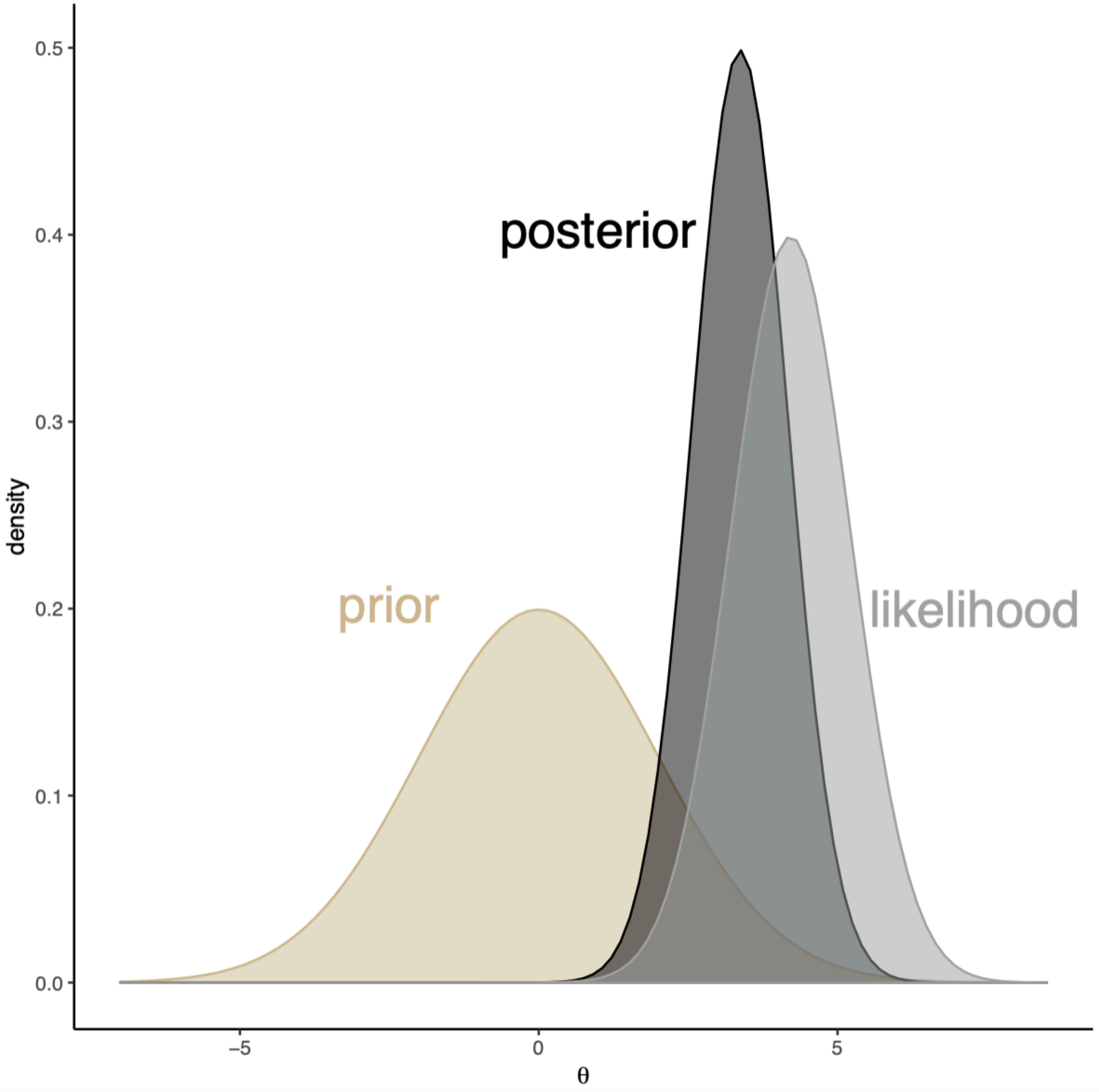


Notes on the prior distribution:

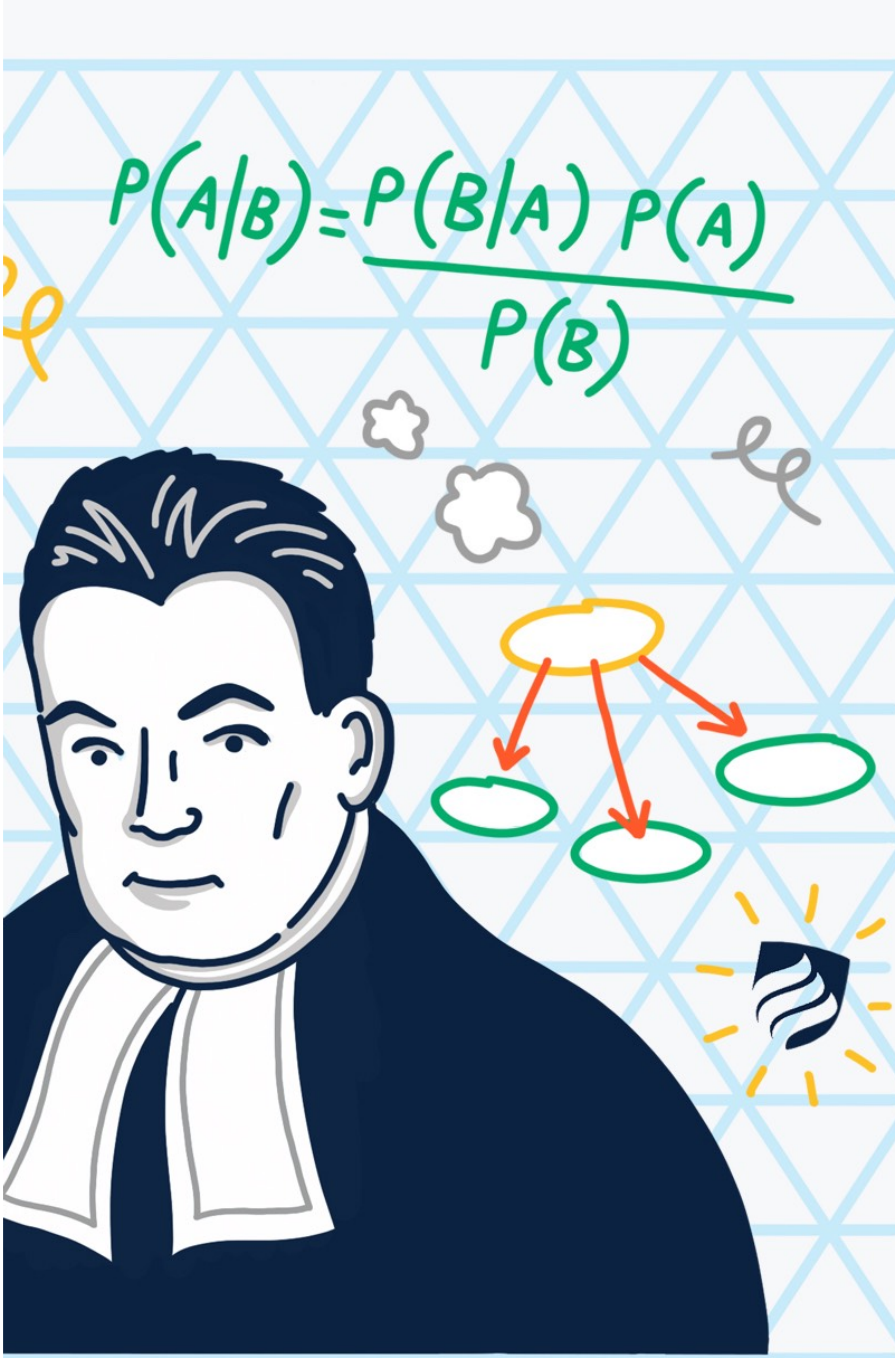
1. The prior distribution is meant to reflect our knowledge/degree of belief in the value of $\boldsymbol{\theta}$ before any new data are observed.

2. It is often the case that we have prior knowledge, an expert opinion, etc. $\pi(\boldsymbol{\theta})$ is the formal way to take that knowledge into account.

3. There is a *ton* to say about what to do in the absence of prior information!

    1. "Uninformative" priors?

    2. "Objective" priors?



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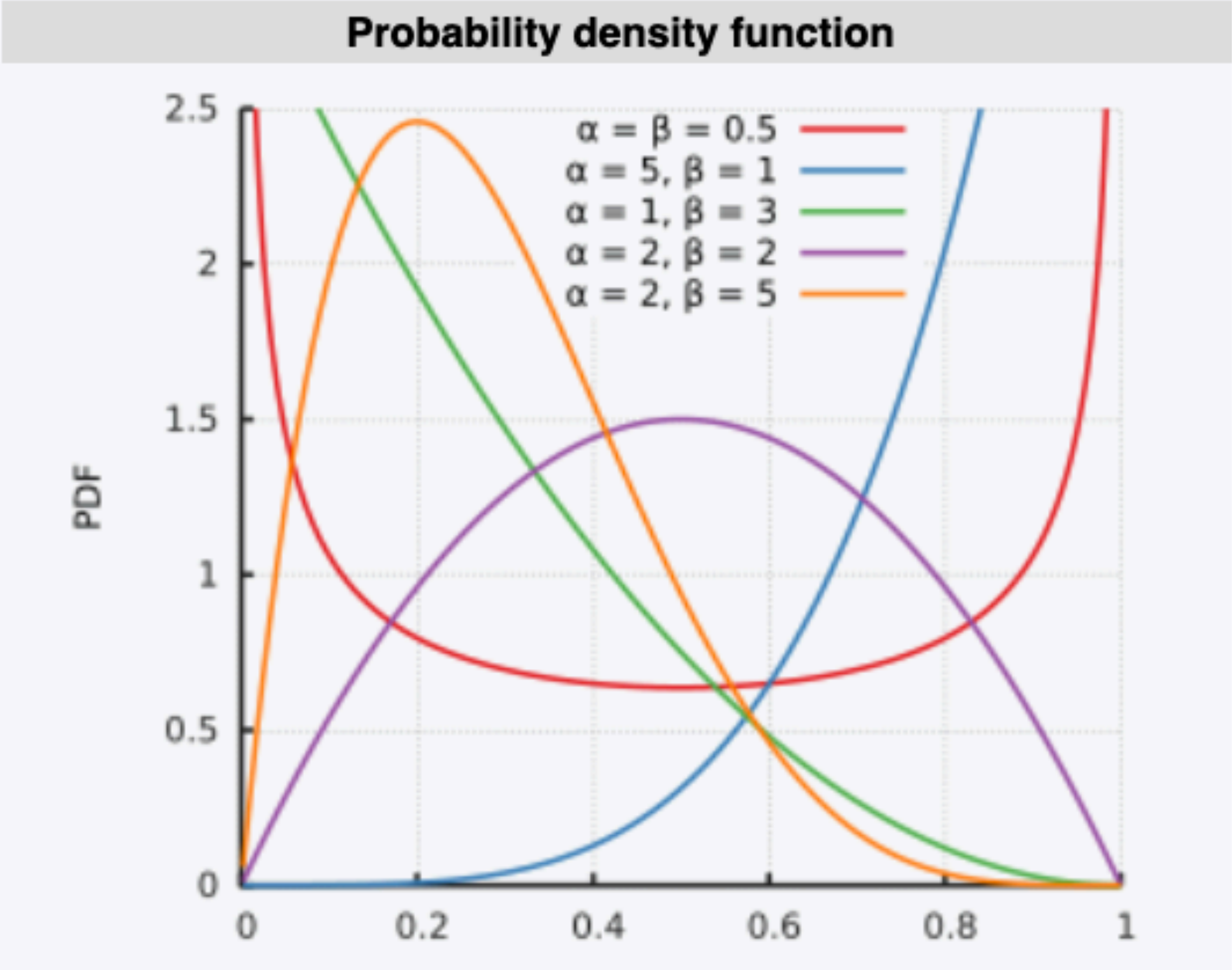

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Example: Binomial Bayes Estimation. Let $X_1, \ldots, X_n \overset{iid}{\sim} Binomial(1,p)$. Let the prior distribution on p be a beta distribution. Find the posterior distribution of $p \mid \mathbf{x}$, i.e., the posterior distribution of the parameter p given that we observed the data $\mathbf{x}$.

- Prior distribution:

- Likelihood:

- Posterior:

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Example: Binomial Bayes Estimation continued.

Let $X_1, \ldots, X_n \overset{iid}{\sim} Binomial(1,p)$. Let the prior distribution on p be a beta distribution. The posterior is given by:

$$p \mid \mathbf{x} \sim Beta\left(n\bar{x} + \alpha, n - n\bar{x} + \beta\right)$$

Estimator of p?

$$\hat{p}_1 = E(p \mid \mathbf{x}) = \frac{n\bar{x} + \alpha}{n + \alpha + \beta}$$

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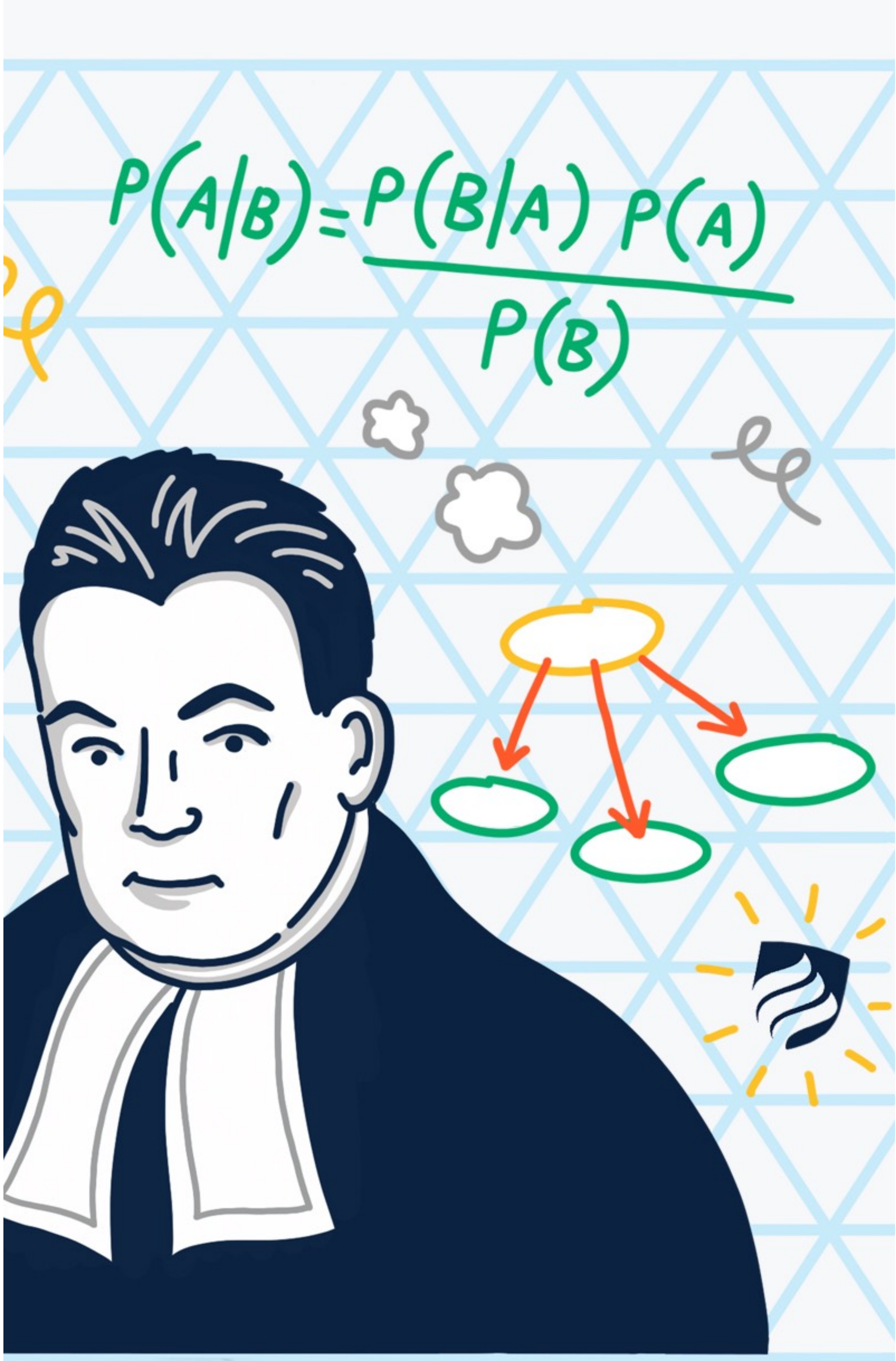
Example: Binomial Bayes Estimation continued.

Let $\widehat{p}_0 = E(p) = \dfrac{\alpha}{\alpha + \beta}$ be the prior mean, and $\widehat{p} = \bar{x}$. Then the posterior mean can also be written as:

$$E(p \mid \mathbf{x}) = \frac{n\bar{x} + \alpha}{n + \alpha + \beta} = (1 - w_n)\widehat{p}_0 + w_n\widehat{p},$$

where $w_n = \dfrac{n}{n + \alpha + \beta}$.



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A class of prior distributions, C, is a **conjugate family** for a likelihood, $f(\mathbf{x} \mid \theta)$, if the posterior distribution is also in class C.

In the previous example, we saw that the beta family of distributions (prior) is a conjugate for the binomial family (likelihood).

Example: Let $X_1, \ldots, X_n \,\Big|\, \mu, \sigma^2 \overset{iid}{\sim} N(\mu, \sigma^2)$. Assume σ^2 is known and we are attempting to learn about μ. Our prior is $\mu \sim N\left(\mu_0, \frac{\sigma^2}{m}\right)$. Find the posterior distribution of $\mu \mid \mathbf{x}$.

- Prior:

- Likelihood:

- Posterior:

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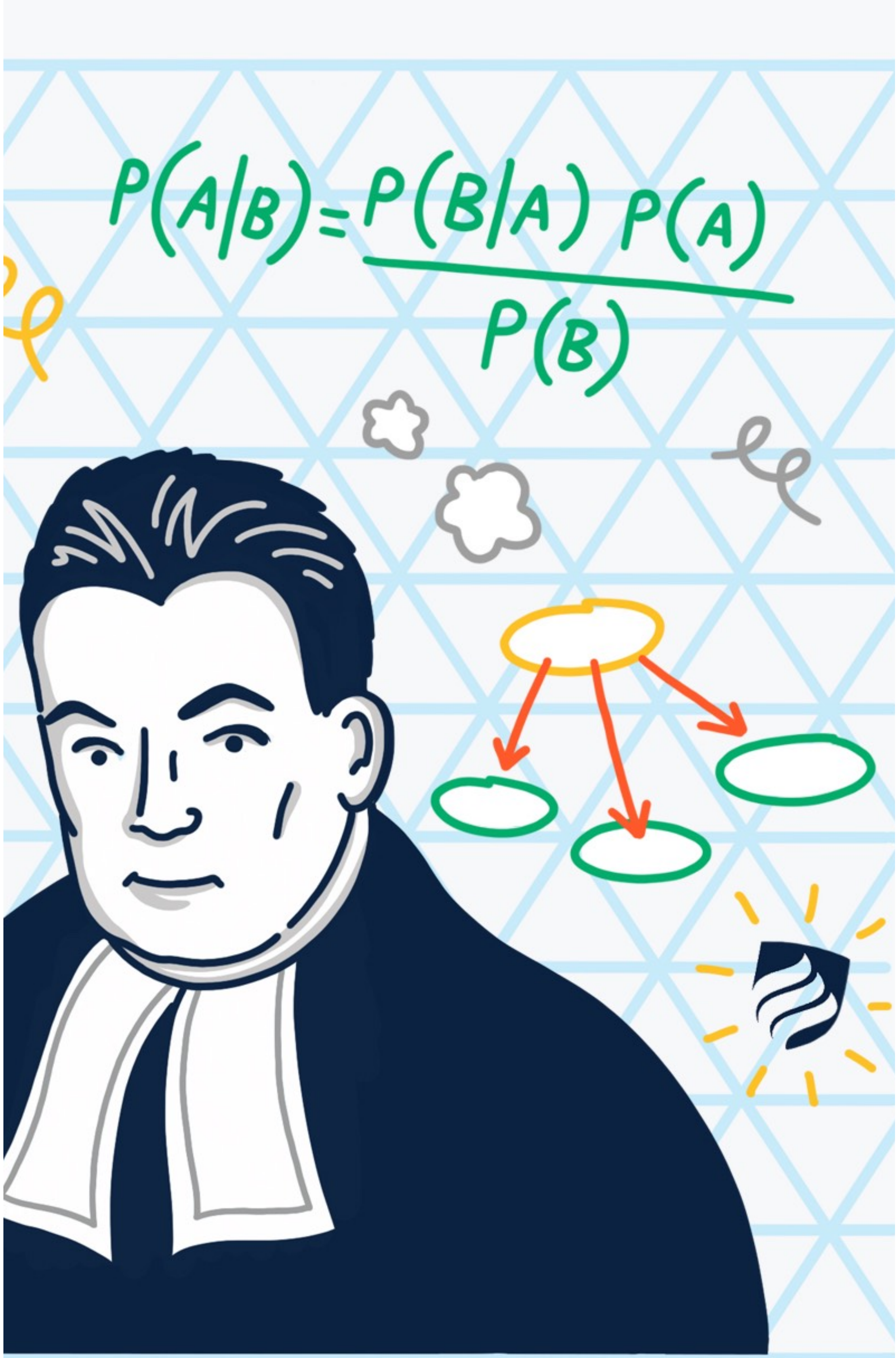
$$\mu_1 = \frac{n\bar{x}}{n+m} + \frac{m\mu_0}{n+m} =$$

$$\sigma_1^2 = \frac{\sigma^2}{n+m} =$$



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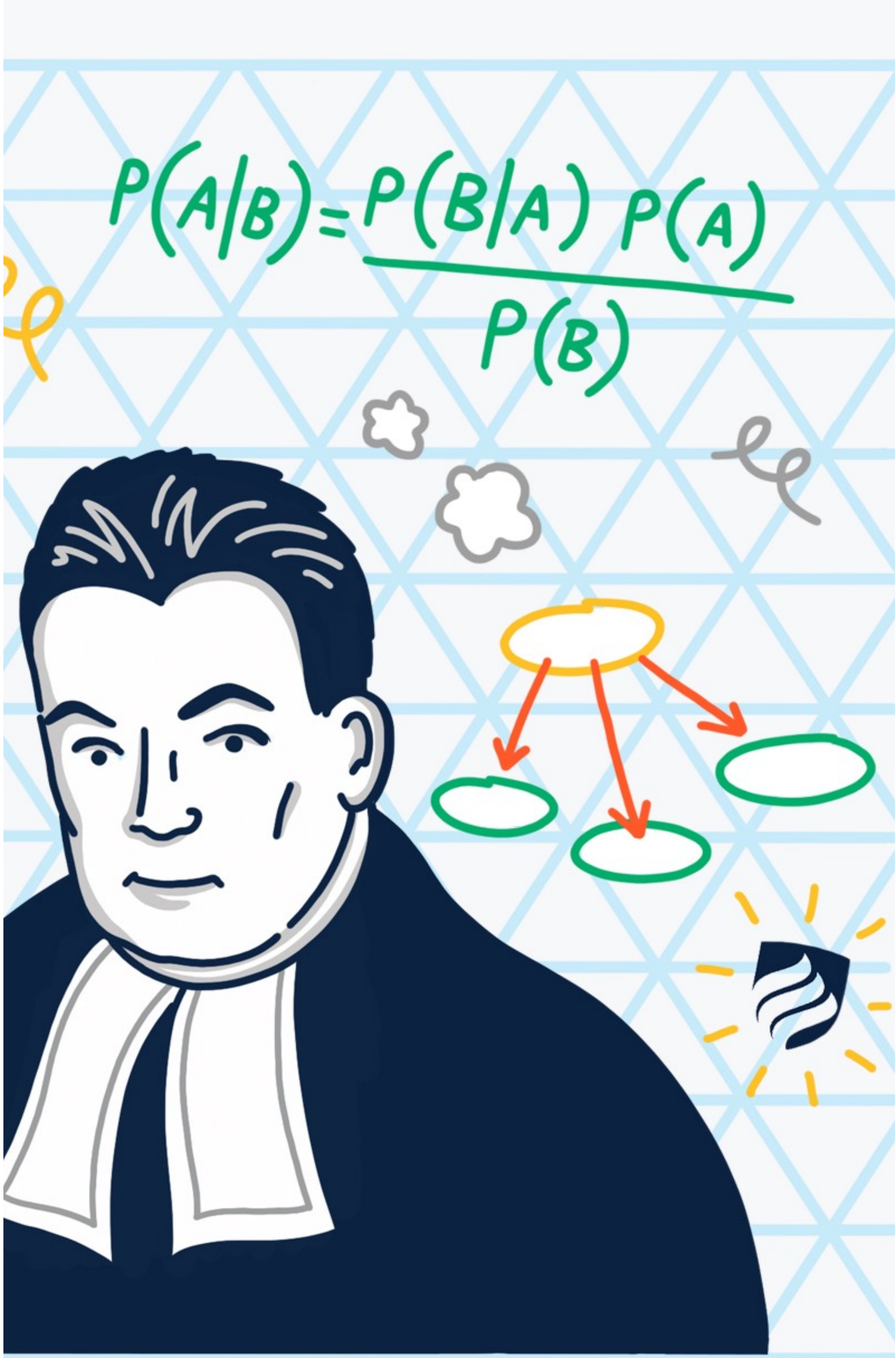
Example: Suppose researchers would like to know the probability, p, that an individual in a given population has a genetic marker that predisposes them for disease D (this genetic marker is such that an individual has it or doesn't; assume that our test for the marker is completely accurate). Researchers collected data in the following way: they tested people for this genetic marker until they found one person who had it. They stopped data collection at that point. Here's the data: $\mathbf{x} = (0,0,0,0,1)$. Suppose your prior beliefs are best represented by a beta$(2,2)$ distribution. What is the posterior distribution for $p \mid x$?



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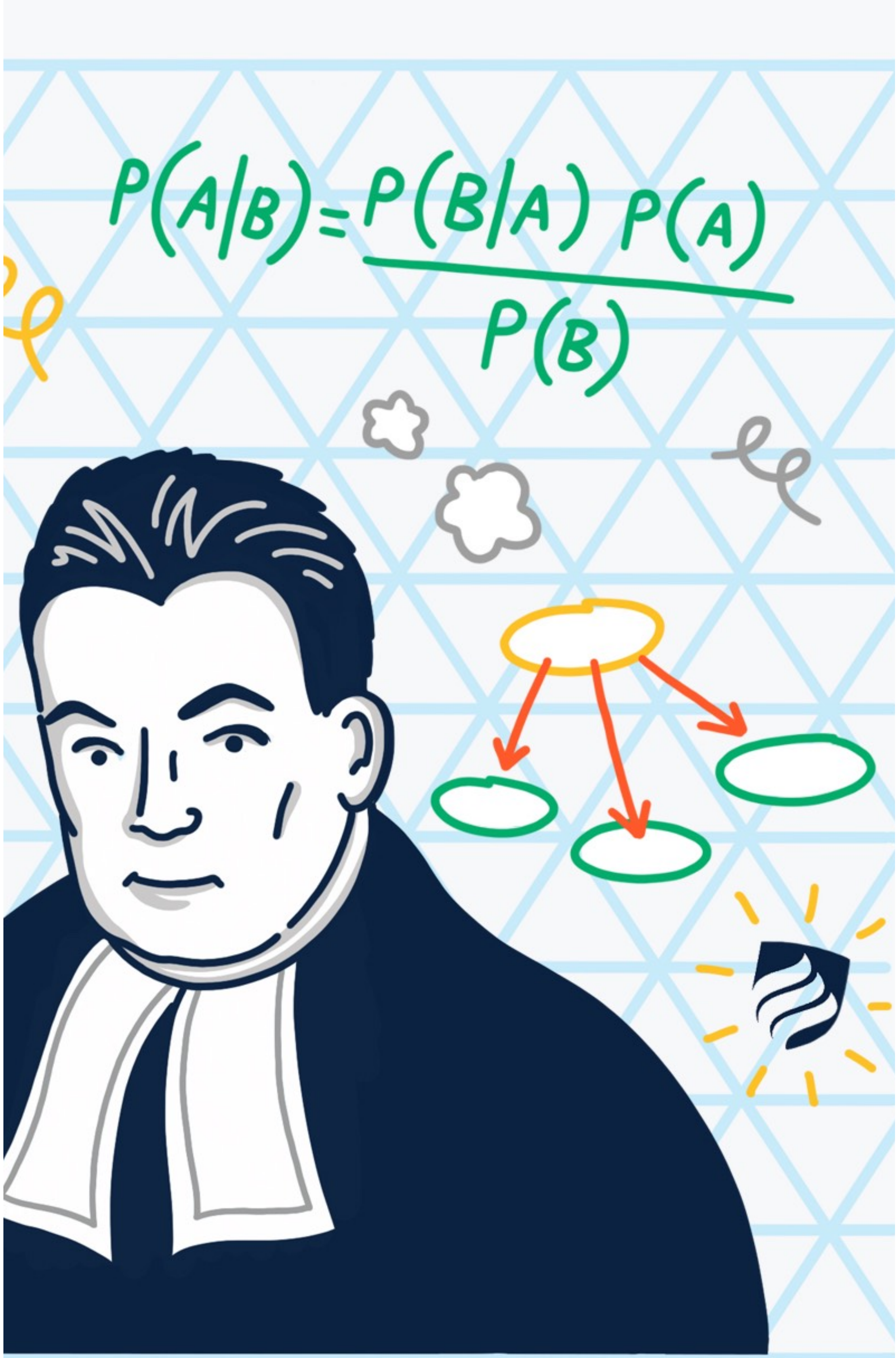
Geometric distribution: The probability distribution of the number of failures, Y, before the first success, supported on the set $\{0,1,2,\ldots\}$



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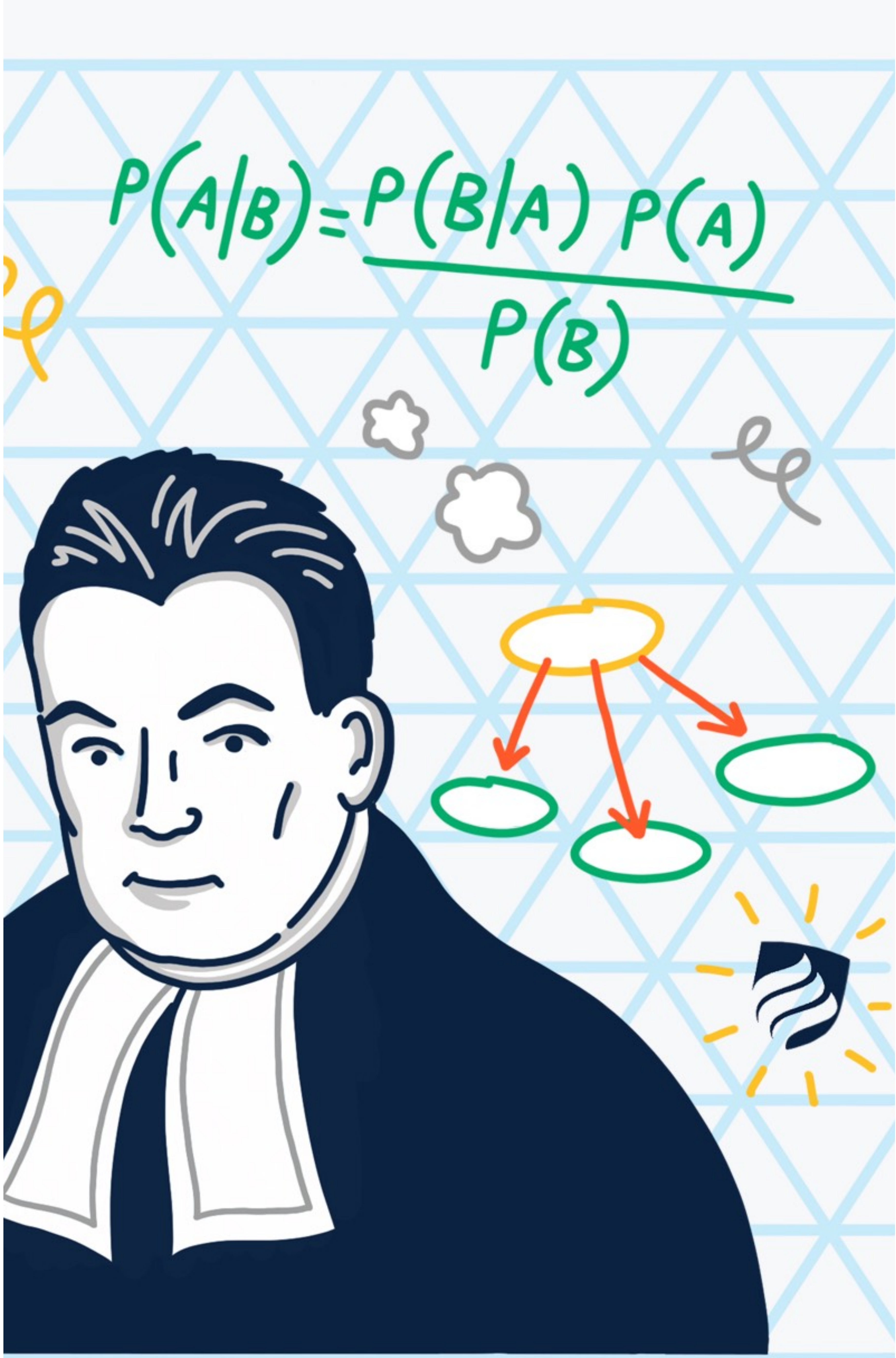
Prior:

Likelihood:

Posterior:



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$95\ \%$ credible interval? Let F^{-1} be the inverse cdf of the posterior distribution for $p \mid x$, i.e., the cdf of beta$(\alpha + 1, y + \beta)$. Then a $95\ \%$ credible interval is given by:

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The inverse gamma distribution will be useful for finding the posterior of σ^2 given iid normal data. The **inverse gamma distribution** is defined as $Y = \frac{1}{X}$, where $X \sim \Gamma(a, b)$ Let's find the pdf of $Y = \frac{1}{X}$.

(Note: in general, to find the pdf of $Y = g(X)$, we can first find the cdf, and then take the derivative.)

Photo by Universal Eye on Unsplash

Example: Let $X_1, \ldots, X_n \,\Big|\, \mu, \sigma^2 \overset{iid}{\sim} N(\mu, \sigma^2)$. Assume μ is known and we are attempting to learn about $\boldsymbol{\sigma}^2$. Our prior is $\sigma^2 \sim Inv\Gamma(a, b)$. Find the posterior distribution of $\boldsymbol{\sigma}^2 \,|\, \mathbf{x}$.