Real Analysis

Walter Johnson Maths Honors Society

Requirements

In order to receive credit for this independent research project and be in good standing with the Walter Johnson Maths Honor Society, you must write a paper which does each of the following:

- Describe, briefly, what Real Analysis is.
- Describe what a manifold is an put this in context of other objects. Provide examples. Explain how curvature is measured on a surface.
- Describe what Fourier series are. Give examples of different basic Fourier series. Provide the Integral representation of a Fourier series of a function f.
- Give an equation for a general first order differential equation. Give examples of these and solve them.
- Explain what Exact Equations are.
- Complete and describe solutions to both problems.

On average, this assignment will take about 3 hours to research and write up. You may **not** work in a group or collaborate with others.
You will be assigned to groups of 6 people, each of which has completed a different independent research project. At the end of the year, you will present your findings to your group, and listen as your peers present their findings. Your presentation must briefly discuss every subject required in your paper along with 1 of the problems you solved, of your choice.

Resources

You are provided with various resources to complete your research. You are welcome to use resources that are not given here.

Functions and Analysis

The **real number line** is denoted $\mathbb{R}$ or $\mathbb{R}^1$, and the cartesian coordinate plane is denoted $\mathbb{R}^2$. This continues to higher, n dimensional space with $\mathbb{R}^n$. A **space** is essentially a multidimensional set of numbers. $\mathbb{R}^3$ is a three-dimensional space, for example.
A **function** f that **maps** a set of points S_1 to another set of points S_2 is notated:

$$f : S_1 \xrightarrow{f} S_2$$

A **functional** is a bit different than a function. A functional maps a space to the set of real numbers $\mathbb{R}$, or another scalar set. Functionals are often represented in their integral forms. An example is the functional J which takes in as its input the **Lagrangian** $L(x, u, u')$ where u is a function:

$$J[f] = \int_1^5 L(x, u, u')dx$$

The **first derivative** of a function $v(t)$ is often notated with a dot above it, and with subsequent dots for subsequent derivatives. The differential notation also sometimes uses a delta δ instead of a d:

$$\frac{dv}{dt} = \frac{\delta v}{\delta t} = \dot{v}$$

Curvature

- Britannica article on curvature and differential geometry. Read through Curvature of Surfaces.
- ETH Zurich Department of Mathematics introduction to Differential Geometry. Start at the preface.

Fourier Series

- Brilliant wiki page on Fourier Series.
- Swarthmore article on the statement of problem solved with Fourier Series and Fourier's determination.

Differential Equations

- Lamar University notes on First Order Differential Equations.
- Cliffs Notes on Exact Equations.
- Lamar University notes on Exact Equations.

Calculus of Variations

- University of Minnesota article on Calculus of Variations.
- Introduction to the Calculus of Variations by Milton Keynes at the Open University. Read through Stationary Paths (1.2).
- Wolfram MathWorld article on Calculus of Variations.

Problem 1

Show that

$$x^4y^3 + x^2y^5 + 2xy = c$$

is a solution to the equation

$$4x^3y^3 + 2xy^5 + 2y + (3x^4y^2 + 5x^2y^4 + 2x)\frac{dy}{dx} = 0$$

Problem 2

Let S be a non-euclidean surface that exists in 5 dimensional space. Let $G[y]$ be a functional that maps a path $y(\vec{x})$ from $\vec{a}$ to $\vec{b}$ on the surface S, where $\vec{x}$ is a five-dimensional variable. If

$$\left.\frac{d}{d\epsilon}G[y + \epsilon g]\right|_{\epsilon=0} = 0$$

For any arbitrary $g(\vec{x})$ such that $g(\vec{a}) = g(\vec{b}) = 0$, what is $y(\vec{x})$?

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