ECE 603
Probability and Random Processes

Lessons 5-6
Chapter 3
Discrete Random Variables

Objectives

- Explore random variables
- Examine Independent Random Variables
- Analyze Probability Mass Function (PMF)
- Apply Special Distributions

Rationale

- In general, to analyze random experiments, we usually focus on some numerical aspects of the experiment.
- For example, in a soccer game we may be interested in the number of goals, shots, shots on goal, corners kicks, fouls, etc.
- In a nutshell, a random variable is a real-valued variable whose value is determined by an underlying random experiment.

Prior Learning

- Basic Concepts
- Counting Methods
- Access to the online textbook: https://www.probabilitycourse.com/

Random Variables

Random experiments have sometimes numerical outputs, such as

- Lifetime of a certain product: $0 \leq T < \infty$
- Amount of money a gambler wins on a trip to the casino
- etc.

Even if the event is not numerical, it can often be considered in terms of numbers (for convenience and mathematical analysis).

Random Variables

Example. Toss a coin five times. Observe the number of heads:

$$S = \{TTTTT, TTTTH, \cdots, HHHHH\}.$$

We define a random variable that gets its value from the outcome of the random experiment:

$$X = 1, 2, 3, 4, \text{or } 5.$$

Definition: A random variable is a real-valued variable that gets its value from a random experiment.

Random Variables

Formal Definition: A random variable is a real-valued function on the sample space:

$$X:\ S \to \mathbb{R}, X(\{HHTHT\}) = 3.$$

Definition: Range of X is the set of possible values for X.

In the above example, $\mathrm{Range}(X) = R = \{1, 2, 3, 4, 5\}.$

We show random variables with capital letters X, Y, Z .

Random Variables

Example. Flip a coin twice, X = the number of heads

$$\text{Range}(X) = R = \{0, 1, 2\}.$$

Example. T: Lifetime of a certain product:

$$\text{Range} = \{X : X \in \mathbb{R}; x \geq 0\} = \mathbb{R}^+ = [0, \infty).$$

Random Variables

Countable set:

a) Finite set

b) One-to-one correspondence with Natural Numbers

$$\mathbb{N} = \{1, 2, 3, 4, \cdots\}$$

i.e., $R = \{a_1, a_2, a_3, \cdots\}$

i.e., I can "list" the elements.

Random Variables

Countably infinite sets:

$$\mathbb{N} = \{1, 2, 3, 4, \cdots\}$$

$$\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$$

List: $\mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3, \cdots\}$

$\mathbb{Q} = \{\frac{a}{b} : \; a, b \in \mathbb{Z}\}$ **Countable**

Random Variables

However $\mathbb{R}$ is Not countable, in fact

$[0, 1] = \{x \in \mathbb{R},\ 0 \leq X \leq 1\}$ **is Not countable.**

Discrete Random Variables

Definition: X is a **discrete random** variable, if its range is countable.

$$R_X = \{x_1, x_2, x_3, \ldots\}.$$

We show the values in the range by lower case letters.

Probability Mass Function

Definition: X is a discrete random variable,

$$\text{Range}(X) = R_X = \{x_1, x_2, x_3, \ldots\}.$$

The function:

$$P_X(x_k) = P(X = x_k), \text{ for } k = 1, 2, 3, \ldots,$$

is called the **probability mass function (PMF)** of X.

Probability Mass Function

Example 1. Toss a fair coin twice, $X =$ # of heads.

Find the range of X, R_X, as well as its probability mass function P_X.

Probability Mass Function

Example 2: $X =$ **# of rolls of a die until the first 6 appears.**

Find the range of $X, R_X,$ **as well as its probability mass function** P_X**.**

Probability Mass Function

Thm. For a discrete random variable with PMF $P_X(x)$ and Range

$$R_X = S_X = \{x_1, x_2, x_3, \cdots\}$$

a) $0 \le P_X(x_k) \le 1 \;\; \text{for all } x_k \in S_X.$

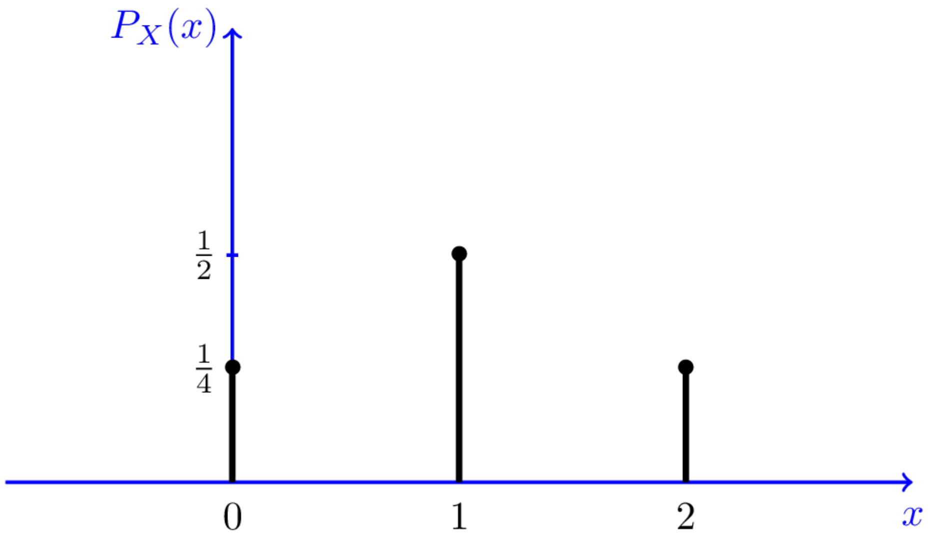
b) $\displaystyle\sum_{k=1}^{\infty} P_X(x_k) = 1.$

c) $\displaystyle A \subset S_X, P(X \in A) = P(A) = \sum_{x_k \in A} \text{Prob}\{X = x_k\} = \sum_{x_k \in A} P_X(x_k)$

Probability Mass Function

If we repeat the experiment over and over and plot the histogram, it will look like

The PMF in example 1



Independent Random Variables

Definition: Consider two discrete random variables X and Y. We say that X and Y are independent if

$$P\Big(X = x, Y = y\Big) = P(X = x)P(Y = y), \quad \text{for all } x, y.$$

In general, if two random variables are independent, then you can write

$$P\Big(X \in A, Y \in B\Big) = P(X \in A)P(Y \in B), \quad \text{for all sets } A \text{ and } B.$$

Independent Random Variables

Definition: Consider n discrete random variables $X_1, X_2, X_3, \ldots, X_n$. We say that $X_1, X_2, X_3, \ldots, X_n$ are independent if

$$P\left(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n\right)$$

$$= P(X_1 = x_1)P(X_2 = x_2)\ldots P(X_n = x_n), \quad \text{for all } x_1, x_2, \ldots, x_n.$$

Summary of Random Variables

- **Random Variables** $X : S \longrightarrow \mathbb{R}$
- **Discrete Random Variable** $R_X = \text{Range}(X)$ **is countable, i.e.,**

$$R_X = \{x_1, x_2, x_3, ...\}.$$

- **PMF:**

$$P_X(x_k) = P(X = x_k)$$

- **Independent Random Variable**

Special Distributions

Families of discrete random variable

Bernoulli RVs:

Example. Flip a coin {H,T}. Take an exam {Pass, Fail}.

$$X \sim Bernoulli(p)$$

PMF:

$$P_X(x) = \begin{cases} p & \text{for } x = 1 \\ 1-p & \text{for } x = 0 \end{cases} \quad \text{Range}(X) = \{0, 1\}.$$

$$P_X(0) = 1 - p, \quad P_X(1) = p.$$

Special Distributions

Geometric RVs:

$$X \sim Geometric(p)$$

$$R_X = \text{Range}(X) = \{1, 2, 3, \cdots\}$$

Random experiment: consider a coin with $P(H) = p$. Toss the coin repeatedly until the first heads is observed.

$X =$ The total number of coin tosses

$$X \sim Geometric(p)$$

Special Distributions

Definition. A random variable X is said to be a *geometric* random variable with parameter p, shown as $X \sim Geometric(p)$, if

$$R_X = \{1, 2, 3, \cdots\} = \mathbb{N},$$

$$P_X(k) = p(1-p)^{k-1}, \qquad k = 1, 2, 3, \ldots$$

Special Distributions

Definition. A random variable X is said to be a **binomial random variable** with parameters n and p $(P(H) = p)$, shown as $X \sim Binomial(n, p)$, if

$$R_X = \{0, 1, 2, ..., n\}$$

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, 2, \cdots, n$$

Special Distributions

Remember:

$$\underbrace{\text{HTH} \quad \ldots \quad \text{H}}_{n \text{ times}} \qquad X = k \;\Rightarrow\; \begin{array}{l} k \;\; \text{Heads } (H) \\ n-k \;\; \text{Tails } (T) \end{array} \longrightarrow \binom{n}{k}$$

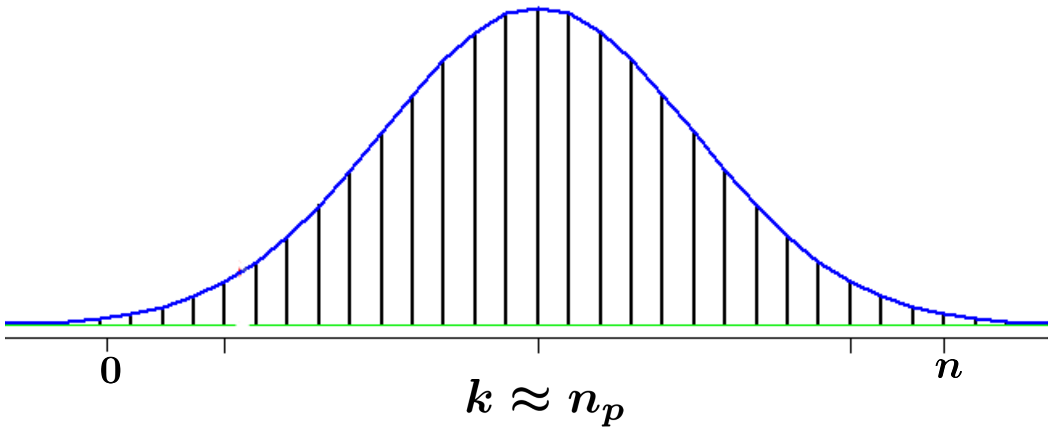
$$\underbrace{\text{HH} \;\ldots\; \text{H}}_{k \text{ Heads}}\underbrace{\text{TT} \;\ldots\; \text{T}}_{n-k \text{ Tails}} \longrightarrow p^k(1-p)^{n-k}$$

If $n = 1,$ **then** $Binomial(1, p) = Binomial(p).$

Special Distributions

Lemma. If $X_1, X_2, \ldots, X_n$ are **independent** $Bernoulli(p)$ random variable, then the random variable X define by $X = X_1 + X_2 + \ldots + X_n$, is a $Binomial(n, p)$ RV.

distribution.



Special Distributions

Pascal Distribution (Negative Binomial):

Example. You flip a coin until you observe m heads.

X **: total number of coin toss**

$$R_X = \{m, m+1, m+2, \cdots\}.$$

Find PMF.

- $A = \{X = k\}, \quad \text{or } A = B \cap C,$
- B **is the event that we observe** $m-1$ **heads in the first** $k-1$ **trials.**
- C **is the event that we observe a heads in the kth (the last) trial.**

Special Distributions

$$P(A) = P_X(k) = P(X = k)$$

$$P(A) = P(B \cap C) = P(B)P(C), \quad B \text{ and } C \text{ are independent events.}$$

$$P(C) = P(H) = p$$

Using binomial formula, $Binomial(n = k - 1, p)$

$$P(B) = \binom{k-1}{m-1} p^{m-1} (1-p)^{\left((k-1)-(m-1)\right)} = \binom{k-1}{m-1} p^{m-1} (1-p)^{k-m}.$$

$$P(A) = P(B \cap C) = P(B)P(C) = \binom{k-1}{m-1} p^{m} (1-p)^{k-m}.$$

Special Distributions

Definition. A random variable X is said to be a **Pascal random variable** with parameters m and p $(P(H) = p)$, shown as $X \sim Pascal(m, p)$, if

$$P_X(k) = \binom{k-1}{m-1} p^m (1-p)^{k-m} \qquad k = m, m+1, m+2, \dots$$

Where $0 < p < 1.$

Special Distributions

Hypergeometric Distribution:

Example. You have a bag that contains b blue marbles and r red marbles. You choose $k \leq b + r$ marbles at random (without replacement).

X : The number of blue marbles in your sample

$$P_X(x) = P(\text{you observe } x \text{ blue marbles}) = \frac{|A|}{|S|} = \frac{\binom{b}{x}\binom{r}{k-x}}{\binom{b+r}{k}}.$$

Special Distributions

Poisson Random Variable:

Poisson RVs are used to model

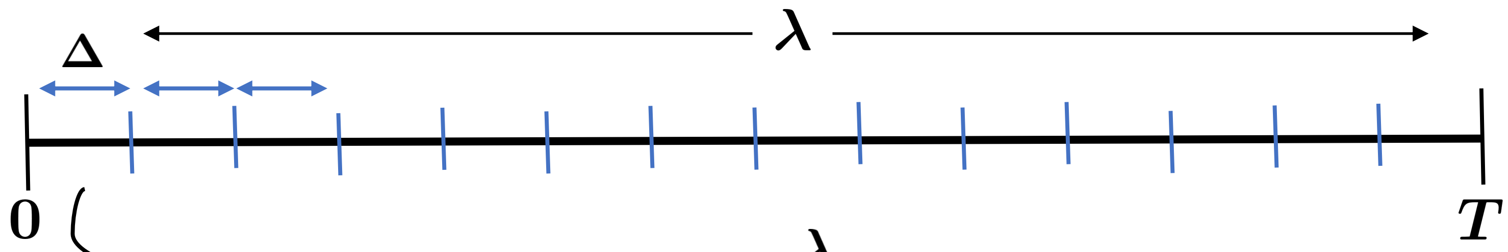
- Arrival of customers at a service facility
- Arrival of information request at a server

Counting the occurrence of certain events in an interval of time or space.

Special Distributions

Arrival of customers in an interval:

λ : the average number of arrivals in that interval



$$X \sim Bernoulli(p = \frac{\lambda}{n})$$

$$\Delta = \frac{T}{n}$$

Special Distributions

X : the total number of customers = $X_1, X_2, \cdots, X_n$

$$X \sim Binomial(n, p = \frac{\lambda}{n})$$

$$P(X = k) = P_X(k) = \binom{n}{k} p^k (1-p)^{n-k} = \binom{n}{k} (\frac{\lambda}{n})^k (1 - \frac{\lambda}{n})^{n-k}.$$

Thus,

$$\lim_{n \to \infty} P_X(k) = \frac{e^{-\lambda} \lambda^k}{k!}.$$

Special Distributions

Definition. A random variable X is said to be a **Poisson random variable** with parameter λ, shown as $X \sim Poisson(\lambda)$, if

$$R_X = \{0, 1, 2, 3, \cdots\},$$

$$P_X(k) = \frac{e^{-\lambda}\lambda^k}{k!}, \qquad k = 0, 1, 2, 3, \ldots$$

Orchestrated Conversation:
Special Distributions

Example. The number of hits to a website is a Poisson with average 2 hits per second.

a) What is the probability of no hits in 0.25 seconds?

b) What is the probability of no more than 2 hits in 1 second?

Special Distributions

Definition. A random variable X is said to be a **Uniform random variable**, shown as $X \sim Uniform(R_X)$, if

$$R_X = \{x_1, x_2, x_3, \cdots\},$$

$$P_X(x_i) = \frac{1}{|R_X|}.$$

Orchestrated Conversation: Cumulative Distribution Function (CDF)

Definition. Let X be any random variable, the function

$$F_X(x) = P(X \leq x), \text{ for all } x \in \mathbb{R},$$

is called the **CDF** of X.

Example. In a family with three children, X is the number of daughters. Find CDF.

Orchestrated Conversation: Cumulative Distribution Function (CDF)

Definition. The cumulative distribution function (CDF) of random variable X is defined as

$$F_X(x) = P(X \leq x), \text{ for all } x \in \mathbb{R}.$$

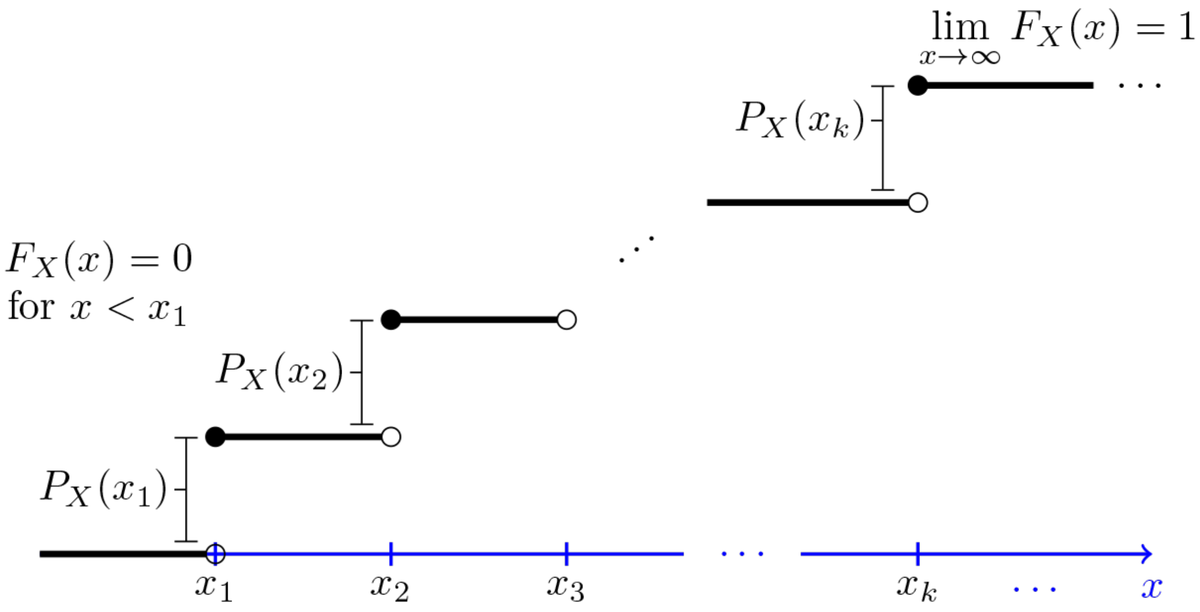
Example. Toss a coin twice, let X be the number of observed heads. Find the CDF of X .

Cumulative Distribution Function (CDF)

Example. Toss a coin twice, let X be the number of observed heads. Find the CDF of X.

Cumulative Distribution Function (CDF)

If X is a discrete random variable with range $R_X = \{x_1, x_2, x_3, \ldots\}$, such that $x_1 \le x_2 \le x_3 \le \cdots$.



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Cumulative Distribution Function (CDF)

Theorem. **Let X be a discrete random variable with range**

$$R_X = \{x_1, x_2, x_3, \dots\}.$$

a) $F_X(-\infty) = P(Y < -\infty) = 0, \quad F_X(+\infty) = 1$

b) $y \geq x \;\Rightarrow\; F_X(y) \geq F_X(x)$

c) $x_i \in R_X, \quad F_X(x_i) - F_X(x_i - \epsilon) = P_X(x_i),$ For $\epsilon > 0$ small enough.

d) $x_i \leq x < x_{i+1} \;\Rightarrow\; F_X(x) = F_X(x_i).$

Cumulative Distribution Function (CDF)

For all $a \leq b,$ we have

$$P(a < X \leq b) = F_X(b) - F_X(a).$$

Cumulative Distribution Function (CDF)

Proof:

$$P(a < X \leq b) = P(X \leq b) - P(X \leq a),$$

$$\underbrace{P(X \leq b)}_{F_X(b)} = \underbrace{P(X \leq a)}_{F_X(a)} + P(a < X \leq b),$$

$$F_X(a) = P(X \leq a) = P(X < a) + P(X = a),$$

$$\Rightarrow \; P(X < a) = F_X(a) - P(X = a).$$

Cumulative Distribution Function (CDF)

Example. Let X be a discrete random variable with range $R_X = \{1, 2, 3, \ldots\}$. Suppose the PMF of X is given by

$$P_X(k) = \frac{1}{2^k} \text{ for } k = 1, 2, 3, \ldots$$

a) Find and plot the CDF of X, $F_X(x)$.

b) Find $P(1 < X \leq 3)$.

Expectation

Expected value (= mean=average):

$$a_1, a_2, \ldots, a_n \;\Rightarrow\; \bar{a} = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Definition. Let X be a discrete random variable with range $R_X = \{x_1, x_2, x_3, \ldots\}$. The expected value of X, denoted by EX is defined as

$$EX = \mu_X = \sum_{x_k \in R_X} x_k P(X = x_k) = \sum_{x_k \in R_X} x_k P_X(x_k).$$

Expectation

Repeat the experiment N times (N large).

$$P(X=x_k) = P_X(x_k) = \frac{(\text{The number of times } X = x_k)}{N} = \frac{N_k}{N},$$

$$\Rightarrow \; N_k \approx N P_X(x_k),$$

$$\begin{aligned}
\text{Average} &= \frac{N_1 x_1 + N_2 x_2 + N_3 x_3 + \ldots}{N} \\
&\approx \frac{x_1 N P_X(x_1) + x_2 N P_X(x_2) + x_3 N P_X(x_3) + \ldots}{N} \\
&= x_1 P_X(x_1) + x_2 P_X(x_2) + x_3 P_X(x_3) + \ldots \\
&= EX.
\end{aligned}$$

Expectation

Example. Let $X \sim Bernoulli(p)$, find EX.

Expectation

Example. Let $X \sim Geometric(p)$, find EX.

Expectation

Example. **Let** $X \sim Poisson(\lambda)$, **find** EX.

Summary

Discrete RVs:

- **Range:** $R_X = \{x_1, x_2, x_3, \ldots\}.$
- **PMF:** $P_X(x_k) = P(X = x_k).$
- **CDF:** $F_X(x) = P(X \leq x), \text{ for all } x \in \mathbb{R}.$
- **Excepted value:** $\mu_X = E[X] = \sum_{x_k \in R_X} x_k P_X(x_k).$

Summary

- $X \sim Bernoulli(p), \quad EX = p.$
- $X \sim Geometric(p), \quad EX = \frac{1}{p}.$
- $X \sim Poisson(\lambda), \quad EX = \lambda.$

Functions of Random Variables

If X is a random variable and $Y = g(X)$, then Y itself is a random variable.

For example: $Y = X^2$.

$$R_X = \{x_1, x_2, x_3, ...\} \rightarrow R_Y = \{g(x)|x \in R_X\} = \{g(x_1), g(x_2), \cdots\}.$$

Then,

$$P_Y(y_k) = P(Y = y_k) = P(g(X) = y_k).$$

Functions of Random Variables

Example. Let X be a discrete random variable uniformly distributed with $P_X(k) = \frac{1}{5}$, for $k = \{-2, -1, 0, 1, 2\}$. Let $Y = |X|$.

a) Find PMF of Y, $P_Y(y)$.

b) Find EY.

Functions of Random Variables

Law of the unconscious statistician (LOTUS) for discrete random variables:

$$E[g(X)] = \sum_{x_k \in R_X} g(x_k) P_X(x_k).$$

In the previous example $g(X) = |X|,$

$$E[|X|] = |-2|.\frac{1}{5} + |-1|.\frac{1}{5} + |0|.\frac{1}{5} + |1|.\frac{1}{5} + |2|.\frac{1}{5}$$
$$= \frac{6}{5}.$$

Functions of Random Variables

Linearity of expectation:

$$Y = aX + b \Rightarrow E[Y] = aEX + b, \quad a, b \in \mathbb{R}$$

Proof: Here $g(X) = aX + b$, so using LOTUS we have

$$\begin{aligned}
E[Y] = E[g(X)] &= \sum g(x_k) P_X(x_k) = \sum_{x_k \in R_X} (ax_k + b) P_X(x_k) \\
&= \underbrace{a \sum_{x_k \in R_X} x_k P_X(x_k)}_{EX} + \underbrace{b \sum_{x_k \in R_X} P_X(x_k)}_{1} \\
&= aEX + b.
\end{aligned}$$

Functions of Random Variables

More generally (Linearity of expectation):

$$Y = X_1, X_2, \cdots, X_n \;\Rightarrow\; E[Y] = E[X_1] + E[X_2] + \cdots + E[X_n].$$

Example. $X \sim Binomial(n, p),\; EX?$

Functions of Random Variables

Example. Let $R_X = \{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\}$, such that

$$P_X(0) = P_X(\frac{\pi}{4}) = P_X(\frac{\pi}{2}) = P_X(\frac{3\pi}{4}) = P_X(\pi) = \frac{1}{5}$$

Find $E[\sin(X)]$.

Variance

The variance is a measure of how spread out the distribution of a random variable is.

$$EX = \mu_X \;\rightarrow\; E[X - \mu_X] = E[X] - E[\mu_X] = \mu_X - \mu_X = 0.$$

The variance of a random variable X, with mean $EX = \mu_X$, is defined as

$$\mathrm{Var}(X) = E\big[(X - \mu_X)^2\big], \quad \mu_X = EX.$$

Variance

The standard deviation of a random variable X is defined as

$$\mathrm{SD}(X) = \sigma_X = \sqrt{\mathrm{Var}(X)}.$$

Variance

Theorem. **Computational formula for the variance:**

$$\mathrm{Var}(X) = E\left[X^2\right] - \left(EX\right)^2.$$

Proof:

$$\begin{aligned}
\mathrm{Var}(X) &= E\left[(X-\mu_X)^2\right] \\
&= E\left[X^2 - 2\mu_X X + \mu_X^2\right] \\
&= E\left[X^2\right] - 2E\left[\mu_X X\right] + E\left[\mu_X^2\right] \qquad \text{by linearity of expectation.} \\
&= E\left[X^2\right] - 2\mu_X^2 + \mu_X^2 \\
&= E\left[X^2\right] - \mu_X^2.
\end{aligned}$$

Variance

Example. $X \sim Bernoulli(p),\ \mathrm{Var}(X)?$

Variance

Theorem. For a random variable X and real numbers a and b,

$$\operatorname{Var}(aX + b) = a^2 \operatorname{Var}(X).$$

Proof:

If $Y = aX + b$, $EY = aEX + b$. Thus,

$$\begin{aligned}
\operatorname{Var}(Y) &= E[(Y - EY)^2] \\
&= E\left[\left((aX + b) - (aEX + b)\right)^2\right] \\
&= E[a^2(X - \mu_X)^2] \\
&= a^2 E[(X - \mu_X)^2] = a^2 \operatorname{Var}(X)
\end{aligned}$$

Variance

Theorem. If $X_1, X_2, \cdots, X_n$ are **independent** random variables and $X = X_1 + X_2 + \cdots + X_n$, then

$$\mathrm{Var}(X) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \cdots + \mathrm{Var}(X_n).$$

Variance

Example. $X \sim Binomial(n, p),\ \mathrm{Var}(X)?$

Post-work for Lessons 5-6

- **Complete homework assignment for Lessons 5-6: HW#3**

Go to the online classroom for details.

To Prepare for the Next Lesson

- Read Chapter 4 in your online textbook:

  https://www.probabilitycourse.com/chapter4/4_0_0_intro.php

- Complete the Pre-work for Lessons 7-9.

Visit the online classroom for details.