Geometry Problem Set #1

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Problems are ordered from easiest to hardest difficulty, with high probability. None of the problems require a calculator, calculus, analysis, or an abacus. If you have any questions, just ask!

1

Regular hexagon $ABCDEF$ has vertices A and C at $(0,0)$ and $(7,1)$, respectively. What is it's area?

2

Triangle ABC is a right triangle with $\angle ACB$ as its right angle, $m\angle ABC = 60°$, and $AB = 10$. Let P be randomly chosen inside $\triangle ABC$, and extend $\overline{BP}$ to meet $\overline{AC}$ at D. What is the probability that $BD > 5\sqrt{2}$?

3

A circle of radius 2 is centered at A. An equilateral triangle with side 4 has a vertex at A. What is the difference between the area of the region that lies inside the circle but outside the triangle and the area of the region that lies inside the triangle but outside the circle?

4

Circles with centers O and P have radii 2 and 4, respectively, and are externally tangent. Points A and B are on the circle centered at O, and points C and D are on the circle centered at P, such that $\overline{AD}$ and $\overline{BD}$ are common external tangents to the circles. What is the area of hexagon $AOBCPD$?

5

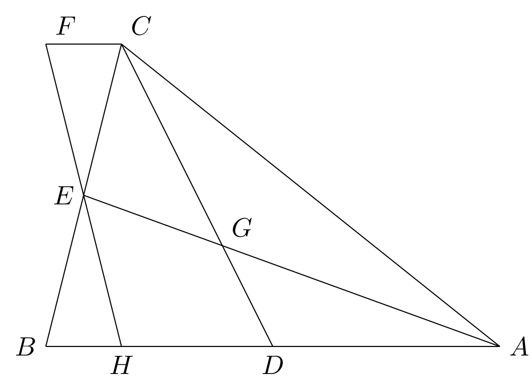
A 2×3 rectangular sheet of paper is folded over a crease through one corner so that the adjacent corner lands precisely on the opposite side. Let α be the measure of the angle marked in the diagram. Calculate $\tan\alpha$

6

Let D be the point on side $\overline{BC}$ of triangle ABC for which $\overline{AD}$ is the angle bisector of $\angle A$. If $AB = AD = 6$ and $CD = 2(BD)$, then find length BD.

7

In $\triangle CBA$, $\overline{CD}$ and $\overline{AE}$ are medians, $\overline{FC}||\overline{AB}$. Lines $\overline{FEH}$, $\overline{CGD}$, and $\overline{AGE}$ are drawn. The area $FCGE = 7$, and the area of $EGDH = 11$. Compute the area of $\triangle CBA$.



8

In triangle ABC, D lies on BC so that $BD : CD = 5 : 2$, and E lies on AB so that $BE : EA = 3 : 4$.F is the intersection of AD and CE. Compute $EF : CF$.

9

In concave hexagon $ABCDEF$, $m\angle A = m\angle B = m\angle C = 90°$, $m\angle D = 100°$, and $m\angle F = 80°$. $CD = FA$, $AB = 7$, $BC = 10$, and $EF + DE = 12$. Compute the area of the hexagon.

10

Isoceles triangle with vertex angle θ has side lengths $\sin\theta, \sqrt{\sin\theta}$, and $\sqrt{\sin\theta}$. What is the area of this triangle?

11

Let G be the centroid of $\triangle ABC$; that is, the point where the segments joining each vertex to the midpoint of the opposite side all meet. If $\triangle ABG$ is equilateral with $AB = 2$, then compute the perimeter of $\triangle ABC$

12

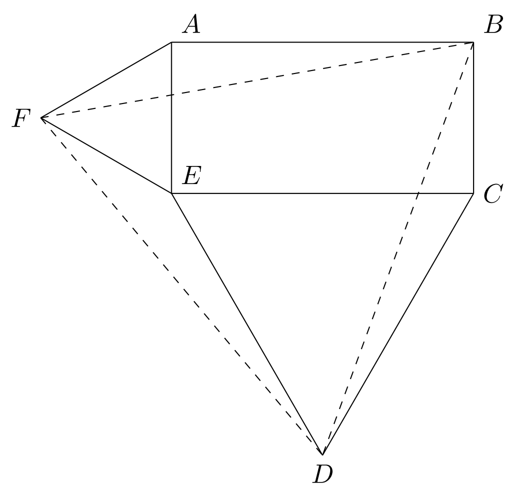
Three mutually tangent spheres of radius 1 rest on a horizontal plane. A sphere of radius 2 rests on them. What is the distance from the plane to the top of the larger sphere?

13

Square $ABCD$ has area 36, and $\overline{AB}$ is parallel to the x-axis. Vertices A, B, and C are on the graphs of $y = \log_a x$, $y = 2\log_a x$, and $y = 3\log_a x$, respectively. What is a?

14

If the sum of the areas of rectangle $ABCE$ and equilateral triangles AEF and DEC exceeds the area of $\triangle BDF$ by 33, compute the area of rectangle $ABCE$

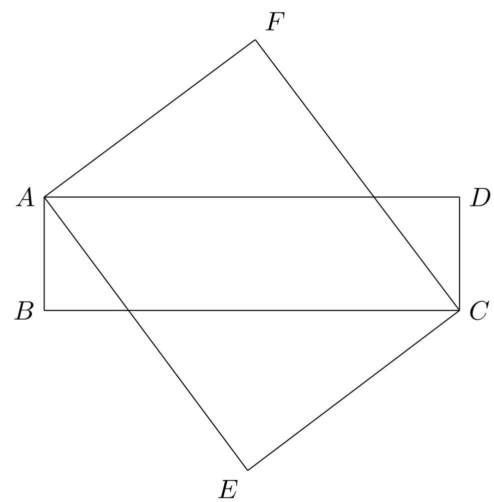


15

Three circles of radius s are drawn in the first quadrant of the xy-plane. The first cirlce is tangent to both axes, the second is tangent to the first circle and the x-axis, and the third is tangent to the first circle and the y-axis. A circle of radius $r > s$ is tangent to both axes and to the second and third circles. What is r/s?

16

In the diagram below, $ABCD$ is a rectangle with side lengths $AB = 3$ and $BC = 11$, and $AECF$ is a rectangle with side lengths $AF = 7$ and $FC = 9$, as shown. The area of the region common to the interiors of both rectangles is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



17

Let ABC be a triangle, and let D, E, F be the midpoints of $BC, AC < AB$ respectively. Let G be the intersection of AD, BE, and CF. Find $\frac{AG}{GD} + \frac{BG}{GE} + \frac{CG}{GF}$.

18

Suppose that in quadrilateral $ABCD$ we have $m\angle ABC = m\angle ACD = 90°$ and $m\angle CBD = m\angle CDB$. Label $m\angle DAC = \alpha$ and $m\angle ACB = \beta$. It follows that one of $\sin\alpha, \cos\alpha, \tan\alpha$ must always be equal to one of $\sin\beta, \cos\beta, \tan\beta$. Which two values are necessarily the same?

19

Let $ABCD$ be a unit square. Let Q_1 be the midpoint of $\overline{CD}$. For $i = 1, 2, \ldots$, let P_i be the intersection of $\overline{AQ_i}$ and $\overline{BD}$, and let Q_{i+1} be the foot of the perpendicular from P_i to $\overline{CD}$. What is

$$\sum_{i=1}^{\infty} [\triangle DQ_iP_i]$$

Where $[\triangle DQ_iP_i]$ denotes the area of that triangle?