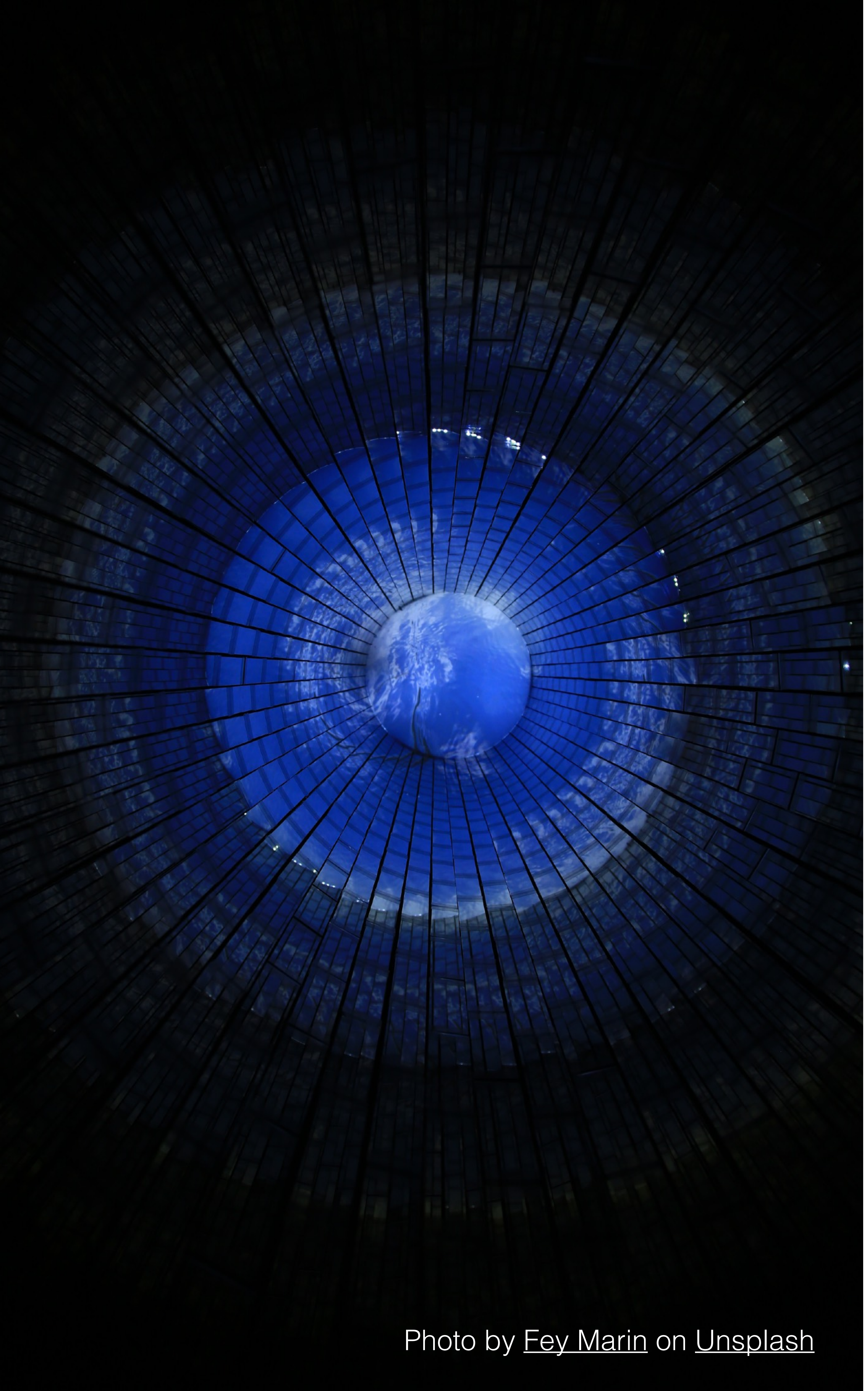
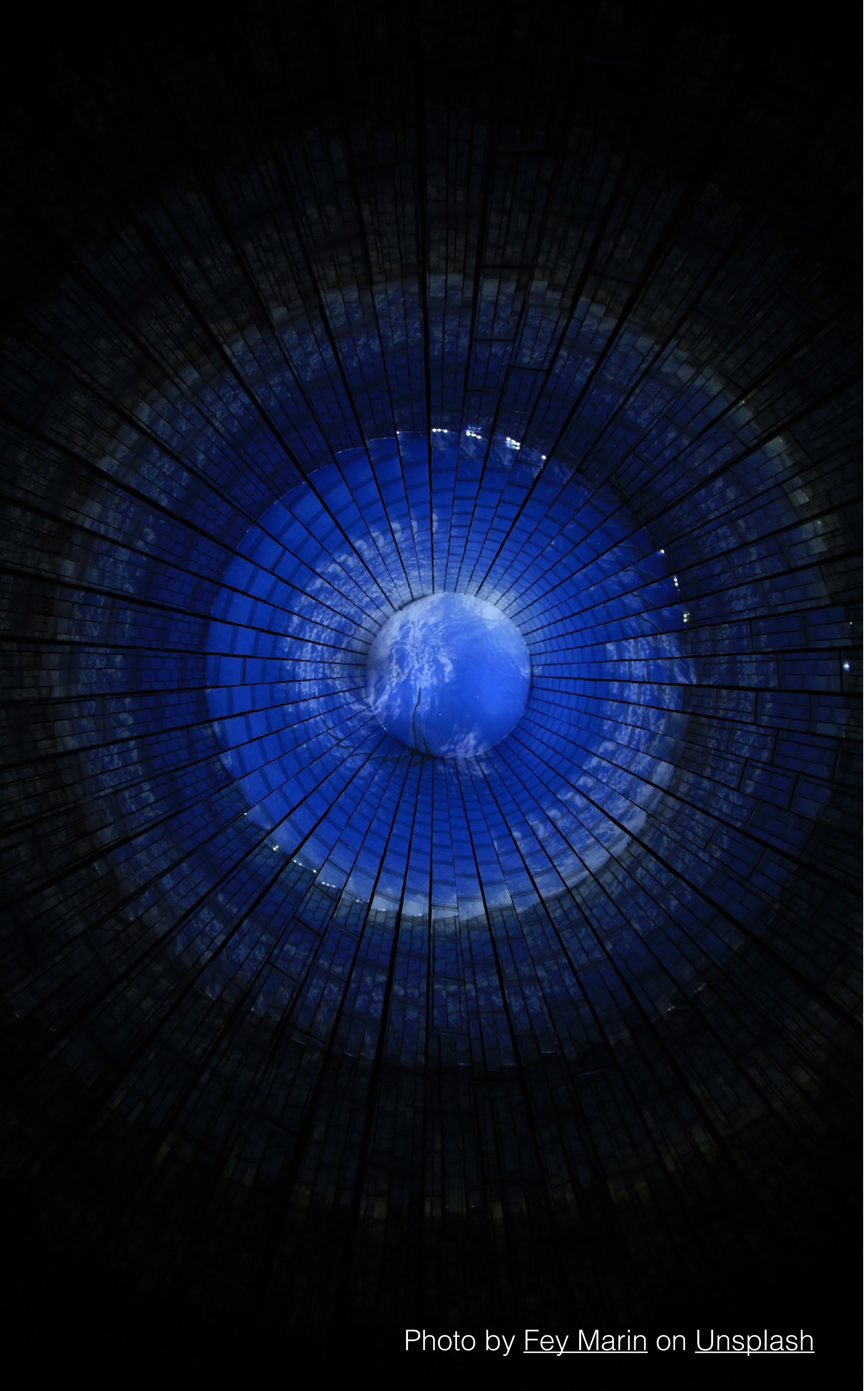
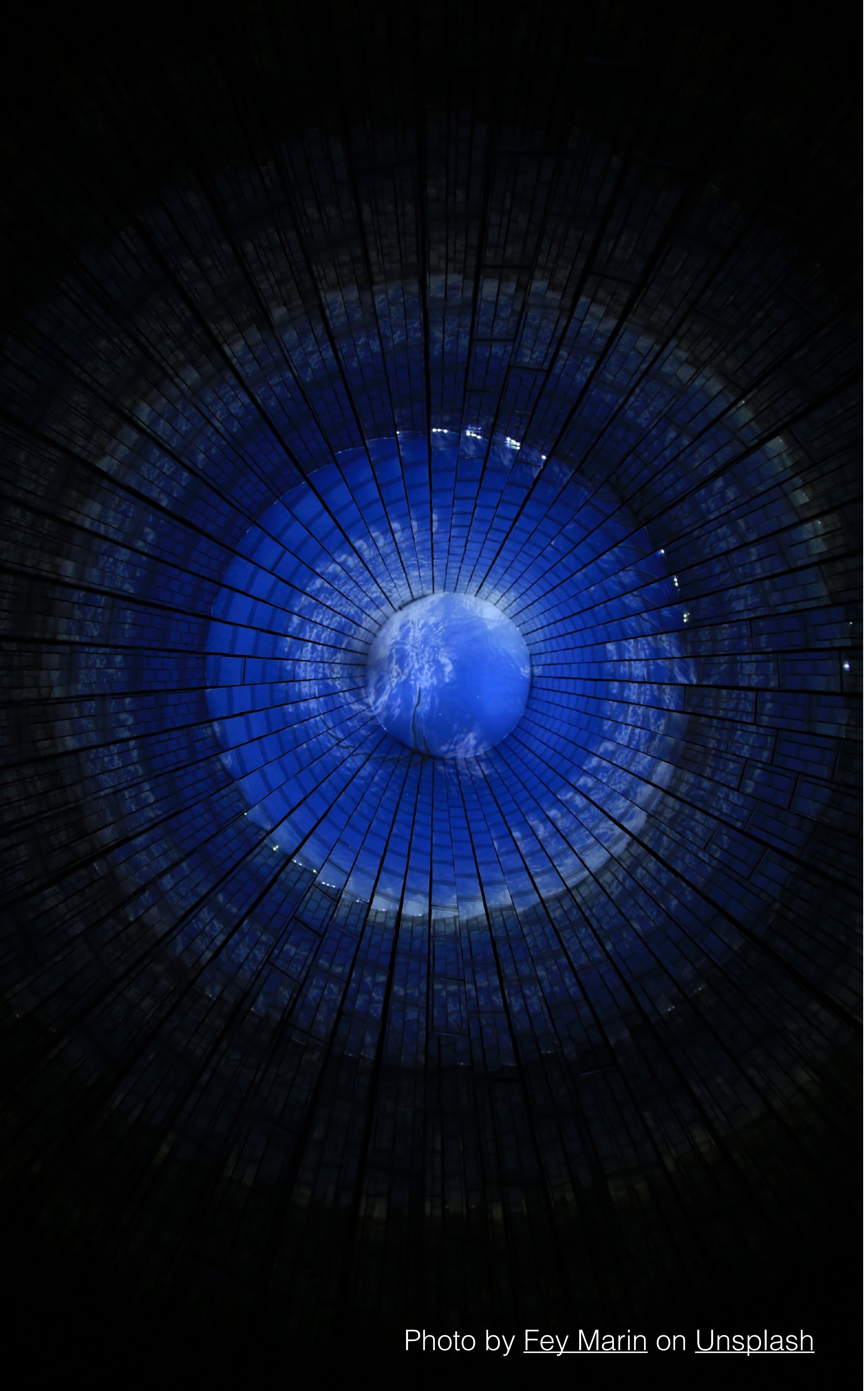


An **improper prior distribution** is any probability density function (or mass function) that is non-negative but does not integrate (or sum) to 1.

Why would we do such a thing?!



Let $X_1, \ldots, X_n \,|\, \mu, \sigma^2 \overset{iid}{\sim} N(\mu, \sigma^2)$ with σ^2 fixed and known, and $\pi(\mu) = c > 0$ for all μ. Clearly this prior is improper. And yet we'll verify numerically in the Unit #2 code that the posterior is proper.



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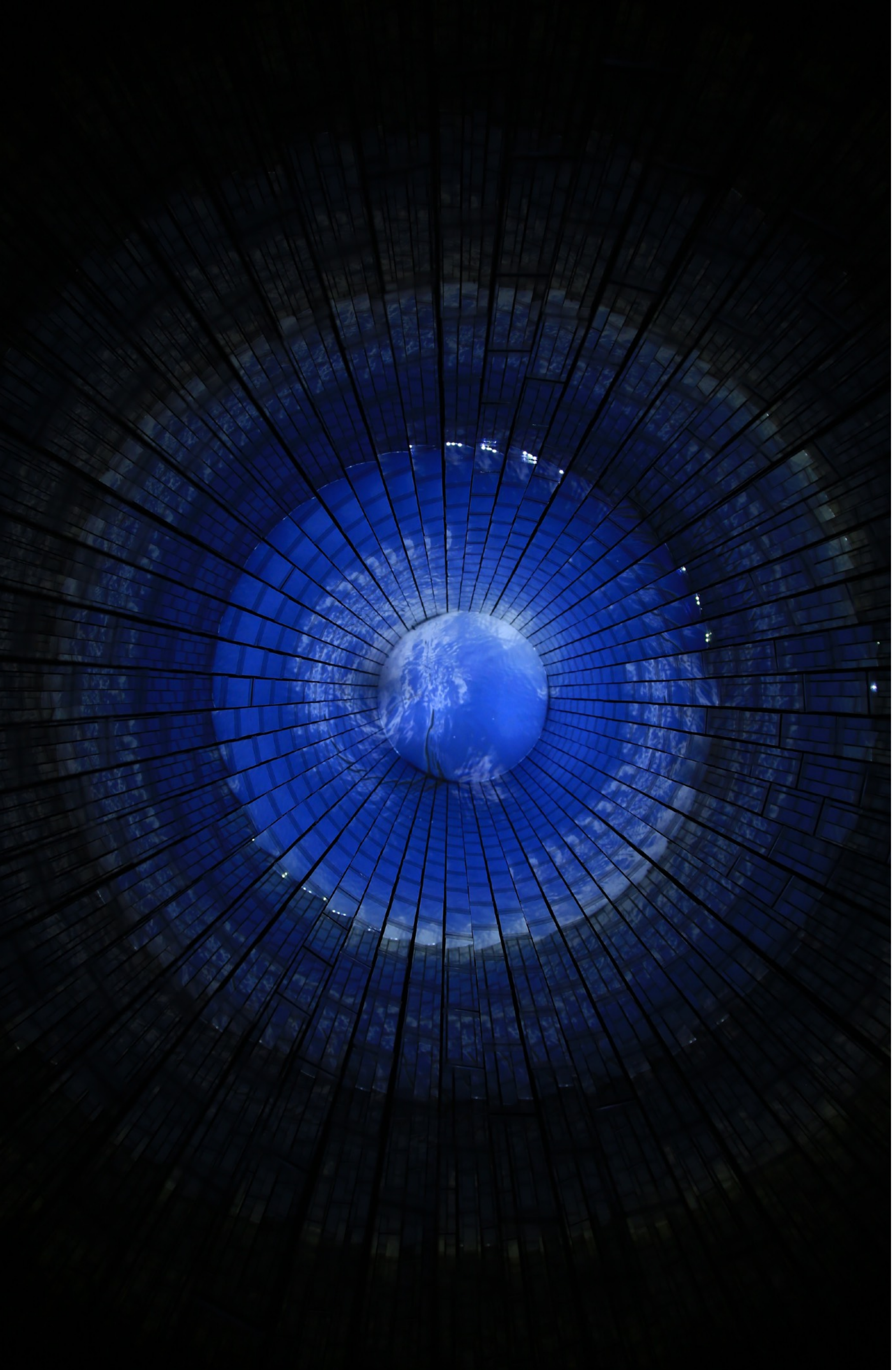
Notes on improper priors:

1. One must verify that an improper prior leads to a proper posterior. Otherwise, the Bayesian inference is not valid!

2. It's not clear what an improper prior means in terms of subjective degrees of belief. An improper prior is not a proper pdf, so how can it summarize an individual's uncertainty?



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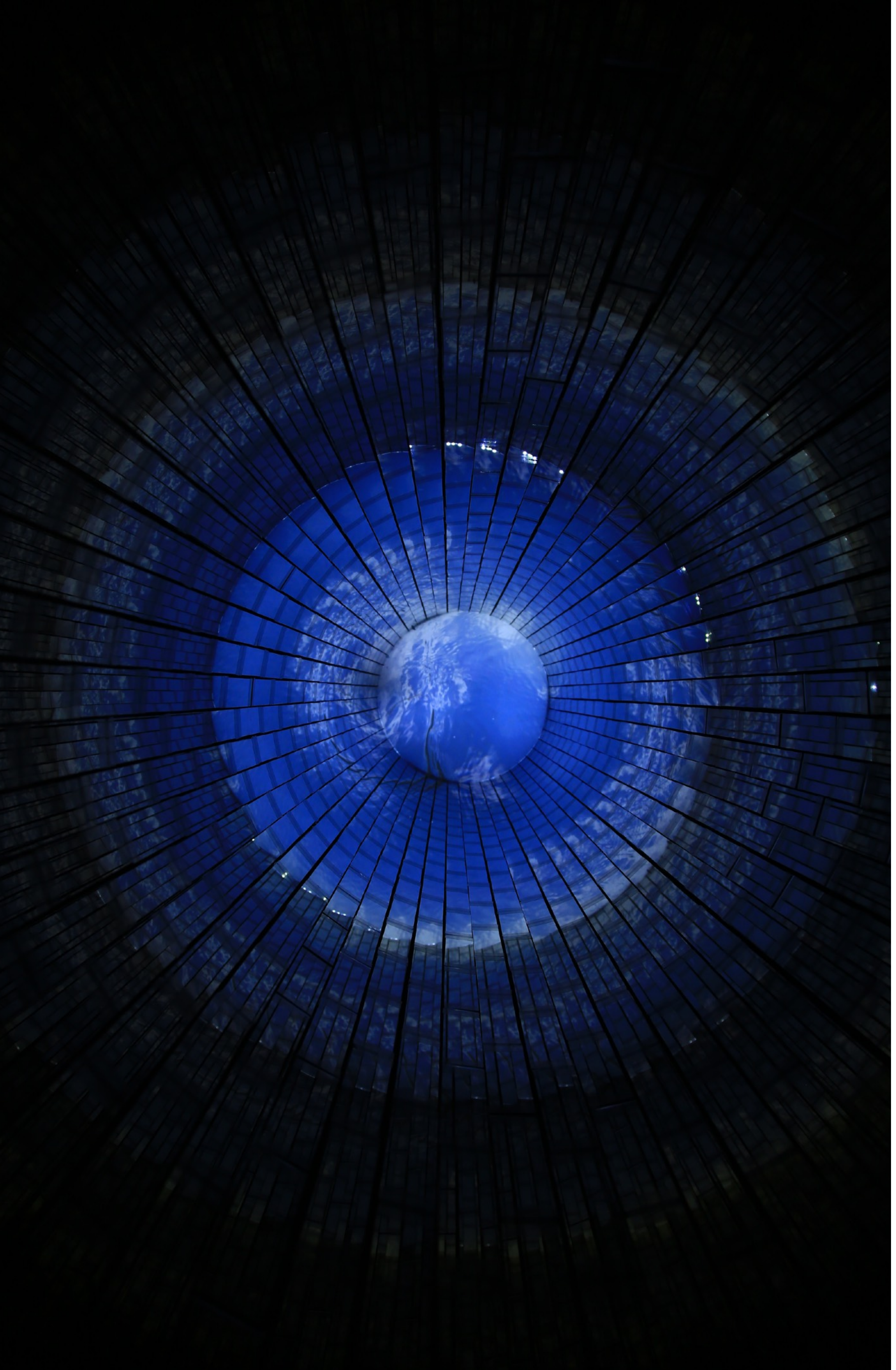
What happens when there isn't any prior information?

The **discrete principle of indifference** states that if there are n possible values in the parameter space, each outcome should be assigned the same prior probability, $1/n$.

The **continuous principle of indifference** states that if there is an interval of possible values in the parameter space, say $\theta \in (a, b)$ then the probability of any subinterval $I = (c, d)$ should have equal probability: $P(I) = \dfrac{d - c}{b - a}$.



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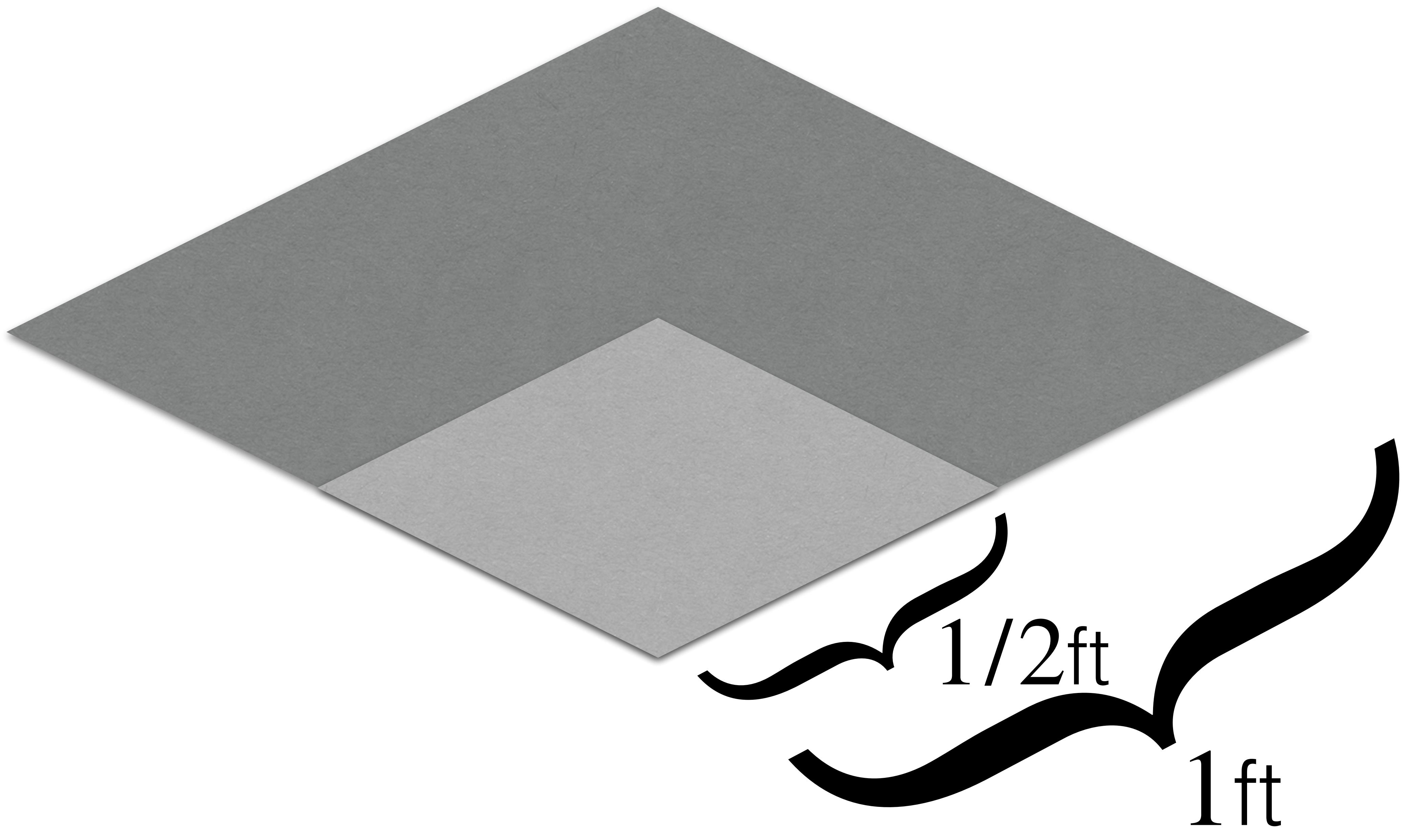
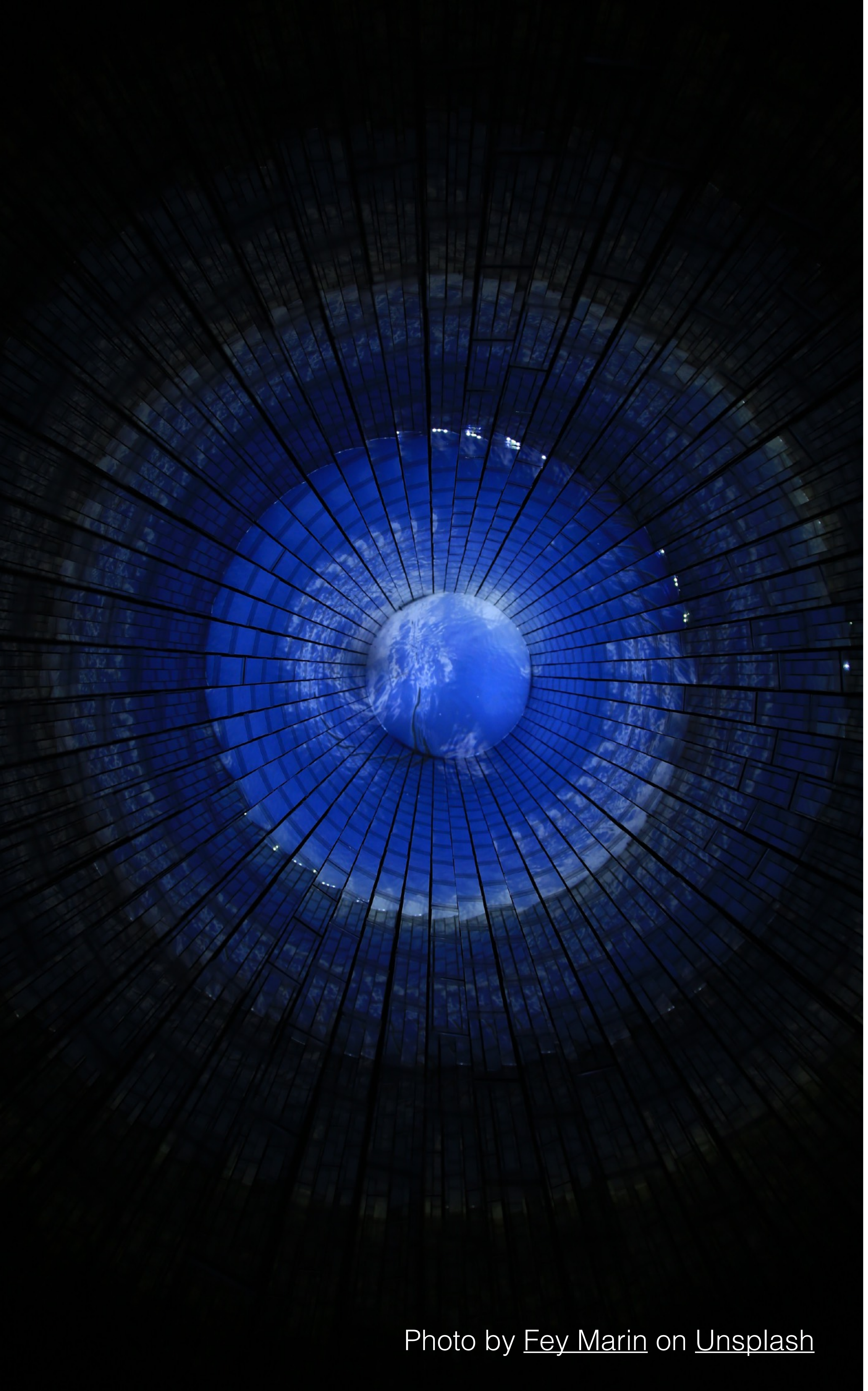


1. A factory produces cubes with side-length between 0 and 1 foot; what is the probability that a randomly chosen cube has side-length between 0 and $1/2$ of a foot?

2. A factory produces cubes with face-area between 0 and 1 square-feet; what is the probability that a randomly chosen cube has face-area between 0 and $1/4$ square-feet?

Example from https://stanford.io/40KyxGc



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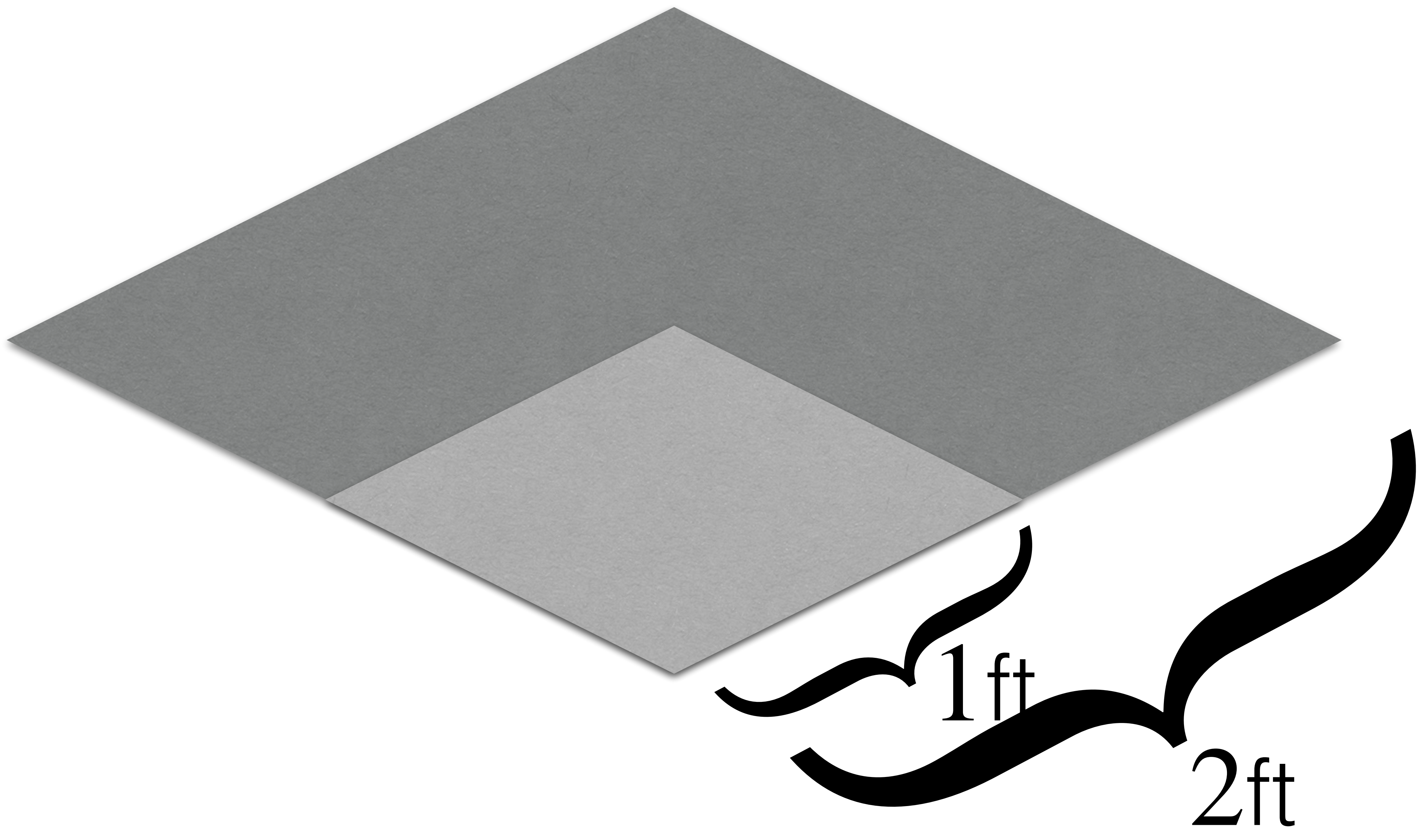
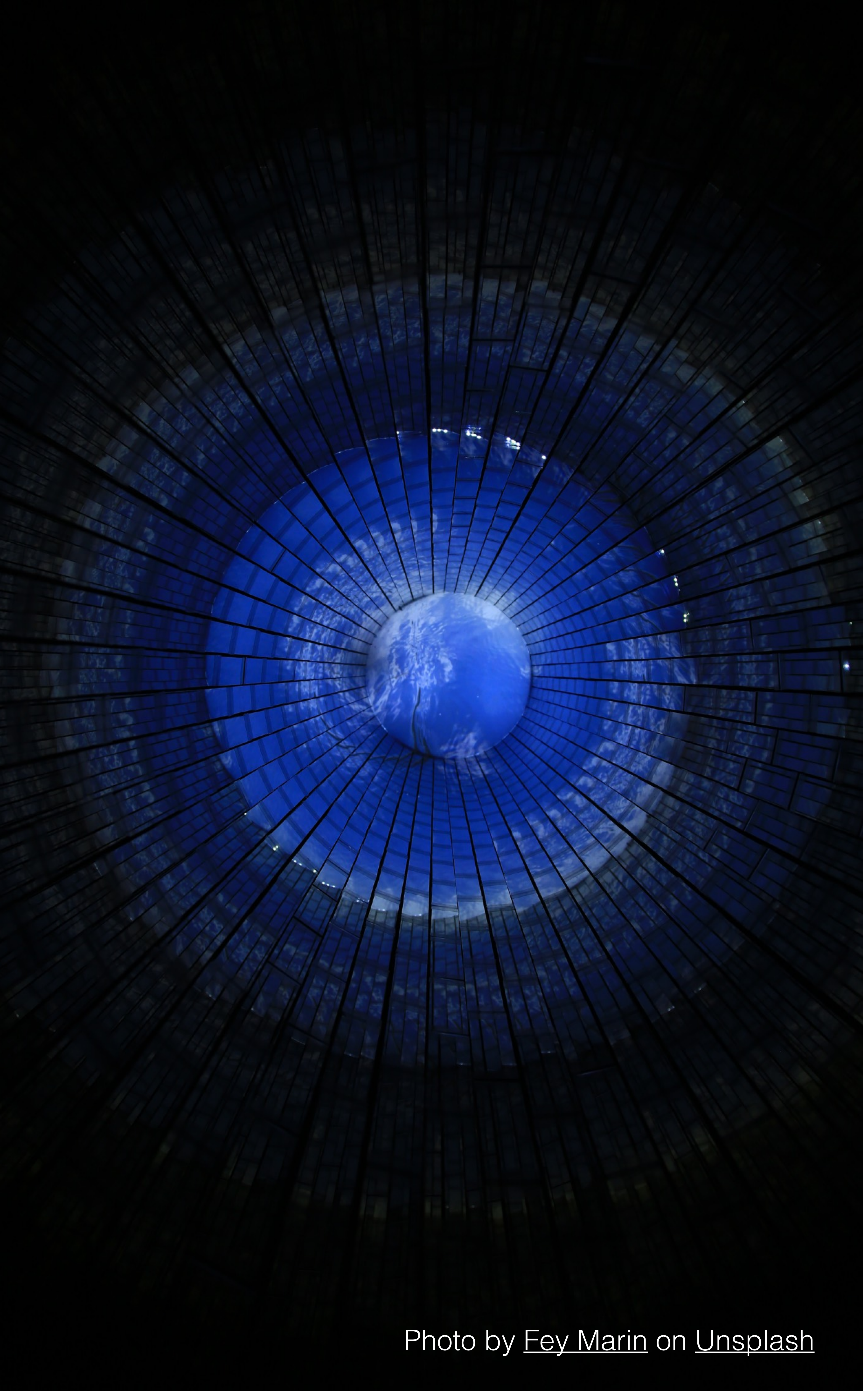


The dimension on which we apply the principle of indifference changes the probability assignments. That is, probabilities assigned according to the principle of indifference are not *invariant* to different parameterizations.



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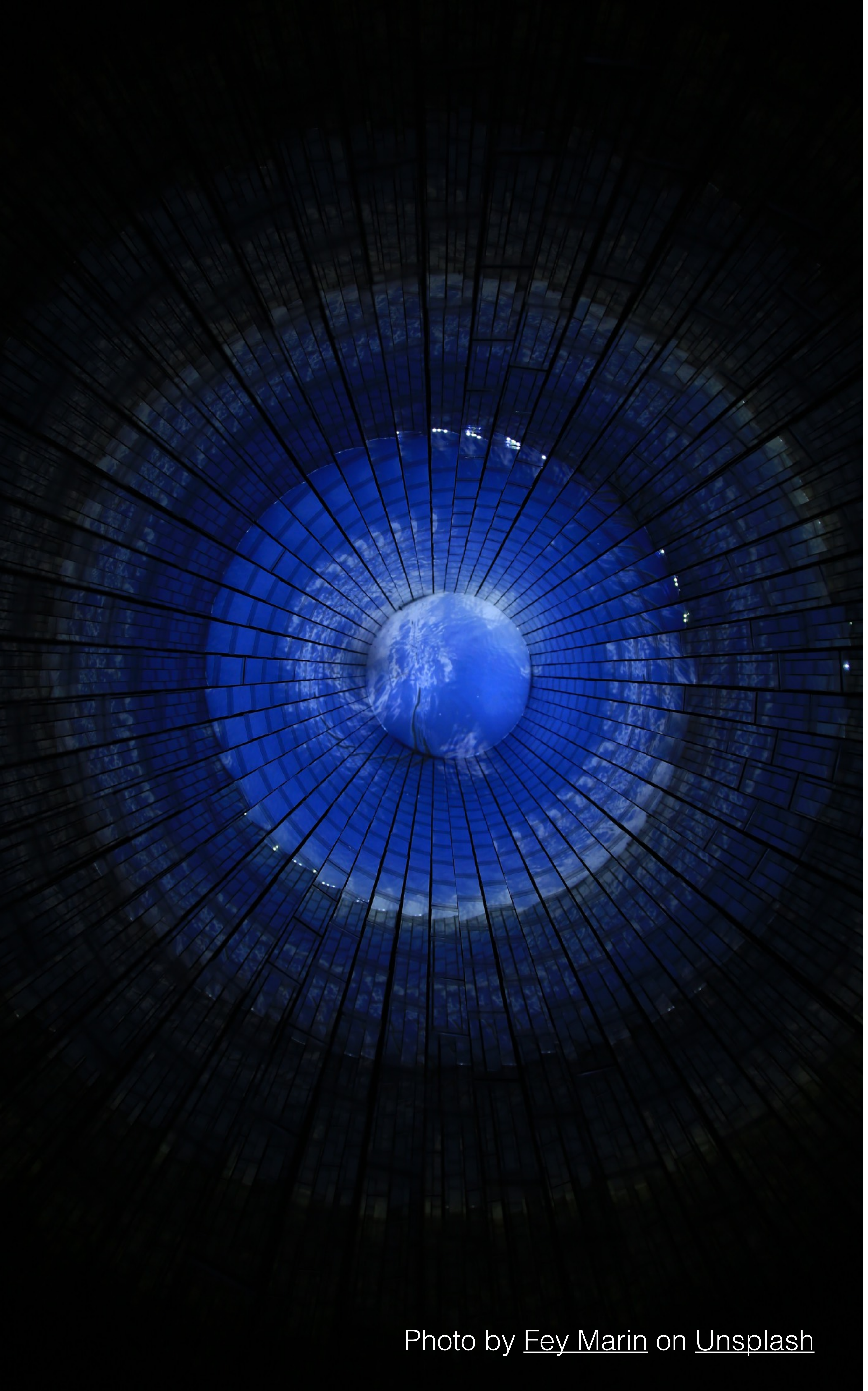
Objective priors are an attempt to avoid personal bias and violations of invariance. They may also provide baselines for comparisons for subjective methods.



The **univariate Jeffrey's prior** is defined as any prior proportional to the square root of the Fisher information number: $\pi(\theta) \propto \sqrt{I(\theta)}$.

Result: Jeffrey's prior is invariant to reparameterization.

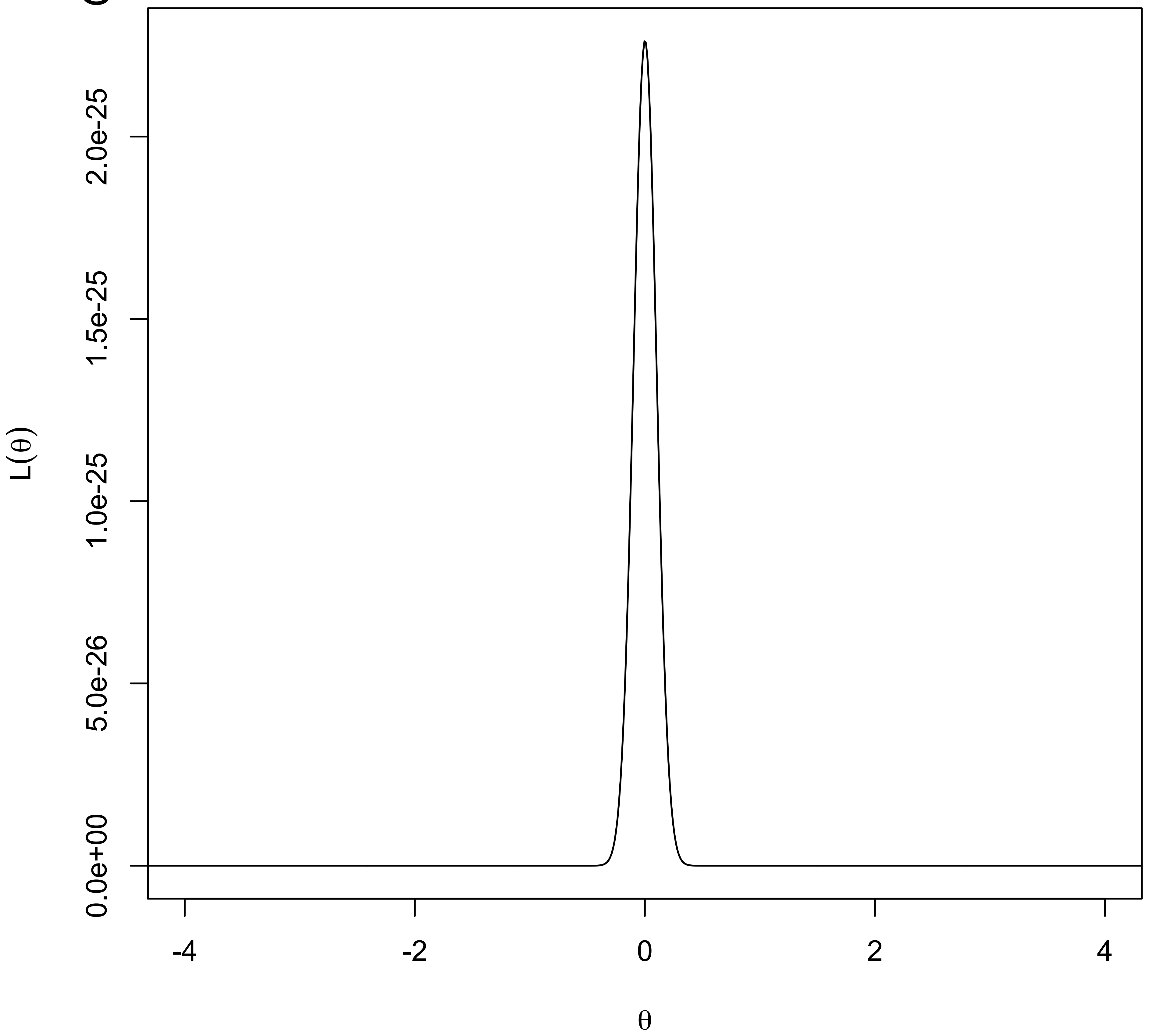
To prove this invariance, we will need to :

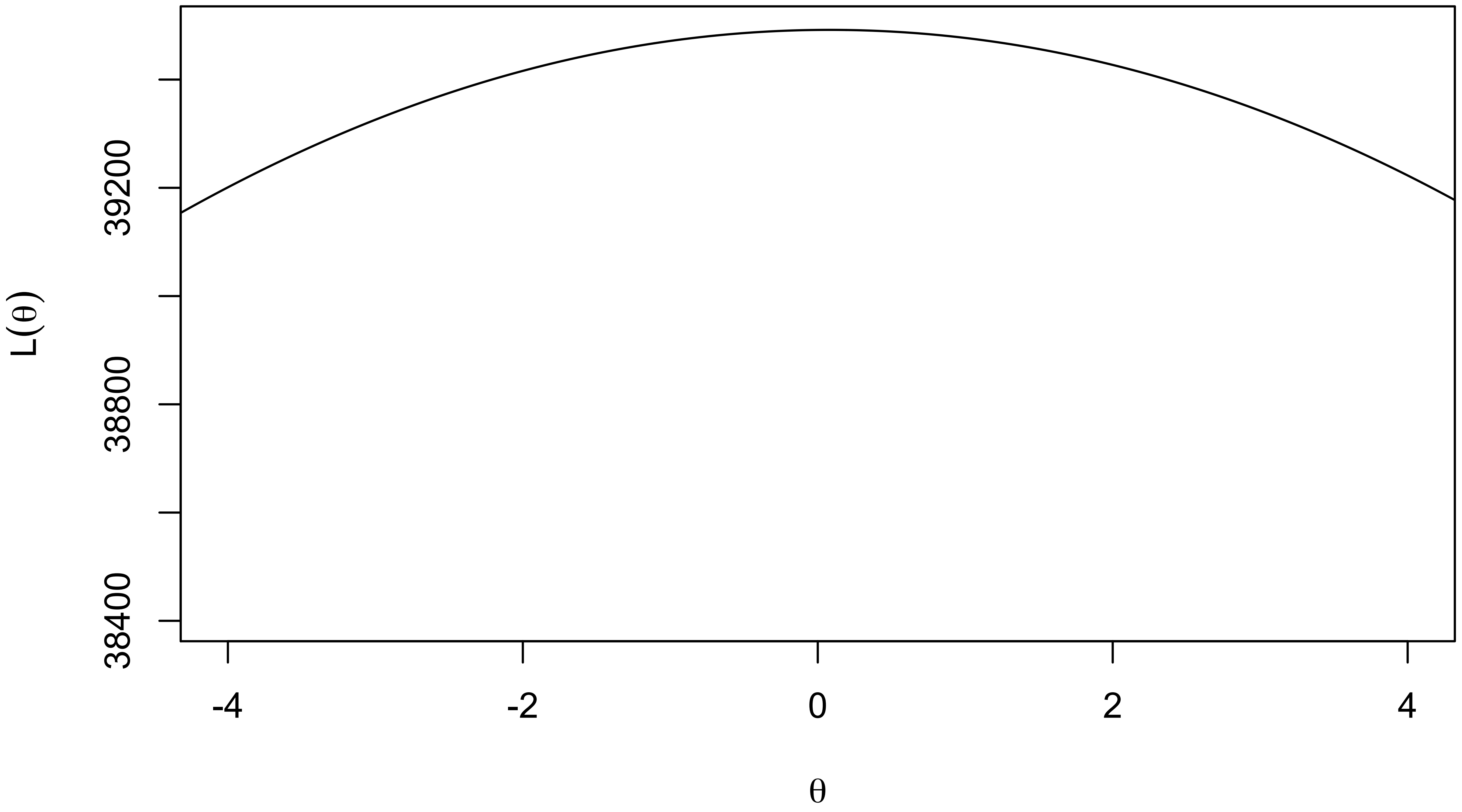
1. Recall how to find pdfs of transformations of random variables (we did this when we derived the inverse gamma distribution a few slides back!)

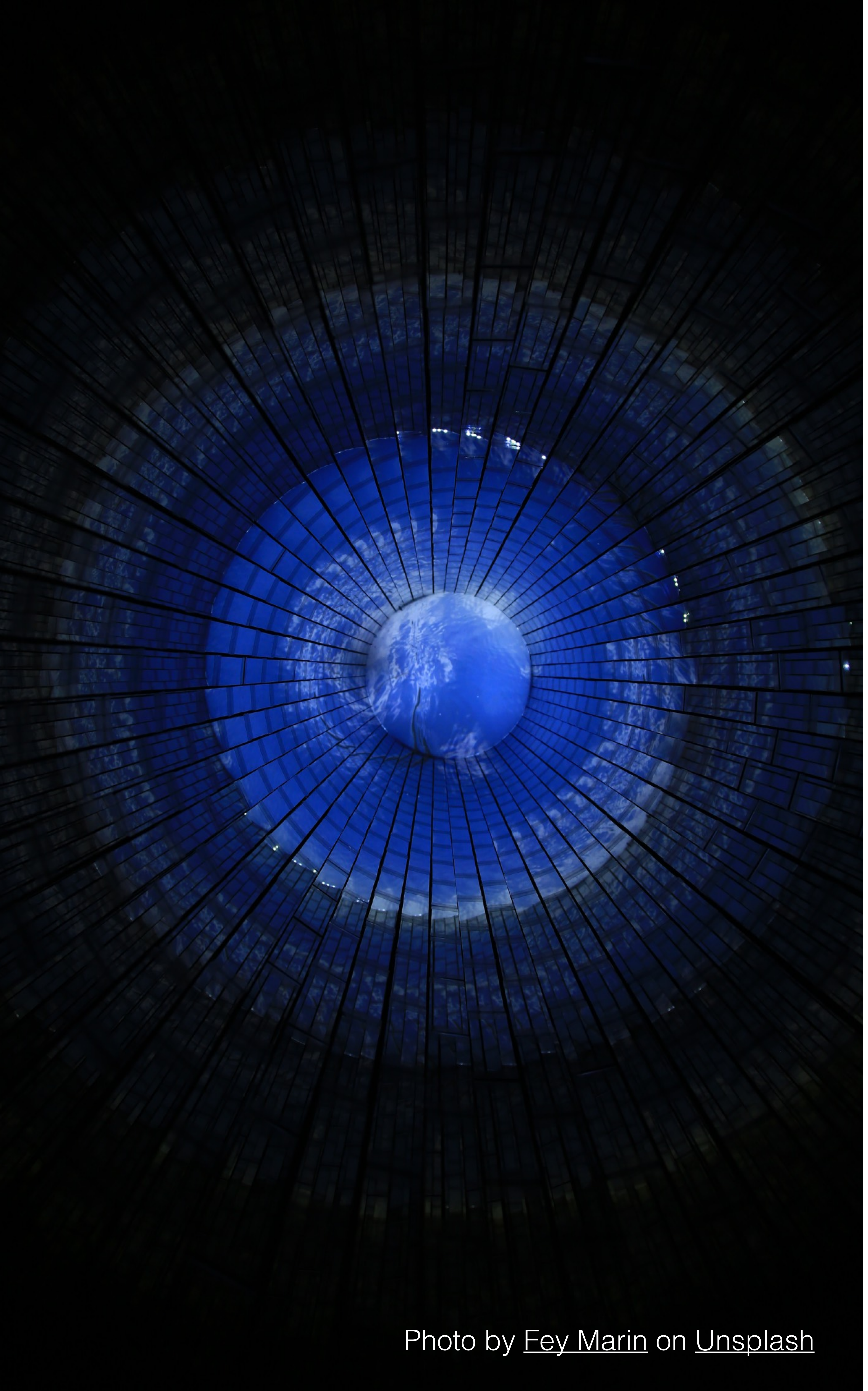
2. Review Fisher's information

Fisher's Information

The Fisher information number (or matrix, when $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)^T$) tells us how much information a sample of data contains about the unknown parameter If $f(\mathbf{x} \mid \theta)$ is sharply peaked with respect to changes in θ, it will be easier to estimate θ from the data than if $f(\mathbf{x} \mid \theta)$ is shallow, it will be harder.







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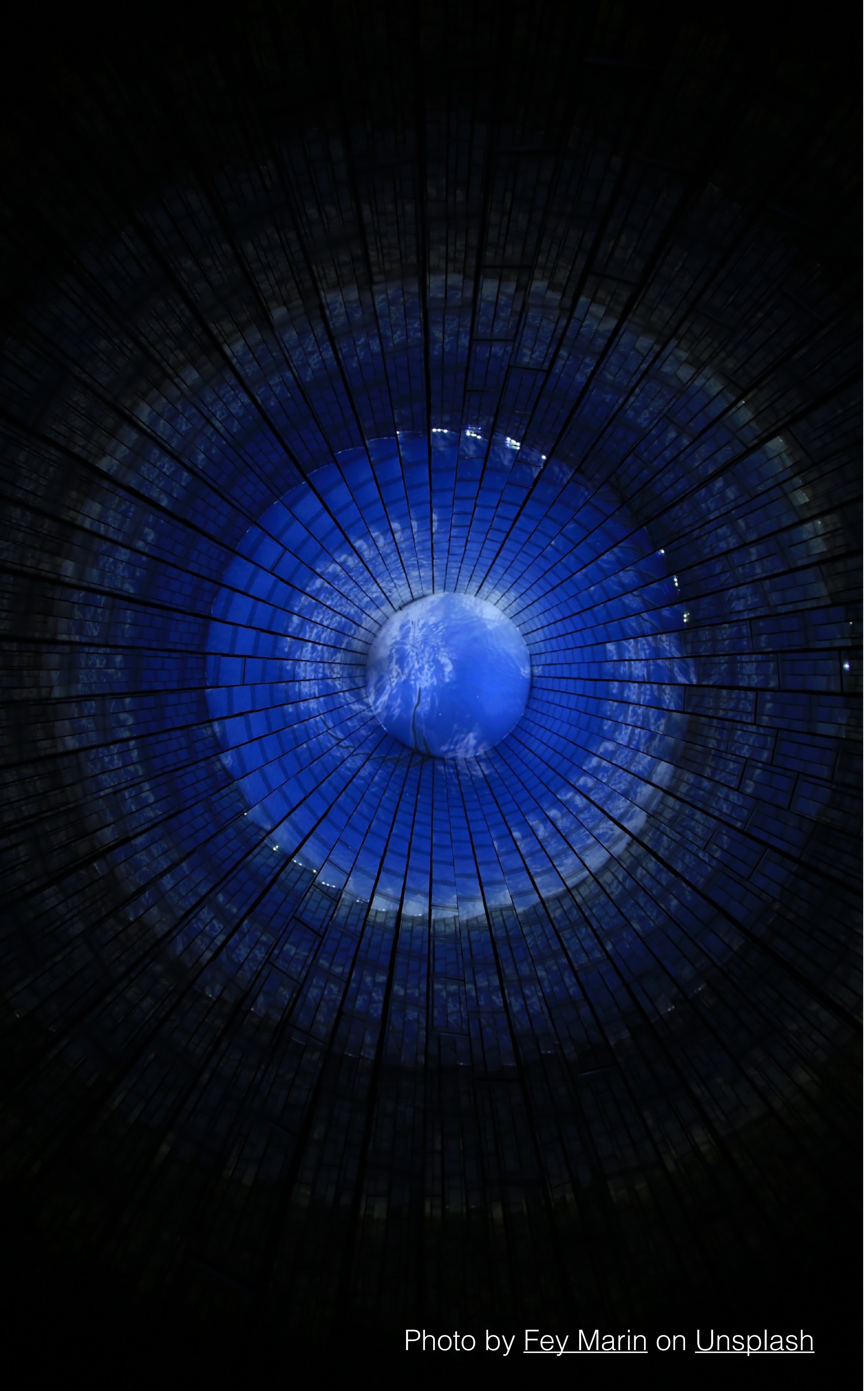
The **score** is the derivative (gradient) of the log-likelihood function with respect to the parameter (vector) $\boldsymbol{\theta}$.

Interpretation: The score indicates how sensitive a likelihood function is to its parameter(s) $\boldsymbol{\theta}$.

Theorem: Under regularity conditions, the expectation of the score is zero.



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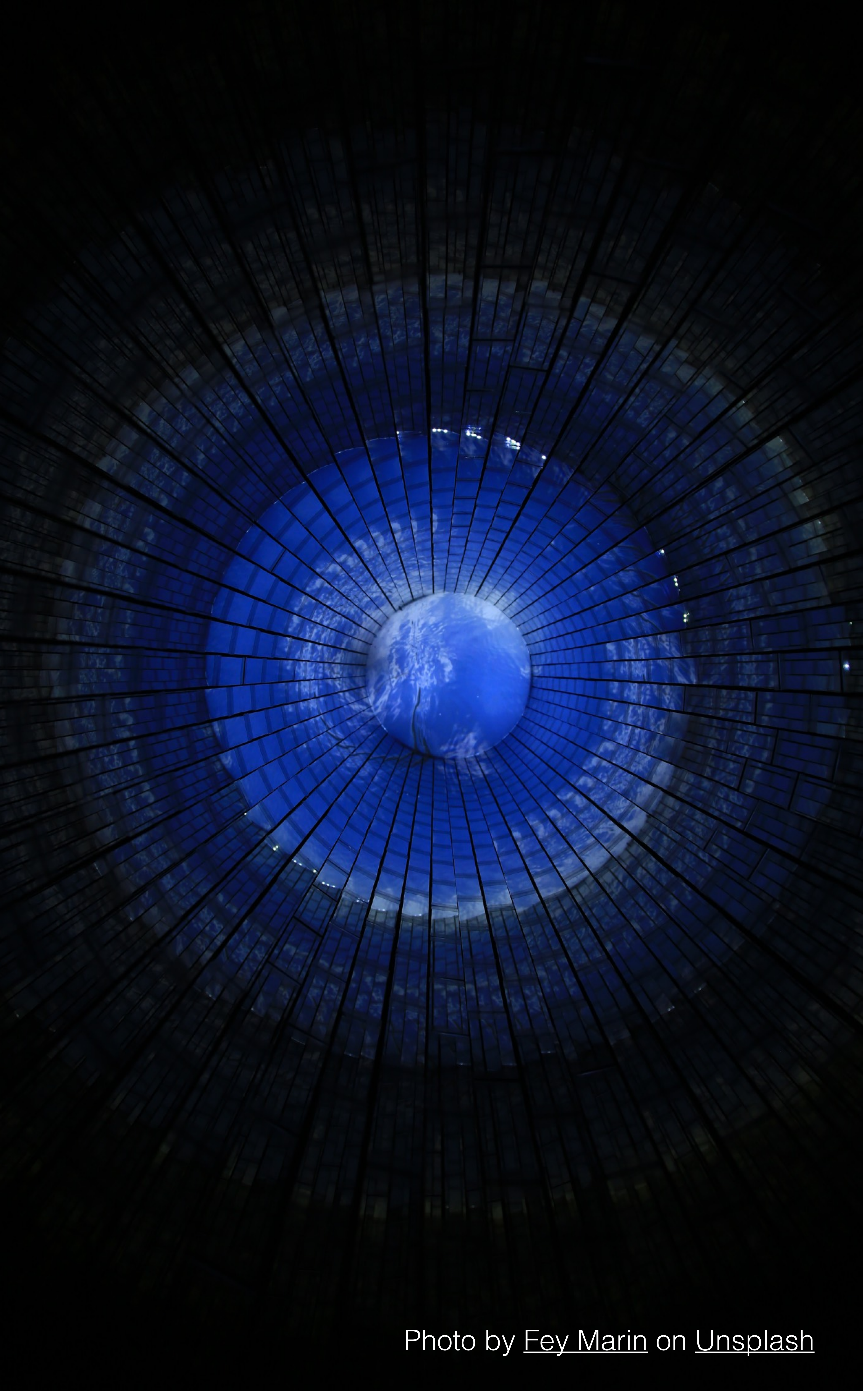
The **Fisher Information** is defined as the variance of the score:

Interpretation: The Fisher Information is a measure of the curvature of the graph of the log likelihood. Near the MLE, high Fisher Information suggests that the max is “sharp”; low Fisher Information suggests that the max is “dull”.

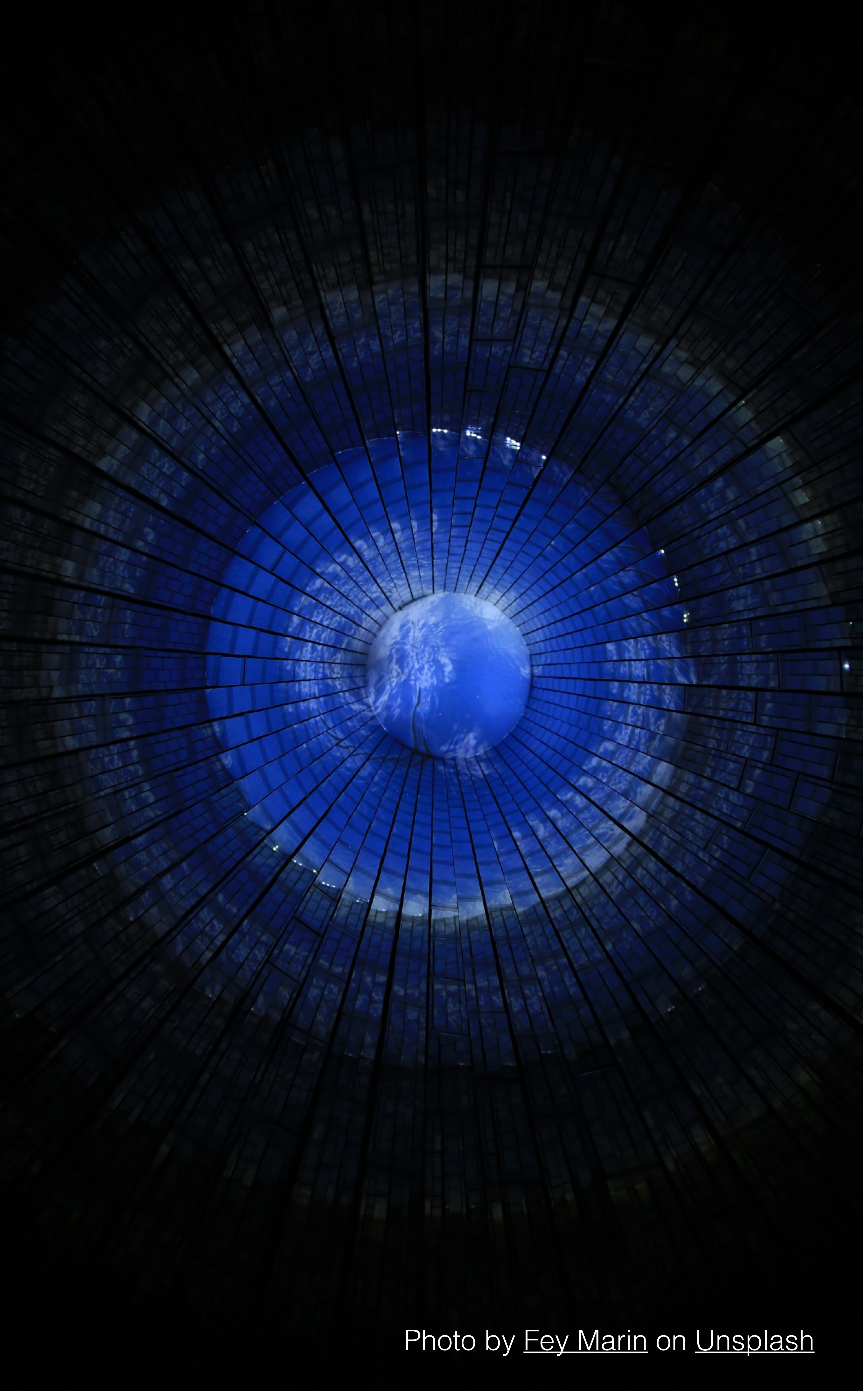
The Fisher information can also be written as:



To show that Jeffrey's prior is invariant to transformation, consider $X_1, \ldots, X_n \overset{iid}{\sim} f(\mathbf{x} \mid \theta)$ and $\pi_\theta(\theta) \propto \sqrt{I_\theta(\theta)}$.

Consider a reparameterization/transformation $\gamma = g(\theta)$. $\pi_\theta(\theta)$ is invariant to transformation if

$\pi_\gamma(\gamma) = \pi_\theta(\theta) \left| \dfrac{d\theta}{d\gamma} \right|$. $\pi_\gamma(\gamma) \propto \sqrt{I_\gamma(\gamma)}$ satisfies this relationship.



Let $Y \,|\, \lambda \sim Poisson(\lambda)$. Derive the Jeffreys' prior for λ.