CSCI 3104: Algorithms
Spring 2022
Recitation #6 - Recursion

Some Helpful Identities

$$a\sum_{i=0}^{n} r^i = a\left(\tfrac{1-r^{n+1}}{1-r}\right) \qquad \text{(Finite Geometric Series)}$$

$$a\sum_{i=0}^{n} i = \frac{an(n+1)}{2} \qquad \text{(Sum of first } n \text{ integers/special case of Finite Arithmetic Series)}$$

$$\sum_{i=0}^{n} ix^i = \frac{x(nx^{n+1}-(n+1)x^n+1)}{(x-1)^2}$$

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Problem 1

For the following algorithms, write the recurrence relation and the base cases. (*note:* we index sets starting from 1)

a. .

Algorithm 1 Recurrence 1.a

```
procedure MAX1(Integer Array A)
    L = |A|

    if L = 1 then return A[1]
    else if L = 0 then return −∞

    m_1 ← MAX1(A[1 : L/2])
    m_2 ← MAX1(A[L/2 + 1 : L])
return m_1
```

b. .

Algorithm 2 Recurrence 1.b

procedure MAX2(Integer Array A)
    $L = |A|$

    if $L = 1$ **then return** $A[1]$
    else if $L = 0$ **then return** $-\infty$

    $m \leftarrow \text{MAX2}(A[1 : L - 3])$

    for $i \leftarrow L$; $i \geq 1$; $i \leftarrow i - 1$ **do**
        if $m \geq A[i]$ **then** $m \leftarrow A[i]$
    return m

Problem 2

For each of the following recurrence relationships, use the **tree method** to obtain a closed-form expression for the runtime by doing the following: 1) draw a tree diagram of the first few layers of the function's recursive calls, 2) determine the depth of your tree, 3) sum work over all vertices, and 4) simplify.

a.

$$T(n) = \begin{cases} 3T(n/6) + n^2 & n > 3 \\ n^2 & n \leq 3 \end{cases}$$

b.

$$T(n) = \begin{cases} 2T(n/5) + n \log_5 n & n > 1 \\ 2 & n \leq 1 \end{cases}$$

Problem 3

For each of the following recurrence relationships, use the **unrolling method** to obtain a closed-form expression for the runtime by doing the following: 1) determine the number of times to unroll, 2) write out several iterations, 3) identify the pattern, and 4) simplify.

a.

$$T(n) = \begin{cases} 3T(n-4) + 2n & n > 4 \\ 3 & n \leq 4 \end{cases}$$

b.

$$T(n) = \begin{cases} 4T(n-5) + n/4 & n > 6 \\ 3 & n \leq 6 \end{cases}$$

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$$T(n) = \begin{cases} \Theta(1) & , \quad n \leq 1 \\ 2T\left(\frac{n}{2}\right) + \Theta(1) & , \quad n > 1 \end{cases}$$

Algorithm 2 Recurrence 1.b

procedure MAX2(Integer Array A)

  $L = |A|$

  if $L = 1$ **then return** $A[1]$

  else if $L = 0$ **then return** $-\infty$

  $m \leftarrow$ MAX2($A[1 : L - 3]$)

  for $i \leftarrow L; i \geq 1; i \leftarrow i - 1$ **do**

    if $m \geq A[i]$ **then** $m \leftarrow A[i]$

  return m

$$T(n) = \begin{cases} \Theta(1) & , \quad n \leq 1 \\ T(n-3) + \Theta(n) & , \quad n > 1 \end{cases}$$

Problem 2

For each of the following recurrence relationships, use the **tree method** to obtain a closed-form expression for the runtime by doing the following: 1) draw a tree diagram of the first few layers of the function's recursive calls, 2) determine the depth of your tree, 3) sum work over all vertices, and 4) simplify.

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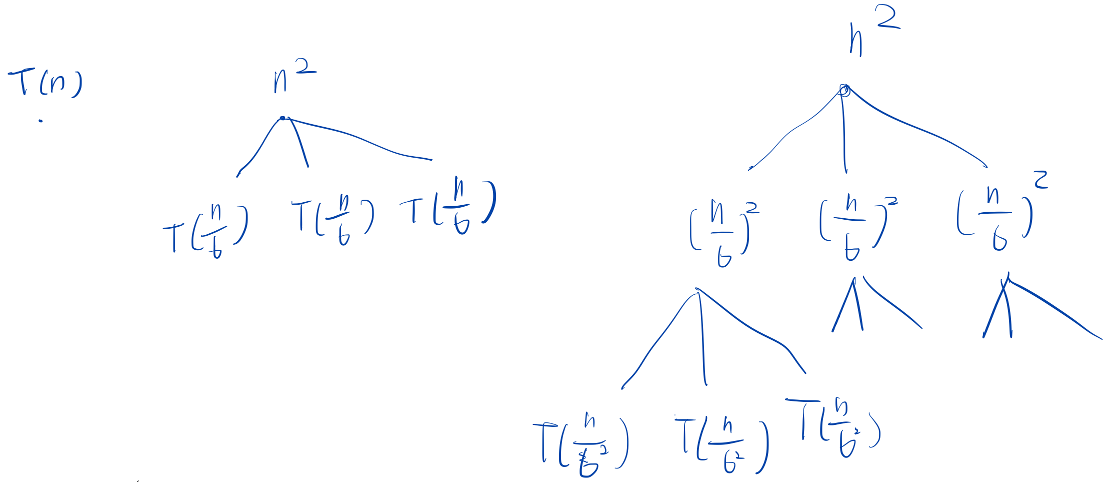
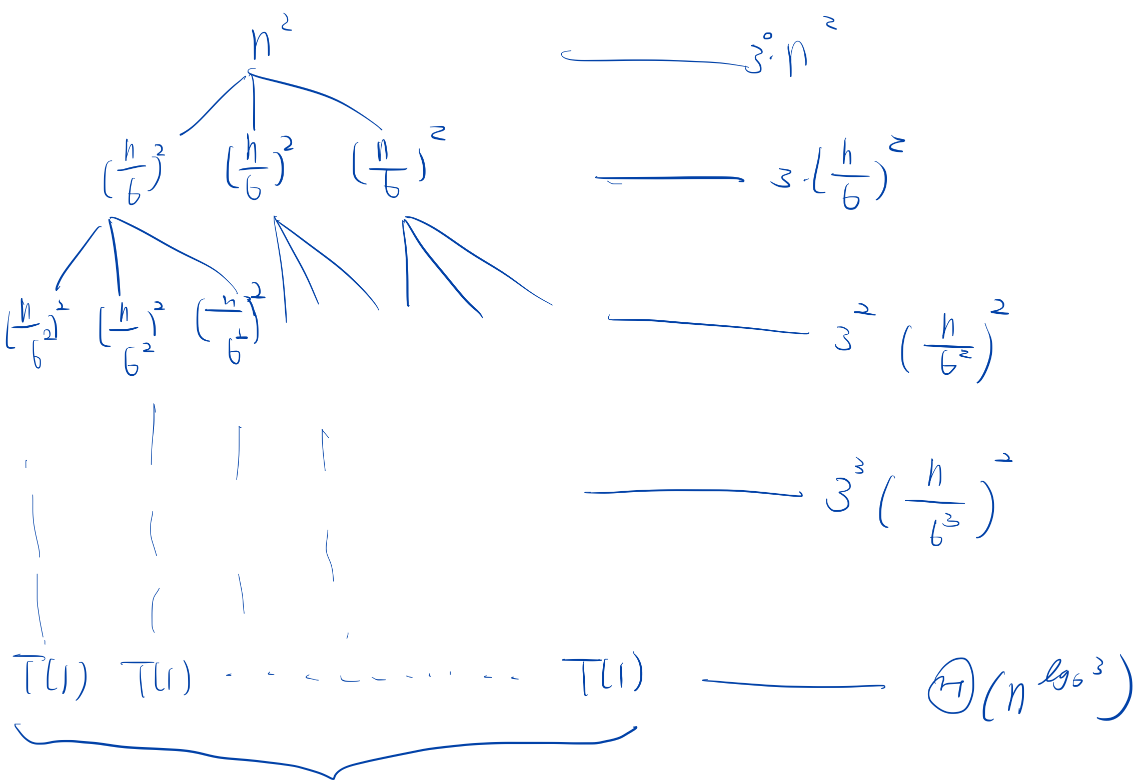
a.



1) Draw a tree



$$3^{\lg_6 n} = n^{\lg_6 3}$$

b) depth of the tree

$$\frac{n}{6^d} \le 3$$

$$n \le 3 \cdot 6^d$$

$$d \ge \lg_6\left(\frac{n}{3}\right)$$

$$d = \left\lceil \lg_6 \frac{n}{3} \right\rceil$$

c) sum over all vertices.

$$3^0\left(\frac{n}{6^0}\right)^2 + 3^1\left(\frac{n}{6^1}\right)^2 + 3^2\left(\frac{n}{6^2}\right)^2 + \cdots + 3^{\lfloor \lg_6 \frac{n}{3} \rfloor}\left(\frac{n}{6^{\lfloor \lg_6 \frac{n}{3} \rfloor}}\right)^2 + \Theta(n^{\lg_6 3})$$

$$= \sum_{i=0}^{\lfloor \lg_6 \frac{n}{3} \rfloor}\left(\left(\frac{3}{6^2}\right)^i n^2\right) + \Theta(n^{\lg_6 3})$$

d) Simplify.

$$= n^2 \sum_{i=0}^{\lfloor \lg_6 \frac{n}{3} \rfloor}\left(12^{-i}\right) + \Theta(n^{\lg_6 3})$$

$$= n^2 \frac{1\left(1-12^{-(\lfloor \lg_6 \frac{n}{3} \rfloor + 1)}\right)}{1-12^{-1}} + \Theta(n^{\lg_6 3})$$

$$\approx n^2 \frac{12}{11}\left(1-12^{-\lg_6 \frac{n}{3}}\right) + \Theta(n^{\lg_6 3})$$

$$= n^2 \frac{12}{11}\left(1-\left(\frac{3}{n}\right)^{\lg_6 12}\right) + \Theta(n^{\lg_6 3})$$

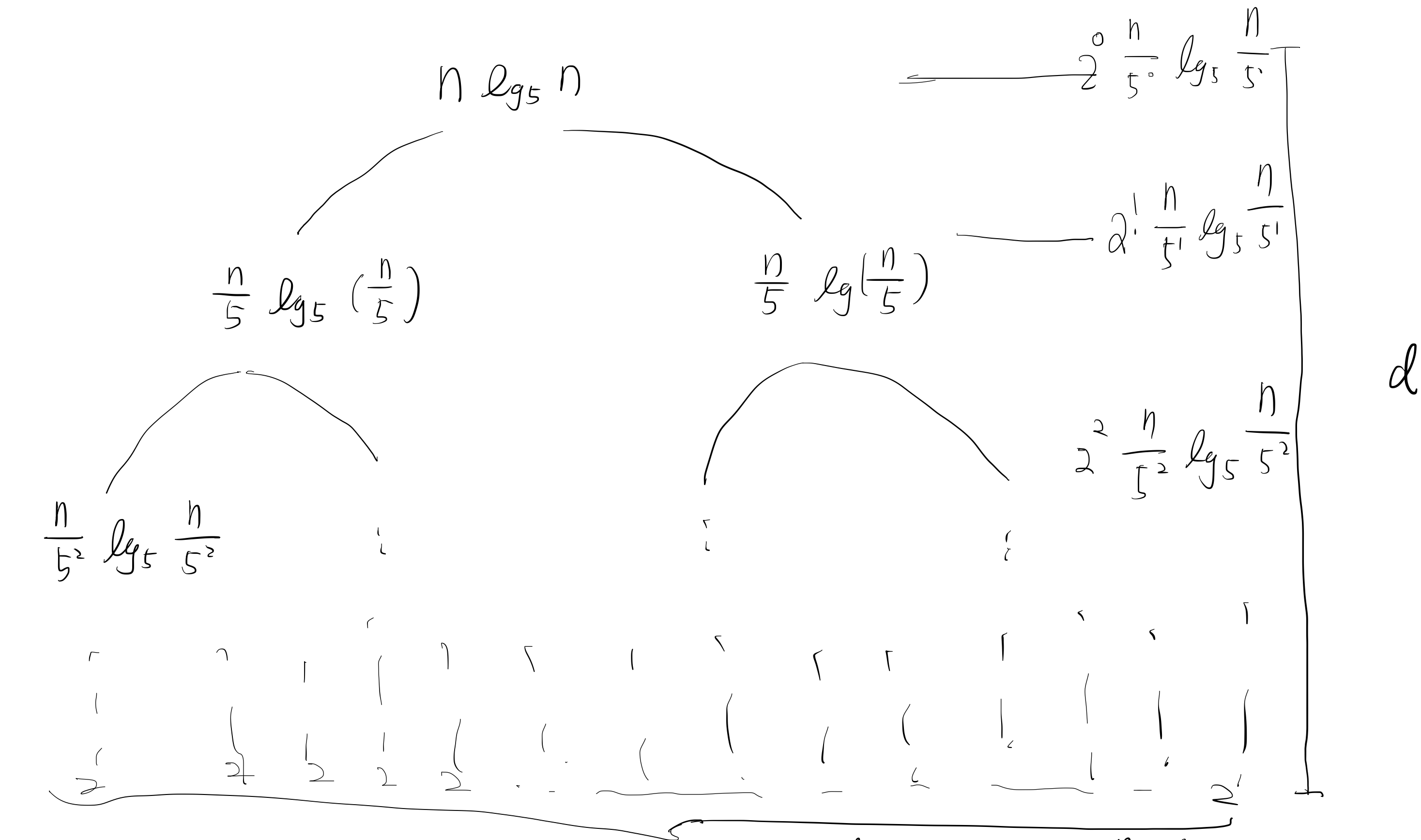
$$= \frac{12}{11}\left(n^2 - 3^{\lg_6 12} \cdot n^{2-\lg_6 12}\right) + \Theta(n^{\lg_6 3})$$

$$= \Theta(n^2) \quad \#$$

b.

$$T(n) = \begin{cases} 2T(n/5) + n\log_5 n & n > 1 \\ 2 & n \le 1 \end{cases}$$

① Draw the tree



$$2(2^d) = 2^{d+1} \approx 2^{lg_5 n + 1}$$

② Depth. d

$$\frac{n}{5^d} \le 1 \Rightarrow d \ge lg_5 n \Rightarrow d = \lceil lg_5 n \rceil \approx lg_5 n$$

③ ④

$$\sum_{i=0}^{\lfloor lg_5 n \rfloor} \left(2^i \frac{n}{5^i}\, lg_5 \frac{n}{5^i}\right) + 2^{lg_5 n + 1}$$

$$= \sum_{i=0}^{\lfloor lg_5 n \rfloor} \left(\frac{2}{5}\right)^i \left(lg_5(n) - i\right) + 2^{lg_5 n + 1}$$

$$\approx lg_5 n \sum_{i=0}^{lg_5 n} \left(\frac{2}{5}\right)^i - \sum_{i=0}^{\lfloor lg_5 n \rfloor} \left(\frac{2}{5}\right)^i i + 2 \cdot n^{lg_5 2}$$

③

$$\lg_5 n \frac{\left(1-\frac{2}{5}^{\lg_5 n+1}\right)}{1-\frac{2}{5}} = \lg_5 n \cdot \left(\frac{5}{3}\right) \cdot \left(1-\frac{2}{5} \cdot n^{(\lg_5 2)-1}\right) \quad \text{—— ①}$$

$$S_t = \sum_{i=0}^{t} i\left(\frac{2}{5}\right)^i = 1\cdot\left(\frac{2}{5}\right) + 2\left(\frac{2}{5}\right)^2 + \cdots\cdots + t\left(\frac{2}{5}\right)^t$$

$$\frac{2}{5} S_t = \left(\frac{2}{5}\right)^2 + 2\left(\frac{2}{5}\right)^3 + \cdots + (t-1)\left(\frac{2}{5}\right)^t + t\left(\frac{2}{5}\right)^{t+1}$$

$$\Rightarrow \quad \frac{3}{5} S_t = \frac{2}{5} + \left(\frac{2}{5}\right)^2 + \cdots\cdots + \left(\frac{2}{5}\right)^t - t\left(\frac{2}{5}\right)^{t+1}$$

$$\frac{3}{5} S_t = \frac{\frac{2}{5}\left(1-\left(\frac{2}{5}\right)^t\right)}{1-\frac{2}{5}} - t\left(\frac{2}{5}\right)^{t+1}$$

$$S_t = \frac{5}{3}\left(\frac{2}{3}\left(1-\left(\frac{2}{5}\right)^t\right) - t\left(\frac{2}{5}\right)^{t+1}\right)$$

$$t = \lg_5 n$$

$$= \frac{5}{3}\left(\frac{2}{3}\left(1-\frac{2}{5}^{\lg_5 n}\right) - \lg_5 n \left(\frac{2}{5}\right)^{\lg_5 n+1}\right)$$

$$= \frac{5}{3}\left(\frac{2}{3} - \frac{2}{3}\cdot n^{\lg_5 \frac{2}{5}} - n^{\lg_5 \frac{2}{5}} \cdot \lg_5 n \cdot \frac{2}{5}\right)$$

$$= c_1 - c_2 \cdot n^{\lg_5 2 - 1} \cdot \left(1 + c_3 \cdot \lg_5 n\right) \quad \text{—— ②}$$

$$① + ② + ③ = \Theta(n)$$

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a.

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a.

① Assume we need unrolling m times.

$n - 4m \leq 4 \Rightarrow m \geq \frac{n-4}{4}$ $\quad m = \lceil \frac{n-4}{4} \rceil \approx \frac{n-4}{4}$

② write out some iterations

$T(n) = 3T(n-4) + 2n$

$= 2n + 3(T(n-4\times 2) + 2(n-4))$

$\vdots$

$= 2n + 3^1(2)(n-4) + 3^2(2)(n-4\cdot 2) + \cdots$

③ Identify the pattern

$$= 2\sum_{i=0}^{\frac{n-4}{4}} 3^i(n-4i)$$

④ Simplify

$$= \underbrace{2\cdot n \sum_{i=0}^{\frac{n-4}{4}} 3^i}_{\text{the prior}} - \underbrace{\sum_{i=0}^{\frac{n-4}{4}} 3^i \cdot 4i}_{\text{the latter}}$$

Assume $m = \frac{n-4}{4}$

the prior.

$$2n \cdot \sum_{i=0}^{\frac{n-4}{4}} 3^i = 2n \cdot \sum_{i=0}^{m} 3^i$$

the latter

$$S_m = 3^1 \cdot 4 \cdot 1 + 3^2 \cdot 4 \cdot 2 + \cdots + 3^m \cdot 4 \cdot m$$

$$3S_m = 3^2 \cdot 4 \cdot 1 + \cdots + 3^m \cdot 4 \cdot (m-1) + 3^{m+1} \cdot 4(m)$$

$$2S_m = \left(4 \sum_{i=1}^{m} 3^i\right) + 4 \cdot 3^{m+1} \cdot m$$

$$S_m = 2\left(\sum_{i=1}^{m} 3^i + 3^{m+1} \cdot m\right)$$

$$= 2n\left(\sum_{i=0}^{m} 3^i\right) + \left(2 \sum_{i=1}^{m} 3^i\right) + 2 \cdot 3^{m+1} \cdot m$$

$$\sum_{i=1}^{m} 3^i = \sum_{i=1}^{\frac{n-4}{4}} 3^i = \frac{3\left(1 - 3^{\frac{n-4}{4}}\right)}{1-3} = \frac{3}{2}\left(3^{\frac{n-4}{4}} - 1\right)$$

$$= 2n + (2n+2) \sum_{i=1}^{m} 3^i + 2 \cdot 3 \cdot 3^m \cdot m$$

$$= 2n + (2n+2)\left(\frac{3}{2}\right)\left(3^{\frac{n-4}{4}} - 1\right) + 2 \cdot 3^{\frac{n-4}{4}+1} \cdot \frac{n-4}{4}$$

$$= \Theta\left(n \cdot 3^{\frac{n-4}{4}}\right)$$

#

b.

$$T(n) = \begin{cases} 4T(n-5) + n/4 & n > 6 \\ 3 & n \le 6 \end{cases}$$

① Assume there're m iterations

$$n - 5m \le 6 \Rightarrow \frac{n-6}{5} \le m \Rightarrow m = \left\lceil \frac{n-6}{5} \right\rceil \approx \frac{n-6}{5}$$

②

$$T(n) = 4T(n-5) + \frac{n}{4}$$

$$= \frac{n}{4} + 4 \cdot \frac{n-5}{4} + 4^2 \cdot \frac{n-5\cdot 2}{4} + \cdots$$

③

$$T(n) = \sum_{i=0}^{m} 4^i \cdot \frac{n-5\cdot i}{4}$$

$$= \frac{1}{4} \sum_{i=0}^{m} \left(4^i \cdot n - 4^i \cdot 5i\right)$$

④ (?)