Chapter 6: Discrete-time (DT) Linear Systems

ECE 481 - Digital Control Systems

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Based on course notes by Professor Chris Nielsen

Outline

[X] Difference equations

[ ] z-Transforms
- [ ] Properties of z-Transforms
- [ ] Final-value theorem for DT Linear Time Invariant (LTI) systems
- [ ] Solving difference equations

[ ] DT models of linear systems
- [ ] Transfer functions
- [ ] State-space models

[ ] Stability of DT systems

[ ] Stability of feedback (DT) systems
- [ ] Internal stability
- [ ] Input-output stability
- [ ] Identifying polymonials with stable roots

[ ] Frequency response

X = The upcoming topic

- = Topic that has been covered

Difference equations

Just like CT systems were represented as ODEs, DT systems are modeled as difference equations.

$$y[k] + a_1 y[k-1] + \cdots + a_n y[k-n] = b_0 u[k] + b_1 u[k-1] + \cdots + b_m u[k-m]$$

Where,
$k,\ n,\ m \in \mathbb{Z}_{\geq 0}$,
$u[k] \in \mathbb{R}$ is the input sequence,
$y[k] \in \mathbb{R}$ is the output sequence,
$a_i,\ b_i \in \mathbb{R}$ are constant coefficients

Example: 5-point moving average

$$y[k] = \frac{1}{5}\left(u[k] + u[k-1] + u[k-2] + u[k-3] + u[k-4]\right)$$

Impulse response:

DT impulse: $\delta[k] = 1$ if $k = 0$, else $\delta[k] = 0$.
Response of system to $u[k] = \delta[k]$ is the system's impulse response.

Question: What is the impulse response of the 5-point moving average ?

$$g[k] = \begin{cases} 1/5 & \text{if } 0 \leq k < 4 \\ 0 & \text{else} \end{cases}$$

Difference equations: Solving for the output sequence

Given input sequence and initial condition(s), how do we solve for the output sequence?

Similar to the response of a constant coefficient ODE (1st order), we can write the system response as:

$$y = y_n + y_f$$

Natural response/ zero-input response

Forced response

Natural response is the general solution to the homogenous equation: $(u[k] = 0 \ \forall k)$

$$y[k] + a_1 y[k-1] + \cdots + a_n y[k-n] = 0$$

Forced response (zero-state response) is obtained by solving:

$$y[k] + a_1 y[k-1] + \cdots + a_n y[k-n] = b_0 u[k] + b_1 u[k-1] + \cdots + b_m u[k-m]$$

while assuming: $y[k] = 0,$ $\cancel{u[k] = 0 \forall k < 0}$

Natural response of a difference equation

Recall that ODEs with constant coefficients have a trial solution that is the complex exponential $e^{\lambda t}$

For example: $\ddot{y} + 3\dot{y} - 2y = 0$

Try $y(t) = \mathrm{e}^{\lambda t}$: $\lambda^2 \mathrm{e}^{\lambda t} + 3\lambda \mathrm{e}^{\lambda t} - 2\mathrm{e}^{\lambda t} = 0$

Holds for all t iff: $\lambda^2 + 3\lambda - 2 = 0 \Rightarrow \lambda = \{-1, -2\}$

This ODE has a general form for the natural response: $y_n(t) = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-2t}$

Similarly for constant coefficient difference equations, we have the trial solution: $y_n[k] = \lambda^k$

For example: $y[k] - y[k-1] - 4y[k-2] + 4y[k-3] = 0$

Substitute $y[k] = \lambda^k$: $\lambda^k - \lambda^{k-1} - 4\lambda^{k-2} + 4\lambda^{k-3} = 0$

This holds for all k iff: $\lambda^3 - \lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda = \{1, 2, -2\}$

Therefore the general form for the natural response of this difference equation is:

$$y_n[k] = c_1(1)^k + c_2 2^k + c_3(-2)^k$$

Next, we'll see how to get the forced response.

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z-Transforms

Let $x[k] \in \mathbb{R}$ (DT), one-sided ZT is:

$$X[z] := \sum_{k=0}^{\infty} x[k]\, z^{-k} \quad , \; z \in \mathbb{C}$$

Not all DT sequences have a ZT, sufficient condition for existence:

$$|x[k]| \leq M \rho^{k} \quad , \; k \geq 0 ,$$

$\uparrow$ Growth bound — here, $M, \rho > 0$ (constants)

$\Rightarrow X[z]$ is bounded.

To see this, let's write $z = re^{j\theta}$, $r > \rho$. Then:

$$|X[z]| = \left| \sum_{k=0}^{\infty} x[k] \left(re^{j\theta}\right)^{-k} \right| \leq \sum_{k=0}^{\infty} \left| x[k] \left(re^{j\theta}\right)^{-k} \right|$$

(Triangle inequality)

$$= \sum_{k=0}^{\infty} |x[k]| r^{-k}$$

Assumption (on suff. condition) that $|x[k]| \leq M \rho^k$

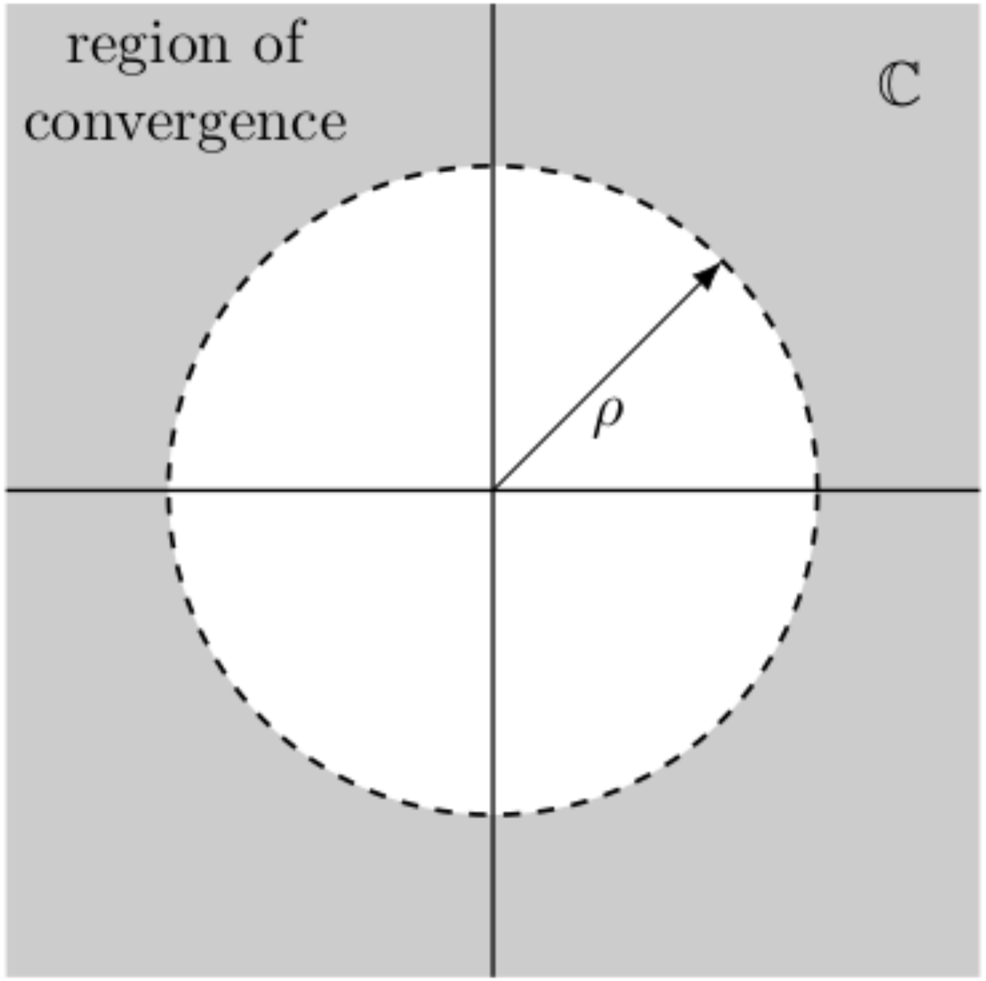
$$\leq M \sum_{k=0}^{\infty} (\rho/r)^k = M \frac{1}{1-\rho/r} \quad \text{since } \rho < r$$

$$< \infty$$

R.O.C of ZT defined by smallest 'ρ' that is a growth bound.

z-Transforms: Region of convergence

The ROC for this series includes the exterior of the disk $|z| > \rho$



Example 6.2.2

Consider the series $x[k] = a^k, \qquad k \geq 0.$

Take its z-Transform:

$$X[z] = \mathscr{Z}\left(a^k\right) = \sum_{k=0}^{\infty} (az^{-1})^k = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \qquad |z| > |a|$$

Special case: If a=1, this is the z-transform of the unit step sequence 1[k].

Example 6.2.3

Consider the series $x[k] = k^k, \qquad k \geq 0.$

Take its z-Transform:

$$X[z] = \sum_{k=0}^{\infty} \left(\frac{k}{z}\right)^k = \lim_{j \to \infty} \sum_{k=0}^{j} \left(\frac{k}{z}\right)^k$$

Summation does not converge for any $z \in \mathbb{C}$

The signal here grows too fast and does not have the growth bound of the form $|x[k]| \leq M\rho^k, \qquad k \geq 0$

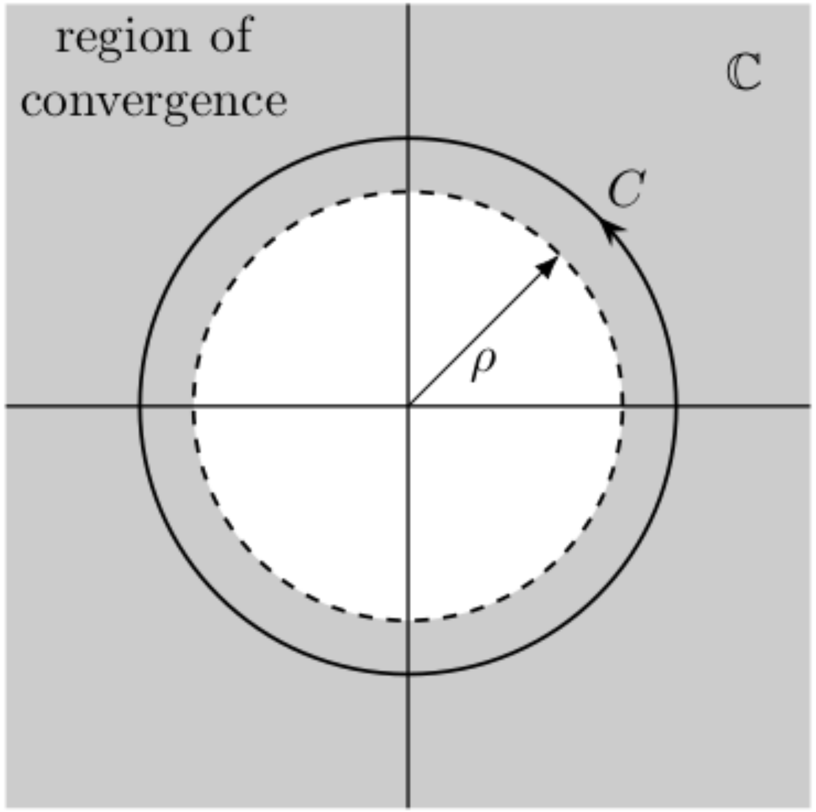
(which is just a sufficient condition but not a necessary one, so is not a proof of lack of existence of Z TF)

Inverse z-Transform

Given X[z], suppose we want to find x[k]. The inversion formula is:

$$x[k] = \frac{1}{2\pi j} \oint_C X[z] z^{k-1} \, \mathrm{d}z$$

The integral is a contour integral in the complex plane around a circle C in the ROC



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Given X[z], suppose we want to find x[k]. The inversion formula is:

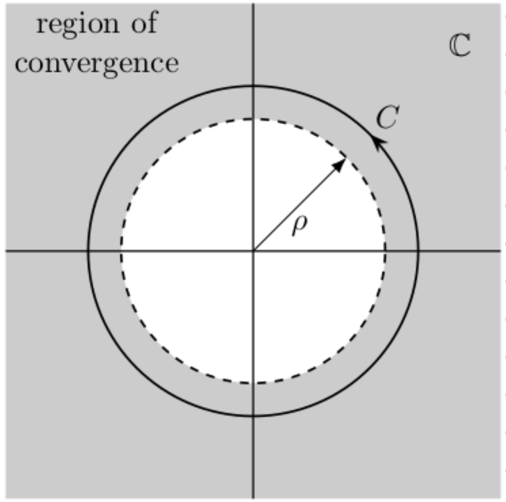
$$x[k] = \frac{1}{2\pi j} \oint_C X[z] z^{k-1} \, \mathrm{d}z$$

In particular, select $z = re^{j\theta}$

$$x[k] = \frac{r^k}{2\pi} \int_0^{2\pi} X[r\mathrm{e}^{j\theta}] \mathrm{e}^{jk\theta} \, \mathrm{d}\theta$$

Note: The ROC is important if we want to use the above formula to compute the z-transform.

Example



Consider the z-transform of 1.3^k, $(k \geqslant 0)$

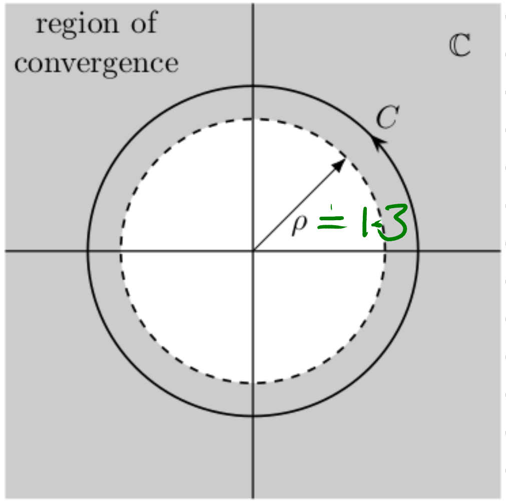
$$X[z] = \frac{z}{z - 1.3}, \qquad \text{ROC} : \{z \in \mathbb{C} : |z| > 1.3\}$$

If $z = re^{j\theta}$, the inversion formula yields:

$$x[k] = \frac{1}{2\pi j}\int_0^{2\pi} X[re^{j\theta}]\left(re^{j\theta}\right)^{k-1} rje^{j\theta}\mathrm{d}\omega = \frac{r^k}{2\pi}\int_0^{2\pi} X[re^{j\theta}]e^{jk\theta}\mathrm{d}\theta$$

What value of r should we use to evaluate this integral?

Example



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With r > 1.3, the circle traced out is inside the ROC. For r=2, we get the original sequence:

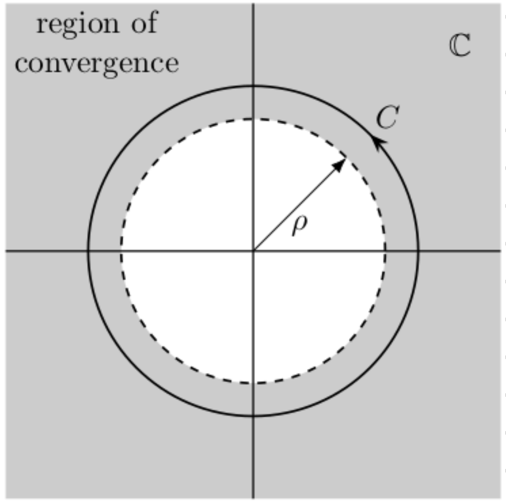
$$x[k] = \begin{cases} 1.3^k & k \geq 0 \\ 0 & k < 0 \end{cases}$$

If we chose r = 1 (resulting contour is not in the ROC), then we get:

$$x[k] = \begin{cases} 0 & k \geq 0 \\ -1.3^k & k < 0 \end{cases}$$

This is not the original sequence! Takeaway is that the ROC must be clearly specified otherwise we might not recover the original sequence.

Example



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$$x[k] = \begin{cases} 0 & k \geq 0 \\ -1.3^k & k < 0 \end{cases}$$

Note: For digital implementations, sequences are normally defined for k>=0, and that gives us z-transforms with a ROC that is the exterior of a disk with a sufficienty large radius (enough to exclude poles of X[z]).

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Properties of z-transforms

Properties of z-transforms

Let $x[k]$ and $y[k]$ be real-valued sequences that are zero for $k < 0$, let a be a real constant and let $m \geq 0$ be an integer.

(i) $\mathscr{Z}\{x[k] + y[k]\} = \mathscr{Z}\{x[k]\} + \mathscr{Z}\{y[k]\}$ (superposition)

(ii) $\mathscr{Z}\{ax[k]\} = a\mathscr{Z}\{x[k]\}$ (homogeneity)

Linearity

(iii) $\mathscr{Z}\{x[k-m]\} = z^{-m}\mathscr{Z}\{x[k]\} + \overbrace{x[-m] + z^{-1}x[-m+1] + \cdots + z^{-m+1}x[-1]}^{\text{For init conditions}}$ (backward shift)

(iv) $\mathscr{Z}\{x[k+m]\} = z^m\mathscr{Z}\{x[k]\} - \left(z^m x[0] + z^{m-1}x[1] + \cdots + zx[m-1]\right)$ (forward shift)

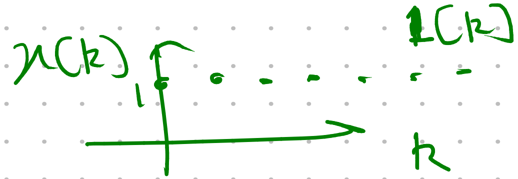
(v) $\mathscr{Z}\{x * y\} = \mathscr{Z}\{x[k]\}\,\mathscr{Z}\{y[k]\}$ (convolution)

(vi) $\mathscr{Z}\{a^k x[k]\} = X[z/a]$. (multiplication by a^k)

See any Signals and Systems book for details, e.g., the one by B. P. Lathi.

Table of z-transforms of popular signals

One-sided z-transforms:



Description	Time-domain $x[k]$	z-Domain $X[z]$
Unit step	$\mathbf{1}[k]$	$\frac{z}{z-1}$
Impulse	$\delta[k]$	1
Ramp	k	$\frac{z}{(z-1)^2}$
Exponential	a^k	$\frac{z}{z-a}$
Exponential I	ka^k	$\frac{az}{(z-a)^2}$
Exponential II	k^2a^k	$\frac{az(z+a)}{(z-a)^3}$
Sine	$\sin(\theta k)$	$\frac{z\sin(\theta)}{z^2-2\cos(\theta)z+1}$
Cosine	$\cos(\theta k)$	$\frac{z(z-\cos(\theta))}{z^2-2\cos(\theta)z+1}$
Growing/decaying sine	$a^k\sin(\theta k)$	$\frac{za\sin(\theta)}{z^2-2a\cos(\theta)z+a^2}$
Growing/decaying cosine	$a^k\cos(\theta k)$	$\frac{z(z-a\cos(\theta))}{z^2-2a\cos(\theta)z+a^2}$

Example: Computing inverse transforms via the table (and partial fractions)

Similar to what we have been doing for Laplace transforms.

Consider the z-transform: $X[z] = \dfrac{z+1}{(z-10)(z+4)}$

Trick: Write down the partial fraction expansion of X[z]/z to get terms that can be inverted (via the table)

$$\frac{X[z]}{z} = \frac{z+1}{z(z-10)(z+4)} = \frac{c_1}{z} + \frac{c_2}{z-10} + \frac{c_3}{z+4}$$

We have,

$$c_1 = \cancel{z} \cdot \frac{(z+1)}{\cancel{z}(z-10)(z+4)}\bigg|_{z=0} = -1/40$$

$$c_2 = (z-10) \cdot \frac{(z+1)}{z(z-10)(z+4)}\bigg|_{z=10} = 11/140$$

$$c_3 = (z-4)\,\frac{(z+1)}{z(z-10)(z+4)}\bigg|_{z=-4} = -3/56$$

$$\Rightarrow X[z] = -\frac{1}{40} + \frac{11}{140}\,\frac{z}{z-10} + \left(-\frac{3}{56}\right)\cdot\frac{z}{z+4}$$

Inv. ZT

$$x[k] = -\frac{1}{40}\,\delta[k] + \frac{11}{140}\,10^k - \frac{3}{56}(-4)^k, \quad k \geq 0$$

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Final value theorem for DT signals

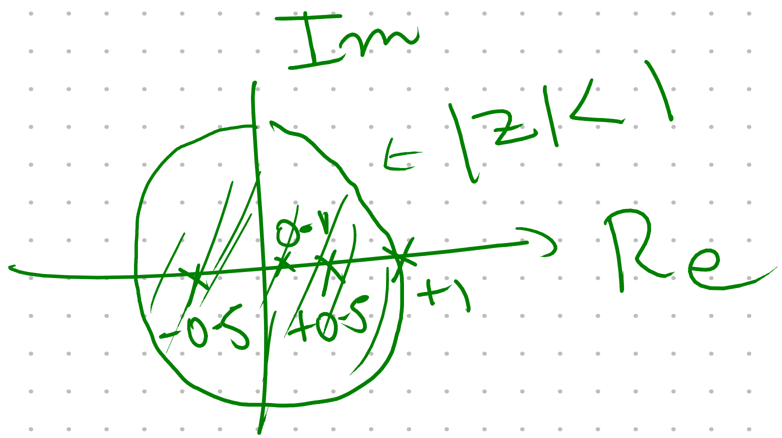
Theorem 6.2.1 (Final-value theorem). *Let $x[k]$ be a signal defined for $k \geq 0$ and let its z-transform $X[z]$ be rational and proper.*

(a) If $X[z]$ has all its poles in $|z| < 1$, then $x[k]$ converges to zero as $k \to \infty$.

(a) If $X[z]$ has all its poles in $|z| < 1$ except for a simple pole at $z = 1$, then

$$\lim_{k \to \infty} x[k] = \lim_{z \to 1} (z - 1) X[z]. \tag{6.4}$$

(c) In all other cases, $x[k]$ does not approach a constant as $k \to \infty$.



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(c) In all other cases, $x[k]$ does not approach a constant as $k \to \infty$.

Compare this to the CT version of FVT:

Theorem 2.5.1 (Final-value theorem). *Let $f(t)$ be a signal defined for $t \geq 0$ and let its Laplace transform $F(s)$ be rational and proper.*

(a) If $F(s)$ has all its poles in $\mathbb{C}^-$, then $f(t)$ converges to zero as $t \to \infty$.

(b) If $F(s)$ has all its poles in $\mathbb{C}^-$ except for a simple pole at $s = 0$, then

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s). \tag{2.4}$$

(c) In all other cases, $f(t)$ does not approach a constant as $t \to \infty$.

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Solving difference equations by z-transforms

Putting together what we've seen in this chapter so far (and just like we did for ODEs via LTs).

Forced response

Example system that also requires us to compute a forced response:

$$y = y_n + y_f$$

$$y[k] - 2y[k-1] = u[k], \quad (1) \qquad y[-1] = 1, \quad u[k] = k\mathbf{1}[k]$$

Natural response

Natural response: y_n will be of the form $c\lambda^k$

Into (1) $\Rightarrow$ $c\lambda^k - 2c\lambda^{k-1} = 0$ {set $u[k]=0$}

$\Rightarrow \lambda - 2 = 0 \Rightarrow \lambda = 2$

$$y_n[k] = c\,2^k$$

Forced response: $u[k] = k\cdot\mathbf{1}[k]$, $U[z] = \dfrac{z}{(z-1)^2}$

Take Z-TF of (1) w/ zero init. conditions:

$$Y - 2z^{-1}Y = U$$

$$\Rightarrow Y(1-2z^{-1}) = U \Rightarrow Y = \frac{z}{z-2}\cdot U$$

$$Y[z] = \frac{z}{z-2} \cdot \frac{z}{(z-1)^2}$$

Via partial fraction expansion:

$$\frac{Y[z]}{z} = \frac{C_1}{z-2} + \frac{C_2}{z-1} + \frac{C_3}{(z-1)^2}$$

Solve, $C_1 = 2$, $C_2 = -2$, $C_3 = -1$

$$Y[z] = 2\frac{z}{z-2} - 2\frac{z}{z-1} - \frac{z}{(z-1)^2}$$

Inv. Z-TF

$$\Rightarrow \quad y_f[k] = 2(2)^k - 2(1)^k - k, \quad k \geq 0.$$

Total response $y = y_n + y_f$

$$y[k] = (c+2)2^k - 2(1)^k - k, \quad k \geq 0 \quad -(2)$$

We can't $y[-1] = 1$ to find 'c' (alone).

Find $y[0]$ that is consistent with $u(k)$ & (1).

Let's use (1) $\Rightarrow y[0] - 2y[-1] = u[0] \quad (k=0)$

$$\Rightarrow y[0] = 2y[-1] + 0 = +2$$

Putting into (2) $\Rightarrow 2 = (c+2) \cdot 1 - 2 - 0 \quad (k=0)$

$$\Rightarrow c = 2$$

$$\therefore y[k] = 4 \cdot 2^k - 2 \cdot 1^k - k, \quad k \geq 0$$

Solving difference equations by z-transforms

Exercise: Re-solve the example simply via z-transforms, this time not setting initial conditions to 0.

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DT Transfer functions

Recall: Transfer function representations exist only for LTI systems (CT or DT). Consider:

$$y[k] + a_1 y[k-1] + \cdots + a_n y[k-n] = b_0 u[k] + b_1 u[k-1] + \cdots + b_m u[k-m]$$

OUTPUT SEQUENCE

INPUT SEQUENCE

Take Z-TF w/ zero-initial conditions:

$$Y + z^{-1} a_1 Y + z^{-2} a_2 Y + \cdots + z^{-n} a_n Y$$

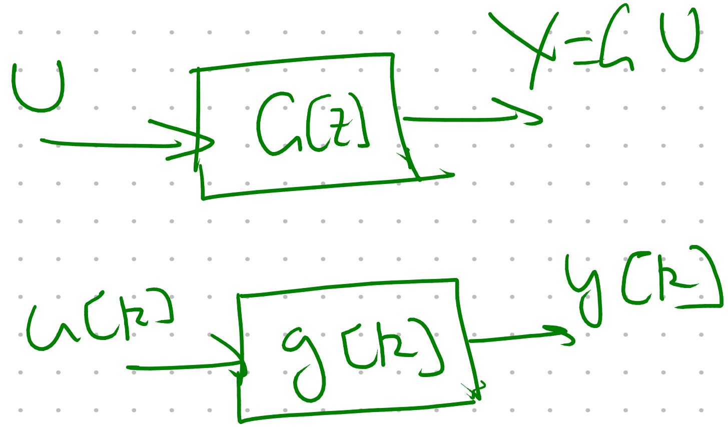
$$= b_0 U + b_1 z^{-1} U + \cdots + b_m z^{-m} U$$

$$\Rightarrow G[z] = \frac{Y}{U} = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

DT Transfer function: Impulse response

$$g[k] = \mathcal{Z}^{-1}\{G[z]\}$$

Impulse resp. TF



$$y[k] = \sum_{m=0}^{k} g[k-m]\,u[m]$$

Convolution in DT.

DT Transfer function and FVT: Example 6.4.2

Consider $\frac{Y}{U} = \frac{z+0.35}{(z-0.5)(z+0.5)(z-0.1)}$, $u[k] = 1[k]$

$$Y[z] = G[z] \cdot U[z] = \frac{z+0.35}{(z-0.5)(z+0.5)(z-0.1)} \cdot \frac{z}{(z-1)}$$

$$\lim_{k\to\infty} y[k] = \lim_{z\to 1} (z-1)Y(z) = \lim_{z\to 1} z\,G[z] = G(1)$$

$$= 2$$

Two forms for DT transfer functions

1) Positive power form,

e.g., $$G[z] = \frac{8z^3 - 24z - 16}{2z^5 + 20z^4 + 98z^3 + 268z^2 + 276z} \quad \text{—} (3)$$

2) Negative power form,

e.g., $$G[z^{-1}] = \frac{8z^{-2} - 24z^{-4} - 16z^{-5}}{2 + 20z^{-1} + 98z^{-2} + 268z^{-3} + 276z^{-4}} \quad \text{—} (4)$$

(3) & (4) are the same thing.

Divide (3) by z^{+5} (& multiply)

Rational TFs and difference equations

Example 6.4.3: Consider $\frac{Y}{U} = \frac{z^2+2z}{2z^3+z^2-0.4z+0.8}$

$$= \frac{z^{-1}+2z^{-2}}{2+z^{-1}-0.4z^{-2}+0.8z^{-3}}$$

$$\Rightarrow (2+z^{-1}-0.4z^{-2}+0.8z^{-3})Y = (z^{-1}+2z^{-2})U$$

Take inv. Z-Transform:

$$2\cdot y[k]+y[k-1]-0.4y[k-2]+0.8y[k-3]$$

$$= u[k-1]+2u[k-2].$$

Note: NPF preferred, consider $\mathcal{Z}\{x[k+1]\} = zX - x[0]$

Need to know in PPF.

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DT state-space models

General DT state-space model: $x[k+1] = f(x[k], u[k]),$

$$y[k] = g(x[k], u[k])$$

here, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$. $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$

$g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$

Special case (LTI):

$$x[k+1] = Ax[k] + Bu[k]$$

$$y[k] = Cx[k] + Du[k]$$

Next state

(f, g linear & TI in x & u)

$A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$

$C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$

DT State-space model from difference equation

Example 6.5.1: $y[k] - 0.5\,y[k-1] + 1.5\,y[k-2] = 0.5\,u[k-2]$

Re-write,

$[y, u \in \mathbb{R}]$ $\quad y[k+2] - 0.5\,y[k+1] + 1.5\,y[k] = 0.5\,u[k]$ $\leftarrow$ (S)

State vector: $x[k] = \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} := \begin{bmatrix} y[k] \\ y[k+1] \end{bmatrix}$

At the next time step: $x_1[k+1] = x_2[k]$

(From (S)) $\quad x_2[k+1] = -1.5\,x_1[k] + 0.5\,x_2[k] + 0.5\,u[k]$

$y[k] = x_1[k] = \underbrace{[1 \;\; 0]}_{C} x[k]$

$$\overbrace{\begin{bmatrix} x_1[k+1] \\ x_2[k+1] \end{bmatrix}}^{x[k+1]} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1.5 & 0.5 \end{bmatrix}}_{A} \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0.5 \end{bmatrix}}_{B} u[k]$$

DT State-space model from difference equation: Relative degree less than n

More (delayed) inputs show up in difference equation/TF.

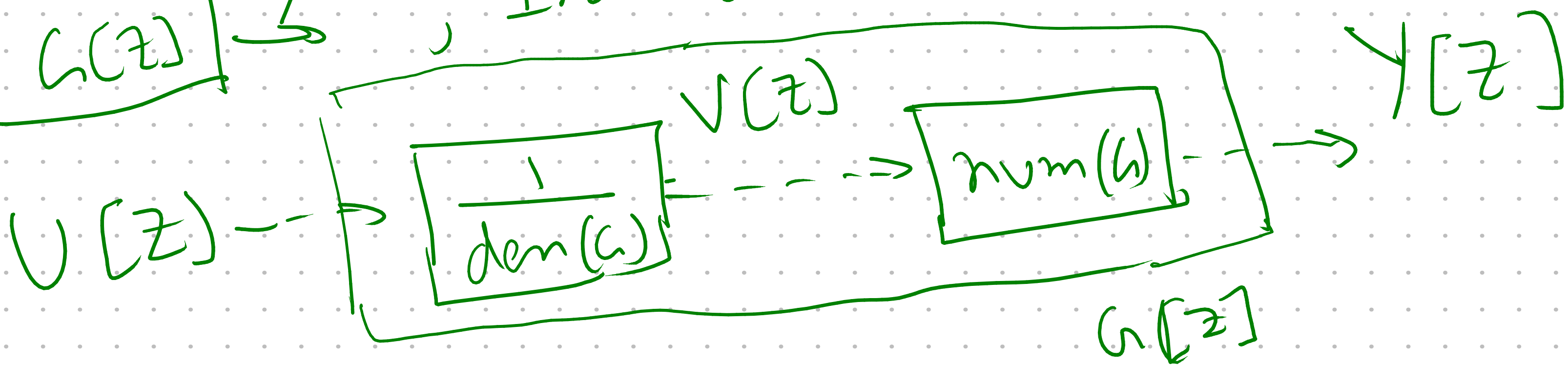
Example 6.5.2: $y[k] - 0.5y[k-1] + 1.5y[k-2] = u[k-1] + 2u[k-2]$

Take Z-TF:

$$Y[z] = \frac{z^{-1} + 2z^{-2}}{1 - 0.5z^{-1} + 1.5z^{-2}} \cdot U[z]$$

$$= \underbrace{\frac{z+2}{z^2 - 0.5z + 1.5}}_{G(z)} U[z]$$

, Introduce intermediate signal $v[k]$



$$V[z] = \frac{1}{den(h)} \cdot U[z] = \frac{1}{z^2 - 0.5z + 1.5} \cdot U[z]$$

Relative degree of 2

$$Y[z] = Num(h)\, V[z] = (z+2)\, V[z]$$

Inv. Z-TF w/ 0 init conditions:

$$\Rightarrow v[k+2] - 0.5\, v[k+1] + 1.5\, v[k] = u[k] \quad (1)$$

$$y[k] = v[k+1] + 2v[k]$$

Define states, $x[k] = (x_1[k], x_2[k]) := [v[k], v[k+1]]$

$$x_1[k+1] = x_2[k],$$

$$x_2[k+1] = -1.5\, x_1[k] + 0.5\, x_2[k] + u[k] \quad \text{(From (1))}$$

$$y[k] = 2x_1[k] + x_2[k]$$

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ -1.5 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y[k] = \begin{bmatrix} 2 & 1 \end{bmatrix} x[k] + [0] u[k]$$

$x \in \mathbb{R}^2$

We had:

$$x_1[k+1] = x_2[k],$$

$$x_2[k+1] = -1.5 x_1[k] + 0.5 x_2[k] + u[k]$$

(From ①)

$$y[k] = 2x_1[k] + x_2[k]$$

Generalizing: TF2SS (SISO)

Consider the difference equation: $y[k] + a_1 y[n-1] + \cdots + a_n y[k-n] = b_1 u[k-1] + \cdots + b_m u[k-m]$

Assume n = m (without loss of generality), and we have the corresponding TF:

Z/TF w/ 0-init cond.

$$G[z] = \frac{b_1 z^{n-1} + b_2 z^{n-2} + \cdots + z b_{n-1} + b_n}{z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n}$$

Using the process in the previous slide(s), we get the resulting state-space model with matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} b_n & \cdots & \cdots & b_1 \end{bmatrix}, \qquad D = 0.$$

CONTROLLABLE CANONICAL FORM.

Note: This also applies to CT systems.

State-space models for proper (but not strictly) DT transfer functions

Such systems have a feedforward component. Example 6.5.4:

$$\frac{Y[z]}{U[z]} = \frac{3z^3 - z^2 + 2z - 6}{z^3 + 2z^2 - 7z} = G[z]$$

Divide the numerator by the denominator and re-write the TF:

$$\frac{Y}{U} = \underbrace{3} + \frac{-7z^2 + 23z - 6}{z^3 + 2z^2 - 7z} \quad G_1[z]$$

strictly proper

show up in $y = Cx + Du$ $(D = 3)$

← Feed forward

See example 6.5.4 in course notes.

If TF is not proper, it is not 'realizable', i.e., there's no corresponding SS model.

Similarity transformations

Recall (from CT chapters) that the state-space realization is not unique. Why?

Consider a realization: $\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases}$ $\equiv H(s)$

Trying to write

Let T be a n x n invertible matrix and define a new (transformed) state vector $\hat{\mathbf{x}} = \mathbf{T}\mathbf{x}$

We now have the model: $\dot{\hat{\mathbf{x}}} = \mathbf{T}\dot{\mathbf{x}} = \mathbf{TAx} + \mathbf{TBu} = \underbrace{\mathbf{TAT}^{-1}}_{\hat{\mathbf{A}}}\hat{\mathbf{x}} + \underbrace{\mathbf{TB}}_{\hat{\mathbf{B}}}\mathbf{u}$

$(\because x = T^{-1}\hat{x})$ $\dot{\hat{x}} = \_\,\hat{x} + \_\,u$

$\mathbf{y} = \mathbf{Cx} = \underbrace{\mathbf{CT}^{-1}}_{\hat{\mathbf{C}}}\hat{\mathbf{x}}$.

$H(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = C(sI-A)^{-1}B + D$

Compute the TF:

$$\begin{aligned}
\hat{\mathbf{H}}(s) &= \hat{\mathbf{C}}(s\mathbf{I} - \hat{\mathbf{A}})^{-1}\hat{\mathbf{B}} = \mathbf{CT}^{-1}(s\mathbf{I} - \mathbf{TAT}^{-1})^{-1}\mathbf{TB} \\
&= \mathbf{CT}^{-1}(s\mathbf{TT}^{-1} - \mathbf{TAT}^{-1})^{-1}\mathbf{TB} \\
&= \underbrace{\mathbf{CT}^{-1}\mathbf{T}}_{I}(s\mathbf{I} - \mathbf{A})^{-1}\underbrace{\mathbf{T}^{-1}\mathbf{T}}_{I}\mathbf{B} \\
&= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \\
&= \mathbf{H}(s)
\end{aligned}$$

$\hat{\mathbf{x}} = \mathbf{T}\mathbf{x}$ is called a similarity transform since it does not change the underlying TF.

How many state-space realizations are there for a given transfer function?

Since T can be any invertible matrix (n x n), therefore an infinite number of transforms and realizations!

Example 6.5.3 and Matlab's TF2SS for the given TF:

(x) $(\hat{x})$

$$\frac{Y}{U} = \frac{0z^3 + \underline{z^2 + 2z + 0z^0}}{z^3 + z^2 - 0.4z + 0.8}$$

$$\hat{x} = Tx$$

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} b_n & \cdots & \cdots & b_1 \end{bmatrix}, \quad D = 0.$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.8 & 0.4 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad D = 0$$

$$C = \begin{bmatrix} 0 & 2 & 1 \end{bmatrix}$$

SS2TF

$x(k+1) = Ax(k) + Bu(k)$

Shorthand:

$x^+ = Ax + Bu$

Start with the state-space model:
$$\begin{aligned} x^+ &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

Take the z-transform(s) with zero initial condition (just like we took the LT for CT models):

$$\begin{aligned} zX[z] &= AX[z] + BU[z] \\ Y[z] &= CX[z] + DU[z]. \end{aligned}$$

Eliminate X[z]

$$\begin{aligned} &(zI - A)X[z] = BU[z] \\ &\Rightarrow X[z] = (zI - A)^{-1}BU[z] \\ &\Rightarrow Y[z] = \underbrace{\left(C(zI - A)^{-1}B + D\right)}_{\text{transfer function } G[z]} U[z]. \end{aligned}$$

plug into $y = Cx + Du$

Shorthand:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right][z] := C\,(zI - A)^{-1}\,B + D.$$

See example 6.5.5 in course notes.

Outline

[-] Difference equations

[-] z-Transforms

- [-] Properties of z-Transforms
- [-] Final-value theorem for DT Linear Time Invariant (LTI) systems
- [X] Solving difference equations

[-] DT models of linear systems

- [-] Transfer functions
- [-] State-space models

[X] Stability of DT systems

[ ] Stability of feedback (DT) systems

- [ ] Internal stability
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Stability of state-space models

For a given Autonomous system $x[k+1] = Ax[k], \qquad x[0] \in \mathbb{R}^n$, will x[k] converge to zero as k goes to infinity?

Definition 6.6.1. System (6.10) is **asymptotically stable** if $x[k] \to 0$ as $k \to \infty$ for every initial condition, i.e., $\forall x(0) \in \mathbb{R}^n$.

How does the state of this autonomous system evolve?

$$x[1] = Ax[0]$$

$$x[2] = Ax[1] = A^2 x[0]$$

$$x[3] = Ax[2] = A^2 x[1] = A^3 x[0]$$

$$\vdots$$

$$x[k] = \underbrace{A^k}_{\text{Transition matrix } (x_0 \mapsto x[k])} x[0]$$

Theorem 6.6.2. *The unique solution to $x[k+1] = Ax[k]$, $x[0] = x_0$, is $x[k] = A^k x_0$.*

Transition matrix and z-transforms

$x[k+1] = Ax[k]$, Z-TF without setting init conditions to zero.

$$zX - zx[0] = AX$$

$$\Rightarrow X[z] = z(zI - A)^{-1}x[0] = (I - z^{-1}A)^{-1}x[0]$$

Compare to $x[k] = A^k x[0]$

Informally, A^k & $(I - z^{-1}A)^{-1}$ are Z-TF pairs,

$$Z\{A^k\} = (I - z^{-1}A)^{-1}$$

Example: computing the transition matrix using z-transforms

Calculate A^k for $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

$$A^k = \mathcal{Z}^{-1}\left\{ z(zI-A)^{-1} \right\}$$ (inv. Z-TF)

$$zI - A = \begin{bmatrix} z-1 & -1 \\ +1 & z-1 \end{bmatrix}$$, invert this:

$$(zI-A)^{-1} = \frac{1}{(z-1)^2+1} \cdot \begin{bmatrix} z-1 & +1 \\ -1 & z-1 \end{bmatrix} = \frac{1}{z^2-2z+2} \begin{bmatrix} z-1 & 1 \\ -1 & z-1 \end{bmatrix}$$

See table of Z-TFs, $z(zI-A)^{-1}$ has 4 elements

$$\Rightarrow A^k = \begin{bmatrix} \cos\left(\frac{\pi}{4}k\right) & \sin\left(\frac{\pi}{4}k\right) \\ -\sin\left(\frac{\pi}{4}k\right) & \cos\left(\frac{\pi}{4}k\right) \end{bmatrix}$$ ← Does not $\to 0$ as $k \to \infty$.

$x[k] = A^k x[0]$, is this AS? No.

Transition matrix for diagonal or diagonalizable A matrices

If A is diagonal, easy. For example: $A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$, $\quad A^k = \begin{bmatrix} 3^k & 0 \\ 0 & 1 \end{bmatrix}$

Otherwise, if the matrix is diagonalizable (we won't conver the theory here), then:

Assume A is s.t. all eigenvectors are linearly independent.

, then $\exists\ V, D \in \mathbb{R}^{n\times n}$ $(A \in \mathbb{R}^{n\times n})$

s.t. D is diagonal, V full-rank

& $AV = VD$ (Recall $Av = \lambda v$)

here, $V = [v_1 | v_2 | \dots | v_n]$, $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \lambda_n \end{bmatrix}$

$$A = VDV^{-1}$$

then $A^k = V D^k V^{-1}$, D^k is easy as D is diagonal.

Stability from eigenvalues

$x^+ = Ax$; $x[k] = A^k x[0]$

Since A^k depends on the eigenvalues of A, we do not need to directly compute A^k and can instead just check the eigenvalues to get the following condition for asymptotic stability:

Proposition 6.6.4. *$A^k \to 0$ as $k \to \infty$ if, and only if, every eigenvalue of A has magnitude less than 1.*

Easier way of checking asymptotic stability without having to compute A^k.

Proof: Exercise, or see course notes.

BIBO stability of DT TFs

$u \longrightarrow \boxed{G} \longrightarrow y$

Same idea as the CT case.

Theorem 6.6.5. *Assume $G[z]$ is proper and rational. Then the following three statements are equivalent:*

1. *The system is BIBO stable.*
2. *The impulse-response function $g[k]$ is absolutely summable, i.e., $\sum_{k=0}^{\infty} |g[k]| < \infty$.*
3. *Every pole of the transfer function $G[z]$ has magnitude less than one.*

Example: Stability of the 5-point moving average

The 5 point moving average y[k] = 0.2(u[k]+u[k-1]+u[k-2]+u[k-3]+u[k-4]) has the TF:

$$\frac{Y[z]}{U[z]} = \frac{1}{5}\frac{z^4 + z^3 + z^2 + z^1 + 1}{z^4}$$

$$\sum_{k=0}^{\infty} |g[k]| = 1$$

What are the poles ? $\{0, 0, 0, 0\}$

Is it BIBO stable ? Yes

Finite impulse response (FIR) systems:

Impulse response of the 5-point moving average filter (previous lecture) was: $g[k] = \begin{cases} 0.2 & 0 \le k \le 4 \\ 0 & \text{otherwise.} \end{cases}$

DT systems who impulse response has only a finite number of non-zero terms are called FIR systems.

Via the second point of theorem 6.6.5, every FIR system is BIBO stable.

Stability of state-space models and BIBO stability

Consider the SISO system
$$x[k+1] = Ax[k] + Bu[k]$$
$$y[k] = Cx[k] + Du[k].$$

$u, y \in \mathbb{R}^1, x \in \mathbb{R}^n$

λ_A, $\det(zI - A) = 0$

It has the TF
$$Y[z]/U[z] = G[z]$$
$$= C(zI - A)^{-1}B + D$$
$$\rightarrow \quad = \frac{1}{\det(zI - A)} C \operatorname{adj}(zI - A) B + D$$

adj(zI−A) is a n×n matrix, where each element is a polynomial in 'z' of degree ≤ n−1

From this, we see that the poles of G[z] are a subset of the eigenvalues of A.

Therefore: asymptotic stability of state-space model $\Longrightarrow$ BIBO stability of TF model.

Usually poles of G[z] are identical to the eigenvalues of A, which happens when

$$\det(zI - A), \qquad C \operatorname{adj}(zI - A)B + D \det(zI - A)$$

are coprime (no common roots).

Num of TF = $C \operatorname{adj}(zI-A)B + D \det(zI-A)$

den " " = $\det(zI-A)$ ⇝ polynomial of degree = n.

May share roots.

Example:

Consider:
$$x[k+1] = \begin{bmatrix} 0 & 1 \\ 0.25 & 0 \end{bmatrix} x[k] + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u[k]$$
$$y[k] = \begin{bmatrix} 1 & 0 \end{bmatrix} x[k] + u[k].$$

Eigenvalues can be found by solving: $\det(zI - A) = z^2 - 0.25 = 0$ to get $z = \pm 0.5$ $\in$ unit circle

Therefore asymptotically stable => BIBO stable.

To verify, compute the TF:

$$G[z] = C(zI - A)^{-1}B + D = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{z}{z^2-0.25} & \frac{1}{z^2-0.25} \\ \frac{0.25}{z^2-0.25} & \frac{z}{z^2-0.25} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 1 = \frac{z^2 + 2z - 0.25}{z^2 - 0.25}$$

Example: BIBO stability does not imply asymptotic stability!

See exercise 6.4 in class notes. Also see Matlab code.

Outline

[-] Difference equations

[-] z-Transforms

- [-] Properties of z-Transforms
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- [-] Solving difference equations

[-] DT models of linear systems

- [-] Transfer functions
- [-] State-space models

[-] Stability of DT systems

[X] Stability of feedback (DT) systems

- [ ] Internal stability
- [ ] Input-output stability
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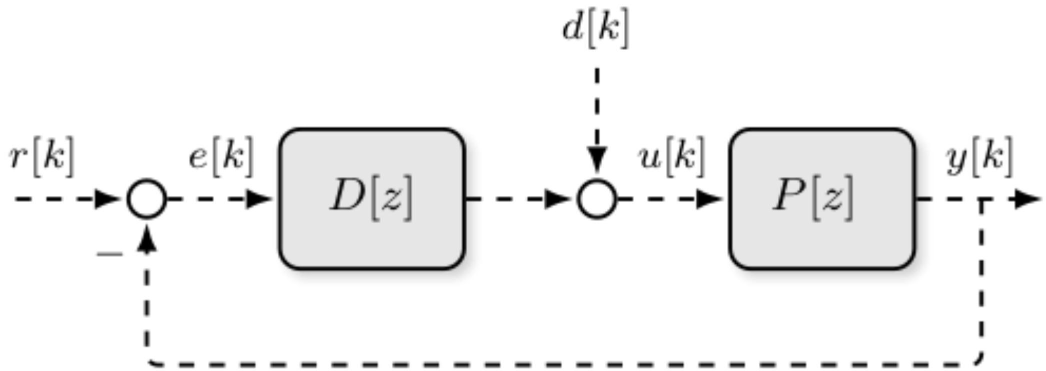
\- = Topic that has been covered

Stability of feedback systems

Follows the same reasoning as for CT systems (see chapter 2).



We'll make the following assumption for well-posedness:

Assumption 6.7.1. The plant and controller TFs are rational, $D[z]$ is proper, and $P[z]$ is strictly proper.
◀

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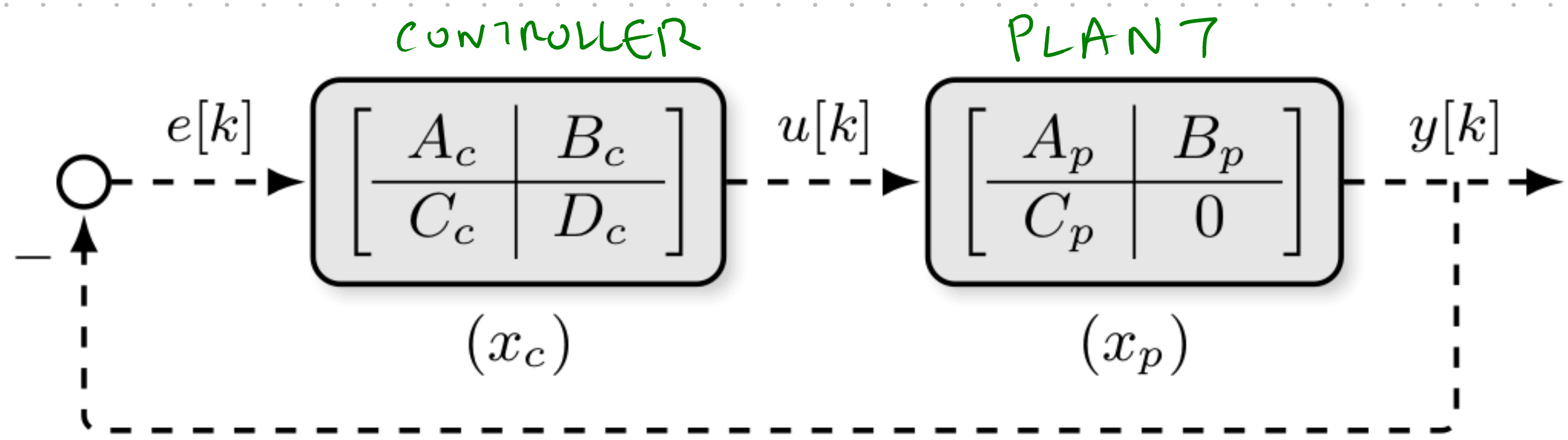
Internal stability

Set external signals r[k] and d[k] to zero for all k. We will study the stability of the closed-loop system with a non-zero initial state (for both/either plant or controller).



By assumption, the plant is a strictly proper TF (Dp = 0) and the controller is a proper TF (realizable):

$$\text{Plant}\begin{cases} x_p^+ = A_p x_p + B_p u \\ y = C_p x_p, \quad e = 0 - y = -C_p x_p \end{cases}$$

$$\text{Controller}\begin{cases} x_c^+ = A_c x_c + B_c e \\ u = C_c x_c + D_c e \end{cases}$$

Plug in expression for u and e into the state-update equations:

$$x_p^+ = (A_p - B_p D_c C_p) x_p + B_p C_c x_c$$

$$(\because \; u = C_c x_c + D_c e, \; e = -C_p x_p)$$

$$x_c^+ = A_c x_c - B_c C_p x_p \quad (\because e = -C_p x_p)$$

$$x_{cl} := \begin{bmatrix} x_p \\ x_c \end{bmatrix} \quad \text{(closed-loop states)}$$

write CL system in the form, $x_{cl}^+ = \underbrace{A_{cl}}_{?} x_{cl}$

$$\begin{bmatrix} x_p^+ \\ x_c^+ \end{bmatrix} = \begin{bmatrix} A_p - B_p D_c C_p & B_p C_c \\ -B_c C_p & A_c \end{bmatrix} \begin{bmatrix} x_p \\ x_c \end{bmatrix}$$

Think of the closed-loop system as the product of the plant and controller.

Let the closed-loop state be $x_{cl} := (x_p, x_c)$ so that

$$x_{cl}^+ = A_{cl} x_{cl} := \begin{bmatrix} A_p - B_p D_c C_p & B_p C_c \\ -B_c C_p & A_c \end{bmatrix} \begin{bmatrix} x_p \\ x_c \end{bmatrix}$$

When is this system stable ?

The feedback system is internally stable if the closed-loop system above is asymptotically stable, i.e., iff all the eigenvalues of Acl have a magnitude less than 1.

Summary:

Internal stability means that with no exogenous input (r=d=0), the internal states of the system (xc and xp) decay to zero for every initial state of the plant xp[0] and controller xc[0].

Internal stability: Example

Consider the following plant: $P[z] = \dfrac{1}{z+2}$

We will find the gains of a DT PI controller such that the closed-loop system is internally stable.

$$D[z] = K_p + K_i \frac{1}{z-1}$$

Convert both the plant and controller to state-space form:

$$x_p^+ = -2x_p + u \qquad\qquad x_c^+ = x_c + e$$
$$y = x_p, \qquad\qquad u = K_i x_c + K_p e$$

{See prev. slide}

Closed-loop state-space model has the matrix: $A_{cl} = \begin{bmatrix} -2-K_p & K_i \\ -1 & 1 \end{bmatrix}$

The eigenvalues of this matrix are the roots of the polynomial:

$$\det(zI - A_{cl}) = z^2 + (1+K_p)z + K_i - K_p - 2$$

Internal stability requires these eigenvalues to be inside the unit circle. Let's pick Kp = -1 and Ki = 1.

What are the resulting eigenvalues ? {0, 0}

Internal stability: Example

Consider the following plant: $P[z] = \dfrac{1}{z+2}$

We will find the gains of a DT PI controller such that the closed-loop system is internally stable.

$$D[z] = K_p + K_i \frac{1}{z-1}$$

Convert both the plant and controller to state-space form:

$$\begin{aligned} x_p^+ &= -2x_p + u \\ y &= x_p, \end{aligned}$$

$$\begin{aligned} x_c^+ &= x_c + e \\ u &= K_i x_c + K_p e \end{aligned}$$

Closed-loop state-space model has the matrix: $A_{cl} = \begin{bmatrix} -2-K_p & K_i \\ -1 & 1 \end{bmatrix}$

The eigenvalues of this matrix are the roots of the polynomial:

$$\det(zI - A_{cl}) = z^2 + (1+K_p)z + K_i - K_p - 2$$

Internal stability requires these eigenvalues to be inside the unit circle. Let's pick Kp = -1 and Ki = 1.

This results in eigenvalues {0,0}. The stabilizing PI controller is:

$$D[z] = -1 + \frac{1}{z-1} = \frac{2-z}{z-1}$$

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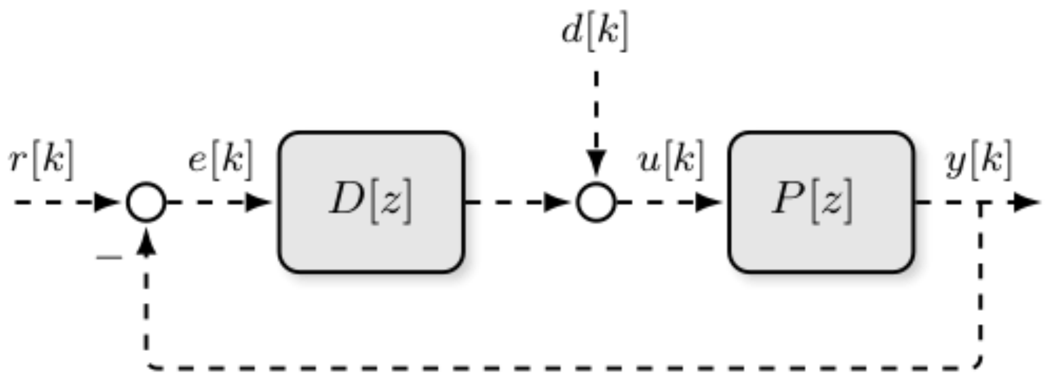
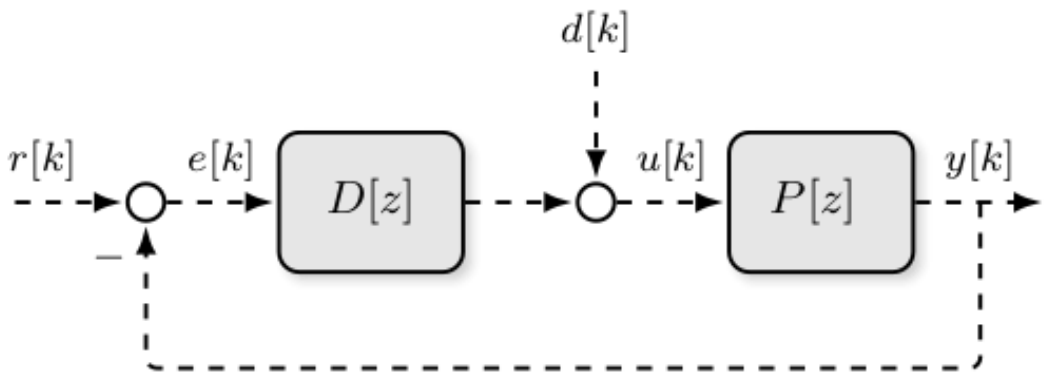
Input-output stability of DT feedback systems

Same concept as we studied for CT systems. How many transfer functions do we have to study?



Input-output stability of DT feedback systems

Same concept as we studied for CT systems.



Definition: The feedback system above is input-output stable if e, u, y are bounded signals when r and d are bounded signals, i.e., the system from (r,d) to (e,u,y) is BIBO stable.

Just like we did in the CT case, there are 6 closed-loop TFs to consider:

$$R \text{ to } E: \frac{1}{1+PD}, \qquad R \text{ to } U: \frac{D}{1+PD}, \qquad R \text{ to } Y: \frac{PD}{1+PD},$$
$$D \text{ to } E: \frac{-P}{1+PD}, \qquad D \text{ to } U: \frac{1}{1+PD}, \qquad D \text{ to } Y: \frac{P}{1+PD}.$$

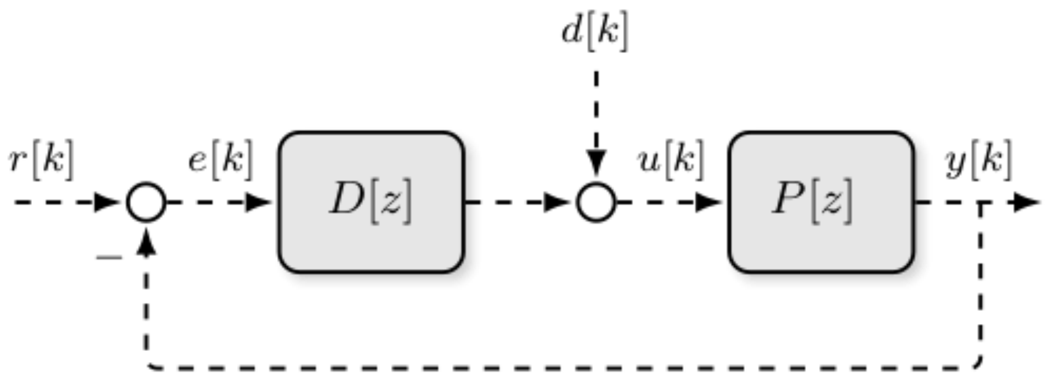
Let $P = \frac{N_p}{D_p}$ and $C = \frac{N_c}{D_c}$. The characteristic polynomial (or LCM of the denominators of the TFs above) is:

$$\pi[z] = D_p[z]D_c[z] + N_p[z]N_c[z]$$

What is the necessary and sufficient condition for BIBO stability (of all these TFs)?

Input-output stability of DT feedback systems

Same concept as we studied for CT systems.



Definition: The feedback system above is input-output stable if e, u, y are bounded signals when r and d are bounded signals, i.e., the system from (r,d) to (e,u,y) is BIBO stable.

Just like we did in the CT case, there are 6 closed-loop TFs to consider:

$$R \text{ to } E: \frac{1}{1+PD}, \qquad R \text{ to } U: \frac{D}{1+PD}, \qquad R \text{ to } Y: \frac{PD}{1+PD},$$
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Let $P = \frac{N_p}{D_p}$ and $C = \frac{N_c}{D_c}$. The characteristic polynomial (or LCM of the denominators of the TFs above) is:

$$\pi[z] = D_p[z]D_c[z] + N_p[z]N_c[z]$$

Theorem 6.7.4. *The feedback system in Figure 6.3 is input-output stable if, and only if, its characteristic polynomial has no roots with magnitude greater than or equal to one.*

Input–output stability: Example

Consider the plant and PI controller from the internal stability example.

$$P[z] = \frac{1}{z+2}, \qquad D[z] = \frac{2-z}{z-1}$$

What is its characteristic polynomial (and its roots)?

$$\pi[z] = D_p[z]D_c[z] + N_p[z]N_c[z] = (z+2)(z-1) + 2 - z = z^2$$

Roots are $\{0,0\}$, both lie inside the unit circle therefore the system is input–output stable.

Can we think about how internal and I/O stability are related?

Internal stability and input-output stability

Internal stability dealt with state-space models, while I/O-stability dealt with TFs. The two are related as follows:

$$\lambda_{A_{cl}} \supseteq \text{Roots of the CP } \pi[z]$$

Therefore, internal stability $\Longrightarrow$ input-output stability.

Usually the roots are identical to the eigenvalues (lambda) of Acl, e.g., in the example of I/O stability.

In such cases, the two concepts are equivalent and we do not (usually) make a distinction in practice when we refer to the stability of a CL system.

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- [-] Solving difference equations

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[-] Stability of DT systems

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- [-] Internal stability
- [-] Input-output stability
- [X] Identifying polymonials with stable roots

[ ] Frequency response

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Identifying stable roots (in the unit circle)

Two reasons to care about what comes next:

1. We do not always know the numerical values of a TF's coefficients, e.g., when the sampling period is not yet fixed, and thus cannot numerically find the roots of the denominator.

2. Back in the old days, numerical software was scarce/slow/non-existent. Even in cases when coefficients are known, these techniques can often make the task of checking stability easier than solving for all the roots.

Identifying stable roots (in the unit circle)

Consider a n-th order polynomial with real-coefficients:

$$\pi[z] = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n, \quad a_i \in \mathbb{R}, \quad n \neq 0.$$

Identifying stable roots (in the unit circle)

CT	DT
$\pi(s)$	$\pi[z]$
$Re(s)<0$	$\lvert z\rvert<1$
	Schur

Consider a n-th order polynomial with real-coefficients:

$$\pi[z] = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n, \ a_i \in \mathbb{R}, \ n \neq 0.$$

(DT)

Definition 6.8.1. A polynomial $\pi \in \mathbb{R}[z]$ is **Schur** if all its roots have magnitude less than one.

Polynomial w/ real-coefficients.

A matrix A is Schur if $\lambda(A)$,

DT: $|\lambda_i(A)| < 1 \ \forall i$

CT: $Re(\lambda_i) < 0 \ \forall i$

Necessary conditions for a polynomial to be Schur

Consider a n-th order polynomial with real-coefficients:

$$\pi[z] = z^n + a_1 z^{n-1} + \cdots + a_{n-1}s + a_n, \ a_i \in \mathbb{R}, \ n \neq 0.$$

We'll first introduce two necessary (but not sufficient) conditions without proof (see notes for a sketch):

(i) $\pi[1] > 0$

(ii) $(-1)^n \pi[-1] > 0$

Necessary conditions for a polynomial to be Schur

Consider a n-th order polynomial with real-coefficients:

$$\pi[z] = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n, \ a_i \in \mathbb{R}, \ n \neq 0.$$

We'll first introduce two necessary (but not sufficient) conditions without proof (see notes for a sketch):

(i) $\pi[1] > 0$

(ii) $(-1)^n \pi[-1] > 0$

(iii) $|a_n| < 1.$

Example

Consider a n-th order polynomial with real-coefficients:

$$\pi[z] = z^n + a_1 z^{n-1} + \cdots + a_{n-1}s + a_n, \; a_i \in \mathbb{R}, \; n \neq 0.$$

The necessary (but not sufficient) conditions are:

(i) $\pi[1] > 0$

(ii) $(-1)^n \pi[-1] > 0$

(iii) $|a_n| < 1$. → i.e. Schur.

Identify which of the following polynomials could have roots inside the unit-circle, and why:

a. $z^4 + 0.3z^3 - z^2 - 0.9$ → $\pi[1] < 0$ → Failed ⇒ Not Schur

b. $z^3 + 0.1z + 1.1$ → Fails ii) & iii) ⇒ Not Schur

c. $z^3 + 0.1z^2 - 0.6z + 0.1$

Not done.

The Routh-Hurwitz criterion: For CT systems, i.e., roots with negative real parts

Consider the polynomial: $\pi(s) = s^n + \underline{a_{n-1}} s^{n-1} + \cdots + a_1 s + a_0,\ a_i \in \mathbb{R},\ n \neq 0.$

We construct the "Routh Array", a table of the form:

From $\pi(s)$

$$
\begin{array}{l|lllll}
\mathbf{s^n} & r_{0,0} = 1 & r_{0,1} = a_{n-2} & r_{0,2} = a_{n-4} & r_{0,3} = a_{n-6} & \cdots \\
\mathbf{s^{n-1}} & r_{1,0} = a_{n-1} & r_{1,1} = a_{n-3} & r_{1,2} = a_{n-5} & r_{1,3} = a_{n-7} & \cdots \\
\mathbf{s^{n-2}} & \boxed{r_{2,0}} & r_{2,1} & r_{2,2} & r_{2,3} & \cdots \\
\mathbf{s^{n-3}} & r_{3,0} & r_{3,1} & r_{3,2} & r_{3,3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & & \\
\mathbf{s^2} & r_{n-2,0} & r_{n-2,1} & 0 & \cdots & \\
\mathbf{s^1} & r_{n-1,0} & 0 & 0 & \cdots & \\
\mathbf{s^0} & r_{n,0} & 0 & 0 & \cdots &
\end{array}
$$

Here, first two rows are computed via coefficients of the polynomial. The third row is computed from the first two as follows: $r_{2,0} = \dfrac{a_{n-1}a_{n-2} - a_{n-3}}{a_{n-1}}$, $\quad r_{2,1} = \dfrac{a_{n-1}a_{n-4} - a_{n-5}}{a_{n-1}}$, $\quad r_{2,2} = \dfrac{a_{n-1}a_{n-6} - a_{n-7}}{a_{n-1}}, \ldots$

The remaining rows are computed via the recursion (same pattern as above):

$$r_{i,j} = \frac{r_{i-1,0}\, r_{i-2,j+1} - r_{i-2,0}\, r_{i-1,j+1}}{r_{i-1,0}}, \quad i \in \{2, \ldots, n\}, \quad j \in \left\{0, \ldots, \left\lfloor \frac{n-i}{2} \right\rfloor \right\}$$

The construction terminates if we get a zero in the first column.
Note: There are at most n+1 rows in the final table (see above).

The Routh–Hurwitz criterion: For CT systems, i.e., roots with negative real parts

Consider the polynomial: $\pi(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0,\ a_i \in \mathbb{R},\ n \neq 0.$

> **Theorem 6.8.2** (Routh-Hurwitz Stability Criterion). *Consider a polynomial $\pi \in \mathbb{R}[s]$ given by* (6.12) *and its associated Routh array in Table* 6.2. *The number of roots of π with real part greater than or equal to zero equals the number of sign changes in the first column of the array.*

Corollary: No sign changes in the first column

$\Rightarrow$ a stable ch. p.

(Roots s.t. $Re(s) < 0$)

Example: $\pi(s) = a_2 s^2 + a_1 s + a_0, \qquad a_2 \neq 0.$

$$
\begin{array}{c|ccc}
s^2 & a_2 & a_0 & 0 \\
s^1 & a_1 & 0 & 0 \\
s^0 & \frac{a_1 a_0}{a_1} = a_0 & 0 & 0
\end{array}
$$

a_i's with the same sign $\iff$ Roots $\pi(s)$ s.t. $Re(s) < 0$.

Routh-Hurwitz for discrete-time systems:

In the form that we have studied, it tells us nothing about a discretization of the system.

What transform should we use to get a DT system for a sampling time of T seconds?

Routh-Hurwitz for discrete-time systems:

In the form that we have studied, it tells us nothing about a discretization of the system.

What transform should we use to get a DT system for a sampling time of T seconds?

We use the bilinear (Tustin) transformation (see chapter 4) since it is stability preserving:

$$s = \frac{2}{T}\frac{z-1}{z+1}, \qquad z = \frac{1+\frac{T}{2}s}{1-\frac{T}{2}s}.$$

T=2

Since it preserves stability for any T, say T=2 for simplicity.

Then, roots of the CT system 'v' and the DT system 'z' are related as follows $z = \dfrac{1+v}{1-v}$.

Rational

That is $v \in \mathbb{C}^-$ iff $|z| < 1$.

The procedure for applying the Routh-Hurwitz test for DT systems is thus:

1. Given a polynomial (DT) $\pi(\cancel{s}\,z) = \cancel{s}^n z + a_{n-1}\cancel{s}^{n-1} z + \cdots + a_1 \cancel{s}\,z + a_0$, form the polynomial $\pi[z]|_{z=\frac{1+v}{1-v}} = \pi\left[\dfrac{1+v}{1-v}\right]$
2. Multiply the polynomial obtained in step (1) by $(1-v)^n$ to obtain the nth degree polynomial in v

$$\hat{\pi}(v) := (1-v)^n \pi\left[\frac{1+v}{1-v}\right].$$

3. The polynomial $\pi[z]$ is Schur if, and only if, the roots of $\hat{\pi}(v)$ all have negative real part and the degree of $\hat{\pi}$ equals the degree of π.

Routh-Hurwitz for discrete-time systems: Example

Consider the second order polynomial: $\pi[z] = z^2 + a_0 z + a_1$

Construct the n-th order polynomial (corresponding to the CT characteristic polynomial):

$$\hat{\pi}(v) = (1-v)^2 \left[\frac{(1+v)^2}{(1-v)^2} + a_0 \frac{(1+v)}{(1-v)} + a_1 \right]$$

$$= (1+v)^2 + a_0(1+v)(1-v) + a_1(1-v)^2$$

$$= (1-a_0+a_1)v^2 + (2-2a_1)v + (1+a_0+a_1)$$

Next, we'll create the Routh array and check for CT stability:

$$\begin{array}{c|cc} v^2 & 1-a_0+a_1 & 1+a_0+a_1 \\ v^1 & 2-2a_1 & 0 \\ v^0 & 1+a_0+a_1 & 0 \end{array}$$

The roots of this CT polynomial are stable (CT sense) iff sign of the first column does not change, i.e.,

$$\mathrm{sgn}(1-a_0+a_1) = \mathrm{sgn}(2-2a_1) = \mathrm{sgn}(1+a_0+a_1)$$

These are equivalent to:

$$|a_1| < 1, \quad |a_0| < |1+a_1|$$

Therefore, the roots of $\pi[z]$ are in the unit circle iff the coefficients satisfy the conditions above.

Takeaway: The Routh-Hurwitz criteria is useful for finding parameter ranges such that a system is stable.

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[-] Difference equations

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[X] Frequency response

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Frequency response of DT LTI systems

We have seen how a CT LTI system changes (possibly) the magnitude and phase of a sinusoidal input to it. The same property holds for a DT system.

Consider a BIBO stable system: $\dfrac{Y[z]}{U[z]} = G[z] = \dfrac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$

Let's apply the (not-real valued) input sequence $u[k] = a\mathrm{e}^{j\theta k}$.

The resulting output is: $Y[z] = G[z]U[z] = G[z]\dfrac{az}{z - \mathrm{e}^{j\theta}}$ ✓ z-TF

Partial fraction expansion of Y[z]/z = $G[z]\dfrac{a}{z - \mathrm{e}^{j\theta}} = \dfrac{c_1}{z - \mathrm{e}^{j\theta}} +$ terms due to poles of $G[z]$, $c_1 = aG[\mathrm{e}^{j\theta}]$

Therefore: $Y[z] = aG[\mathrm{e}^{j\theta}]\dfrac{z}{z - \mathrm{e}^{j\theta}} + z \times (\text{terms due to poles of } G[z])$ (Multiply above by z)

Since we assumed G to be BIBO stable: Response due to poles approaches 0 as $k \to \infty$

Thus, the steady-state output of this DT system to the input sequence is: $\lim_{k\to\infty} y[k] = aG[\mathrm{e}^{j\theta}]\mathrm{e}^{j\theta k}$

$G[\mathrm{e}^{j\theta}]$ is a complex number which we can write in polar form: $G[\mathrm{e}^{j\theta}] = \left|G[\mathrm{e}^{j\theta}]\right| \mathrm{e}^{j\angle G[\mathrm{e}^{j\theta}]}$

Therefore the steady state output in polar form is $\lim_{k\to\infty} y[k] = a\left|G[\mathrm{e}^{j\theta}]\right| \mathrm{e}^{j(\theta k + \angle G[\mathrm{e}^{j\theta}])}$

magnitude

Phase shift.

Frequency response of DT LTI systems: Some special cases.

The following theorem covers some special cases of the previous derivation.

Theorem 6.9.1. *Assume $G[z]$ is rational, proper, and all its poles have magnitude less than one. Then the steady-state response to the input $u[k] = a\mathrm{e}^{j\theta k}$ is*

$$y[k] = aG[\mathrm{e}^{j\theta}]\mathrm{e}^{j\theta k}. \tag{6.15}$$

In particular

- *The steady-state response to the input $u[k] = a\cos(\theta k)$ is*

$$y[k] = a|G[\mathrm{e}^{j\theta}]|\cos\left(\theta k + \measuredangle G[\mathrm{e}^{j\theta}]\right). \tag{6.16}$$

- *The steady-state response to the input $u[k] = a\sin(\theta k)$ is*

$$y[k] = a|G[\mathrm{e}^{j\theta}]|\sin\left(\theta k + \measuredangle G[\mathrm{e}^{j\theta}]\right). \tag{6.17}$$

- *The steady-state response to the input $u[k] = a\mathbf{1}[k]$ is*

$$y[k] = aG[1]. \tag{6.18}$$

Therefore the steady-state response of a BIBO stable DT LTI system to a DT sinusoid input is also a DT sinusoid of the same frequency. The DT system can also impact the amplitude and phase shift of the output sinusoid.

Frequency response of DT LTI systems: Definition

Definition 6.9.2. Assume $G[z]$ is rational, proper, and all its poles have magnitude less than one.

(a) The function $(-\pi, \pi] \to \mathbb{C}$, $\theta \mapsto G[\mathrm{e}^{j\theta}]$ is the **frequency response** of G.

(b) The function $(-\pi, \pi] \to \mathbb{R}$, $\theta \mapsto \left|G[e^{j\theta}]\right|$ is the **amplitude** or **magnitude response** of G.

(c) The function $(-\pi, \pi] \to (-\pi, \pi]$, $\theta \mapsto \measuredangle G[\mathrm{e}^{j\theta}]$ is the **phase response** of G.

Hence the frequency response of a DT (BIBO stable) LTI system is determined by setting $z = e^{j\theta}$ and varying θ.

Note: By assuming the G is BIBO stable (all poles inside unit circle), the ROC of the z-transform contains the unit circle, therefore the statement above is fine.

Setting $z = e^{j\theta}$ in G[z] to get $G[e^{j\theta}]$ equals the discrete-time Fourier Transform of the impulse response of the system, g[k].

Frequency response of DT LTI systems: Definition

Definition 6.9.2. Assume $G[z]$ is rational, proper, and all its poles have magnitude less than one.

(a) The function $(-\pi, \pi] \to \mathbb{C}$, $\theta \mapsto G[\mathrm{e}^{j\theta}]$ is the **frequency response** of G.

(b) The function $(-\pi, \pi] \to \mathbb{R}$, $\theta \mapsto \left|G[e^{j\theta}]\right|$ is the **amplitude** or **magnitude response** of G.

(c) The function $(-\pi, \pi] \to (-\pi, \pi]$, $\theta \mapsto \measuredangle G[\mathrm{e}^{j\theta}]$ is the **phase response** of G.

Hence the frequency response of a DT (BIBO stable) LTI system is determined by setting $z = e^{j\theta}$ and varying θ.

Remark 6.9.3. If the impulse response $g[k] = \mathscr{Z}^{-1}\{G[z]\}$ is a real-valued function, then the complex conjugate of $G[\mathrm{e}^{j\theta}]$ is $G[\mathrm{e}^{-j\theta}]$. This implies that

$$|G[\mathrm{e}^{j\theta}]| = |G[e^{-j\theta}]| \qquad \text{(even function)}$$

and

$$\measuredangle G[\mathrm{e}^{j\theta}] = -\measuredangle G[e^{-j\theta}] \qquad \text{(odd function)}.$$

Thus for the frequency response of a DT system with a real rational TF, we only need to plot the response for $\theta \in [0, \pi]$. This is similar to us only considering positive frequencies for CT systems.

Example

Consider the TF: $G[z] = \dfrac{z+1}{10z - 0.8}$

Its frequency response is: $G[e^{j\theta}] = \dfrac{e^{j\theta}+1}{10e^{j\theta} - 0.8}$

We can compute the response in Matlab:

```
n=[1 1];d=[10 -8];
w=-10:0.01:10;
G=freqz(n,d,w);
plot(w,abs(G),w,angle(G),':', 'LineWidth', 2);
```

Note: We see the 2*pi periodicity of the frequency response and the even (odd) nature of the magnitude (phase) response (see remark 6.9.3).

Remark 6.9.4. MATLAB always assumes that a discrete-time system arises from sampling a continuous-time system with sampling rate T. Therefore, by default, the `bode` command will plot the magnitude and phase Bode plots over the range $[0, \pi/T]$ where T is the sampling rate of the DT plant. For purely DT systems like the ones we've studied in this chapter, you can set $T = 1$. ♦

Finally, see example 6.9.2 from the course notes.

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