Diagnostic And Problem Set #1

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May 2021

Don't be intimidated by these questions - if you can't do some, or all, of them right now, that's ok! You will be able to do all of them by the end of the year!

Within each section, problems are ordered from easiest to hardest difficulty. None of the problems require a calculator, calculus, analysis, or an abacus.

1 Algebra

1.1

Find the largest value of x for which $x^2 + y^2 = x + y$ has a solution, if x and y are real.

1.2

All the roots of polynomial

$$z^6 - 10z^5 + Az^4 + Bz^3 + Cz^2 + Dz + 16$$

are positive integers. What is the value of B?

1.3

Let $g(x)$ be a polynomial with leading coefficient 1, whose three roots are the reciprocals of the three roots of $f(x) = x^3 + ax^2 + bx + c$, where $1 < a < b < c$. What is $g(1)$ in terms of a, b and c?

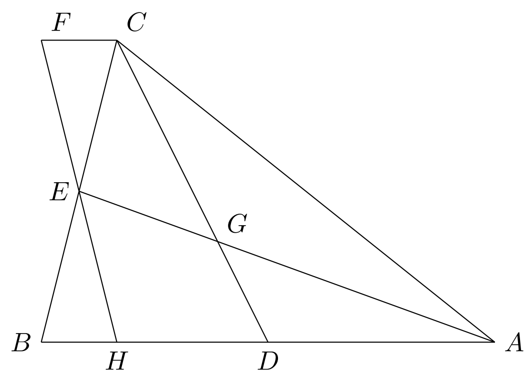
2 Geometry

2.1

Let D be the point on side $\overline{BC}$ of triangle ABC for which $\overline{AD}$ is the angle bisector of $\angle A$. If $AB = AD = 6$ and $CD = 2(BD)$, then find length BD.

2.2

In $\triangle CBA$, $\overline{CD}$ and $\overline{AE}$ are medians, $\overline{FC} || \overline{AB}$. Lines $\overline{FEH}$, $\overline{CGD}$, and $\overline{AGE}$ are drawn. The area $FCGE = 7$, and the area of $EGDH = 11$. Compute the area of $\triangle CBA$.



2.3

Let $ABCD$ be a unit square. Let Q_1 be the midpoint of $\overline{CD}$. For $i = 1, 2, \ldots$, let P_i be the intersection of $\overline{AQ_i}$ and $\overline{BD}$, and let Q_{i+1} be the foot of the perpendicular from P_i to $\overline{CD}$. What is

$$\sum_{i=1}^{\infty} [\triangle DQ_iP_i]$$

Where $[\triangle DQ_iP_i]$ denotes the area of that triangle?

3 Complex Numbers

3.1

A function f is defined by $f(z) = i\overline{z}$, where $i = \sqrt{-1}$ and $\overline{z}$ is the complex conjugate of z. How many values of z satisfy both $|z| = 5$ and $f(z) = z$?

3.2

Let $P(z) = z^8 + (4\sqrt{3}+6)z^4 - (4\sqrt{3}+7)$. What is the minimum perimeter among all the 8-sided polygons in the complex plane whose vertices are precisely the zeros of $P(z)$?

3.3

Let $\xi = \cos(\frac{2\pi}{7}) + i\sin(\frac{2\pi}{7})$ be a seventh root of unity. Compute the value of

$$(2\xi+\xi^2)(2\xi^2+\xi^4)(2\xi^3+\xi^6)(2\xi^4+\xi^8)(2\xi^5+\xi^{10})(2\xi^6+\xi^{12})$$

4 Number Theory

4.1

Let $k = 2008^2 + 2^{2008}$ Find the units digit of k^2

4.2

Find $3m^2n^2$ if m, n are integers such that $m^2+3m^2n^2 = 30n^2 + 517$

4.3

One of Euler's conjectures was disproved in the 1960s by three mathematicians when they showed there was a positive integer such that

$$133^5 + 110^5 + 84^5 + 27^5 = n^5$$

Find the value of n

5 Combinatorics

5.1

How many ways are there to choose a 5-letter word from the 26-letter English alphabet with replacement, where words that are anagrams are considered the same?

5.2

Given that

$$\binom{23}{3} = 1771, \binom{23}{4} = 8855, \binom{23}{5} = 33649$$

Find

$$\binom{25}{5}$$

5.3

Row 1 of pascal's triangle consists of two 1s. Let $\{a_i\}$, $\{b_i\}$, and $\{c_i\}$ be the sequence, from left to right, of the elements in the 2005th, 2006th, and 2007th row, respectively, with the leftmost element occurring at $i = 0$. Compute:

$$\sum_{i=0}^{2006} \frac{b_i}{c_i} - \sum_{i=0}^{2005} \frac{a_i}{b_i}$$

6 Proof Based

6.1

Prove that $n! > 2^n$ for all $n \geq 4$.

6.2

Let $r_1, r_2, r_3, \ldots, r_n$ be n real numbers each greater than zero. Prove that for any real number $x > 0$,

$$(x+r_1)(x+r_2)\cdots(x+r_n) \leq \left(x + \frac{r_1+r_2+r_3+\cdots+r_n}{n}\right)^n$$

7 Miscellaneous

7.1

A polyhedron has 12 vertices. At 6 of them, 4 edges come together; at the other 6, 3 edges come together. Compute the number of faces that the polyhedron has.

7.2

For positive real numbers a, b, c, d find the minimum value of the expression

$$(a+b+c+d)\left(\frac{25}{a} + \frac{36}{b} + \frac{81}{c} + \frac{144}{d}\right)$$