Combinatorial Games

June 15, 2021

Games

Based on the title slide, I predict some of you are thinking this is going to be about Game Theory, or something of the sort. But interestingly enough, there are so many different kinds of games out there, the kinds of games we'll be looking at today are not considered to be a part of Game Theory. They are instead of Combinatorial Game Theory, which, given the vastly different nature of the games studied, constitute a different field.

Games

Game Theorists might look at games such as Chess or Go. These games are very complex, and computationally considered Hard, for the most part (Chess is P-complete but Generalized Chess is EXP-Hard. Go is NP-Hard at least).
But there are some other games, particularly what are called FISP games, which fall into another interesting category of analysis, Combinatorial Game Theory. This is what we will examine today.

FISP Games

FISP stands for

- Finite: The game will only last so long. It will end and one player will win.
- Impartial: Both players have access to the same set of moves.
- Standard Play: This means that the winner and loser are determined by who runs out of moves first. The first player who cannot make a valid move is the loser.

Players also alternate turns in a FISP game, where legal moves are specified by rules.

FISP Games

FISP stands for

- Finite: The game will only last so long. It will end and one player will win.
- Impartial: Both players have access to the same set of moves.
- Standard Play: This means that the winner and loser are determined by who runs out of moves first. The first player who cannot make a valid move is the loser.

Players also alternate turns in a FISP game, where legal moves are specified by rules.
Why isn't Chess a FISP game?

NIM

The most important game in Combinatorial Game Theory is called NIM. In NIM, two players alternate taking any number of stones off of a pile until there are no stones left on the pile.

NIM

The most important game in Combinatorial Game Theory is called NIM. In NIM, two players alternate taking any number of stones off of a pile until there are no stones left on the pile. An example graph of a NIM game which starts with 3 stones is shown below:

$$
\begin{tikzcd}
 & & & 3* \arrow[dll] \arrow[ddlll] \arrow[dddll] \\
 & 2* \arrow[dl] \arrow[ddr] & & \\
1* \arrow[drr] & & & \\
 & & 0* &
\end{tikzcd}
$$

The asterisk next to the number means that number is a pile size.

NIM

We'll look at one of the most common ways to play NIM, which is with 3-piles (3-pile NIM). We will also play this game *Normally*, which means the player to take the last stone wins (this is opposed to playing it *Misere*, where the player to take the last stone loses).

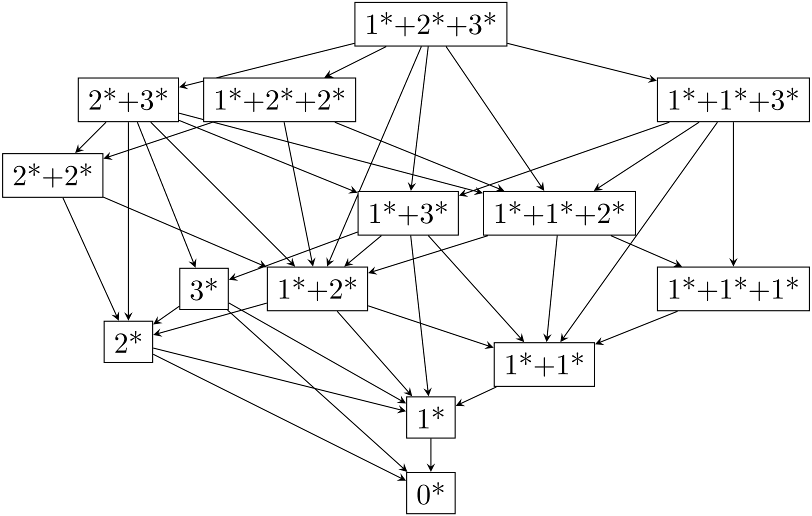
The way NIM works is as such: Two players alternate taking any number of stones off a pile until there are no stones left. In 3-pile NIM, we start with 3 piles of stones.

NIM

Let's play an example game with piles of 1, 2 and 3 stones.

NIM

One technique we have to analyze these games is via graphs. Let's look at the graph for the NIM game starting with three piles of sizes 1,2 and 3.



N- and P-positions

An interesting thing about FISP games is that we can classify every position as one of two kinds:

- N-positions: has status "N". The player whose turn it is to move (the Next player) at that position has a winning strategy. If they play correctly, they will win regardless of what the opponent does.
- P-positions: has status "P". The player whose turn just ended (the Previous player) has a winning strategy.

N-positions are ones where you want to find yourself.
P-positions are those where you want to get your opponent to.

N- and P-positions

Now that we have some vocabulary to describe the FISP games, we want to figure out how to win them.
The first thing we focus on is the fact that *terminal positions* are always P positions. What this means is that if a player is at the terminal position, they have lost, and the previous player has won.

N- and P-positions

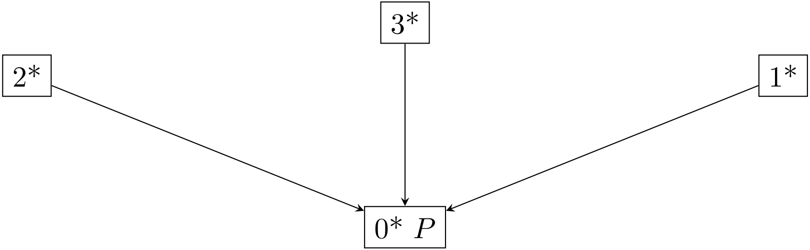
The first thing we focus on is the fact that *terminal positions* are always P positions. What this means is that if a player is at the terminal position, they have lost, and the previous player has won.
Now what we do is work our way backwards from the terminal position to generate the N- and P-status of the other positions. What makes an N position a winning position?

N- and P-positions

Now what we do is work our way backwards from the terminal position to generate the N- and P-status of the other positions. A position is only an N position (a winning position) if you can move to a losing position from it (at least one). This means that the player at the N position can move to a position which is P, forcing the other player to be in a losing position.

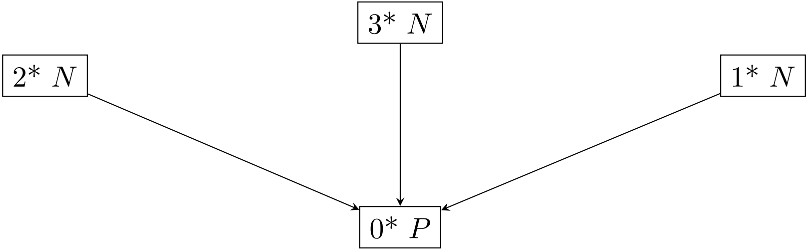
N- and P-positions

A position is only an N position (a winning position) if you can move to a losing position from it. We see that all of the 1-pile positions can move to a P position, so they are all winning positions.



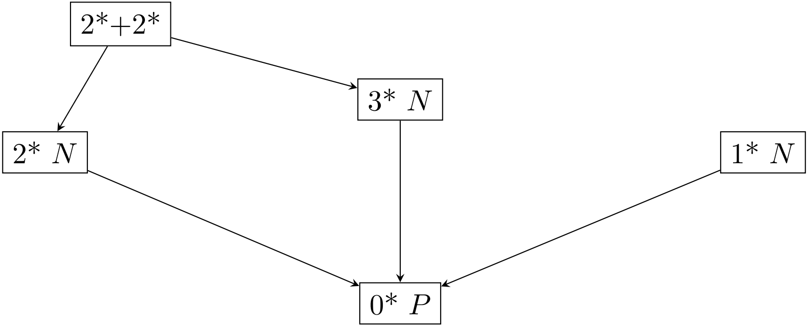
N- and P-positions

A position is only an N position (a winning position) if you can move to a losing position from it. We see that all of the 1-pile positions can move to a P position, so they are all winning positions.



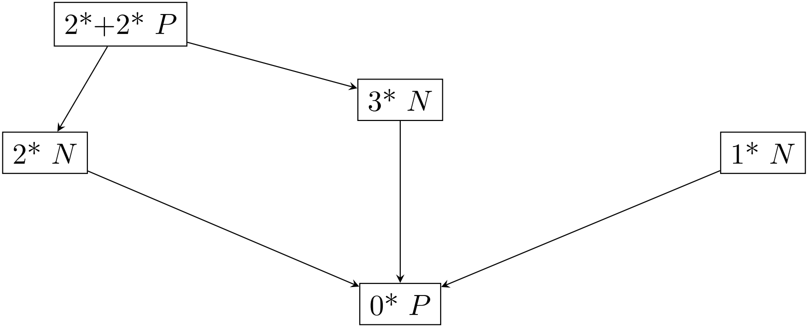
N- and P-positions

Now let's look at some of the previous positions. We see that 2*+2* can only reach N positions, which means wherever a player moves to from 2*+2* will be a winning position for the player whose turn it is then.



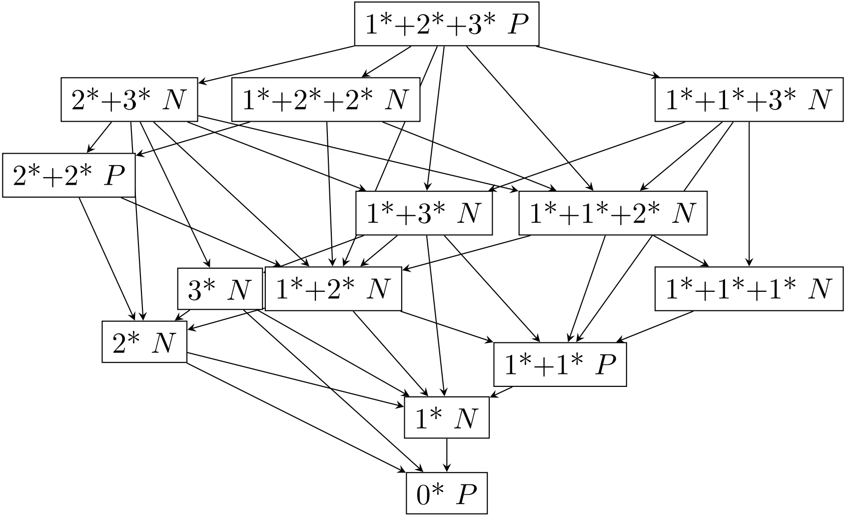
N- and P-positions

Now let's look at some of the previous positions. We see that 2*+2* can only reach N positions, which means wherever a player moves to from 2*+2* will be a winning position for the player whose turn it is then. This means 2*+2* is a losing position. Wherever you go from there, the other player can win.



N- and P-positions

Let's label the positions in our graph from earlier.



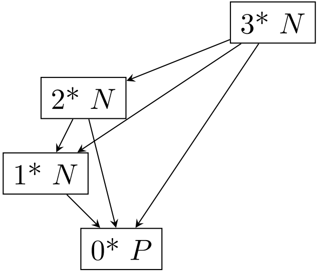
N- and P-positions

We have determined a few things about N and P positions now:

- An N position is a position that *can* move to a losing position. At least 1 is a P position.
- A P position is a position that *cannot* move to a losing position. Every position it can move to is an N position.

1-pile NIM

Let's backtrack a bit. Consider a game of 1-pile NIM. Let's start with 3 stones:



Because any non-terminal position can reach the terminal position, any non-terminal position is a winning position. This makes 1-pile NIM very boring.

1-pile NIM

In essense, 1-pile NIM is a game defined as such:

- $X*$ is an N position if $X > 0$. The next player can simply win by removing all of the stones from the heap.
- $X*$ is a P position if $X = 0$.

This is a very simple game, but when you play with multiple NIM heaps simultaneously, interesting results occur.

Addition of Games

So we now can consider what it really means to have 2-pile NIM. What we are really doing is combining two 1-pile NIM games!
In order to use our information regarding 1-pile NIM to gain information regarding 2-pile NIM, we consider a variety of arbitrary positions in different arbitrary FISP games.

Addition of Games

If we define a game through it's positions and possible moves, we can combine multiple NIM games with the following rules, given two games G_1 and G_2, and the game composed by the two $G_1 + G_2$:

- A position of $G_1 + G_2$ consists of position g_1 from G_1 and g_2 from G_2. This new position is considered $g_1 + g_2$.
- A valid move from $g_1 + g_2$ is *either* a valid move in G_1 from g_1 to some other position g_1' or a valid move in G_2 from g_2 to some other position g_2'
- A player loses the game $G_1 + G_2$ if it is their turn and they are unable to move. They have no valid moves in either G_1 or G_2

Addition of Games

Regarding the N/P status of positions in a game, there are a few important lemmas we will show:

- For any position g in any FISP game, $g+g$ is always a P-position.
- If g is a P-position in a FISP game G, then for any other position h in a FISP game H, $h+g$ has the same N/P status as h.

Addition of Games

Regarding the N/P status of positions in a game, there are a few important lemmas we will show:

- For any position g in any FISP game, $g + g$ is always a P-position.
- If g is a P-position in a FISP game G, then for any other position h in a FISP game H, $h + g$ has the same N/P status as h.

Notice that the composition of two states in two games has an N/P status, just like a single game. This means composing greater numbers of games is equivalent to composing two games, and then composing that new game with another game. In general, these positions are associative.

Addition of Games

Let's try to show that $g + g$ is always a P position in the game $G + G$. The intuition behind this is simple; let's say the player moving from $g + g$, player 2, moves to $g' + g$. Then player 1 can move to $g' + g'$, which is either two terminal positions, or two identical positions. Because FISP games are finite, player 1 can just repeatedly immitate player 2's moves, thus securing a win.

Addition of Games

Now let's try to show that if g is a P-position, then for any other position h, $h+g$ has the same status as h.
Well first let's suppose h is a P-position. Then the Previous player, the player whose turn it is *not*, has a winning strategy in either game H or G. This means that wherever the current player moves to from $h+g$, the previous player can continue their cooresponding winning strategy in that game. Thus, for any move made, the previous player will always have a winning move. So if h has status P, $h+g$ is P.
Now suppose that h is an N position. By the definition of a N position, the next player can move to a P position $h'+g$. This position is a P position because h' and g are P, and thus $h+g$ is N because it can move to a P position.

Addition of Games

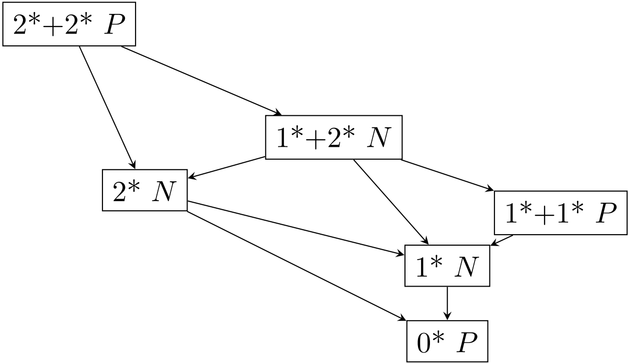
So now, we know what the N/P status of $h + g$ is if one is N and the other is P, or if they are both P. But what about when they are both N?
The answer is, it could be either!

Addition of Games

If they are both N, the composition of the two positions could be either P or N. Consider our previous diagram. 2*+2* is P, but it is the composition of two N positions. 1*+2* is an N position, but it is also the composition of two N positions.

Addition of Games

Let's look at this in the context of our previous game. We see 2^*+2^* is P and 1^*+2^* is N, although 1^* and 2^* individually are both N.



What Makes 3-pile NIM Special?

Theorem. For any two numbers X and Y, there exists at most one number Z such that $X^*+Y^*+Z^*$ is a P position.

This is one of two very important theorems we will cover today.

Proving Z Is Unique

Theorem. For any two numbers X and Y, there exists at most one number Z such that $X^*+Y^*+Z^*$ is a P position.

Notice that to prove this, we don't even have to prove the existence of Z. In other words, Z is not guaranteed to exist. We only aim to prove that there is *atmost* one Z such that $X^*+Y^*+Z^*$ is a P position.

Proving Z Is Unique

Theorem. For any two numbers X and Y, there exists at most one number Z such that $X^*+Y^*+Z^*$ is a P position.

We prove by contradiction. Suppose $X^*+Y^*+Z_1^*$ and $X^*+Y^*+Z_2^*$ are both P positions such that $Z_1 > Z_2$. One can easily move from the former to the latter by subtracting $Z_1 - Z_2$ stones from pile Z_1. But this means one can move from a P position to a P position, which is a contradiction. The first position would have had to be an N position. Thus we have a contradiction.

Winning 3-pile NIM

Now, we can construct a table whose entry in row X and column Y is the unique value of Z such that $X^* + Y^* + Z^*$ is a P-position. That way, whatever assortment of piles we find ourselves faced with, we know where to move if we look two of our three piles' values up in the table.

Winning 3-pile NIM

Now, we can construct a table whose entry in row X and column Y is the unique value of Z such that $X^* + Y^* + Z^*$ is a P-position:

	0	1	2	3	4
0	0	1	2	3	4
1	1	0	3	2	
2	2	3	0	1	
3	3	2	1	0	
4	4				0

MEX Rule

Now how are we going to figure this table out implicitly?

	0	1	2	3	4
0	0	1	2	3	4
1	1	0	3	2	
2	2	3	0	1	
3	3	2	1	0	
4	4				0

I'll just tell you. Interestingly enough, the rule we use to determine these values is the MEX rule, or Minimum Excluded value. The number in row X and column Y is actually going to be the smallest number that is *excluded* from the same row and column as X and Y. Implicitly:

$$[X, Y] = \text{MEX}(\{[X', Y] : X' < X\} \cup \{[X, Y'] : Y' < Y\})$$

MEX Rule

Interestingly enough, the rule we use to determine these values is the MEX rule, or Minimum Excluded value. The number in row X and column Y is actually going to be the smallest number that is *excluded* from the same row and column as X and Y. Implicitly:

$$[X, Y] = \mathrm{MEX}(\{[X', Y] : X' < X\} \cup \{[X, Y'] : Y' < Y\})$$

Let's take a moment to figure out what this means mathematically, and contextually.

MEX Rule

Interestingly enough, the rule we use to determine these values is the MEX rule, or Minimum Excluded value. The number in row X and column Y is actually going to be the smallest number that is *excluded* from the same row and column as X and Y. Implicitly:

$$[X,Y] = \mathrm{MEX}(\{[X',Y] : X' < X\} \cup \{[X,Y'] : Y' < Y\})$$

Let's prove that this is really the value to make $X^* + Y^* + [X,Y]^*$ a P-position.

Proof of MEX Rule

$$M = \text{MEX}(\{[X', Y] : X' < X\} \cup \{[X, Y'] : Y' < Y\})$$

Let M be the value that makes $X^* + Y^* + M^*$ a P-position. Consider the three ways the Next player can move from this position:

- Suppose the player moves to $X_1^* + Y^* + M^*$. By the definition of MEX, $M \neq [X_1, Y]$, thus $X_1^* + Y^* + M^*$ is not a P position, and is an N position.

Proof of MEX Rule

$$M = \text{MEX}(\{[X', Y] : X' < X\} \cup \{[X, Y'] : Y' < Y\})$$

Let M be the value that makes $X^* + Y^* + M^*$ a P-position. Consider the three ways the Next player can move from this position:

- Suppose the player moves to $X_1^* + Y^* + M^*$. By the definition of MEX, $M \neq [X_1, Y]$, thus $X_1^* + Y^* + M^*$ is not a P position, and is an N position.
- Suppose the player moves to $X^* + Y_1^* + M^*$. By the definition of MEX, $M \neq [X, Y_1]$, thus $X^* + Y_1^* + M^*$ is not a P position, and is an N position.

Proof of MEX Rule

$$M = \mathrm{MEX}(\{[X', Y] : X' < X\} \cup \{[X, Y'] : Y' < Y\})$$

Let M be the value that makes $X^* + Y^* + M^*$ a P-position.

- Finally, the player can move to $X^* + Y^* + M_1^*$ with $M_1 < M$. By the definition of MEX, M_1 must be an element of

$$M_1 \in \{[X', Y] : X' < X\} \cup \{[X, Y'] : Y' < Y\}$$

Proof of MEX Rule

$$M = \mathrm{MEX}(\{[X', Y] : X' < X\} \cup \{[X, Y'] : Y' < Y\})$$

Let M be the value that makes $X^* + Y^* + M^*$ a P-position.

- Finally, the player can move to $X^* + Y^* + M_1^*$ with $M_1 < M$. By the definition of MEX, M_1 must be an element of

$$M_1 \in \{[X', Y] : X' < X\} \cup \{[X, Y'] : Y' < Y\}$$

Since we can suppose $M_1 = [X_1, Y]$ with $X_1 < X$, we have that $X_1^* + Y^* + M_1^*$ is a P-position, which means that $X_1^* + Y^* + M^*$ must be an N-position, because it can move to the lower P position.

Nimsum

We have shown that $[X, Y]$ is indeed the MEX rule of the set of moves that one can move to. This gives us the full table below:

	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	0	3	2	5	4	7	6	9
2	2	3	0	1	6	7	4	5	10
3	3	2	1	0	7	6	5	4	11
4	4	5	6	7	0	1	2	3	12
5	5	4	7	6	1	0	3	2	13
6	6	7	4	5	2	3	0	1	14
7	7	6	5	4	3	2	1	0	15
8	8	9	10	11	12	13	14	15	0

Nimsum

We have shown that $[X, Y]$ is indeed the MEX rule of the set of moves that one can move to. This gives us the full table below:

	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	0	3	2	5	4	7	6	9
2	2	3	0	1	6	7	4	5	10
3	3	2	1	0	7	6	5	4	11
4	4	5	6	7	0	1	2	3	12
5	5	4	7	6	1	0	3	2	13
6	6	7	4	5	2	3	0	1	14
7	7	6	5	4	3	2	1	0	15
8	8	9	10	11	12	13	14	15	0

This table tells us everything we need to know in order to win a game of 3-pile NIM. Now, suppose we want to compute $[2019, 2020]$. That would take a long time with the formulas we have now. Is there an explicit formula for this table?

Nimsum

	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	0	3	2	5	4	7	6	9
2	2	3	0	1	6	7	4	5	10
3	3	2	1	0	7	6	5	4	11
4	4	5	6	7	0	1	2	3	12
5	5	4	7	6	1	0	3	2	13
6	6	7	4	5	2	3	0	1	14
7	7	6	5	4	3	2	1	0	15
8	8	9	10	11	12	13	14	15	0

Turns out, there is an explicit form for this table, known as the Nimsum.

Nimsum

The Nimsum is essentially a base-2 addition *without carry*. This means we do not carry the “1” when we add. Take the Nimsum of 5 and 7 base-2, which is denoted $5 \oplus 7$:

$$\begin{array}{cccc} & 1 & 0 & 1 \\ \oplus & 1 & 1 & 1 \\ \hline & & 1 & 0 \end{array}$$

This is equivalent to the XOr operation programatically. Turns out, this formula fills our whole table!