

# Discrete Analysis

## Walter Johnson Maths Honors Society

### Requirements

In order to receive credit for this independent research project and be in good standing with the Walter Johnson Maths Honor Society, you must write a paper which does each of the following:

- Describe, briefly, what Discrete Analysis.
- Introduce  $\mathbb{Z}/m\mathbb{Z}$  and describe modular arithmetic.
- Define the Permutation function, and derive the Choose function from it.
- Derive and describe Pascal's Triangle.
- Describe and prove the Hockey Stick Identity, and explain it's naming.
- Describe the concept of Block Walking, and it's relevance to Pascal's Triangle. Explain the geometric significance, in 2 and  $n$  dimensions.
- Describe the usage of Generating Functions.
- Complete and describe solutions to all three problems.

On average, this assignment will take about 3 hours to research and write up. You may **not** work in a group or collaborate with others.

You will be assigned to groups of 6 people, each of which has completed a different independent research project. At the end of the year, you will present your findings to your group, and listen as your peers present their findings. Your presentation must briefly discuss every subject required in your paper along with 1 of the problems you solved, of your choice.

### Resources

You are provided with various resources to complete your research. You are welcome to use resources that are not given here.

### Modular Arithmetic

- Brilliant wiki page on modular arithmetic.
- Stanford Cryptology department article on Modular Arithmetic. More detailed look at arithmetic under a modulus.
- Wolfram MathWorld wiki page on Modular Arithmetic.

### Permutations and Combinations

- Introduction to Counting from University of Wollongong.
- Article by Jon Dreyer on combinations.

### Pascal's Triangle

- Cornell paper on Pascal's Triangle.
- Math Centre UK introduction to Pascal's Triangle.

### Hockey Stick Identity

- Wikipedia page article on Hockey Stick Identity.
- Brilliant wiki page on the Hockeystick Identity

### Block Walking

- Combinatorial Identities: Binomial Coefficients, Pascal's Triangle, and Stars & Bars by Molly Maxwell at Colorado College.
- Art of Problem Solving wiki page on Block Walking.

### Generating Functions

- Art of Problem Solving wiki page on Generating Functions.
- MIT lecture notes on Generating Functions and counting with them.

### Problem 1

Consider all 1000 element subsets of the set  $\{1, 2, 3, \dots, 2015\}$ . From each subset pick the least element. The arithmetic mean of all of these least elements is  $\frac{p}{q}$ , where  $p$  and  $q$  are relatively prime positive integers. Using the Hockey Stick Identity, find  $\frac{p}{q}$ .

### Problem 2

Prove the Hat Stacking Identity:

$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$$

With a block-walking argument.

### Problem 3

Prove that the generating function for the Fibonacci numbers is the following:

$$F_f(x) = f_0 + f_1x + f_2x^2 + \dots = \frac{x}{1 - x - x^2}$$

Where  $f_0 = 0, f_1 = 1$ , and  $f_n = f_{n-1} + f_{n-2}$