Combinatorics Homework

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1 Review

1.1 Combinatorics and Choosing

Combinatorics is the study of counting. Combinatorics is traditionally a harder subject in competition mathematics for a variety of reasons, but primarily that there is some casework and case-specific approaches to problems. This means the techniques used to handle some of these problems need to be understood well enough at their core to be applied in a wider variety of scenarios. There are two main combinatorial approaches we will look at, first being Stars and Bars, a technique used to create bijections between combinatorial problems and sets, and Pascal's Triangle. These are both intricate ideas and are hard to apply rigorously.
The first thing we should discuss, which is the basis of everything we will do in combinatorics, is the Choose function:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

This function counts the amount of ways to choose k objects from a set of n indistinguishable objects. Try this with some small values n to become comfortable with it, counting manually and comparing your answer.
We will not discuss the real and complex analysis behind this function, but there is quite a lot interesting about this function and it's non-integer and non-real counterparts.

1.2 Pascal's Triangle

As you should be familiar with, Pascal's Triangle is a triange starting with a 1, with each row defined as the sum of the two elements proceeding it.

$$
\begin{array}{lccccccccccccc}
n=0 & & & & & & & 1 & & & & & & \\
n=1 & & & & & & 1 & & 1 & & & & & \\
n=2 & & & & & 1 & & 2 & & 1 & & & & \\
n=3 & & & & 1 & & 3 & & 3 & & 1 & & & \\
n=4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
n=5 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
n=6 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

This triangle ties together almost all of the concepts relevant to combinatorics, and is the focus of incredible findings in combinatorics, graph theory, and number theory.

The primary finding in this triangle is that each number in the triangle is equivalent to the choose function with an input of the row and the element number. Consider the top 1, this is equal to 0 choose 0. Consider the first ($n = 1$) row, the first number. This is equal to 1 choose 0. The second number in this row is equal to 1 choose 1. The second row has 2 choose 0, 2 choose 1, and 2 choose 2 as it's entries, and so forth. This gives us:

$$
\begin{array}{lccccccccccccc}
n=0 & & & & & & & \binom{0}{0} & & & & & & \\
n=1 & & & & & & \binom{1}{0} & & \binom{1}{0} & & & & & \\
n=2 & & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
n=3 & & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
n=4 & & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
n=5 & & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
n=6 & \binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}
$$

Try this with some small values to convince yourself of this.

Another finding to note, if you recall from our Polynomials and Algebra unit, each row in this table is equivalent to the coefficients for the binomial expansion of $(a + b)^n$. Consider the value $2^n = (1 + 1)^n$. We see that this gives us simply the sum

$$\binom{n}{0} \times 1 + \binom{n}{1} \times 1 + \cdots + \binom{n}{n} \times 1 = 2^n$$

This shows that the elements in each row of Pascal's Triangle sum to 2^n.

1.3 Combinatorial Identities

Given our algebraic definition for the choose function, as well as Pascal's triangle, we have a variety of identities derived. The first being the symmetry

$$\binom{n}{k} = \binom{n}{n-k}$$

This can be shown algebraically. Then, Pascal's Identity:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

We see that this is simply rewriting the fundamental construction of Pascal's triangle in terms of the choose function. Repeated application of Pascal's identity gives us the Hockeystick Identity:

$$\sum_{n=k}^{n=j} \binom{n}{k} = \binom{j+1}{k+1}$$

Named as such because of it's resemblance to a hockystick on Pascal's Triangle. This can be reflected on Pascal's Triangle given our symmetric identity to obtain

$$\sum_{n=k}^{n=j} \binom{n}{n-k} = \binom{j+1}{j-k}$$

This identity is extremely powerful, and has fascinating double-applications (hint-hint, 3.4). An example of this is:

$$\binom{4}{0} + \binom{5}{1} + \binom{6}{2} + \binom{7}{3} = \binom{8}{3}$$

All of these techniques are fair game on tougher combinatorical problems, and being able to identify the applications of certain identities will come with time.

1.4 Stars and Bars

Stars and Bars is a specific application of using Bijections to solve problems. Making Bijections are a brilliant way to go about combinatorical problems,

because counting different arrangements of sets is significantly easier than case-work solutions to combinatorical problems.
The claim made in Stars and Bars is that there is a bijection between the amount of ways to have n objects in k groups and a set with n stars and $k-1$ bars. This is not as intuitive as you think it is: what this bijection means is that for every such set of stars and bars, it uniquely cooresponds to a combination of n objects into k groups. Because there is a bijection, counting all the possible ways to put n objects into k groups is equal to the amount of sets with n stars and $k-1$ bars, which is

$$\binom{n+k-1}{k-1}$$

Consider an example where you must count how many possible ways there are to have 15 dogs with (at most) three different breeds. This is equivalent to having a set with 15 stars and 2 bars, each bar denoting the beginning of a new breed. This gives us a set somewhat resembling

$$\{*****|********|******\}$$

How many possible such sets are there? Well, how many ways can you choose 2 slots of 17 slots to place a bar in?

$$\binom{2}{17}$$

How would you go about this if it was mandatory that there has to be at least one dog of each breed?

2 Introductory Problems

If counting like this is new to you, just take a step back. You will need to take some time to model how exactly to count all of the different variables, which to multiply and which to add.

2.1

How many ways are there to choose a 5-letter word from the 16-letter English alphabet with replacement, where words that are anagrams are considered the same?

2.2

In the expansion

$$(2x + \frac{k}{x})^8$$

Where k is a positive constant, the term independent of x is 700000. Find k.

2.3

A scanning code consists of a 7×7 grid of squares, with some of its squares colored black and the rest colored white. There must be at least one square of each color in this grid of 49 squares. A scanning code is called "symmetric" if its look does not change when the entire square is rotated by a multiple of 90° counterclockwise around its center, nor when it is reflected across a line joining opposite corners or a line joining midpoints of opposite sides. What is the total number of possible symmetric scanning codes?

2.4

3 red balls and 12 blue balls are randomly placed in a row. What is the probability that no two red balls are adjacent?

2.5

Suppose 5 distinct numbers are chosen from nine integers $1, 2, \ldots, 9$ to create a 5-digit number. Given that the digits chosen must consist of 3 odd numbers and 2 even numbers, how many distinct 5-digit numbers can be created?

2.6

Find the sum

$$\binom{100}{1} + 2\binom{100}{2} + 3\binom{100}{3} + \cdots + 100\binom{100}{100}$$

2.7

Suppose a 4-digit number is created from 4 distinct integers chosen among

$$\{1, 2, 3, 4, 5, 7, 9\}$$

How many such 4-digit numbers include both 1 and 2 but neither begin nor end with 1?

3 Further Application

3.1

Suppose 10 points are drawn on a plane such that exactly 4 of the points are collinear and among the remaining points, no three points are collinear. How many distinct lines can be drawn by connecting any 2 among these 10 points?

3.2

There are n different cards, where n is an integer and $n \geq 6$. The number of possible sets of 6 cards that can be chosen is 6 times the number of possible sets of 3 cards that can be chosen. Find n.

3.3

A parking lot has 16 spaces in a row. Twelve cars arrive, each of which requires one parking space, and their drivers chose spaces at random from among the available spaces. Auntie Em then arrives in her SUV, which requires 2 adjacent spaces. What is the probability that she is able to park?

3.4

Consider all 1000 element subsets of the set

$$\{1, 2, 3, \ldots, 2015\}$$

From each of these such subsets choose the least element. The arithmetic mean of all of these least elements is

$$\frac{p}{q}$$

Where p and q are relatively prime positive integers. Find $p + q$

3.5

Let S denote the set of all triples (a, b, c) of positive integers where

$$a + b + c = 15$$

Compute

$$\sum_{(a,b,c)\in S} abc$$