Guided Discussion: Group Theory

Slide Components

Problems

Walter Johnson Math Team

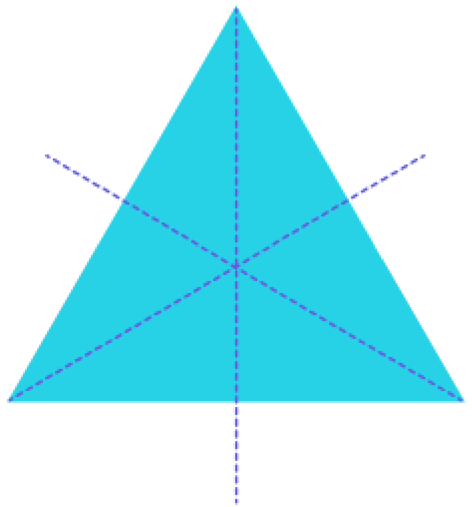
Guided Discussion: Looking at Symmetry

Yes, we're starting off simple. You know what that means! Really high acceleration take-off!

Rigid Transformation *maps a shape back to itself.*

Identity Transformation, I *returns the same value that was used in its argument*

- *Examples of Rigid Transformations*
  - *Rotating ET by 120 degrees*
  - *Reflecting ET by certain lines*
- *To start, consider regular polygons and their symmetries*



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- *How many symmetries does a square have?*

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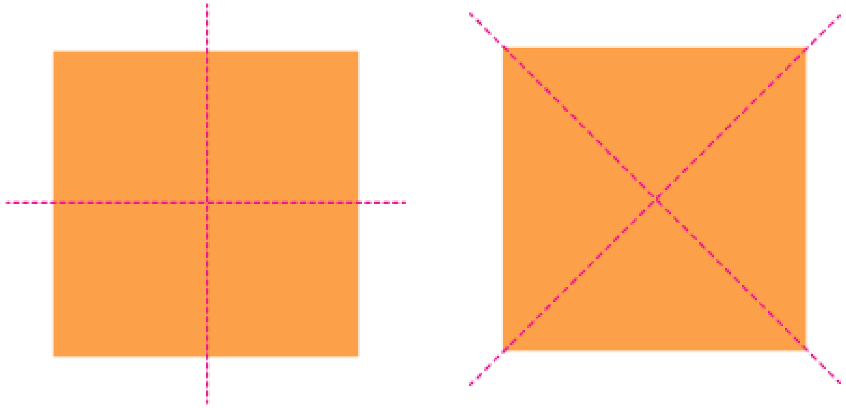
- *How many symmetries does a square have?*
  - *3 rotational symmetries*
  - *4 reflective symmetries*
  - *1 identity symmetry*
- *8 total symmetries*



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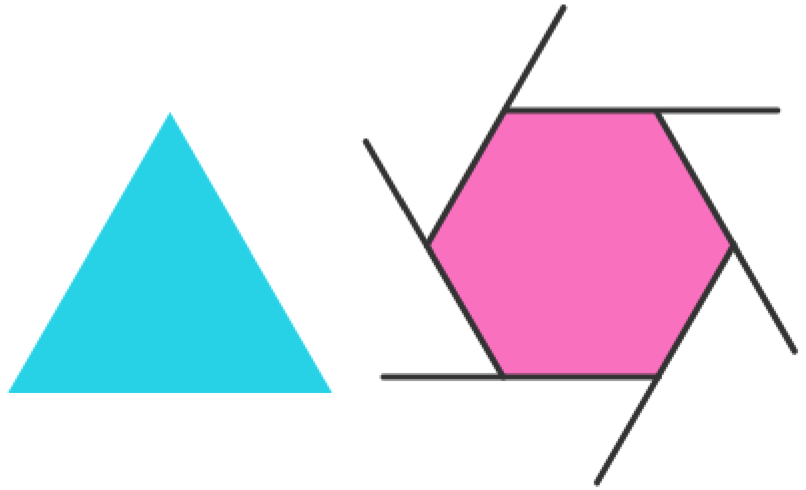
- *Which of these two has more symmetries?*



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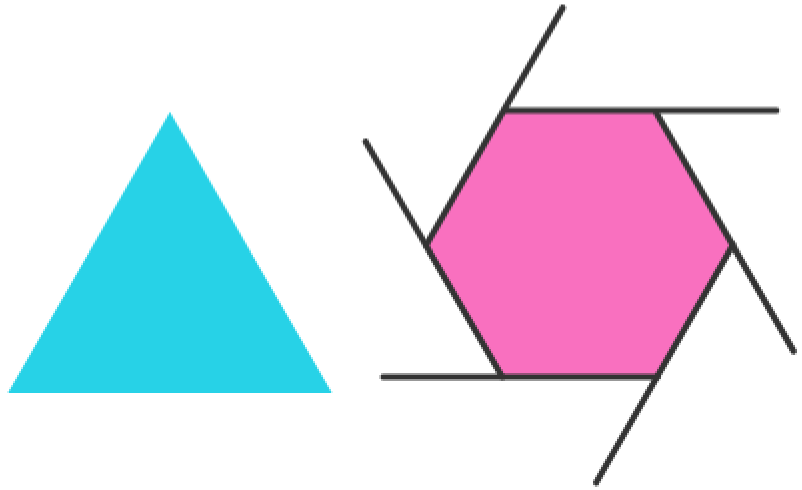

- *They have the same number of symmetries! But what is different about those symmetries?*

Guided Discussion: Looking at Symmetry

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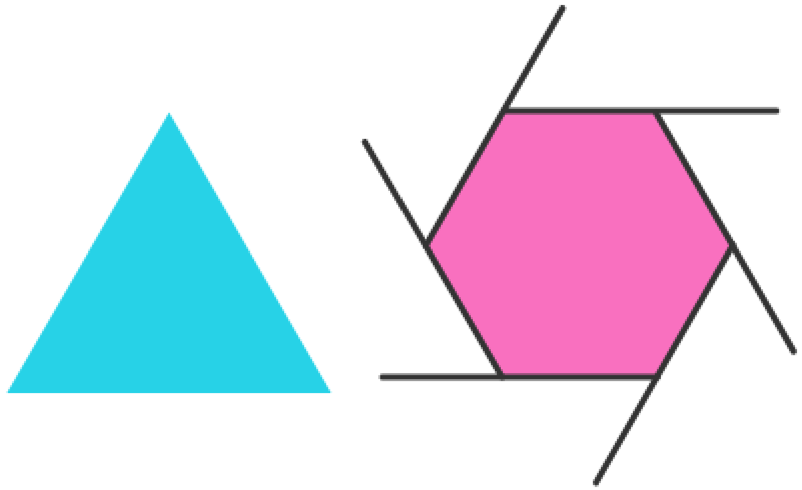
What happens when we do transformations multiple times?

*Is some multiple of a transformation **T** going to equal the identity transformation?*



- *One key difference is that the hexagon has symmetries, that, if applied less than 6 times, does not equal the identity transformation.*
- *Does the triangle have this?*

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What happens when we do transformations multiple times?

*Is some multiple of a transformation **T** going to equal the identity transformation?*

- *Group Theory is an area of algebra, which means it's a study of how combining objects can make new ones.*
- *To use new notation:*
- *Only the Hexagon had symmetry* S *such that* $I \neq S^1, I \neq S^2, \cdots, I \neq S^5$*, but* $I = S^6$
- *Did the triangle have a symmetry* S *such that* $S^6 = I$*?*

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*****Group Theory gets a little looser with it's notation than you're used to. The product sign * is commonly used to denote an operation, not just multiplication. And sometimes, it's another operator denoting it.***

If A and B are symmetries, we express the combination of the two as

$$A * B = AB$$

Which denotes doing B first, and then A

What are the symmetries for say, the letter I?

Guided Discussion: *Looking at Symmetry*

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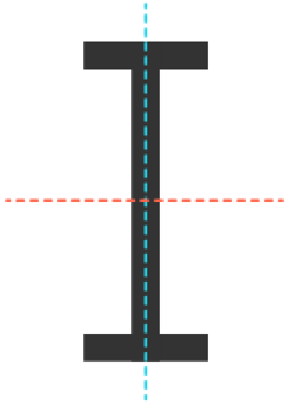

- 2 reflections, H, V
- 1 rotation by $180°$, R
- 1 identity transformation, I

Let's consider what happens when we multiply these symmetries in set T

$$T = \{H, V, R, I\}$$

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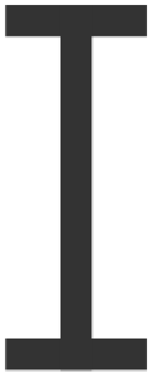
$$T = \{H, V, R, I\}$$

Let's make a basic multiplication table

	I	H	V	R
I	I			
H		I		
V			I	
R				I



Do we notice that, for any element in T, once we apply it twice, it's the equivalent of the identity transformation, I?

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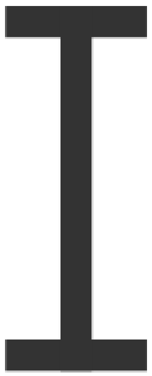
$$T = \{H, V, R, I\}$$

Let's fill it all out!

	I	H	V	R
I	I	H	V	R
H	H	I	R	V
V	V	R	I	H
R	R	V	H	I



What is this table the same as? (Not intuitive – don't try to answer this one)

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$$T = \{H, V, R, I\}$$

Let's fill it all out!

	I	**H**	**V**	**R**
I	I	H	V	R
H	H	I	R	V
V	V	R	I	H
R	R	V	H	I

	(0, 0)	**(1, 0)**	**(0, 1)**	**(1, 1)**
(0, 0)	(0, 0)	(1, 0)	(0, 1)	(1, 1)
(1, 0)	(1, 0)	(0, 0)	(1, 1)	(0, 1)
(0, 1)	(0, 1)	(1, 1)	(0, 0)	(1, 0)
(1, 1)	(1, 1)	(0, 1)	(1, 0)	(0, 0)

This is isomorphic to an addition table of ordered pairs mod 2

Guided Discussion: *Defining Groups*

*****Symmetries are an example of functions in groups. We're no longer going to call them symmetries, and now call them functions.***

Identity Function, I *returns the same value that was used in its argument. "Does nothing", equivalent of $f(x) = x$*

Group Theory *is an area of algebra, which means it's a study of how combining objects can make new ones*

Inverses *if element T is in a group, then the inverse of T and T are equal to I*

What happens when you apply the identity symmetry I by any other symmetry?

$$S * I = S$$
$$I * S = S$$

Also, suppose there is a symmetry S of a given shape. There is some way to undo this transformation, right?

We call this the inverse, so that

$$T * S = I$$

And S and T are inverses.

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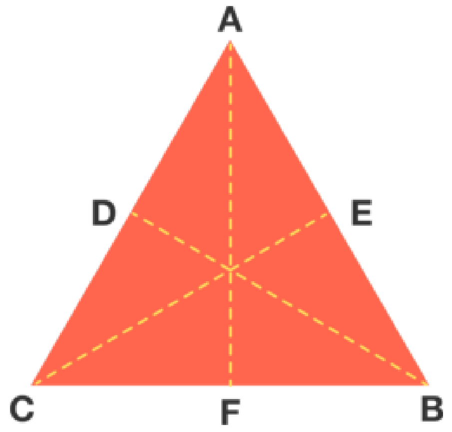
Commutativity *is where order does not matter for an operation*

Some algebraic systems have commutativity. This means that when you perform an operation, the order doesn't matter. For example, multiplication.

$$x * y = y * x$$

But what about symmetries of an ET?



Guided Discussion: Defining Groups

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Group Theory *is an area of algebra, which means it's a study of how combining objects can make new ones*

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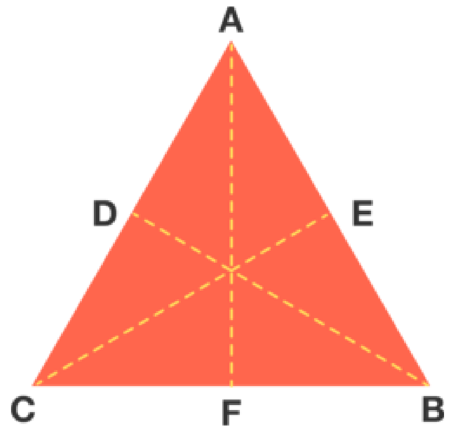

As we see, order does matter in this case.

Associativity (that $(x * y) * z = x * (y * z)$*) does hold.*

Guided Discussion: Defining Groups

There is a rule that maps every pair of elements from G to G

$$G \times G \to G$$

Or that "G cross G maps to G"

Known as being "closed" in *

Every element has an inverse

*And so finally, we define a **Group**.*

- *Set G, together with binary operation $*$*
- *Such that any for any two elements* x and y in G, $x * y \in G$
- *There is an identity element $e \in G$ such that for any element x in G,*
  $$e * x = x * e = x$$
- *For every element x in G, there is an inverse such that $x * x^{-1} = e$*
- *The operation is associative*
  $$(x * y) * z = x * (y * z)$$

Guided Discussion: Defining Groups

*A **Group** is set **G** with binary operation * that satisfies the following **4 axioms***

- ***G** is **Closed** in **
  - *Every pair of elements has a mapping to an of elements in G*
- *There exists an identity element, **e** in G*
  - $e * x = x * e = x$
- *For each x in G, **there exists an inverse** of x such that $x^{-1} * x = e$*
- ***Associative** $x * (y * z) = (x * y) * z$*

Examples of Groups!

- *Is set $\mathbb{Z}$ and operation addition $+$ a group?*

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Examples of Groups!

- *Is set $\mathbb{Z}$ and operation addition $+$ a group?*
- *Every two integers sum to another integer*
- *The identity is 0*
- *There is an inverse of each element (10 and -10 are inverses)*
- *There is associativity*

Yes!!!

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Examples of Groups!

- *Is set* $\mathbb{Z}$ *and operation multiplication a group?*

No, because although it is closed and there is an identity element, 1*, there are not inverses such that*

$3 * x = 1$ *where* $x \in \mathbb{Z}$

Guided Discussion: Types of Groups

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There are three main important types of groups:

- *Dihedral Groups*
- *Symmetric Groups*
- *Cyclic Groups*

Guided Discussion: *Dihedral Groups*

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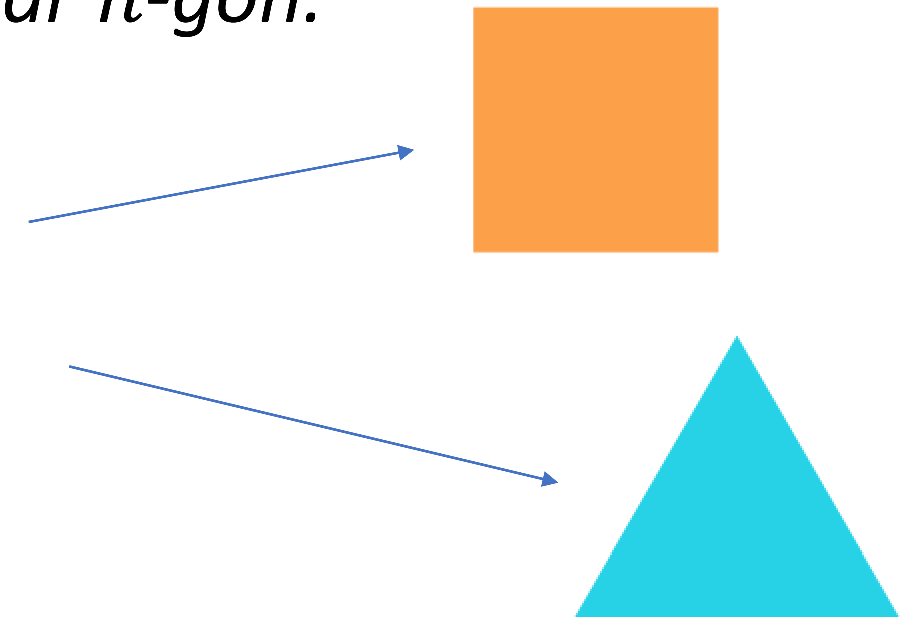
*A **Dihedral Group, $\boldsymbol{D_n}$** is a group corresponding to regular n-gon*

Dihedral Groups, denoted D_n are the corresponding groups to a regular n-gon.

- D_4
- D_3



What is the size of D_n?

Guided Discussion: *Dihedral Groups*

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What is the size of D_n?

$2n$

As there are $n - 1$ rotations, n reflective symmetries, and 1 identity transformation

Guided Discussion:
Symmetric Groups

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*A **Symmetric Group, $\boldsymbol{S_n}$** are the permutations on $\mathbb{Z}/n\mathbb{Z}$*

Symmetric Groups, S_n, is the set of permutations on $\{0,1,2 \cdots n\}$, which is a bijective function from that set to itself.

Consider group S_3 and element ϕ which maps 1 to 2, 2 to 3, and 3 to 1. This permutation, ϕ, is an element of S_3

Where does ϕ^2 map 3?

Guided Discussion: Symmetric Groups

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*****Note! The Symmetric Group itself does not contain 1,2…n. It instead contains all the possible permutations of that set.***

Where does ϕ^2 map 3?

$$3 \to 1, 1 \to 2$$

So it maps $3 \to 2$

Guided Discussion: *Symmetric Groups*

What is a good application/example of a symmetric/permutation group?

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What is a good application/example of a symmetric/permutation group?

Shuffling a deck of cards! Every way to shuffle a deck of cards would be an element s

$$s \in S_{52}$$

Guided Discussion:
Cyclic Groups

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*A **Cyclic Group, $\boldsymbol{Z_n}$** is the group of modular addition $\mathrm{mod}\ n$*

Cyclic Groups, Z_n are, put simply, the group of modular addition $\mathrm{mod}\ n$.

The set is $\mathbb{Z}/n\mathbb{Z}$ and the operation is addition modulo n

For Z_4, for example:

$$\{0,1,2,3\}$$

And the operation is addition $\mathrm{mod}\ 4$

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Let's take Z_4 for example.

The identity element is 0

The group operation of 3 applied to the group operation of 3 gives 2

The inverse of group operation 3 is 1 since $3 + 1 = 0\ mod\ 4$

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It's worth noting that Z_n is also the set of rotations of an n-gon

Then what's the difference between Z_n and D_n?

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*A **Cyclic Group,*** $\boldsymbol{Z_n}$ *is the group of modular addition mod* n

It's worth noting that Z_n *is also the set of rotations of an* n*-gon*

Then what's the difference between Z_n *and* D_n*?*

D_n *is all the symmetries on an* n*-gon.* Z_n *is isomorphic only to the rotations of an* n*-gon*

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A little exercise:

Other than 0, does Z_n contain an element that is its own inverse?

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A little exercise:

Other than 0, does Z_n contain an element that is its own inverse?

Yes! But only if n is even!

$$4 + 4 = 0 \; mod \; 8$$

$$5 + 4 = 0 \; mod \; 9$$

Guided Discussion:
Filling Some Gaps

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*Let G be a group. A subset $H \subseteq G$ is a **subgroup** if it forms a group under the same operation already defined in G*

*Not every subset of a group can be a group itself, you need to check and see if there is still **closure** and if every element has an **inverse***

Guided Discussion:
Filling Some Gaps

*A **Group** is set $\boldsymbol{G}$ with binary operation * that satisfies the following **4 axioms***

- ***G is Closed** in **
- $\boldsymbol{e} \in G$
- $\forall x \in G, \exists\, x^{-1}$ *such that* $x^{-1} * x = e$
- ***Associative*** $x * (y * z) = (x * y) * z$

*A **Dihedral Group, $\boldsymbol{D_n}$** is a group corresponding to regular n-gon*

*A **Symmetric Group, $\boldsymbol{S_n}$** are the permutations on $\mathbb{Z}/n\mathbb{Z}$*

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Two large ideas:

- *The Order of an element*
- *Group Isomorphism*

Guided Discussion: Order

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Let G be a group, and let $g \in G$. Suppose there is some positive integer k for which

$$g^k = e$$

Then, there must be an infinite amount of such integers.

Why?

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***Order** of $g \in G$ is smallest such integer such that $g^k = e$*

Let G be a group, and let $g \in G$. Suppose there is some positive integer k for which

$$g^k = e$$

Because

$$g^k g^k = g^{2k} = e$$

And thus all integer multiples of k also satisfy.

We say the order of $\boldsymbol{g}$ is the smallest such integer

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We say the order of $\boldsymbol{g}$ is the smallest such integer.

What if there is no positive integer k such that $g^k = e$?

Then we say that g has infinite order

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Take some examples.

In D_3, what is the order of a reflection r?

What about the order of rotation s by $120°$?

What is the order of group identity e?

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Take some examples.

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Guided Discussion: *Complex Numbers*

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19th century mathematician W.R. Hamilton was looking to generalize the complex numbers to three dimensions but was having trouble.

He iconically realized the solution was to add a fourth dimension.

The "second dimension" in complex numbers is generated by multiples of imaginary number i

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Complex Numbers

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The "second dimension" in complex numbers is generated by multiples of imaginary number i, but his useful insight was that there was a pleasingly symmetric multiplication operation defined on 3 symbols, i, j, k

$$i^2 = j^2 = k^2 = ijk = -1$$

**Note! Not commutative!

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Complex Numbers

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$$i^2 = j^2 = k^2 = ijk = -1$$

*The set $\{\pm 1, \pm i, \pm j, \pm k\}$ and operation multiplication form the **Quaternion Group,** $\boldsymbol{Q_8}$*

How many elements in this group have order 4?

Guided Discussion: *Isomorphisms*

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An Isomorphism between two groups is a bijective map preserving group operations.

Essentially, the groups are the same.

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Looking back, we saw that the group of symmetries on the letter I is isomorphic to the group of pairs of integers $\bmod 2$ and the operation addition.

This underlying structure between these two is called the Klein Group. Both groups previously examined are isomorphic to the Klein Group.

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Practice!

The group Z_{12} *is isomorphic to?*

- S_3
- S_4
- *Rotational symmetries of a regular dodecagon*
- *Symmetry group of a regular dodecagon*

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Practice!

The group of symmetries on a rhombus is isomorphic to?

- Z_4
- D_4
- *The Klein Group*
- S_4

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Practice!

The group D_3 is isomorphic to what group?

- Z_6
- D_6
- S_3
- S_4

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- S_3