Number Theory Problem Set #1

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May 2021

Problems are ordered from easiest to hardest difficulty, with high probability. None of the problems require a calculator, calculus, analysis, or an abacus. If you have any questions, just ask!

1

A faulty odometer proceeds from digit 3 to digit 5, always skipping the digit 4, regardless of position. If the odometer now reads 002005, how many miles has the car actually traveled?

2

In the equation below, A and B are consecutive positive integers, and A, B, and $A+B$ represent number bases:

$$132_A + 43_B = 69_{A+B}$$

What is $A+B$?

3

Call a positive integer *fibbish* if each digit, after the leftmost two, is at least the sum of the previous two digits. Compute the greatest fibbish number.

4

Find the number of digits in $(4^5)(5^{13})$

5

Find the smallest 3-digit number such that both of the following are true:

1. The number formed by any two of its digits (in either order) is a prime.
2. The number formed by its three digits, in any order, is a prime.

6

A certain positive integer requires four digits when written in base 5, but has only seven digits when written in base 2. Furthermore, this number is not a palindrome when written in either base 2 or base 5. Find this number, writing your answer in base 7.

7

Determine the smallest positive integer c such that for any positive integer n, the decimal representation of the number $c^n + 2014$ has digits all less than 5.

8

Let

$$k = 2008^2 + 2^{2008}$$

Find the units digit of k^2

9

Six distinct positive integers are randomly chosen between 1 and 2006, inclusive. What is the probability that some pair of these integers has a difference that is a multiple of 5?

10

if a, b, c are non-negative integers less than 7 such that

$$a + 2b + 3c \equiv 0 \ (\text{mod} 7)$$
$$2a + 3b + c \equiv 4 \ (\text{mod} 7)$$
$$3a + b + 2c \equiv 4 \ (\text{mod} 7)$$

determine the remainder when abc is divided by 7.

11

Find the least positive integer such that when its left-most digit is deleted, the resulting integer is 1/29 of the original integer.

12

A palindrome between 1000 and 10,000 is chosen at random. What is the probability that it is divisible by 7?

13

A sequence of numbers is defined by

$$D_0 = 0, D_1 = 0, D_2 = 1$$

and

$$D_n = D_{n-1} + D_{n-3}$$

for $n \geq 3$. What are the parities (evenness or oddness) of the triple of numbers

$$D_{2021}, D_{2022}, D_{2023}$$

14

Find $3m^2n^2$ if m, n are integers such that

$$m^2 + 3m^2n^2 = 30n^2 + 517$$

15

What is the sum of the last two digits of

$$8^{25} + 12^{25}$$

16

For all positive integers n less than 2002, let

$$a_n = \begin{cases} 11 & \text{if } n \text{ is divisible by 13 and 14} \\ 13 & \text{if } n \text{ is divisible by 11 and 14} \\ 14 & \text{if } n \text{ is divisible by 11 and 13} \\ 0 & \text{otherwise} \end{cases}$$

Calculate

$$\sum_{n=1}^{2001} a_n$$

17

Find the smallest positive integer m above 2010 such that the difference $\frac{1}{2010} - \frac{1}{m}$ does *not* reduce to a fraction of the form $\frac{1}{n}$ for some integer n.

18

Find the remainder when the number $1^{40}2^{39}3^{38}\cdots 39^2 40^1$ is divided by 41.

19

Let f be a function with the following properties:
$(i) f(1) = 1$, and
$(ii) f(2n) = n \times f(n)$, for any positive integer n.
What is the value of $f(2^{100})$?

20

The numbers $1, 2, \ldots, 10$ are written on a board. Every minute, one can select three numbers a, b, c on the board, erase them, and write $\sqrt{a^2 + b^2 + c^2}$ in their place. This process continues until no more numbers can be erased. What is the largest possible number that can remain on the board at this point?

21

One of Euler's conjectures was disproved in the 1960s by three mathematicians when they showed there was a positive integer such that

$$133^5 + 110^5 + 84^5 + 27^5 = n^5$$

Find the value of n

22

If

$$f(x) = x^{x^{x^x}}$$

Find the last two digits of

$$f(17) + f(18) + f(19) + f(20)$$