theorem (Appendix 3.A. in notes for details)

Theorem 3.A.1. *The following statements are equivalent*

(a) $a(s)$ *and* $b(s)$ *are coprime.*

(b) $\det(\mathbf{S}(a(s), b(s))) \neq 0$. $\Leftrightarrow$ Full rank $\Leftrightarrow \exists S^{-1}$

(c) There exist unique polynomials $x, y \in \mathbb{R}[s]$ *with* $\deg(x(s)) < m$ *and* $\deg(y(s)) < n$ *such that*

$$a(s)x(s) + b(s)y(s) = 1.$$

Bezout equation.

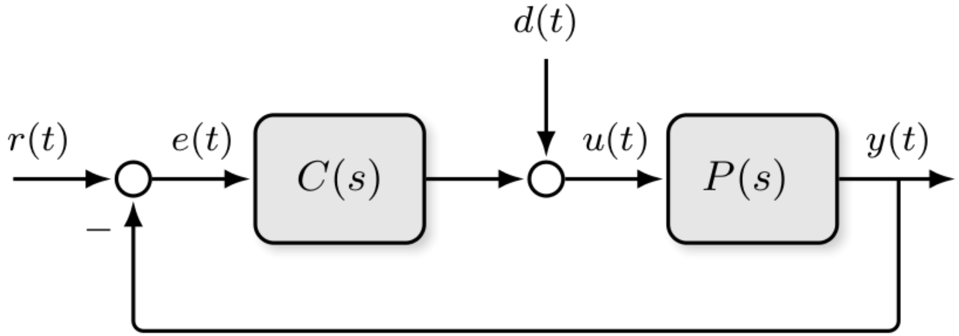
Assumption 3.1.1

Plant num $(b(s))$,

den $(a(s))$

are coprime.

An example: Second order plant



$$P(s) = \frac{1}{s(s+1)} = \frac{1}{s^2+s+0} =: \frac{b_2 s^2 + \cancel{b_1 s} + b_0}{s^2 + a_1 s + a_0}$$

Requirements: Closed-loop stability and settling time <=2 seconds. $\quad \mathrm{Re}(s) \leq -\dfrac{4}{T_s^{max}}$

Good region for CL poles: $\mathbb{C}_g = \{s \in \mathbb{C} : \mathrm{Re}(s) \leq -2\} \subseteq \mathbb{C}^-$

Controller (order 2-1=1): $\quad C(s) = \dfrac{g_1 s + g_0}{f_1 s + f_0}$

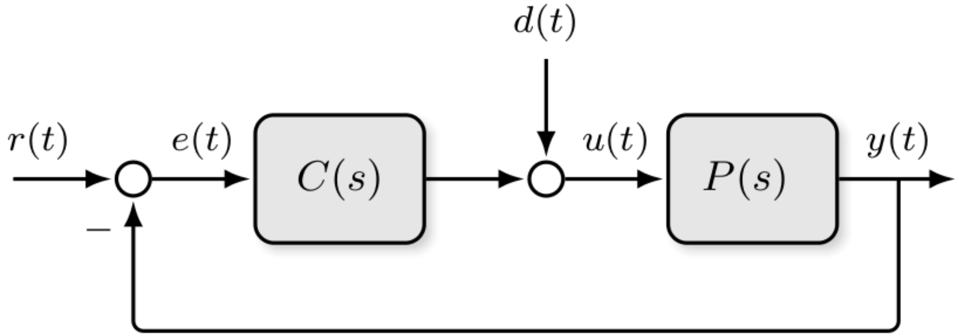
3 Poles to be placed; Dominant: s = -3-j, -3+j (complex conjugates), non-dominant: s = -10

$$\begin{aligned}
& = \prod_{i=1}^{3}(s-\lambda_i) \\
\pi_{\mathrm{des}}(s) &= (s+3+j)(s+3-j)(s+10) \\
&= s^3 + 16s^2 + 70s + 100 \\
&=: s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0.
\end{aligned}
\quad \Longrightarrow \quad
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} f_1 \\ f_0 \\ g_1 \\ g_0 \end{bmatrix} =
\begin{bmatrix} 1 \\ 16 \\ 70 \\ 100 \end{bmatrix} \leftarrow \alpha_i\text{'s}$$

$$\Rightarrow \quad f_1 = 1, \quad f_0 = 15, \quad g_1 = 55, \quad g_0 = 100$$

An example: Second order plant

$$P(s) = \frac{1}{s(s+1)} = \frac{1}{s^2+s+0} =: \frac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0}$$



Requirements: Closed-loop stability and settling time <=2 seconds.

Pole placement gives us the controller: $C(s) = (55s+100)/(s+15)$

Closed-loop TF from R to Y:

$$\frac{Y(s)}{R(s)} = \frac{C(s)P(s)}{1+C(s)P(s)} = \frac{55s+100}{s^3+16s^2+70s+100} = \frac{55s+100}{(s+3+j)(s+3-j)(s+10)}$$

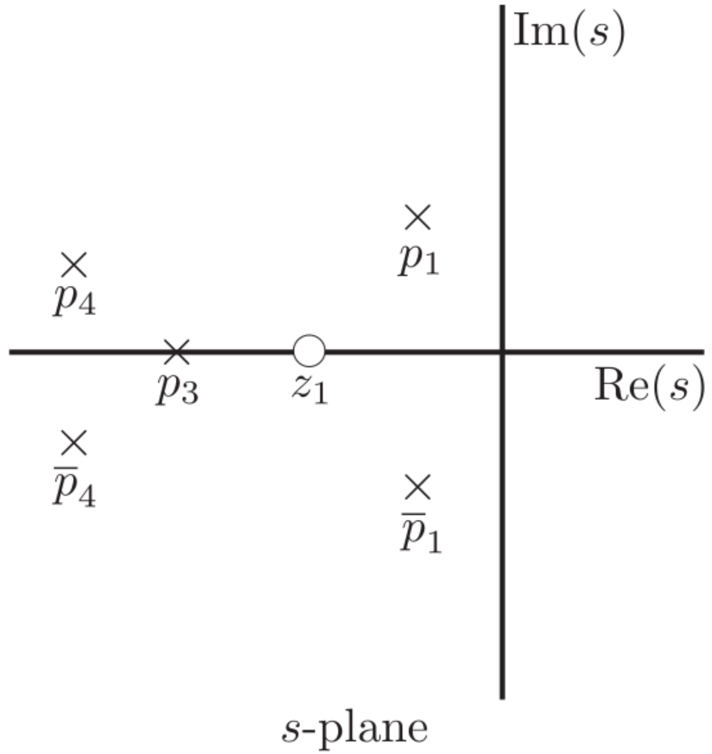
Resulting system has a settling time of 1.73s (and is of course, stable).

Some characteristics of the pole-placement technique

- If Np and Dp are co-prime, the Sylvester matrix is invertible and there exists a unique solution for the controller parameters.

- The good region is obtained for a prototype second order system. The closed-loop system is not 2nd order, e.g., consider a n=2 (plant order), so the controller order has to be m=n-1=1.

- To make the exhibit the characteristics of a 2nd order system (to some extent), the desired pole locations should include two dominant complex conjugate poles.



Recall: Dominant poles and zeros from chapter 2.

Pole placement with strictly proper controllers

- We have designed controllers of order n-1 to place poles at arbitrary desired locations.
- The transfer function for these controllers is not necessarily proper, e.g., see below

$$P(s) = \frac{1}{s(s+1)} = \frac{1}{s^2+s+0} =: \frac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0} \qquad C(s) = (55s+100)/(s+15) = C_0 + C_1(s)$$

C_1 is a strictly proper T.F.

- Consider a discontinuous reference signal for the system to track

Control signal can be discontinuous (why? recall feed-forward control), which might damage actuators!

(For proper, but not strictly proper controllers)

Pole placement with strictly proper controllers

Theorem 3.2.3. *Suppose that Assumption 3.1.1 holds. There exists an mth order strictly proper controller that solves the PPP for any symmetric set* $\{\lambda_1, \ldots, \lambda_{n+m}\}$ *if, and only if,* $m \geq n$.

Compare this to result for pole placement with proper controllers.

Designing the controller: Pole placement via strictly proper controllers

Say controller order is n (so is plant order), therefore we now have 2n closed-loop pole locations.

$$\pi_{\text{des}}(s) = \prod_{i=1}^{2n}(s-\lambda_i) =: s^{2n} + \alpha_{2n-1}s^{2n-1} + \alpha_{2n-2}s^{2n-2} + \cdots + \alpha_1 s + \alpha_0$$

Designing the controller: Pole placement via strictly proper controllers

Say controller order is n (so is plant order), therefore we now have 2n closed-loop pole locations.

$$\pi_{\text{des}}(s) = \prod_{i=1}^{2n}(s - \lambda_i) =: s^{2n} + \alpha_{2n-1}s^{2n-1} + \alpha_{2n-2}s^{2n-2} + \cdots + \alpha_1 s + \alpha_0$$

Controller transfer function is: $C(s) = \dfrac{g_{n-1}s^{n-1} + g_{n-2}s^{n-2} + \ldots + g_1 s + g_0}{\underbrace{f_n s^n}_{\text{monic}} + f_{n-1}s^{n-1} + \ldots + f_1 s + f_0}$ Let $f_n = 1$

Strictly proper controller TF.

Designing the controller: Pole placement via strictly proper controllers

Say controller order is n (so is plant order), therefore we now have 2n closed-loop pole locations.

$$\pi_{\text{des}}(s) = \prod_{i=1}^{2n}(s - \lambda_i) =: s^{2n} + \alpha_{2n-1}s^{2n-1} + \alpha_{2n-2}s^{2n-2} + \cdots + \alpha_1 s + \alpha_0$$

Controller transfer function is: $C(s) = \dfrac{g_{n-1}s^{n-1} + g_{n-2}s^{n-2} + \ldots + g_1 s + g_0}{f_n s^n + f_{n-1}s^{n-1} + \ldots + f_1 s + f_0}$ Let $f_n = 1$

Compare coefficients between π_{des} and $\pi(s) = D_p(s)D_c(s) + N_p(s)N_c(s)$

Designing the controller: Pole placement via strictly proper controllers

Say controller order is n (so is plant order), therefore we now have 2n closed-loop pole locations.

$$\pi_{\text{des}}(s) = \prod_{i=1}^{2n}(s-\lambda_i) =: s^{2n} + \alpha_{2n-1}s^{2n-1} + \alpha_{2n-2}s^{2n-2} + \cdots + \alpha_1 s + \alpha_0$$

Controller transfer function is: $C(s) = \dfrac{g_{n-1}s^{n-1} + g_{n-2}s^{n-2} + \ldots + g_1 s + g_0}{f_n s^n + f_{n-1}s^{n-1} + \ldots + f_1 s + f_0}$ Let $\underline{f_n = 1}$

Compare coefficients between π_{des} and $\pi(s) = D_p(s)D_c(s) + N_p(s)N_c(s)$

SYLVESTER MATRIX.   ← Unknowns

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & b_n & \cdots & \cdots & 0 \\ a_{n-1} & 1 & & \vdots & b_{n-1} & b_n & & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & \vdots & & \ddots & b_n \\ a_0 & \cdots & \cdots & a_{n-1} & b_0 & \cdots & \cdots & b_{n-1} \\ 0 & \ddots & & \vdots & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & a_0 & 0 & \cdots & \cdots & b_0 \end{bmatrix} \begin{bmatrix} f_{n-1} \\ \vdots \\ \vdots \\ f_0 \\ g_{n-1} \\ \vdots \\ \vdots \\ g_0 \end{bmatrix} = \begin{bmatrix} \alpha_{2n-1} - a_{n-1} \\ \alpha_{2n-2} - a_{n-2} \\ \vdots \\ \alpha_n - a_0 \\ \alpha_{n-1} \\ \vdots \\ \alpha_0 \end{bmatrix}$$

Compare this to what we had for the (not strictly) proper controller case.

Example, and comparison with a (not strictly) proper controller

Same plant as the earlier example: $P(s) = \dfrac{1}{s(s+1)} = \dfrac{1}{s^2+s+0} =: \dfrac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0}$

Controller (same order as plant, strictly proper): $C(s) = \dfrac{g_1 s + g_0}{f_2 s^2 + f_1 s + f_0}$

Example, and comparison with a (not strictly) proper controller

Same plant as the earlier example: $P(s) = \frac{1}{s(s+1)} = \frac{1}{s^2+s+0} =: \frac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0}$

Controller (same order as plant, strictly proper): $C(s) = \frac{g_1 s + g_0}{f_2 s^2 + f_1 s + f_0}$

Desired poles: $\{-3-j, -3+j, -10, -11\}$

$$\Rightarrow \pi_{\text{des}}(s) = (s+3+j)(s+3-j)(s+10)(s+11) = s^4 + 27s^3 + 246s^2 + 870s + 1100$$

Compare coefficients and write the equations in matrix form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_0 \\ g_1 \\ g_0 \end{bmatrix} = \begin{bmatrix} 27-1 \\ 246-0 \\ 870 \\ 1100 \end{bmatrix} \quad \Rightarrow$$

Gives us the controller:

Closed-loop transfer function (R to Y): $\frac{Y(s)}{R(s)} = \frac{C(s)P(s)}{1+C(s)P(s)} = \frac{630s + 990}{(s^2+6s+10)(s+10)(s+11)}$

Compare to (non strictly) proper controller and resulting CL TF:

$$\boxed{C(s) = (55s+100)/(s+15)}$$

$$\frac{55s + 100}{(s+3+j)(s+3-j)(s+10)}$$

Frequency domain comparison between proper and strictly proper controller design

No free lunch.

Outline

[-] Converting design specifications into desired pole locations

<-> Assumptions, notation, good regions for poles

<-> Step-response characteristics into constraints for closed-loop poles

[-] Pole placement design

<-> Plant and controller model

<-> Pole Placement Problem (PPP) and existence of controllers

<-> Placing poles where we need them to be: polynomial Diophantine equation

<X> Limitations of pole placement

[ ] Pole placement for reference tracking control

<> Asymptotic step tracking

<> Practical (relaxed) step tracking

X = The upcoming topic

- = Topic that has been covered

Limitations of the pole placement method

- In most cases, the resulting system is not 2nd order and may not really behave like one.

- Higher the order of the plant, the less effective our approximations of the good region become.

- No idea where closed-loop zeros will lie (at the design stage)!

- Consequently, little idea about what is happening in the frequency domain and the closed-loop system may have bad characteristics such as poor sensitivity, low bandwidth etc., limiting use in practice.

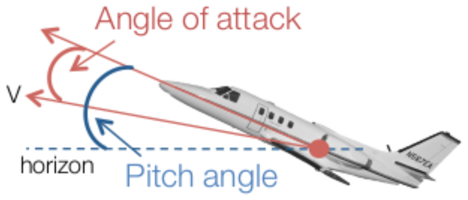
- Not all real-world systems will meet the assumptions that we have made (e.g., assumption 3.1.1).

Example: Cessna Citation Aircraft

Linearized continuous-time model:
(at altitude of 5000m and a speed of 128.2 m/sec)

$$\dot{x} = \begin{bmatrix} -1.2822 & 0 & 0.98 & 0 \\ 0 & 0 & 1 & 0 \\ -5.4293 & 0 & -1.8366 & 0 \\ -128.2 & 128.2 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} -0.3 \\ 0 \\ -17 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x$$



- Input: elevator angle
- States: x_1: angle of attack, x_2: pitch angle, x_3: pitch rate, x_4: altitude
- Outputs: pitch angle and altitude
- Constraints: elevator angle ± 0.262rad ($\pm 15°$), elevator rate ± 0.524rad ($\pm 60°$), pitch angle ± 0.349 ($\pm 39°$)

Open-loop response is unstable (open-loop poles: 0, 0, $-1.5594 \pm 2.29i$)

Objective: Design a controller to track a reference pitch angle.

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<-> Pole Placement Problem (PPP) and existence of controllers
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<-> Limitations of pole placement

[X] Pole placement for reference tracking control
<> Asymptotic step tracking
<> Practical (relaxed) step tracking

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Pole placement and tracking error

- So far we have seen how to place poles in "good regions" to satsify design specifications.

- An important design specification is in terms of tracking error, i.e., given a positive constant $e_{\text{ss}}^{\max}$, we want the tracking performance to satisfy the steady-state specification:

$$|e_{\text{ss}}| = |r(t) - y_{\text{ss}}| \leq e_{\text{ss}}^{\max}$$

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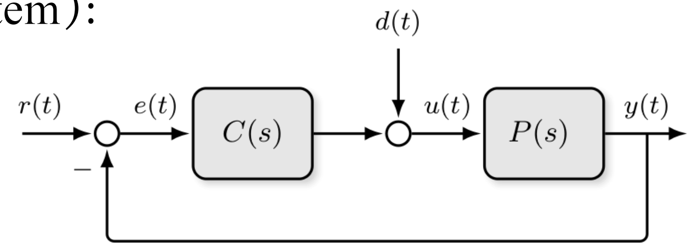
Asymptotic step tracking

Asymptotic tracking occurs when the steady-state error goes to zero, i.e., $e_{\text{ss}}^{\max} = 0$

The steady-state tracking error for a step input (assuming stable CL-system):



$$
\begin{aligned}
e_{\text{ss}} &= \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) \\
&= \lim_{s\to 0} s\frac{1}{1+PC}\frac{1}{s} \\
&= \lim_{s\to 0} \frac{1}{1+PC}.
\end{aligned}
$$

Recall (from the internal model principle) that asymptotic step tracking requires PC have a pole at s=0.

Asymptotic step tracking: Cases

Case 1: P(s) has a pole at s=0. The "regular" pole-placement method is sufficient to get perfect tracking.

Case 2: P(s) has a zero at s=0. Asymptotic step tracking is not possible?

Why? Can't we take a controller C(S) = (1/s)K(s)?

Case 3: P(s) has neither a pole or zero at s=0.

Recall the internal model principle and introduce a pole at s=0 in the loop-gain (PC)

Choose $C(s) = C_1(s)/s$

$\Rightarrow P(s)C(s) = P(s)C_1(s)/s$, has a pole at s=0

Can now use regular pole placement to design a controller C1(s) for the augmented plant P(s)/s.

Note: If P(s) had order n, then the augmented plant has order n+1. This implies that we need a controller C1(s) of order n to solve the pole placement problem. The final controller C = C1(s)/s this has order n+1.

Example

$$P(s) = \frac{1-s}{s^2+1}$$

Definition: Minimum phase system.

An LTI system is minimum phase if the system and its inverse are both causal and stable.

Example

$$P(s) = \frac{1-s}{s^2+1}$$

Design a controller such that the closed loop system has:

1. Maximum settling time of 2 seconds
2. Maximum overshoot of $e^{-\pi}$
3. Asymptotic step tracking.

Example

$$P(s) = \frac{1-s}{s^2+1}$$

Design a controller such that the closed loop system has:

1. Maximum settling time of 2 seconds
2. Maximum overshoot of $e^{-\pi}$
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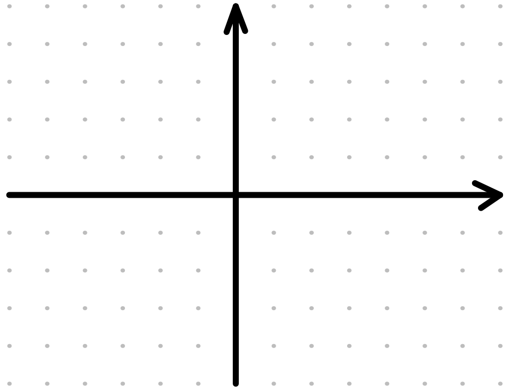

Transient response specifications (1 and 2) give the good region: $\mathbb{C}_g = \{s \in \mathbb{C} : \mathrm{Re}\,(s) \leq -2\} \cap \{s \in \mathbb{C} : |\arg\,(s)| \geq 3\pi/4\}$

For asymptotic step tracking, we will augment the plant with a pole at s=0: $\dfrac{P(s)}{s} = \dfrac{1-s}{s^3+s}$

We have to design a controller C(s) = C1(s)/s, where:

$$C_1(s) = \frac{g_2 s^2 + g_1 s + g_0}{f_2 s^2 + f_1 s + f_0}$$

Place the closed-loop poles (5 of them) at: $\{-2 \pm j, -8, -9, -10\}$

$\pi_{\text{des}}(s) = s^5 + 31s^4 + 355s^3 + 1823s^2 + 4090s + 3600$ The pole placement equation (Sylvester matrix):

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_2 \\ f_1 \\ f_0 \\ g_2 \\ g_1 \\ g_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 31 \\ 355 \\ 1823 \\ 4090 \\ 3600 \end{bmatrix}$$

Solving this gives us the controller C(s) = C1(s)/s, where : $C_1(s) = \dfrac{4564s^2 + 2772s + 3600}{s^2 + 31s + 4918}$

Outline

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<X> Practical (relaxed) step tracking

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Practical step tracking

Assume the feedback system is stable, the reference is r(t) = 1(t) (a step).

Suppose the tracking requirements are relaxed: $e_{ss}^{\max} \neq 0$

Recall: $e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1+PC}\frac{1}{s} = \lim_{s\to 0}\frac{1}{1+PC}$

If $P(0) \neq \infty$ and $P(0) \neq 0$, then:

$$e_{ss} = \frac{1}{1+P(0)C(0)}$$

Depends on C(0), want abs(e_ss) below max.

We need to place poles 'arbitrarily' (requires a n-1 order controller), and also adjust C(0).

Approach: Increase the order of the controller to get the extra degree of freedom.

$$C(s) = \frac{g_n s^n + \cdots + g_1 s + g_0}{f_n s^n + \cdots + f_1 s + f_0}$$

Practical step tracking: 2nd order systems

The plant model: $$P(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0}$$

Practical step tracking: 2nd order systems

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Controller (of order n=2): $$C(s) = \frac{g_2 s^2 + g_1 s + g_0}{f_2 s^2 + f_1 s + f_0}$$

Practical step tracking: 2nd order systems

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Assume that the desired pole locations are: $\Lambda = \{\lambda_1, \ldots, \lambda_4\}$

Write down the Polynomial Diophantine equation (match the CL and desired characteristic polynomials)

Practical step tracking: 2nd order systems

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\begin{aligned}
\pi(s) &= D_p D_c + N_p N_c \\
&= (f_2 + b_2 g_2) s^4 + (a_1 f_2 + b_1 g_2 + f_1)\, s^3 + (f_0 + a_0 f_2 + a_1 f_1 + b_0 g_2 + b_1 g_1)\, s^2 \\
&\quad + (a_0 f_1 + a_1 f_0 + b_0 g_1 + b_1 g_0)\, s + a_0 f_0 + b_0 g_0 .
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$$\pi_{\text{des}}(s) = (s - \lambda_1)(s - \lambda_2)(s - \lambda_3)(s - \lambda_4) =: s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0$$

Practical step tracking: 2nd order systems

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$$\pi_{\text{des}}(s) = (s - \lambda_1)(s - \lambda_2)(s - \lambda_3)(s - \lambda_4) =: s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0$$

How many equations do we have ? How many unknowns ? What is missing ?

Practical step tracking: 2nd order systems

The plant model: $P(s) = \dfrac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0}$

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Take the steady-state error (bound) requirement: $|e_{\text{ss}}| = \left| \dfrac{1}{1 + P(0)C(0)} \right| \leq e_{\text{ss}}^{\max}$

Practical step tracking: 2nd order systems

The plant model: $P(s) = \dfrac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0}$

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\end{aligned}
$$

$$\pi_{\text{des}}(s) = (s-\lambda_1)(s-\lambda_2)(s-\lambda_3)(s-\lambda_4) =: s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0$$

Take the steady-state error (bound) requirement: $|e_{\text{ss}}| = \left|\dfrac{1}{1+P(0)C(0)}\right| \leq e_{\text{ss}}^{\max}$

More than one equation, so let us satisfy it with an equality:

System of equations for relaxed step tracking ('modified' Sylvester matrix)

5 equations from the CL and desired characteristic polynomials:

$$\begin{aligned}\pi(s) &= D_p D_c + N_p N_c \\ &= (f_2 + b_2 g_2)s^4 + (a_1 f_2 + b_1 g_2 + f_1)\, s^3 + (f_0 + a_0 f_2 + a_1 f_1 + b_0 g_2 + b_1 g_1)\, s^2 \\ &\quad + (a_0 f_1 + a_1 f_0 + b_0 g_1 + b_1 g_0)\, s + a_0 f_0 + b_0 g_0.\end{aligned}$$

$$\pi_{\text{des}}(s) = (s-\lambda_1)(s-\lambda_2)(s-\lambda_3)(s-\lambda_4) =: s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0$$

1 equation from the steady-state error requirement: $b_0 g_0 - \delta a_0 f_0 = 0$

Put them together to get a system of 6 equations with 6 unknowns:

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & b_2 & 0 & 0 \\ a_1 & 1 & 0 & b_1 & b_2 & 0 \\ a_0 & a_1 & 1 & b_0 & b_1 & b_2 \\ 0 & a_0 & a_1 & 0 & b_0 & b_1 \\ 0 & 0 & a_0 & 0 & 0 & b_0 \\ \hdashline 0 & 0 & -\delta a_0 & 0 & 0 & b_0 \end{array}\right] \begin{bmatrix} f_2 \\ f_1 \\ f_0 \\ g_2 \\ g_1 \\ g_0 \end{bmatrix} = \left[\begin{array}{c} 1 \\ \alpha_3 \\ \alpha_2 \\ \alpha_1 \\ \alpha_0 \\ \hdashline 0 \end{array}\right]$$

Note:

- Num and den of plant are co-prime and P(s) has no poles/zeros at s=0, implies unique solution.
- Approach generalizes to systems of order n.

An example (2nd order)

$$P(s) = \frac{1}{(s+0.5)^2} = \frac{1}{s^2+s+0.25}$$

Requirements:
- CL stability
- Max. step tracking error of 5%
- Step response has: a) Max. overshoot of 20%, b) Max settling time of 4s.

Characterizing the "good region":

- Stability and setting time: $\mathbb{C}_g \subseteq \{s \in \mathbb{C} : \mathrm{Re}\,(s) \leq -1\}$
- Max overshoot: $\%\mathrm{OS} \leq 0.2$

$$\Rightarrow \zeta \geq -\frac{\ln \%\mathrm{OS}}{\sqrt{\pi^2 + (\ln \%\mathrm{OS})^2}} = 0.4459 \qquad \Rightarrow \theta \leq \arccos(0.4459)$$

$$\theta \leq 62.9^\circ \ (\approx 1.1 \text{ rad}).$$

Recall that arg(s) = $\pi - \theta$ (rad)

Intersect these to get the good region for placing poles in the complex plane:

$$\mathbb{C}_g = \{s \in \mathbb{C} : \mathrm{Re}\,(s) \leq -1\} \cap \{s \in \mathbb{C} : |\arg(s)| \geq 2.04\}$$

An example (2nd order), continued

$$\mathbb{C}_g = \{s \in \mathbb{C} : \mathrm{Re}\,(s) \le -1\} \cap \{s \in \mathbb{C} : |\arg\,(s)| \ge 2.04\}$$

Pick 4 CL poles (2nd order plant, 2nd order controller): $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{-1+j1.95, -1-j1.95, -5-5\}$

CL CP is: $\pi_{\mathrm{des}}(s) = (s-\lambda_1)(s-\lambda_2)(s-\lambda_3)(s-\lambda_4) = s^4 + 12s^3 + 49.8s^2 + 98.03s + 120.1$

Steady-state error bound (desired) gives us: $\delta = \dfrac{1}{e_{\mathrm{ss}}^{\mathrm{max}}} - 1 = \dfrac{1}{0.05} - 1 = 19$

The controller is 2nd order: $C(s) = \dfrac{g_2 s^2 + g_1 s + g_0}{f_2 s^2 + f_1 s + f_0}$

The design equations for matching the CP coeffs and respecting the steady-state tracking requirement are:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & 1 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 1 \\ 0 & 0 & -\frac{19}{4} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_2 \\ f_1 \\ f_0 \\ g_2 \\ g_1 \\ g_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 12 \\ 49.8 \\ 98.03 \\ 120.1 \\ 0 \end{bmatrix}$$

The resulting controller is:

$$C(s) = \frac{14.53s^2 + 71.26s + 114.1}{s^2 + 11s + 24.02}$$

An example (2nd order), simulation results

The CL step-response characteristics are:

- %OS = 25% > 20% (not satisfied!)
- Ts = 3.8s
- e_ss = 0.05

An example (2nd order), simulation results

Recall that max overshoot (%) = 100*(max magnitude - steady state value)/(steady state value)

Overshoot > 20% (why?)

An example (2nd order), simulation results

Recall that max overshoot (%) = 100*(max magnitude - steady state value)/(steady state value)

Overshoot > 20% (why?)

$$\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{-1+j1.95, -1-j1.95, -5-5\}$$

Non-dominant poles are not entirely non-dominant!

An example (2nd order): Improved tracking with stable pole cancellations

$$P(s) = \frac{1}{(s+0.5)^2} = \frac{1}{s^2+s+0.25}$$

Let us, via the controller, cancel the two plant poles at -0.5 while keeping the two complementary ones.

$$C(s) = \frac{g_2\,(s+0.5)^2}{f_2 s^2 + f_1 s + f_0}$$

The closed-loop characteristic polynomial is thus:

$$\pi(s) = g_2\,(s+0.5)^2 + (s+0.5)^2\,(f_2 s^2 + f_1 s + f_0) = (s+0.5)^2\,(f_2 s^2 + f_1 s + f_0 + g_2)$$

Note, the cancelled poles will show up in the CP (as expected).

Place the two complementary poles in the good region as before: $\{\lambda_1, \lambda_2\} = \{-1 + j1.95, -1 - j1.95\}$

The design equations are now simply:

$$f_2 s^2 + f_1 s + f_0 + g_2 = (s+1-j1.95)(s+1+j1.95) = s^2 + 2s + 4.803$$

And the steady-state requirements are satisfied via:

$$\frac{g_2}{4} - \frac{19}{4} f_0 = 0.$$

An example (2nd order): Improved tracking with stable pole cancellations

$$P(s) = \frac{1}{(s+0.5)^2} = \frac{1}{s^2+s+0.25} \qquad C(s) = \frac{g_2\,(s+0.5)^2}{f_2 s^2 + f_1 s + f_0}$$

Combining the constraints gives us the system of equations:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -\frac{19}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} f_2 \\ f_1 \\ f_0 \\ g_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4.803 \\ 0 \end{bmatrix}$$

Solving this gives us: $(f_2, f_1, f_0, g_2) = (1, 2, 0.24, 4.56)$

The resulting controller is:

$$C(s) = 4.56\frac{(s+0.5)^2}{s^2+2s+0.24}$$

The CL step-response characteristics are now:

- %OS = 19.8% < 20% (requirement now satisfied!)
- Ts = 3.8s
- e_ss = 0.05

Outline

[X] Converting design specifications into desired pole locations

<-> Assumptions, notation, good regions for poles

<-> Step-response characteristics into constraints for closed-loop poles

[-] Pole placement design

<-> Plant and controller model

<-> Pole Placement Problem (PPP) and existence of controllers

<-> Placing poles where we need them to be: polynomial Diophantine equation

<-> Limitations of pole placement

[-] Pole placement for reference tracking control

<-> Asymptotic step tracking

<-> Practical (relaxed) step tracking

X = The upcoming topic

- = Topic that has been covered