Abstract Algebra

Walter Johnson Maths Honors Society

Requirements

In order to receive credit for this independent research project and be in good standing with the Walter Johnson Maths Honor Society, you must write a paper which does each of the following:

- Define, briefly, what Abstract Algebra is.
- Define a group. State and explain the four group axioms.
- Define, describe, and give examples of Cyclic Groups and Symmetric Groups.
- Define and describe the Quaternion Group and it's relation to complex numbers.
- Define Homomorphism.
- Explain Lagrange's Theorem and give an example of it by exhausting all of the possible subgroups of another group.
- Complete and describe solutions to both problems.

On average, this assignment will take about 3 hours to research and write up. You may **not** work in a group or collaborate with others.

You will be assigned to groups of 6 people, each of which has completed a different independent research project. At the end of the year, you will present your findings to your group, and listen as your peers present their findings. Your presentation must briefly discuss every subject required in your paper along with 1 of the problems you solved, of your choice.

Resources and Definitions

You are provided with various resources to complete your research. You are welcome to use resources that are not given here.

The sentences describing the article are hyperlinks to the article, you can click on them to access the article. We also provide you with some brief definitions in order to get you started on your research.

Functions and Operations

A **function** is something which **maps** a set of numbers or elements, S_1, to another set of numbers, S_2. An example of this is the $\tan^{-1} x$ function, which maps the reals, $\mathbb{R}$ to the set $(\frac{-\pi}{2}, \frac{\pi}{2})$. We notate this as

$$f(x) = \tan^{-1} x$$

and

$$\mathbb{R} \xrightarrow{f(x)} \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

A **cartesian product** of two sets, notated $S_1 \times S_2$, is simply the set with all elements which are ordered pairs of the two elements of the first two. Let

$$S_1 = \{2, 3, 4\} \text{ and } S_2 = \{4, 5\}$$

then

$$S_3 = S_1 \times S_2 = \{(2, 4), (2, 5), (3, 4), (3, 5), (4, 4), (4, 5)\}$$

where the ordered pair $(2, 5)$ exists in the set $S_1 \times S_2$.

Group Axioms

- MIT Intro to Group Theory using Permutation Puzzles. Start at page 3 where a Group is defined.
- Brilliant wiki page on Group Theory. Start at the definition of a Group.

Cyclic Groups

- Purdue Mathematics chapter on Cyclic Groups. Defined and explained using both multiplication and addition as a group operator.
- Lecture on Cyclic and Abelian Groups by Mathew Macauley at Clemson University. Includes a Geometric interpretation of the group structure.

Symmetric Groups

- University of Michigan article on the Symmetric Group by Karen E. Smith. Intricate article but well put.
- Brilliant wiki page on Symmetric Groups.

The Quaternion Group

- Article on the Quaternion Group by Stefan Porubsky at the Institute of Computer Science of the Academy of Sciences of the Czech Republic.
- Mathematics Stack Exchange question on Quaternions and Complex Numbers.

Subgroups

- Brilliant wiki page on Subgroups.

Homomorphism

- Britannica article on group homomorphisms.
- Brilliant wiki page on group homomorphisms. Start at definition.

An **Isomorphism** is simply when a homomorphism creates a bijection between the sets of the two groups.

Lagrange's Theorem

- Stanford cryptography notes on Lagrange's Theorem.
- Brilliant wiki page on Lagrange's Theorem. Start at the preliminary.

Problem 1

Prove there is an isomorphism between Z_6 and $Z_2 \times Z_3$, given that the cross product of two groups $G_1 \times G_2$ is a piece-wise mapping of the operations of ordered pairs of elements from each set.

Problem 2

Prove Fermat's Little Theorem:

$$1 \equiv a^{p-1} \pmod{p}$$

using the cyclic group on p with operation multiplication.

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