ECE 603
Probability and Random Processes

Lessons 21-24
Chapter 11
Some Important Random Processes

Objectives

- Explore Poisson processes.
- Examine Markov chains.
- Explore Brownian Motion, also known as the Wiener Process

Rationale

In this lesson, you will consider some of the specific random processes that are frequently found in various applications.

You begin with a review of one of the most widely-used counting processes, the Poisson process, which is used to calculate the probability of the number of events occurring in a fixed time or space.

You then explore the Markov chain, which considers the impact one event has one the next and so on, in a “chain-event”.

Brownian motion, also known as Wiener process, is used most frequently in engineering, finance, and physical science. You will explore its role in probability theory.

Prior Learning

- Basic Concepts
- Counting Methods
- Random Variables
- Access to the online textbook: https://www.probabilitycourse.com/

Some Important Random Processes

1) Poisson Processes
2) Markov chains
   - DTMC
   - CTMC
3) Brownian Motion

Poisson Processes

Counting Processes

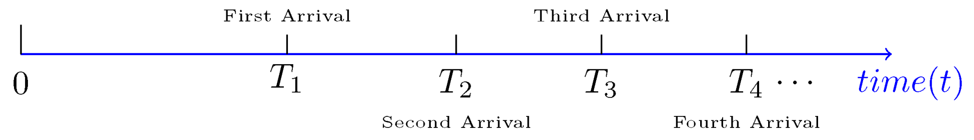
$N(t)$ **: The number of events (arrivals) by time t, starting from time 0.**

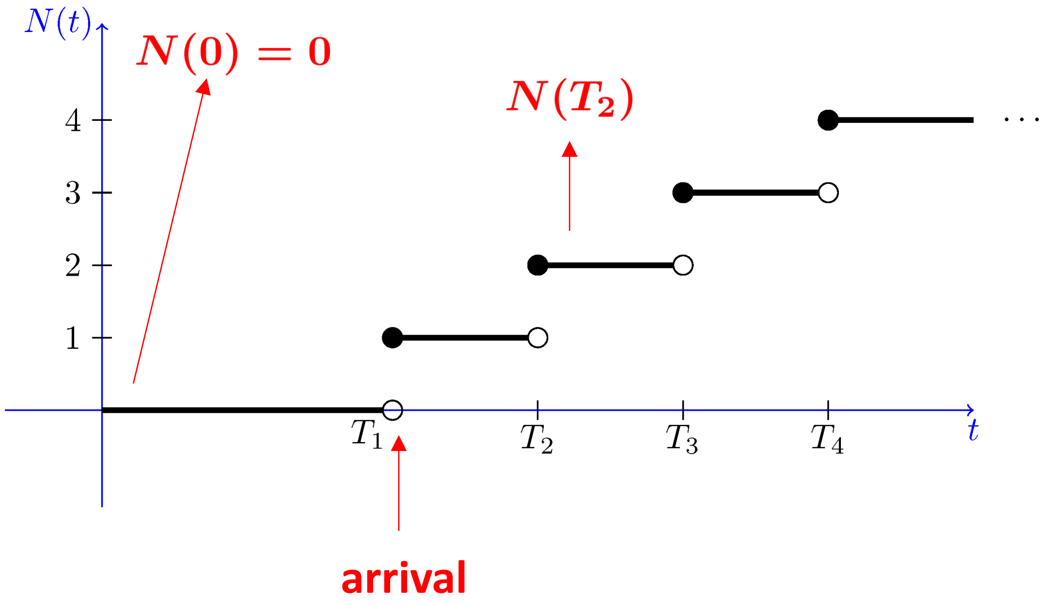
Example. The number of earthquakes in an area during $\in [0, t)$.

Example. The number of customers arriving at a store by time t.

Poisson Processes

$N(t) - N(s)$ **: the number of arrivals in** $(s,\ t)$





Poisson Processes

Definition. A random process $\{N(t), t \in [0, \infty)\}$ is said to be a **counting process** if $N(t)$ is the number of events occurred from time 0 up to and including time t. For a counting process, we assume

1. $N(0) = 0$.
2. $N(t) \in \{0, 1, 2, \cdots\}$, for all $t \in [0, \infty)$.
3. For $0 \leq s < t$, $N(t) - N(s)$ shows the number of events that occur in the interval $(s, t]$.

Poisson Processes

A counting process has **independent increments** if the numbers of arrivals in non-overlapping (disjoint) intervals are independent.

A counting process has **stationary increments** if, for all $t_2 > t_1 \geq 0$,
$N(t_2) - N(t_1)$ has the same distribution as $N(t_2 - t_1)$.

Poisson Processes

Poisson Processes: has **independent** and **stationary** increments = events occur completely at random.

Review of Poisson random variable:

$$X \sim Poisson(\mu)$$

$$P_X(k) = \frac{e^{-\mu}\mu^k}{k!} \quad \text{for } k = 0, 1, 2, \cdots$$

Poisson Processes

Properties:

1. If $X \sim Poisson(\mu)$, then $EX = \mu$, and $\mathrm{Var}(X) = \mu$.
2. If $X_i \sim Poisson(\mu_i)$, for $i = 1, 2, \cdots, n$, and the X_i's are independent, then
$$X_1 + X_2 + \cdots + X_n \sim Poisson(\mu_1 + \mu_2 + \cdots + \mu_n).$$
3. The Poisson distribution can be viewed as the limit of binomial distribution.

Poisson Processes

Theorem. Let $Y_n \sim Binomial\big(n, p = p(n)\big)$. Let $\mu > 0$ be a fixed real number, and $\lim\limits_{n \to \infty} np = \mu.$ Then, the PMF of Y_n converges to a $Poisson(\mu)$ PMF, as $n \to \infty$. That is, for any $k \in \{0, 1, 2, \ldots\}$, we have

$$\lim_{n \to \infty} P_{Y_n}(k) = \frac{e^{-\mu}\mu^k}{k!}.$$

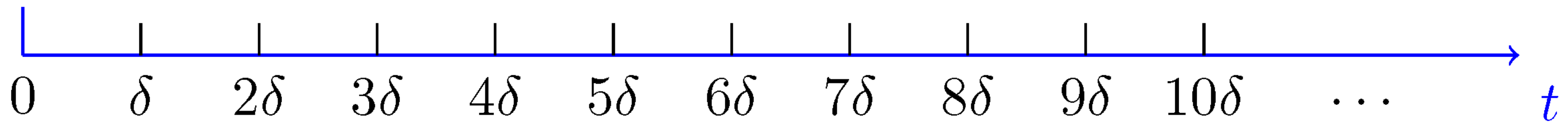
Poisson Processes

Mathematical construction:

Toss a coin in every δ seconds with $P(H) = \lambda\delta$

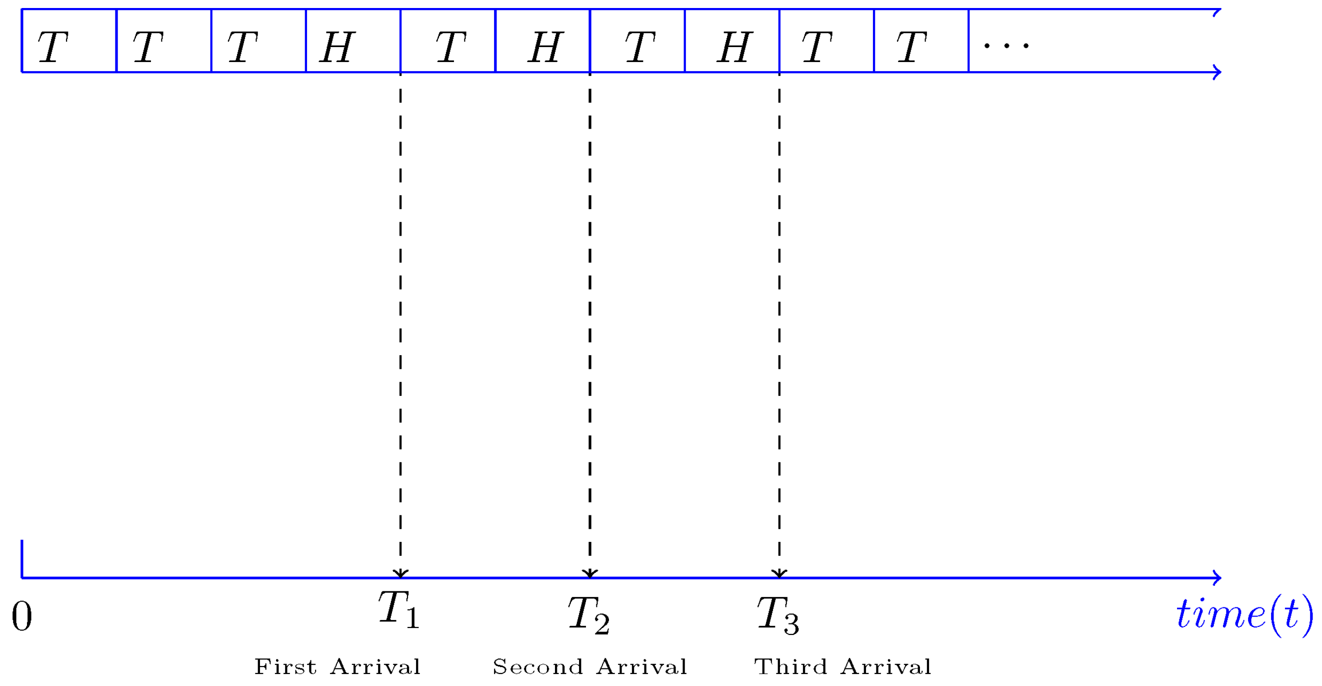
Let $\delta \to 0$

$N(t)$: Number of observed H's by time t



Poisson Processes

Mathematical construction:



Poisson Processes

$$P(H) = p = \lambda\delta$$

$N(t)$: **the number of arrivals (number of heads) from time 0 to time** t

$$n = \frac{t}{\delta}, \quad (0, t]$$

Poisson Processes

Toss the coin n time, the number of heads

$$N(t) \sim Binomial(n, p), \quad n = \frac{t}{\delta}, \;\; p = \lambda\delta$$

$$\delta \to 0 \qquad\qquad P(N(t) = k) = \binom{n}{k} P^k (1-p)^{n-k} \;\to\; \frac{e^{-\lambda^t}(\lambda^t)^k}{k!}$$
$$n \to \infty$$

$$(0, t] \;\to\; \lambda t$$

Poisson Processes

The Poisson Process:

Let $\lambda > 0$ be fixed. The counting process $\{N(t), t \in [0, \infty)\}$ is called a Poisson process with rates λ if all the following conditions hold:

1. $N(0) = 0.$
2. $N(t)$ has independent increments;
3. The number of arrivals in any interval of length $\tau > 0$ has $Poisson(\lambda\tau)$ distribution.

Poisson Processes

Poisson Processes:

1) Independent Increments
2) Stationary increments

Poisson Processes

Example. **The number of customers arriving at a grocery store can be modeled by a Poisson process with intensity $\lambda = 10$ customers per hour.**

1) **Find the probability that there are 2 customers between 10:00 and 10:20.**
2) **Find the probability that there are 3 customers between 10:00 and 10:20 and 7 customers between 10:20 and 11.**

Poisson Processes

Consider the interval $(0, \Delta]$ where Δ is small.

$$X = N(\Delta) \sim Poisson(\lambda\Delta)$$

$$\begin{aligned} P(N(\Delta) = 0) &= \frac{e^{-\lambda^{\Delta}}(\lambda^{\Delta})^{0}}{0!} = e^{-\lambda\Delta} \\ &= 1 - \lambda\Delta + \frac{\lambda^2}{2}\Delta^2 - \cdots \quad \text{(Taylor Series)} \\ &\approx 1 - \lambda\Delta + o(\Delta). \end{aligned}$$

Poisson Processes

$o(\Delta)$ shows a function that is negligible compared to Δ , as $\Delta \to 0$. More precisely, $g(\Delta) = o(\Delta)$ means that

$$\lim_{\Delta \to 0} \frac{g(\Delta)}{\Delta} = 0.$$

$$P(N(\Delta) = 0) = 1 - \lambda\Delta + o(\Delta).$$

UMASS AMHERST

Poisson Processes

$$\begin{aligned} P(N(\Delta)=1) &= \frac{e^{-\lambda\Delta}\lambda\Delta}{1!} \\ &= \lambda\Delta\left(1-\lambda\Delta+\frac{\lambda^2}{2}\Delta^2-\cdots\right) \quad \text{(Taylor Series)} \\ &= \lambda\Delta+o(\Delta). \end{aligned}$$

$$P(N(\Delta)=1)=\lambda\Delta+o(\Delta).$$

$$P(N(\Delta)\geq 2)=o(\Delta).$$

Poisson Processes

The Second Definition of the Poisson Process:

Let $\lambda > 0$ be fixed. The counting process $\{N(t), t \in [0, \infty)\}$ is called a Poisson process with rate λ if all the following conditions hold:

1. $N(0) = 0$.
2. $N(t)$ has **independent** and **stationary** increments;
3. We have

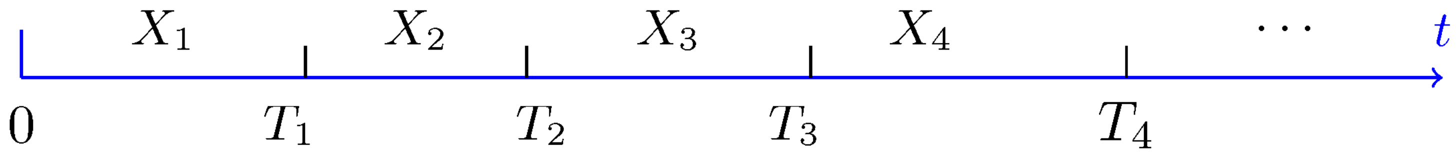
Poisson Processes

The Second Definition of the Poisson Process

$$P(N(\Delta) = 0) = 1 - \lambda\Delta + o(\Delta),$$
$$P(N(\Delta) = 1) = \lambda\Delta + o(\Delta),$$
$$P(N(\Delta) \geq 2) = o(\Delta).$$

Poisson Processes

Arrival and Inter-arrival times



$$P(X_1 > t) = P\Big(\underbrace{\text{no arrival in } (0,t]}_{Poisson(\lambda t)}\Big) = e^{-\lambda t}.$$

$$F_{X_1}(t) = P(X_1 \le t) = 1 - e^{-\lambda t} \quad t > 0$$

Poisson Processes

$$X_1 \sim Exponential(\lambda)$$

$$X_2 \sim Exponential(\lambda)$$

$\longrightarrow$ **Independent**

Poisson Processes

Inter-arrival Times for Poisson Processes

if $N(t)$ is a Poisson process with rate λ, then the inter-arrival times $X_1, X_2, \cdots$ are independent and

$$X_i \sim Exponential(\lambda), \quad \text{for } i = 1, 2, 3, \cdots .$$

Poisson Processes

Example. Let $N(t)$ be a Poisson process with intensity $\lambda = 2$, and let $X_1, X_2, \cdots$ be the corresponding inter-arrival times.

a. Find the probability that the first arrival occurs after $t = 0.5$, i.e., $P(X_1 > 0.5)$.

b. Given that we have had no arrivals before $t = 1$, find $P(X_1 > 3)$.

c. Given that the third arrival occurred at time $t = 2$, find the probability that the fourth arrival occurs after $t = 4$.

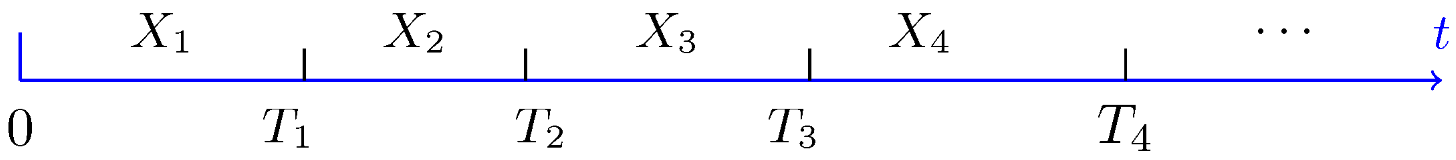
Poisson Processes

d. I start watching the process at time $t = 10$. Let T be the time of the first arrival that I see. In other words, T is the first arrival after $t = 10$. Find ET and $\mathrm{Var}(T)$.

e. I start watching the process at time $t = 10$. Let T be the time of the first arrival that I see. Find the conditional expectation and the conditional variance of T given that I am informed that the last arrival occurred at time $t = 9$.

Poisson Processes

Arrival Times:



$$
\begin{aligned}
T_1 &= X_1, \\
T_2 &= X_1 + X_2, \\
T_3 &= X_1 + X_2 + X_3, \\
&\vdots
\end{aligned}
$$

Poisson Processes

Arrival Times:

n^{th} arrival

$$T_n = X_1 + X_2 + \cdots + X_n$$

$$X_i \sim Exponential(\lambda)$$

$$\Rightarrow \; T_n \sim Gamma(n, \lambda)$$

Merging and Splitting Poisson Processes

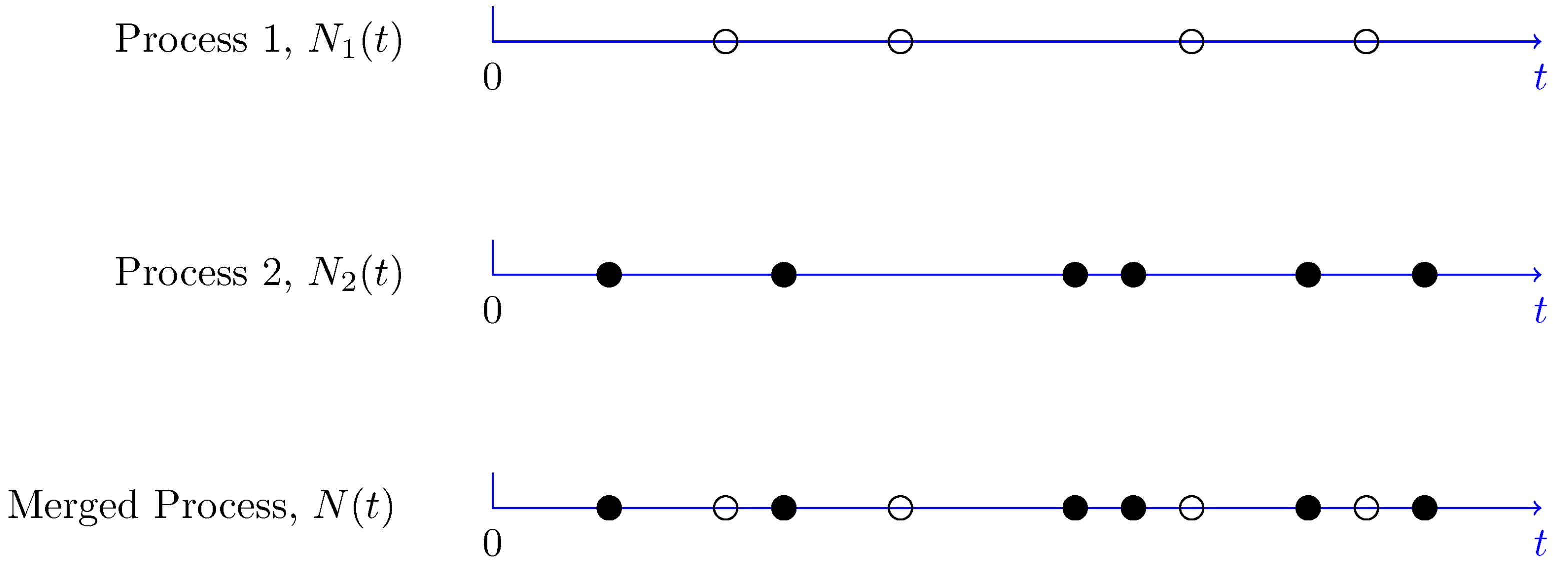
Merging Independent Poisson Processes:

Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rates λ_1 and λ_2 respectively.

$$N_1(t) \sim Poisson(\lambda_1), \qquad N_2(t) \sim Poisson(\lambda_2)$$

$$N(t) = N_1(t) + N_2(t)$$: **Merged Process**

Merging and Splitting Poisson Processes



$$N(t) \sim Poisson(\lambda = \lambda_1 + \lambda_2)$$

Merging and Splitting Poisson Processes

Merging Independent Poisson Processes

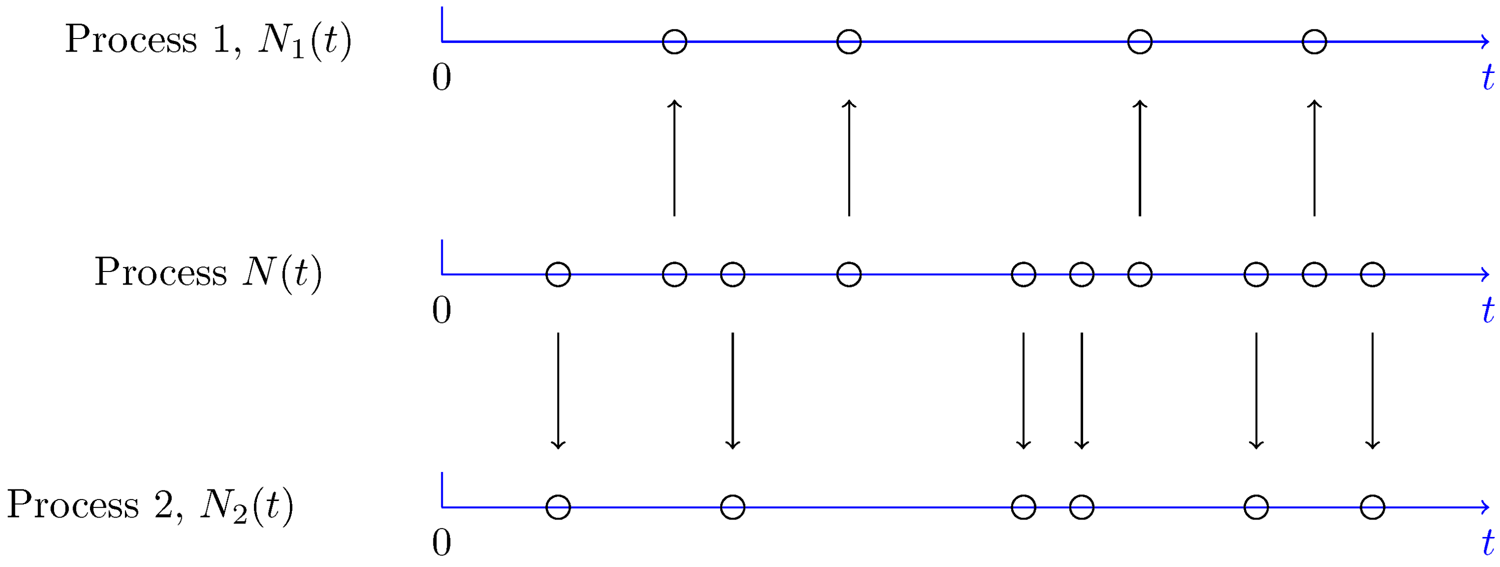
let $N_1(t), N_2(t), \cdots, N_m(t)$ be m independent Poisson processes with rates $\lambda_1, \lambda_2, \cdots, \lambda_m$. Let also

$$N(t) = N_1(t) + N_2(t) + \cdots + N_m(t), \quad \text{for all } t \in [0, \infty).$$

Then, $N(t)$ is a Poisson process with rate $\lambda_1 + \lambda_2 + \cdots + \lambda_m$.

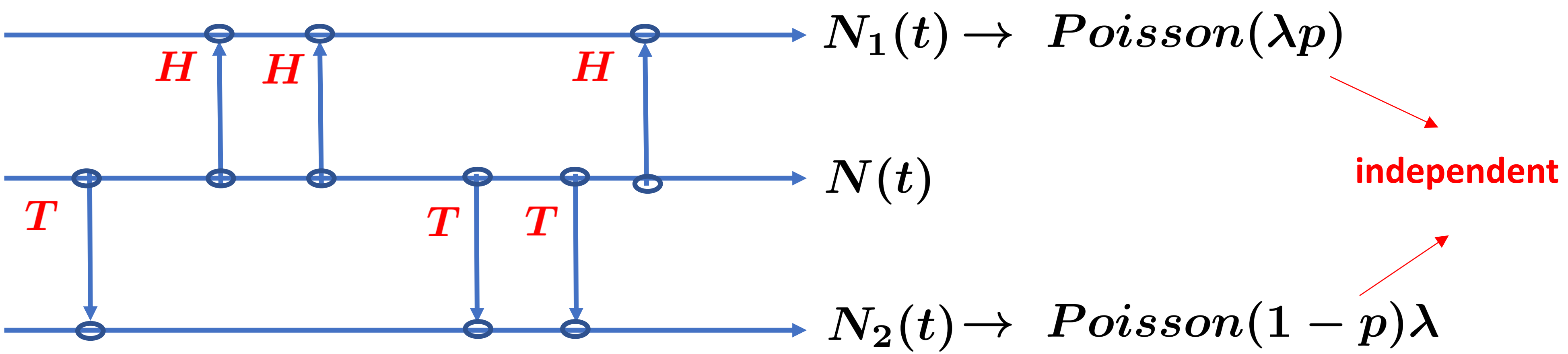
Merging and Splitting Poisson Processes

Splitting (Thinning) of Poisson Processes:



Merging and Splitting Poisson Processes

Toss a coin to decide up or down: $P(H) = p$



Merging and Splitting Poisson Processes

Splitting a Poisson Processes

Let $N(t)$ be a Poisson process with rate λ . Here, we divide $N(t)$ to two processes $N_1(t)$ and $N_2(t)$ in the following way. For each arrival, a coin with $P(H) = p$ is tossed. If the coin lands heads up, the arrival is sent to the first process ($N_1(t)$), otherwise it is sent to the second process. The coin tosses are independent of each other and are independent of $N(t)$. Then,

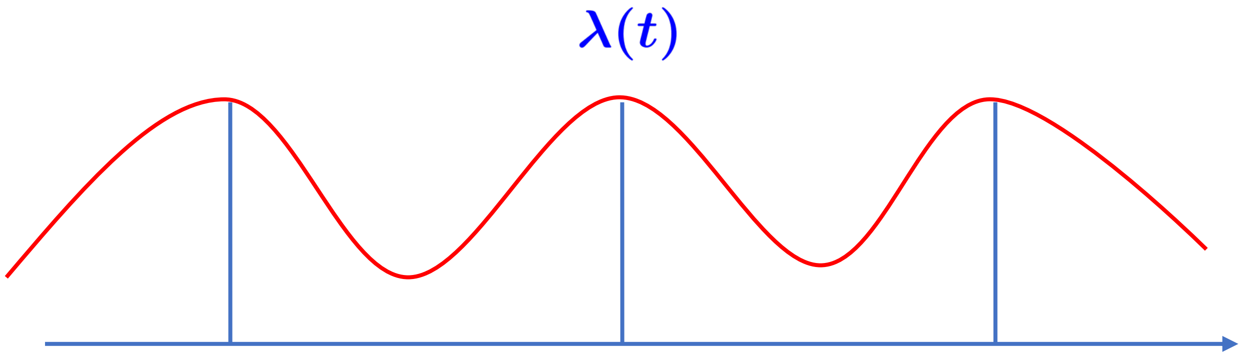
Merging and Splitting Poisson Processes

Splitting a Poisson Processes

1. $N_1(t)$ is a Poisson process with rate λp ;
2. $N_2(t)$ is a Poisson process with rate $\lambda(1-p)$;
3. $N_1(t)$ and $N_2(t)$ are independent.

Nonhomogeneous Poisson Processes

Similar to Poisson process but $\lambda = \lambda(t)$.



$$P(H) = \lambda\delta = \lambda(t)\delta$$

Nonhomogeneous Poisson Processes

Nonhomogeneous Poisson Process

Let $\lambda(t) : [0, \infty) \mapsto [0, \infty)$ be an integrable function. The counting process .$\{N(t), t \in [0, \infty)\}$ is called a nonhomogeneous Poisson process with rate $\lambda(t)$ if all the following conditions hold.

1. $N(0) = 0.$
2. $N(t)$ **has independent increments;**

Nonhomogeneous Poisson Processes

3. for any $t \in [0, \infty)$, we have

$$\begin{aligned}
P(N(t+\Delta) - N(t) = 0) &= 1 - \lambda(t)\Delta + o(\Delta), \\
P(N(t+\Delta) - N(t) = 1) &= \lambda(t)\Delta + o(\Delta), \\
P(N(t+\Delta) - N(t) \geq 2) &= o(\Delta).
\end{aligned}$$

Nonhomogeneous Poisson Processes

Thus

1. $N(0) = 0.$

2. **Independent**

3. $N(t+s) - N(t) \sim Poisson\left(\int_t^{t+s} \lambda(\alpha) d\alpha\right).$

Nonhomogeneous Poisson Processes

Example. Let $\{N(t), t \in [0, \infty)\}$ be a Poisson process with rate λ , and X_1 be its first arrival time. Show that given $N(t) = 1$, then X_1 is uniformly distributed in $(0, t]$. That is, show that

$$\boldsymbol{P(X_1 \leq x | N(t) = 1) = \frac{x}{t}, \quad \text{for } 0 \leq x \leq t.}$$

Nonhomogeneous Poisson Processes

More generally:

Given $N(t) = n$ the locations of arrivals are uniformly and independently distribution in $(0, t]$.

Nonhomogeneous Poisson Processes

Example. **Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rates $\lambda_1 = 1$ and $\lambda_2 = 2$, respectively. Find the probability that the second arrival in $N_1(t)$ occurs before the third arrival in $N_2(t)$. *Hint* : One way to solve this problem is to think of $N_1(t)$ and $N_2(t)$ as two processes obtained from splitting a Poisson process.**

Discrete-Time Markov Chains

$X_0, X_1, X_2, X_3, \cdots \leftarrow$ a random process

In real life usually there is dependence between X_i's .

Markov chain:

X_m depends on X_{m-1} but not on the other pervious values.

Usually X_i shows the state of the system at time i .

W.L.G $\quad X_i \in \{0, 1, 2, \cdots\}$

Discrete-Time Markov Chains

Discrete-Time Markov Chains

Consider the random process $\{X_n, n = 0, 1, 2, \cdots\}$, where $R_{X_i} = S \subset \{0, 1, 2, \cdots\}$. We say that this process is a Markov chain if

$$
\begin{aligned}
P(X_{m+1} = j | X_m = i, X_{m-1} = i_{m-1}, \cdots, X_0 = i_0) \\
= P(X_{m+1} = j | X_m = i),
\end{aligned}
$$

for all $m, j, i, i_0, i_1, \cdots i_{m-1}$. If the number of states is finite, e.g., $S = \{0, 1, 2, \cdots, r\}$, we call it a finite Markov chain.

Discrete-Time Markov Chains

Transition probabilities

$$p_{ij} = P(X_{m+1} = j | X_m = i).$$

State Transition Matrix and Diagram

State Transition Matrix:

We often list the transition probabilities in a matrix. The matrix is called the state transition matrix or transition probability matrix and is usually shown by P.

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1r} \\ p_{21} & p_{22} & \cdots & p_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ p_{r1} & p_{r2} & \cdots & p_{rr} \end{bmatrix}.$$

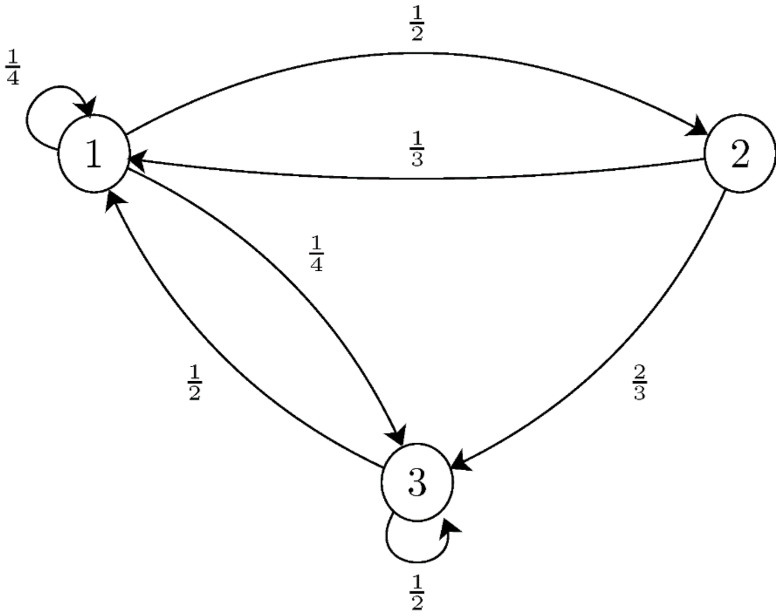
State Transition Matrix and Diagram

State Transition Diagram:

A Markov chain is usually shown by a state transition diagram.

Example.

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$



State Transition Matrix and Diagram

Example. **Consider the Markov chain shown in pervious example.**

a) **Find** $P(X_4 = 3|X_3 = 2)$.

b) **Find** $P(X_3 = 1|X_2 = 1)$.

c) **If we know** $P(X_0 = 1) = 1/3$, **find** $P(X_0 = 1, X_1 = 2)$.

d) **If we know** $P(X_0 = 1) = 1/3$, **find** $P(X_0 = 1, X_1 = 2, X_2 = 3)$.

Probability Distributions

State Probability Distributions:

Consider a Markov chain $\{X_n, n = 0, 1, 2, ...\}$, where $X_n \in S = \{1, 2, \cdots, r\}$.

$$\pi^{(0)} = \begin{bmatrix} P(X_0 = 1) & P(X_0 = 2) & \cdots & P(X_0 = r) \end{bmatrix}.$$

time 0

$$\pi^{(1)} = \begin{bmatrix} P(X_1 = 1) & P(X_1 = 2) & \cdots & P(X_1 = r) \end{bmatrix}.$$

In general:

$$\pi^{(n)} = \begin{bmatrix} P(X_n = 1) & P(X_n = 2) & \cdots & P(X_n = r) \end{bmatrix}.$$

Probability Distributions

If we have $\pi^{(0)}$, how do we find $\pi^{(1)}$?

$$P(X_1 = 1) = \sum_{k=1}^{r} P(X_1 = 1 | X_0 = k) P(X_0 = k)$$
$$= \sum_{k=1}^{r} P_{k1} P(X_0 = k)$$

Probability Distributions

Generally:

$$P(X_1 = j) = \sum_{k=1}^{r} P(X_0 = k) P_{kj}$$

$$\begin{bmatrix} P(X_0 = 1) & P(X_0 = 2) & \cdots & P(X_0 = r) \end{bmatrix} \cdot \begin{bmatrix} P_{1j} \\ P_{2j} \\ \cdot \\ \cdot \\ \cdot \\ P_{rj} \end{bmatrix}$$

Probability Distributions

In matrix form:

$$\pi^{(1)} = \begin{bmatrix} P(X_0 = 1) & P(X_0 = 2) & \cdots & P(X_0 = r) \end{bmatrix} \cdot \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1r} \\ p_{21} & p_{22} & \cdots & p_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ p_{r1} & p_{r2} & \cdots & p_{rr} \end{bmatrix}$$

$$= \pi^{(0)} P.$$

Probability Distributions

More generally:

$$\pi^{(n+1)} = \pi^{(n)}P, \quad \text{for } n = 0, 1, 2, \cdots ;$$
$$\pi^{(n)} = \pi^{(0)}P^n, \quad \text{for } n = 0, 1, 2, \cdots .$$

Probability Distributions

Example. Consider a system that can be in one of two possible states, $S = \{0, 1\}$. In particular, suppose that the transition matrix is given by

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

Suppose that the system is in state 0 at time $n = 0$, i.e., $X_0 = 0$.

a) Draw the state transition diagram.

b) Find the probability that the system is in state 1 at time $n = 3$.

Probability Distributions

n -Step Transition Probabilities:

LOTP

$$p_{ij}^{(2)} = P(X_2 = j | X_0 = i) = \sum_{k \in S} p_{ik} p_{kj}$$

Probability of going from state i to state j in exactly two transitions.

Probability Distributions

Generally:

p_{ij}^{m} : Probability of going from state i to state j in exactly m transitions.

$$
\begin{aligned}
p_{ij}^{(m+n)} &= P(X_{m+n} = j | X_0 = i) \\
&= \sum_{k \in S} p_{ik}^{(m)} p_{kj}^{(n)}.
\end{aligned}
$$

LOTP

Probability Distributions

Two step transition matrix:

$$P^{(2)} = \begin{bmatrix} p_{11}^{(2)} & p_{12}^{(2)} & \cdots & p_{1r}^{(2)} \\ p_{21}^{(2)} & p_{22}^{(2)} & \cdots & p_{2r}^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ p_{r1}^{(2)} & p_{r2}^{(2)} & \cdots & p_{rr}^{(2)} \end{bmatrix}$$

$$= \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1r} \\ p_{21} & p_{22} & \cdots & p_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ p_{r1} & p_{r2} & \cdots & p_{rr} \end{bmatrix} \cdot \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1r} \\ p_{21} & p_{22} & \cdots & p_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ p_{r1} & p_{r2} & \cdots & p_{rr} \end{bmatrix} = P^2.$$

Probability Distributions

The Chapman-Kolmogorov equation can be written as

$$\begin{aligned} p_{ij}^{(m+n)} &= P(X_{m+n} = j | X_0 = i) \\ &= \sum_{k \in S} p_{ik}^{(m)} p_{kj}^{(n)}. \end{aligned}$$

The n-step transition matrix is given by

$$P^{(n)} = P^n, \quad \text{for } n = 1, 2, 3, \cdots .$$

Classification of States

Two states i and j are said to **communicate**, written as $i \leftrightarrow j$, if they are **accessible** from each other. In other words,

$$i \leftrightarrow j \quad \text{means} \quad i \to j \text{ and } j \to i.$$

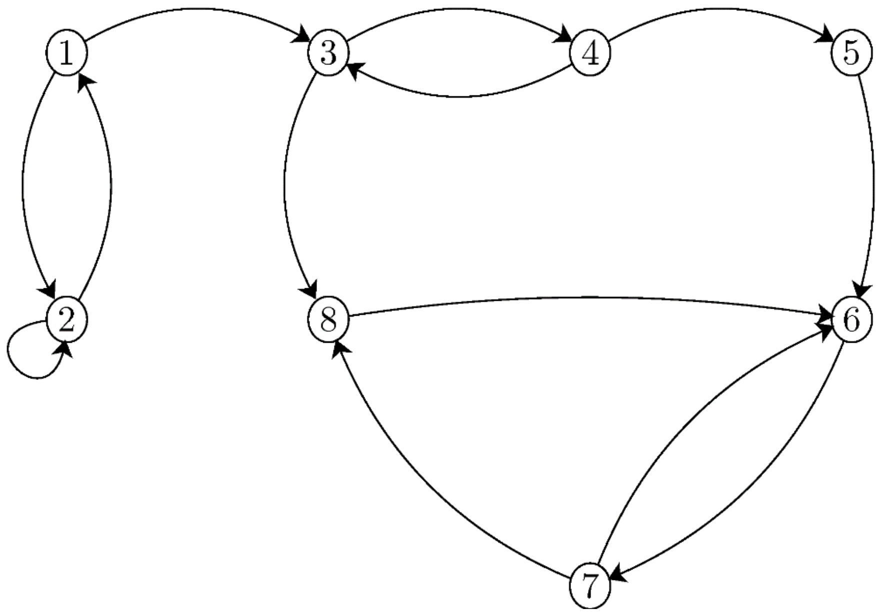
Classification of States

Communication is an equivalence relation. That means that

- **Every state communicates with itself** $i \leftrightarrow i$;
- **If** $i \leftrightarrow j$, **then** $j \leftrightarrow i$;
- **If** $i \leftrightarrow j$ **and** $j \leftrightarrow k$, **then** $i \leftrightarrow k$.

Classification of States

Example. **Consider the Markov chain shown in the following Figure. It is assumed that when there is an arrow from state i to state j, then $p_{ij} > 0$. Find the equivalence classes for this Markov chain.**



Classification of States

A Markov chain is said to be irreducible if all states communicate with each other.

Recurrent and transient states:

$$f_{ii} = P(X_n = i, \text{ for some } n \geq 1 | X_0 = i).$$

recurrent if $f_{ii} = 1;$

transient if $f_{ii} < 1.$

Classification of States

Consider a discrete-time Markov chain. Let V be the total number of visits to state i.

a. If i is a **recurrent** state, then

$$P(V = \infty | X_0 = i) = 1.$$

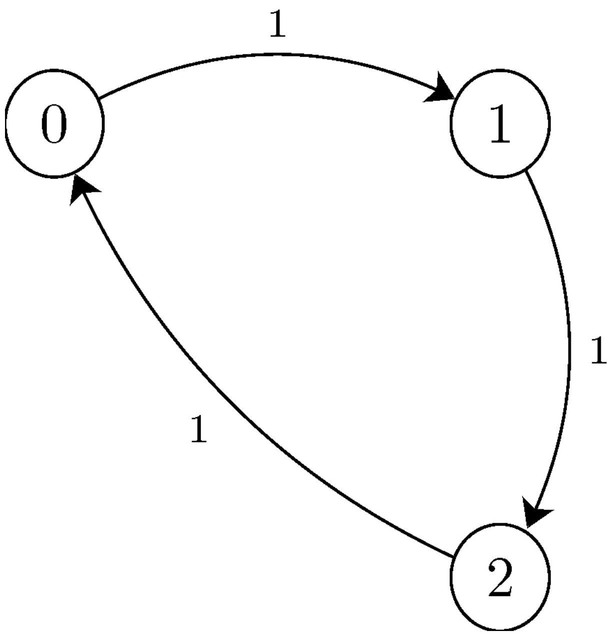
b. If i is a **transient** state, then

$$V | X_0 = i \sim Geometric(1 - f_{ii}).$$

Classification of States

Periodicity:

Consider the Markov chain shown in the following Figure. There is a periodic pattern in this chain.



Classification of States

The period of a state i is the largest integer d satisfying the following property: $p_{ii}^{(n)} = 0$, where n is not divisible by d. The period of i is shown by $d(i)$. If $p_{ii}^{(n)} = 0$, for all $n > 0$, then we let $d(i) = \infty$.

- If $d(i) > 1$, we say that state i is periodic.
- If $d(i) = 1$, we say that state i is aperiodic.

➢ If $i \leftrightarrow j$, then $d(i) = d(j)$.

Classification of States

Consider a finite irreducible Markov chain X_n:

a) If there is a self-transition in the chain ($p_{ii} > 0$ for some i), then the chain is aperiodic.

b) Suppose that you can go from state i to state i in l steps, i.e., $p_{ii}^{(l)} > 0$. Also suppose that $p_{ii}^{(m)} > 0$. If $\gcd(l, m) = 1$, then state i is aperiodic.

Classification of States

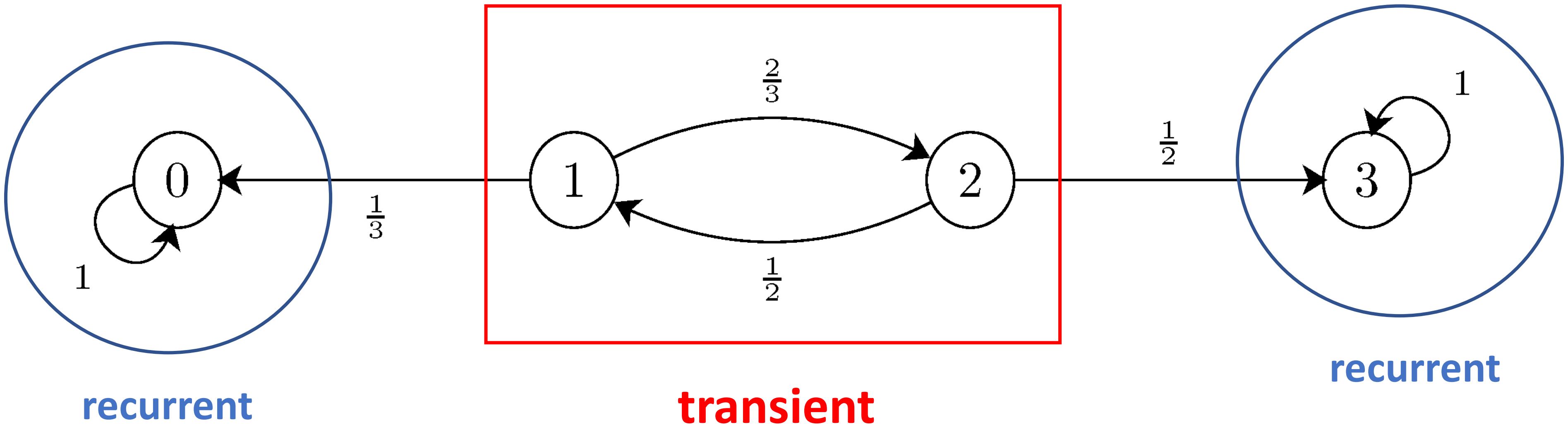
c) The chain is aperiodic if and only if there exists a positive integer n such that all elements of the matrix P^n are strictly positive, i.e.,

$$p_{ij}^{(n)} > 0, \quad \text{for all } i, j \in S.$$

Using the Law of Total Probability with Recursion

Absorption Probabilities:

Example.



Using the Law of Total Probability with Recursion

$$
\begin{aligned}
a_0 &= P(\text{absorption in } 0 | X_0 = 0), \\
a_1 &= P(\text{absorption in } 0 | X_0 = 1), \\
a_2 &= P(\text{absorption in } 0 | X_0 = 2), \\
a_3 &= P(\text{absorption in } 0 | X_0 = 3).
\end{aligned}
$$

Using the Law of Total Probability with Recursion

How do we find a_i ?

Main Idea: Apply the law of total probability

$$a_i = \sum_k a_k p_{ik}, \qquad \text{for } i = 0, 1, 2, 3$$

Using the Law of Total Probability with Recursion

Thus,

$$
\begin{aligned}
a_0 &= a_0, \\
a_1 &= \frac{1}{3}a_0 + \frac{2}{3}a_2, \\
a_2 &= \frac{1}{2}a_1 + \frac{1}{2}a_3, \\
a_3 &= a_3.
\end{aligned}
$$

We also know $a_0 = 1$ and $a_3 = 0$. Then we have

$$a_1 = \frac{1}{2}, \qquad a_2 = \frac{1}{4}.$$

Using the Law of Total Probability with Recursion

$$b_i = P(\text{absorption in } 1 | X_0 = i)$$

Since $a_i + b_i = 1$, we conclude

$$b_0 = 0, \quad b_1 = \frac{1}{2}, \quad b_2 = \frac{3}{4}, \quad b_3 = 1.$$

Using the Law of Total Probability with Recursion

Similar idea (using LOTP) can be used to find.

- **Mean Hitting Times:**

  The expected time until the process hits a certain set of state for the first time.

- **Mean Return Times:**

  The expected time until returning to state i .

Using the Law of Total Probability with Recursion

Mean Hitting Times

Consider a finite Markov chain $\{X_n, n = 0, 1, 2, \cdots\}$ with state space $S = \{0, 1, 2, \cdots, r\}$. Let $A \subset S$ be a set of states. Let T be the first time the chain visits a state in A. For all $i \in S$, define

$$t_i = E[T | X_0 = i].$$

Using the Law of Total Probability with Recursion

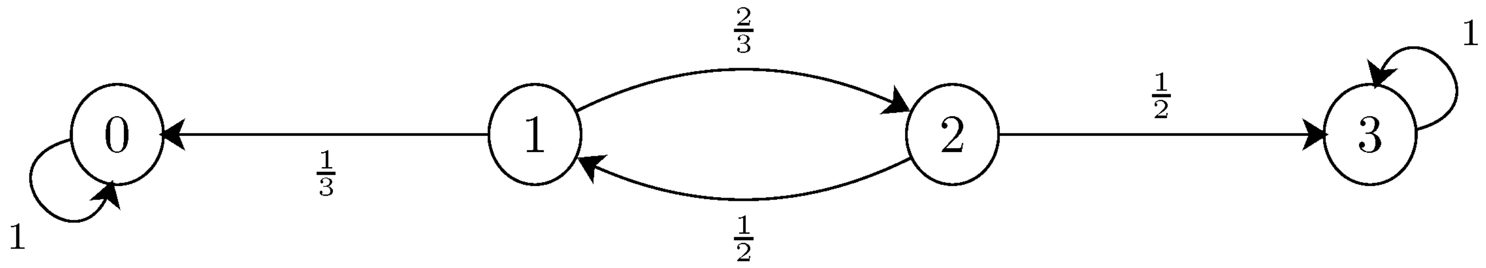
Mean Hitting Times

By the above definition, we have $t_j = 0$, for all $j \in A$. To find the unknown values of t_i's, we can use the following equations

$$t_i = 1 + \sum_k t_k p_{ik}, \qquad \text{for } i \in S - A.$$

Using the Law of Total Probability with Recursion

Example.



t_i : The number of steps needed until the chain hits the state 0 or 3 given $X_0 = i.$

$t_0 = t_3 = 0,$

Using the Law of Total Probability with Recursion

$$t_1 = 1 + \frac{1}{3}t_0 + \frac{2}{3}t_2 = 1 + \frac{2}{3}t_2.$$

Similarly, we can write

$$t_2 = 1 + \frac{1}{2}t_1 + \frac{1}{2}t_3 = 1 + \frac{1}{2}t_1.$$

Solving the above equations, we obtain

$$t_1 = \frac{5}{2}, \quad t_2 = \frac{9}{4}.$$

Using the Law of Total Probability with Recursion

Mean Return Times:

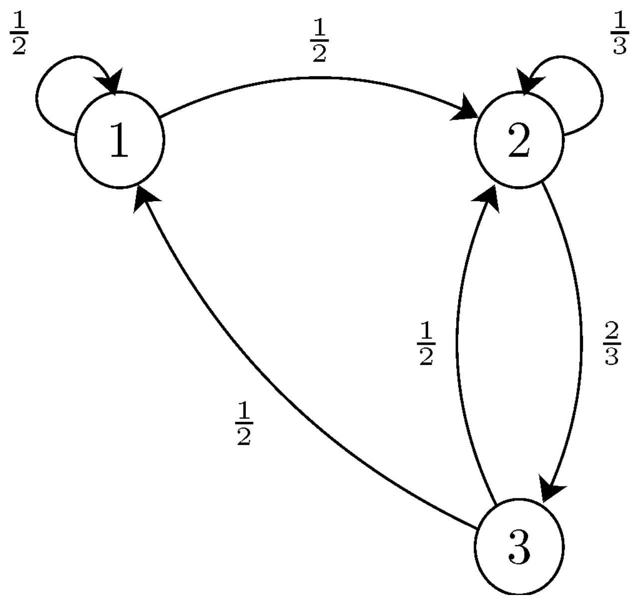
$$r_l = E[R_l | X_0 = l].$$

R_l : **return to state** l.

Using the Law of Total Probability with Recursion

Example. **Consider the Markov chain shown in the following Figure. Let t_k be the expected number of steps until the chain hits state 1 for the first time, given that $X_0 = k$. Clearly, $t_1 = 0$. Also, let r_1 be the mean return time to state 1.**

1. **Find t_2 and t_3.**
2. **Find r_1.**



Using the Law of Total Probability with Recursion

Mean Return Times

Consider a finite irreducible Markov chain $\{X_n, n = 0, 1, 2, \cdots\}$ with state space $S = \{0, 1, 2, \cdots, r\}$. Let $l \in S$ be a state. Let r_l be the **mean return time** to state l. Then

$$r_l = 1 + \sum_k t_k p_{lk},$$

Using the Law of Total Probability with Recursion

Mean Return Times

where t_k is the expected time until the chain hits state l given $X_0 = k$. Specifically,

$$t_l = 0,$$

$$t_k = 1 + \sum_j t_j p_{kj}, \qquad \text{for } k \neq l.$$

Stationary and Limiting Distributions

long-term behavior of Markov chains

The fraction of time that the Markov chain spends at state i as time $n \to \infty$.

$$\pi^{(n)} = \begin{bmatrix} P(X_n = 0) & P(X_n = 1) & \cdots \end{bmatrix}$$

Stationary and Limiting Distributions

Example. Consider a Markov chain with two possible states, $S = \{0, 1\}$. In particular, suppose that the transition matrix is given by

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix},$$

where a and b are two real numbers in the interval $[0, 1]$ such that $0 < a + b < 2$. Suppose that the system is in state 0 at time $n = 0$ with probability α, i.e.,

Orchestrated Conversation: Stationary and Limiting Distributions

$$\pi^{(0)} = \begin{bmatrix} P(X_0 = 0) & P(X_0 = 1) \end{bmatrix} = \begin{bmatrix} \alpha & 1-\alpha \end{bmatrix},$$

Where $\alpha \in [0, 1]$.

a) Using induction (or any other method), show that

$$P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}.$$

Orchestrated Conversation: Stationary and Limiting Distributions

b) Show that

$$\lim_{n\to\infty} P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix}.$$

c) Show that

$$\lim_{n\to\infty} \pi^{(n)} = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}.$$

Stationary and Limiting Distributions

The initial state (X_0) does not matter as n becomes large. Thus, we can write

$$\lim_{n\to\infty} P(X_n = 0 | X_0 = i) = \frac{b}{a+b},$$
$$\lim_{n\to\infty} P(X_n = 1 | X_0 = i) = \frac{a}{a+b}.$$

Stationary and Limiting Distributions

Limiting Distributions

The probability distribution $\pi = [\pi_0, \pi_1, \pi_2, \cdots]$ is called the limiting distribution of the Markov chain X_n if

$$\pi_j = \lim_{n \to \infty} P(X_n = j | X_0 = i)$$

for all $i, j \in S$, and we have

$$\sum_{j \in S} \pi_j = 1.$$

Stationary and Limiting Distributions

How to find the limiting distribution?

Finite Markov chains:

$$\pi^{(n+1)} = \pi^{(n)} P = \pi^{(n)}, \quad n \to \infty$$

Steady-state

So solve $\pi = \pi P \rightarrow \pi$ **Steady-state** (limiting) distribution

Stationary and Limiting Distributions

Theorem. Consider a finite Markov chain $\{X_n, n = 0, 1, 2, ...\}$ where $X_n \in S = \{0, 1, 2, \cdots, r\}$. Assume that the chain is irreducible and aperiodic. Then,

1. The set of equations

$$\pi = \pi P,$$
$$\sum_{j \in S} \pi_j = 1$$

has a unique solution.

Stationary and Limiting Distributions

2. The unique solution to the above equations is the limiting distribution of the Markov chain, i.e.,

$$\pi_j = \lim_{n\to\infty} P(X_n = j | X_0 = i),$$

for all $i, j \in S$.

3. We have

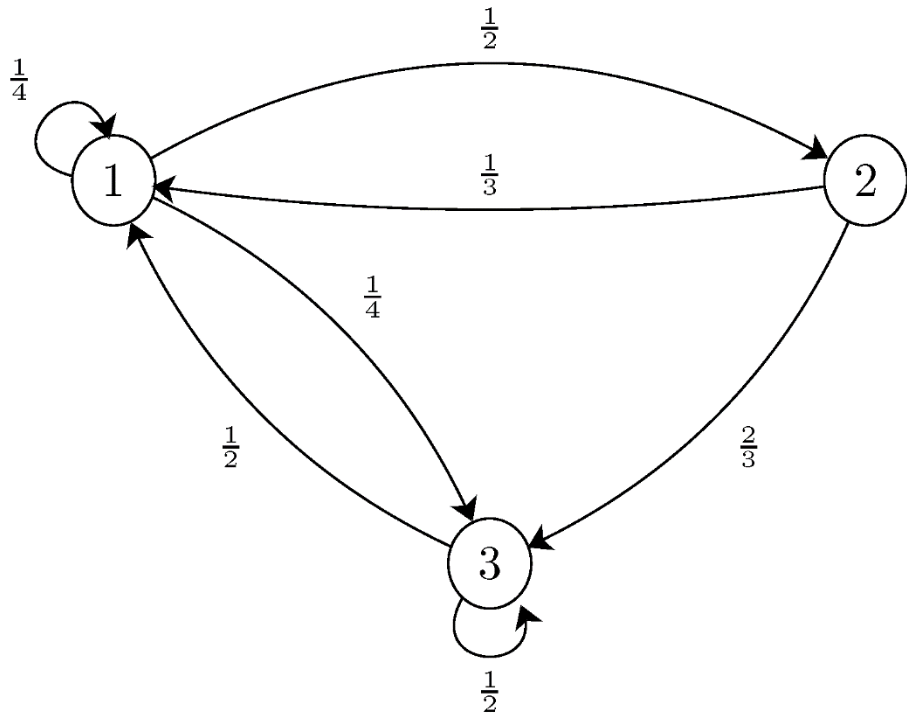
$$r_j = \frac{1}{\pi_j}, \quad \text{for all } j \in S,$$

where r_j is the mean return time to state j .

Orchestrated Conversation: Stationary and Limiting Distributions

Example. Consider the Markov chain shown in the following Figure.

a) Is this chain irreducible?
b) Is this chain aperiodic?
c) Find the stationary distribution for this chain.
d) Is the stationary distribution a limiting distribution for the chain?



Stationary and Limiting Distributions

Countably Infinite Markov Chains:

Let i be a recurrent state. Assuming $X_0 = i$, let R_i be the number of transitions needed to return to state i, i.e.,

$$R_i = \min\{n \geq 1 : X_n = i\}.$$

If $r_i = E[R_i | X_0 = i] < \infty$, then i is said to be positive recurrent. If $E[R_i | X_0 = i] = \infty$, then i is said to be null recurrent.

UMASS AMHERST

Stationary and Limiting Distributions

Theorem. Consider an infinite Markov chain $\{X_n, n = 0, 1, 2, ...\}$ where $X_n \in S = \{0, 1, 2, \cdots\}$. Assume that the chain is irreducible and aperiodic. Then, one of the following cases can occur:

1. All states are transient, and

$$\lim_{n \to \infty} P(X_n = j | X_0 = i) = 0, \text{ for all } i, j.$$

2. All states are null recurrent, and

$$\lim_{n \to \infty} P(X_n = j | X_0 = i) = 0, \text{ for all } i, j.$$

Stationary and Limiting Distributions

3. All states are positive recurrent. In this case, there exists a limiting distribution, $\pi = [\pi_0, \pi_1, \cdots]$, where

$$\pi_j = \lim_{n \to \infty} P(X_n = j | X_0 = i) > 0,$$

for all $i, j \in S$. The limiting distribution is the unique solution to the equations

Stationary and Limiting Distributions

$$\pi_j = \sum_{k=0}^{\infty} \pi_k P_{kj}, \qquad \text{for } j = 0, 1, 2, \cdots,$$

$$\sum_{j=0}^{\infty} \pi_j = 1.$$

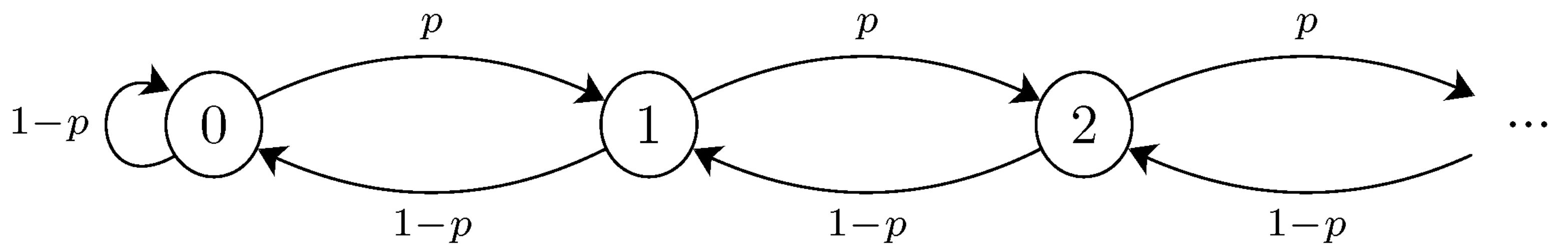
We also have

$$r_j = \frac{1}{\pi_j}, \quad \text{for all } j = 0, 1, 2, \cdots,$$

where r_j is the mean return time to state j .

Stationary and Limiting Distributions

Example. **Consider the Markov chain shown in the following Figure. Assume that $0 < p < 1/2$. Does this chain have a limiting distribution?**



Practicing some problems on Markov Chains

Absorption Probabilities

Consider a finite Markov chain $\{X_n, n = 0, 1, 2, \cdots\}$ with state space $S = \{0, 1, 2, \cdots, r\}$. Suppose that all states are either absorbing or transient. Let $l \in S$ be an absorbing state. Define

$$a_i = P(\text{absorption in } l | X_0 = i), \quad \text{for all } i \in S.$$

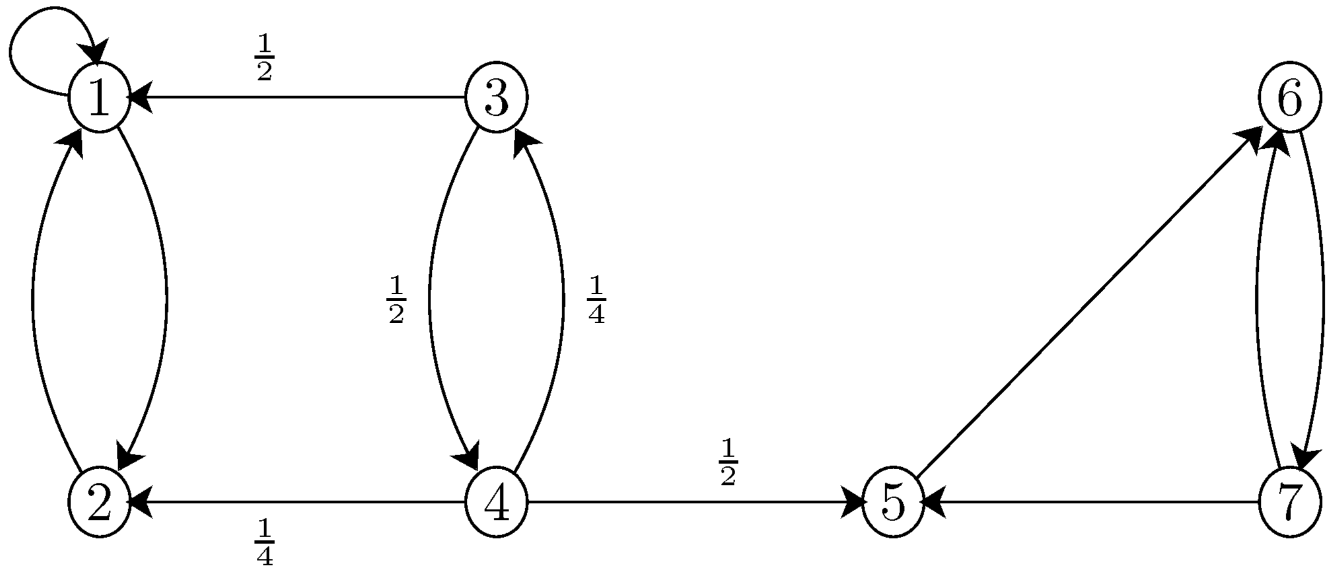
Practicing some problems on Markov Chains

By the above definition, we have $a_l = 1$, and $a_j = 0$ if j is any other absorbing state. To find the unknown values of a_i's, we can use the following equations

$$a_i = \sum_k a_k p_{ik}, \qquad \text{for } i \in S.$$

Practicing some problems on Markov Chains

Example 2. **Consider the Markov chain in the following Figure. There are two recurrent classes, $R_1 = \{1, 2\}$, and $R_2 = \{5, 6, 7\}$. Assuming $X_0 = 3$, find the probability that the chain gets absorbed in R_1.**



Practicing some problems on Markov Chains

Mean Hitting Times

Consider a finite Markov chain $\{X_n, n = 0, 1, 2, \cdots\}$ with state space $S = \{0, 1, 2, \cdots, r\}$. Let $A \subset S$ be a set of states. Let T be the first time the chain visits a state in A. For all $i \in S$, define

$$t_i = E[T|X_0 = i].$$

Practicing some problems on Markov Chains

Mean Hitting Times

By the above definition, we have $t_j = 0$, for all $j \in A$. To find the unknown values of t_i's, we can use the following equations

$$t_i = 1 + \sum_k t_k p_{ik}, \quad \text{for } i \in S - A.$$

Practicing some problems on Markov Chains

Example. Consider the Markov chain of previous Example. Again assume .$X_0 = 3$. We would like to find the expected time (number of steps) until the chain gets absorbed in R_1 or R_2 . More specifically, let T be the absorption time, i.e., the first time the chain visits a state in R_1 or R_2. We would like to find

$$E[T|X_0 = 3].$$

Practicing some problems on Markov Chains

Mean Return Times

Consider a finite irreducible Markov chain $\{X_n, n = 0, 1, 2, \cdots\}$ with state space $S = \{0, 1, 2, \cdots, r\}$. Let $l \in S$ be a state. Let r_l be the the mean return time to state l. Then

$$r_l = 1 + \sum_k t_k p_{lk},$$

Practicing some problems on Markov Chains

Mean Return Times

where t_k is the expected time until the chain hits state l given $X_0 = k$. Specifically,

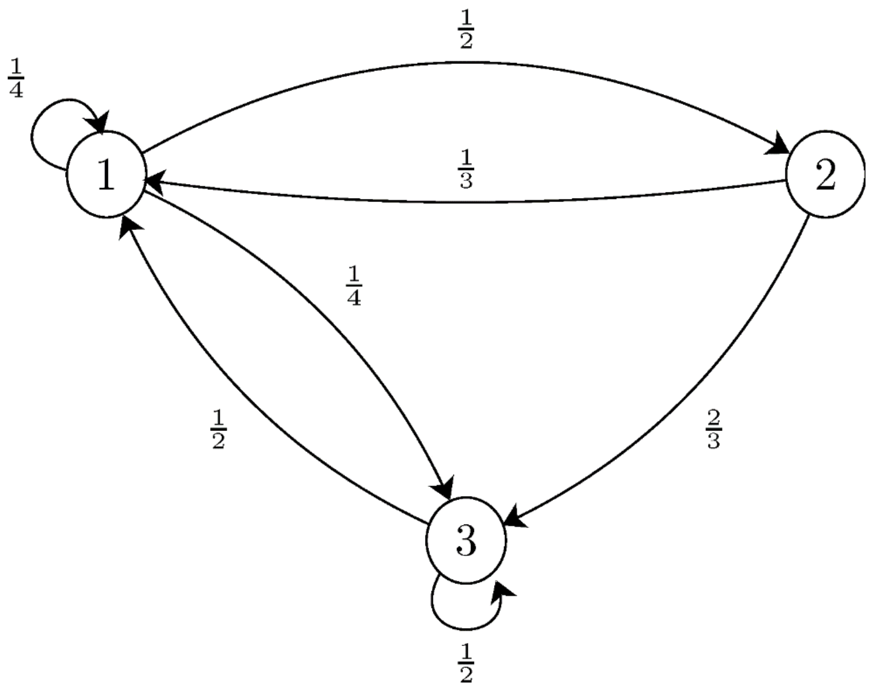
$$
\begin{aligned}
t_l &= 0, \\
t_k &= 1 + \sum_j t_j p_{kj}, \qquad \text{for } k \neq l.
\end{aligned}
$$

Practicing some problems on Markov Chains

Example. **Consider the Markov chain shown in the following Figure. Assume** $X_0 = 1$**, and let** R **be the first time that the chain returns to state 1, i.e.,**

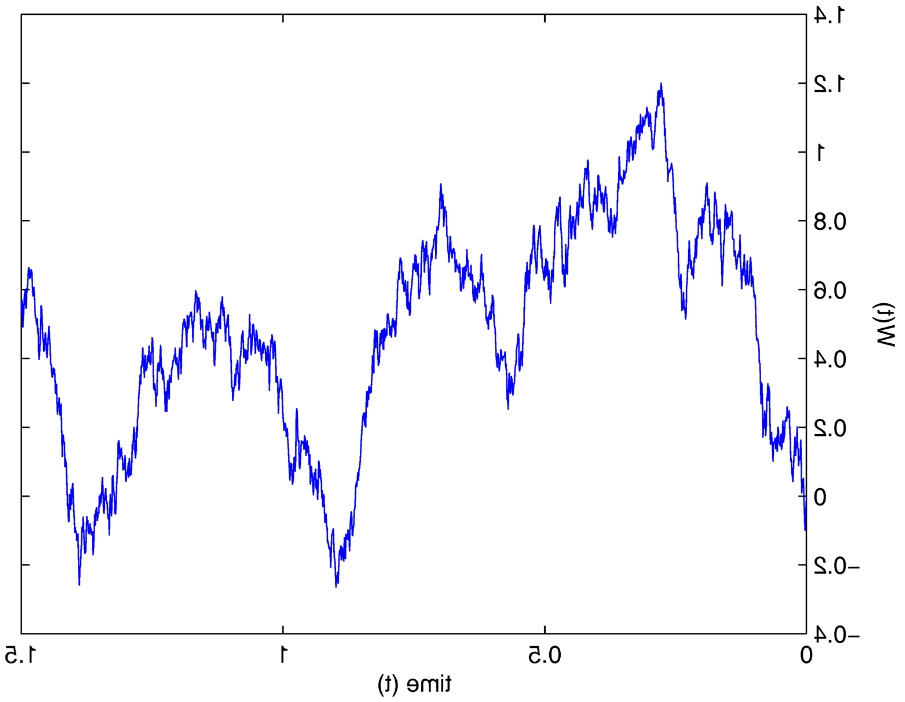
$$R = \min\{n \geq 1 : X_n = 1\}.$$

Find $E[R|X_0 = 1]$.



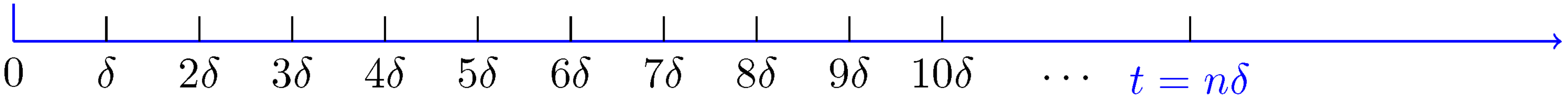
Brownian Motion (Wiener Process)

Brownian motion is another widely-used random process. The following Figure shows a sample path of Brownian motion.



Brownian Motion (Wiener Process)

Divide the half-line $[0, \infty)$ to tiny subintervals of length δ



We define the random variables X_i as follows. $X_i = \sqrt{\delta}$ if the kth coin toss results in heads, and $X_i = -\sqrt{\delta}$ if the kth coin toss results in tails. Thus,

$$\boldsymbol{X_i = \begin{cases} \sqrt{\delta} & \text{with probability } \frac{1}{2} \\ -\sqrt{\delta} & \text{with probability } \frac{1}{2} \end{cases}}$$

Brownian Motion (Wiener Process)

Now, we define the process $W(t)$ as follows. We let $W(0) = 0$. At time $t = n\delta$ the value of $W(t)$ is given by

$$\boldsymbol{W(t) = W(n\delta) = \sum_{i=1}^{n} X_i.}$$

Brownian Motion (Wiener Process)

Moreover, the X_i's are independent. Note that

$$E[X_i] = \frac{1}{2}\sqrt{\delta} - \frac{1}{2}\sqrt{\delta} = 0,$$

$$\mathrm{Var}(X_i) = EX_i^2 = \frac{1}{2}\delta + \frac{1}{2}\delta = \delta.$$

Brownian Motion (Wiener Process)

We know that,

$$W(t) = \sum_{i=1}^{n} X_i,$$

Thus,

$$E[W(t)] = \sum_{i=1}^{n} E[X_i] = 0$$

$$\mathrm{Var}(W(t)) = \sum_{i=1}^{n} \mathrm{Var}(X_i) = n\mathrm{Var}(X_i) = n\delta = t.$$

Brownian Motion (Wiener Process)

BM: Gaussian

$$W(t) \sim N(0, \sigma^2 = t)$$

Brownian Motion (Wiener Process)

Standard Brownian Motion

A Gaussian random process $\{W(t), t \in [0, \infty)\}$ is called a (standard) **Brownian motion** or a (standard) **Wiener process** if

1. $W(0) = 0.$

2. For all $0 \leq t_1 < t_2,\ W(t_2) - W(t_1) \sim N(0, t_2 - t_1).$

Brownian Motion (Wiener Process)

3. $W(t)$ has independent increments. That is, for all $0 \le t_1 < t_2 < t_3 \cdots < t_n$, the random variables

$$W(t_2) - W(t_1),\ W(t_3) - W(t_2),\ \cdots,\ W(t_n) - W(t_{n-1})$$

are independent;

4. $W(t)$ has continuous sample paths.

Brownian Motion (Wiener Process)

Example. Let $W(t)$ be a standard Brownian motion. For all $s, t \in [0, \infty)$, find

$$C_W(s,t) = \mathrm{Cov}(W(s), W(t)).$$

Let's assume $s \le t$:

$$\begin{aligned}
\mathrm{Cov}\big(W(s), W(t)\big) &= \mathrm{Cov}\big(W(s), W(s) + W(t) - W(s)\big) \\
&= \mathrm{Cov}\big(W(s), W(s)\big) + \mathrm{Cov}\big(W(s), W(t) - W(s)\big) \\
&= \mathrm{Var}\big(W(s)\big) + \mathrm{Cov}\big(W(s), W(t) - W(s)\big) \\
&= s + \mathrm{Cov}\big(W(s), W(t) - W(s)\big).
\end{aligned}$$

Brownian Motion (Wiener Process)

$$\mathrm{Cov}\big(W(s), W(t) - W(s)\big) = 0$$

Then,

$$\mathrm{Cov}\big(W(s), W(t)\big) = s.$$

For $t \leq s$:

$$\mathrm{Cov}\big(W(s), W(t)\big) = t.$$

We conclude

$$\mathrm{Cov}\big(W(s), W(t)\big) = \min(s, t), \qquad \text{for all } s, t.$$

Brownian Motion (Wiener Process)

Example. **Let** $W(t)$ **be a standard Brownian motion.**

a. **Find** $P(1 < W(1) < 2)$.

b. **Find** $P(W(2) < 3 | W(1) = 1)$.

Post-work for Lesson

- **Complete homework assignment for Lessons 18-20:**

HW#11 and HW#12

Go to the online classroom for details.