Chapter 2: Review of Signals, Systems and Analog Control

ECE 481 - Digital Control Systems

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Based on course notes by Professor Chris Nielsen.

Outline

[X] Representing Continuous-time (CT) Signals and Transforms
- Laplace Transform
- Fourier Transform

[ ] Transfer functions for modeling dynamical systems
- Linear Time Invariant (LTI) systems
- Transfer function representations

[ ] Stability of LTI systems: Bounded-input bounded-ouput (BIBO)

[ ] Stability of feedback systems (LTI)

[ ] Time-response of dynamical systems

[ ] Frequency response of dynamical systems
- Graphical representations

[ ] Dominant Poles and Zeros

[ ] Reference tracking control

X - The upcoming topic

An introduction to signals

Notation:

Time domain, $t \geq 0$

Continuous time (CT) signals; $x(t)$, $x: \mathbb{T} \rightarrow \mathbb{R}$, $x: \mathbb{T} \rightarrow \mathbb{C}$

Vector valued signals: $\vec{x}(t)$, $x: \mathbb{T} \rightarrow \mathbb{R}^n$, $x: \mathbb{T} \rightarrow \mathbb{C}^n$

$$x(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad x_i: \mathbb{T} \rightarrow \mathbb{R} \ \forall i$$

Discrete signals: $x(kT)$, $k \in \mathbb{Z}_{\geq 0}$, $T \Rightarrow$ Sampling time

$x[k]$, $x: \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}^n$

$\mathbb{C}^- : \{ s \in \mathbb{C} : Re(s) < 0 \}$, $\mathbb{C}_{<0}$

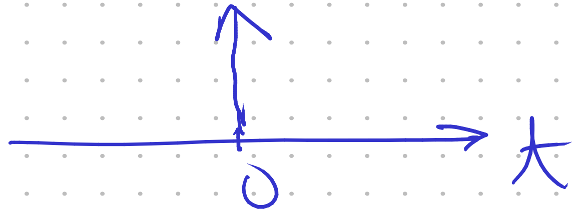
Element of a set.

$\bar{\mathbb{C}}^- : \{ s \in \mathbb{C} ; Re(s) \leq 0 \}$,

$\mathbb{C}^+$, $\bar{\mathbb{C}}^+$

An introduction to signals

The continuous-time delta, or impulse:



$\delta(t)$ {distribution}

Sifting formula: $\int_{-\infty}^{\infty} x(t)\,\delta(t) = x(0)$

Applicable if $x(t)$ is "smooth" at $t=0$.

$f \in C^{\infty}$

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-\tau)\,d\tau = x(\tau)$$

define: $\lambda := t-\tau$

$$\Rightarrow \int_{-\infty}^{\infty} x(\lambda+\tau)\,\delta(\lambda)\,d\lambda = x(\tau)$$

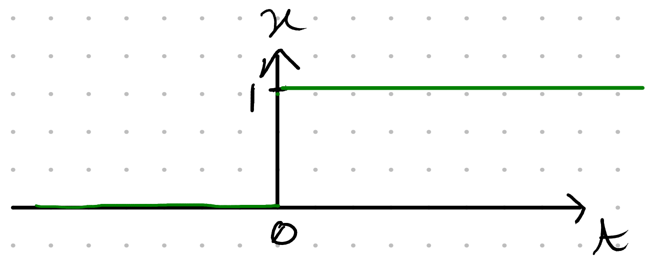
The Laplace Transform

One sided Laplace transform (LT)

$$\mathcal{L}\{x(t)\} = X(s) = \int_0^{\infty} e^{-st}\, x(t)\, dt$$

ROC: $\mathrm{Re}(s) > a$

Example: Laplace Transform of the unit step signal



$$x(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$X(s) = \int_{-\infty}^{\infty} e^{-st} x(t)\, dt = \int_{0}^{\infty} e^{-st}\, dt$$

$$= -\frac{e^{-st}}{s}\Big|_{0}^{\infty} = 1/s$$

Aside: Why do we care about the unit step signal so much in classical control theory?

- Many practical 'reference' signals for controllers are step signals, or a composition of step signals.
- Response to step signals gives useful information to control and system designers.

e.g., Double Lane Change maneuver in cars.
https://youtu.be/zaYFLb8WMGM?t=23

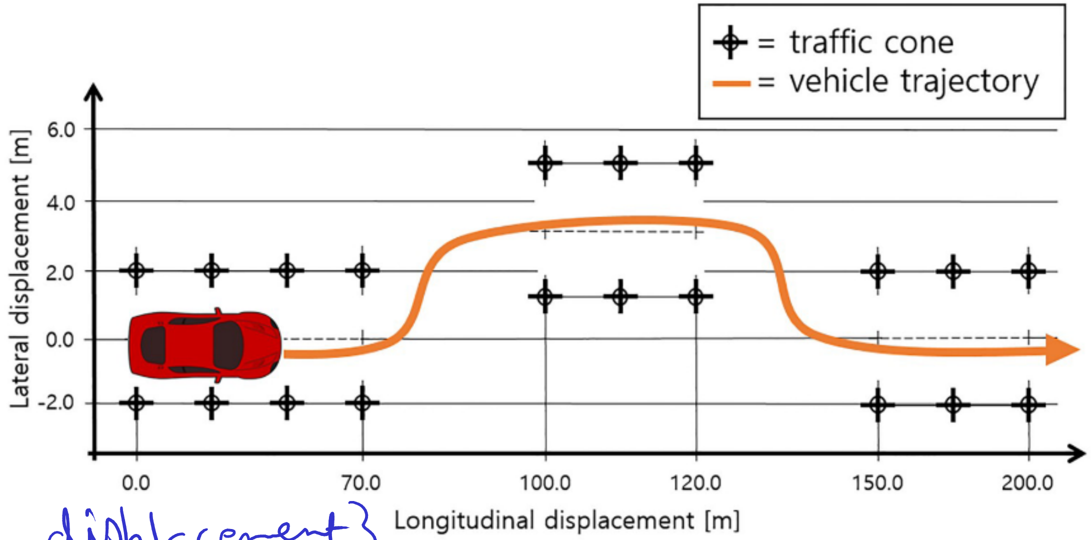
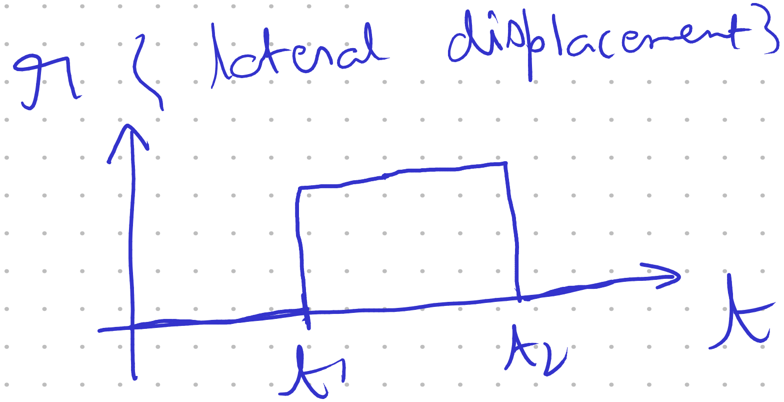


Figure from:
J. Lee, H-J. Chang,
Explicit model predictive control for linear time-variant systems with application to double-lane-change maneuver. 2018.



Existence of the Laplace transform

The Laplace transform of x(t) exists if x(t) satisifes:

a) $x(t)$ must be piece-wise continuous, i.e., in a given time interval, $x(t)$ only has finite "jumps".

e.g. step, pulse, delta

b) $x(t)$ must be of exponential order, i.e,

$$|x(t)| \le M e^{ct} \quad \forall t \geq 0 \ (M, c \in \mathbb{R}_{\geq 0})$$

e.g. $f(t) = e^{t^2} \rightsquigarrow$ Does LT exist?

Some important one-sided Laplace Transforms

Description	Time domain $f(t)$	s-Domain $F(s)$
Unit step	$\mathbf{1}(t)$	$\frac{1}{s}$
Impulse	$\delta(t)$	1
Ramp	t	$\frac{1}{s^2}$
Exponential	e^{at}	$\frac{1}{s-a}$
Sine	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
Cosine	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
Generalized exponential	$t^n \mathrm{e}^{at}$	$\frac{n!}{(s-a)^{n+1}}$
Generalized sine	$\mathrm{e}^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$
Generalized cosine	$\mathrm{e}^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
Sine with linear growth	$t\sin(\omega t)$	$\frac{2\omega s}{(s^2+\omega^2)^2}$
Cosine with linear growth	$t\cos(\omega t)$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$

$\xrightarrow{LT}$

$\xleftarrow{\mathcal{L}^{-1}\{F(s)\}}$

Source: Course notes, Table 2.1

Useful properties of Laplace Transforms

Properties of Laplace transforms

Let f and g be real-valued univariate functions, continuously differentiable at $t = 0$, and let a be a real constant.

(i) $\mathscr{L}\{f + g\} = \mathscr{L}\{f\} + \mathscr{L}\{g\}$

(ii) $\mathscr{L}\{af\} = a\mathscr{L}\{f\}$ } Linearity

(iii) $\mathscr{L}\left\{\frac{\mathrm{d}f}{\mathrm{d}t}\right\} = s\mathscr{L}\{f\} - f(0)$ (Recall, high order)

(iv) $\mathscr{L}\{f * g\} = \mathscr{L}\{f\}\mathscr{L}\{g\}$

(v) $\mathscr{L}\left\{\int_0^t f(\tau)\mathrm{d}\tau\right\} = \frac{1}{s}\mathscr{L}\{f\}$

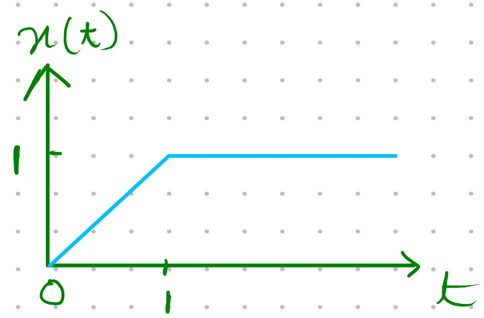
← NOT RATIONAL TF (PADÈ APPROXIMATION)

(vi) $\mathscr{L}\{f(t - T)\} = \mathrm{e}^{-sT}\mathscr{L}\{f\}, \qquad T \geq 0.$ (DELAY)

$$h = f * g = \int_0^t f(t-\tau)\,g(\tau)\,d\tau = \int_0^t f(\tau)\,g(t-\tau)\,d\tau, \quad t \geq 0$$

[CONVOLUTION]

The Laplace Transform: Example



The Fourier Transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

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Transfer functions

PLANT

$$U(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$

$U = \mathcal{L}\{u(t)\}$ INPUT

$Y = \mathcal{L}\{y(t)\}$ OUTPUT

$$Y(s) = G(s)\,U(s)$$

$$y = g * u$$

g: IMPULSE RESPONSE

TF are only defined for Linear Time Invariant (LTI) systems.

EXAMPLE: Which of the following is LTI:

✓ a) $\ddot{y} + a_1 \dot{y} + a_2 y = b_1 \dot{u} + b_2 u$

b) $\dot{y} + (\sin(t))\, y = c\, u(t)$

Example: Write down the TF of

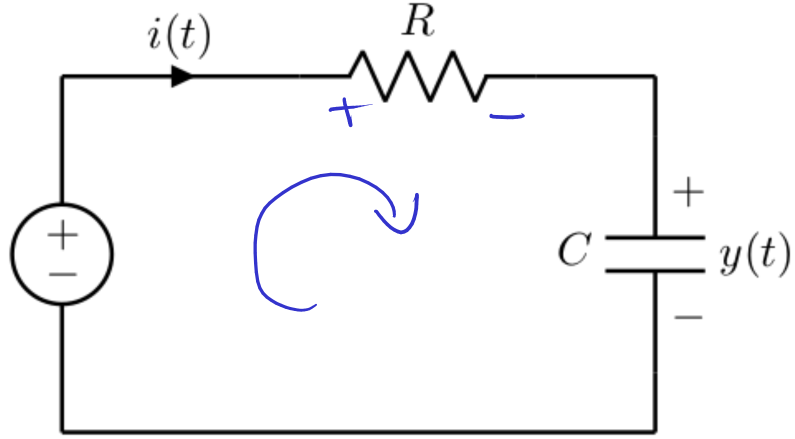
$$\ddot{y} + a_1\dot{y} + a_2 y = b_1\dot{u} + b_2 u$$

Take LT of the equation above with 0 initial conditions.

$$\Rightarrow s^2 Y + a_1 s Y + a_2 Y = b_1 s U + b_2 U$$

$$\Rightarrow \frac{Y}{U} = G(s) = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2}$$

An example: A low-pass filter



KVL

$$-u(t) + Ri(t) + y(t) = 0$$

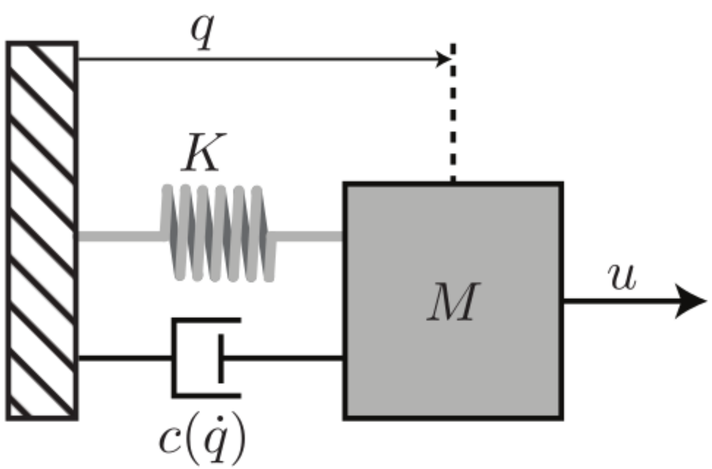
Capacitor: $i(t) = C\dot{y}(t)$

$$\Rightarrow u(t) = RC\dot{y}(t) + y(t) \quad \{\text{ODE}\}$$

Take LT w/ zero initial conditions:

$$\frac{Y}{U} = \frac{1}{RCs+1}$$

Mass-spring-(linear) damper system



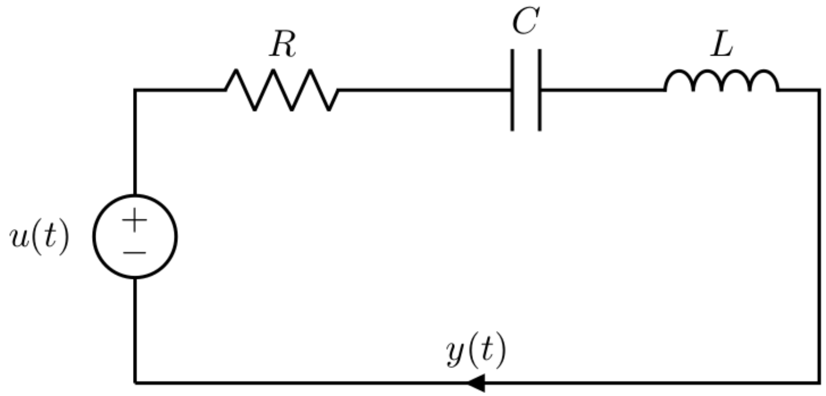
$$M\ddot{q} = \sum \text{Forces}$$

$$= u - Kq - c(\dot{q})$$

Assume Linear damper,

$$c(\dot{q}) = c_1 \dot{q} \quad , \quad c_1 \in \mathbb{R}$$

Series RLC circuit



Transfer functions of commonly used blocks in feedback-control

Description	Governing Equation	Transfer Function
Pure gain	$y(t) = u(t)$	1
Integrator	$\dot{y}(t) = u(t)$	$\frac{1}{s}$
Double integrator	$\ddot{y}(t) = u(t)$	$\frac{1}{s^2}$
Ideal differentiator	$y(t) = \dot{u}(t)$	s
Time delay	$y(t) = u(t-T),\ T > 0$	e^{-sT} (irrational)
Prototype second order system	$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = K\omega_n^2 u(t)$	$\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
Proportional-integral-derivative controller	$y(t) = K_p u(t) + K_i \int_0^t u(\tau)\mathrm{d}\tau + K_d \dot{u}(t)$	$K_p + \frac{K_i}{s} + K_d s$

Definition and types of transfer functions

Definition 2.2.1. (i) A transfer function $G(s)$ is **(real) rational** if it is the quotient of two polynomials

$$G(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \frac{N_G(s)}{D_G(s)}$$

where the coefficients a_i, b_i are real constants. The numbers m, n are the **degrees** of the numerator and denominator polynomials. Let $\mathbb{R}(s)$ denote the set of rational functions in $s \in \mathbb{C}$ with coefficients in $\mathbb{R}$.

(ii) A rational transfer function is **proper** if $n \geq m$. This is equivalent to the condition

$$\lim_{s\to\infty} G(s) \quad \text{exists in } \mathbb{C}.$$

(iii) A rational transfer function is **strictly proper** if $n > m$. This is equivalent to the condition

$$\lim_{s\to\infty} G(s) = 0.$$

(iv) A transfer function is **improper** if it is not proper.

$n < m$

Improper transfer functions

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}, \; m > n$$

Some definitions:

Causality: An input-output system is causal if its current (at time t) output depends only on the current and past values of the input.

Realizability: The process of converting a system from Laplace domain (TF) to a state-space model.
In general, realizability corresponds to creating a physical system from a mathematical model.

Note: Systems that are not causal are not realizable.

Improper transfer functions are not:
1. Causal (i.e., the output at time t may depend on future values of the input to the system).
2. Realizable

Exercise: Take the ideal differentiator G(s) = s, input a sine wave and write the output as a sine wave.

Definitions: Poles and zeros of transfer functions

Assumption: Numerator N(s) and Denominator D(s) of the transfer function G(s) = N(s)/D(s) are co-prime.

e.g.: $G = \dfrac{s+1}{s+2}$, $H = \dfrac{(s+1)(s-1)}{(s+2)(s-1)}$

> **Definition 2.2.2.** A complex number $p \in \mathbb{C}$ is a **pole** of a transfer function $G(s)$ if
>
> $$\lim_{s \to p} |G(s)| = \infty.$$
>
> A complex number $z \in \mathbb{C}$ is a **zero** of a transfer function $G(s)$ if
>
> $$\lim_{s \to z} G(s) = 0.$$

Poles of TF above: -2

Zeros " " " : -1

$s = +1 \rightsquigarrow$ Root of Num(H(s)), Den(H(s))

but not Pole or zero.

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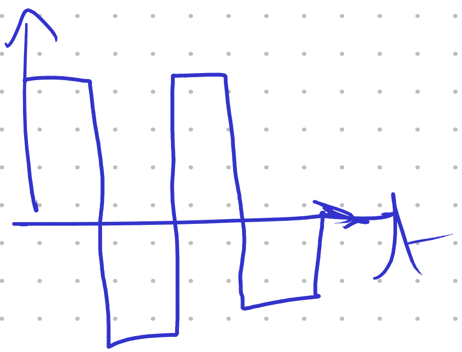
Example: Phugoid in aircrafts

$$u(t) \longrightarrow \boxed{\;G(s)\;} \longrightarrow y(t)$$

Video: https://www.youtube.com/watch ?v=ysdU4mnRYdM

u(t)



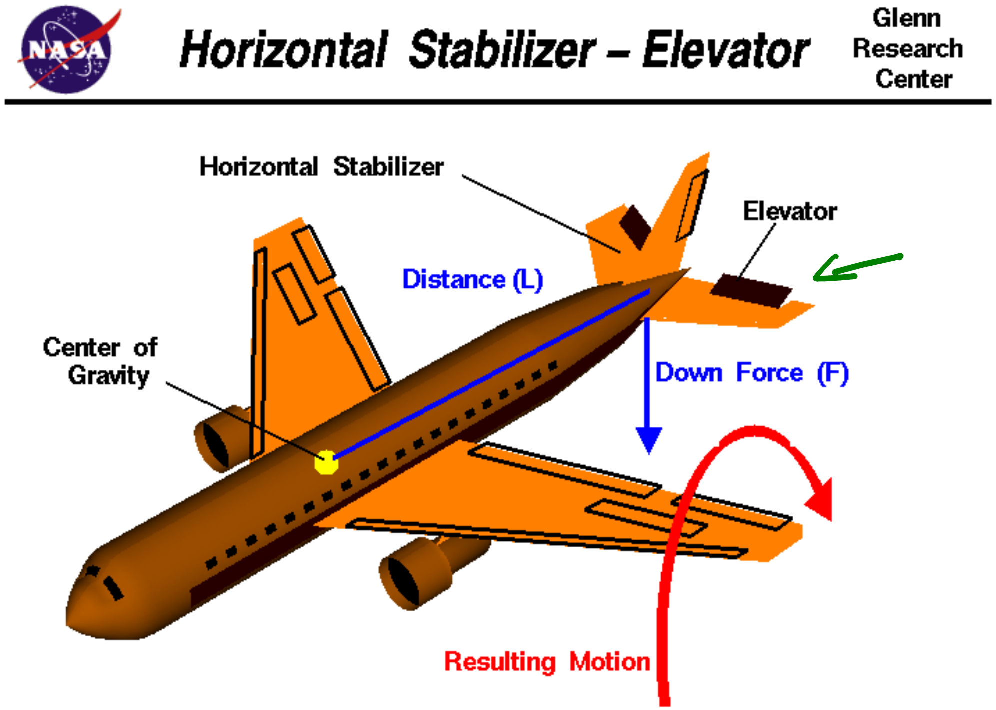


Image from: https://www.grc.nasa.gov/www/k-12/airplane/elv.html

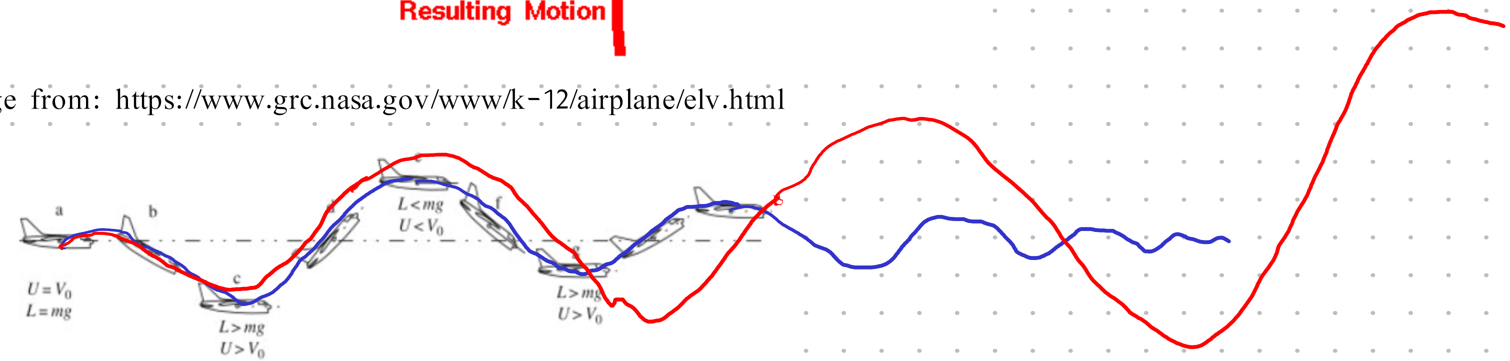
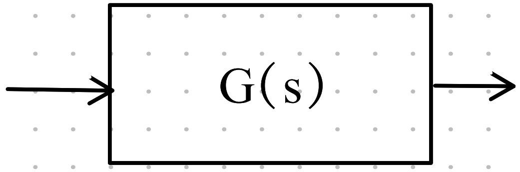


Image from: M. V., Cook, Flight Dynamics Principles (Third Edition), 2013

Bounded-input bounded-output (BIBO) stability: Definition

Does a bounded input always produce a bounded output?



Bounded signal:

$u(t)$ is bounded if $\|u\|_\infty \le C$, $C \in \mathbb{R}$

Inf-norm: $\|u\|_\infty := \sup_{t \geq 0} |u(t)|$

[Think of $\sup_{t\geq 0}$ as $\max_{t \geq 0}$]

BIBO: $\|u\|_\infty$ is finite $\Rightarrow$ $\|y\|_\infty$ is finite

Bounded-input bounded-output (BIBO) stability: Theorem

If $G(s)$ proper (but not strictly), then, 2) $g(t)$ is square integrable: $\int_0^\infty |g(t)|^2 dt < \infty$

Theorem 2.3.3. *Assume $G(s)$ is strictly proper, rational. Then the following three statements are equivalent:*

1. *The system is BIBO stable.*

   $y = g * u$

2. *The impulse-response function $g(t)$ is absolutely integrable, i.e., $\int_0^\infty |g(t)|\mathrm{d}t < \infty$.*
3. *Every pole of the transfer function $G(s)$ has negative real part.*

Note: If the TF G(s) is improper, it is not BIBO stable.

Theorem 2.3.4. *If $G(s) \in \mathbb{R}(s)$ is improper, then $G(s)$ is not BIBO stable.*

See course notes for proof sketch.

Example: BIBO-stability of a Low Pass Filter

$RC > 0$

Transfer function, $G(s) = \frac{1}{RCs+1}$ → Pole at $s = -1/RC$ Impulse response, $g(t) = \frac{1}{RC}\exp(-t/RC)\mathbf{1}(t)$

Let u be a bounded input, the system's response to u is: $u(t) \leq \|u\|_\infty \leq C \in \mathbb{R}$

$y = g * u$ $|y(t)| = |\int_0^t g(\tau)u(t-\tau)d\tau| \leq \int_0^t |g(\tau)| \cdot |u(t-\tau)| d\tau$

{Cauchy-Schwarz}

Bounded input $\leq \int_0^t |g(\tau)| d\tau \, \|u\|_\infty$

$\leq \int_0^\infty |g(\tau)| d\tau \, \|u\|_\infty$

$= \int_0^\infty \frac{1}{RC} e^{-\tau/RC} d\tau \, \|u\|_\infty$

$= \|u\|_\infty$

$\Rightarrow \|y\|_\infty \leq \|u\|_\infty \leq C \Rightarrow$ BIBO

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Let u be a bounded input, the system's response to u is:

$$
\begin{aligned}
|y(t)| &= \left|\int_0^t g(\tau)u(t-\tau)d\tau\right| \\
&\leq \int_0^t |g(\tau)|\,|u(t-\tau)|d\tau \quad \text{(via the Cauchy-Schwarz inequality)} \\
&\leq \int_0^t |g(\tau)|d\tau\,||u||_\infty \\
&\leq \int_0^\infty |g(\tau)|d\tau\,||u||_\infty \\
&= \int_0^\infty \frac{1}{RC}\exp(-\tau/RC)d\tau\,||u||_\infty \quad \text{plug the impulse response} \\
&= ||u||_\infty
\end{aligned}
$$

Therefore $\forall\, t \geq 0$, $|y(t)| \leq ||u||_\infty$ or $||y||_\infty \leq ||u||_\infty$; i.e., system is BIBO stable.

Example: BIBO-stability of an integrator

$$u \longrightarrow \boxed{1/s} \longrightarrow y$$

* Marginally stable.

$G(s) = 1/s$

→ Pole at $s = 0$ * $Re(s) \geq 0$

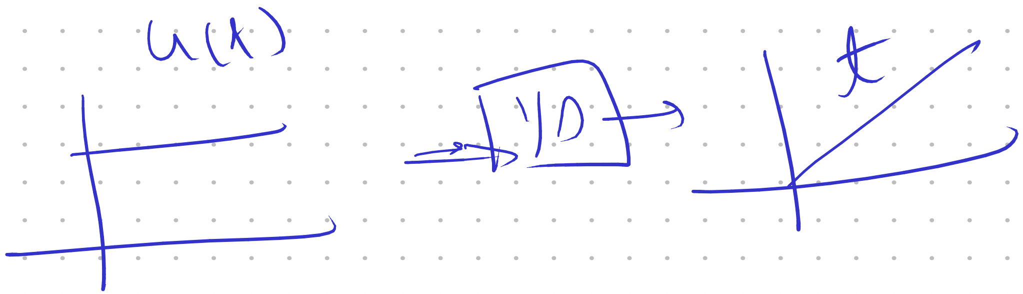
e.g., $u(t) = \cos(t)$, $y(t) = -\sin(t)$

↳ Bounded $\|u\|_\infty \leq 1$

↳ Bounded output $\|y\|_\infty \leq 1$

e.g., $u(t) = 1(t)$ (unit step) , $y(t) = ?$

$= t$



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<> Necessary and sufficient conditions

<> Unstable pole-zero cancellations

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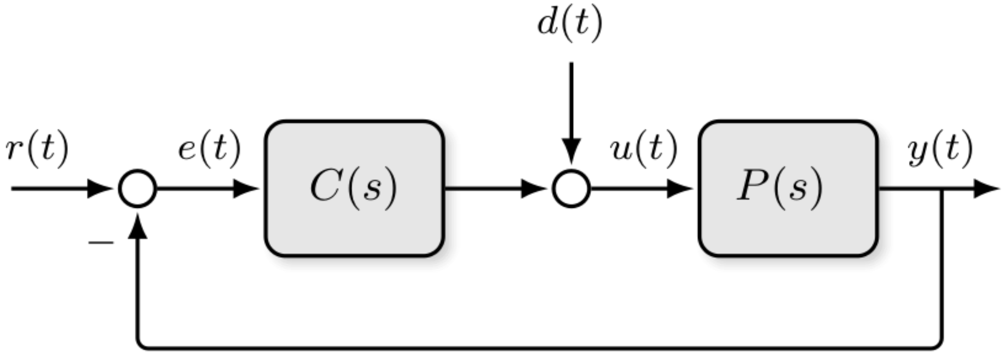
X - The upcoming topic

Stability of feedback systems

Assumption (for well-posedness*):

The plant and controller are rational transfer functions, C(s) is proper and P(s) is strictly proper.



* (r, d) pairs $\Rightarrow$ unique solutions for (e, u, y)

Stability of feedback systems

Definition 2.4.2. The feedback system in Figure 2.6 is **input-output stable** provided e, u, and y are bounded signals whenever r and d are bounded signals; briefly, the system from (r, d) to (e, u, y) is BIBO stable.

How many TFs? 6

$$\left.\begin{aligned} E &= R - Y = R - PU \\ U &= D + CE \end{aligned}\right\} \text{Summing junctions}$$

$\Downarrow$ In vector form

$$\underbrace{\begin{bmatrix} 1 & P \\ -C & 1 \end{bmatrix}}_{M} \begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} R \\ D \end{bmatrix} \quad \text{— (1)}$$

Assumption $\Rightarrow$ $\det(M) \neq 0$

($\Rightarrow$ $1 + PC$ is not identically zero) → Prove

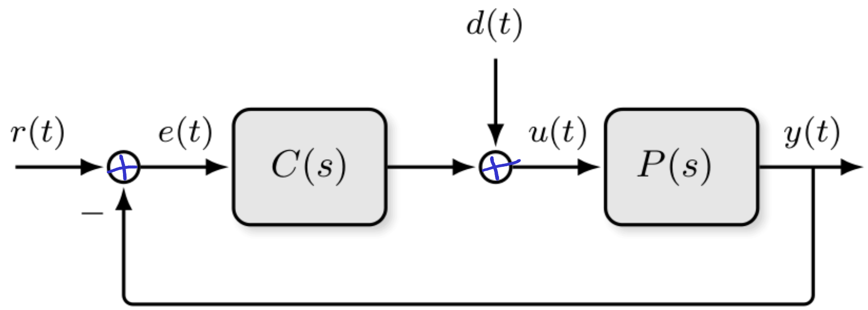


Figure 2.6: Unity feedback system.

Solve for E and U:

$$\textcircled{1} \Rightarrow \begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} 1 & P \\ -C & 1 \end{bmatrix}^{-1} \begin{bmatrix} R \\ D \end{bmatrix} = \frac{\text{adj}(M)}{\det(M)} \cdot \begin{bmatrix} R \\ D \end{bmatrix}$$

$$\begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} \frac{1}{1+PC} & \frac{-P}{1+PC} \\ \frac{+C}{1+PC} & \frac{1}{1+PC} \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix} \leftarrow \text{4 TFs}$$

$$Y = PU = \frac{PC}{1+PC} \cdot R + \frac{P}{1+PC} \cdot D$$

$Y = R - E$, if R is Bounded & E is bounded

$\Rightarrow Y$ bounded

4 TFs (in matrix above) BIBO stable => Closed-loop system is input-output stable.

Necessary and sufficient condition for input-output stability

Characteristic Polynomial: Least common multiple of the denominators of the transfer functions from all inputs to all outputs, where $P = N_p/D_p$ and $C = N_c/D_c$

R, D E, U

4 TFs :

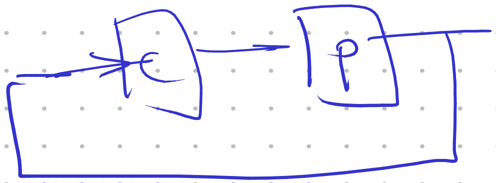
$$\begin{bmatrix} \frac{1}{1+PC} & \frac{-P}{1+PC} \\ \frac{C}{1+PC} & \frac{1}{1+PC} \end{bmatrix} = \frac{1}{D_pD_c + N_pN_c} \begin{bmatrix} D_pD_c & -N_pD_c \\ D_pN_c & D_pD_c \end{bmatrix}$$

Ch. p. , $\pi(s) = D_p D_c + N_p N_c$

> **Theorem 2.4.5.** *The feedback system is input-output stable if, and only if, the characteristic polynomial has no roots with* $\mathrm{Re}(s) \geq 0$.

Example: Unstable open-loop but stable with feedback for what values of K ?

$P(s) = \frac{1}{s-1} = \frac{N_p}{D_p}$ $\quad C(s) = K = \frac{N_c}{D_c}$



↳ BIBO stable? No.

$\Pi(s) = N_p N_c + D_p D_c = s - 1 + K$

Roots of ch.p. , $1 - K$

$Re(s) < 0 \Rightarrow K > 1$

Example:

$$P(s) = \frac{1}{s^2 - 1} \qquad C(s) = \frac{s-1}{s+1}$$

$$PC = \frac{1}{\cancel{(s-1)}(s+1)} \cdot \frac{\cancel{(s-1)}}{(s+1)}$$

Pole-zero cancellation at $s = +1$.

$$\begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} \frac{1}{1+PC} & \frac{-P}{1+PC} \\ \frac{C}{1+PC} & \frac{1}{1+PC} \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}$$

$\Rightarrow$ Not I/O stable

$$= \begin{bmatrix} \frac{(s+1)^2}{s^2+2s+2} & \frac{s+1}{(s-1)(s^2+2s+2)} \\ \frac{(s+1)(s-1)}{s^2+2s+2} & \frac{(s+1)^2}{s^2+2s+2} \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}$$

$\pi(s) = (s-1)(s^2+2s+2)$, has root in $\mathrm{Re}(s) \geq 0$

$\Rightarrow$ Not I/O stable.

Unstable pole-zero cancellations (and why they are bad news)

Example: $P(s) = 1/(s^2 - 1), \quad C(s) = (s-1)/(s+1)$

Exercise: Work out the TFs from r and d to e and u, are they all BIBO stable?

Definition 2.4.6. The plant $P(s)$ and controller $C(s)$ have a **pole-zero cancellation** if there exists a $\lambda \in \mathbb{C}$ such that

$$N_p(\lambda) = D_c(\lambda) = 0 \qquad \text{(controller pole cancels a plant zero)}$$
$$D_p(\lambda) = N_c(\lambda) = 0 \qquad \text{(controller zero cancels a plant pole).}$$

It is called an **unstable pole-zero cancellation** if $\operatorname{Re}(\lambda) \geq 0$.

Unstable pole-zero cancellations (and why they are bad news)

Corollary 2.4.7. *If there is an unstable pole-zero cancellation, then the feedback system is not input-output stable.*

Proof: Let's say the p-z cancellation is at $\lambda \in \overline{\mathbb{C}}^+$ ($Re(\lambda) \geq 0$)

Ch.p: $\pi(\lambda) = N_p(\lambda) N_c(\lambda) + D_p(\lambda) D_c(\lambda) = 0 + 0$

$\Rightarrow \lambda$ is a root of ch.p.,

$Re(\lambda) \geq 0 \Rightarrow$ Not I/O stable.

With the above statement in mind, we now have an alternate test for input-output stability.

Theorem 2.4.9. *The feedback system is input-output stable if, and only if,*

1. *The transfer function* $1 + PC$ *has no zeros in* $\mathrm{Re}(s) \geq 0$, *and*
2. *the product* PC *has no unstable pole-zero cancellations.*

As an exercise, prove the theorem above.

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[X] Time-response of dynamical systems

[] Frequency response of dynamical systems

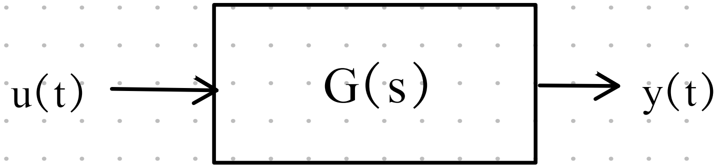
- Graphical representations

[] Dominant Poles and Zeros

[] Reference tracking control

X - The upcoming topic

Time-domain response of LTI systems



Stable plant

Example. Phugoid: Aircraft's response to an elevator singlet,
video: https://www.youtube.com/watch ?v=ysdU4mnRYdM

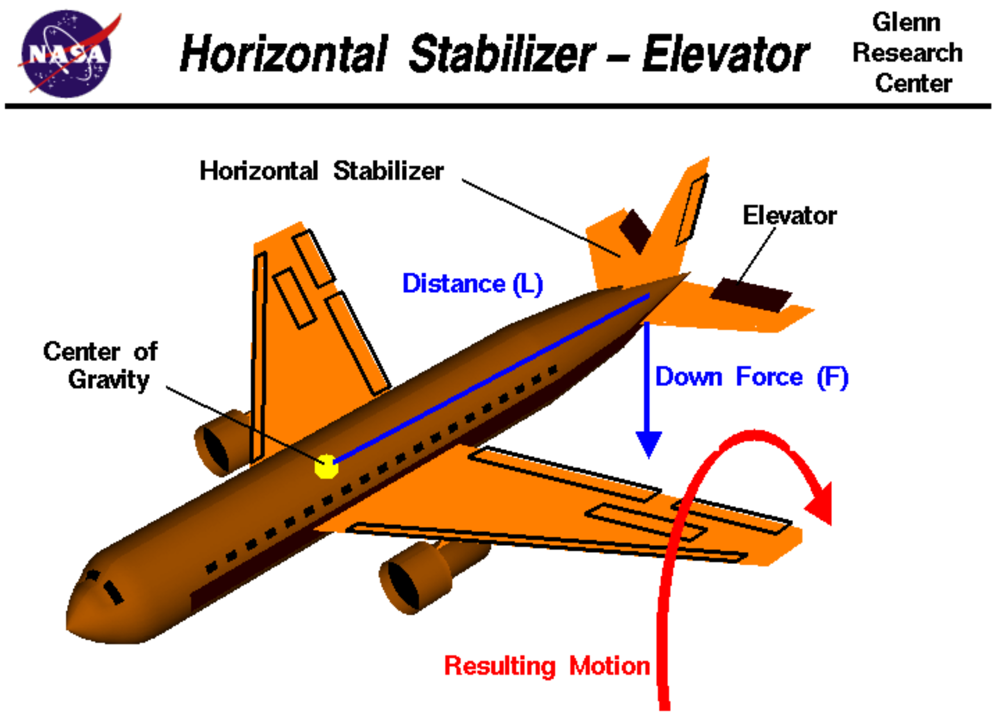


Image from: https://www.grc.nasa.gov/www/k-12/airplane/elv.html

Elevation (θ)

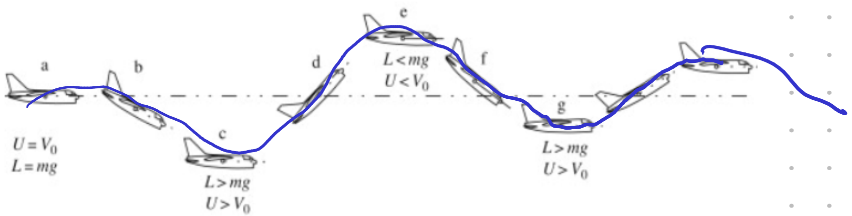
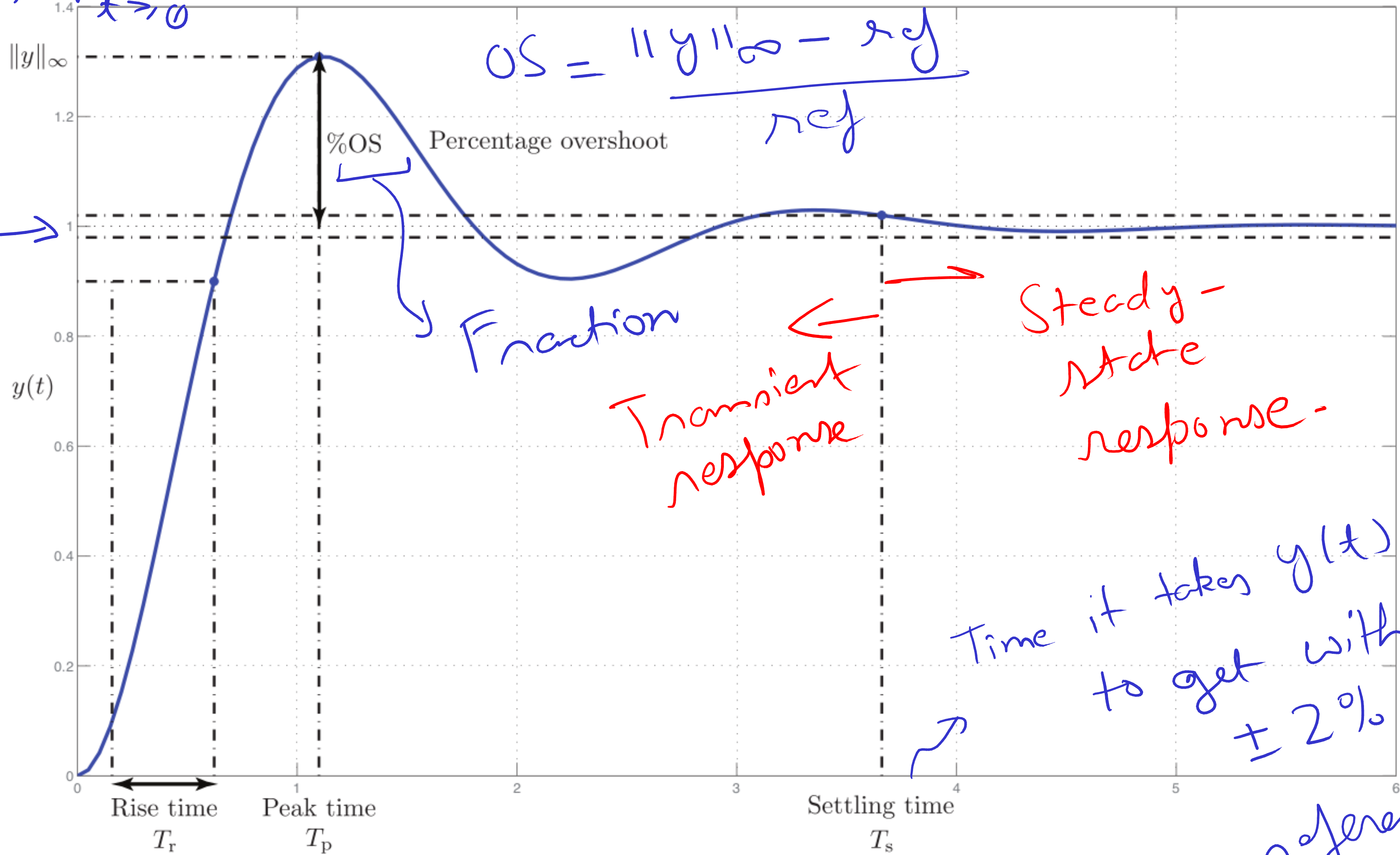


Image from: M. V., Cook, Flight Dynamics Principles (Third Edition), 2013

Time response to unit-step inputs

$u(t) = 1(t)$ [unit step]

$\|y\|_\infty = \sup_{t>0} |y(t)|$



$OS = \dfrac{\|y\|_\infty - ref}{ref}$

Fraction

ref. $= 1$

Transient response

Steady-state response.

Time it takes $y(t)$ to get within $\pm 2\%$ of reference value.

Time it takes $y(t)$ to get from 0% to 90% of reference value.

Steady-state response and the final value theorem (FVT)

Theorem 2.5.1 (Final-value theorem). *Let $f(t)$ be a signal defined for $t \geq 0$ and let its Laplace transform $F(s)$ be rational and proper.*

(a) If $F(s)$ has all its poles in $\mathbb{C}^-$, then $f(t)$ converges to zero as $t \to \infty$.

(b) If $F(s)$ has all its poles in $\mathbb{C}^-$ except for a simple pole at $s = 0$, then

$$y_{ss} = \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s). \tag{2.4}$$

(c) In all other cases, $f(t)$ does not approach a constant as $t \to \infty$.

BIBO stability and steady-state response to a unit-step input

Recall the step response of an integrator (unbounded output).

> **Corollary 2.5.2.** *A linear time-invariant system with a proper and rational transfer function is BIBO stable if, and only if, its unit step response approaches a constant value.*

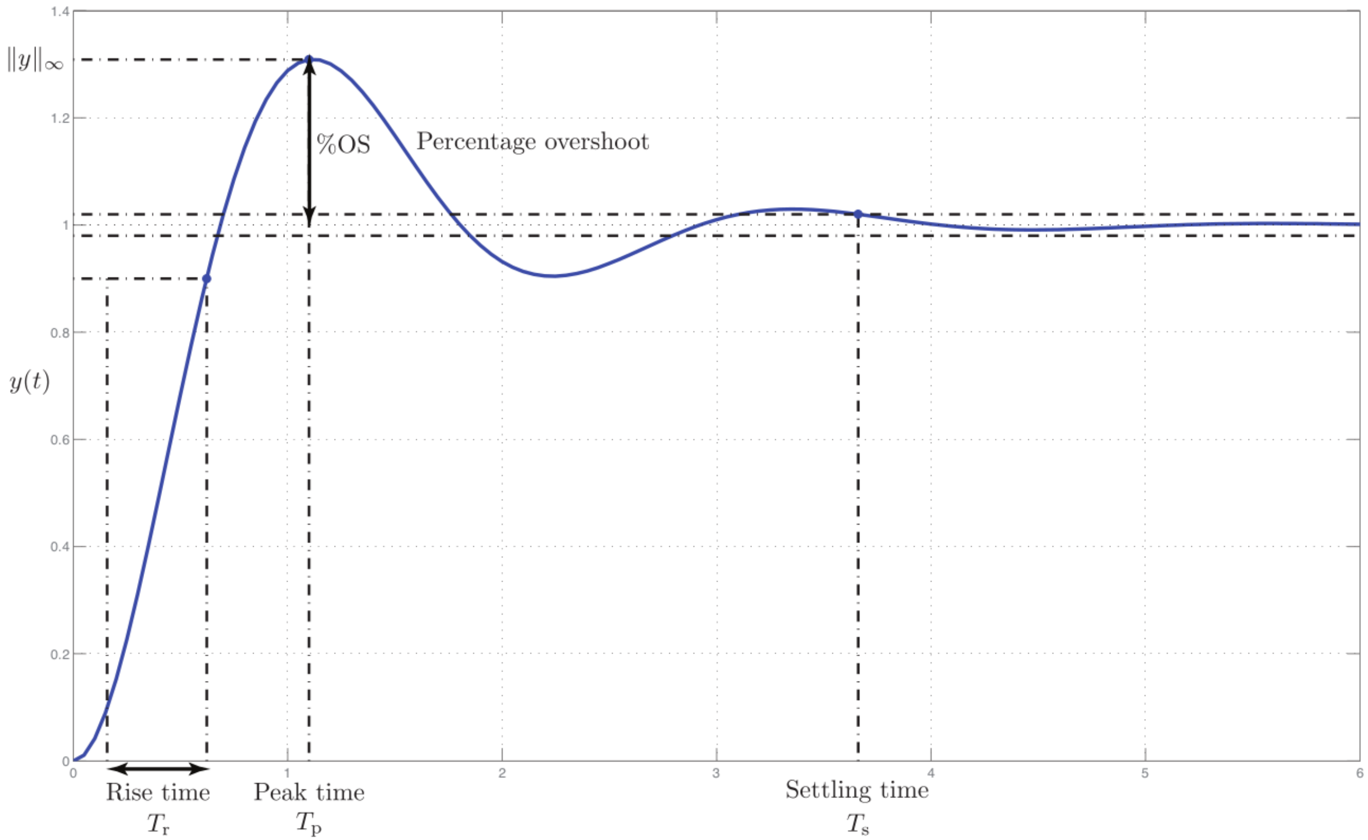
Example: Consider a system with the TF $G(s) = \dfrac{s+80}{(s+4)(s+10)}$. Is it BIBO stable? Yes, poles at $-4, -10$

$u(t) = \mathbb{1}(t)$, $y_{ss} = \lim\limits_{t \to \infty} y(t)$

Because all poles of $Y(s) = G(s) \cdot \frac{1}{s}$ ← $R(s)$ are in $\mathbb{C}^-$ except one.

$$y_{ss} = \lim_{s \to 0} s\,Y(s) = \left. \frac{\cancel{s} \cdot (s+80)}{(s+4)(s+10)} \cdot \frac{1}{\cancel{s}} \right|_{s=0} = 2$$

Transient response to unit-step inputs



Prototype second-order systems:

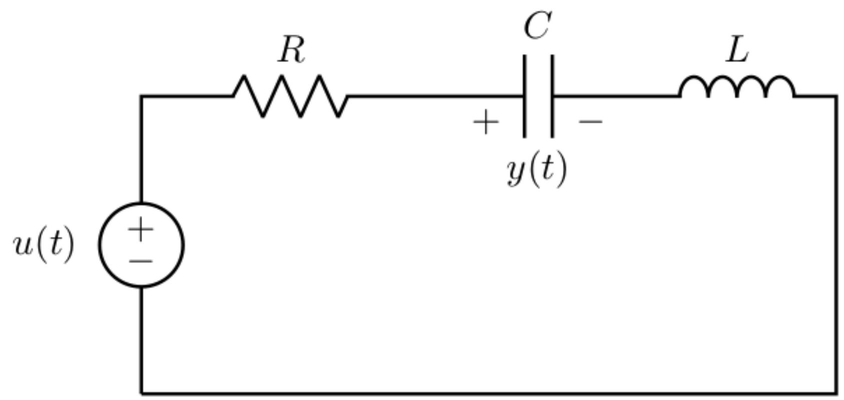
ODE: $\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = k\omega_n^2 u,$

$k, \zeta, \omega_n \in \mathbb{R}$

TF: $\dfrac{Y}{U} = \dfrac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

Examples of second-order systems

Example 1: Series RLC circuit (output is voltage across capacitor).



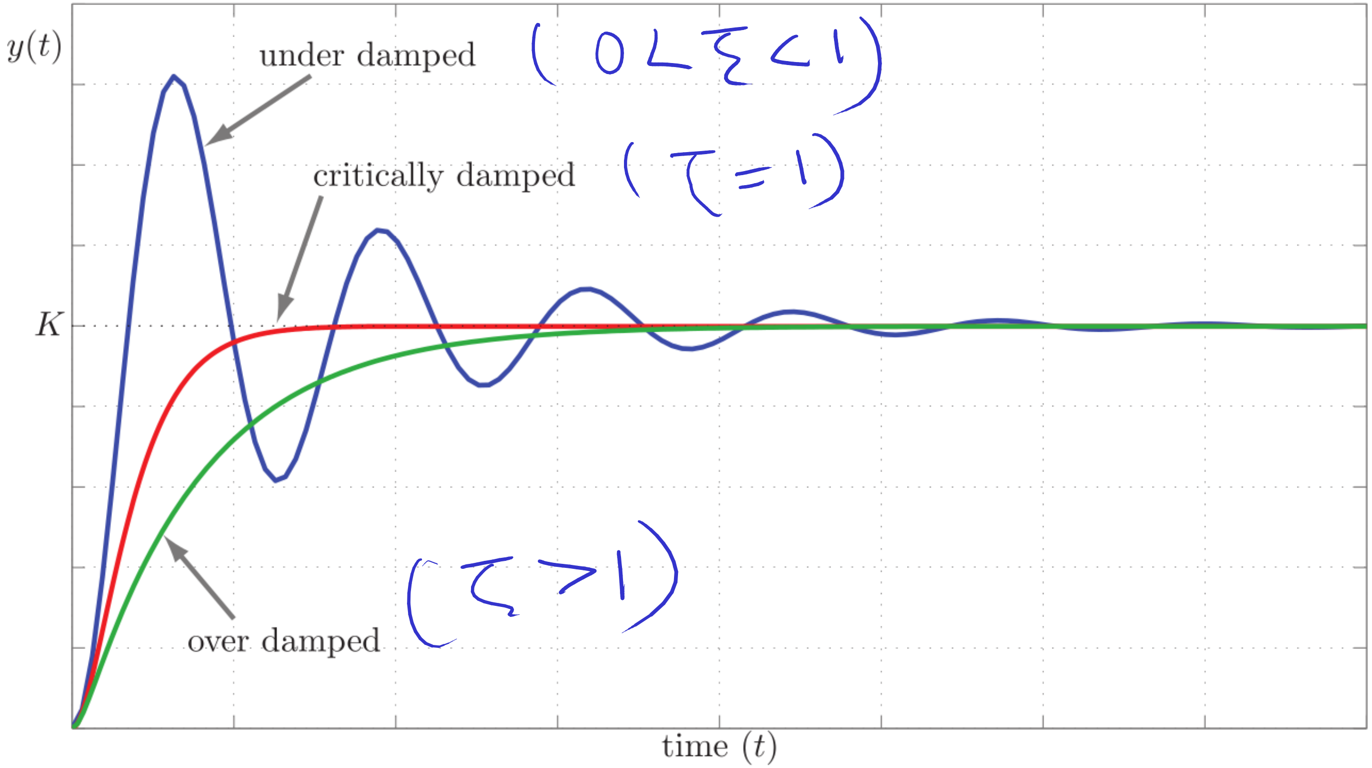
$$\frac{Y(s)}{U(s)} = G(s) = \frac{1}{s^2LC + sRC + 1} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Compare to prototype second order transfer function: $G(s) = \dfrac{K\omega_n^2}{s^2 + 2\zeta\omega_n^2 s + \omega_n^2}$

$$\omega_n = \frac{1}{\sqrt{LC}}, \qquad K = 1, \qquad \zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$$

Damping ratio (ζ)

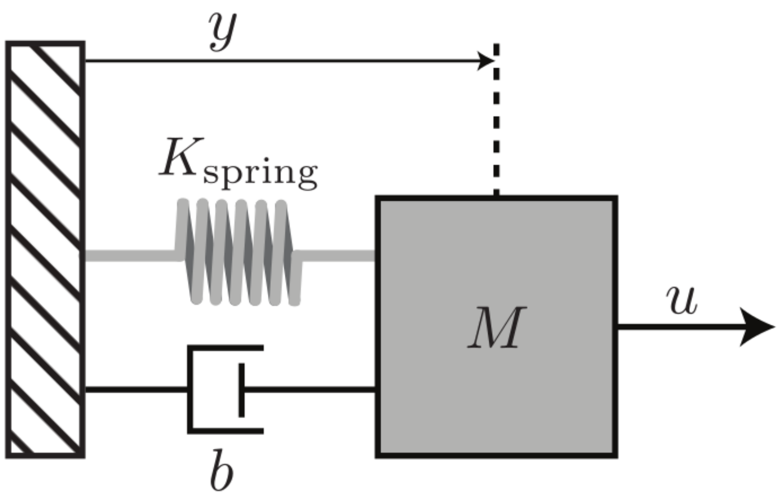
undamped, $\zeta = 0$.



What type of damping does the aircraft motion (phugoid) exhibit?

Examples of second-order systems

Example 2: Mass-(linear) Spring-Damper system

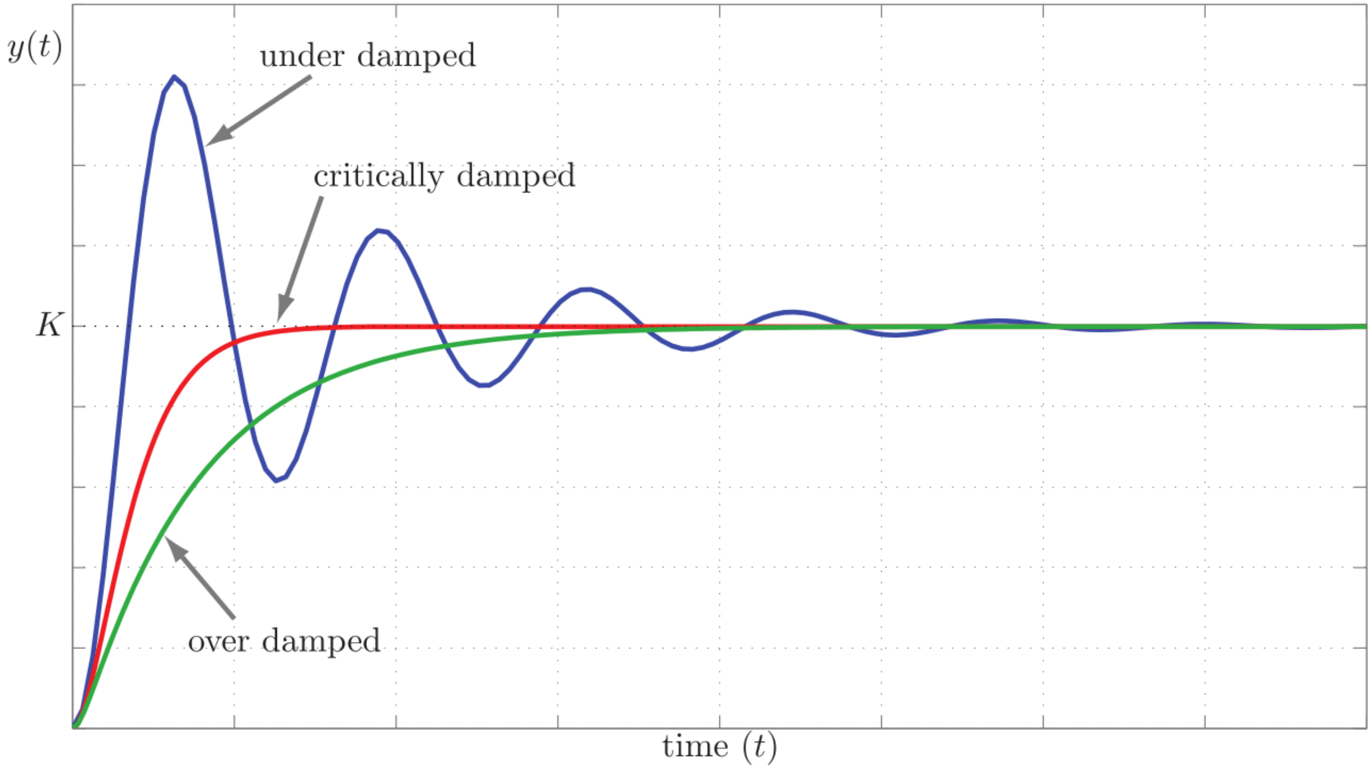


$$\frac{Y(s)}{U(s)} = \frac{1}{Ms^2 + bs + K_{\text{spring}}}$$

Exercise: Work out the natural frequency, gain (K) and damping ratio for this system.

Recall the Jeep bouncing around during a DLC (previous lecture). Consider (just for now) the each wheel as a mass-spring-damper system. Why did reducing the mass not work?
(Note: This is not the complete picture, but just a very weak approximation of what was going on.)

Damping ratio and properties of a 2nd order systems step response



Name	Poles	Zeros	Steady-state gain	Step response
Underdamped	$-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$	None	K	Oscillatory
Critically damped	$-\omega_n$ (repeated)	None	K	Not oscillatory
Overdamped	$-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$	None	K	Not oscillatory

Damping ratio and pole locations

Poles of underdamped standard second order system:

$$-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$\arg(s) = \pi - \theta$

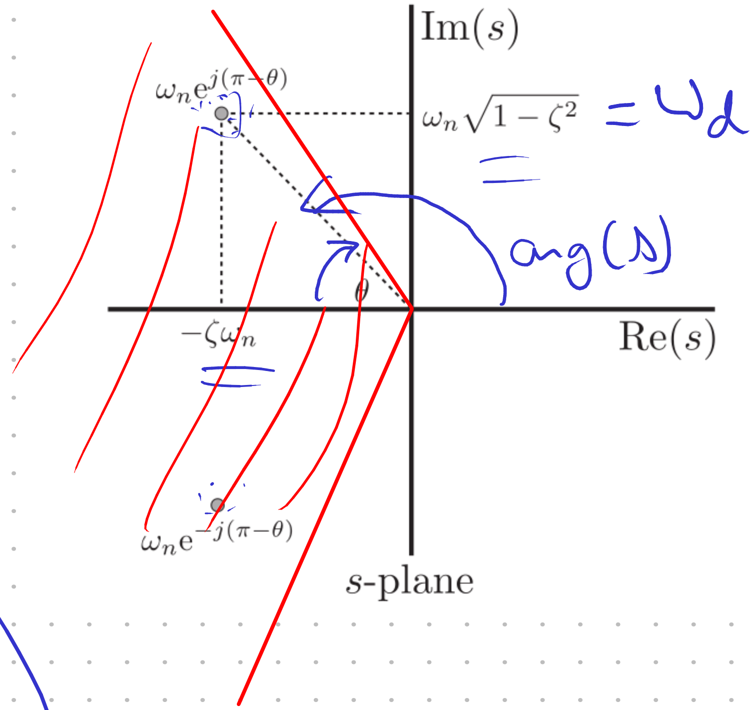
$s = a + jb$

$\arg(s) = ?$

$= \tan^{-1}\left(\frac{b}{a}\right)$

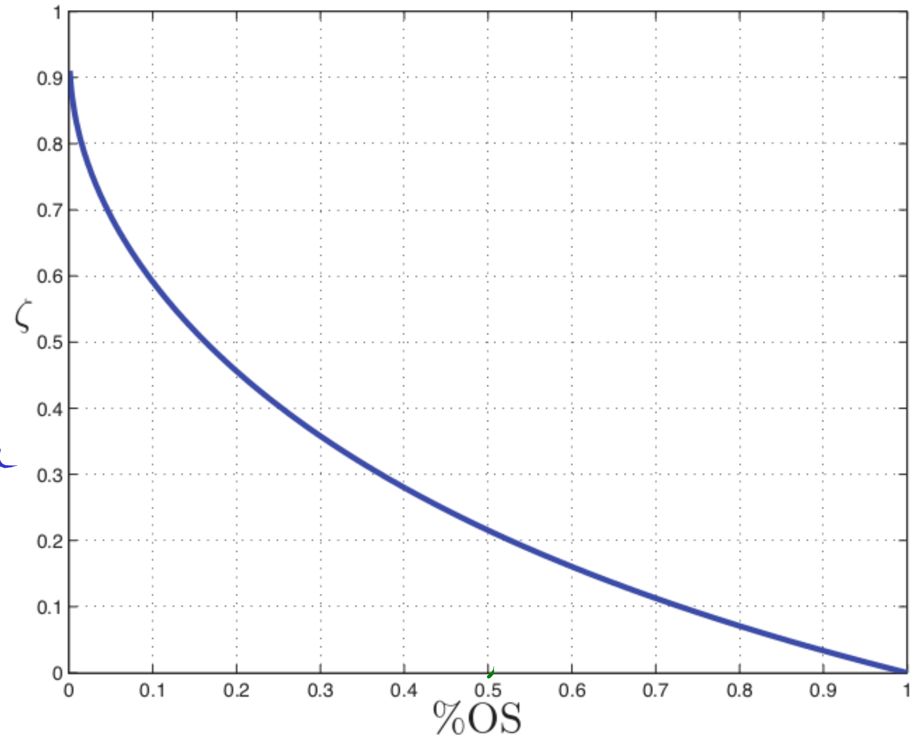
$\cos(\theta) = \zeta$

$\%OS \le \%OS^{max}$



Damping ratio its and relation to overshoot

$$\%OS = \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$



Not a percentage!

$$\Rightarrow \zeta = \frac{\ln(\%OS)}{\sqrt{\pi^2 + (\ln(\%OS))^2}}$$

Control specs: $\%OS \leq \%OS^{max}$

Recall $\cos(\theta) = \zeta$

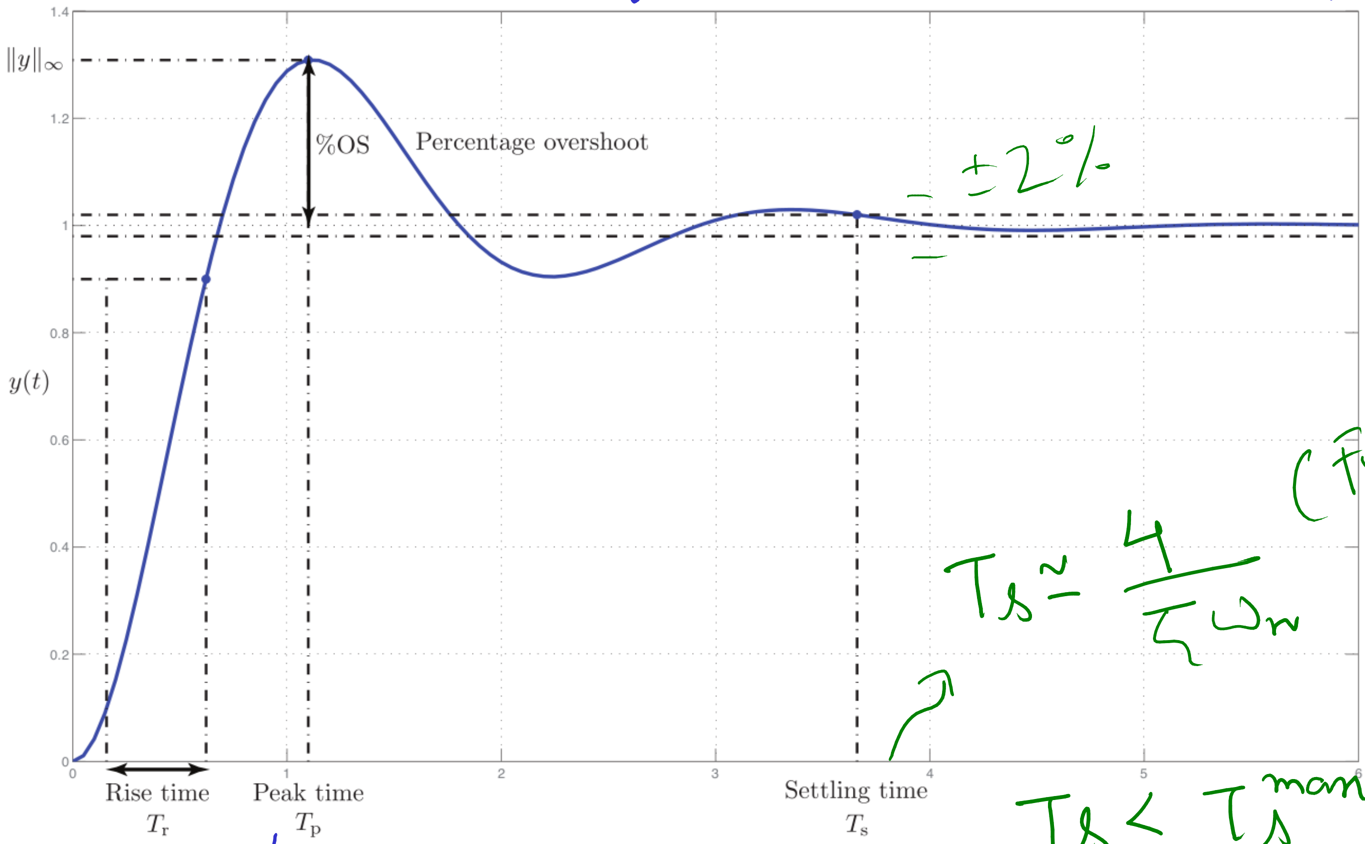
$$\Rightarrow \quad \%OS \leq \%OS^{max}$$

$$\Rightarrow \text{Poles, } \{ s \in \mathbb{C}, |\arg(s)| \geq \pi - \theta_{max}$$

$$\theta^{max} = \cos^{-1}\left(\frac{\ln(\%OS^{max})}{\sqrt{\pi^2 + (\ln(\%OS^{max}))^2}}\right)$$

Settling time and Peak time as functions of the damping ratio

Poles of 2nd order (std.) system = $-\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$



$T_s \simeq \dfrac{4}{\xi\omega_n}$ (for standard 2nd order systems).

$T_s \leq T_s^{max}$

$\Rightarrow \underbrace{\xi\omega_n}_{\text{-ve real part of poles.}} \geq \dfrac{4}{T_s^{max}}$

→ '-'ve real part of poles.

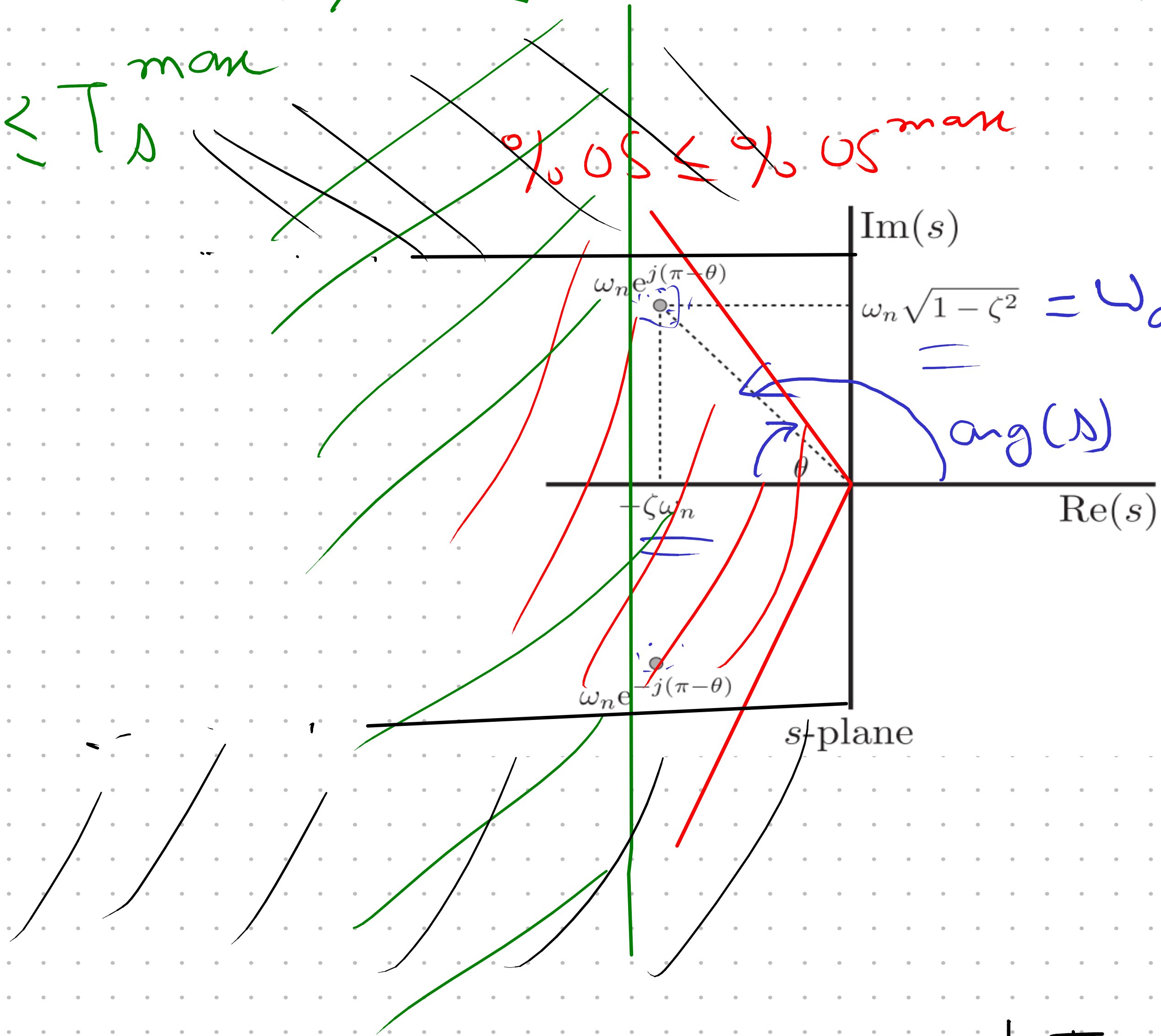
$T_p = \dfrac{\pi}{\omega_n\sqrt{1-\xi^2}}$

$T_p \leq T_p^{max} \Rightarrow |Im(poles)| \geq \dfrac{\pi}{T_p^{max}}$

$$\Rightarrow \quad \Leftarrow \quad \mathrm{Re}(\text{poles}) \le -\frac{4}{T_s^{max}}$$

$$T_s \le T_s^{max}$$

$$\%OS \le \%OS^{max}$$



$$T_p \le T_p^{max}$$

$$|\mathrm{Im}(\text{poles})| \ge \pi / T_p^{max}$$

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[-] Stability of feedback systems (LTI)

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[X] Frequency response of dynamical systems
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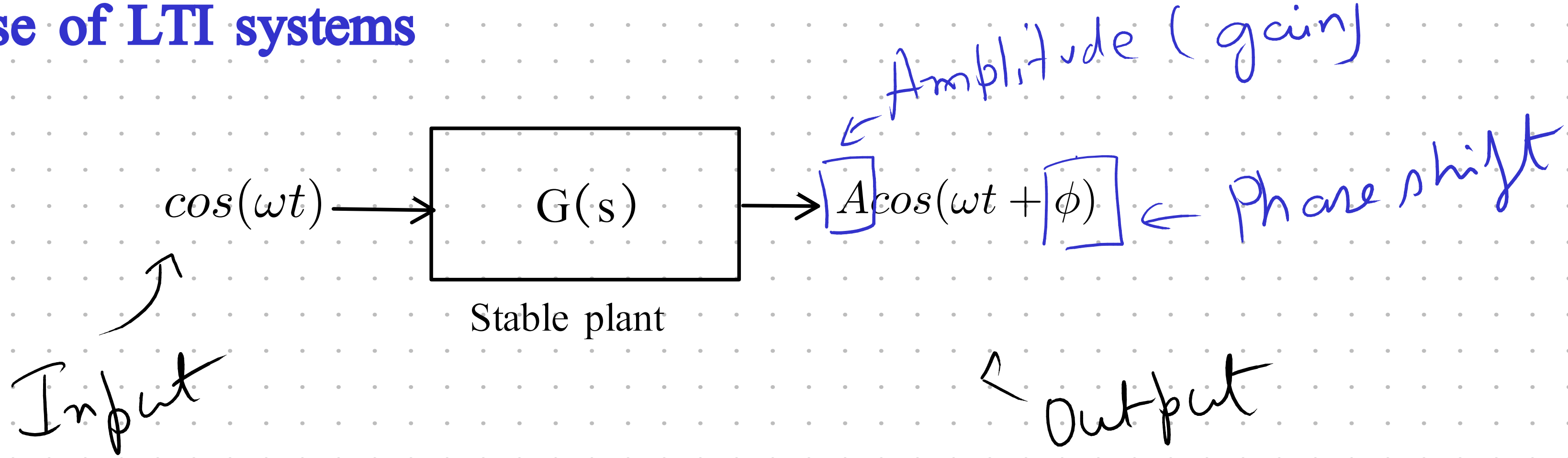
[] Dominant Poles and Zeros

[] Reference tracking control

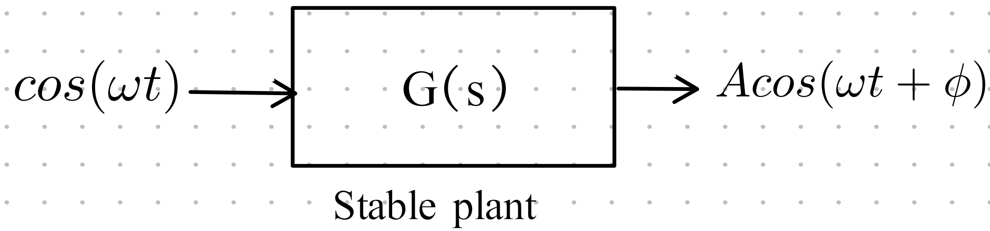
X - The upcoming topic

Frequency response of LTI systems



Steady-state response of LTI systems

$u(t) = a e^{j\omega t}$
$= a \cos \omega t + j a \sin \omega t$



Theorem 2.6.1. *Assume $G(s)$ is rational, proper, and has all its poles in $\mathbb{C}^-$. Then the steady-state response to the input $u(t) = a\mathrm{e}^{j\omega t}$ is*

$$y(t) = aG(j\omega)\mathrm{e}^{j\omega t}. \tag{2.12}$$

Freq. response of plant.

In particular

- *The steady-state response to the input $u(t) = a\cos(\omega t)$ is*

Magnitude of G.

$$y(t) = a\,|G(j\omega)|\cos(\omega t + \angle G(j\omega)). \tag{2.13}$$

Phase of G

- *The steady-state response to the input $u(t) = a\sin(\omega t)$ is*

$$y(t) = a\,|G(j\omega)|\sin(\omega t + \angle G(j\omega)). \tag{2.14}$$

- *The steady-state response to the input $u(t) = a\mathbf{1}(t)$ is*

$$y(t) = aG(0). \tag{2.15}$$

DC Gain of G(s).

Defining the frequency, magnitude and phase response of LTI systems

$$\cos(\omega t) \longrightarrow \boxed{\;G(s)\;} \longrightarrow A\cos(\omega t + \phi)$$

Stable plant

Assume G(s) is rational, proper and has poles such that Re(s)<0

Definition 2.6.2. Assume $G(s)$ is rational, proper, and has all its poles in $\mathbb{C}^-$.

(a) The function $\mathbb{R} \to \mathbb{C}$, $\omega \mapsto G(j\omega)$ is the **frequency response** of G.

(b) The function $\mathbb{R} \to \mathbb{R}$, $\omega \mapsto |G(j\omega)|$ is the **amplitude** or **magnitude response** of G.

(c) The function $\mathbb{R} \to (-\pi, \pi]$, $\omega \mapsto \measuredangle G(j\omega)$ is the **phase response** of G.

Wrap-around.

Graphical representation of frequency response

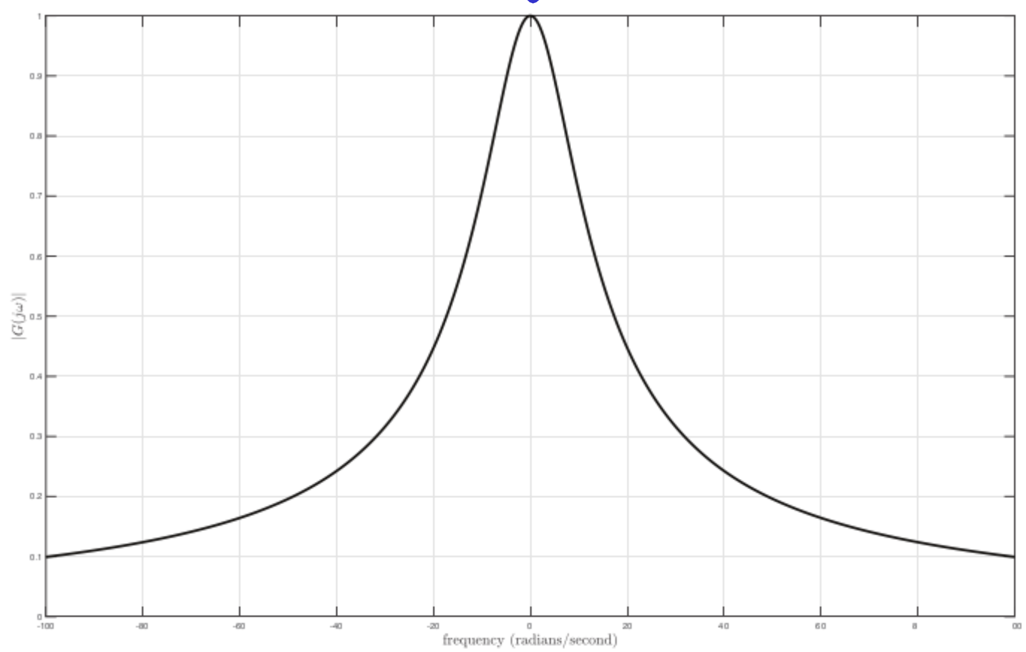
(LTI)

Consider the transfer function $G(s) = \dfrac{-10}{s+10}$, $s = j\omega$

To get freq. response: $\Rightarrow G(j\omega) = \dfrac{-10}{10+j\omega}$

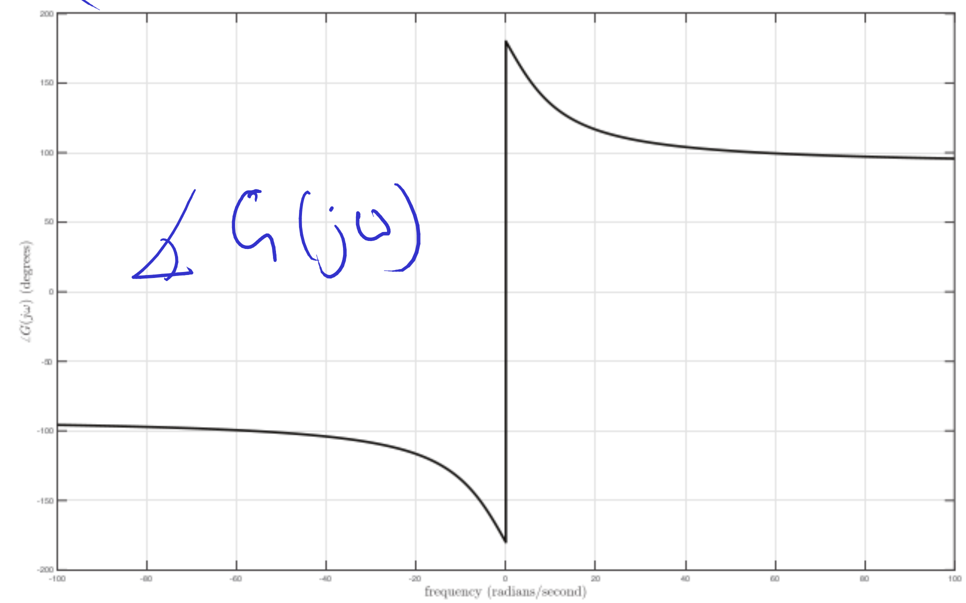
$= \dfrac{-10(10-j\omega)}{(10+j\omega)(10-j\omega)} = a + jb$

$|G(j\omega)|$



(a) *Magnitude response.*

$\angle G(j\omega)$



(b) *Phase response.*

Symmetric in ω

$|G(j\omega)| = |G(-j\omega)|$

Anti-Symmetric

$\angle G(j\omega) = -\angle G(-j\omega)$

Graphical representation of frequency response: Bode plots

$\rightsquigarrow$ Magnitude $= 20\log_{10}(|G(j\omega)|)$

Example, the RC low pass filter, RC = 0.0001 $\quad G(s) = \dfrac{1}{RCs+1}$

Bandwidth of a system

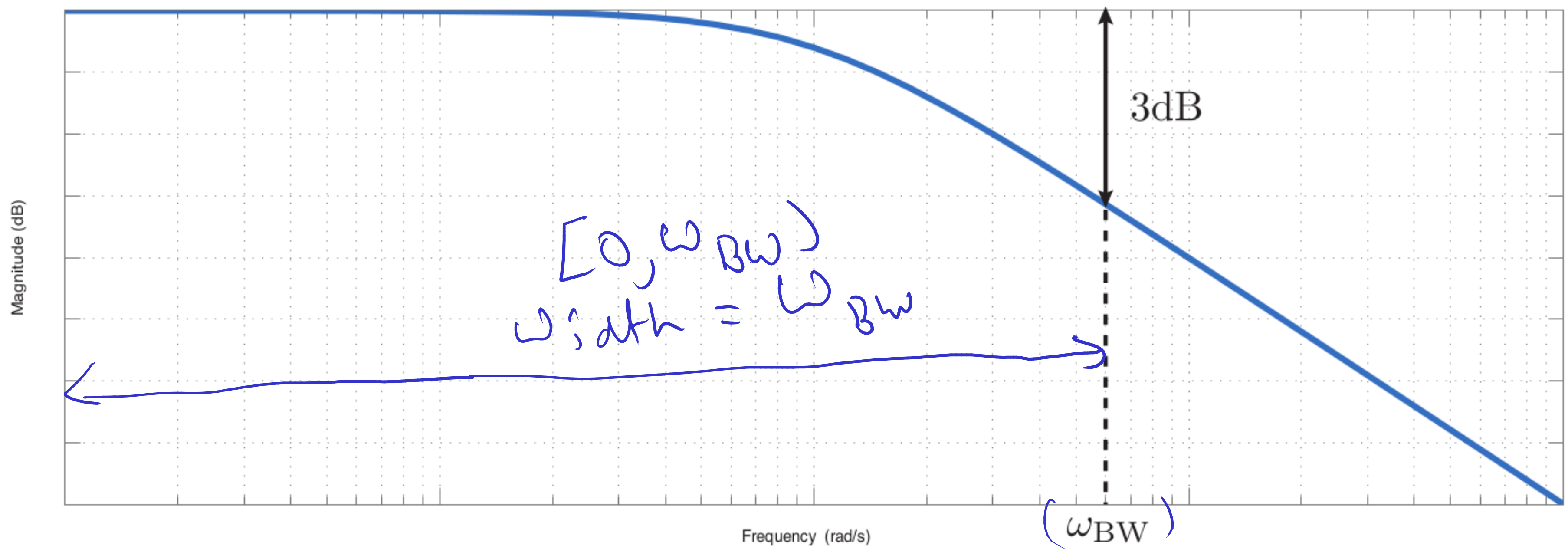
inf-norm

Definition 2.6.4. Let $G(s)$ be a rational, proper transfer function with all its poles in $\mathbb{C}^-$. Let $\|G\|_\infty := \sup_{\omega\in[0,\infty)} |G(j\omega)|$ denote the maximum magnitude of $G(j\omega)$. The **bandwidth** of G is the width of the frequency range in $[0,\infty)$ in which, for every ω in this range

$$|G(j\omega)| \geq \frac{1}{\sqrt{2}}\|G\|_\infty.$$

In terms of decibels, for every ω in this frequency range, $20\log|G(j\omega)| - 20\log\|G\|_\infty \geq -3\text{dB}$.

Higher the bandwidth, the better. In general, high bandwidth implies a fast system response.



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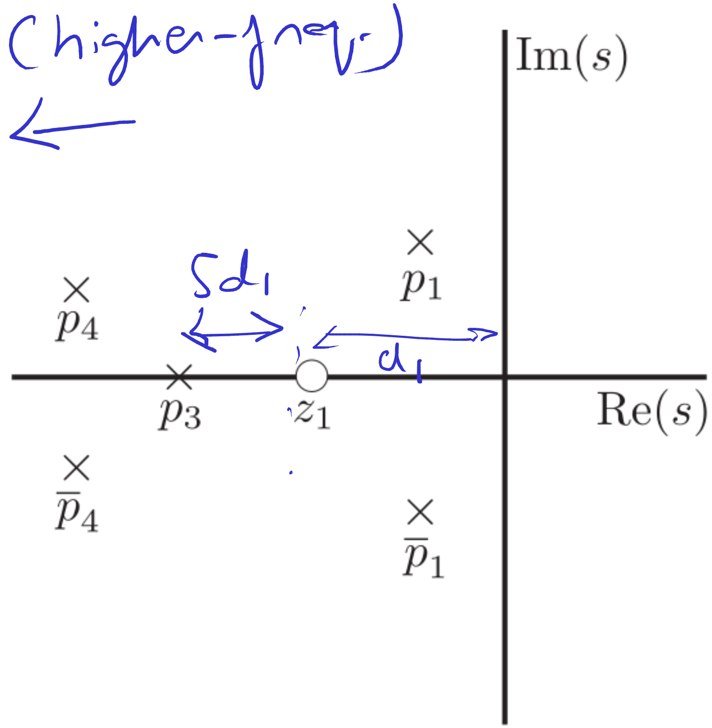
X - The upcoming topic

Poles and zeros that dominate the system response

Not all systems are second-order, and not all poles/zeros contribute equally to the system response.

- Poles and zeros far to the left of the Imaginary line do not have a significant impact on the low-frequency response. (Recall: Drawing bode plots by hand).

- Rule of thumb: Ignore poles and zeros that are at least 5 times away from the imaginary axis (non-dominant) compared to the ones closer to the axis (dominant).



s-plane

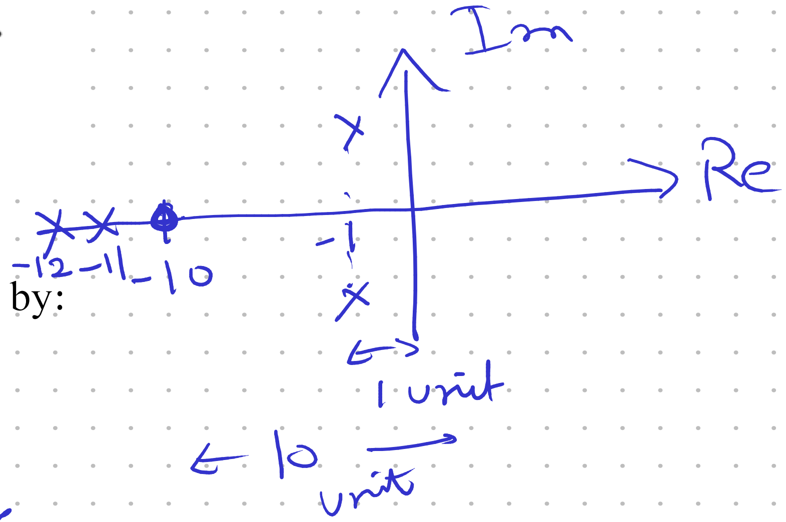
Example: Reducing model order

← 4th order system.

$$G(s) = \frac{s+10}{(s+11)(s+12)(s^2+2s+2)}$$
$$= \underbrace{\frac{s+10}{(s+11)(s+12)}}_{=:G_{\text{fast}}(s)} \underbrace{\frac{1}{s^2+2s+2}}_{=:G_{\text{slow}}(s)}.$$

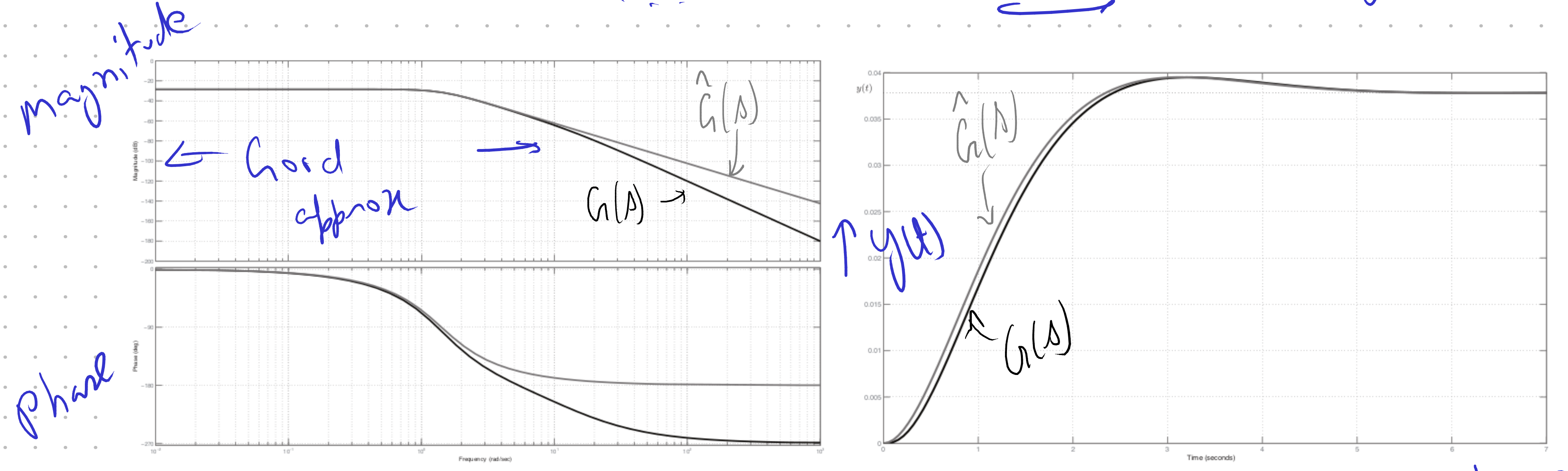
DC Gain:

$$G_{\text{fast}}(0) = \frac{10}{132}$$



Hence, we can approximate the transfer function by:

$$G(s) \approx \underbrace{\frac{10}{132}}_{\to G_{\text{fast}}(0)} \frac{1}{s^2+2s+2} =: \hat{G}(s)$$



(a) Bode plot. → ω

(b) Step response. → time

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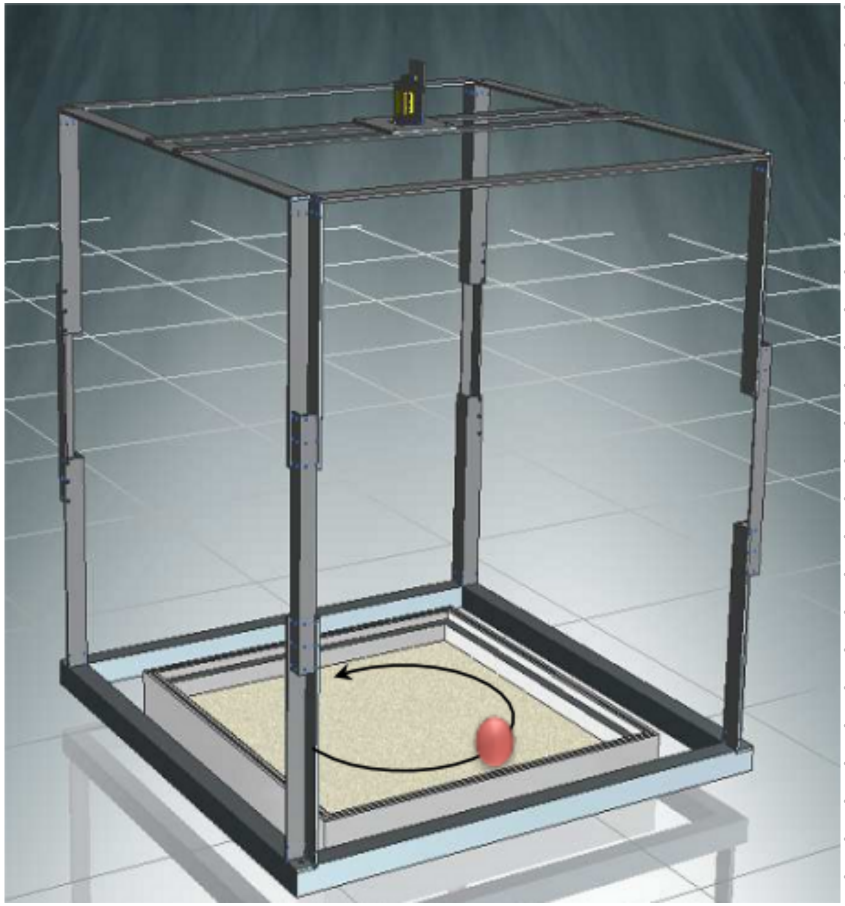
X - The upcoming topic

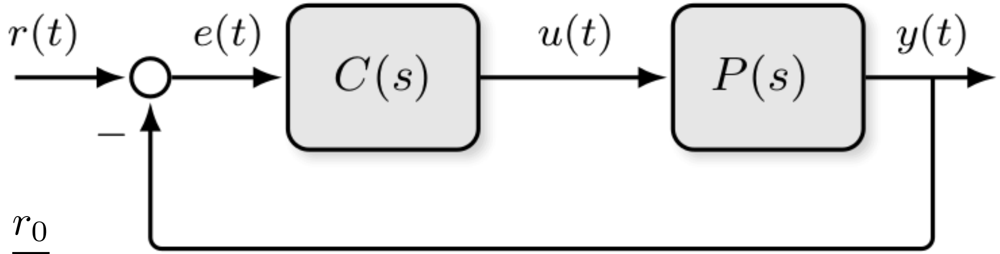
Tracking reference signals

We have seen many examples of this in the first lecture, for example the ball-on-plate system.



Reference signal is a path for the ball to track.
Control algorithm actuates the plate (motors) to move the ball and track the reference path.

Tracking reference signals



Example: $P(s) = \dfrac{1}{s+1} \quad C(s) = \dfrac{1}{s}$

Constant reference, i.e., $r(t) = r_0\mathbf{1}(t)$, or, $R(s) = \dfrac{r_0}{s}$

What is the steady-state error?

$$e_{ss} = \operatorname*{Lt}_{t\to\infty} e(t)$$

$$E(s) = \frac{1}{1+P(s)C(s)} \cdot R(s) = \frac{\cancel{s}(s+1)}{s^2+s+1} \cdot \frac{r_0}{\cancel{s}}$$

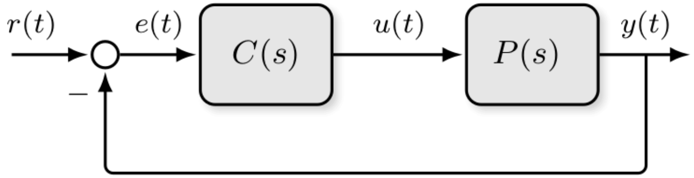
Internal model principle

$$= \frac{s+1}{s^2+s+1} \cdot r_0$$

FVT $\Rightarrow$ $e_{ss} = 0$.

(Asymptotically perfect tracking)

Internal Model Principle



Generalize the example, and write: $\dfrac{E(s)}{R(s)} = \dfrac{1}{1+PC} = \dfrac{D_pD_c}{D_pD_c + N_pN_c}$

Assume the reference is a step-signal, the steady-state error is:

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \cancel{s} \cdot \frac{1}{1+P(s)C(s)} \cdot \frac{r_0}{\cancel{s}}$$

$$= \lim_{s\to 0} \frac{1}{1+P(s)C(s)} = \frac{1}{1+P(0)C(0)}$$

When is $e_{ss} = 0$? $P(0)C(0) = \infty \iff$ Pole at $s=0$ in $P(s)C(s)$

$\rightsquigarrow$ Integrator in PC

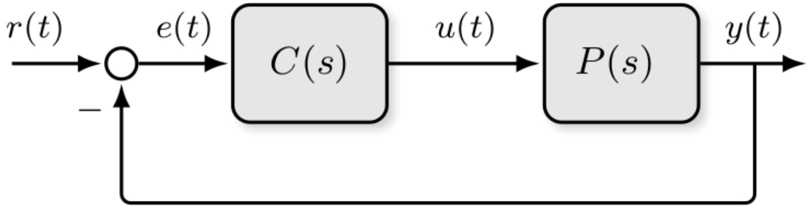
$(1/s)$

Internal Model Principle

Theorem (internal model principle). Assume $P(s)$ is strictly proper, $C(s)$is proper and the feedback systems is stable.
If $C(s)P(s)$ contains an internal reference model of the unstable parts of $R(s)$, then the tracking error goes to 0 asymptotically.

$$R(s) = \frac{N_r(s)}{D_r(s)}$$

$$= \frac{N_r(s)}{\underbrace{D_r^+(s)}_{\text{has poles in } Re(s) \geq 0} \; \underbrace{D_r^-(s)}_{\text{poles are in } Re(s) < 0}}$$



Example: Tracking a sinusoid reference

Let $P(s) = \frac{1}{s+1}$, the reference signal is $r(t) = r_0 \sin(t)$ or $R(s) = \frac{r_0}{s^2+1}$ ⇝ poles ? : $\pm j$

Let the reference signal be written as $R(s) = \frac{N_r}{D_r^+ D_r^-}$

Internal reference model requires the closed-loop transfer function to have:

$CP = \frac{N}{DD_r^+}$ → copy of ref's unstable part.

Assume stable CL system.

The error term is therefore: $E = \frac{1}{1+CP}R = \frac{D\cancel{D_r^+}}{\pi(s)} \cdot \frac{N_r}{\cancel{D_r^+} D_r^-}$, where $\pi(s)$ is the characteristic polynomial

$= \frac{D}{\pi(s)} \cdot \frac{N_r}{D_r^-}$, which only has stable poles ⇒ FVT :)

The controller therefore should be of the form: $C(s) = \frac{1}{s^2+1} C_1(s)$ → D_r^+

Here, $C_1(s)$ is an extra factor to ensure closed-loop stability, e.g., $C_1(s) = s$

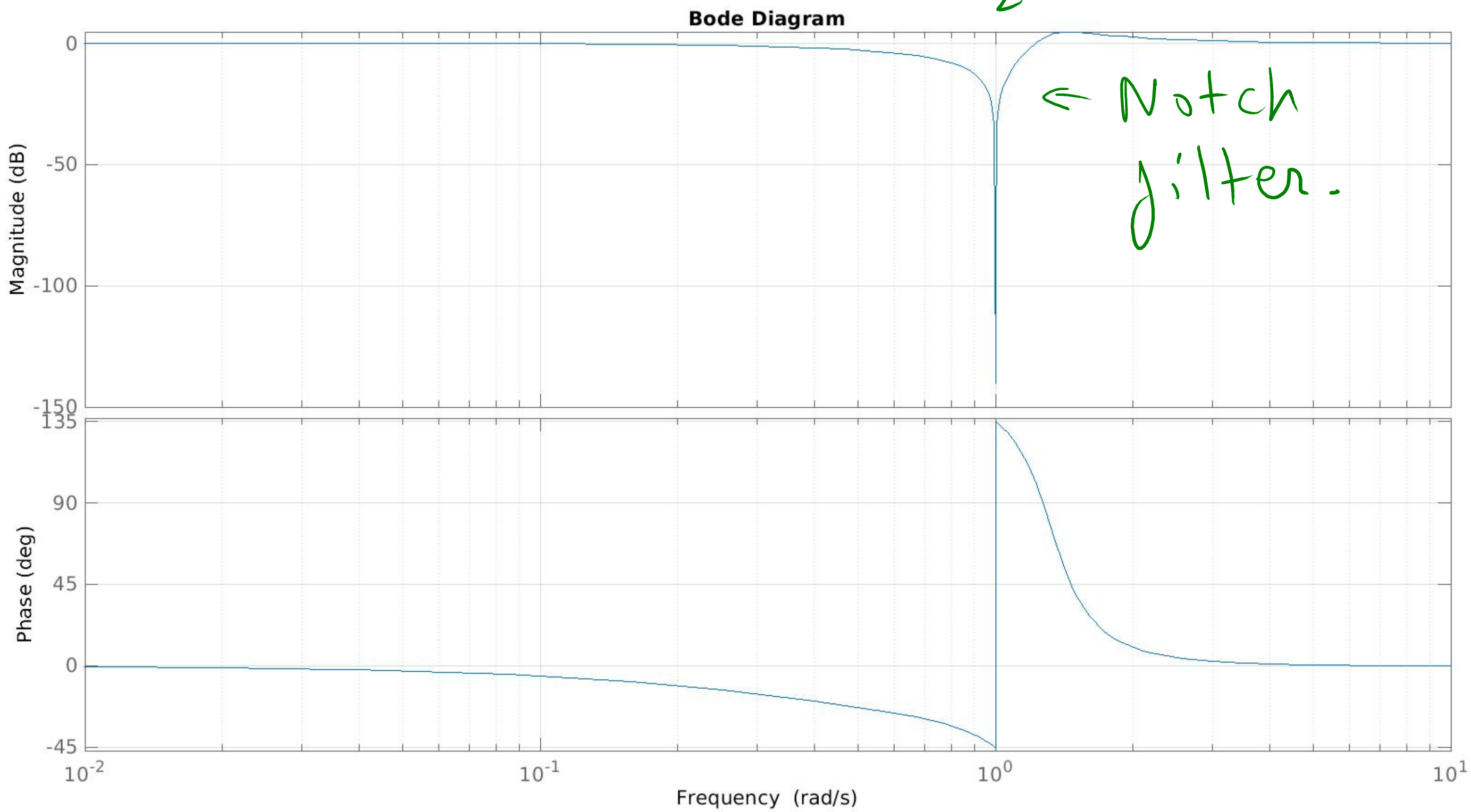
Note: This is not an unstable pole-zero cancellation! We are not cancelling anything from the plant, only from the reference and so the full system is still input-output stable (given properly designed C).
Exercise: Verify this by writing down the characteristic polynomial of the system.

A frequency domain interpretation

Consider the previous example of the sinusoidal reference signal. The transfer function from R to E is:

$$\frac{E(s)}{R(s)} = \frac{1}{1 + C(s)P(s)} = \frac{s^3 + s^2 + s + 1}{s^3 + s^2 + 2s + 1}$$

← 1 rad s^{-1}, $r(t) = r_0 \sin(t)$ $(\omega = 1)$



← Notch filter.

Summary (reference tracking control):

- If the closed-loop system is unstable, then we can't even talk about steady-state tracking performance since there will likely not be a steady-state response.
- Even if the closed-loop system is stable, we have to be careful applying the FVT to compute the steady-state tracking error because, depending on the reference signal, $sE(s)$ may have poles in $\mathrm{Re}\,(s) \geq 0$.
- If the closed-loop system is stable and $P(s)C(s)$ contains an internal model of the unstable part of $R(s)$, then perfect asymptotic tracking occurs.
- It does not matter if $P(s)$ or $C(s)$ provide the internal model. Perfect asymptotic tracking occurs when the product $C(s)P(s)$ has an internal model of the unstable part of $R(s)$.

  → Mag. of loop gain

- If $C(s)P(s)$ does not contain an internal model, then increasing the gain $|C(j\omega)P(j\omega)|$ over the range of frequencies that $r(t)$ contains decreases the steady-state tracking error. This is consistent with the idea that high-gain leads to good performance.

Outline

[-] Representing Continuous-time (CT) Signals and Transforms
- Laplace Transform
- Fourier Transorm

[-] Transfer functions for modeling dynamical systems
- Linear Time Invariant (LTI) systems
- Transfer function representations

[-] Stability of LTI systems: Bounded-input bounded-ouput (BIBO)

[-] Stability of feedback systems (LTI)

[-] Time-response of dynamical systems

[-] Frequency response of dynamical systems
- Graphical representations

[-] Dominant Poles and Zeros

[-] Reference tracking control

X - The upcoming topic

Summary

Through our quick recap of analog control systems, we have seen:

- We can represent time domain signals in the s-domain (frequency) via Laplace transforms.

- We can go from ODEs to transfer functions to model LTI systems.

- Notions of stability for open-loop and closed-loop systems.

- How to compute the characteristic polynomial (ch.p.) of a CL system and use it to check input-output stability.

- Time domain response of systems, and closed-form expressions for the step-response characteristics of a 2nd order system.

- The final value theorem and steady-state response of a system.

- Frequency response of a LTI system.

- How to reduce higher order systems to lower order systems when possible.

- The internal model principle for asymptotically perfect tracking of apriori known reference signals.