Guided Discussion: Inequalities

Jenson's, Cauchy Schwartz, Power Means, AM-GM Problems

Walter Johnson Math Team

Guided Discussion: Derivation

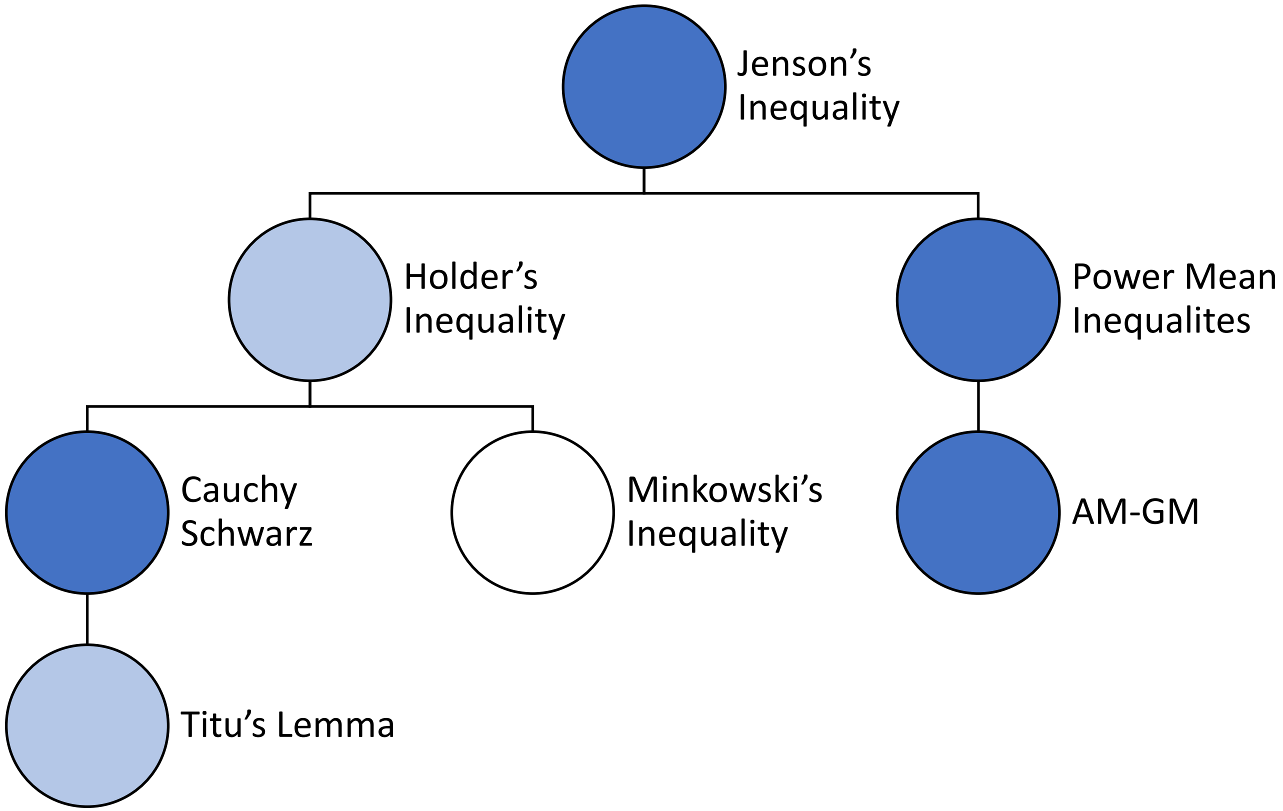
A little background on these inequalities is useful.

The premise is we have a very, very general inequality.

And all of our more specific, more useful inequalities are special cases of these general inequalities.

This creates the hierarchy shown to the right.

The ones in blue are the ones we will focus on for Competition Math.

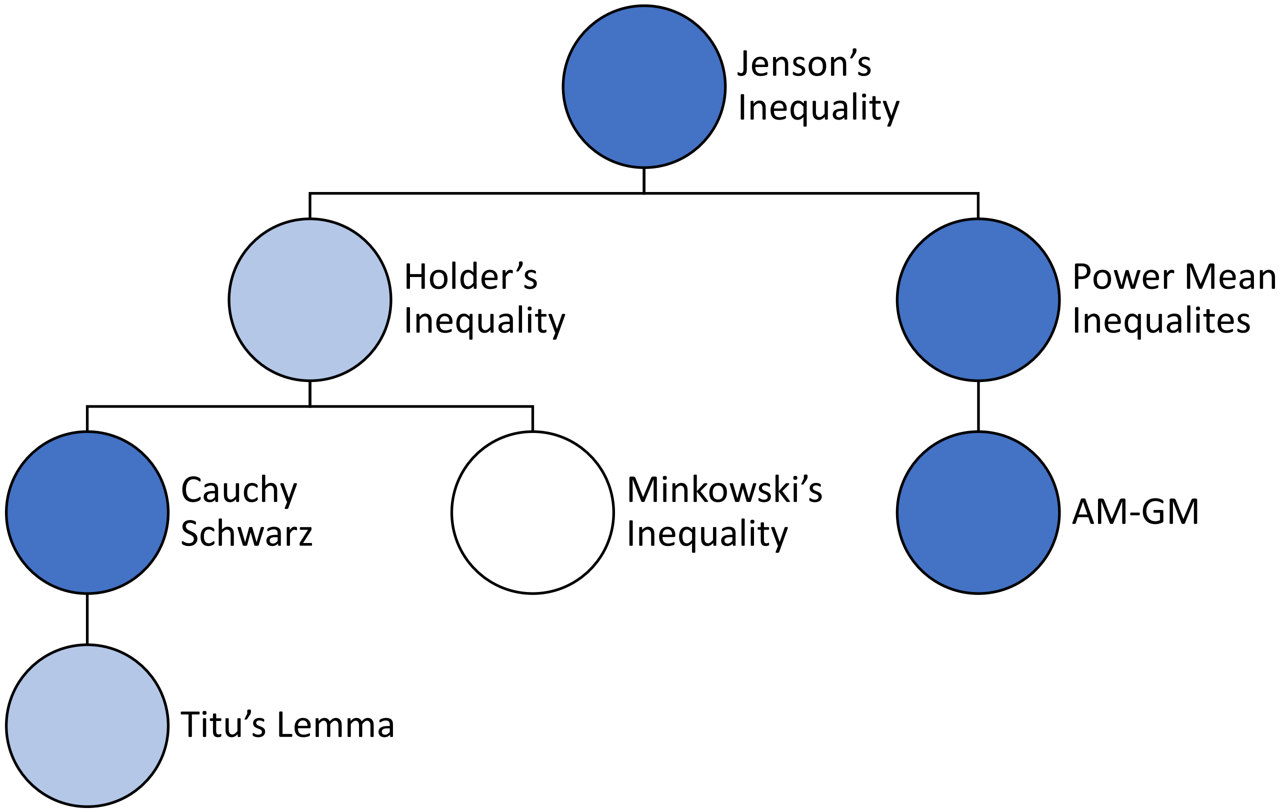


Guided Discussion: Derivation

A little background on these inequalities is useful.

*I want you to note, that memorizing the **formula** for each inequality **is not as important** as remembering the **structure** of the relationship being described by the inequality. The equations will follow smoothly.*

*We will look at the **Intuition** behind each inequality, then see the formula.*

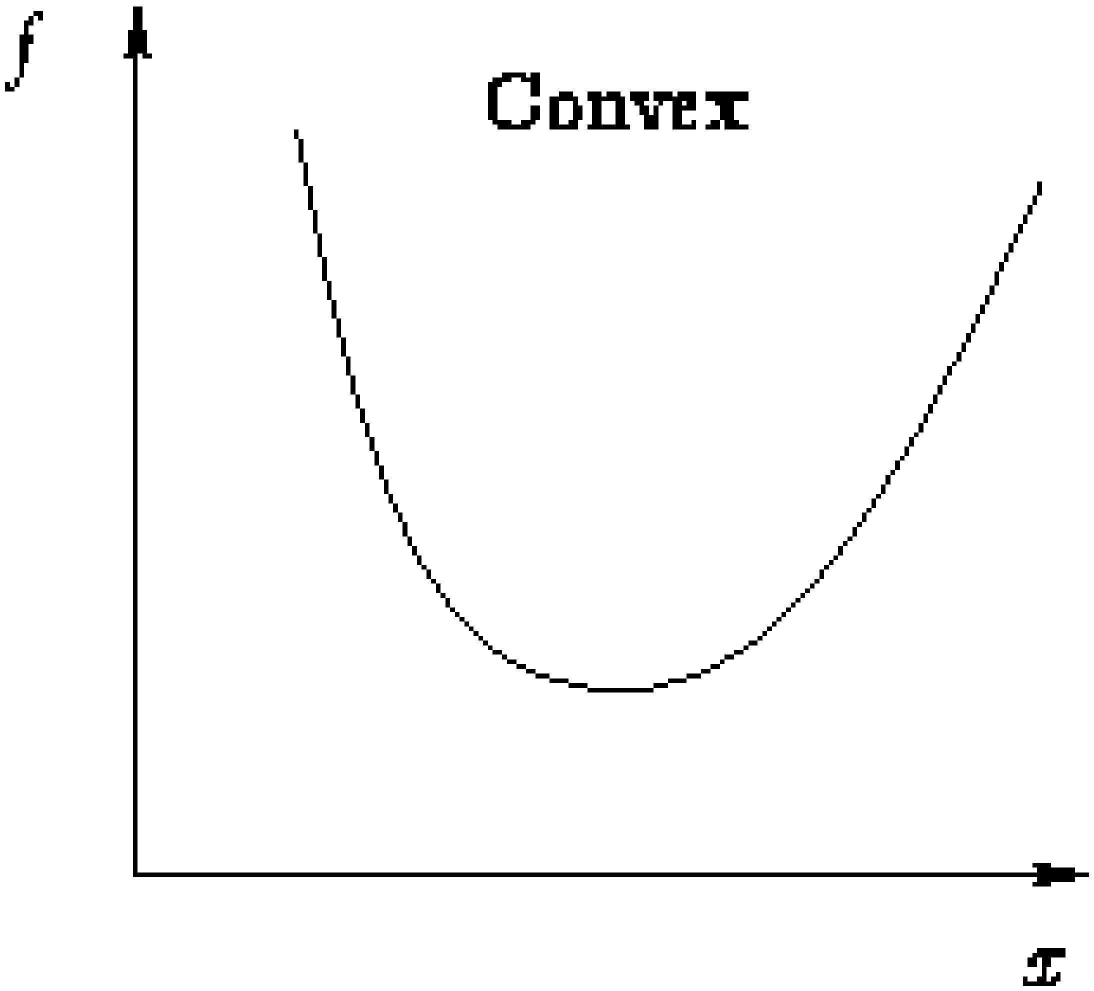


Guided Discussion: Jenson's Inequality

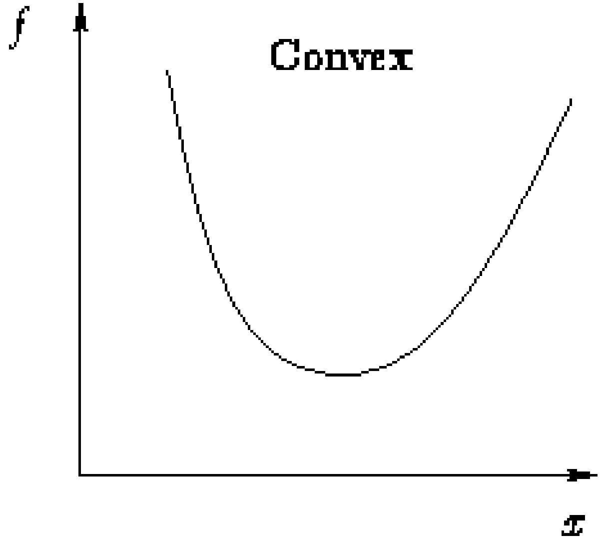
The derivation for Jenson's Inequality leads from general function concavity.

Say we have a function, f, which is convex on interval I, and we have sequence $x_1, x_2 \cdots x_n$

*Jenson's Inequality states that the **Average of the Outputs of the Sequence** is greater than or equal to the **Output of the Average of the Sequence***



Guided Discussion: Jenson's Inequality



Jenson's Inequality states that the ***Average of the Outputs of the Sequence*** *is greater than or equal to the* ***Output of the Average of the Sequence***

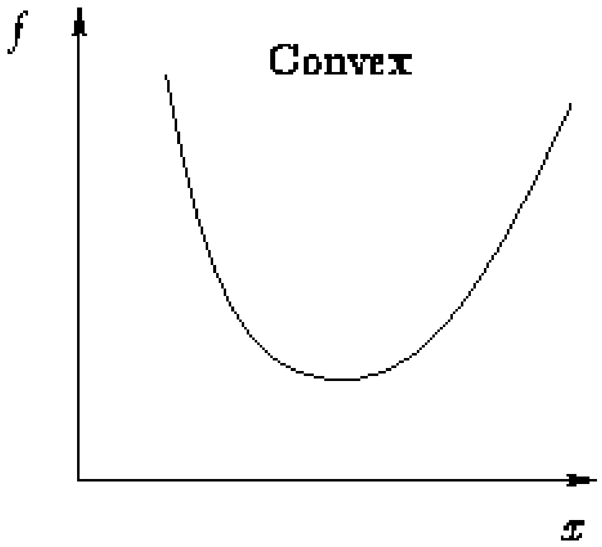
$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \geq f\left(\frac{x_1 + x_2 + \cdots x_n}{n}\right)$$

This is intuitive if you imagine the case where X contains two elements.

Notice that this inequality reverses when the function is Concave down.

Convince yourself this is true generally.

Guided Discussion:
Jenson's Inequality



Jenson's Inequality states that the ***Average of the Outputs of the Sequence*** *is greater than or equal to the* ***Output of the Average of the Sequence***

Furthermore*, this generalizes to a* ***Weighted Jenson's Inequality****, which satisfies for a weighted mean.*

Convince yourself this satisfies for a weighted mean average as well.

$$\frac{\omega_1 f(x_1) + \omega_2 f(x_2) + \cdots + \omega_n f(x_n)}{\omega_1 + \omega_2 + \cdots + \omega_n} \geq f\left(\frac{\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_n x_n}{\omega_1 + \omega_2 + \cdots + \omega_n}\right)$$

Guided Discussion:
Jenson's Inequality

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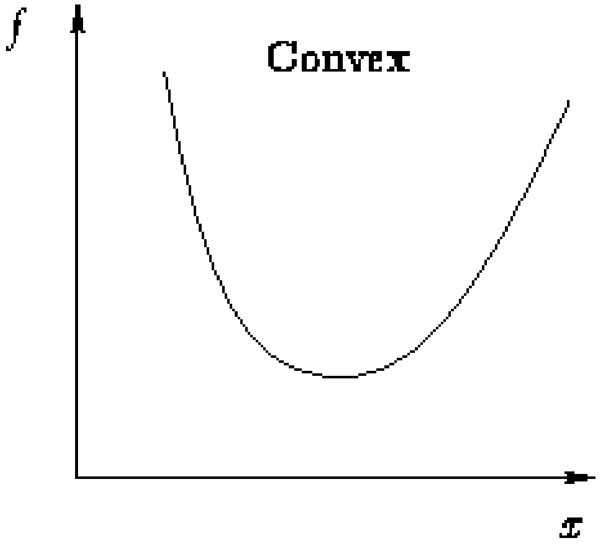
See that the General Jenson's Inequality is a special case where all the weights ω_i are 1



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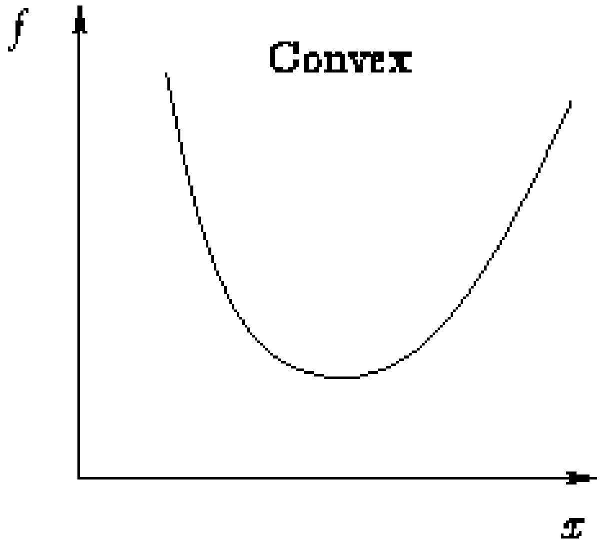
Guided Discussion:
Jenson's Inequality



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$$\frac{\omega_1 f(x_1) + \omega_2 f(x_2) + \cdots + \omega_n f(x_n)}{\omega_1 + \omega_2 + \cdots + \omega_n} \geq f\left(\frac{\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_n x_n}{\omega_1 + \omega_2 + \cdots + \omega_n}\right)$$

$$\frac{\sum_{i=1}^{n} \omega_i f(x_i)}{\sum_{i=1}^{n} \omega_i} \geq f\left(\frac{\sum_{i=1}^{n} \omega_i x_i}{\sum_{i=1}^{n} \omega_i}\right)$$

Guided Discussion: Power Mean Inequalities

The Power Mean Inequalities derive from Jenson's, but we won't show a proof of that here.

The Power Mean Inequalities actually are an infinite amount of inequalities.

Let's start by defining the first three power means inequalities.

The Arithmetic Mean:

$$\frac{a_1 + a_2 + \cdots a_n}{n} = \frac{\sum_1^n a_i}{n}$$

The Quadratic Mean:

$$\sqrt[2]{\frac{a_1^2 + a_2^2 + \cdots a_n^2}{n}} = \sqrt[2]{\frac{\sum_1^n a_i^2}{n}}$$

The Harmonic Mean:

$$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots \frac{1}{a_n}} = \frac{n}{\sum_1^n \frac{1}{a_i}}$$

Guided Discussion: Power Mean Inequalities

We can easily see the Arithmetic mean inequality is a special case where our power variable p equals 1, then our Quadratic mean is a case where $p = 2$, and our Harmonic Mean is a case where $p = -1$. We generalize a power mean below:

$$\sqrt[p]{\frac{a_1^p + a_2^p + \cdots a_n^p}{n}}$$

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Power Mean Inequalities

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$$\sqrt[p]{\frac{a_1^p + a_2^p + \cdots a_n^p}{n}}$$

What happens when $p = 0$?

Obviously, this formula does not "work" when $p = 0$. But interestingly enough, we substitute the Geometric Mean for the case when $p = 0$

Geometric Mean

$$\sqrt[n]{\frac{a_1 a_2 \cdots a_n}{n}}$$

Note, this is an inherently different structure.

Guided Discussion: *Power Mean Inequalities*

We can easily see the Arithmetic mean inequality is a special case where our power variable p equals 1, then our Quadratic mean is a case where $p = 2$, and our Harmonic Mean is a case where $p = -1$. We generalize a power mean below:

$$\sqrt[p]{\frac{a_1^p + a_2^p + \cdots a_n^p}{n}}$$

Now you're probably wondering what the relationship between these are?

The "Inequality" comes about as the following is true:

$$a_i \in \mathbb{R}^+$$
$$m \geq k$$

$$\sqrt[m]{\frac{a_1^m + a_2^m + \cdots a_n^m}{n}} \geq \sqrt[k]{\frac{a_1^k + a_2^k + \cdots a_n^k}{n}}$$

Guided Discussion: Power Mean Inequalities

We can easily see the Arithmetic mean inequality is a special case where our power variable p equals 1, then our Quadratic mean is a case where $p = 2$, and our Harmonic Mean is a case where $p = -1$.

We generalize the Power Mean Inequality Below:

$$\sqrt[m]{\frac{a_1^m + a_2^m + \cdots a_n^m}{n}} \geq \sqrt[k]{\frac{a_1^k + a_2^k + \cdots a_n^k}{n}}$$

Now we have that

$$\cdots \sqrt[2]{\frac{a_1^2 + a_2^2 + \cdots a_n^2}{n}} \geq \frac{a_1 + a_2 + \cdots a_n}{n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots \frac{1}{a_n}} \cdots$$

Furthermore, that

$$\cdots \frac{a_1 + a_2 + \cdots a_n}{n} \geq \sqrt[n]{\frac{a_1 a_2 \cdots a_n}{n}} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots \frac{1}{a_n}} \cdots$$

With our Geometric Mean substitution for $p = 0$

Guided Discussion: Holder's Inequality

Holder's inequality essentially states that for

$$\lambda_a + \lambda_b + \cdots \lambda_z = 1$$

$$a_i \geq 0, b_i \geq 0, \cdots z_i \geq 0$$

Generally, Holder's Inequality will not appear in our level of comp math. However, Cauchy Schwartz, which is a specific case of the Holder's Inequality, definitely will.

$$(a_1 + a_2 + \cdots a_n)^{\lambda_a}(b_1 + b_2 + \cdots b_n)^{\lambda_b} \cdots (z_1 + z_2 + \cdots z_n)^{\lambda_z} \geq a_1^{\lambda_a} b_1^{\lambda_b} \cdots z_1^{\lambda_z} + \cdots + a_n^{\lambda_a} b_n^{\lambda_b} \cdots z_n^{\lambda_z}$$

I really want you guys to just get an understanding of the structure for this inequality, and see how Cauchy Schwartz is a special case of this.

This is a lot! But bear with me for a second.

Guided Discussion: Holder's Inequality

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Note, that if there are only two lists, and $\lambda_1 = \lambda_2 = {}^1/_2$, the we have C-S

$$(a_1 + \cdots + a_n)^{1/2}(b_1 + \cdots + b_n)^{1/2} \geq a_1^{1/2} b_1^{1/2} + \cdots + a_1^{1/2} b_1^{1/2}$$

Guided Discussion: Holder's Inequality

Holder's inequality essentially states that for

$$\lambda_a + \lambda_b + \cdots \lambda_z = 1$$
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$$(a_1 + \cdots + a_n)^{1/2}(b_1 + \cdots + b_n)^{1/2} \geq a_1^{1/2} b_1^{1/2} + \cdots + a_n^{1/2} b_n^{1/2}$$

Although we are not used to seeing it written this way, this is an equivalent statement to the C-S we are used to:

$$\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2$$

Guided Discussion:
Cauchy Schwartz

This leads us to a smooth transition into Cauchy Schwartz, which has a variety of applications and derivations.

$$\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2$$

Real vector equivalent:

$$(\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v}) \geq (\vec{u} \cdot \vec{v})^2$$

Complex Vector equivalent:

$$\langle u|u\rangle\langle v|v\rangle \geq \langle u|v\rangle^2$$

Guided Discussion:
Cauchy Schwartz

$$\left(\frac{x_1^2}{y_1}+\cdots+\frac{x_n^2}{y_n}\right)(y_1+\cdots+y_n)\geq(x_1+\cdots+x_n)^2$$

Titu's Lemma is simply a certain arrangement of sets for a Cauchy Schwartz Inequality. We have

$$a_i=\frac{x_i}{\sqrt{y_i}},b_i=\sqrt{y_i}$$

And with C-S we have the inequality to the right

Which yields

$$\left(\frac{x_1^2}{y_1}+\cdots+\frac{x_n^2}{y_n}\right)\geq\frac{(x_1+\cdots+x_n)^2}{y_1+\cdots+y_n}$$

Problems:

The positive real numbers w, x, y and z satisfy $w + 2x + 3y + 4z = 5$. What is the minimum possible value of $w^2 + \frac{1}{2}x^2 + \frac{1}{3}y^2 + \frac{1}{4}z^2$?

For positive real numbers a, b, c, d find the minimum value of the expression

$$(a+b+c+d)\left(\frac{25}{a} + \frac{36}{b} + \frac{81}{c} + \frac{144}{d}\right)$$

Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$. Show that

$$\frac{a_1^2}{a_1+b_1} + \frac{a_2^2}{a_2+b_2} + \cdots + \frac{a_n^2}{a_n+b_n} \geq \frac{a_1 + a_2 + \cdots + a_n}{2}$$

Show that for

$$f(x) = \frac{(x+k)^2}{x^2+1}$$

that

$$f(x) \leq k^2 + 1$$

Show that for any two positive real numbers a and b,

$$\frac{a+b}{2} - \sqrt{ab} \geq \sqrt{\frac{a^2+b^2}{2}} - \frac{a+b}{2}$$

by showing that this inequality is equivalent to

$$\frac{(a+b)^2}{2} \leq \sqrt{(2ab)(a^2+b^2)}$$

and then using the AM-GM Inequality.

Let $r_1, r_2, r_3, \ldots, r_n$ be n real numbers each greater than zero. Prove that for any real number $x > 0$,

$$(x+r_1)(x+r_2)\cdots(x+r_n) \leq \left(x + \frac{r_1 + r_2 + r_3 + \cdots + r_n}{n}\right)^n$$

What is the smallest possible value of the expression

$$\frac{x^2+361}{x}$$

Prove that for positive reals a, b, c summing up to 1, we have

$$\frac{1}{a+b} + \frac{16}{c} + \frac{81}{a+b+c} \geq 98$$

Problems:

The positive real numbers w, x, y and z satisfy $w + 2x + 3y + 4z = 5$. What is the minimum possible value of $w^2 + \frac{1}{2}x^2 + \frac{1}{3}y^2 + \frac{1}{4}z^2$?

C-S

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C-S

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AMgeqGM

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AMgeqGM

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AMgeqGM

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Titu's