Chapter 5: Nonlinear systems and control

ECE 481 - Digital Control Systems

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Based on notes by Prof. Chris Nielsen

Outline

[X] Introduction

[ ] State-space models

[ ] Linearization
- [ ] Equilibrium configuration
- [ ] Transfer function from a linear state-space model
- [ ] Control design using linearization

[ ] Inverting static nonlinearities
- [ ] Conditions for inverting static nonlinearities
- [ ] Local cancellation of static nonlinearities
- [ ] Inverting dynamic nonlinearities (feedback linearization), an example.

[ ] Static friction
- [ ] Dither signals
- [ ] Deadzone non-linearity

[ ] Describing functions
- [ ] Optimal Quasi-Linearization
- [ ] Constructing describing functions
- [ ] Periodic solutions and their stability

Introduction

- Most, if not all, dynamical systems are non-linear.

- Non-linearities arise not just from the physics, but also from effects such as saturation, static-friction, non-linear sensors etc.

- Bad news: Non-linear systems do not (directly) have a transfer function representation, so our control design techniques up to now cannot be applied directly.

- Good news: With some neat tricks, we can get back to a point where we can apply the control techniques that we have seen up to now, such as pole-placement, or even frequency-domain design.

Beyond this course: There exist a whole host of techniques for directly controlling non-linear systems with guarantees on properties like stability and robustness, while minimizing the number of simplifying assumptions being made.

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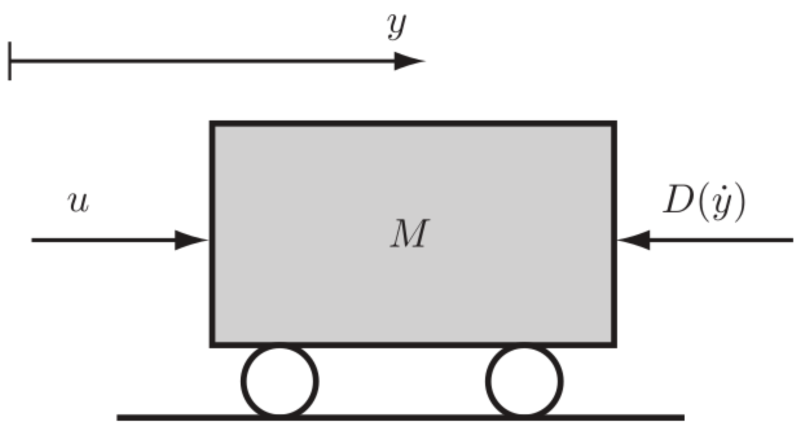
State-space models for representing non-linear systems

$y = 0$



Example: A cart facing wind resistance.

Input: A driving force (u)
Output: Displacement (y)
Wind resistance: $D(\dot{y})$, $D : \mathbb{R} \to \mathbb{R}$

From Newton's second law: $M\ddot{y} = \sum_i \text{forces}$

$$\Rightarrow M\ddot{y} = u - D(\dot{y}) \quad — (1)$$

Define "state" variables: $x_1 := y$, $x_2 := \dot{y}$

Then,

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{M}u - \frac{1}{M}D(x_2) \end{aligned}\right\} \text{State/dynamic eq}^{ns}$$

$$y = x_1 \quad \Big\} \text{ output}$$

Here,

$$f : \overset{(x)}{\mathbb{R}^2} \times \overset{(u)}{\mathbb{R}} \longrightarrow \overset{(\dot{x})}{\mathbb{R}^2}$$

$$f(x_1, x_2, u) := \begin{bmatrix} x_2 \\ -\frac{D}{m}(x_2) + \frac{u}{m} \end{bmatrix}$$

$$h : \overset{(x)}{\mathbb{R}^2} \times \overset{(u)}{\mathbb{R}} \longrightarrow \overset{(y)}{\mathbb{R}} \qquad \{ y = h(x_1, x_2, u) = x_1 \}$$

$$x \in \mathbb{R}^2 := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad u \in \mathbb{R}$$

$\Leftarrow$ 2 1st order ODEs.

$$\Rightarrow \quad \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = f(x_1, x_2, u)$$

$$y = h(x_1, x_2, u) = x_1$$

State-space models: General form for non-linear systems, and a special form for LTI

Let $x \in \mathbb{R}^n$ be the state, $u \in \mathbb{R}^m$ (input), $y \in \mathbb{R}^p$ (output)

The general state space form is

<u>NL:</u>

$$\dot{x} = f(x,u) \quad , \text{where } f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

$$y = h(x,u) \qquad h : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^p$$

<u>Special case!</u> LTI systems, where,

$$f(x,u) = Ax + Bu; \quad h(x,u) = Cx + Du$$

$$A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}$$

$$C \in \mathbb{R}^{p \times n}, \quad D \in \mathbb{R}^{p \times m}$$

Example: Wind resistance is linear, { (cont example) }

$$M\ddot{y} = u - D(\dot{y})$$

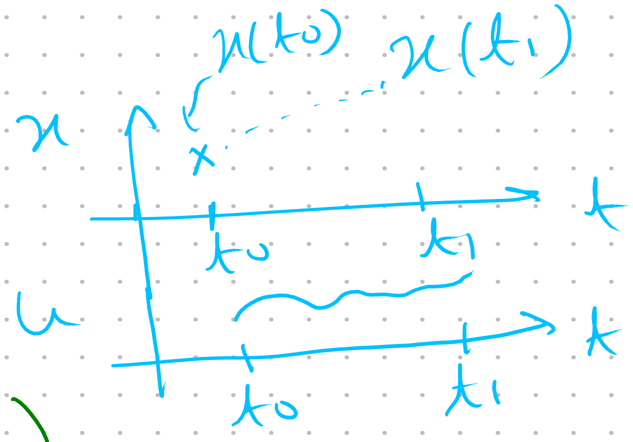
Let $D(\dot{y}) = d\dot{y}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{d}{m}x_2 + \frac{u}{m} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -d/M \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ 1/M \end{bmatrix}}_{B} \underbrace{u}_{u}$$

$$y = h(x,u) = x_1 = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D} u$$

The state of a system

$x(t_0) \in \mathbb{R}^n$ captures all system dynamics upto time t_0, i.e., no prior info is required.



For any t_0, t_1, $t_0 < t_1$,
knowing $x(t_0)$ & $u(t)$ $\forall t \in [t_0, t_1)$
is sufficient for computing $x(t_1)$ {or $y(t_1)$}

Consider the cart w/ no input & no air resistance,

i.e., $M\ddot{y} = 0$ — ②

Let's consider $x := y \in \mathbb{R}$. Given $x(t_0)$, can we compute $x(t_1)$, $t_1 > t_0$, on the position of the cart at time t_1? No.

Similarly, $x := \dot{y} \in \mathbb{R}$ won't work either.

For mechanical systems, $\vec{x} = \begin{bmatrix} \vec{\text{positions}} \\ \vec{\text{velocities}} \\ \vec{\Delta \text{ positions}} \\ \vec{\Delta \text{ velocities}} \end{bmatrix}$

Eq^n ② : $x_1 := y$, $x_2 := \dot{y}$

$$\Rightarrow \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 0 \end{aligned}$$

Note: there exists an explicit solution for x(t) in terms of x(0) and the signal u(t) (t from 0 to t) when the state-space model is LTI.

States for discrete-time systems: An example (LTI)

CT: $\dot{x} = Ax + Bu$, $y = Cx + Du$, DT: $x_{k+1} = Ax_k + Bu_k$, $y_k = Cx_k + Du_k$

e.g.: Weighted average (difference equation): 3

$$y_{k+2} = 3y_{k+1} + 2y_k + u_k, \quad y, u \in \mathbb{R}$$

→ Can compute $\{y_k\}_{k \geq 0}$ recursively given, y_{-1}, y_{-2} & $\{u_k\}_{k \geq -2}$.

Define $x(k_0)$ captures all dynamics upto k, & knowing $x[k_0]$ & $\{u_k\}_{k \leq k_1}$, can compute $x[k_1]$

$x_1[k] := y[k]$, $x_2[k] := y[k+1]$ ← 4

From (3) & (4)

$$x_1[k+1] = x_2[k]$$

$$x_2[k+1] = 3x_2[k] + 2x_1[k] + u[k]$$

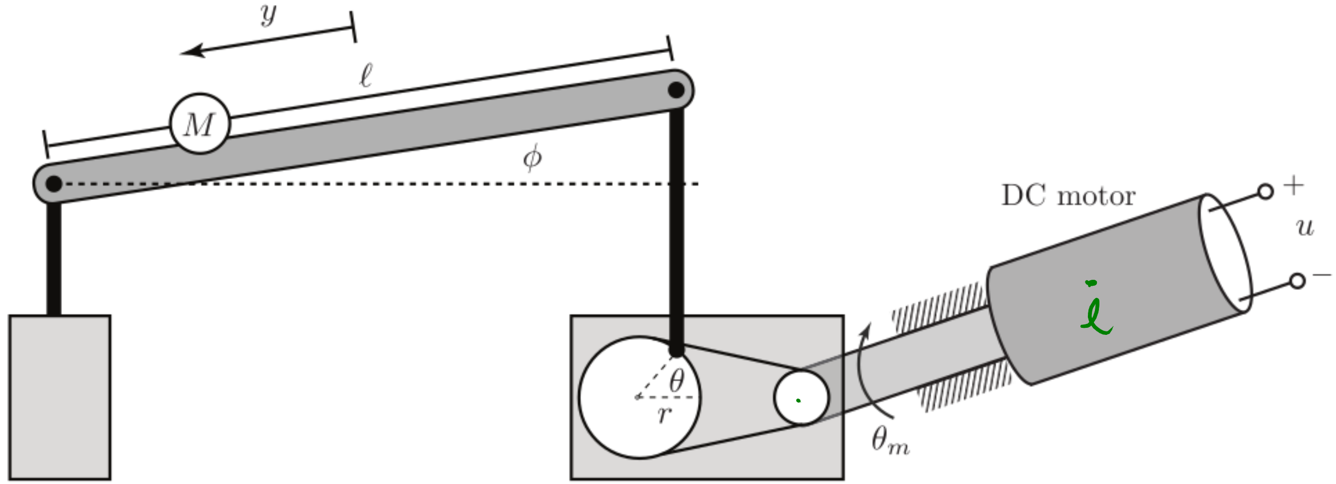
$$y[k] = x_1[k]$$

$$\Rightarrow \quad x_{k+1} := \begin{bmatrix} x_1[k+1] \\ x_2[k+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[k]$$

$$x_{k+1} = Ax_k + Bu_k \qquad \{ x^+ = Ax + Bu \}$$

$$y_k = Cx_k + Du_k$$

Example: Ball and beam system



Ball:

Newton's 2nd law:

$$M\ddot{y} = Mg\sin\phi \quad \text{— (1)}$$

Relate ϕ to θ:

$$l\sin\phi = r\sin\theta \quad \text{— (1a)}$$

Gear ratio K_g, $\Rightarrow$ $\theta = K_g\theta_m$ — (1b)

Motor model: $J\ddot{\theta}_m = -b\dot{\theta}_m + K_\tau i$ — (2)

Inertia, viscous friction, motor torque constant

$$-u + Ri + L\frac{di}{dt} + K_e\dot{\theta}_m = 0 \quad \text{— (3)}$$

(1), (2), (3) are the dynamics of this system.

States: $\vec{x} = (x_1, x_2, x_3, x_4, x_5) := (y, \dot{y}, \theta_m, \dot{\theta}_m, i)$

$\dot{x} = f(x, u) \qquad u \in \mathbb{R}, x \in \mathbb{R}^5$

1, 1a, 1b

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_2 \\ (g^*/\ell)\sin(k_g x_3) \\ x_4 \\ \frac{1}{J}(-b x_4 + k_\tau x_5) \\ \frac{1}{L}(-k_e x_4 - R x_5) + \frac{u}{L} \end{bmatrix} = f(x, u)$$

2

3

$$= \underbrace{f(x) + g(x)u}$$

Control affine NL systems $\leftarrow$ Special case of NL systems

$y = h(x, u) = x_1$

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Linearization

General form of state-space models (non-linear):

$$\dot{x} = f(x,u)$$
$$y = h(x,u)$$

[Usually obtained via first-principles modeling]

How do we get an LTI representation (approximation) from NL dynamics?

LTI systems are a special case:

$$f(x,u) = Ax + Bu$$
$$h(x,u) = Cx + Du$$

[Good for control design/analysis]

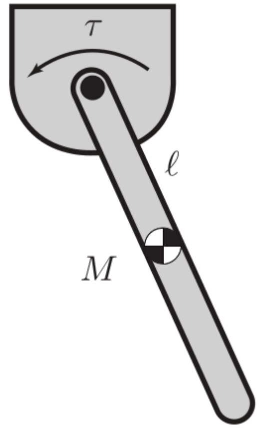
Example 5.2.1: A physical pendulum

Consists of a rod made of homogenous material.

Lenth = l, mass = M, center of mass at l/2

Actuator is a motor at the pivot point, control input is Torque.

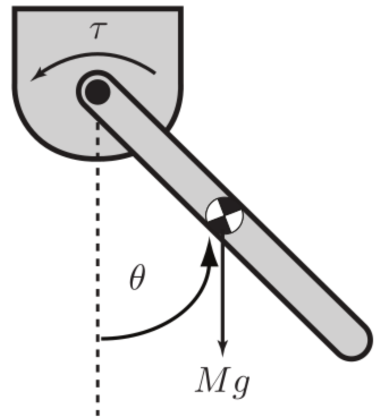
Dynamics (from first-principles physics modeling) are: $\ddot{\theta} = \dfrac{3}{Ml^2}\tau - 1.5\dfrac{g}{l}\sin(\theta)$



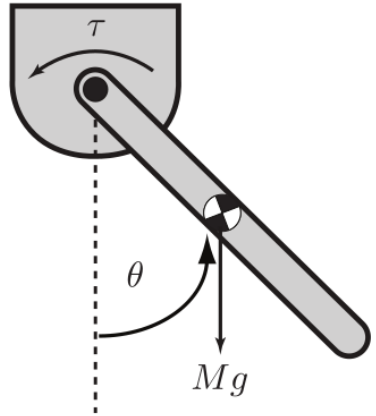
Let $x_1 := \theta$, $x_2 := \dot{\theta}$, $u = \tau$, $y = \theta$

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -1.5\frac{g}{l}\sin(x_1) + \frac{3}{Ml^2}u \end{aligned}\right\} f(x,u)$$

$$y = x_1 = h(x,u)$$



Equlibrium configuration and equilibrium points



Definition 5.2.1. Consider a system in state-variable form (5.3). A constant pair $(\overline{x}, \overline{u})$ is called an **equilibrium configuration of** (5.3) if $f(\overline{x}, \overline{u}) = (0, 0, \ldots, 0)$. The constant $\overline{x}$ is called an **equilibrium point**.

e.g., Pendulum, $x_1, x_2; u$ s.t.

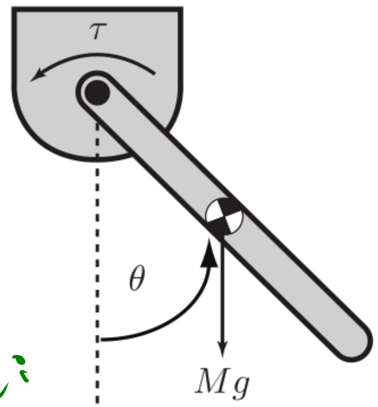
$$\dot{x} = f(x, u) = 0$$

$$\theta = \dot{\theta} = 0, \quad \tau = 0$$

$$\bar{x}_1 = \bar{x}_2 = 0 = u$$

A lot of systems can be well approximated by linear dynamics "reasonably well" in a neighborhood of the equilibrium configuration. => Can do linear control.

Equlibrium configuration and equilibrium points: Example



$$\dot{x} = f(x,u) = \begin{bmatrix} x_2 \\ -\frac{1.5g}{l}\sin x_1 + \frac{3}{ml^2}u \end{bmatrix}, \text{ say}$$

$\bar{u} = 0$

Case 1:

Equi. config, $(\bar{x},\bar{u})$ s.t. $f(\bar{x},\bar{u}) = 0$

$$\Rightarrow \bar{x}_2 = 0, \quad \bar{x}_1 = k\pi, \quad k \in \mathbb{Z}$$

when $\tau = 0 = \bar{u}$, $\bar{x} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix}$, $\bar{u} = 0$

Physically, 2 equi configs when $\bar{u} = 0$.

Case 2: $u = \bar{u} \neq 0$ (constt. torque), what is $\bar{x}$?

$$\frac{3}{ml^2}\bar{u} = 0$$

$$\bar{x}_2 = 0, \quad -1.5\frac{g}{l}\sin(\bar{x}_1) + \frac{3}{ml^2}$$

4

$$\Rightarrow \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} \sin^{-1}\left(\frac{2\bar{u}}{gmL}\right) + 2\pi k \\ 0 \end{bmatrix}$$

<u>Case 3:</u> $\bar{x}_1 \in (-\pi, \pi]$, $\bar{x}_2 = 0$

From equation (4) $\Rightarrow \bar{u} = \frac{1}{2} Mgl \sin(\bar{x}_1)$

can make the pendulum rest at a any angle by applying a suitable constant torque.

Linearization around an equilibrium point

Assume $\bar{x}, \bar{u}$ is an equilibrium config, i.e., $f(\bar{x},\bar{u}) = 0$.

Consider small perturbations around this:

$\delta x(t) := x(t) - \bar{x}$, $\delta u(t) := u(t) - \bar{u}$, $\delta x(t), \delta u(t)$ "small".

Taylor expansion of $f(x,u)$ around $\bar{x},\bar{u}$ (assume $x \in \mathbb{R}^n, u \in \mathbb{R}^m$)

$$\Rightarrow \dot{x}(t) = f(x,u) \in \mathbb{R}^n$$

$$= \underbrace{\cancel{f(\bar{x},\bar{u})}}_{\in \mathbb{R}^n = 0 \text{ (Equi.)}} + \underbrace{\underbrace{\frac{\partial f}{\partial x}\Big|_{(\bar{x},\bar{u})}}_{\in \mathbb{R}^{n\times n}} (x-\bar{x})}_{\in \mathbb{R}^{n\times 1}} + \underbrace{\underbrace{\frac{\partial f}{\partial u}\Big|_{(\bar{x},\bar{u})}}_{\in \mathbb{R}^{n\times m}} (u-\bar{u}) + \underbrace{\text{H.O.T}}_{\text{IGNORE!}}}_{\in \mathbb{R}^{n\times 1}}$$

$$A := \frac{\partial f}{\partial x}\Big|_{(\bar{x},\bar{u})} \in \mathbb{R}^{n\times n} \text{ (STATE JACOBIAN)}, \quad B := \frac{\partial f}{\partial u}\Big|_{(\bar{x},\bar{u})} \in \mathbb{R}^{n\times m} \text{ (INPUT JACOBIAN)}$$

$\Rightarrow \quad \delta \dot{x} = A\,\delta x + B\,\delta u$ { In terms of perturbations }

(we used $\delta \dot{x} = \frac{d}{dt}(x - \bar{x}) = \frac{d}{dt} x$)

Similarly, $y = h(x, u)$

can represent as $\delta y = y(t) - \bar{y}$

$= C\,\delta y + D\,\delta u$

$$C := \left.\frac{\partial h}{\partial x}\right|_{(\bar{x}, \bar{u})} ,$$

$$D := \left.\frac{\partial h}{\partial u}\right|_{(\bar{x}, \bar{u})}$$

Linearization around an equilibrium point

Example: The pendulum (again)

Say $\theta = \pi$, $(\bar{y} = \pi = \bar{x}_1)$, $\bar{x}_2 = 0$, $\bar{u} = 0$

$$\dot{x} = f(x,u) = \begin{bmatrix} x_2 \\ -1.5\frac{g}{l}\sin x_1 + \frac{3}{ml^2}u \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \quad y = x_1$$

Linearize around $(\bar{x},\bar{u})$,

$$A = \frac{\partial f}{\partial x}\bigg|_{(\bar{x},\bar{u})} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\bigg|_{(\bar{x},\bar{u})}$$

$$= \begin{bmatrix} 0 & 1 \\ -1.5\frac{g}{l}\cos(x_1) & 0 \end{bmatrix}\bigg|_{(\bar{x},\bar{u})} = \begin{bmatrix} 0 & 1 \\ 1.5\frac{g}{l} & 0 \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}\Bigg|_{(\bar{x},\bar{u})} = \begin{bmatrix} 0 \\ \frac{3}{ml^2} \end{bmatrix}$$

$$\Rightarrow \delta \dot{x} = A\delta x + B\delta u$$

Similarly, $y = x_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 \cdot u$

$$= Cx + Du$$

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Transfer function from (LTI) state-space: SS2TF

Let's start with the LTI system:
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$

Take LT w/ 0-init conditions.

$$X(s) = \mathcal{L}\{x(t)\} := \begin{bmatrix} \mathcal{L}\{x_1(t)\} \\ \mathcal{L}\{x_2(t)\} \\ \vdots \\ \mathcal{L}\{x_n(t)\} \end{bmatrix}$$

$$\Rightarrow \quad sX = AX + BU$$
$$Y = CX + DU$$

Identity matrix $(n \times n)$

Eliminate $X(s)$,
$$(sI - A)X = BU$$
$$\Rightarrow X = (sI - A)^{-1} BU$$

$$Y = CX + DU = C(sI-A)^{-1}BU + DU$$

$$\Rightarrow G = \frac{Y}{U} = \underbrace{C}_{\mathbb{R}^{p\times n}}\underbrace{(sI-A)^{-1}}_{\in \mathbb{R}^{n\times n}}\underbrace{B}_{\mathbb{R}^{n\times m}} + D \rightarrow \mathbb{R}^{p\times m}$$

$= p\times m$ transfer functions

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](s) := C(sI-A)^{-1}B + D$$

G is a matrix of size $p\times m$, each element is a TF.

Transfer function from (LTI) state-space: SS2TF

Why? Re-write $(sI-A)^{-1} = \frac{\mathrm{adj}(sI-A)}{\det(sI-A)}$

$\mathrm{adj}(sI-A) \rightsquigarrow$ Matrix of size $n \times n$, each element is a polynomial of degree $< n$.

$\det(sI-A) \rightarrow$ Polynomial of order n.

$$G := C(sI-A)^{-1}B + D = C \underbrace{\frac{\mathrm{adj}(sI-A)}{\det(sI-A)}}_{\text{Strictly proper TFs}} B + D$$

Strictly proper TFs $\Longleftrightarrow D = 0$

Realization problem: TF2SS

Given $G(s)$, find A, B, C, D such that the resulting state-space model has a TF that is $G(s)$.

Note: The solution is not unique.

Example: Pendulum (yet again)

From earlier,we obtained the linearized dynamics for the pendulum (about upright position):

$$\delta\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 3\frac{g}{\ell} & 0 \end{bmatrix}}_{A}\delta x + \underbrace{\begin{bmatrix} 0 \\ \frac{3}{M\ell^2} \end{bmatrix}}_{B}\delta u$$

$$\delta y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C}\delta x. \qquad D = 0$$

The transfer function (LT w/ zero initial conditions):

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B + D \\ &= \frac{1}{\det(sI - A)} C \,\mathrm{adj}\,(sI - A)B + D \\ &= \frac{1}{s^2 - 3\frac{g}{\ell}} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & 1 \\ 3\frac{g}{\ell} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{3}{M\ell^2} \end{bmatrix} + 0 \\ &= \frac{3}{M\ell^2}\frac{1}{s^2 - 3\frac{g}{\ell}}. \end{aligned}$$

Here, $sI - A = \begin{bmatrix} s & -1 \\ -3g/l & s \end{bmatrix}$, $\det(sI-A) = s^2 - 3g/l$

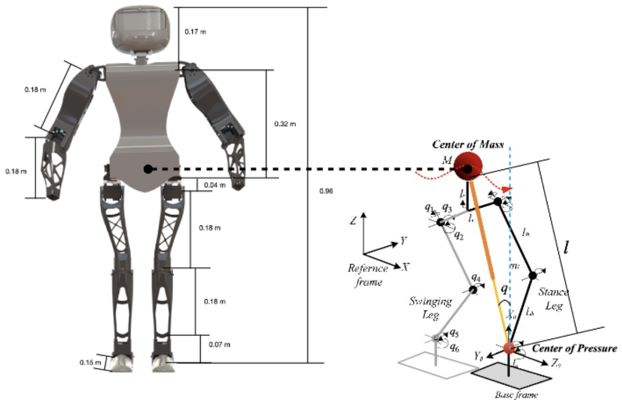
$$\mathrm{adj}(sI-A) = \begin{bmatrix} s & 1 \\ 3\frac{g}{\ell} & s \end{bmatrix}$$

Is this stable? No (see poles). In fact we don't expect the upright pendulum to be stable.

Theorem (see Non-Linear systems, H. Khalil): If a linearized system (around an equilibrium point) is asymptotically stable, then the non-linear system is locally stable around that equilibrium point.
If the linearized system is unstable, then the non-linear system is also unstable around that equilibrium.

Stabilize (via control design) the linearized system, and you have a locally stable non-linear system!

Summary of what we have covered so far (in this chapter)



Physical System

Figure from A. Bhaskar et al., Adaptive Artificial Time Delay Control for Bipedal Walking with Robustification to State-dependent Constraint Forces, IEEE ICAR, 2021.

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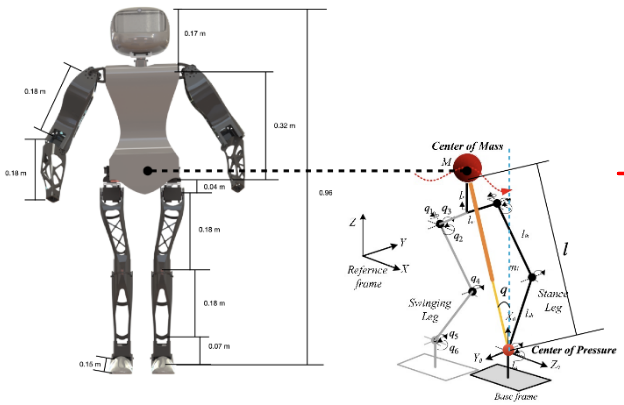

Physical System

First principles

$$M(q)\ddot{q} + H(q,\dot{q}) = \tau$$

ODE (non-linear)

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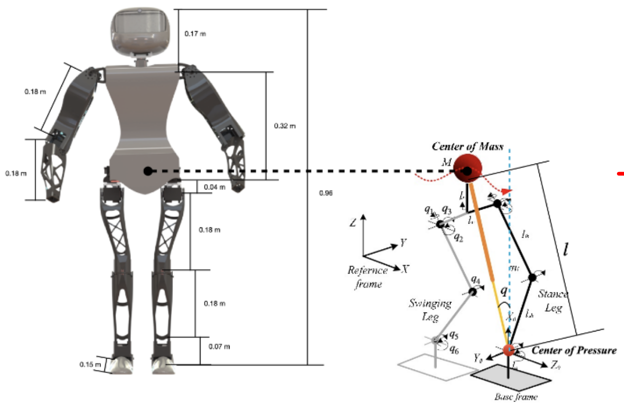

Physical System

First principles

$$M(q)\ddot{q} + H(q, \dot{q}) = \tau$$

ODE (non-linear)

Define state variables

e.g., $x_1 = q$, $x_2 = \dot{q}$

$$\dot{x} = f(x, u),\ y = h(x, u)$$

Non-linear State-space model

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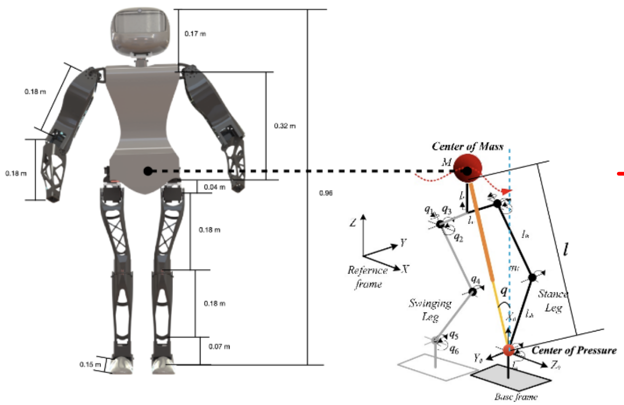

Physical System

First principles

$$M(q)\ddot{q} + H(q, \dot{q}) = \tau$$

ODE (non-linear)

Define state variables

e.g., $x_1 = q$, $x_2 = \dot{q}$

$$\dot{x} = f(x, u), \; y = h(x, u)$$

Non-linear State-space model

Linearize (pick an equilibrium point)

$$\dot{x} = Ax + Bu, \; y = Cx + Du$$

LTI State-space model

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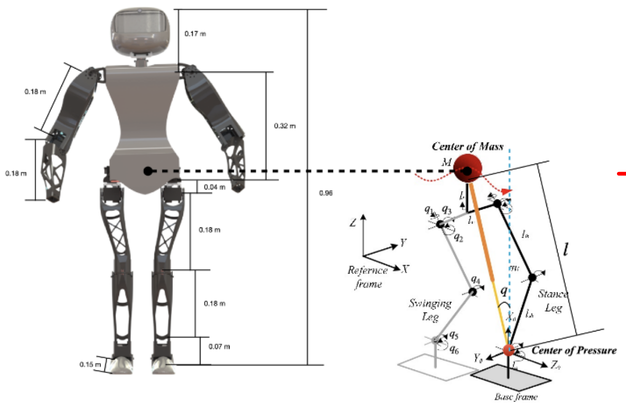


Physical System

First principles

$$M(q)\ddot{q} + H(q,\dot{q}) = \tau$$

ODE (non-linear)

Define state variables

e.g., $x_1 = q$, $x_2 = \dot{q}$

$$\dot{x} = f(x,u),\ y = h(x,u)$$

Non-linear State-space model

Linearize

$$\dot{x} = Ax + Bu,\ y = Cx + Du$$

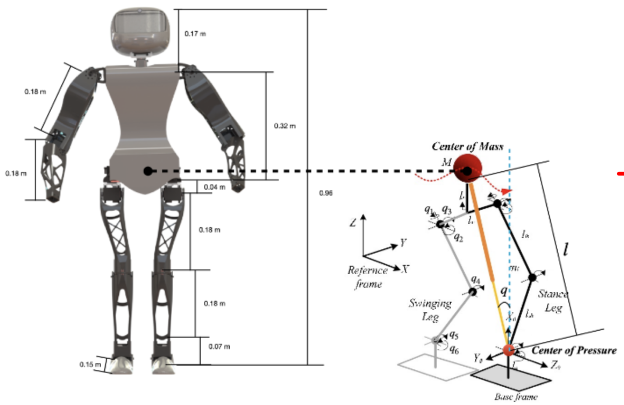
LTI State-space model

Laplace Transform

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

Transfer function

Summary of what we have covered so far (in this chapter)



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?

Outline

[-] Introduction

[-] State-space models

[-] Linearization
    [-] Equilibrium configuration
    [-] Transfer function from a linear state-space model
    [X] Control design using linearization

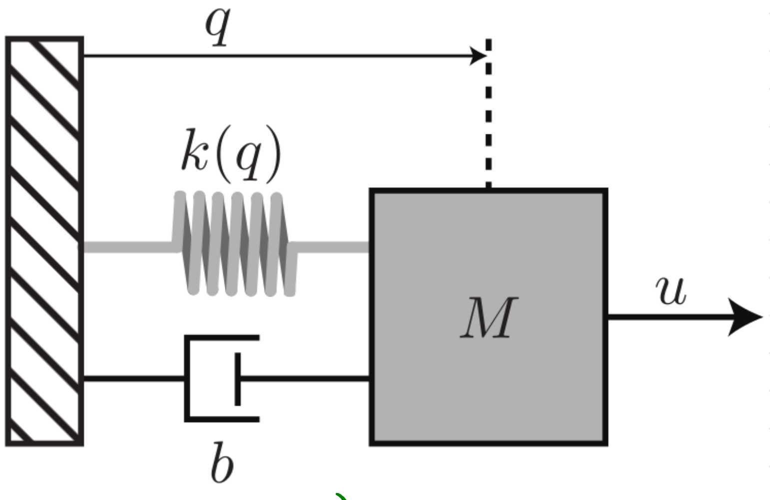
[ ] Inverting static nonlinearities
    [ ] Conditions for inverting static nonlinearities
    [ ] Local cancellation of static nonlinearities
    [ ] Inverting dynamic nonlinearities (feedback linearization), an example.

[ ] Static friction
    [ ] Dither signals
    [ ] Deadzone non-linearity

[ ] Describing functions
    [ ] Optimal Quasi-Linearization
    [ ] Constructing describing functions
    [ ] Periodic solutions and their stability

Control design using linearization: Using what we have seen so far

Example: Mass-spring-damper



We want to control the system around the position $\bar{q} = 0.5m$

Dynamics: $M\ddot{q} = u - b\dot{q} - k(q)$ — (1)

Spring restoring force: $k(q) = K(1 - a^2q^2)q,$ (2) $\quad |aq| < 1, \quad K > 0$

Define states: $x = (x_1, x_2) := (q, \dot{q})$

State-space model:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{M}\Big(- K\left(1 - a^2x_1^2\right)x_1 - bx_2 + u\Big) \\ y &= x_1. \end{aligned}$$

(from (1) & (2))

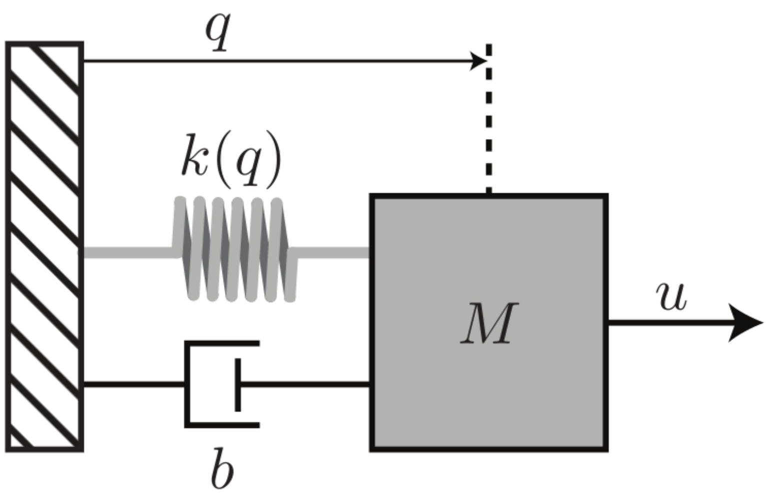
Solve for equilibrium configuration around $\bar{y} = 0.5$

$$\begin{aligned} 0 &= \overline{x}_2 \\ 0 &= \frac{1}{M}\left(-K\left(1 - a^2\overline{x}_1^2\right)\overline{x}_1 - b\overline{x}_2 + \overline{u}\right) \\ 0.5 &= \overline{x}_1. \end{aligned} \quad \Rightarrow \quad \begin{aligned} \overline{x}_1 &= 0.5 \\ \overline{x}_2 &= 0 \\ \overline{u} &= \frac{K}{2}\left(1 - 0.25a^2\right) \end{aligned}$$

Control design using linearization: Using what we have seen so far

Example: Mass-spring-damper

We want to control the system around the position $\bar{q} = 0.5m$



State-space (NL):
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{M}\Big(- K\left(1 - a^2 x_1^2\right) x_1 - b x_2 + u\Big) \\ y &= x_1. \end{aligned}$$

Linearize around equilibrium configuration:

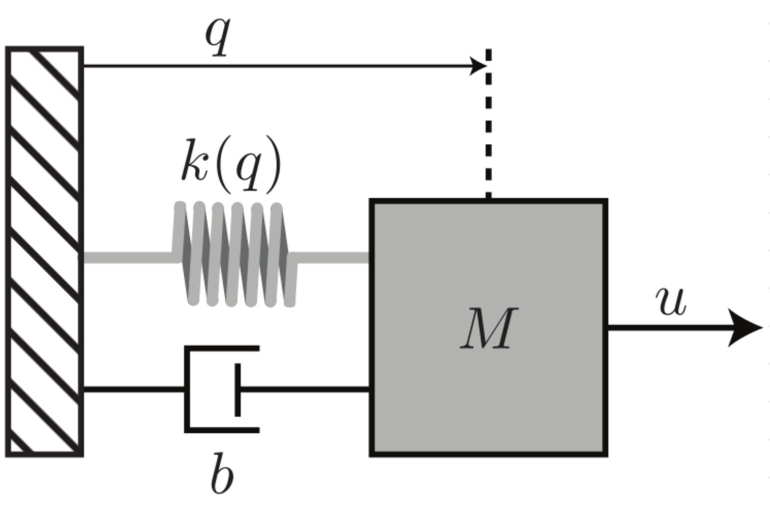
$$A = \left.\frac{\partial f}{\partial x}\right|_{(\overline{x},\overline{u})} = \left.\begin{bmatrix} 0 & 1 \\ -\frac{K}{M}\left(1 - 3a^2 x_1^2\right) & -\frac{b}{M} \end{bmatrix}\right|_{(\overline{x},\overline{u})} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M}\left(1 - 0.75a^2\right) & -\frac{b}{M} \end{bmatrix}$$

$$B = \left.\frac{\partial f}{\partial u}\right|_{(\overline{x},\overline{u})} = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}$$

$$C = \left.\frac{\partial h}{\partial x}\right|_{(\overline{x},\overline{u})} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = \left.\frac{\partial h}{\partial u}\right|_{(\overline{x},\overline{u})} = 0$$

Control design using linearization: Using what we have seen so far

Example: Mass-spring-damper



We want to control the system around the position $\bar{q} = 0.5m$

State-space (NL): $\dot{x}_1 = x_2$

$$\dot{x}_2 = \frac{1}{M}\Big(-K\left(1 - a^2 x_1^2\right) x_1 - b x_2 + u \Big)$$

$$y = x_1.$$

Linearized system: $\delta\dot{x} = A\delta x + B\delta u,\ \delta y = C\delta x + D\delta u$

Where, $\delta x = x - \overline{x}$, $\delta u = u - \overline{u}$ and $\delta y = y - \overline{y}$

Next, let us get a transfer function from input (deviation) $\Delta U := \mathscr{L}\{\delta u\}$ to output (deviation) $\Delta Y := \mathscr{L}\{\delta y\}$

$$\begin{aligned}
\frac{\Delta Y}{\Delta U} = P(s) &= C(sI - A)^{-1} B \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ \frac{K}{M}\left(1 - 0.75a^2\right) & s + \frac{b}{M} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} \\
&= \frac{1}{s^2 + \frac{b}{M}s + \frac{K}{M}\left(1 - 0.75a^2\right)} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s + \frac{b}{M} & 1 \\ -\frac{K}{M}\left(1 - 0.75a^2\right) & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} \\
&= \frac{1/M}{s^2 + \frac{b}{M}s + \frac{K}{M}\left(1 - 0.75a^2\right)}.
\end{aligned}$$

Consider a system with M = K = b = 1 and a^2 = 1/3

Control design using linearization: Using what we have seen so far

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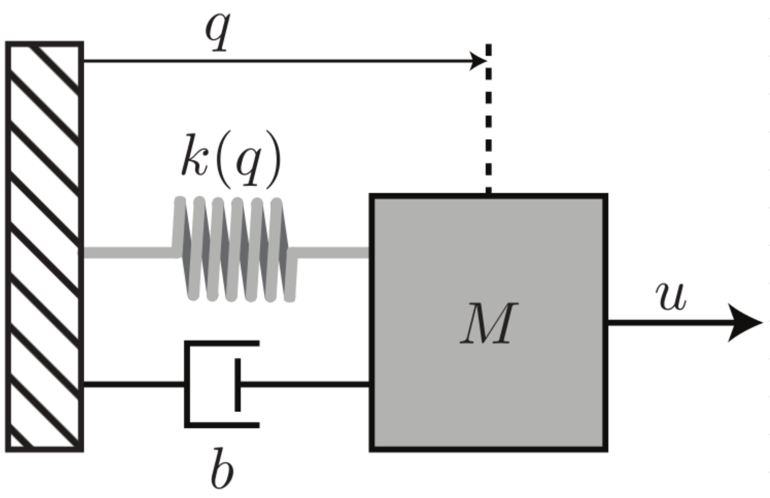

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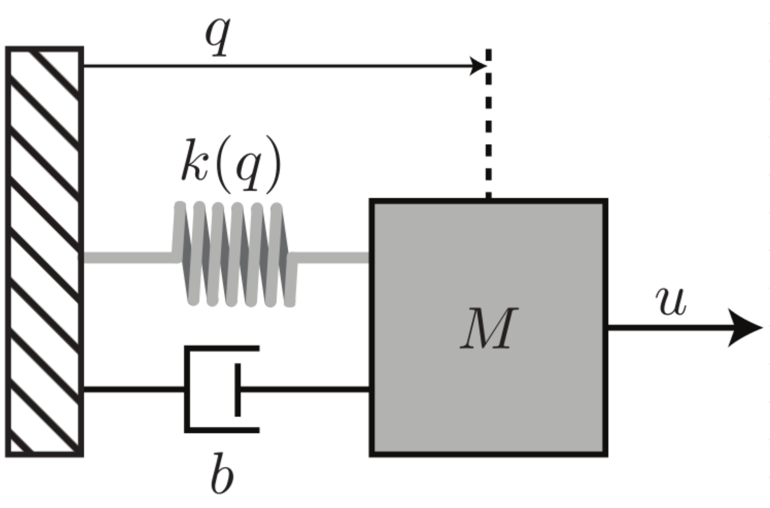
$$\frac{\Delta Y}{\Delta U} = \frac{1}{s^2 + s + 0.75}$$

Control design using linearization: Using what we have seen so far



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PI controller to stabilize the plant and track the desired reference position:

$$C(s) = K_p + \frac{K_i}{s}, \qquad K_p = 1, \qquad K_i = 0.5$$

Control design using linearization: Using what we have seen so far

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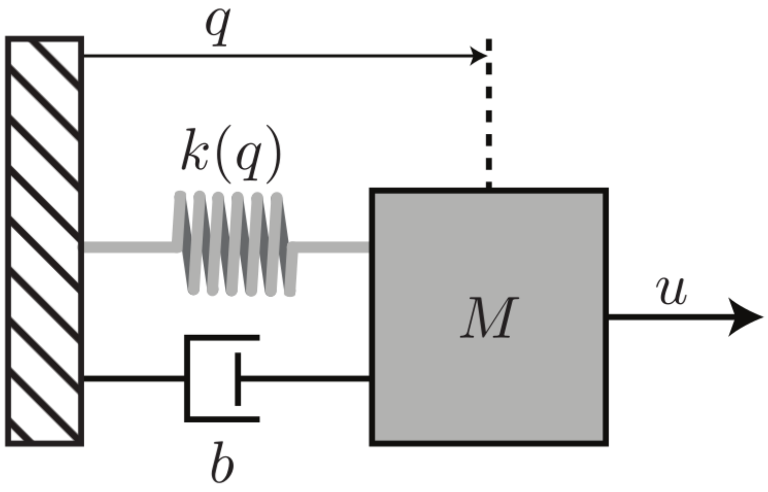

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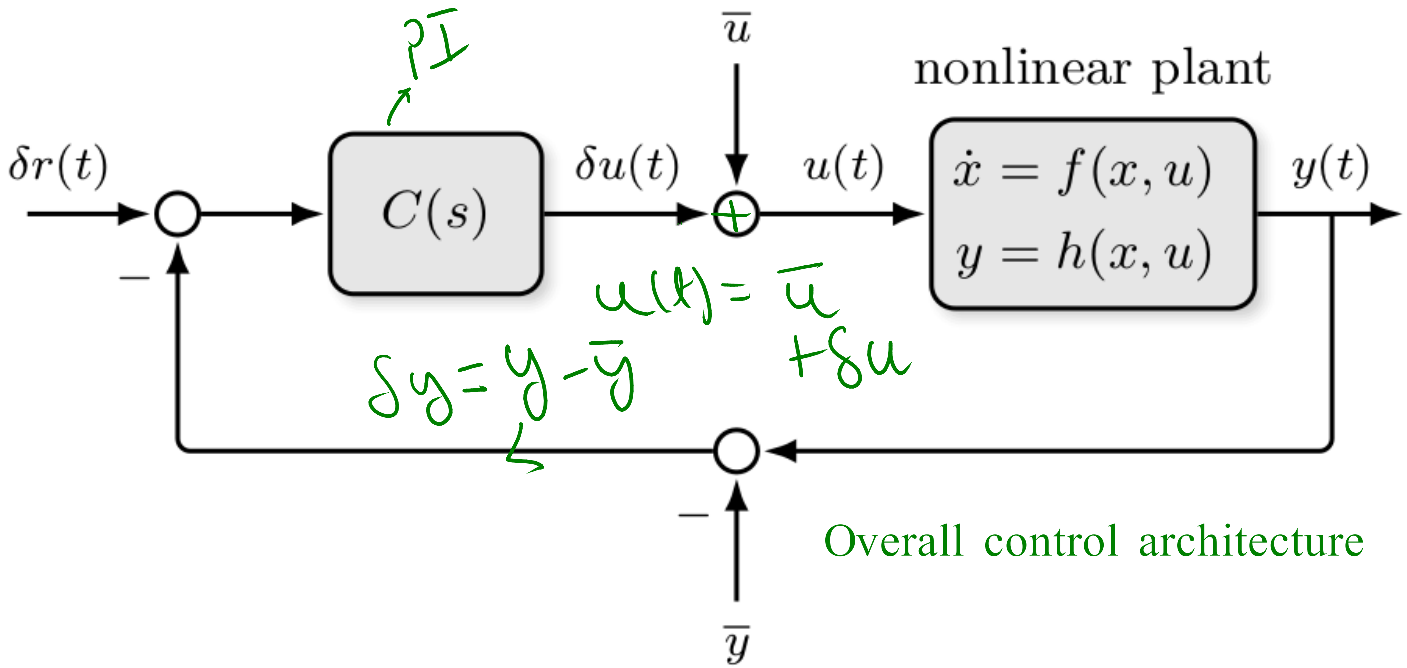
Where, $\delta x = x - \bar{x}$, $\delta u = u - \bar{u}$ and $\delta y = y - \bar{y}$

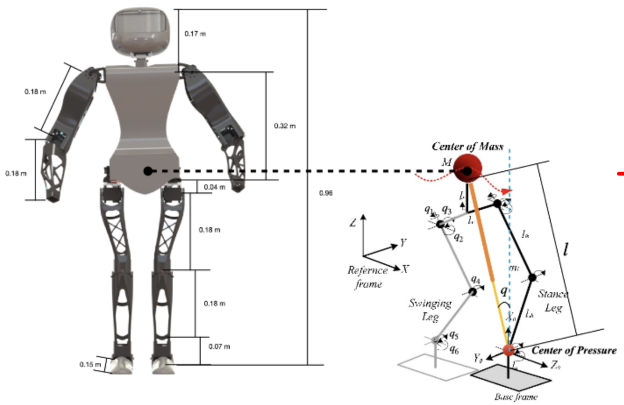
Remember that the control law is based on linearization around the equilibrium configuration, therefore the final implementation will have to account for the definition of $\delta u,\ \delta y$

$G = \dfrac{\Delta Y}{\Delta U}$



Overall control architecture

Summary of what we have covered so far (in this chapter)



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First principles

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Define state variables

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Non-linear State-space model

Linearize

$$\dot{x} = Ax + Bu,\; y = Cx + Du$$

LTI State-space model

Laplace Transform

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

Transfer function

Design a controller

(e.g., via LQR, Lead-Lag, Pole-placement etc.)

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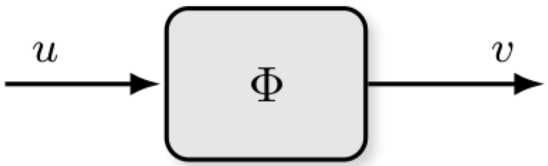
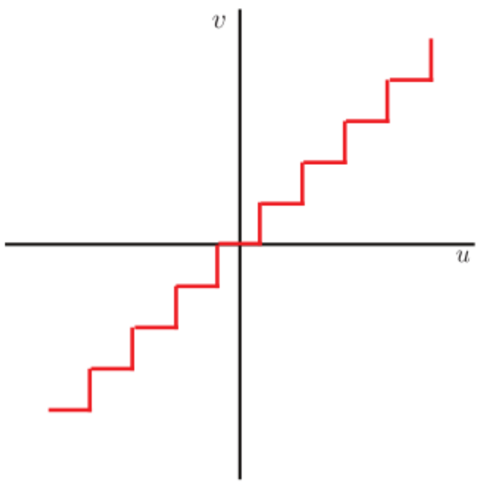
Static nonlinearities and inverting them

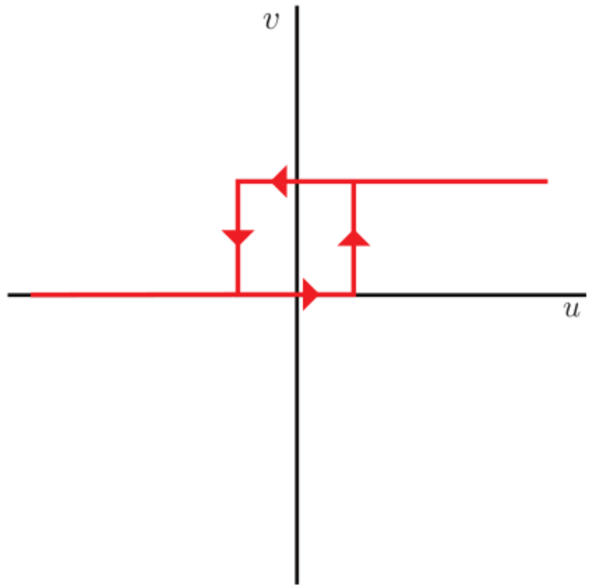
Static non-linearity: A non-linear function that does not have memory, i.e., of the form $v(t) = \Phi(u(t))$



Example of a static nonlinearity: Quantization.



Example of a non-static nonlinearity: Relay with hysterisis



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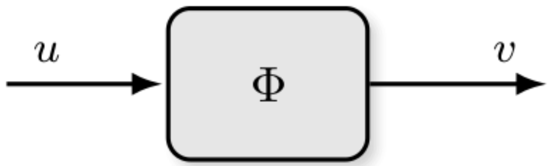
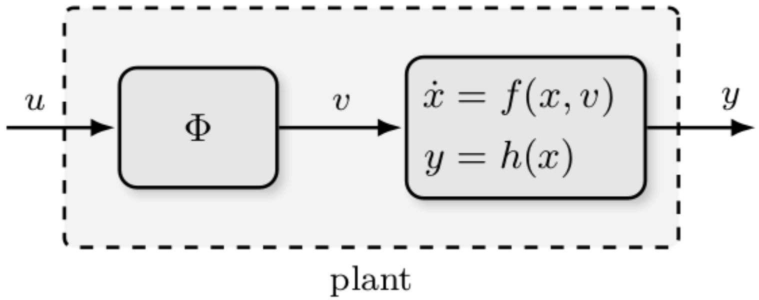
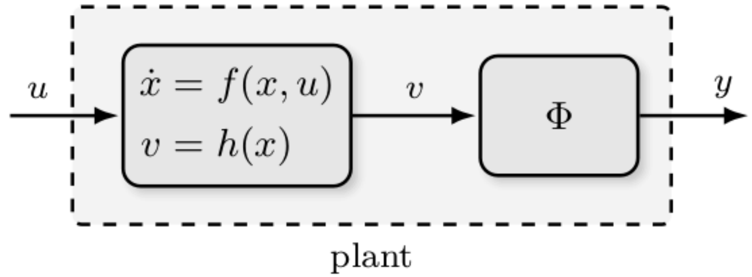


Static non-linearities could show up at the input or output (or both) of a plant.



(a) Static nonlinearity at the input.

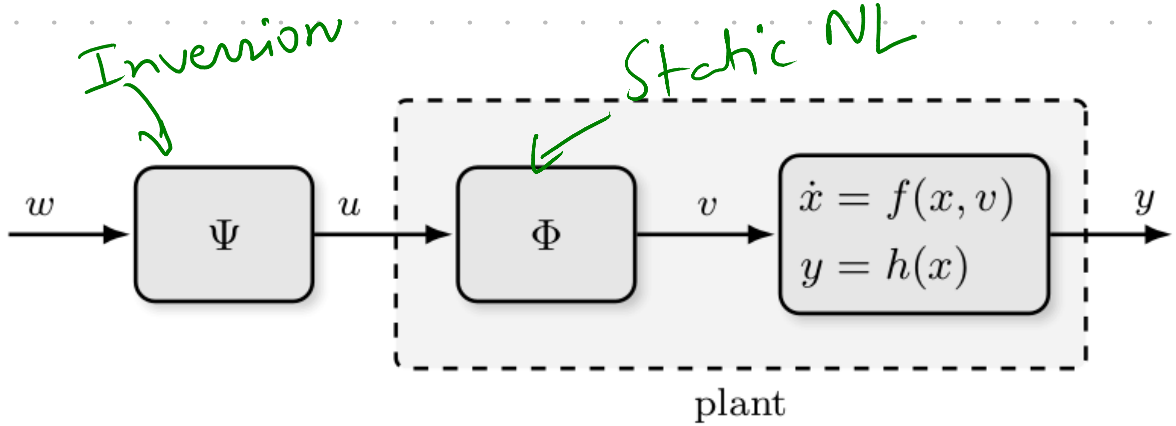


(b) Static nonlinearity at the output.

Inverting static nonlinearities

Sometimes possible to 'perfectly' cancel the effect of static nonlinearity by properly choosing a control input.

For example, inverting an input static nonlinearity



Suppose $\exists\, \Psi$ s.t. $\Phi(\Psi(w)) = w$, on

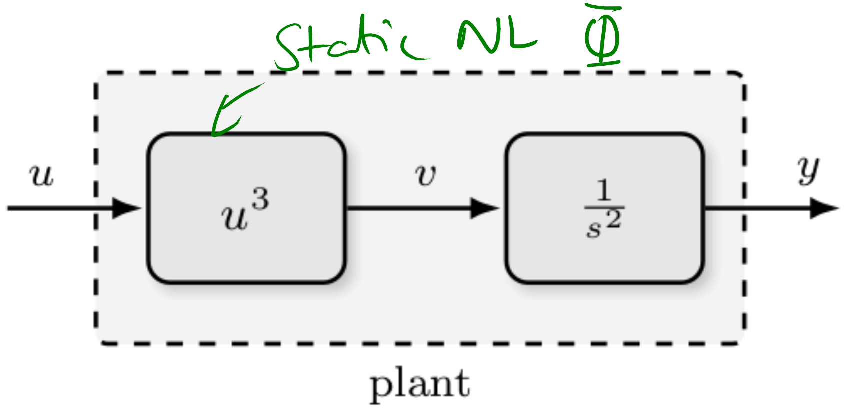
$$v = \Phi(u) = \Phi(\Psi(w)) = w$$

$\therefore$ System: $\dot{x} = f(x, w)$

$$y = h(x)$$

if f, h are LTI $\Rightarrow$ TF from w to y

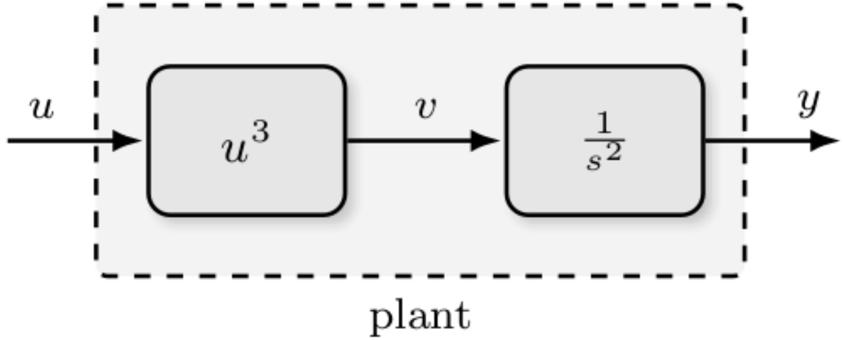
Example: Control design using nonlinear inversion

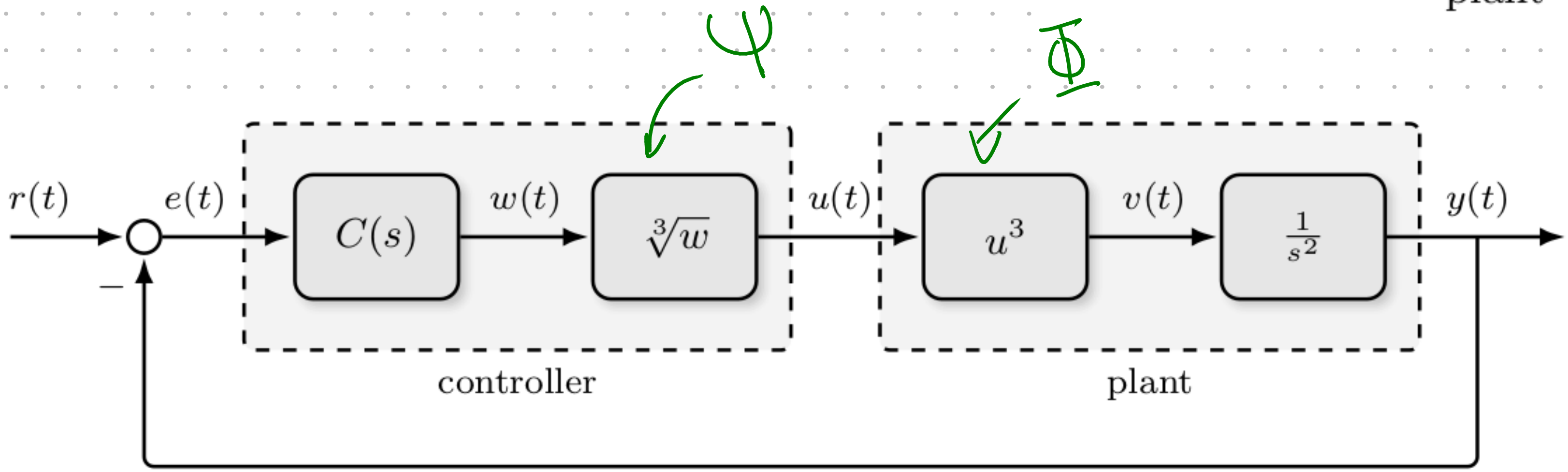


Example: Control design using nonlinear inversion

Setting the preliminary control $u = \Psi(w) = w^{1/3}$

Transfer function from w to y is $\dfrac{Y}{W} = \dfrac{1}{s^2}$





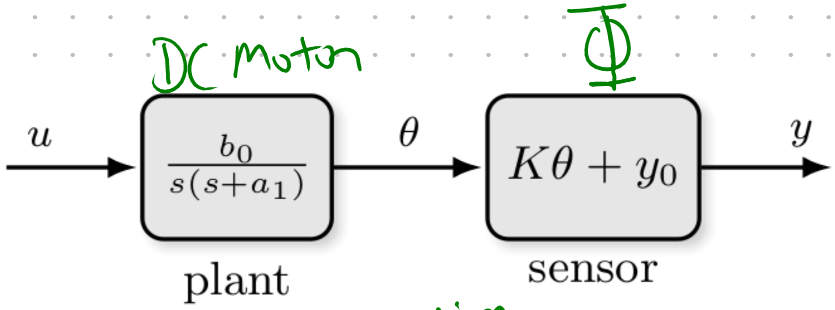
$\Phi(\Psi(w)) = w \Rightarrow v(t) = w(t) \quad \forall t$

$\Rightarrow \dfrac{Y}{V} = \dfrac{Y}{W} = \dfrac{1}{s^2}$ (LTI)

Nonlinear inversion: Output nonlinearity

Example: DC motor with nonlinear sensor for angular position.

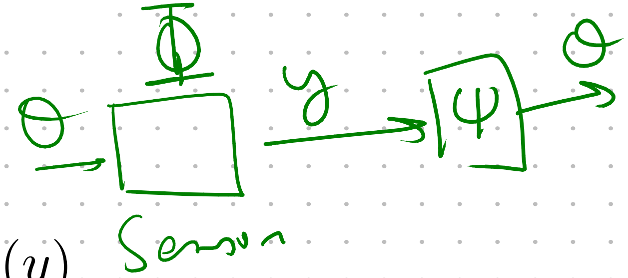
$$y = K\theta + y_0 \Rightarrow \theta = (y - y_0)/K$$



$$\Phi(\theta) = K\theta + y_0, \qquad K, y_0 \in \mathbb{R},$$

Inversion

Define $\quad \Psi(y) := \dfrac{1}{K}(y - y_0) \quad$ then, $\quad \Psi(\Phi(\theta)) = \theta$



Use this as feedback, and we have a LTI system from u to $\Psi(y)$

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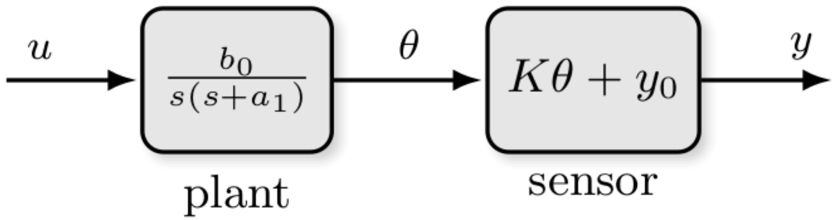
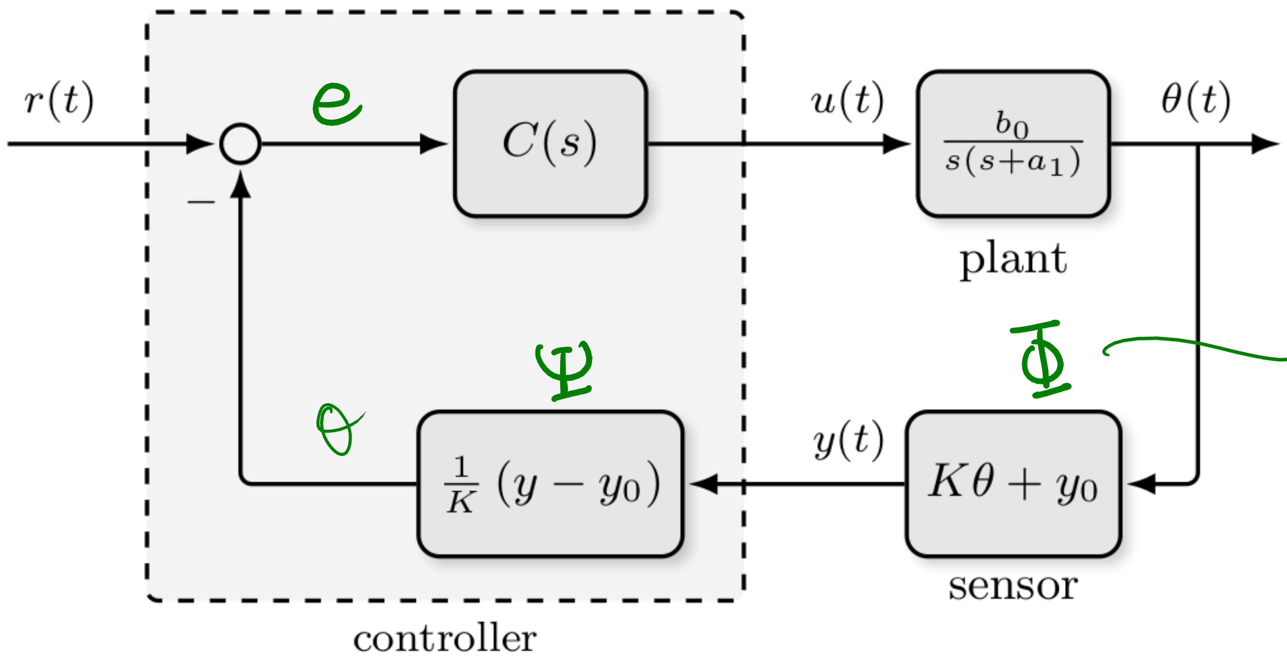


$$\Phi(\theta) = K\theta + y_0, \qquad K, y_0 \in \mathbb{R},$$

Define $\quad \Psi(y) := \dfrac{1}{K}(y - y_0) \quad$ then, $\quad \Psi(\Phi(\theta)) = \theta$ → valid $\forall\, y, \theta \in \mathbb{R}$

Use this as feedback, and we have a LTI system from u to $\Psi(y)$

We can then implement a controller with the following CL system architecture:



Static NL

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Conditions for inverting static non-linearities

What allowed us to cancel (invert) the non-linearities in the previous examples?

Conditions for inverting static non-linearities

What allowed us to cancel (invert) the non-linearities in the previous examples?

1. The nonlinearity $\Phi : \mathbb{R} \to \mathbb{R}$ is static and appears at either the input/output.

Conditions for inverting static non-linearities

What allowed us to cancel (invert) the non-linearities in the previous examples?

1. The nonlinearity $\Phi : \mathbb{R} \to \mathbb{R}$ is static and appears at either the input/output.

$$
\begin{aligned}
\dot{x} &= f(x, v) \\
y &= h(x) \\
v &= \Phi(u) \text{ static nonlinearity at input}
\end{aligned}
\qquad\qquad
\begin{aligned}
\dot{x} &= f(x, u) \\
v &= h(x) \\
y &= \Phi(v) \text{ static nonlinearity at output}
\end{aligned}
$$

Conditions for inverting static non-linearities

What allowed us to cancel (invert) the non-linearities in the previous examples?

1. The nonlinearity $\Phi : \mathbb{R} \to \mathbb{R}$ is static and appears at either the input/output.

2. If the nonlinearity Φ is at the input, it has a right inverse. If it is at the output, then it has a left inverse.

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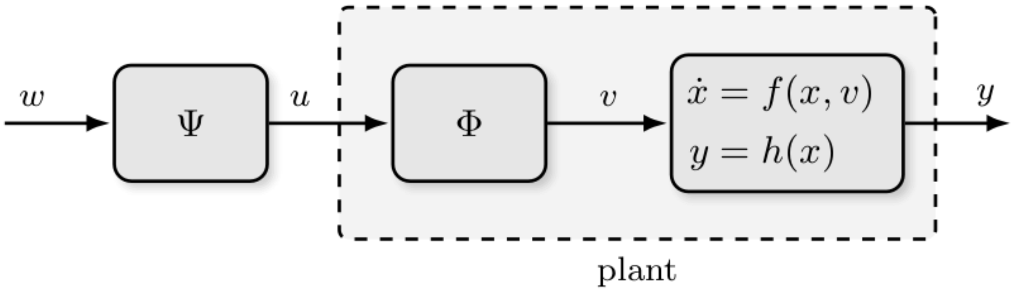
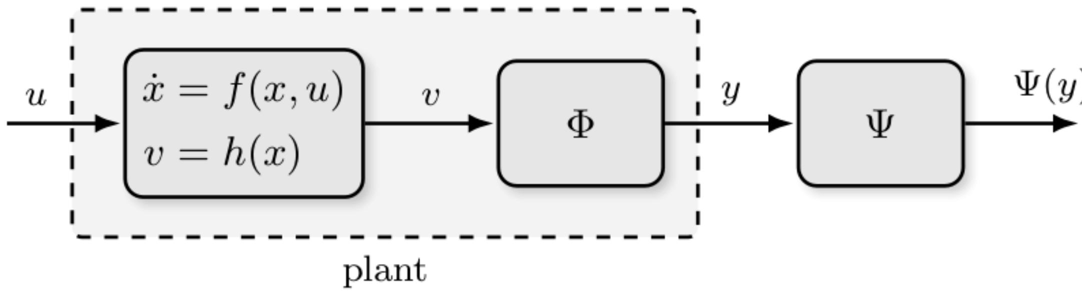
INPUT



OUTPUT



$$\Phi(\Psi(w)) = w$$

$$\Phi \circ \Psi = I$$

$$\Psi(\Phi(v)) = v$$

$$\Psi \circ \Phi = I$$

Definition 5.3.4. Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a real-valued function of a real variable.

(i) A function Ψ is a **right inverse** of Φ if, for all $w \in \mathbb{R}$, $\Phi(\Psi(w)) = w$.

(ii) A function Ψ is a **left inverse** of Φ if, for all $y \in \mathbb{R}$, $\Psi(\Phi(y)) = y$.

Conditions for inverting static non-linearities

What allowed us to cancel (invert) the non-linearities in the previous examples?

1. The nonlinearity $\Phi : \mathbb{R} \to \mathbb{R}$ is static and appears at either the input/output.
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When does the inverse exist?

Conditions for inverting static non-linearities

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For input nonlinearities Φ must be *onto*.
For output nonlinearities Φ must be *one-to-one*

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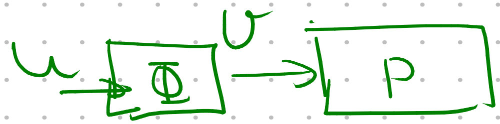

When does the inverse exist?

For input nonlinearities Φ must be *onto*.
For output nonlinearities Φ must be *one-to-one*

Definition 5.3.3. Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a real-valued function of a real variable.

(i) The function Φ is **one-to-one** if $y_1 \neq y_2$ implies $\Phi(y_1) \neq \Phi(y_2)$.

(ii) The function Φ is **onto** if for any $v \in \mathbb{R}$ there exists a $u \in \mathbb{R}$ such that $v = \Phi(u)$.

Proposition 5.3.5 ([MacLane and Birkhoff, 1999, Theorem I.1]). *A function $\Phi : \mathbb{R} \to \mathbb{R}$ has a right inverse if, and only if, Φ is onto. A function $\Phi : \mathbb{R} \to \mathbb{R}$ has a left inverse if, and only if, Φ is one-to-one.*

Conditions for inverting static non-linearities

What allowed us to cancel (invert) the non-linearities in the previous examples?

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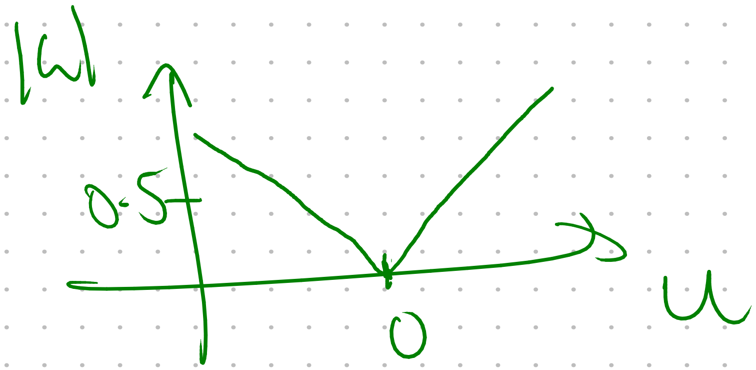
2. If the nonlinearity Φ is at the input, it has a right inverse. If it is at the output, then it has a left inverse.

When does the inverse exist?

For input nonlinearities Φ must be *onto*.
For output nonlinearities Φ must be *one-to-one*

> **Definition 5.3.3.** Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a real-valued function of a real variable.
>
> (i) The function Φ is **one-to-one** if $y_1 \neq y_2$ implies $\Phi(y_1) \neq \Phi(y_2)$.
>
> (ii) The function Φ is **onto** if for any $v \in \mathbb{R}$ there exists a $u \in \mathbb{R}$ such that $v = \Phi(u)$.

Examples:
$\Phi(u) = u^3$ is both onto and one-to-one.
$\Phi(u) = |u|$ is neither onto nor one-to-one.
$\Phi(u) = \arctan(u)$is one-to-one but not onto.

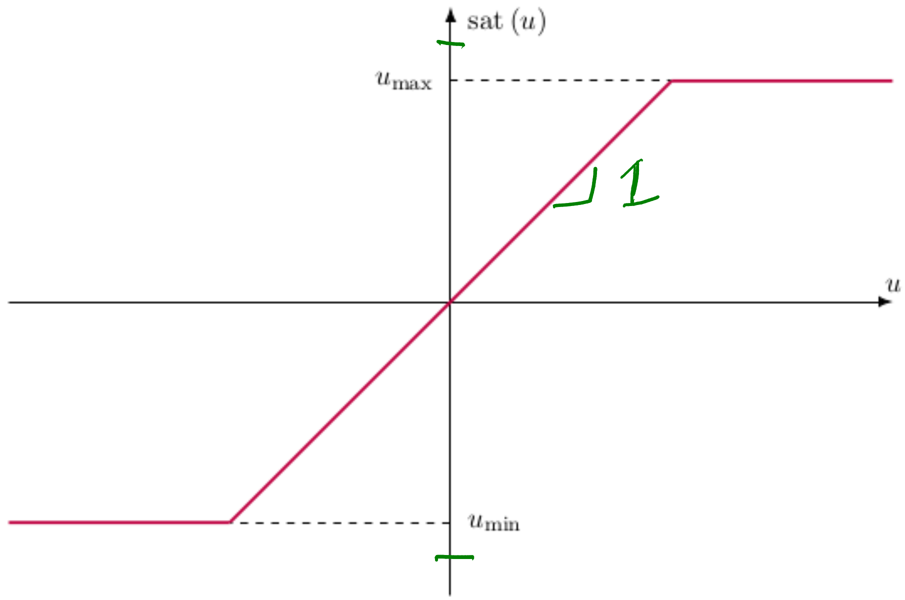


Saturation: A static nonlinearity, but not invertible.

Example 5.3.6. (Saturation Nonlinearity) Consider the saturation nonlinearity

$$\mathrm{sat} : \mathbb{R} \to \mathbb{R}$$
$$u \mapsto \begin{cases} u & \text{if } u_{\min} \leq u \leq u_{\max}, \\ u_{\min} & \text{if } u < u_{\min}. \\ u_{\max} & \text{if } u > u_{\max} \end{cases}$$

whose graph is shown in Figure 5.23.



Neither one-to-one nor onto, so cannot be inverted if it shows up at either the input or the output.

Note: Saturation is a common actuator nonlinearity, i.e., shows up at the input. Advanced control techniques such as Model Predictive Control can deal with this (in most cases) for LTI systems and beyond.

Outline

[-] Introduction

[-] State-space models

[-] Linearization
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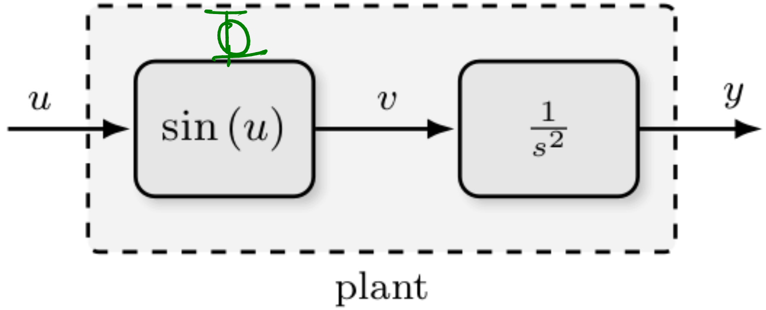
- [-] Control design using linearization

[-] Inverting static nonlinearities
- [-] Conditions for inverting static nonlinearities
- [X] Local cancellation of static nonlinearities
- [ ] Inverting dynamic nonlinearities (feedback linearization), an example.

[ ] Static friction
- [ ] Dither signals
- [ ] Deadzone non-linearity

[ ] Describing functions
- [ ] Optimal Quasi-Linearization
- [ ] Constructing describing functions
- [ ] Periodic solutions and their stability

Local cancellation of static nonlinearities: Example



Input nonlinearity is static but not onto, i.e. not right invertible.

$$\Phi(u) = \sin(u)$$

Find a part of the function's domain where it is. in the range $[-1,1]$

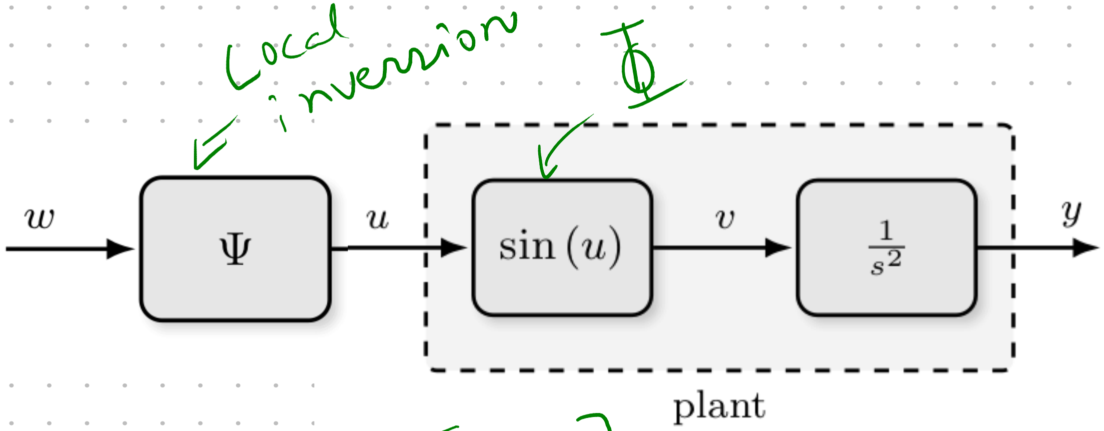
If $u \in [-\pi/2, \pi/2]$, $\sin(u)$ is onto in this domain.

Assume we can guarantee that the control lies in this interval (how?), then

If $\Psi(w) = \arcsin(w) \Rightarrow \Phi(\Psi(w)) = w, \ \forall w \in [-1, 1]$

Linear system from w to y, as long as w is guaranteed to be in [-1,1]

Local inversion



$v = w$ iff $w \in [-1,1]$

$$\frac{Y}{W} = \frac{1}{s^2}, \quad w(t) \in [-1,1]$$

Outline

[-] Introduction

[-] State-space models

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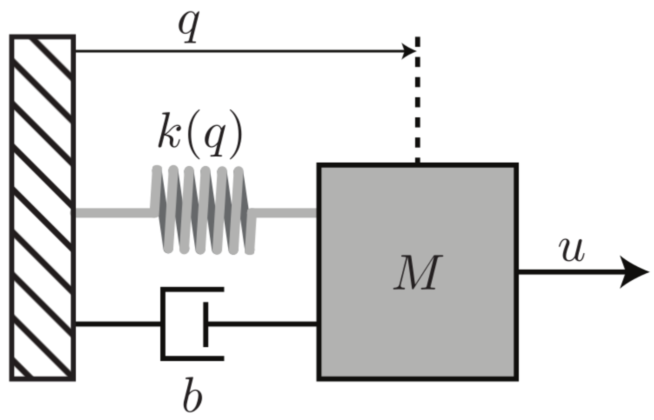
A taste of feedback linearization: Inverting dynamic nonlinearities



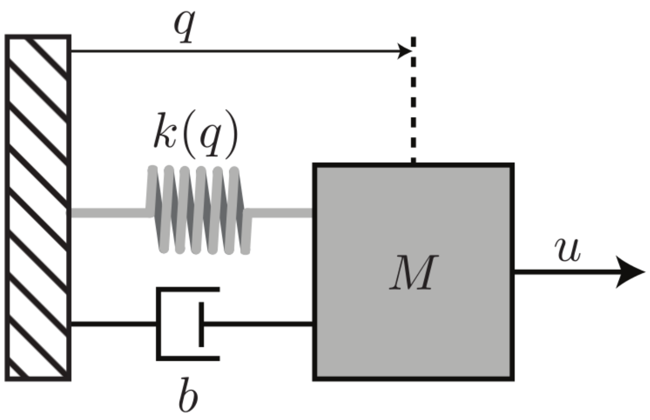
Example: Nonlinear mass-spring-damper

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{M}\Big(-K\left(1 - \underbrace{a^2 x_1^2}_{}\right)x_1 - bx_2 + u\Big)$$
$$y = x_1.$$

NL

A taste of feedback linearization: Inverting dynamic nonlinearities

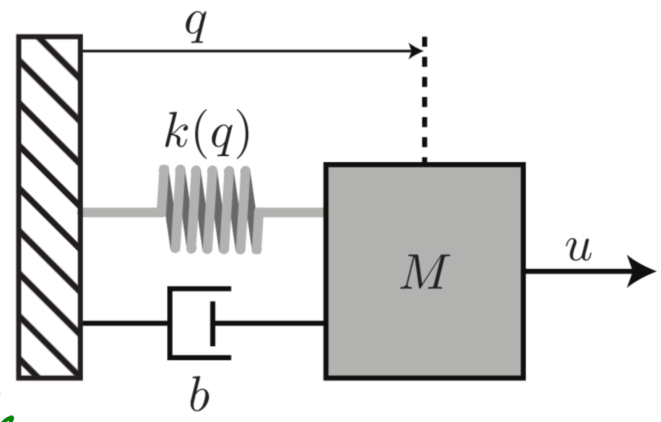
Example: Nonlinear mass-spring-damper



$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{M}\Big(- K\left(1 - a^2 x_1^2\right) x_1 - b x_2 + u \Big) \\
y &= x_1 .
\end{aligned}
$$

Consider the feedback control law: $u = -K a^2 y^3 + w$

A taste of feedback linearization: Inverting dynamic nonlinearities



Example: Nonlinear mass-spring-damper

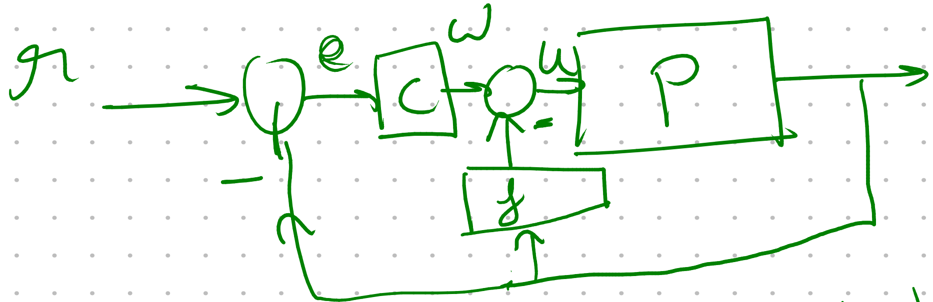
$$\left.\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \frac{1}{M}\Big(-K\left(1-a^2x_1^2\right)x_1 - bx_2 + u\Big)\\ y &= x_1.\end{aligned}\right\} P$$

Cancels

Consider the feedback control law: $u = -Ka^2y^3 + w$ NL Feedback

The dynamics from the new input (w) to the output (y) are now LTI:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \frac{1}{M}\left(-Kx_1 - bx_2 + w\right)\\ y &= x_1.\end{aligned}$$

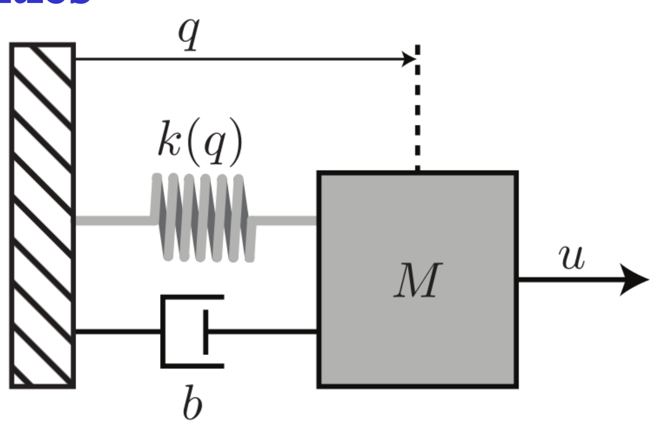


$y(t) = x_1(t)$

$$u = w - f(y) = w - Ka^2y^3$$

A taste of feedback linearization: Inverting dynamic nonlinearities

Example: Nonlinear mass-spring-damper



$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{M}\Big(-K\left(1 - a^2 x_1^2\right)x_1 - bx_2 + u\Big)$$
$$y = x_1.$$

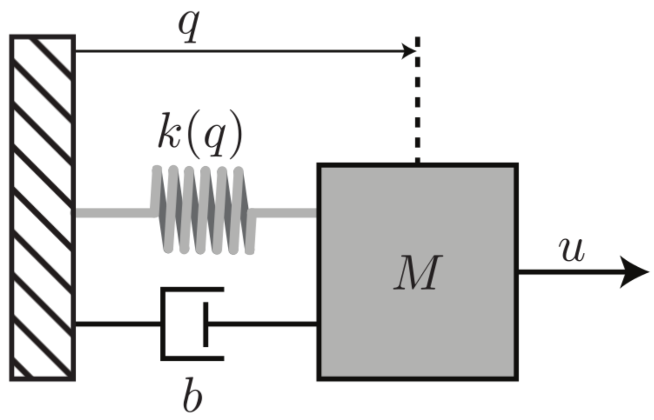
Consider the feedback control law: $u = -Ka^2y^3 + w$

The dynamics from the new input (w) to the output (y) are now LTI:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{M}\left(-Kx_1 - bx_2 + w\right)$$
$$y = x_1.$$

We can now get a TF from this LTI state-space model: $\dfrac{Y(s)}{W(s)} = \dfrac{1/M}{s^2 + \frac{b}{M}s + \frac{K}{M}}$

A taste of feedback linearization: Inverting dynamic nonlinearities



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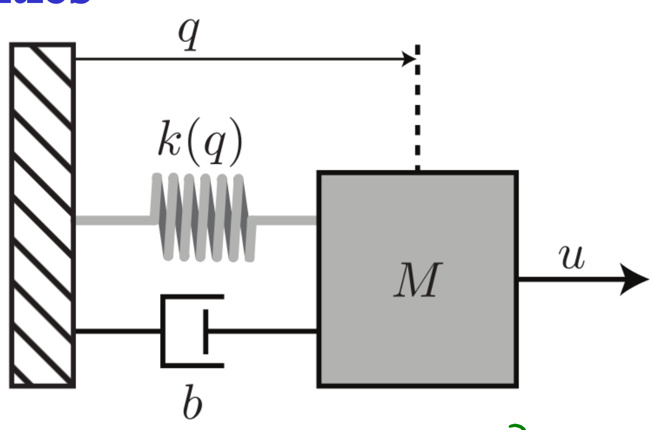
We can now get a TF from this LTI state-space model: $\dfrac{Y(s)}{W(s)} = \dfrac{1/M}{s^2 + \frac{b}{M}s + \frac{K}{M}}$

We can design a controller to get the desired system (y) response.

A taste of feedback linearization: Inverting dynamic nonlinearities

Example: Nonlinear mass-spring-damper



$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{M}\Big(-K\left(1-a^2x_1^2\right)x_1 - bx_2 + u\Big) = \underbrace{f(x) + g(x)u}_{\text{CONTROL AFFINE}}$$
$$y = x_1.$$

(NECESSARY FOR FEEDBACK LINEARIZATION BUT NOT SUFFICIENT).

Consider the feedback control law: $u = -Ka^2y^3 + w$

The dynamics from the new input (w) to the output (y) are now LTI:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{M}\left(-Kx_1 - bx_2 + w\right)$$
$$y = x_1.$$

We can now get a TF from this LTI state-space model: $\dfrac{Y(s)}{W(s)} = \dfrac{1/M}{s^2 + \frac{b}{M}s + \frac{K}{M}}$

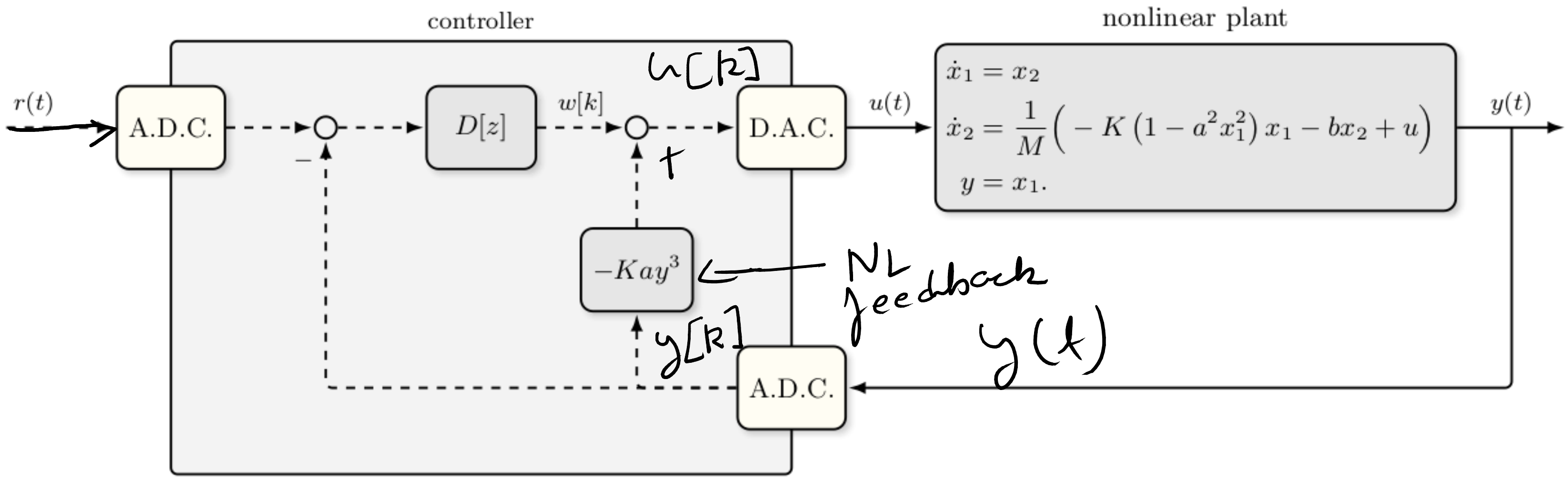
We can design a controller to get the desired system (y) response.

For many feedback-affine nonlinear systems, we can get a feedback linearizing control law automatically via well defined operations, i.e., do not need to handcraft nonlinear feedback laws (think MIMO systems). See the book on non-linear control by Khalil for more details.

Inverting dynamic nonlinearities: Discretization can be problematic

Consider a DT PI controller for 'w', implemented via the architecture below:

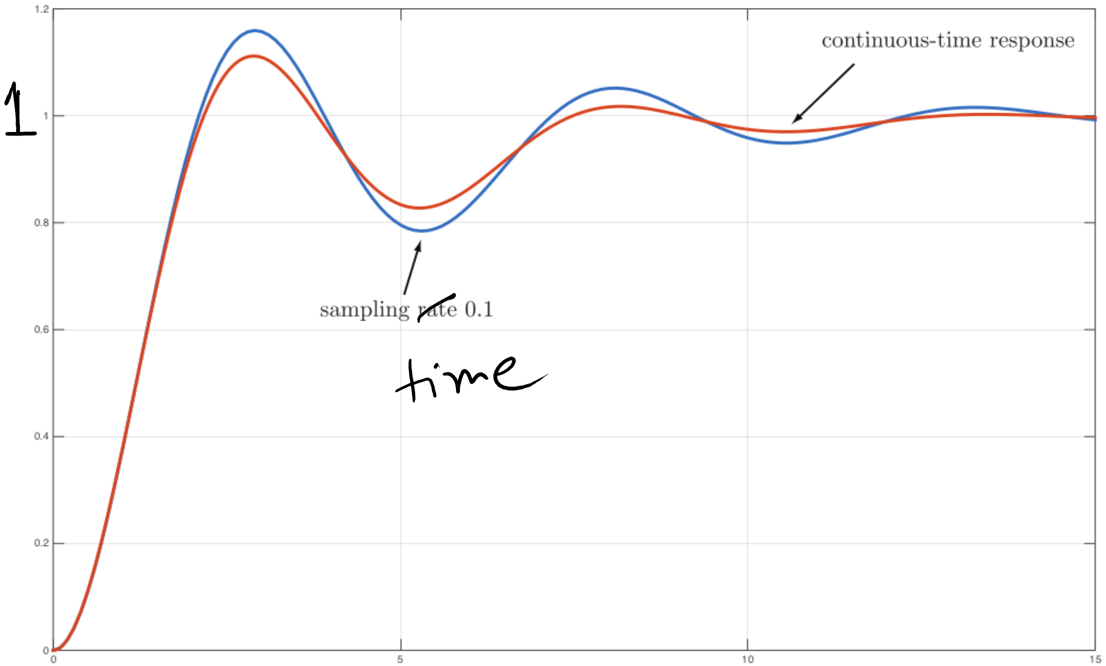
$$u[k] = w[k] - Ka\,y[k]^3$$



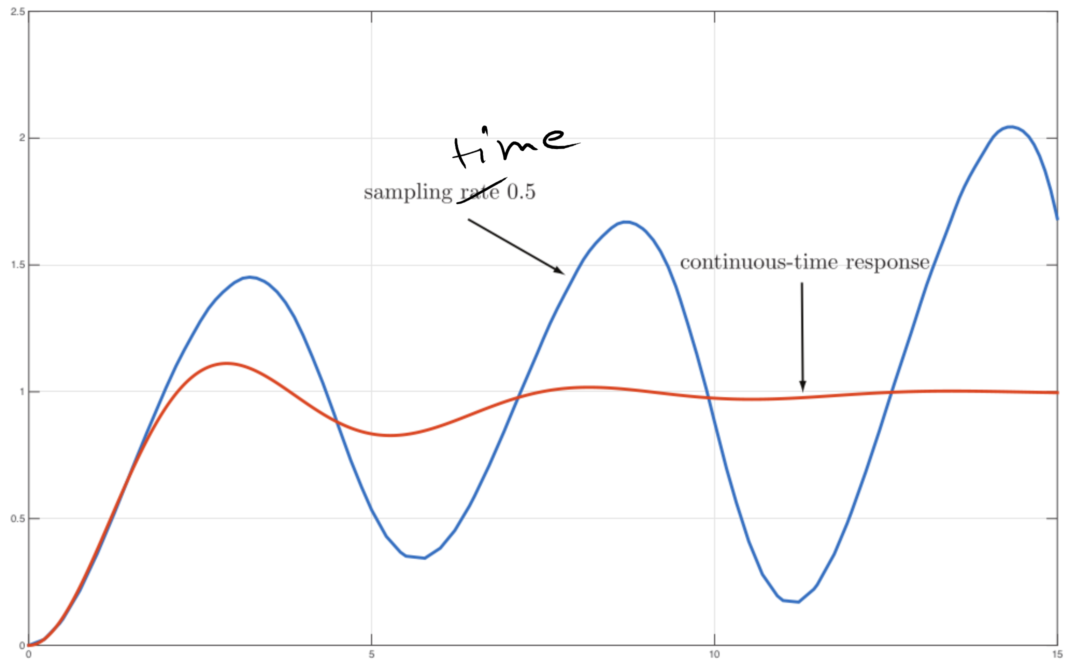
The A.D.C/D.A.C now play a role since the value of y is not actually constant between sampling instants.

Inverting dynamic nonlinearities: Discretization can be problematic

Larger the sampling time, the more the system deviates from its CT LTI model, e.g., the step response with Kp = 1, Ki = 0.5



(a) Sampling period $T = 0.1$ seconds.



(b) Sampling period $T = 0.5$ seconds.

Note: We do not face this issue with inverting static nonlinearities when we use a DAC modeled as zero-order hold since the value of v(t) = w[k] for kT<=t<=(k+1)T, i.e., the mapping (inversion) does not add any modeling error due to sampling time.

Outline

[-] Introduction

[-] State-space models

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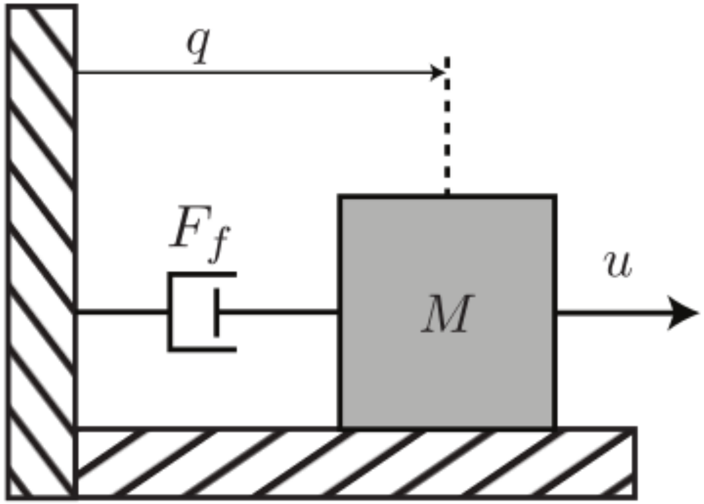
- [ ] Optimal Quasi-Linearization
- [ ] Constructing describing functions
- [ ] Periodic solutions and their stability

Static friction

A common nonlinearity found in systems with moving components.

Example: Simplified Mass-spring-damper and types of friction models.



Consider a nonlinear damper which models the force due to friction (Ff) and the applied force (u). The system model is:

$$M\ddot{q} + F_f = u$$

The friction force F_f has the following components:
– Static friction F_s
– Coulomb friction F_C
– Viscous friction F_v

Static friction

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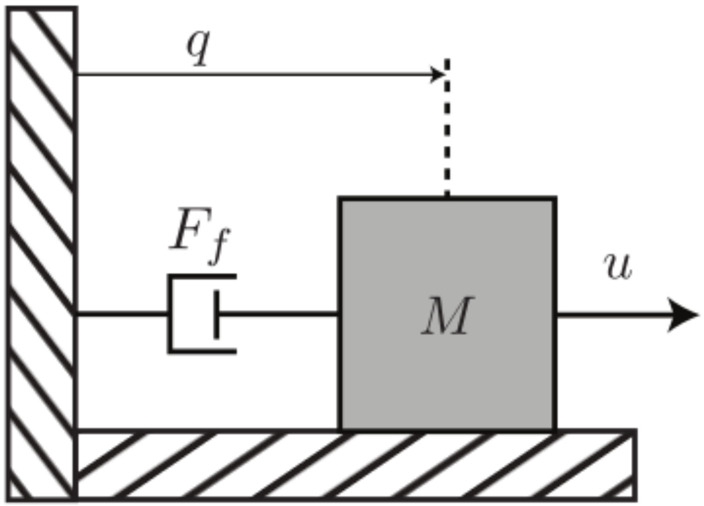

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– Static friction F_s
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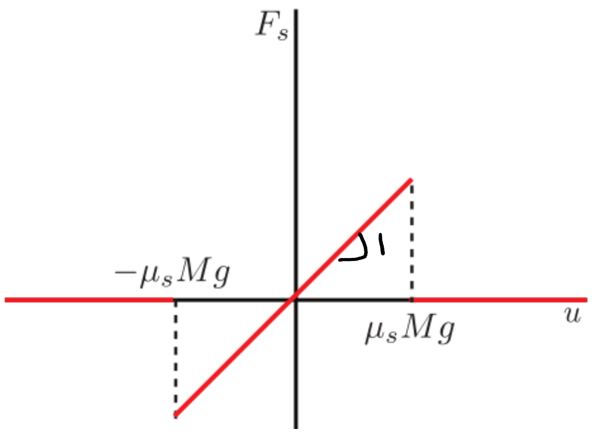
– Viscous friction F_v

Static friction: Acts when the mass is at rest, and acts parallel to the surface.

$F_s \in [-\mu_s Mg,\ \mu_s Mg]$, where $\mu_s \in (0, 1)$ is the coefficient of static friction.



Static friction

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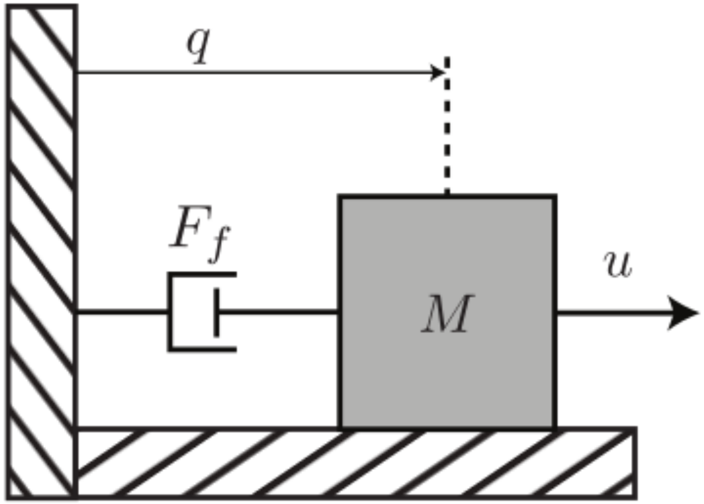

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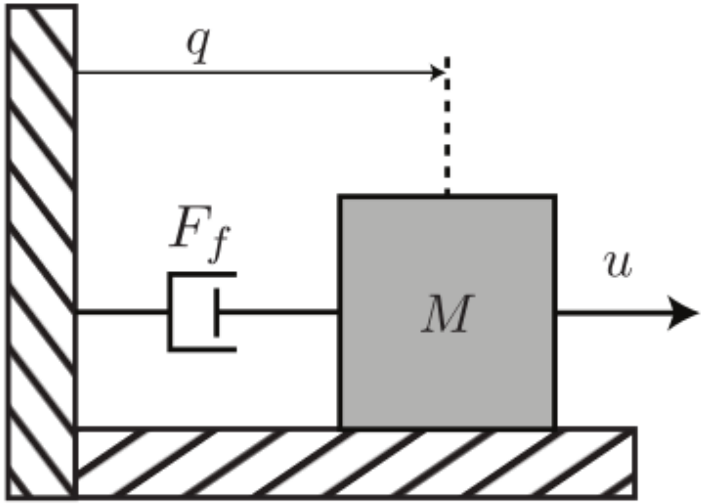
$F_s \in [-\mu_s M g,\ \mu_s M g]$, where $\mu_s \in (0, 1)$ is the coefficient of static friction.

Once the mass starts to move, the friction is modeled as Coulomb friction (constant when the object is moving) and the resistance due to air or the medium is modeled viscous friction (modeled as being proportional to velocity of object).
See class notes for more details. Also see the Stribeck effect.

Static friction

A common nonlinearity found in systems with moving components.



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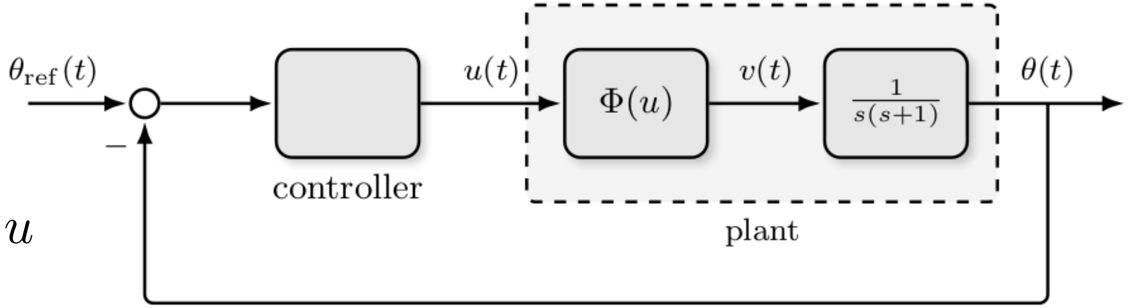
Static friction: Acts when the mass is at rest, and acts parallel to the surface.

$F_s \in [-\mu_s Mg,\ \mu_s Mg]$, where $\mu_s \in (0, 1)$ is the coefficient of static friction.

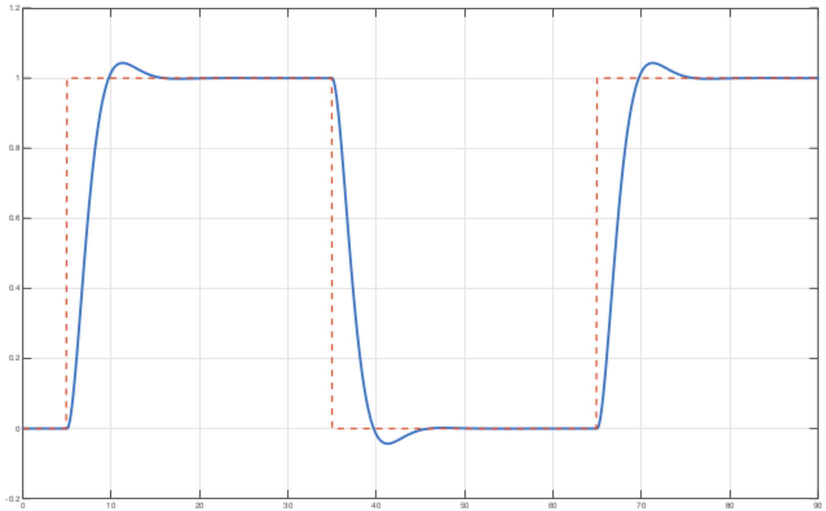
Static friction is of particular concern when motors are involved and can:
i) Increase steady-state error.
ii) Induce oscillations when there is an integral control component in the controller.
iii) Lead to CL instability for open-loop unstable plants.

Example: DC motor with static friction



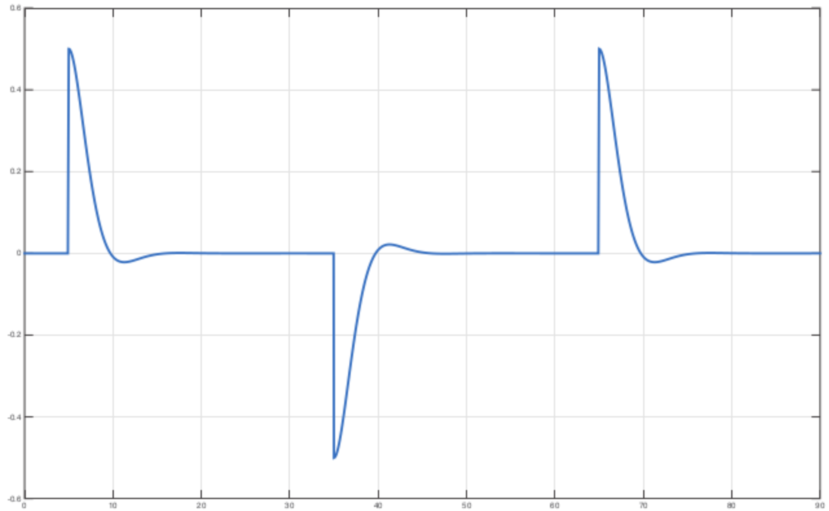
$\Phi(u)$ is the static friction.

First, assume there is no static friction, i.e. $\Phi(u) = u$ and we have a proportional controller $C(s) = 0.5$



(a) $\theta(t)$ and $\theta_{\text{ref}}(t)$.

0

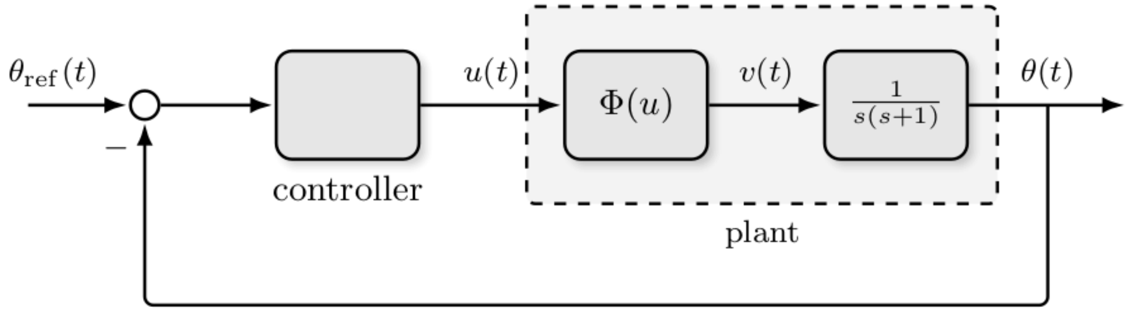


(b) Control signal $u(t)$.

Plant has an integrator, therefore we get asymptotic step tracking in simulation.

Example: DC motor with static friction

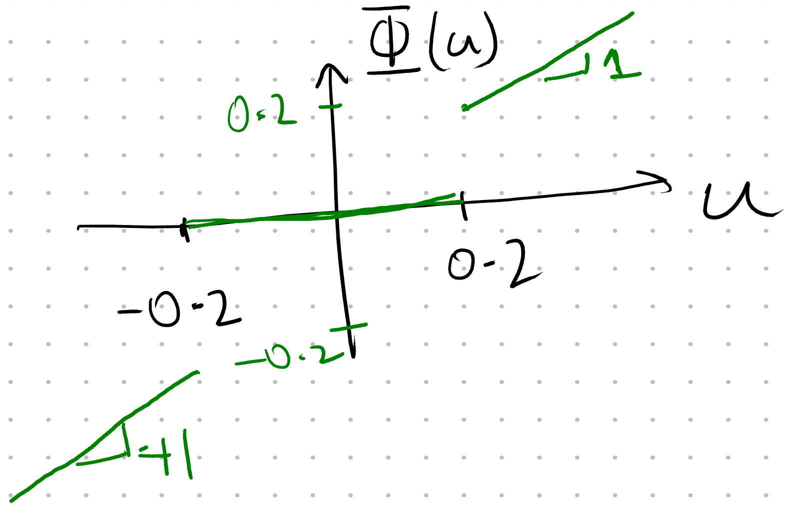
$\Phi(u)$ is the static friction.



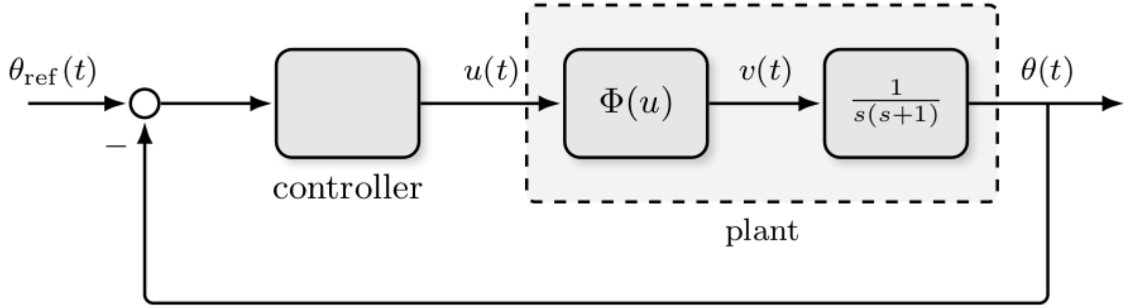
Next, suppose we have static friction modeled as:

$$\Phi(u) = \begin{cases} u & |u| \geq 0.2 \\ 0 & |u| < 0.2 \end{cases}$$

Is this invertible ? No.



Example: DC motor with static friction

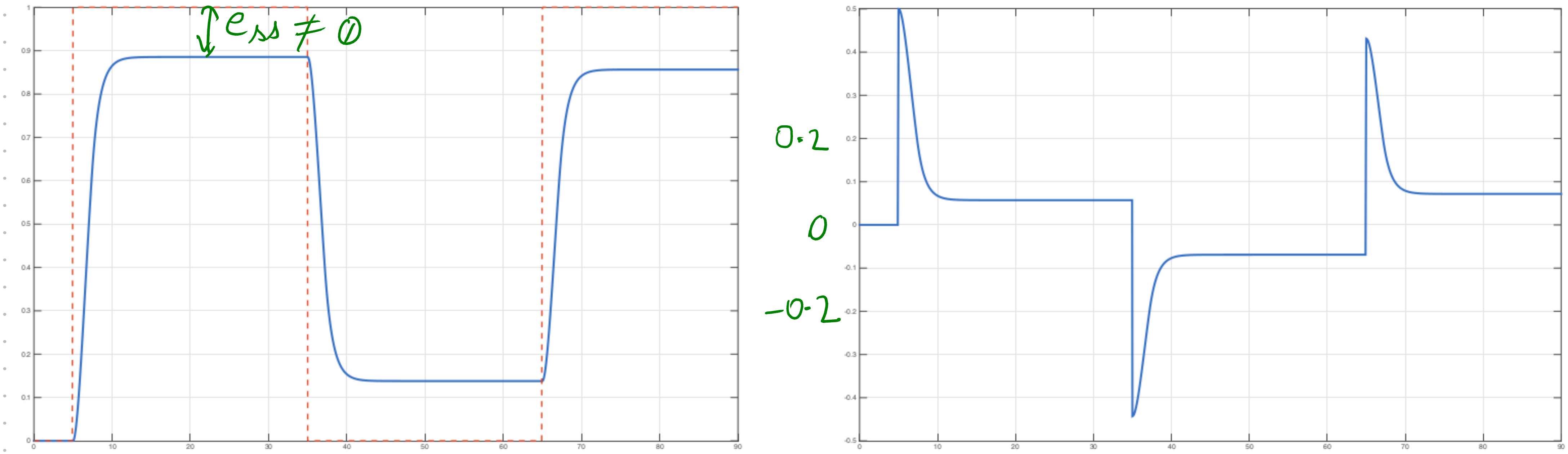


$\Phi(u)$ is the static friction.

Next, suppose we have static friction modeled as:

$$\Phi(u) = \begin{cases} u & |u| \geq 0.2 \\ 0 & |u| < 0.2 \end{cases}$$

Consider the same proportional controller $C(s) = 0.5$, and the same reference signal as before:



(a) $\theta(t)$ and $\theta_{\text{ref}}(t)$.

(b) Control signal $u(t)$.

Control performance is worse: We have a steady-state error and the controller is putting in more effort (higher L2 norm of the control signal compared to when there was no static friction).

Outline

[-] Introduction

[-] State-space models

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    [-] Equilibrium configuration
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    [X] Dither signals
    [ ] Deadzone non-linearity

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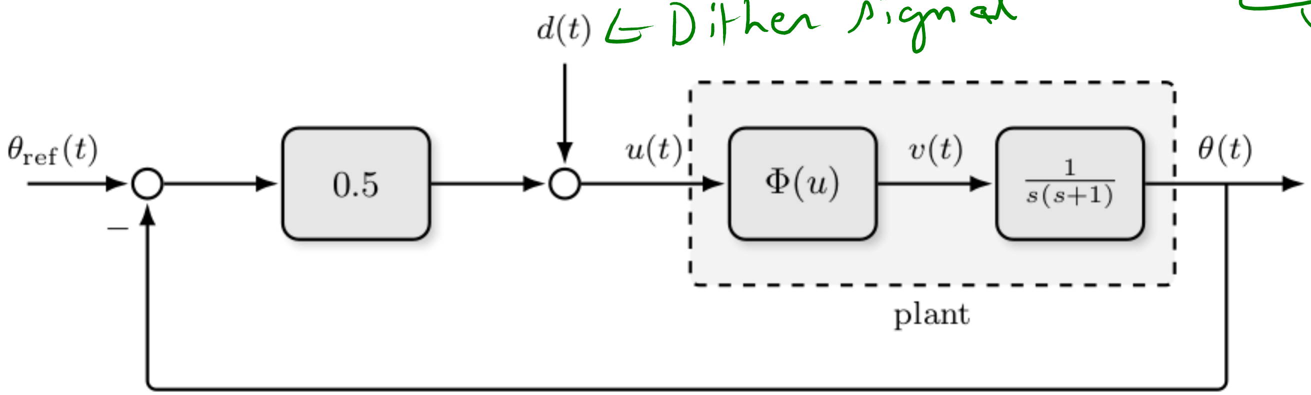
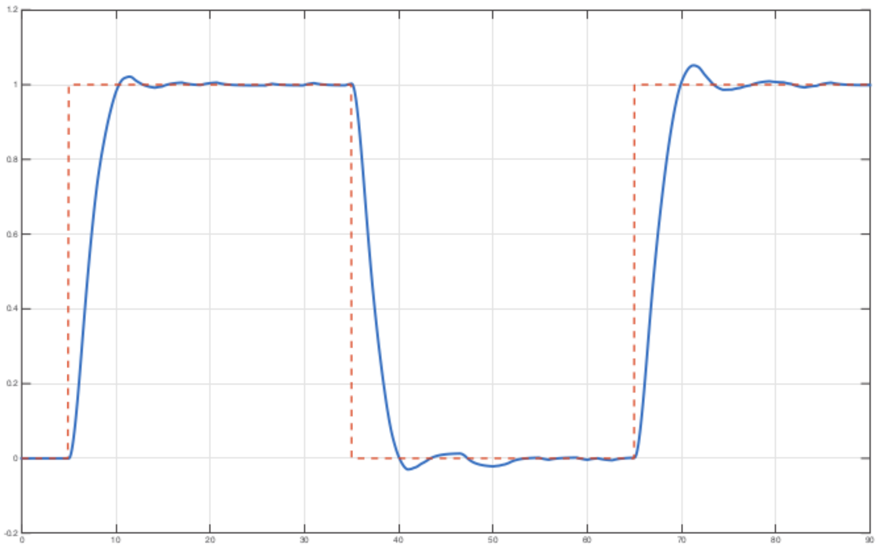
    [ ] Periodic solutions and their stability

Dealing with static friction: a) Dither signals

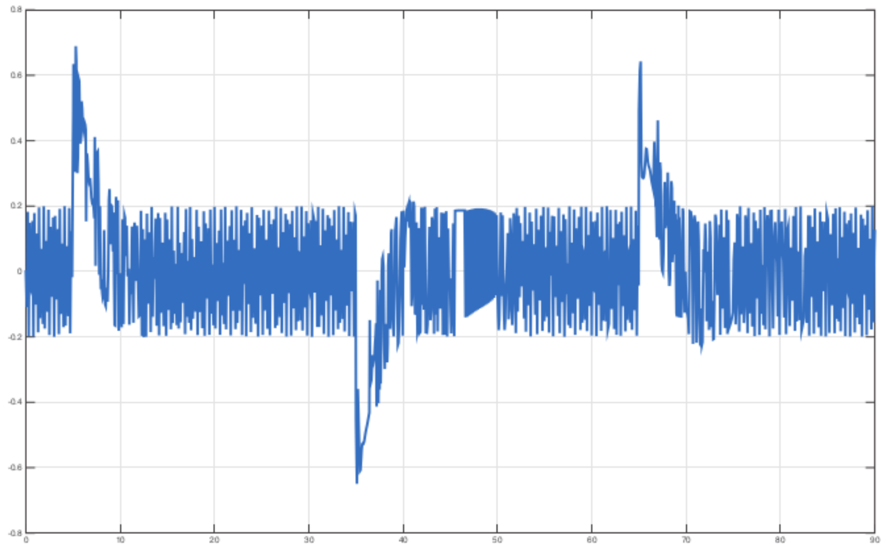
A dither signal is a periodic signal with small amplitude that oscillates around zero (zero-mean). Tries to make systems overcome static friction by making all parts of the system "jitter", or persistently excited enough to avoid static friction. This jitter/dither is added to the control signal.

Example: Sinusoidal dither signal added to the controller of a DC motor. $d(t) = \underbrace{0.2}_{\text{Chosen empirically!}}\sin(100t)$





(a) $\theta(t)$ and $\theta_{\mathrm{ref}}(t)$.



(b) Control signal $u(t)$.

Dealing with static friction: a) Dither signals

A dither signal is a periodic signal with small amplitude that oscillates around zero (zero-mean).
Tries to make systems overcome static friction by making all parts of the system "jitter", or persistently excited enough to avoid static friction. This jitter/dither is added to the control signal.

Drawbacks:
1. Wears out actuators and other moving parts due to constant movement.
2. Consumes more energy for control.
3. Excites unmodeled plant dynamics (e.g., high-frequency response).
4. Requires careful tuning, and static friction can change over time.

Outline

[-] Introduction

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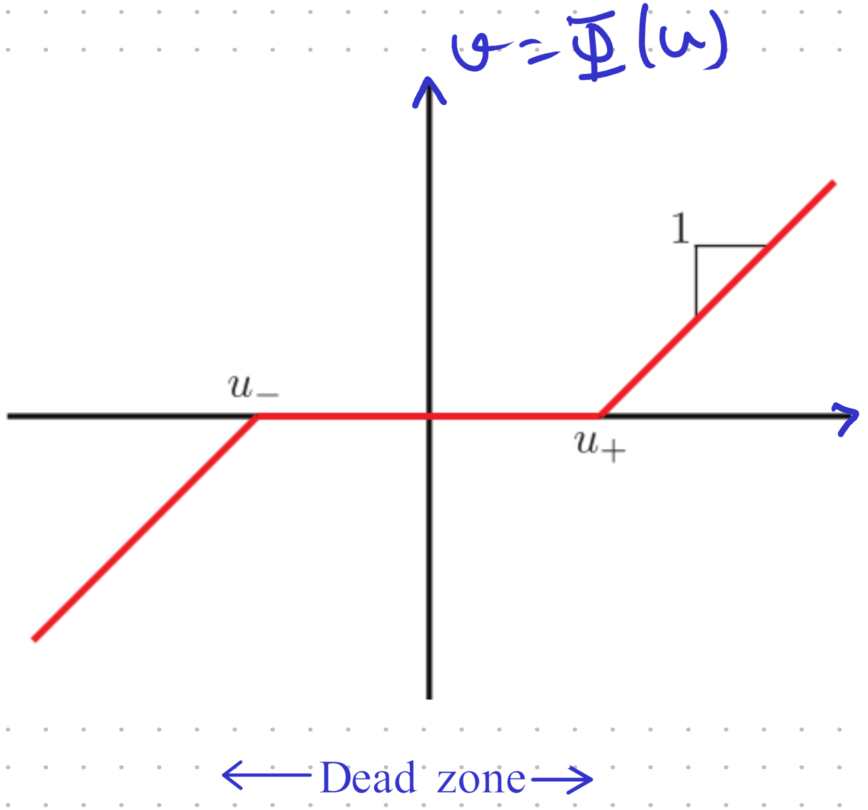
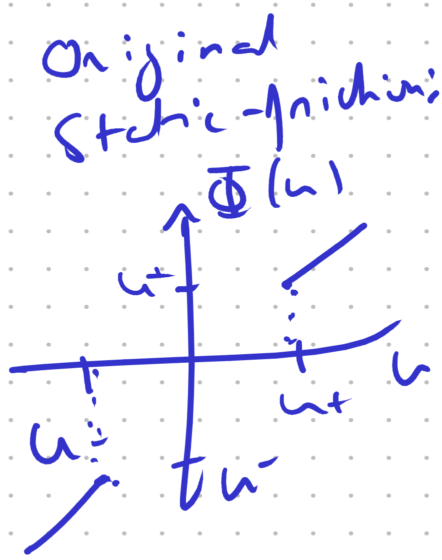
Dealing with static friction: b) approximating it as a dead zone nonlinearity

Our crude approximation of static friction is ~~neither static nor~~ not right invertible. How do we overcome this?

Dealing with static friction: b) approximating it as a dead zone nonlinearity

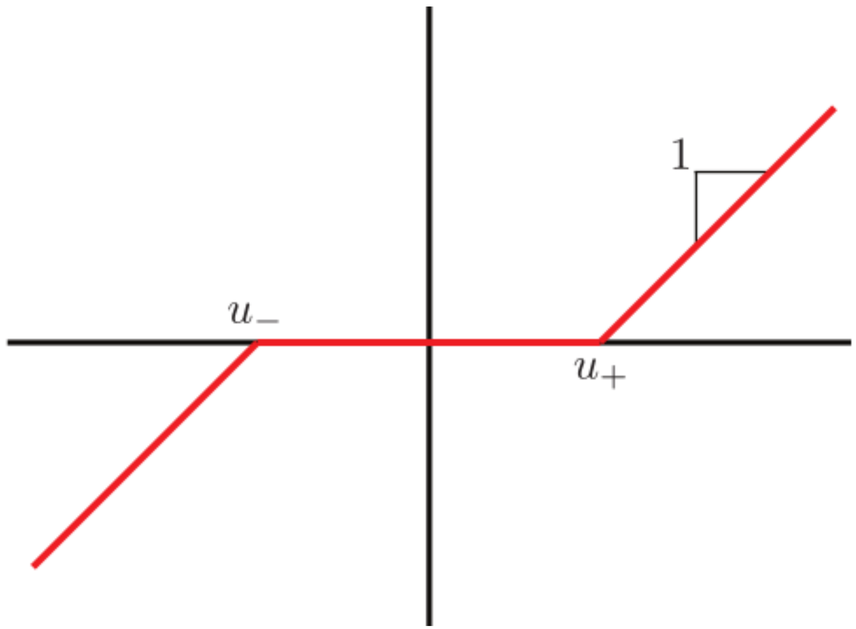
Our crude approximation of static friction is neither static nor right invertible. How do we overcome this?
By approximating static friction in an even more crude manner: As a dead zone nonlinearity.





Dealing with static friction: b) approximating it as a dead zone nonlinearity

Our crude approximation of static friction is neither static nor right invertible. How do we overcome this? By approximating static friction in an even more crude manner: As a dead zone nonlinearity.



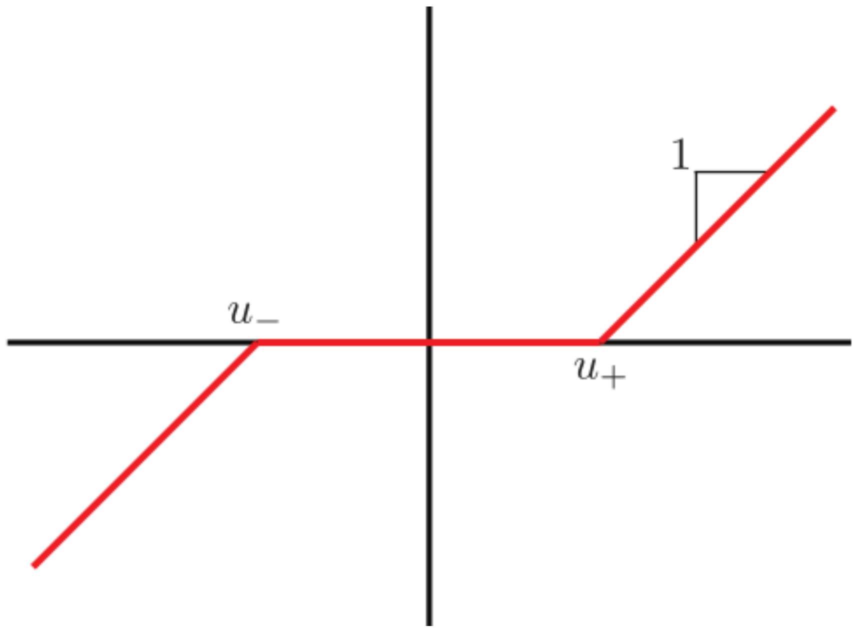
⟵Dead zone⟶

By using this approximation as a static input nonlinearity, we approximate static friction where the control input to the system is zero for values of u between u+ and u-, and then linearly increases (unit slope).

u+ and u- are functions of the coefficient of friction and need to be (re)determined empirically.

Dealing with static friction: b) approximating it as a dead zone nonlinearity

Our crude approximation of static friction is neither static nor right invertible. How do we overcome this? By approximating static friction in an even more crude manner: As a dead zone nonlinearity.



Is this static nonlinearity onto?

Is it one-to-one?

⟵ Dead zone ⟶

By using this approximation as a static input nonlinearity, we approximate static friction where the control input to the system is zero for values of u between u+ and u-, and then linearly increases (unit slope).

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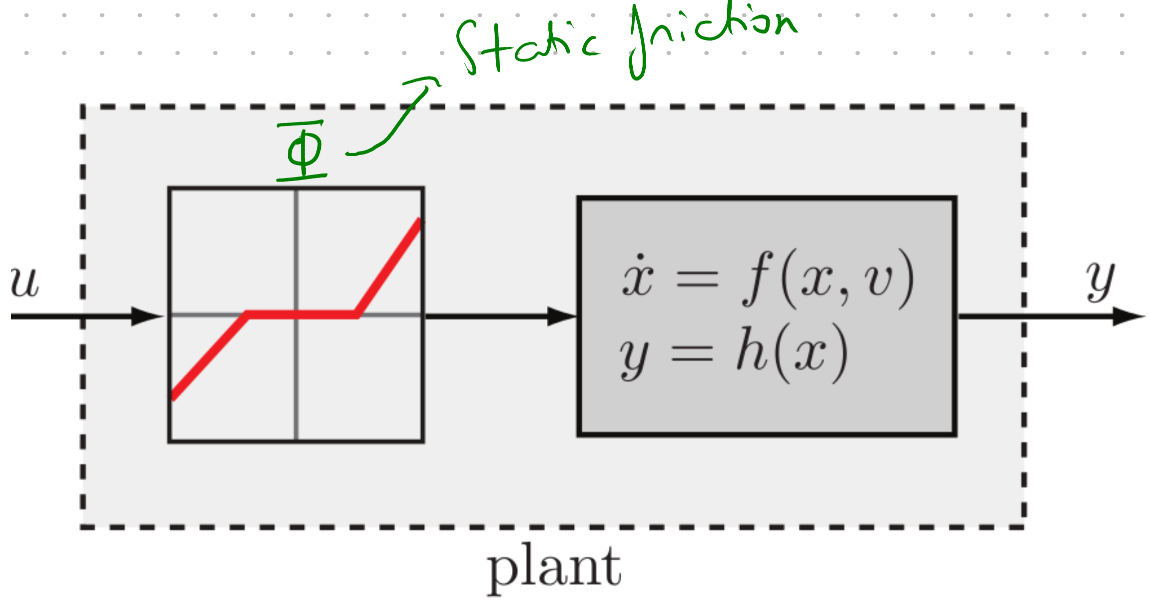
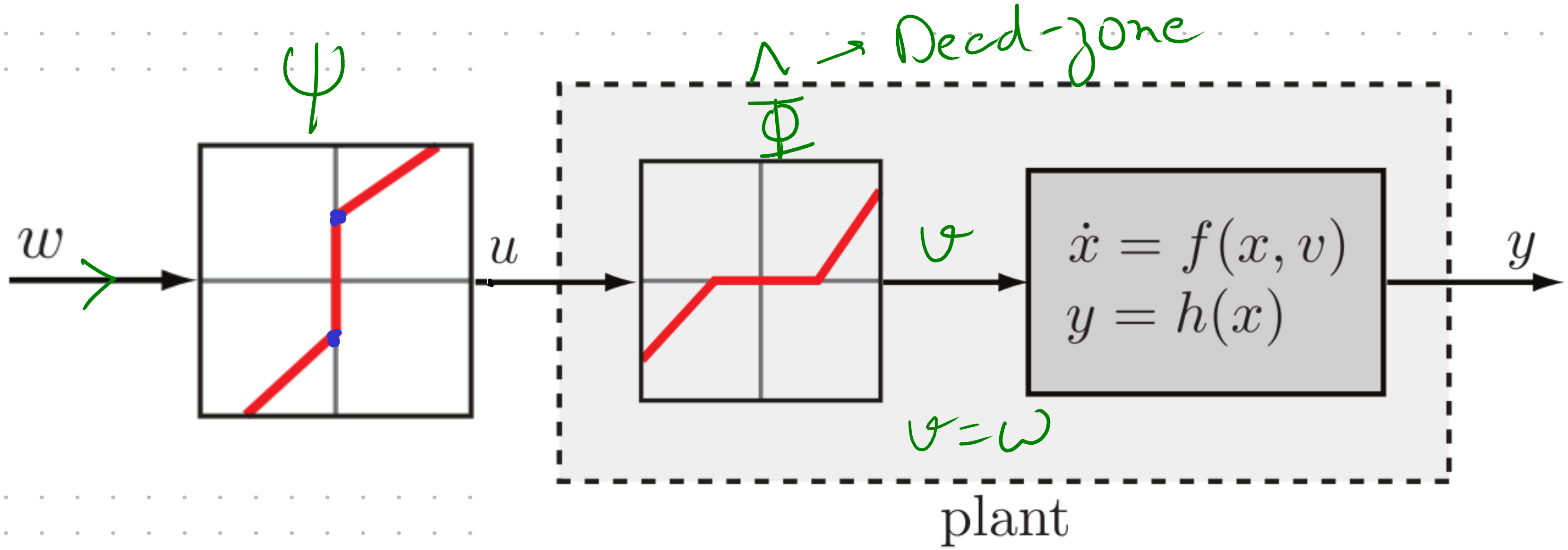
Dealing with static friction: b) approximating it as a dead zone nonlinearity

We now have the following approximation of a system with static friction.



Inverting a deadzone nonlinearity

The deadzone linearity is onto, so it can be inverted (if it is an input nonlinearity).

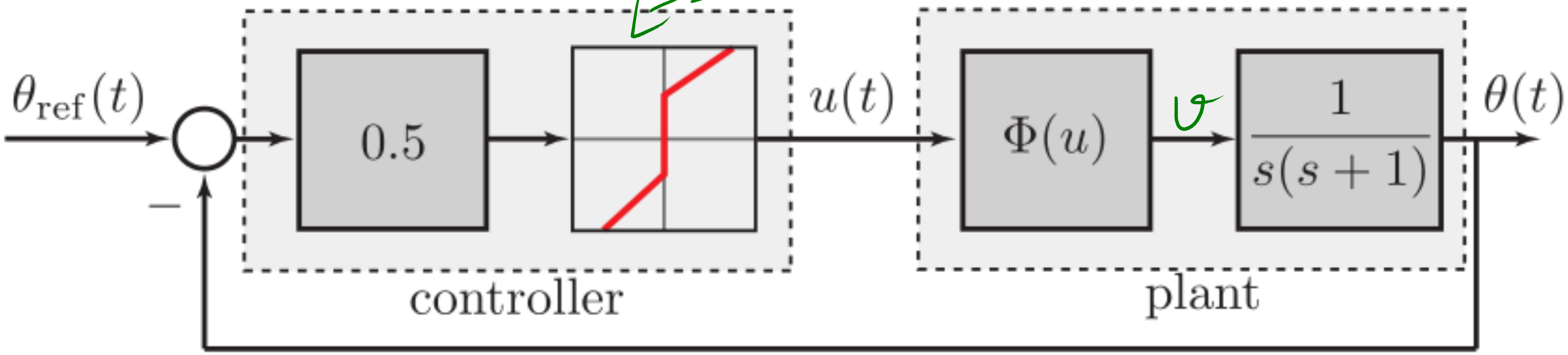


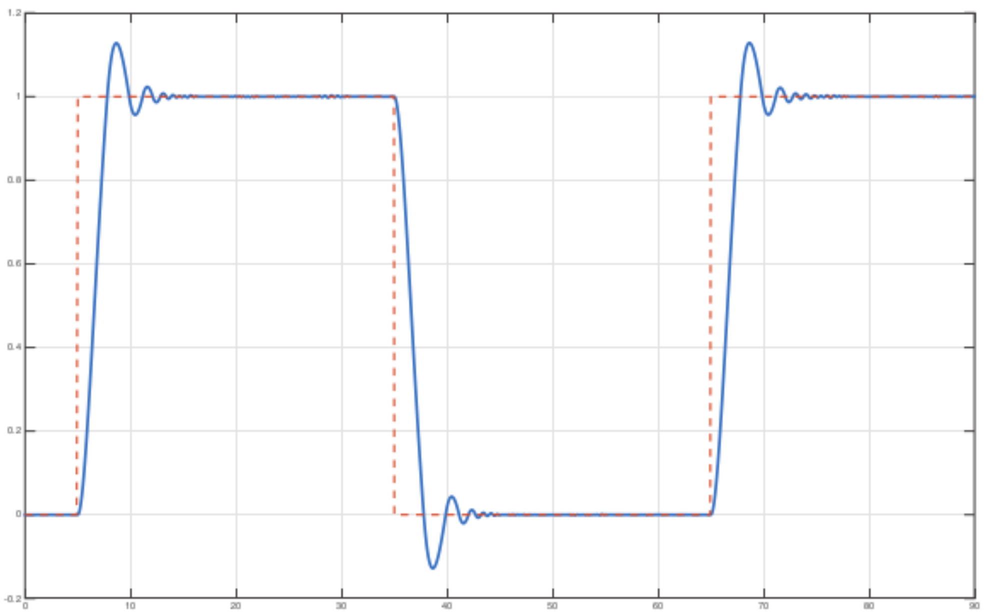
Note, this is still a crude approximation of static-friction.

Inverting a deadzone nonlinearity

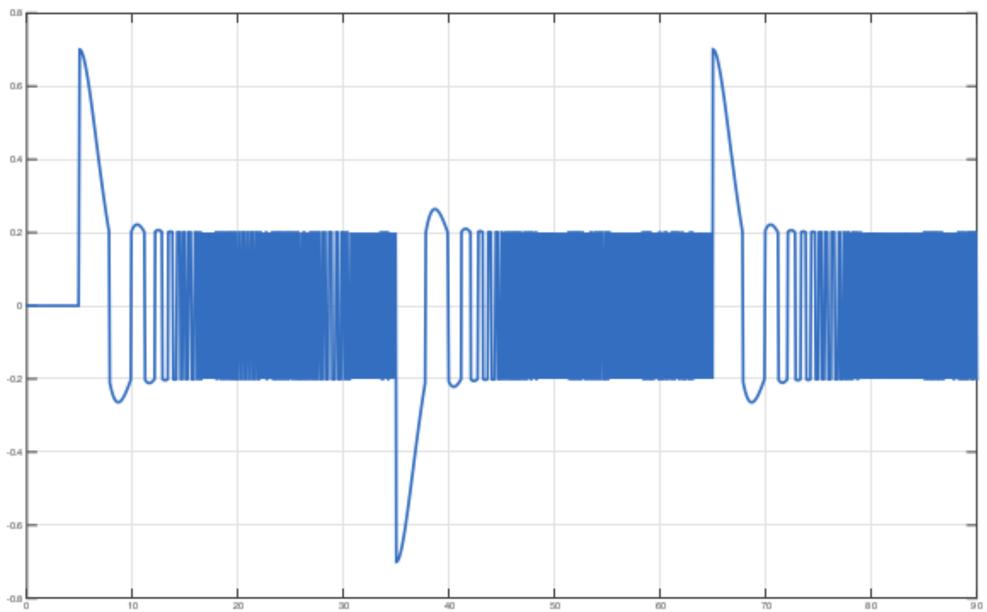
Controller performance while inverting the deadzone nonlinearity

← Inversion of deadzone





(a) $\theta(t)$ and $\theta_{\text{ref}}(t)$.



(b) Control signal $u(t)$.

Tracking is improved, but the control signal can't settle at zero and can be potentially unkind to actuators.

Outline

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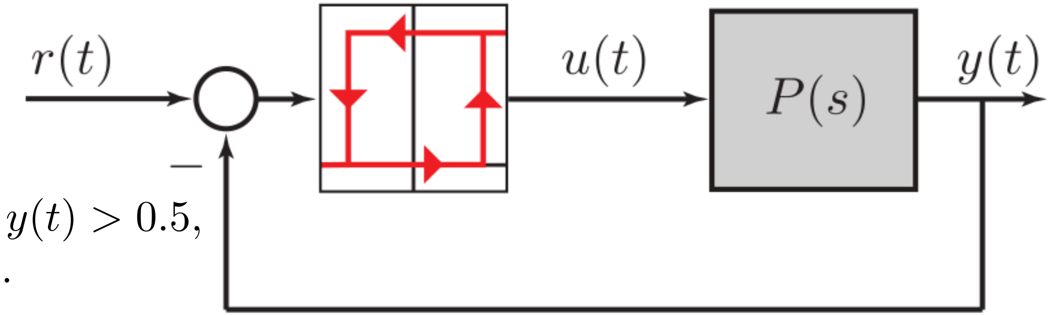
- [-] Dither signals
- [-] Deadzone non-linearity

[X] Describing functions
- [ ] Optimal Quasi-Linearization
- [ ] Constructing describing functions
- [ ] Periodic solutions and their stability

Describing functions

A method for linear approximation of NL systems with perioidic behaviors.

Example: Thermostat (on-off control) with hysterisis. Hystersis prevents "chattering" of the control signal.



$$P(s) = \frac{2}{s+1}$$

Furnace is on when: $r(t) - y(t) > 0.5$,
off when $r(t) - y(t) < -0.2$.

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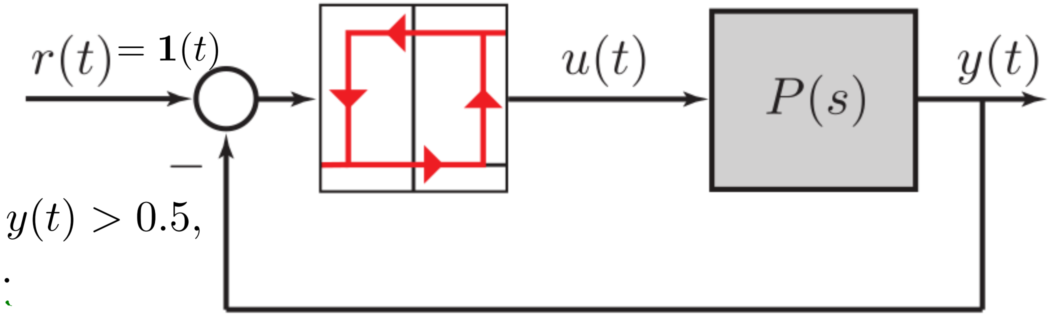

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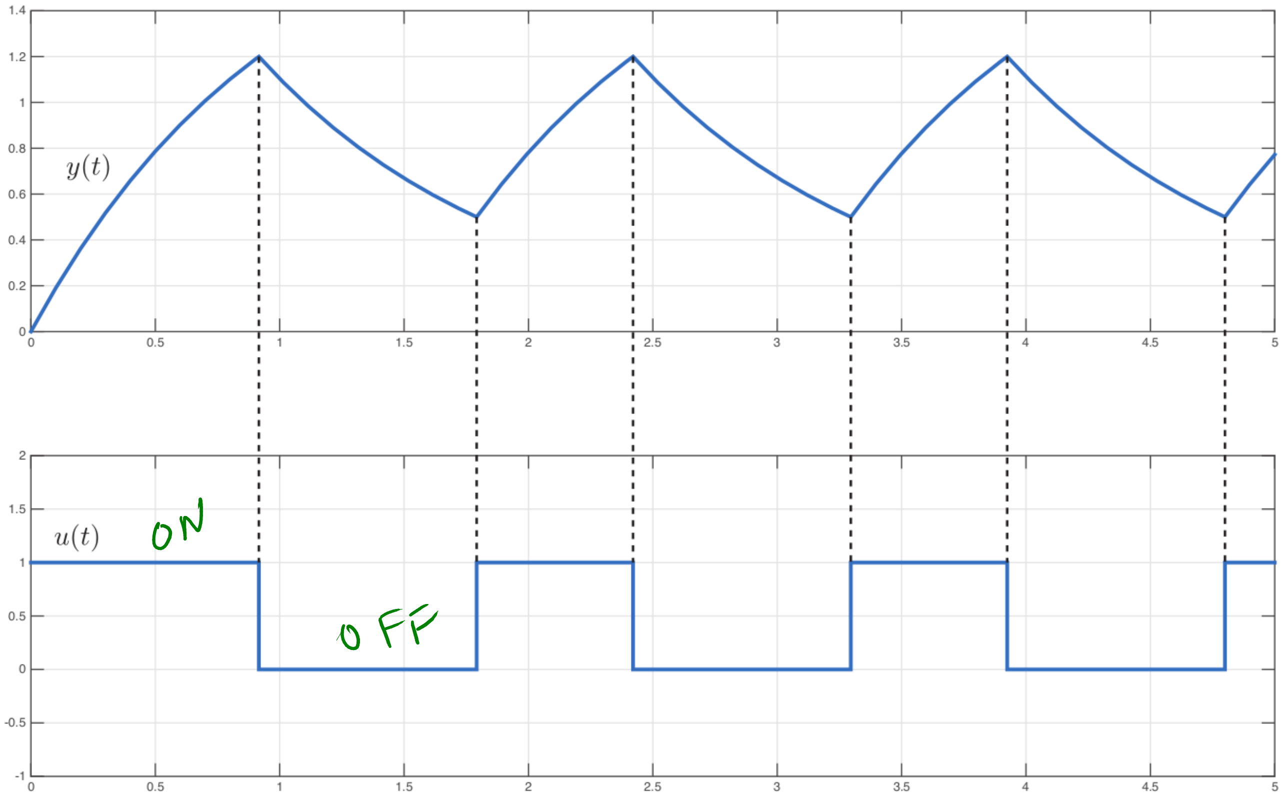

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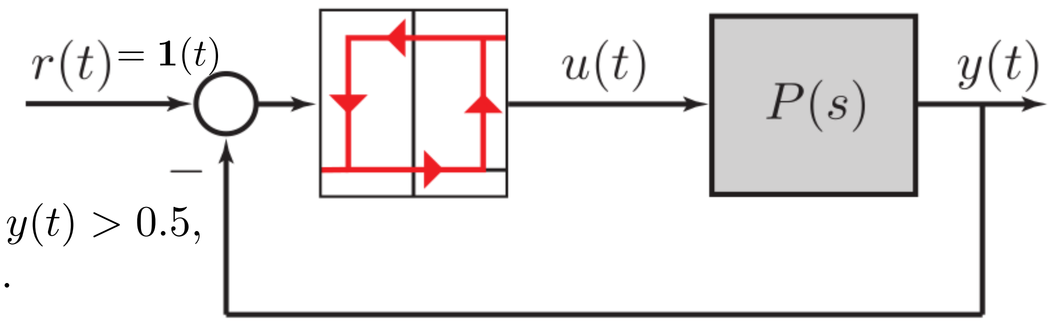

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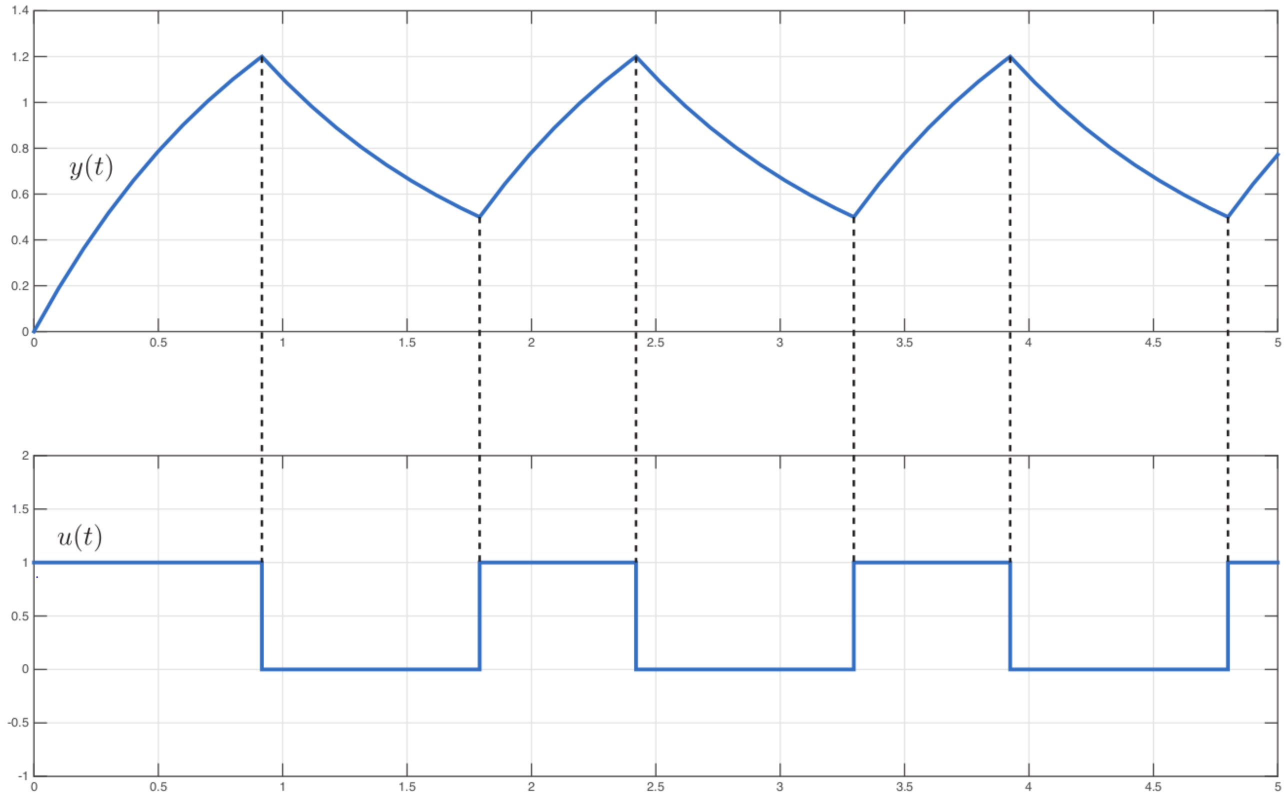

We want to prove the existence, and find the expression, for a perdioic solution.

The problem is a hard one, and we rely on approximations.

Benefit: Can use linear systems tools.
Drawback: Relies on approximations, and works for very specific cases.

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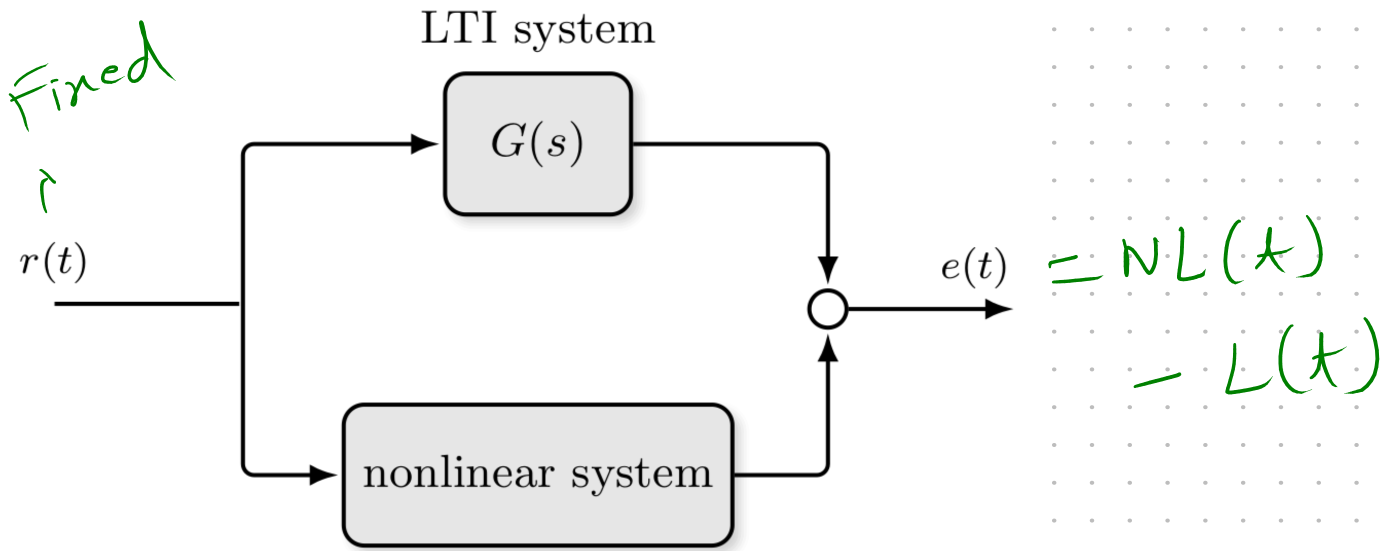
  [X] Optimal Quasi-Linearization
  [ ] Constructing describing functions
  [ ] Periodic solutions and their stability

Optimal Quasi-Linearization

Find the "best" LTI system that approximates the NL system for a given reference input.



Given r(t), we want the G(s) that minimizes the error $E(G) = \lim_{T\to\infty} \frac{1}{T} \int_0^T e^2(t)\ dt$

G(s) is called the optimal quasi-linearization of the nonlinear system, and the solution is known if some conditions are met (next slide).

Note: The approximation G(s) is "best" only for the given reference input. If the reference changes, G(s) also changes. G(s) is also not unique.

Optimal Quasi-Linearization: Sinusoidal reference signal.

A common form of reference signals is: $r(t) = a\sin(\omega t)$, a, $\omega > 0$

Theorem 5.5.2 ([Vidyasagar, 2002, Theorem 4.1.36]). *Let $r(t) = a\sin(\omega t)$ and let y be the output of a nonlinear system under the input $r(t)$. Assume that*

(i) The output of the nonlinear system has the form $y(t) = y_{\mathrm{tr}}(t) + y_{\mathrm{ss}}(t)$ where $y_{\mathrm{tr}}(t)$ is the transient response and $y_{\mathrm{ss}}(t)$ is the steady-state response.

(ii) The steady-state response y_{ss} of the nonlinear system due to the input $r(t)$ is periodic with period $2\pi/\omega$ and

$$\int_0^\infty y_{\mathrm{tr}}^2(t)\,\mathrm{d}t < \infty.$$

$\exists\, t'$ s.t. $\forall t > t'$
$y_{tr}(t) = 0$.

(iii) The transfer function $G(s)$ is proper and has all its poles in the open left half complex plane.

Under these conditions, G is an optimal quasi-linearization of the nonlinear system for the input $r(t)$ if, and only if

$$G(j\omega) = \frac{a_1(a,\omega) + jb_1(a,\omega)}{a}$$

where

$$a_1(a,\omega)\sin(\omega t) + b_1(a,\omega)\cos(\omega t)$$

is the first harmonic of $y_{\mathrm{ss}}(t)$.

Optimal Quasi-Linearization: Sinusoidal reference signal.

The condition on $G(jw)$ is called the principle of harmonic balance.

The result states that the optimal choice of G is one whose steady-state response due to the sinusoidal input $r(t)$ precisely matches the first harmonic of the NL system's steady-state response to $r(t)$.

Optimal Quasi-Linearization: Sinusoidal reference signal.

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Definition 5.5.4. Let $r(t)$ and $y(t)$ be as in Theorem 5.5.2. The **describing function** of the nonlinear system is the complex-valued function

$r(t) = a \sin(\omega t)$

$$\eta(a,\omega) = \frac{a_1(a,\omega) + jb_1(a,\omega)}{a}. \tag{5.8}$$

Optimal Quasi-Linearization: Sinusoidal reference signal.

The condition on G(jw) is called the principle of harmonic balance.

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$$\eta(a,\omega) = \frac{a_1(a,\omega) + jb_1(a,\omega)}{a}. \tag{5.8}$$

Note: The function is unique once a and ω are fixed.
Describing functions for static nonlinearities do not depend on the frequency ω.

Proposition 5.5.5. *The describing function of a static nonlinearity of the form $y(t) = \Phi(r(t))$ where $\Phi : \mathbb{R} \to \mathbb{R}$, is independent of ω.*

Proposition 5.5.6. *The describing function of an odd, static nonlinearity of the form $y(t) = \Phi(r(t))$ where $\Phi : \mathbb{R} \to \mathbb{R}$, is real-valued.*

$\Phi : \mathbb{R} \to \mathbb{R}$ is odd iff $-\Phi(-x) = \Phi(x)$

Take away: Describing functions are a way to find a linear model that approximates the response of a nonlinear system for a fixed reference signal.

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[-] Introduction

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  [X] Constructing describing functions

  [ ] Periodic solutions and their stability

Constructing describing functions

Let $r(t) = a\sin(\omega t)$, $a, \omega > 0$.
Assume the nonlinear system satisifes assumptions of theorem 5.5.2

$y_{ss}(t)$ is the SS response to $r(t)$ & $2\pi/\omega$ - periodic

$y_{ss}(t) = y_{ss}(t + 2\pi/\omega)$ can be represented via a trigonometric Fourier series:

$$y_{ss}(t) = a_0 + \sum_{n=0}^{\infty} a_n \sin(n\omega t) + \sum_{n=1}^{\infty} b_n \cos(n\omega t), \text{ where,}$$

$$a_0 := \frac{1}{T}\int_{-T/2}^{T/2} y_{ss}(t)\,dt, \quad a_n := \frac{2}{T}\int_{-T/2}^{T/2} y_{ss}(t)\sin(n\omega t)\,dt$$

$$b_n := \frac{2}{T}\int_{-T/2}^{T/2} y_{ss}(t)\cos(n\omega t)\,dt$$

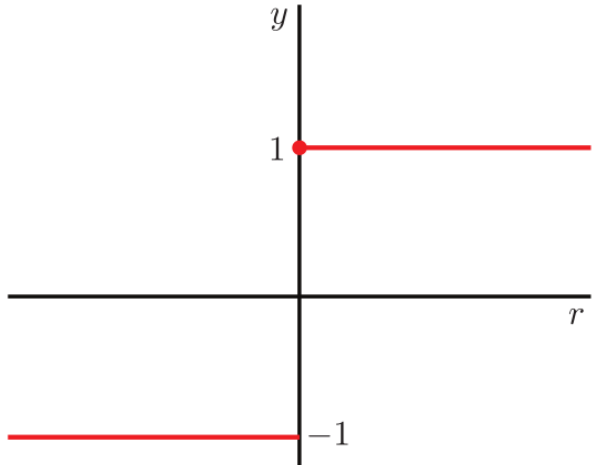
Fourier coefficients

Recall, a_1, b_1 show up in the describing function.

Constructing describing functions: Example

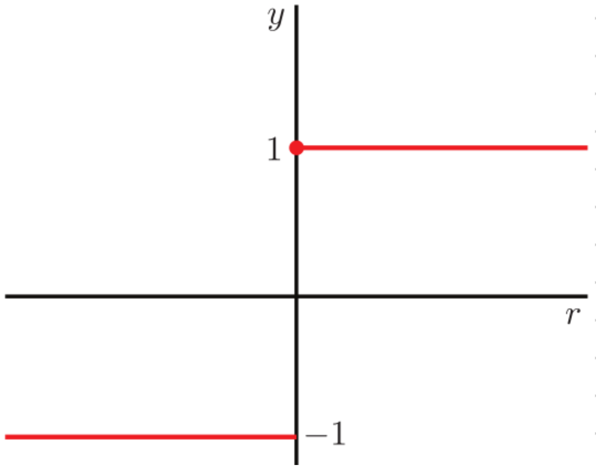
Sign non-linearity.

Is it a static nonlinearity ?



Constructing describing functions: Example

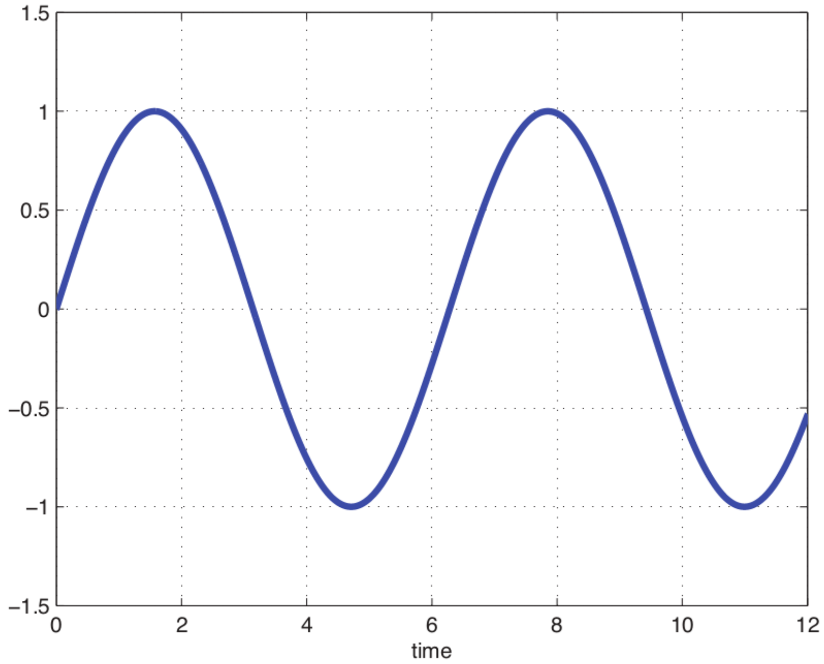
Sign non-linearity.



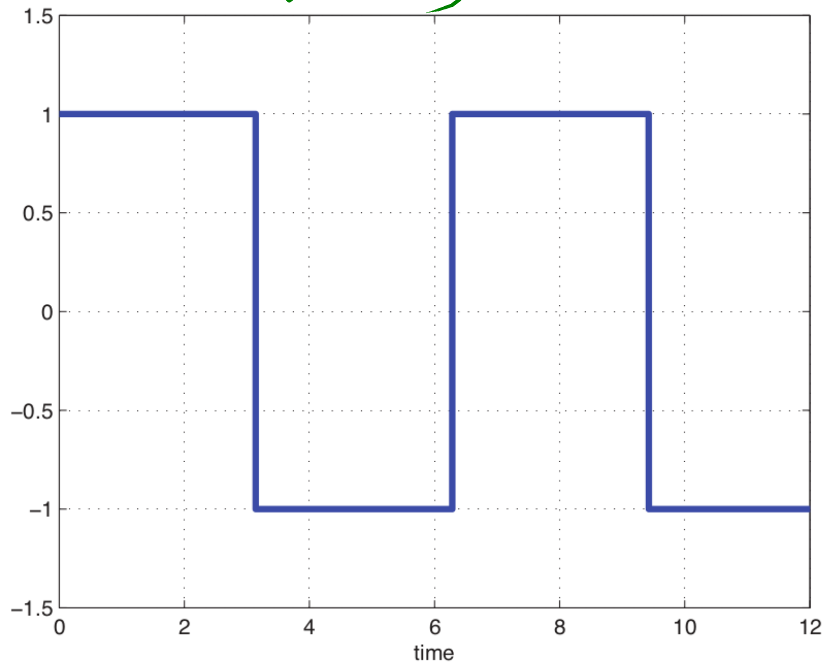
Response to a sinusoidal input has no transient component and is periodic:

$$T = 2\pi/\omega$$

$y_{ss}(t)$



(a) $r(t) = a\sin(\omega t)$, $a = 1$, $\omega = 1$.

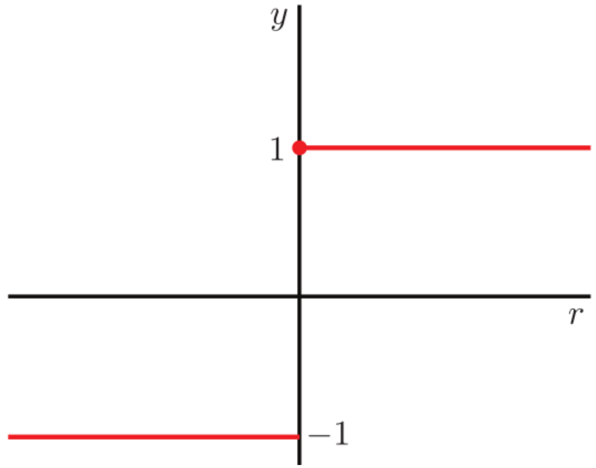


(b) Output $y(t) = \Phi(r(t))$.

Constructing describing functions: Example

Sign non-linearity (static).

Steady state response is: $y_{\mathrm{ss}}(t) = \begin{cases} -1 & -T/2 < t < 0 \\ 1 & 0 \leq t \leq T/2. \end{cases}$



Constructing describing functions: Example

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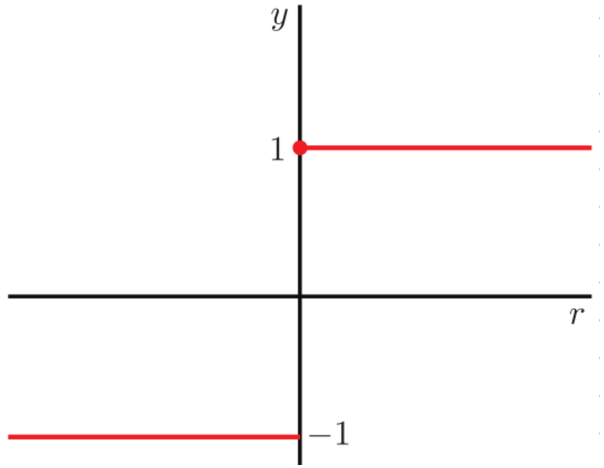


Steady state response is: $y_{ss}(t) = \begin{cases} -1 & -T/2 < t < 0 \\ 1 & 0 \leq t \leq T/2. \end{cases}$

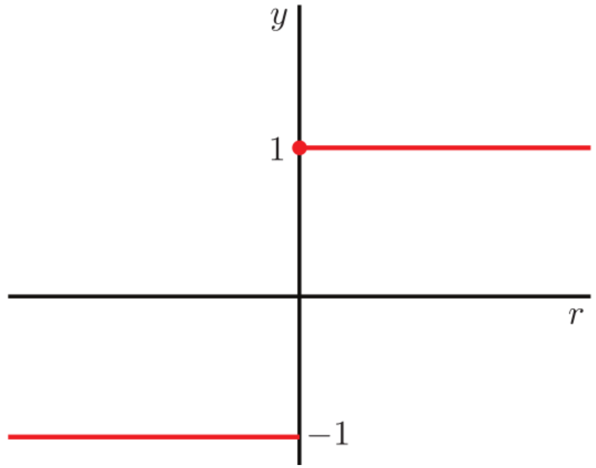
Computing the first components of the Fourier series expansion of $y_{ss}(t)$ we get

$$a_1 = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} y_{ss}(t) \sin(\omega t) \; \mathrm{d}t = \frac{4}{T} \int_{0}^{\frac{T}{2}} \sin(\omega t) \; \mathrm{d}t = \frac{4}{\pi}$$

$$b_1 = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} y_{ss}(t) \cos(\omega t) \; \mathrm{d}t = \frac{2}{T} \int_{-\frac{T}{2}}^{0} -\cos(\omega t) \; \mathrm{d}t + \frac{2}{T} \int_{0}^{\frac{T}{2}} \cos(\omega t) \; \mathrm{d}t = 0 + 0 = 0.$$

Constructing describing functions: Example

Sign non-linearity (static).



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$$b_1 = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} y_{ss}(t)\cos(\omega t)\ \mathrm{d}t = \frac{2}{T}\int_{-\frac{T}{2}}^{0} -\cos(\omega t)\ \mathrm{d}t + \frac{2}{T}\int_{0}^{\frac{T}{2}} \cos(\omega t)\ \mathrm{d}t = 0 + 0 = 0.$$

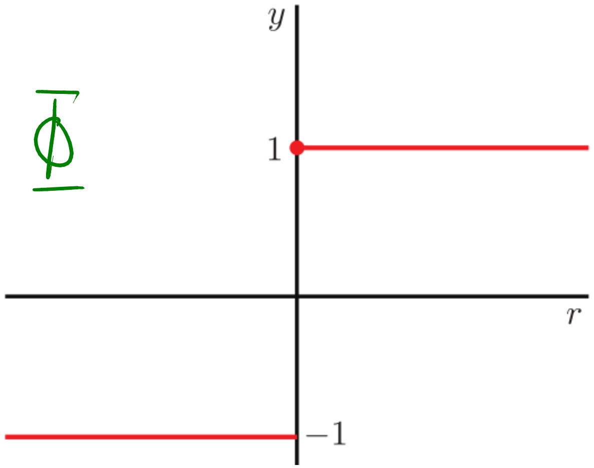
Therefore the describing function is

$$\begin{aligned} \eta(a,\omega) &= \frac{a_1(a,\omega) + jb_1(a,\omega)}{a} \\ &= \frac{a_1(a,\omega)}{a} \\ &= \frac{4}{a\pi}. \end{aligned}$$

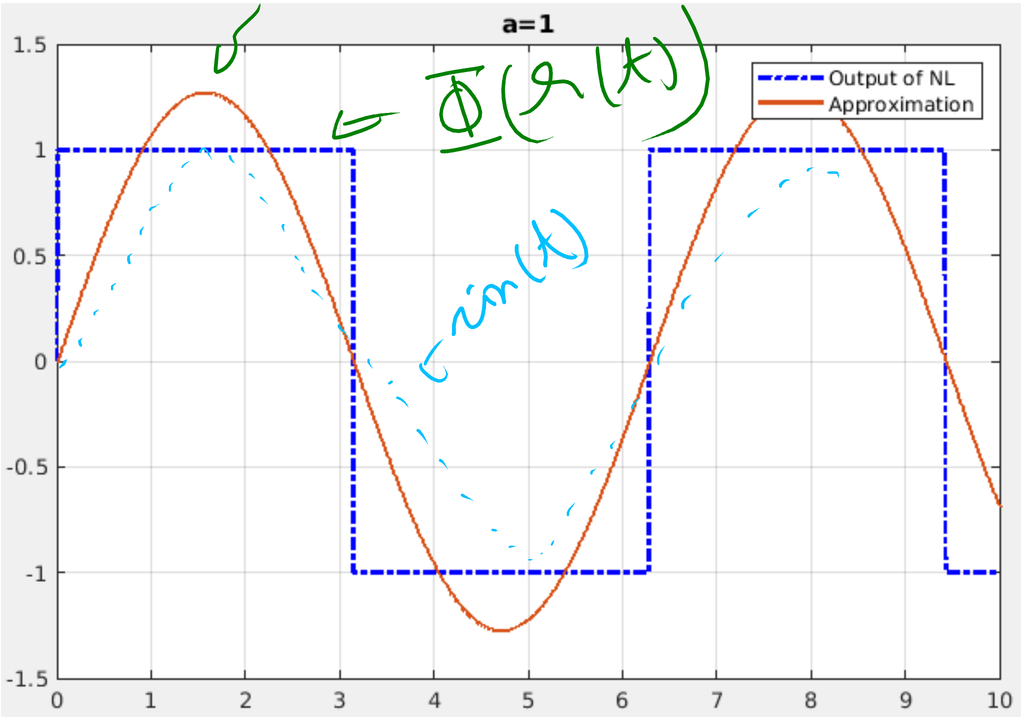
$y_{ss}(t) \simeq \eta(a,\omega)\, r(t)$

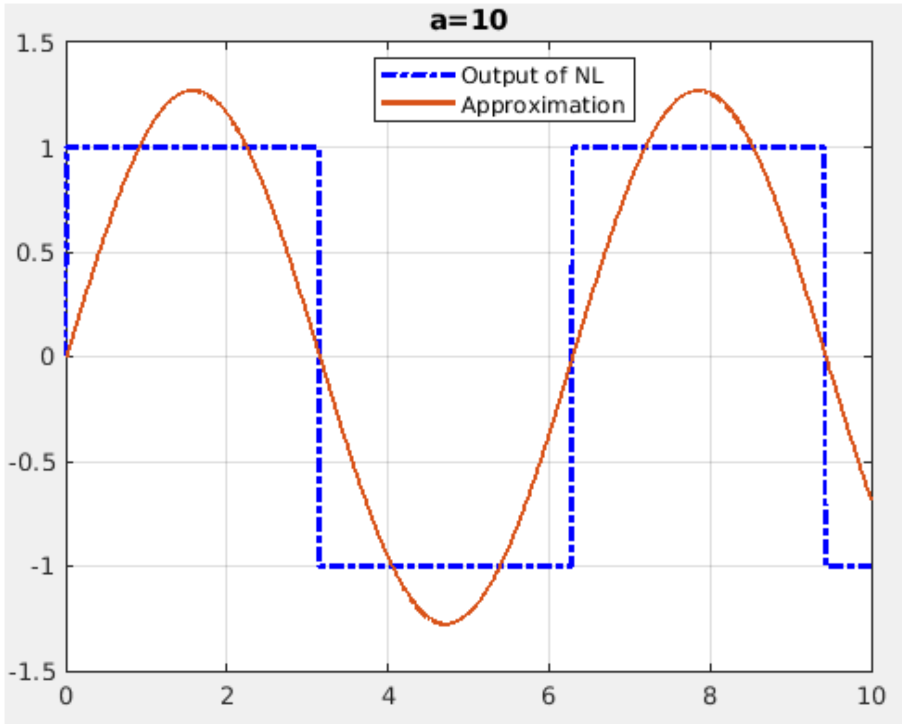
Constructing describing functions: Example

Describing function is: $\eta(a, \omega) = \dfrac{4}{\pi a}$



$\simeq \eta(a,\omega) r(t)$

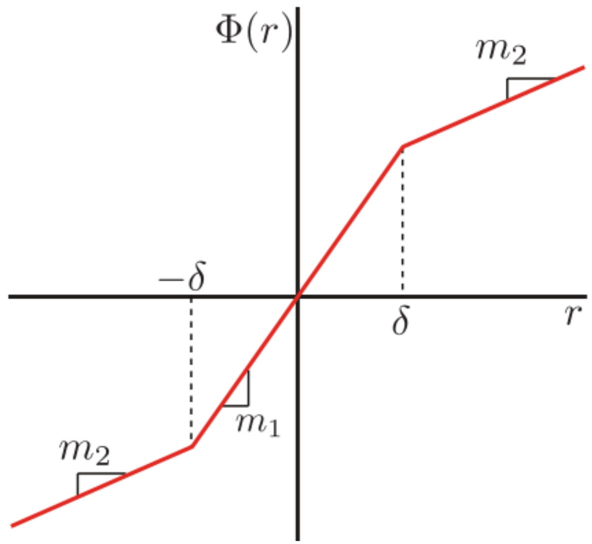




Constructing describing functions: Example

Piecewise Linear Memoryless System

Let the input be $r(t) = a\sin(\omega t)$.

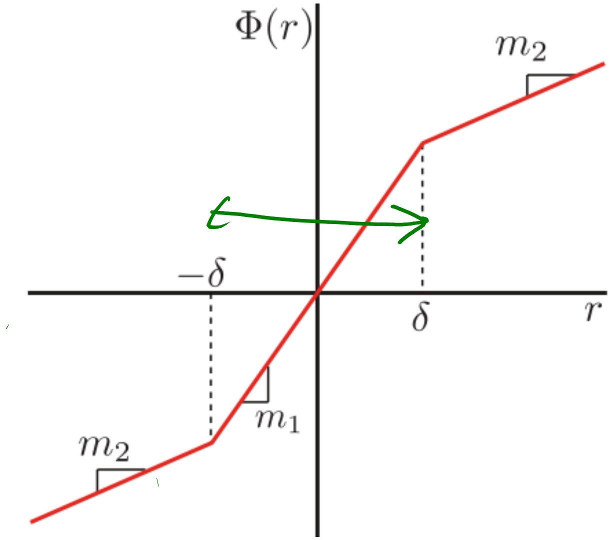


Constructing describing functions: Example

Piecewise Linear Memoryless System

Let the input be $r(t) = a\sin(\omega t)$.

If $|a| \leq \delta$, then $\eta(a, \omega) = m_1$ else



Constructing describing functions: Example

Piecewise Linear Memoryless System

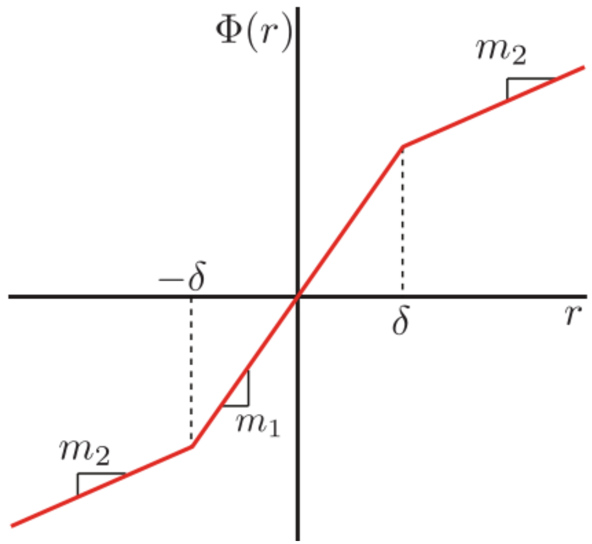
Let the input be $r(t) = a\sin(\omega t)$.

If $|a| \leq \delta$, then $\eta(a,\omega) = m_1$ else

$$\eta(a,\omega) = (m_1 - m_2) f(\delta/a) + m_2$$

where,

$$f(x) = \frac{2}{\pi}\left(\arcsin(x) + x\sqrt{1-x^2}\right)$$



Constructing describing functions: Example

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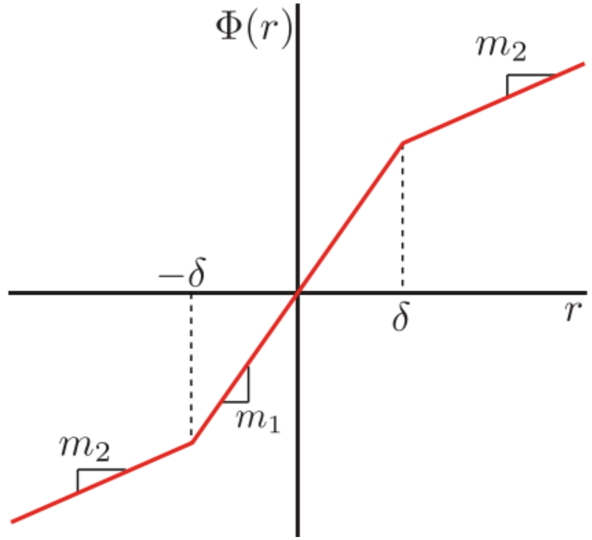

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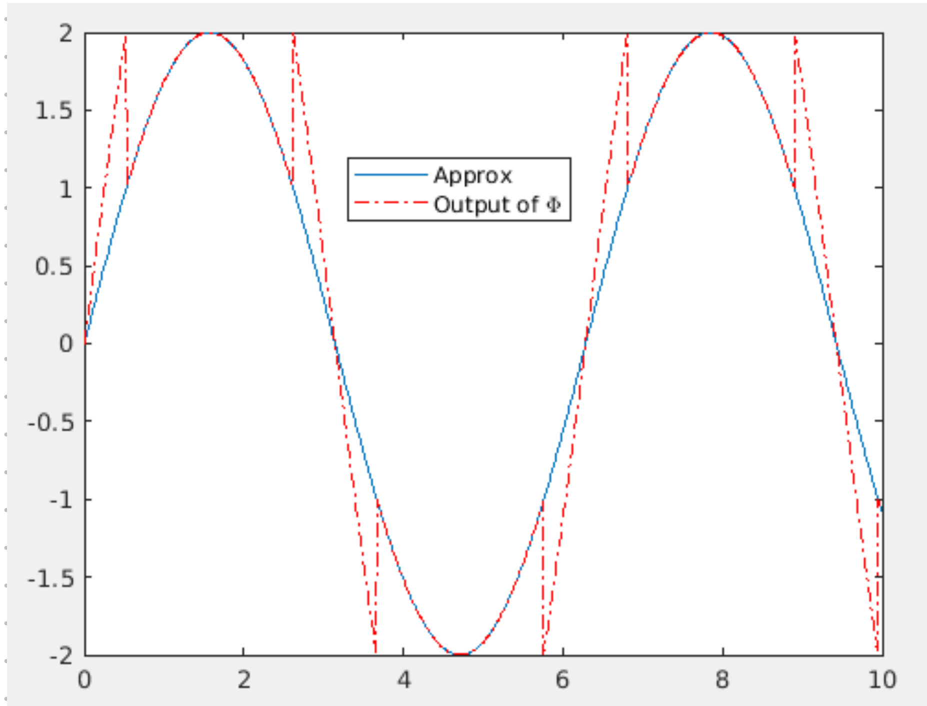
→ Indep -endent of ω

e.g., $m_1 = 2$, $m_2 = 1$, $\delta = 1$, $a = 2 \Rightarrow \eta = 0.6090$



Constructing describing functions: Example

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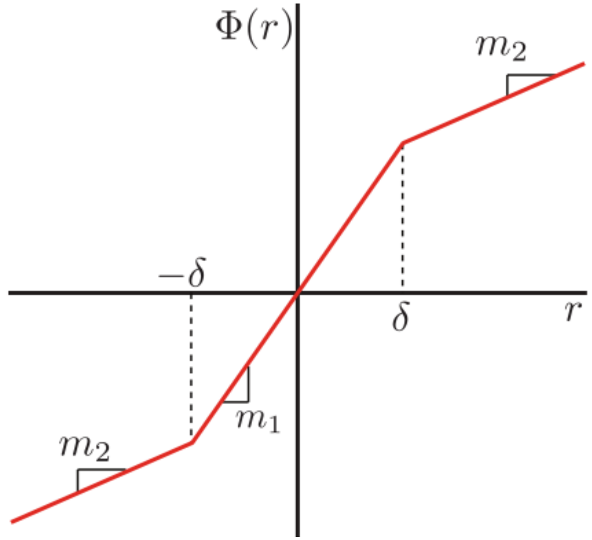
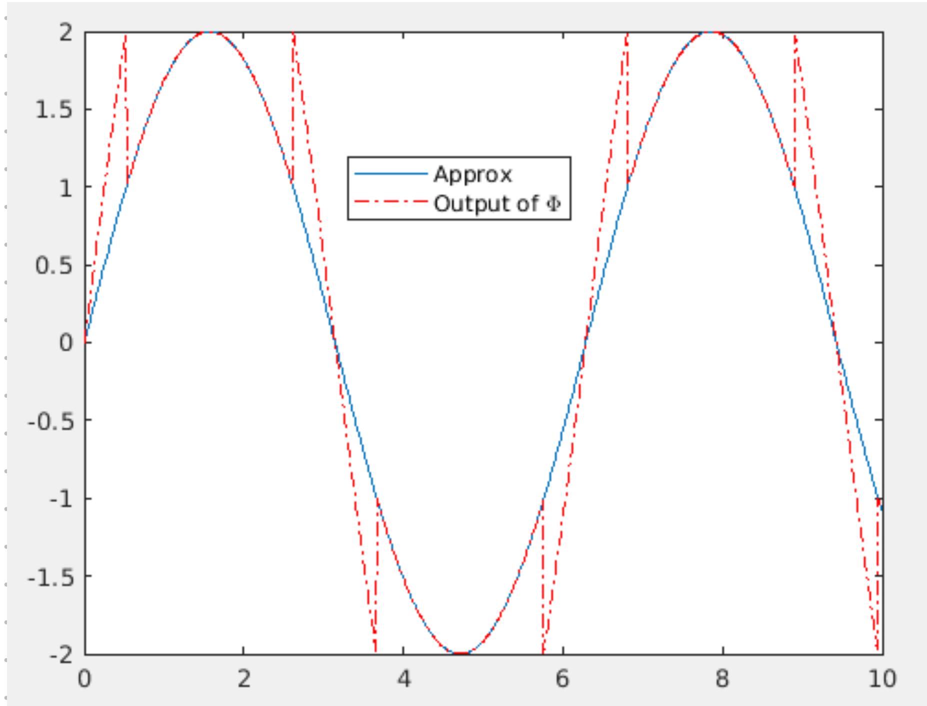


e.g., $m_1 = 2,\ m_2 = 1,\ \delta = 1,\ a = 2 \Rightarrow \eta = 0.6090$



Note: Saturator is a special case of this.

Also see Remark 5.5.8 in the course notes.

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[-] Introduction

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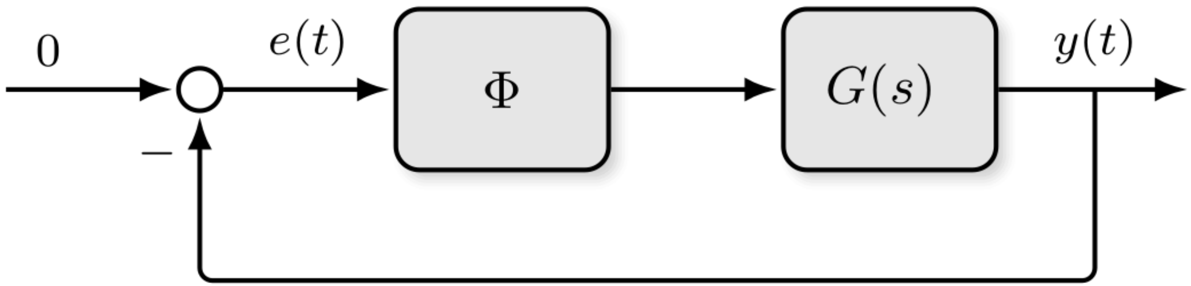
- [X] Periodic solutions and their stability

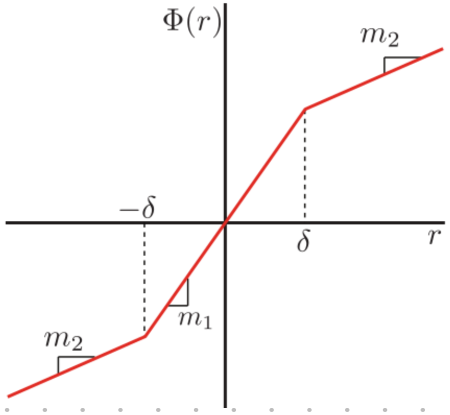
Periodic solutions of non-linear systems via describing functions

Assume G(s) is a BIBO stable LTI plant, and the non-linearity satisfies assumptions of Theorem 5.5.2.



Example:

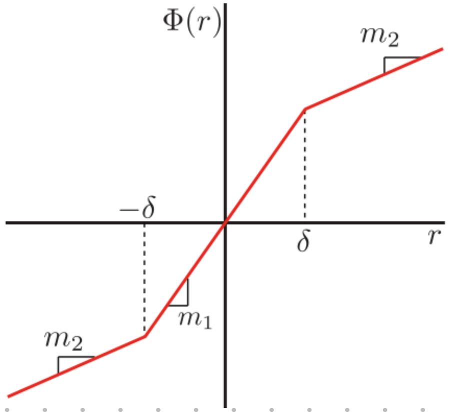
Φ is the saturator non-linearity with $m_1 = 4$, $m_2 = 0, \delta = 1.$

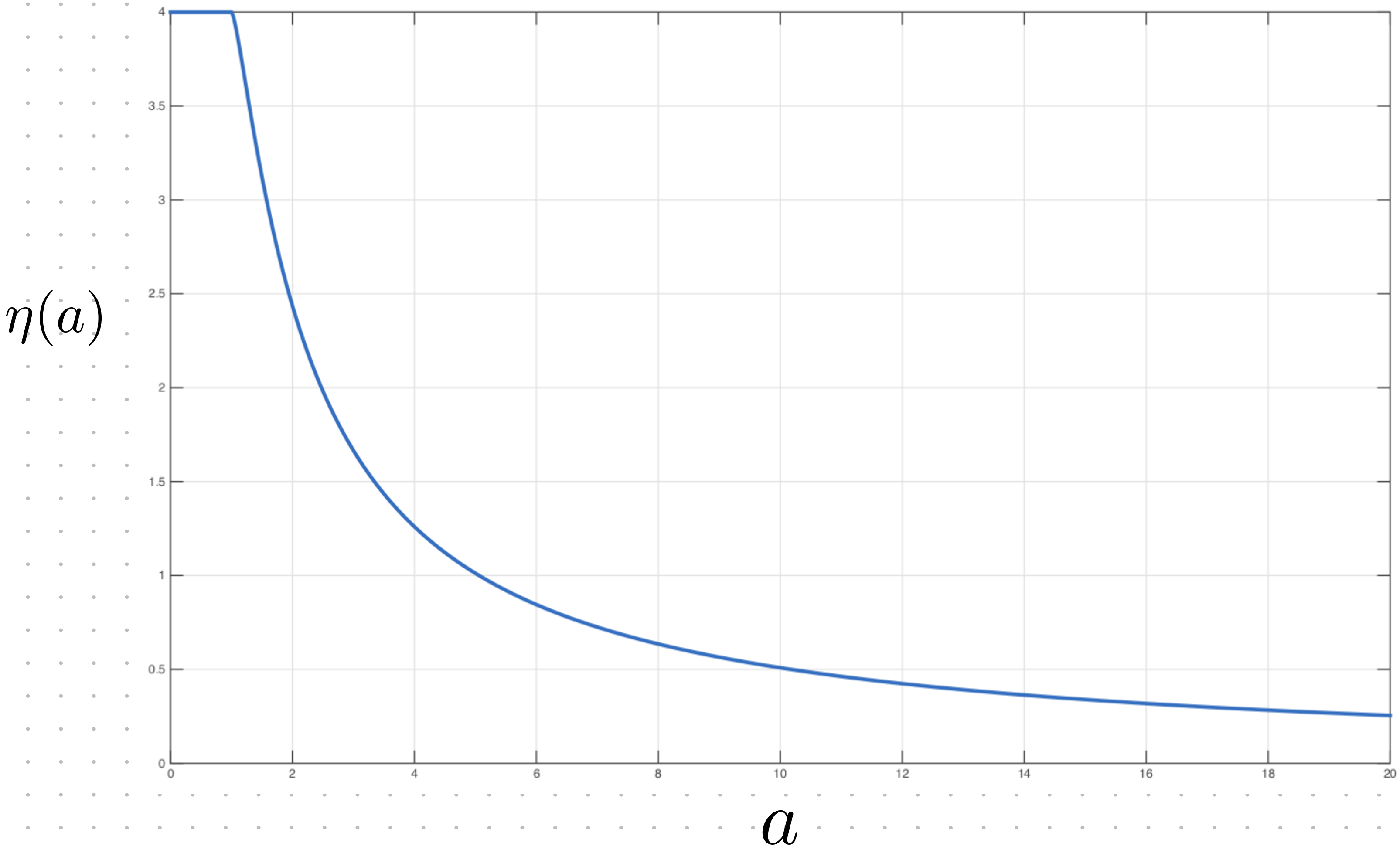


Example:

Φ is the saturator non-linearity with $m_1 = 4$, $m_2 = 0, \delta = 1$.

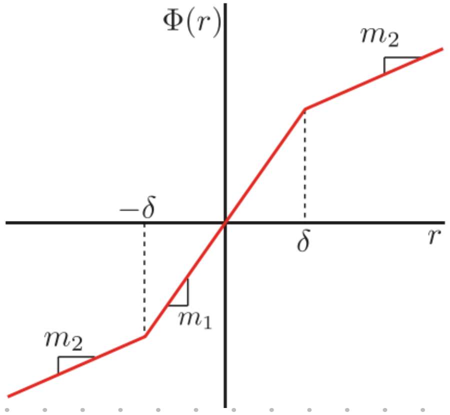
$$\eta(a,\omega) = \begin{cases} 4 & |a| \leq 1 \\ \frac{8}{\pi}\left(\arcsin(1/a) + \frac{1}{a}\sqrt{1-\frac{1}{a^2}}\right) & |a| > 1 \end{cases}$$





Example:

Φ is the saturator non-linearity with $m_1 = 4$, $m_2 = 0, \delta = 1$.



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Let the LTI plant be:

$$G(s) = \frac{(s+20)^2}{(s+1)(s+2)(s+3)}$$

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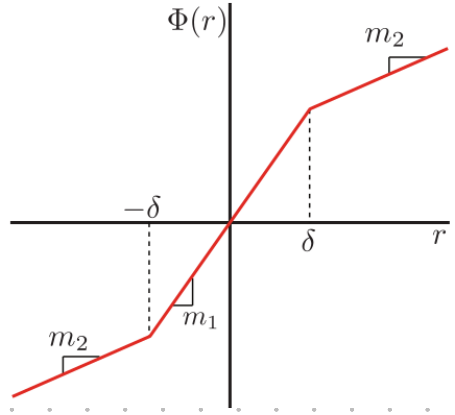

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We thus want to find values of a such that $G(j\omega) = -\dfrac{1}{\eta(a)}$

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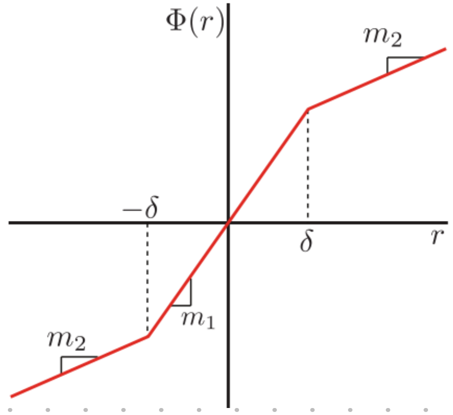

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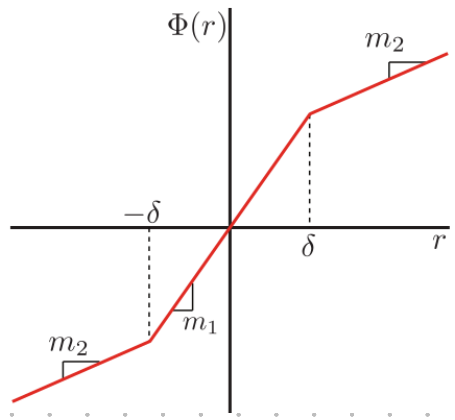

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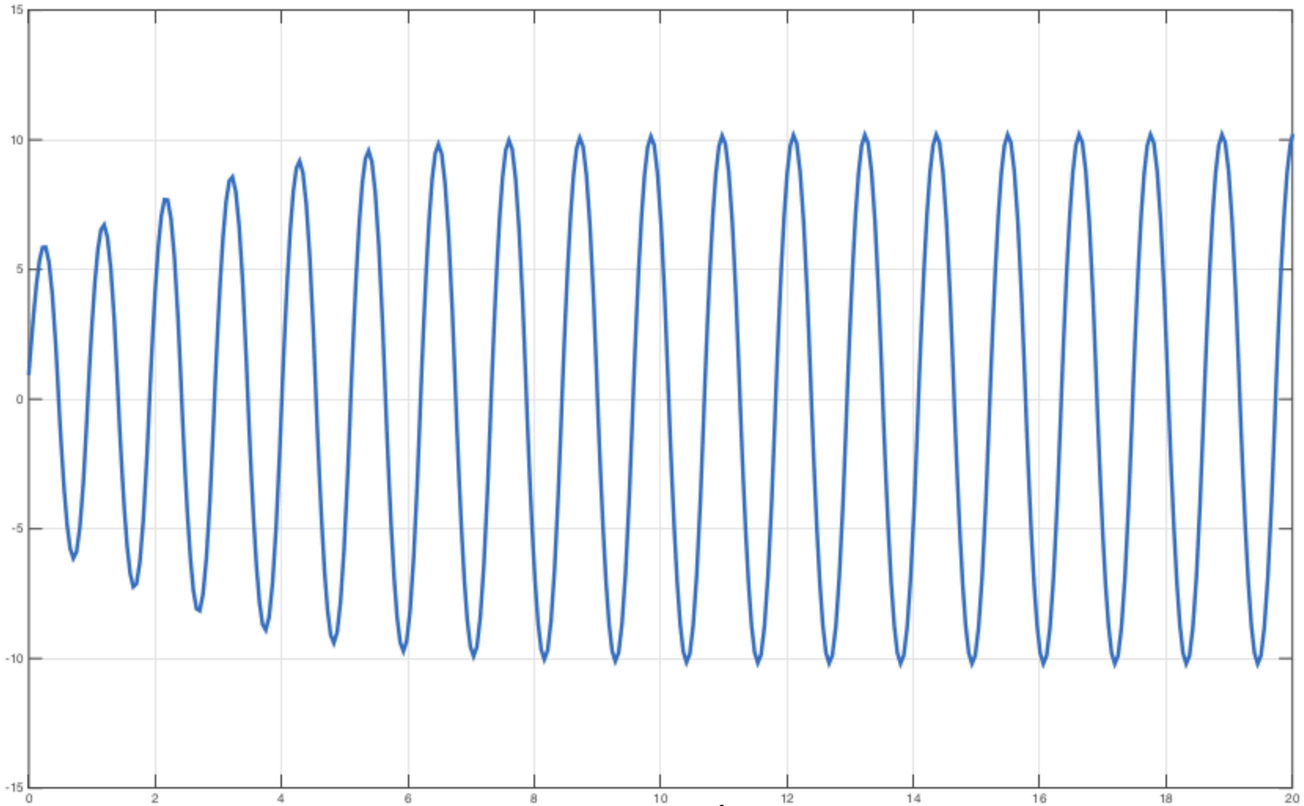
We can solve for a: $a_1 \approx 11.2, \qquad a_2 \approx 1.4$

Therefore we expect two periodic solutions $11.2\sin(5.4t)$ and $1.4\sin(11.9t)$

Example: Simulate the system

Non-zero initial condition.

$y(t)$



t

Actual solution is periodic and has amplitude $= 10.16$
and frequency around 5.56.
This is close to the predicted: $11.2 \sin(5.4t)$

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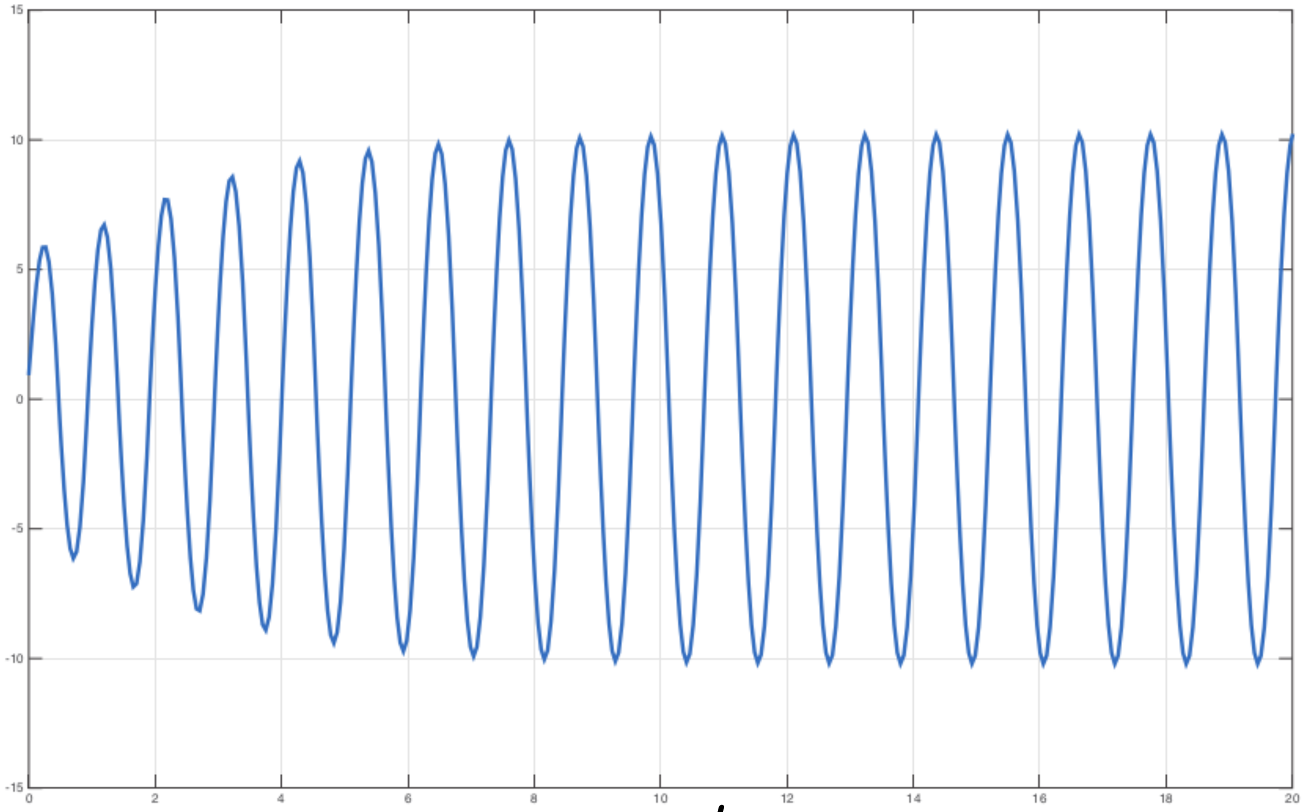


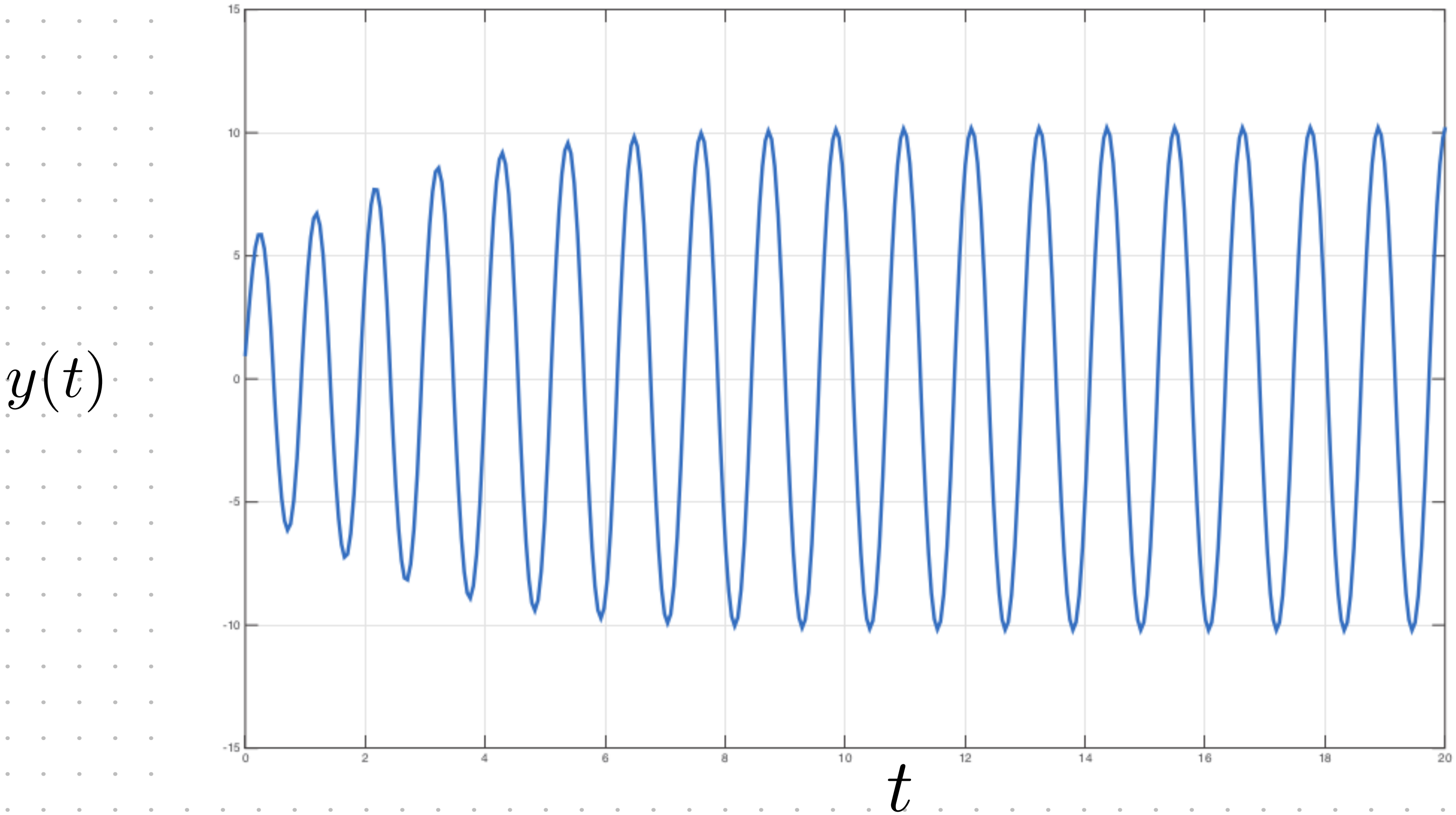
t

Actual solution is periodic and has amplitude $= 10.16$
and frequency around 5.56.
This is close to the predicted: $11.2 \sin(5.4t)$

The other computed periodic solution is not a stable one, i.e., if the system is perturbed, the response is no longer the computed periodic one (see Section 5.5.4 in text).

Example: Simulate the system

Non-zero initial condition.



Remark 5.5.9. Be careful using describing function analysis to predict the existence of periodic solutions. Describing function analysis may predict the existence of periodic solutions when there are none. A periodic solution may exist even if describing function analysis doesn't predict one. Finally, the predicted amplitude and frequency are only approximations and can be far from the true values. ♦

Outline

[-] Introduction

[-] State-space models

[-] Linearization
- [-] Equilibrium configuration
- [-] Transfer function from a linear state-space model
- [-] Control design using linearization

[-] Inverting static nonlinearities
- [-] Conditions for inverting static nonlinearities
- [-] Local cancellation of static nonlinearities
- [-] Inverting dynamic nonlinearities (feedback linearization), an example.

[-] Static friction
- [-] Dither signals
- [-] Deadzone non-linearity

[-] Describing functions
- [-] Optimal Quasi-Linearization
- [-] Constructing describing functions
- [-] Periodic solutions and their stability