Linear Algebra

Walter Johnson Maths Honors Society

Requirements

In order to receive credit for this independent research project and be in good standing with the Walter Johnson Maths Honor Society, you must write a paper which does each of the following:

- Describe, briefly, what Linear Algebra is.
- Describe a vector space and provide an example.
- Define and describe the four fundamental vector spaces of a matrix, and the Rank-Nullity Theorem.
- Define and describe what Eigenvalues and Eigenvectors are.
- Describe what the diagonalization of a matrix is.
- Define and describe $A = PDP^{-1}$ and $A^n = PD^nP^{-1}$.
- Complete and describe solutions to both problems.

On average, this assignment will take about 3 hours to research and write up. You may **not** work in a group or collaborate with others.

You will be assigned to groups of 6 people, each of which has completed a different independent research project. At the end of the year, you will present your findings to your group, and listen as your peers present their findings. Your presentation must briefly discuss every subject required in your paper along with 1 of the problems you solved, of your choice.

Resources

You are provided with various resources to complete your research. You are welcome to use resources that are not given here.

Vectors and Matrices

An n-dimensional **vector** is often denoted as such:

$$\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$$

The **dot-product** of two vectors is a scalar value:

$$\vec{v} \cdot \vec{u} = v_1u_1 + v_2u_2 + \cdots + v_nu_n$$

A **matrix** is often represented as such:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where M is a 2×2 matrix.

Identity Matrix

The **identity matrix** is a diagonal matrix with only 1 entries along its diagonal:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is an example of the 3×3 identity matrix. Multiplying this matrix by any other matrix results in the initial matrix, hence, I is the solution to the equation

$$MI = M$$

for another matrix M.

Vector Spaces

- Khan Academy video about vector spaces and linear combinations of vectors.
- Article by University of Toronto professor on vector spaces.

Unlike topological spaces, vector spaces are always embedded in at most a 1 higher dimensional Euclidean space than the dimension of the vector space itself. Take a vector space V of dimension 5. We know that V can be directly embedded in $\mathbb{R}^6$

$$V \in \mathbb{R}^6$$

Four Fundamental Vector Spaces

- Brilliant wiki page on the four fundamental subspaces.
- MIT article on the four fundamental subspaces by the legendary Gilbert Strang.
- Rank-Nullity Theorem textbook excerpt from Purdue University.
- Brilliant wiki page on the Rank Nullity Theorem.

Eigenvalues and Eigenvectors

- Swarthmore site describing a derivation of eigenvalues and eigenvectors.
- Visually-aided explanation to Eigenvalues and Eigenvectors from Dan Margalit and Joseph Rabinoff at Georgia Tech.

Matrix Diagonalization

- Brilliant wiki page on Matrix Diagonalization.
- An example from Oklahoma State University on finding the diagonal matrix from a given matrix.
- In-depth walkthrough of matrix diagonalization.

Problem 1

Google became famous for their high-speed internet searching algorithm, PageRank. This algorithm took an n by n matrix. n pages were chosen because these n pages were found to contain a keyword as described by the user. Each entry (i, j) of the n by n matrix contains the integer number of references that page i references page j. To order these webpages based on which pages are the most nested-linked among all of the other n pages, this matrix needs to be exponentiated a few thousand times. How is Google able to do this so fast?

Problem 2

Given A is a matrix and λ is an eigenvalue of A, prove that the determinant of $\lambda I - A$ is 0:

$$\det(\lambda I - A) = 0$$

Hint: If the determinant of a matrix is zero, it is not invertible.

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