Series and Recursion Homework

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1 Review

1.1 Recursive and Modular Series

You are bound to encounter the following kinds of problems:

1. Given the series $a_1, a_2, \ldots$ find a_{2021}
2. Given the function $f(x)$, find $f^{2021}(z)$
3. Find the sum of $a_1, a_2, \cdots, a_{2021}$

There are occasions when the problems will require some more complicated manuvering work, but for the most part, they require patience and an understanding that there is recursion in the series or modularity in the order of the function. It is also not a coincidence that I put 2021 as the bounds of those sequences. Because recursion and modularity make for arbitrary final values, writers of math competitions love to throw in the year number. Think of it as a way for them to timestamp their problems. This is not always reliable though, as there are times when the year number is used due to a unique property of the number. But we save this discussion for another time.

1.2 Infinite Geometric Series

Infinite geometric series pop up in a variety of competitive scenarios. The primary two forms to apply the formula are the r formula:

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$$

and

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

Which are best when using either a real number or a fraction with a numerator not equal to 1. The p form is simply the substitution $r = 1/p$:

$$\sum_{n=1}^{\infty} r^n = \frac{1}{p-1}$$

Once a proper substitution is apparently applicable, utilize it.

2 Introductory Problems

2.1

If

$$f(z) = \frac{z+1}{z-1}$$

Find $f^{1991}(2+i)$

2.2

Let

$$\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{5^2}\right)\cdots\left(1-\frac{1}{1991^2}\right) = \frac{x}{1991}$$

Compute the integer x

2.3

Solve for x given the following:

$$e^{x^{e^{x^{e^{x^{\cdot^{\cdot^{\cdot}}}}}}}} = 2$$

2.4

Let $f(x) = x^2(1-x)^2$. What is the value of the sum

$$f\left(\frac{1}{2019}\right) - f\left(\frac{2}{2019}\right) + f\left(\frac{3}{2019}\right) - \cdots + f\left(\frac{2017}{2019}\right) - f\left(\frac{2018}{2019}\right)$$

2.5

A sequence of numbers is defined recursively by $a_1 = 1$, $a_2 = \frac{3}{7}$, and

$$a_n = \frac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}}$$

for all $n \geq 3$. Then a_{2019} can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$?

3 Further Application

3.1

Evaluate

$$\left(1 + \frac{2019}{2 + \frac{2019}{2 + \frac{2019}{2 + \ddots}}}\right)^2$$

3.2

If $b = 100$, what is the infinite sum below?

$$(\log_b 2)^0 \left(\log_b 5^{2^0}\right) + (\log_b 2)^1 \left(\log_b 5^{2^1}\right) + (\log_b 2)^2 \left(\log_b 5^{2^2}\right) + \cdots$$

3.3

Let $a + ar_1 + ar_1^2 + ar_1^3 + \cdots$ and $a + ar_2 + ar_2^2 + ar_2^3 + \cdots$ be two different infinite geometric series of positive numbers with the same first term. The sum of the first series is r_1, and the sum of the second series is r_2. What is $r_1 + r_2$?

3.4

Let $ABCD$ be a unit square. Let Q_1 be the midpoint of $\overline{CD}$. For $i = 1, 2, \ldots,$ let P_i be the intersection of $\overline{AQ_i}$ and $\overline{BD}$, and let Q_{i+1} be the foot of the perpendicular from P_i to $\overline{CD}$. What is

$$\sum_{i=1}^{\infty} [\triangle DQ_iP_i]$$

Where $[\triangle DQ_iP_i]$ denotes the area of that triangle?

3.5

For some positive integer k, the repeating base-k representation of the (base-ten) fraction $\frac{7}{51}$ is

$$0.\overline{23}_k = 0.23232323232323\ldots_k$$

What is k?

3.6

Amelia has a coin that lands on heads with probability $\frac{1}{3}$, and Blaine has a coin that lands on heads with probability $\frac{2}{5}$. Amelia and Blaine alternately toss their coins until someone lands head and 'wins'. Amelia goes first. The probabilty that Amelia wins is $\frac{p}{q}$. What is $q - p$?

3.7

For each integer $n \geq 4$, let a_n denote the base-n number $0.\overline{133}_n$. The product $a_4 a_5 \cdots a_{99}$ can be expressed as $\frac{m}{n!}$ where m and n are positive integers and n is as small as possible. What is the value of m?