Combinatorics Problem Set #1

Walter Johnson Mathematics Team

June 2021

Problems are ordered from easiest to hardest difficulty. None of the problems require a calculator, calculus, analysis, or an abacus.

1

How many solutions (in ordered pairs of positive integers) does the equation

$$xy = 2(x + y)$$

have?

2

Let S be the set of lattice points $P(x, y)$, where x and y are integers from 0 to 3, inclusive. We call the set $\{P_1, P_2, P_3, P_4\}$ a SQUARESET if P_1, P_2, P_3, and P_4 are the vertices of a square (order is not important). How many distinct SQUARESETS are there that are subsets of S?

3

Jacob has 4 indistinguishable gold coins and 4 indistinguishable silver coins. Each coin has an engraving of a face on one side, but not on the other. He wants to stack the eight coins on a table into a single stack so that no two adjacent coins are face to face. Find the number of possible distinguishable arrangements of the 8 coins.

4

In a regular 20-gon, three distinct vertices are chosen at random. Compute the probability that the triangle formed by these three vertices is a right triangle.

5

What is the sum of all integer values n for which

$$\binom{20}{n} + \binom{20}{10} = \binom{21}{11}$$

6

Find the number of non-negative integer solutions to

$$a + b + c + d + e + f = 23$$

7

Find the sum

$$\binom{100}{1} + 2\binom{100}{2} + 3\binom{100}{3} + \cdots + 100\binom{100}{100}$$

8

Suppose 6 points A, B, C, D, E, F are chosen uniformly at random on the circumference of some circle. The probability that line segments $\overline{AB}$, $\overline{CD}$, $\overline{EF}$ do not intersect is $\frac{p}{q}$ with p,q coprime. Compute $p + q$.

9

At Easter-Egg Academy, each student has two eyes, each of which can be eggshell, cream, or cornsilk. It is known that 30% of the students have at least one eggshell eye, 40% of the students have at least one cream eye, and 50% of the students have at least one cornsilk eye. What percentage of the students at Easter-Egg Academy have two eyes of the same color?

10

Let a and b be lengths of the legs of a right triangle with the following properties: All three sides of the triangle are integers, and the perimeter of the triangle is numerically equal to the area of the triangle. Compute all possible ordered pairs (a, b) where $a < b$.

11

How many ordered sets of positive integers $(a_1, a_2, a_3, a_4, a_5, a_6)$ are there such that $a_i \geq i$ for $i = 1, 2, \ldots, 6$ and

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \leq 100$$

12

A dot is marked at each vertex of a triangle ABC. Then 2,3, and 7 more dots are marked on the sides $\overline{AB}, \overline{BC}, \overline{CA}$, respectively. How many triangles have their vertices at these dots?

13

Suppose 10 points are drawn on a plane such that exactly 4 of the points are collinear and amoung the remaining points, no three points are collinear. How many distinct lines can be drawn by connecting any 2 among these 10 points?

14

3 red balls and 12 blue balls are randomly placed in a row. What is the probability that no two red balls are adjacent?

15

How many degree 7 polynomials $f(x)$ are there with positive integer coefficients such that $f(1) = 15$, $f(-1) = 3$, and $f(0) = 2$?

16

Michelle is at the bottom-left corner of a 6×6 lattice grid, at $(0, 0)$. The grid also contains a pair of one-time-use teleportation devices at $(2, 2)$ and $(3, 3)$; the first time Michelle moves to one of these points she is instantly teleported to the other point and the devices disappear. If she can only move up or to the right in unit increments, in how many ways can she reach the point $(6, 6)$?

17

A parking lot has 16 spaces in a row. Twelve cars arrive, each of which requires one parking space, and their drivers chose spaces at random from among the available spaces. Auntie Em then arrives in her SUV, which requires 2 adjacent spaces. What is the probability that she is able to park?

18

Find the number of ways to distribute 100 pieces of candy to five children, given the youngest two children will accept at most one piece of candy, the middle child will accept any amount of candy, and the oldest two will only accept an odd amount of candy.

19

Chelsea, Barbara, and Jenna go to dinner, and have to pay a \$50 dollar bill. Chelsea insists on paying a number of dollars that is a multiple of 3. Barbara and Jenna each insist on paying an odd number of dollars. How many ways can they pay the bill?

20

Consider all 1000 element subsets of the set $\{1, 2, 3, \ldots, 2015\}$. From each such subset choose the least element. The arithmetic mean of all of these least elements is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$

21

Row 1 of pascal's triangle consists of two 1s. Let $\{a_i\}$, $\{b_i\}$, and $\{c_i\}$ be the sequence, from left to right, of the elements in the 2005th, 2006th, and 2007th row, respectively, with the leftmost element occurring at $i = 0$. Compute:

$$\sum_{i=0}^{2006} \frac{b_i}{c_i} - \sum_{i=0}^{2005} \frac{a_i}{b_i}$$

22

Compute

$$\sum_{n_{60}=0}^{2} \sum_{n_{59}=0}^{n_{60}} \cdots \sum_{n_2=0}^{n_3} \sum_{n_1=0}^{n_2} \sum_{n_0=0}^{n_1} 1$$

23

Let S denote the set of all triples (a, b, c) of positive integers where $a + b + c = 15$. Compute

$$\sum_{(a,b,c)\in S} abc$$