Chapter 7: Sampled-data Systems

ECE 481 - Digital Control Systems

Yash Vardhan Pant

Based on course notes by Professor Chris Nielsen

Outline

[X] Sampled-data systems: Introduction (again)

[ ] State-space analysis

- [ ] Solution to Continuous-time state-space models
- [ ] Step-invariant transformations (C2D) in the state-space
- [ ] Direct step-invariant transformations (transfer function)

[ ] The effect of sampling on discretization

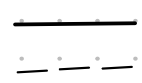
- [ ] Example and definition of pathological sampling
- [ ] Frequency domain

[ ] Selecting sampling time

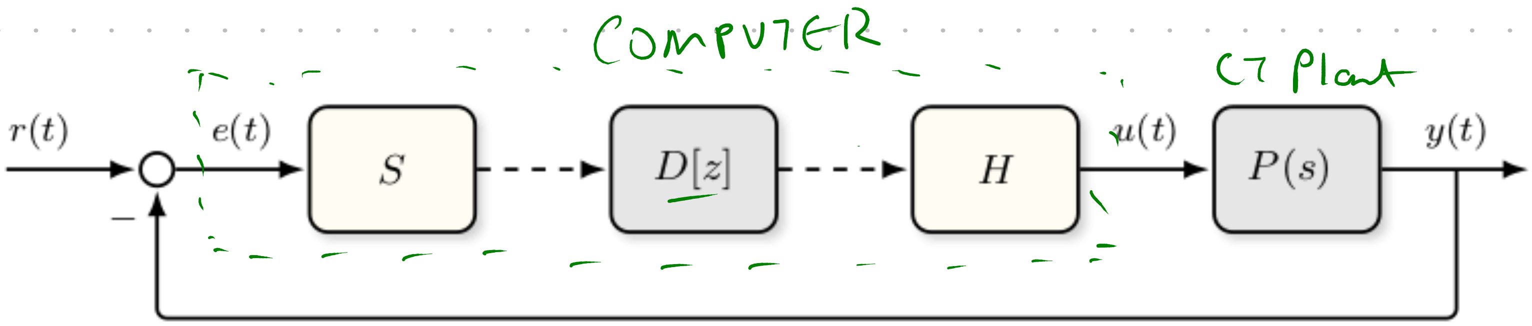
X = The upcoming topic

- = Topic that has been covered

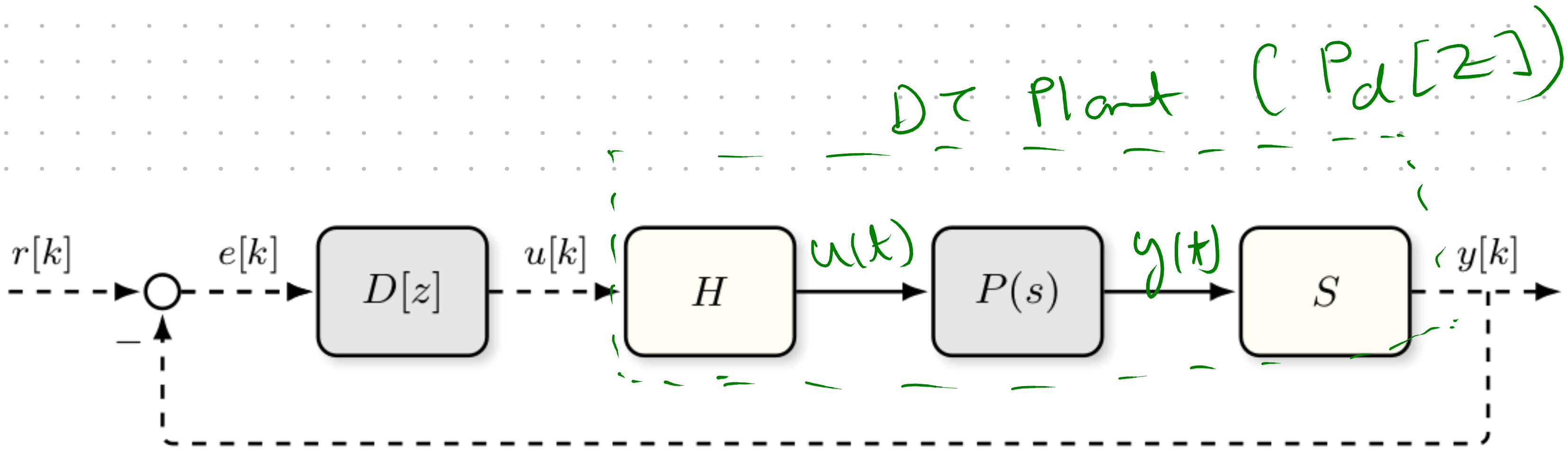
Sampled-data systems



From the view of the microcontroller/digital controller:



Sample operator is linear (recall chapter 4), so we can rearrange to get an equivalent configuration:



Here, Pd[z] := S P H is the step-invariant transform of the CT system into a DT system (again, chapter 4).

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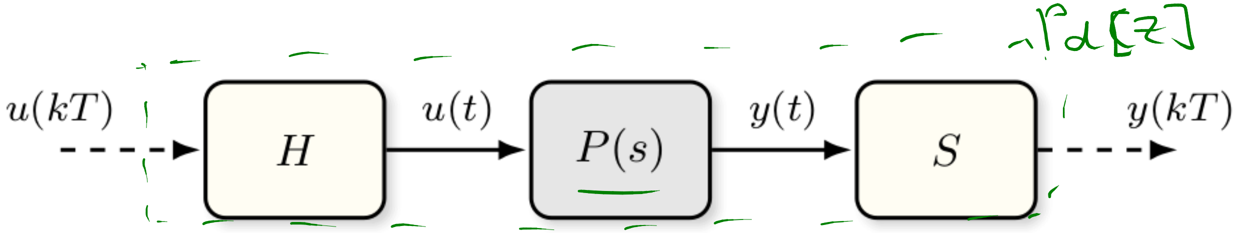
State-space analysis



Step-invariance method applied to a LTI system

$P(s)$ corresponds to

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$u, y \in \mathbb{R}$, $x \in \mathbb{R}^n$

Solution for CT LTI SS systems for a given $x(0) = x_0 \in \mathbb{R}^n$, $u(\tau)\ \forall\ \tau \in [0, t)$ is:

$$x(t) = \underbrace{e^{At}}_{\text{CT State Transition matrix.}} x_0 + \underbrace{\int_0^t e^{(t-\tau)A} B u(\tau)\, d\tau}_{\text{Convolution}}$$

$$y(t) = Cx(t) + Du(t)$$

The matrix exponential

Definition 7.2.1. The **matrix exponential** of $A \in \mathbb{R}^{n\times n}$ is

$$e^A := \sum_{k=0}^{\infty} \frac{1}{k!} A^k = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots .$$

Power series

- The sum converges $\forall$ A (square).
- Size(e^A) = Size(A), i.e., if $A \in \mathbb{R}^{n\times n}$, $e^A \in \mathbb{R}^{n\times n}$
- Only defined for square matrices.

Properties of the matrix exponential (function e^{At}), $t \mapsto e^{tA}$

$\mathbb{R} \mapsto \mathbb{R}^{n\times n}$

1. $e^{tA}\Big|_{t=0} = I_{n\times n}$

2. $e^{t_1 A} e^{t_2 A} = e^{(t_1+t_2)A}$

Note: $e^{tA_1} e^{tA_2} \neq e^{t(A_1+A_2)}$, unless A_1 & A_2 commute, i.e. $A_1 A_2 = A_2 A_1$

3. $(e^{A})^{-1} = e^{-A} \Rightarrow (e^{tA})^{-1} = e^{-tA}$

4. e^{At} & A commute

5. $\frac{d}{dt} e^{At} = A e^{tA} = e^{tA} A$ (using 4)

6. $\mathcal{L}\{e^{tA}\} = (sI - A)^{-1}$ [Similar to 2-7F of A^k],

Example

Calculate e^{At} for $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ using LTs.

Using property 6, $e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\}$

$$sI-A = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}, \quad (sI-A)^{-1} = \frac{1}{s^2+3s+2}\begin{bmatrix} s+3 & +1 \\ -2 & s \end{bmatrix}$$

For each element, do a partial fraction expansion:

$$(sI-A)^{-1} = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & -\frac{1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

Take inv. LT

$$e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & \cdots \\ \cdots & \end{bmatrix}$$

Exercise: compute the matrix exponential for the previous example without using LTs.

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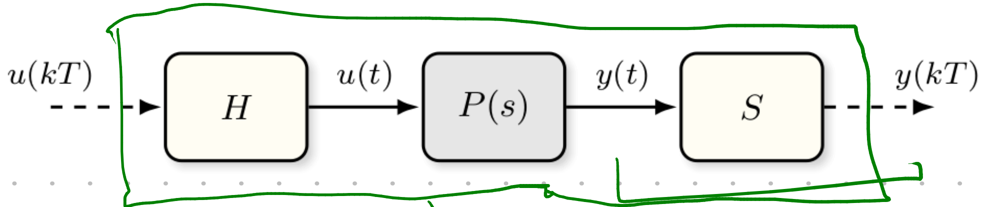
Step-invariant transformation (C2D)



Pd[z]

C, D matrices unchanged.

Given the CT system:
$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$
(Eq. 7.1 in the notes.)

and its solution: $$x(t) = e^{tA}x_0 + \int_0^t e^{(t-\tau)A}Bu(\tau)d\tau$$

Theorem 7.2.2. *The step-invariant transformation of (7.1) with sampling period T is the discrete-time system*

$$\begin{aligned} x[k+1] &= A_d x[k] + B_d u[k] \\ y[k] &= Cx[k] + Du[k] \end{aligned} \tag{7.3}$$

← Note, C & D are unchanged

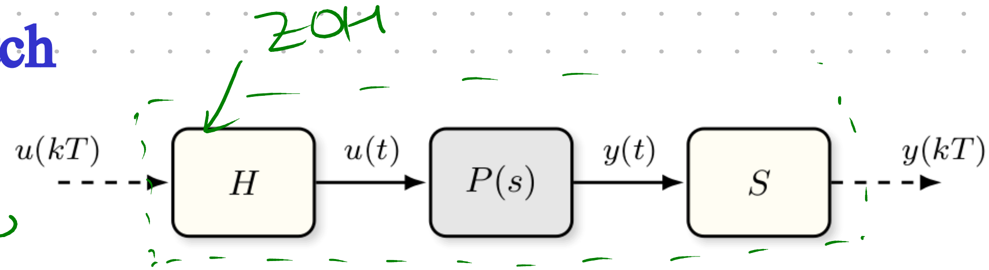
where $x[k] = x(kT)$, $u[k] = u(kT)$, $y[k] = y(kT)$ and

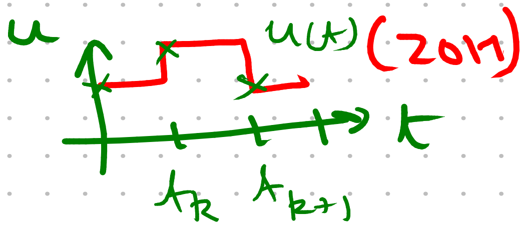
$$A_d := e^{TA}. \tag{7.4a}$$

$$B_d := \int_0^T e^{\tau A}d\tau B. \tag{7.4b}$$

Step-invariant transformation (C2D): Proof sketch

A formal version is in the class notes, Appendix 7.A





Let's say that start time is t_0,

$x(t_0)$ is given,

then, $x(t) = e^{A(t-t_0^{*})}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$

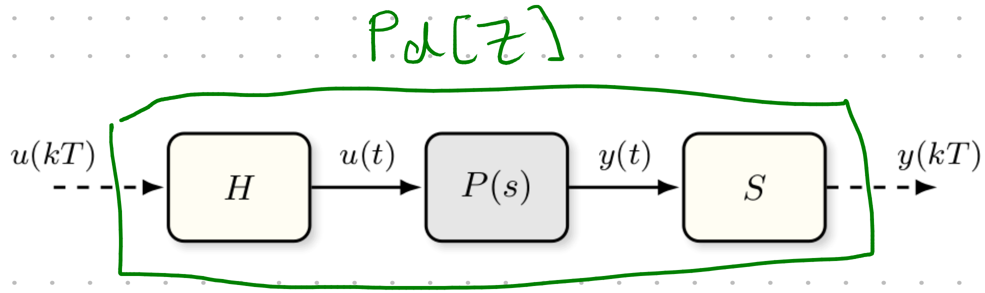
Let $t_0 = t_k = kT$, let $t = t_{k+1} = (k+1)T$

Note, ZOH on input, $u(t) = u(t_k)\ \forall\, t \in [t_k, t_{k+1})$

$$x(t_{k+1}) = \underbrace{e^{AT \leftarrow \text{why?}*} x(t_k)}_{A_d x[k]} + \underbrace{\int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}B\,d\tau}_{B_d}\ \underbrace{u[t_k]}_{\text{since ZOH}}$$

Do a change of variables to turn the integral into $\int_0^T$ $\Rightarrow$ Theorem on prev. page.

Step-invariant transformation (C2D): Remarks



Matlab uses this on the inside the function c2d (default).

It is a mapping from (A, B, C, D) to (Ad, Bd, C, D)

In terms of transfer functions:

From Theorem 7.2.2.

$$P(s) = C(sI - A)^{-1}B + D \quad \text{is mapped to} \quad P_d[z] = C(zI - A_d)^{-1}B_d + D$$

If A is non-singular/full-rank/invertible, then we can explicitly integrate the matrix exponential:

$$\int_0^T e^{\tau A} d\tau = A^{-1}\left(e^{AT} - I\right) \qquad \text{(to compute Bd)}$$

Proof sketch:

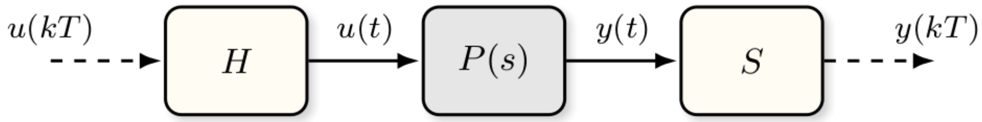
$$\int_0^T A e^{\tau A} d\tau = \int_0^T \frac{d}{d\tau} e^{\tau A} d\tau \quad \text{(From property 5)}$$

$$= e^{\tau A}\Big|_0^T = e^{TA} - I \quad \text{(property 1)}$$

Multiply with A^{-1} $\Rightarrow \int e^{\tau A} d\tau = A^{-1}(e^{TA} - I)$

Step-invariant transformation (C2D): Remarks

Matlab uses this on the inside the function c2d (default).



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Unlike Euler/Tustin etc., this is an exact discretization and is valid for all T.

Computing Ad and Bd requires computing the matrix exponential exp(At); can be done via Laplace trf.

w/ $u[k] = 0 \ \forall k$

Finally, since $A_d = e^{AT}$ and the state solution of DT state-space systems is $x[k] = A_d^k x[0]$, we can also compute A_d^k (DT transition matrix) using the computed CT matrix exponential as follows:

$$A_d^k = \left(e^{AT}\right)^k = e^{AkT}$$

Example: Scalar state-space system

$$\dot{x} = ax + bu, \quad y = cx + du, \quad a \neq 0. \quad x, y, u \in \mathbb{R}$$

Get the step-invariant DT system.

$a_d = e^{aT}$, to compute b_d, exploit a is full-rank.

$$b_d = a^{-1}(e^{aT} - 1)\, b = \frac{b}{a}(e^{aT} - 1)$$

DT system:

$$x[k+1] = \underbrace{e^{aT}}_{a_d} x[k] + \underbrace{\frac{b}{a}(e^{aT} - 1)}_{b_d} u[k]$$

$$y[k] = c\,x[k] + d\,u[k]$$

eigenvalues:

CT: $\lambda = a$

DT: $\lambda = a_d = e^{aT}$

Example: Double integrator $(\ddot{y} = u)$

State space CT: $\dot{x} = \overbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}^{A} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

(TF: $1/s^2$)

$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$

To get the C2D (step-invariant)

A is $\underbrace{\text{nilpotent}}$, $A^2 = 0$.

$\exists\, k \geq 0$, s.t. $A^k = 0$

iff $\lambda_i(A) = 0 \;\forall i$

Using definition (power-series) of e^A, $e^{At} = I + At + 0 \ldots$ ($\because A$ is nilpotent, $A^2 = 0$)

$$= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$A_d = e^{TA} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \quad \{T \text{ is sampling time}\}$$

$$B_d = \int_0^T e^{\tau A}\, d\tau B = \int_0^T \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau$$

$$= \int_0^T \begin{bmatrix} \tau \\ 1 \end{bmatrix} d\tau = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}$$

DT system:

$$x^+ = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Step-invariant transform: Mapping the eigenvalues from c2d

In the scalar example, the CT pole at $s = \lambda$ is mapped to $e^{\lambda T}$

This relation holds in general when we do the step-invariant discretization of a CT LTI system!

Note: No such (simple) relation between the zeros of the CT and step-invariant DT transformation.

Eigenvalue mapping: Let λ be an eigenvalue of A, i.e. $\exists v, Av = \lambda v$

$$A_d v = \underbrace{e^{AT}}_{\substack{A_d \\ \text{Step-inv. c2d}}} v = \left(I + TA + \frac{T^2}{2}A^2 + \frac{T^3}{3!}A^3 + \cdots\right) v$$

$$= v + TAv + \frac{T^2}{2}A^2 v + \cdots \qquad (A^n v = \lambda^n v)$$

$\because Av = \lambda v$

$$\Rightarrow A_d v = \Big(1 + \underbrace{T\lambda + \frac{T^2}{2}\lambda^2 + \cdots}_{e^{\lambda T}}\Big) v$$

$$A_d v = \underbrace{e^{\lambda T}}_{\text{eigenvalue of } A_d} v$$

Example

The mapping is not just for eigenvalues, but also for the poles (since poles are a subset of eigenvalues).

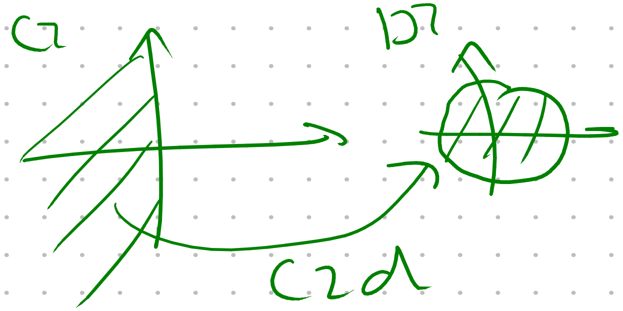
Let $P(s) = \dfrac{1}{s^2(s+1)}$ and T = 0.1 seconds. See Matlab example for poles of corresponding DT TF.

Note: Given what we've seen so far, we would:

a) Get the corresponding ODE
b) Turn that into a CT state-space model
c) Compute the DT state-space model via the step-invariant TF.
d) Get a DT TF from the DT state-space model.

Step-invariant transform: Preserving stability

As we saw in Chapter 4, the step-invariant transform also preserves stability.



Let's say for the CT system, we have an eigenvalue $\lambda = \alpha + j\beta$

Stability $\Rightarrow$ $Re(\lambda) < 0 \Rightarrow \alpha < 0$.

For the DT system, eigenvalue would be $e^{\lambda T}$ $(T>0)$

$$|e^{\lambda T}| = |e^{\alpha T}| \cdot \underbrace{|e^{j\beta T}|}_{=1} = |e^{\alpha T}| < 1 \quad (\because \alpha < 0)$$

$\Rightarrow$ DT stable.

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Step-invariant transformation: A view from the transfer function side of things

$$A_d = e^{A\tau}$$
$$B_d = \int \cdots$$

CT state-space

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

DT state-space

$$x^+ = A_d x + B_d u$$
$$y = Cx + Du$$

$$\frac{Y(s)}{U(s)} = G(s)$$

CT TF

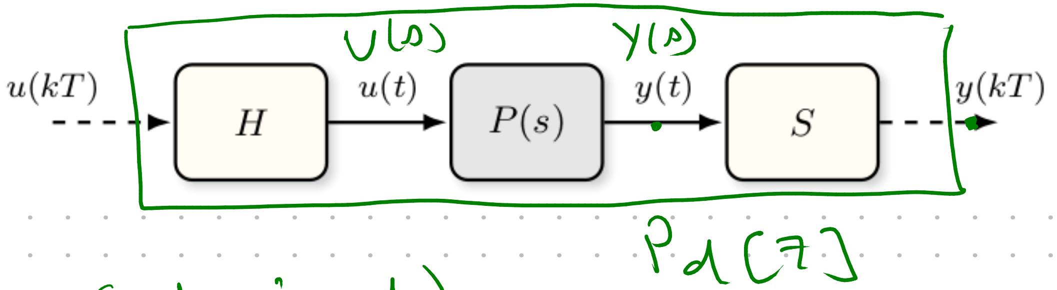
?

$$\frac{Y[z]}{U[z]} = G_d[z]$$

DT TF

Step-invariant transformation: A view from the transfer function side of things

How do we go directly from CT TF to DT TF via the step-invariant transformation?



Follow the recipe:

1. Let $u[k] = u(kT) = \mathbf{1}[k]$. Then $U[z] = z/(z-1)$. (Step signal)

2. Then $u(t)$ is a continuous-time unit step $\mathbf{1}(t)$ so $U(s) = 1/s$.

3. Then $Y(s) = P(s)U(s)$. Get $y(t)$, the step response of P.

4. Sample $y(t)$ to get $y(kT) = y[k]$ and then $Y[z]$.

5. Finally, the step-invariant transformation of P is

Y(s)

Y[z]

$$\frac{Y[z]}{U[z]} = \frac{z-1}{z}\mathscr{Z}\left(\mathscr{L}^{-1}\left\{\frac{P(s)}{s}\right\}\Big|_{t=kT}\right).$$

ZTF

sampling y(t) to get y(kT)

y(t)

y[kT]

1 / ZTF of unit step

Example

Recipe for the step-invariant transform (TF)

$$\boxed{\frac{Y[z]}{U[z]} = \frac{z-1}{z}\mathscr{Z}\left(\mathscr{L}^{-1}\left\{\underbrace{\frac{P(s)}{s}}_{Y(s)}\right\}\Big|_{t=kT}\right)}$$

Let $P(s) = \dfrac{3}{s+2}$, get $P_d[z]$ via step-invariant transform (T).

$$Y(s) = P(s)\,U(s) = P(s)\cdot\frac{1}{s} = \frac{3}{s(s+2)}$$

Take inv LT, but first: $Y(s) = \dfrac{a}{s} + \dfrac{b}{s+2}$ (PF expansion)

$a = 1.5,\ b = -1.5$

$$\Rightarrow Y(s) = \frac{1.5}{s} - \frac{1.5}{s+2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 1.5\left(1 - e^{-2t}\right)\mathbb{1}(t)$$

Sample at every T seconds: $y[k] = \dfrac{3}{2}\left(1 - e^{-2kT}\right),$

$k \in \mathbb{Z}_{\geq 0}$

Take Z-TF

$$Y[z] = \mathcal{Z}\{y[n]\} = 1.5 \frac{z}{z-1} - \frac{3}{2} \frac{z}{z-e^{-2T}}$$

$$= \frac{3}{2} \frac{z(1-e^{-2T})}{(z-1)(z-e^{-2T})}$$

Finally $P_d[z] = \frac{z-1}{z} Y[z]$

$$= \frac{3}{2} \left(\frac{1-e^{-2T}}{z-e^{-2T}} \right)$$

Step-invariant transformation: summary

CT state-space

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Ad, Bd using the matrix exponential

DT state-space

$$x^{+} = A_d x + B_d u$$
$$y = Cx + Du$$

→ Matlab's c2d path.

$$\frac{Y(s)}{U(s)} = G(s)$$

CT TF

Recipe we just saw

$$\frac{Y[z]}{U[z]} = G_d[z]$$

DT TF

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Pathological sampling: an example

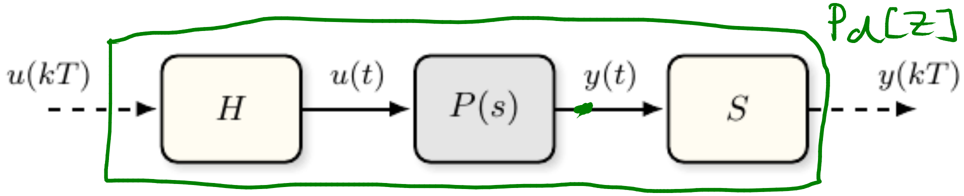

Figure 7.3: Step-invariance method applied to an LTI system.

What impact do the S and H have on the C2D of P(s)?

Consider the case where the plant is an oscillator which oscillates as the sampling frequency, i.e.,

$$P(s) = \frac{\omega_s s}{s^2 + \omega_s^2}, \qquad \omega_s = \frac{2\pi}{T}. \leftarrow \text{Sampling time}$$

What is the unit-step response of this system? Consider u(t) = 1(t), and thus:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{P(s)}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{\omega_s}{s^2+\omega_s^2}\right\} = \sin(\omega_s t)$$

Pass this y(t) through the sampling (S) block, we get:

$$y[k] = y(t)\Big|_{t=kT} = \sin(\omega_s kT) = \sin\Big(\overbrace{\frac{2\pi}{T}}^{\omega_s} \cdot kT\Big) = \sin(2\pi k) = 0 \;\; \forall k.$$

The discretized system is now a "zero" system, i.e., Pd[z] = 0, even when P(s) is not. P(s) = 0 would also map to this same discretized "zero" system, i.e., the C2D mapping is not one-to-one.

This example where get a DT system that does not accurately represent the CT system is an example of "pathological sampling", where the continuous time behavior of the system is lost even for a step-invariant transform. However, this would only happen for some sampling frequencies, as seen above.

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Frequency response and sampling time

Consider the setup shown on the right.

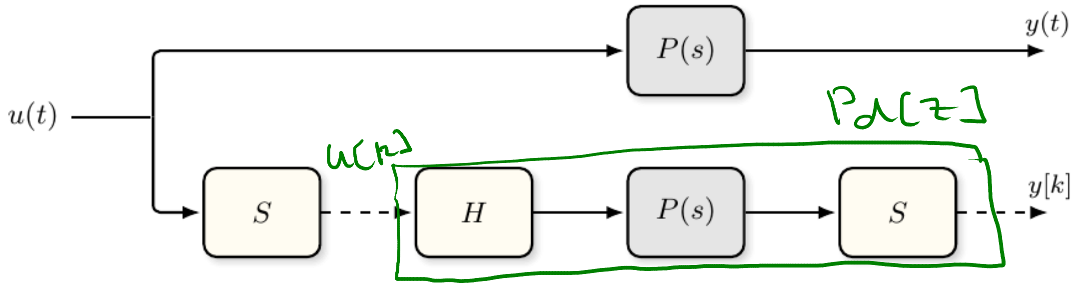


Figure 7.5: Comparing the frequency responses of $P(s)$ and $P_d[z]$.

We are interested in analyzing when the frequency response of the DT system is close to that of the CT system.

Let the input to this system be a sinusoid $u(t) = e^{j\omega t}$.

Now consider the sampled version of the input to generate: $u[k] = e^{j\omega T k} = e^{j\theta k}$ $(\theta = \omega T)$

Let us divide the frequency (radians) axis into non-overlapping intervals of width $2\pi/T$:

Baseband

$$\ldots, \left(-\frac{\pi}{T}, \frac{\pi}{T}\right], \left(\frac{\pi}{T}, \frac{3\pi}{T}\right], \ldots$$

Assume $\omega \in (\pi/T, 3\pi/T]$. We can rewrite $\omega = \omega_0 + 2\pi/T$, where $\omega_0 \in (-\pi/T, \pi/T]$, i.e. is in the "baseband".

$(\because \theta = \omega T)$

Then we can rewrite the sampled input as $u[k] = e^{j(\omega_0 T + 2\pi)k} = e^{j\omega_0 T k} = e^{j\theta_0 k}$ where $\theta_0 := \omega_0 T \in (-\pi, \pi]$.

For the comparison with the CT system, we'll consider the case when u(t) is at a low frequency. See appendix 7.B of the course notes for a general case.

Frequency response and sampling time

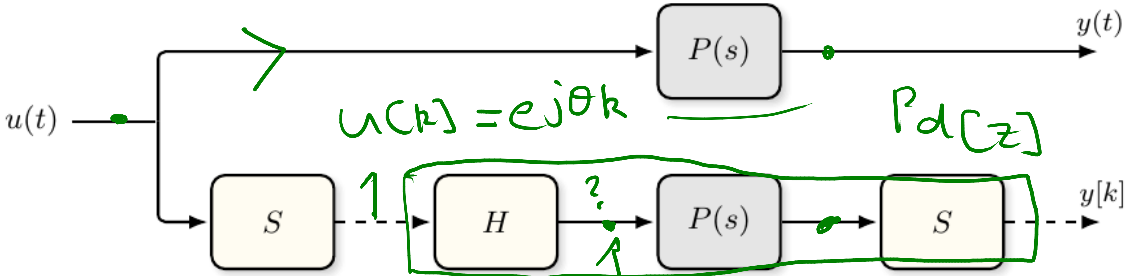


Figure 7.5: Comparing the frequency responses of $P(s)$ and $P_d[z]$.

Continuing from the previous slide, let the sampling frequency be $\omega_s = 2\pi/T$. (radians)

We'll assume that the Nyquist frequency is $\omega_N = \omega_s/2 = \pi/T$.

Considering the low-frequency input case, the CT input is $u(t) = e^{j\omega t}$, $\omega \ll \omega_N$.

What is the steady-state output from P(jw) to this input? $y(t) = P(j\omega)\, e^{j\omega t}$

$\tilde{u} \simeq u \Rightarrow P(s)u = P(s)\tilde{u}$ (at low freqs.)

Similary, what is the steady-state output from Pd to the sampled input $e^{j\theta k}$, where $\theta = \omega T \ll \pi$?

$y[k] = P_d[e^{j\theta}]\, e^{j\theta k}$

Focusing on the bottom (DT) path, since $\omega \ll \omega_N$, the hold block's output is close to $e^{j\omega t}$.

Therefore, the steady state outputs in CT and DT are close to each other, i.e., $P_d[\mathrm{e}^{j\theta}]\mathrm{e}^{j\theta k} \approx P(j\omega)\mathrm{e}^{j\omega T k}$.

Thus at low frequencies, the two plants are similar to each other: $P_d[\mathrm{e}^{j\omega T}] \approx P(j\omega)$ or $P_d[\mathrm{e}^{j\theta}] \approx P\left(j\frac{\theta}{T}\right)$.

At higher frequencies, our assumption about the ZOH doesn't work anymore (due to aliasing), so the above statement doesn't hold.

See Ch. 4.

Rule of thumb: $P_d[\mathrm{e}^{j\omega T}] \approx P(j\omega)$ when $\omega \leq \omega_\mathrm{N}/5$. $= \omega_s/10$ $(\omega_N = \omega_s/2)$

Example 7.4.2

Consider the CT plant $P(s) = \dfrac{1}{s^2 + 0.1s + 1}$, and its DT version via the step-invariant transform, discretized at a sampling frequency of 10 rad/s: $P_d[z] = \mathtt{c2d}(P) = \dfrac{0.19(z + 0.98)}{z^2 - 1.57z + 0.94}$

Compare their Bode plots, and see if they are consistent with the discussion in the previous slides.

Example 7.4.3: Mismatch as frequency of input increases

Consider our setup as earlier (on the right).

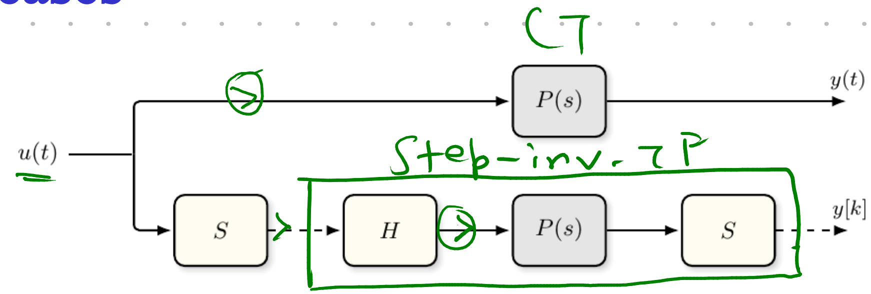


Figure 7.5: Comparing the frequency responses of $P(s)$ and $P_d[z]$.

$$P(s) = \frac{1}{s+1}, \qquad P_d[z] = \text{c2d}(P(s)) = \frac{0.09516}{z - 0.9048}.$$

Here, sampling time is T = 0.1 seconds to get the DT plant.

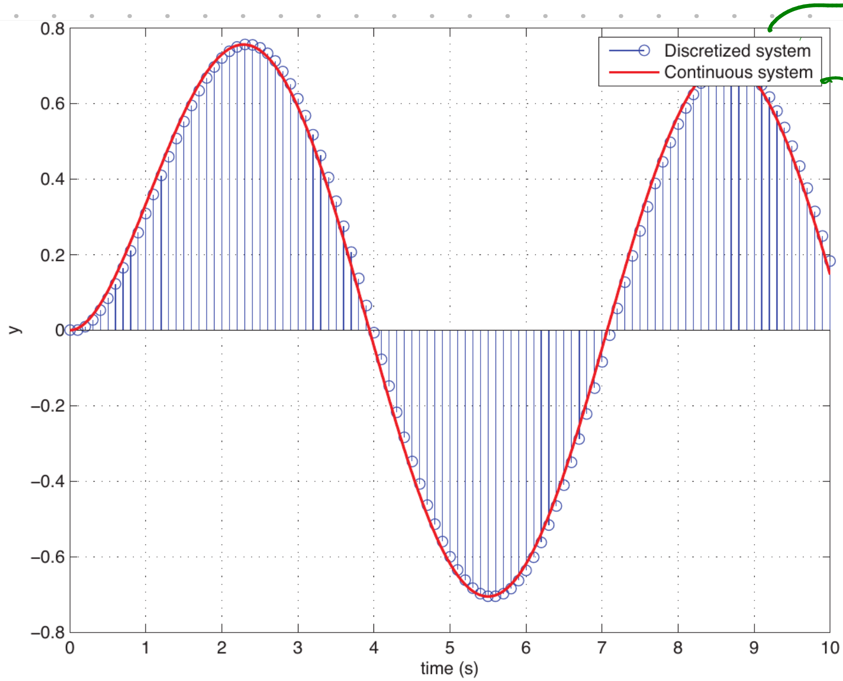
Let's assume u(t) = cos(wt). We expect steady-state responses to be similar when $\omega \ll \omega_N = \omega_s/2 = 10\pi$.

We can see this (and how the responses diverge) in action as the frequency of the input increases.

Case 1:

y[k]

y(t)

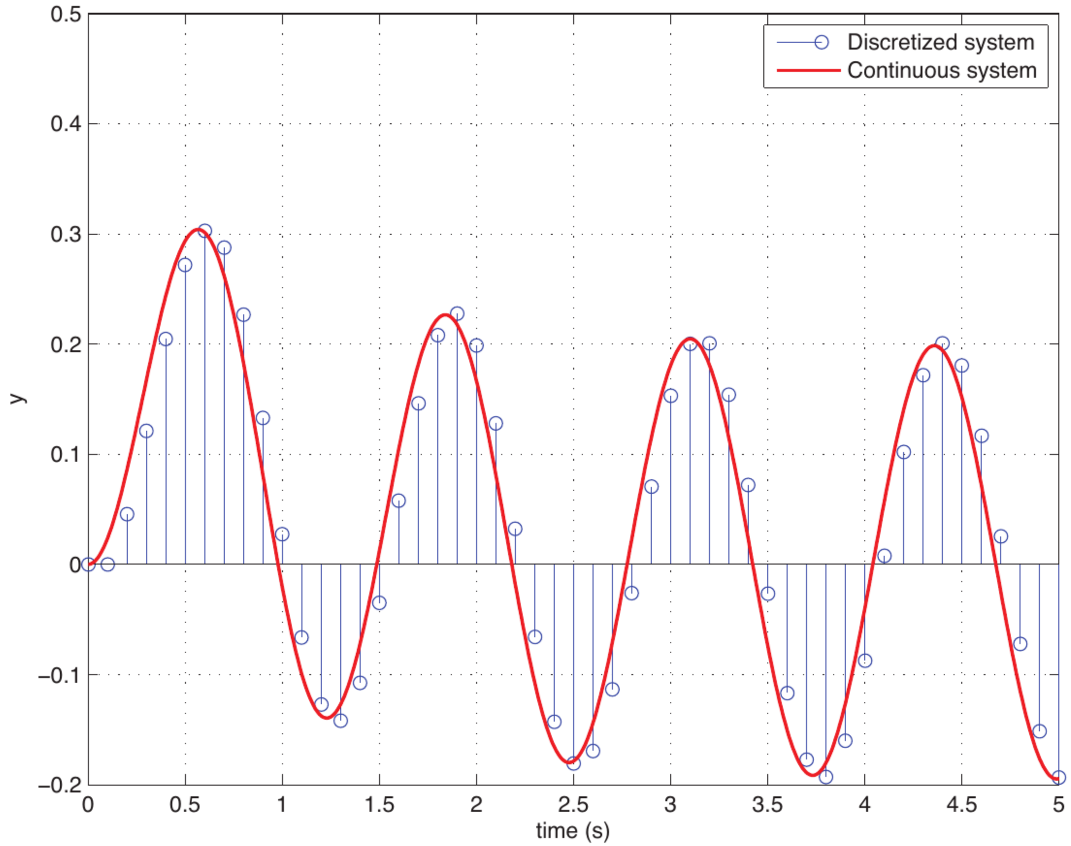


(a) $\omega = 1 \ll \omega_s/2 = 10\pi$.

← Stem plot.

Responses match.

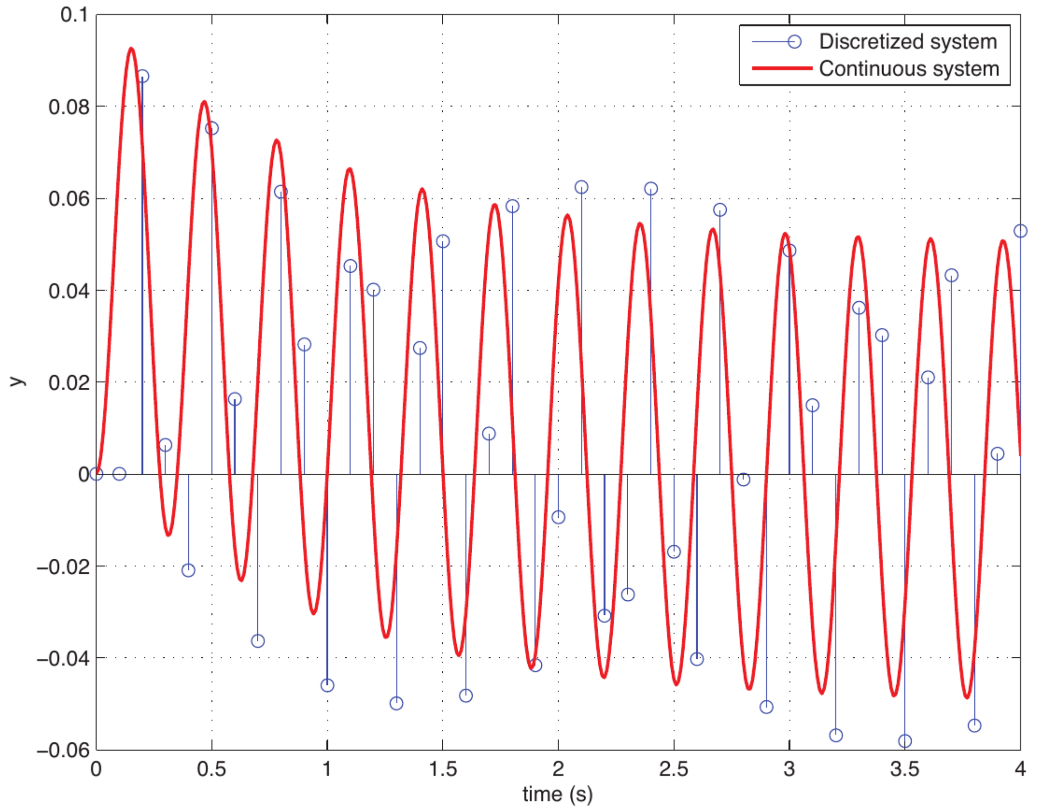
Case 2:



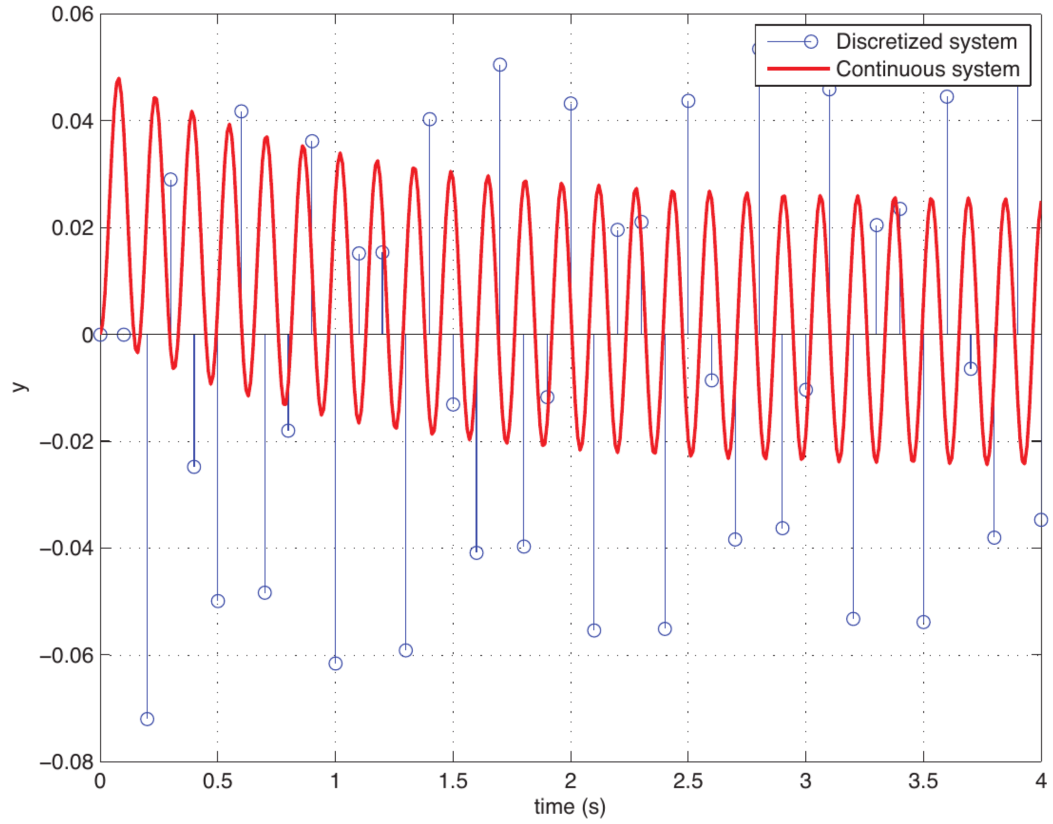
(b) $\omega = 5 \ll \omega_s/2 = 10\pi$.

Responses are still close.

Cases 3 and 4:



(c) $\omega = 20 < \omega_s/2 = 10\pi$.



(d) $\omega = 40 > \omega_s/2 = 10\pi$.

Response in CT and DT diverge as the input has high frequency, and unsurprisingly looks bad when the signal is faster than the Nyquist frequency (figure on right).

Outline

[-] Sampled-data systems: Introduction (again)

[-] State-space analysis
- [-] Solution to Continuous-time state-space models
- [-] Step-invariant transformations (C2D) in the state-space
- [-] Direct step-invariant transformations (transfer function)

[-] The effect of sampling on discretization
- [-] Example and definition of pathological sampling
- [-] Frequency domain

[X] Selecting sampling time

X = The upcoming topic

- = Topic that has been covered

Comments on selecting the sampling period

TLDR: Sample as fast as physically possible!

- Select a sampling time T that avoid pathological sampling. Then we can do "direct design" of controllers on the DT model.

- For design via emulation, select a sampling time T so that the closed-loop bandwidth of the system satisfies $\omega_s \geq 10\omega_{BW}$. This is a direct consequence of the rule of thumb that we had introduced.

- Additionally, T should be small enough so that inter-sample behavior is "good", i.e., no wild jumps between sample values, otherwise safety constraints might be violated, e.g., temperature control of a room at 1-hour sampling intervals. This is best addressed through simulation, or analytically via the Lipschitz constant of the CT (possibly non-linear) system.

- On the flip side, T should be large enough (without violating above points) so that the processor speed is kept low to allow for real-time implementations. Think how much time it takes to compute u[kT] from y[kT]. This time should be ideally an order of magnitude less than your sampling time.

- Finally, a slow CT plant (e.g., chemical reactions) with fast sampling can lead to outputs where very little changes across multiple time steps. This could cause issues with finite precision arithmetic.

- Note: The sampling period is mostly limited by the update rates of sensors and actuators (e.g., over a controller area network for cars), and not so much by "control-specific" issues that we have studied here.

- When possible: low-pass filters to all analog signals to satisfy the rule of thumb (cutoff freq <= ws/5).