Combinatorics

Counting and Pascal's Triangle,
Complementary and Constructive Counting,
Stars and Bars,
States, Generating Functions

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What is Combinatorics?

Combinatorics is the study of counting, as well as finite and discrete structures.
We're going to start by looking at some of the fundamental structures in Combinatorics, particularly **Pascal's Triangle**, which we will build a lot of our combinatorical techniques off of.
Then, we'll look at broader strategies like **Complementary and Constructive Counting**, and lastly we'll look at some more intricate strategies such as **State**-work, **Generating Functions**, and **Block Walking**.

Basics and Choosing

Some quick vocabulary:

- The **Factorial operator**:

$$\boxed{n! = 1 \cdot 2 \cdot 3 \cdots n = \prod_{i=1}^{n} i}$$

  and

$$\boxed{(k+1) \cdot (k+2) \cdots (n-1) \cdot (n) = \frac{n!}{k!}}$$

- A **Permutation** of a set of elements is a certain ordering of those elements. $BACD$ is a permutation of the letters A, B, C, and D. Of n elements, there are $n!$ ways to permute them.
- A **Choosing** of a set of elements is a certain selection of those elements. B and D is a choice of the letters A, B, C, and D. Choices are typically unordered.

Choosing

A fundamental building block of Combinatorics is the Choose operator. This answers the question: How many ways can we pick k objects from n objects, if the order doesn't matter?

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$$\boxed{\frac{n!}{k!(n-k)!} = \binom{n}{k}}$$

We derive this with the definition of a permutation. If we have n objects, the ways we permute them is $n!$, but if we're only taking k objects, then the last few objects, which take $(n-k)!$ ways to permute, we divide out. The $k!$ division comes from dividing out the ways we permute the chosen k objects, to make sure no arrangement is chosen multiple times.

Pascal's Triangle

We're all familiar with **Pascal's Triangle**, which is constructed downward, adding numbers down the rows:

$$
\begin{array}{lccccccccccccc}
n=0 & & & & & & & 1 & & & & & & \\
n=1 & & & & & & 1 & & 1 & & & & & \\
n=2 & & & & & 1 & & 2 & & 1 & & & & \\
n=3 & & & & 1 & & 3 & & 3 & & 1 & & & \\
n=4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
n=5 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
n=6 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Each of these numbers are determined by adding the numbers diagonally above them.

Pascal's Triangle

An important note on Pascal's Triangle that connects it to the rest of combinatorics is the fact that each entry is a binomial coefficient:

$$
\begin{array}{lccccccccc}
n=0 & & & & & \binom{0}{0} & & & & \\
n=1 & & & & \binom{1}{0} & & \binom{1}{1} & & & \\
n=2 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
n=3 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
n=4 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
$$

This opens our manipulation of Choose functions up to a world of new possibilities! The first being Pascal's Identity and Symmetry:

$$\boxed{\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}} \text{ and } \boxed{\binom{n}{k} = \binom{n}{n-k}}$$

Which follow directly from the triangle.

Pascal's Triangle

There are a variety of other relationships we develop with Pascal's Triangle. Namely, the coefficients of Pascal's Triangle are the coefficients for a binomial $(x+1)^n$, hence, why the numbers are called the binomial coefficients:

$$\begin{array}{lccccccccccc}
n=0 & & & & & & 1 & & & & & \\
n=1 & & & & & 1 & & 1 & & & & \\
n=2 & & & & 1 & & 2 & & 1 & & & \\
n=3 & & & 1 & & 3 & & 3 & & 1 & & \\
n=4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
n=5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Notice, $(x+1)^3 = x^3 + 3x^2 + 3x + 1$, which are the numbers in the third row of Pascal's Triangle. This connects binomial coefficients to our choose function.

Pascal's Triangle

Let $n = 3$. We see that

$$(x+1)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2 + \binom{3}{2}x + \binom{3}{3}$$

Pascal's Triangle

Let $n = 3$. We see that

$$(x+1)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2 + \binom{3}{2}x + \binom{3}{3}$$

What happens when $x = 1$? We have

$$(1+1)^3 = \binom{3}{0}1^3 + \binom{3}{1}1^2 + \binom{3}{2}1 + \binom{3}{3}$$

And thus

$$2^3 = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3}$$

Pascal's Triangle

We have

$$(1+1)^3 = \binom{3}{0}1^3 + \binom{3}{1}1^2 + \binom{3}{2}1 + \binom{3}{3}$$

And thus

$$2^3 = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3}$$

This generalizes for all n:

$$\boxed{2^n = \sum_{k=0}^{n} \binom{n}{k}}$$

Pascal's Triangle

Another development from Pascal's Triangle gives us the **Hockey Stick Identity**:

$n = 0$						1						
$n = 1$					1		1					
$n = 2$				1		2		1				
$n = 3$			1		3		3		1			
$n = 4$		1		4		6		4		1		
$n = 5$	1		5		10		10		5		1	

Which gives us:

$$\boxed{\sum_{k=0}^{n} \binom{m+k}{k} = \binom{m+n+1}{n}}$$

Questions?

Counting Techniques

Sometimes you will run across problems that don't have straight-forward solutions. In these cases, there are a variety of strategies for us to keep in mind:

1. **Complementary Counting** is counting the *opposite* of what you want. If it's too hard to find the probability that P happens, maybe it is easier to find the probability that P does not happen, and subtract $1 - P$. This applies for non-probabalistic scenarios as well.
2. **Constructive Counting**, which involves counting cases *constructively*; starting off with small base cases and building up to the total cases.
3. **Casework**, which involves breaking a problem down into manageable cases.
4. Create a **Bijection**. Which means creating an alternate scenario which encapsulates the same information, but is easier to calculate

Questions?

Stars and Bars

Stars and Bars is an incredibly useful technique in a lot of combinatorics problems, but it has a nuanced derivation.
Sometimes called **Balls and Urns**, because of the statement of the technique:
The number of ways to place n indistinguishable balls in k urns is

$$\boxed{\binom{n+k-1}{k-1}}$$

We will derive this in the following slides.

Stars and Bars

Let's say we have 9 balls and we want to put them in 4 urns. We're going to represent this as such: We line all of the balls up in a row:

$$* * * * * * * * *$$

This 9 or n balls.

Stars and Bars

Let's say we have 9 balls and we want to put them in 4 urns. We're going to represent this as such: We line all of the balls up in a row:

$$* * * * * * * * *$$

We then insert 3 dividers to mark our 4 different Urns:

$$*|*|*****|**$$

Notice we have $k = 4$ urns, but $k - 1 = 3$ dividers. Now, we're going to stretch our brains a bit.

Stars and Bars

What we need to do here is convince you that we have just made a **Bijection** between all of the possible ways to allocate the $n = 9$ balls into $k = 4$ urns and all of the possible ways to place our $k - 1 = 3$ dividers:

$$| * * | * * * * * * | *$$

This represents the first urn with 0 balls, the second with 2, the third with 6, and the last with 1.

$$* * * | * * | * * | * *$$

This represents the first urn with 3, the second with 2, the third with 2, and the last with 2. Convince yourself there is a bijection.

Stars and Bars

Well now that we know there is a bijection between all of the possible ways we can put $n = 9$ balls into $k = 4$ urns and the ways we can insert $k - 1 = 3$ sliders into an assortment of $n = 9$ balls. So how do we count how many possible ways there are to do this?

Stars and Bars

Well now that we know there is a bijection between all of the possible ways we can put $n = 9$ balls into $k = 4$ urns and the ways we can insert $k - 1 = 3$ sliders into an assortment of $n = 9$ balls. The amount of ways to do this is equivalent to the amount of ways to choose the $k - 1$ ways to pick the dividers from the $n + k - 1$ slots:

$$\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_$$

Since we have $n + k - 1$ slots, and $k - 1$ dividers to place, the number of ways to choose to place the $k - 1$ dividers into the $n + k - 1$ slots is

$$\boxed{\binom{n+k-1}{k-1}}$$

Stars and Bars

$$\boxed{\binom{n+k-1}{k-1}}$$

This solution also helps to solve problems like this:
How many ordered pairs of positive integers (a, b, c) satisfy

$$a + b + c = 17?$$

We notice in this scenario there are 17 balls, 3 urns, and thus our solution is

$$\binom{17+3-1}{3-1} = \binom{19}{2} = 171$$

Stars and Bars

$$\boxed{\binom{n+k-1}{k-1}}$$

This solution also helps to solve problems like this:
How many ordered pairs of positive integers (a, b, c) satisfy

$$a + b + c = 17?$$

We notice in this scenario there are 17 balls, 3 urns, and thus our solution is

$$\binom{17+3-1}{3-1} = \binom{19}{2} = 171$$

How could you solve the same problem if it had a less than sign?

$$a + b + c \leq 17?$$

Stars and Bars

$$a + b + c \leq 17?$$

For something like this, we can see that the problem is equivalent to

$$a + b + c + d = 17$$

for some unique d, for which we can then solve this with Stars and Bars.

Stars and Bars comes in handy in a variety of scenarios, but a few more techniques are crucial as well.

Questions?

Generating Functions

A **Generating Function** is a function which, by the laws of algebra, give us **coefficients** of importance. The variable x in the equation has no real meaning to us.
Generating functions are important because their coefficients can encapsulate a lot of information in one algebraic expression.

Generating Functions

For example, the generating function for a dice being rolled once is

$$x + x^2 + x^3 + x^4 + x^5 + x^6$$

What this means, is that the *number of ways* you can roll a 1 is equal to the coefficient of x^1 and the number of ways you can roll a 2 is equivalent to the coefficient of x^2, etc. In this case, there is only one way to roll a 1 or a 2, as seen in the function.

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Generating Functions

This is where Generating Functions become especially powerful! The Generating Function for rolling a two dice is equivalent to the product of the generating functions of each die:

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)(x + x^2 + x^3 + x^4 + x^5 + x^6)$$

Which gives us:

$$x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}$$

Notice that each of these coefficients gives us the number of ways to get that number from rolling two dice!

Generating Functions

This is the power of generating functions: when we're dealing with multiple scenarios acting at the same time, **multiplying two generating functions** often results in the generating function **for the two scenarios combined**.

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The other advantage of Generating Functions are they enable us to analyze scenarios with **infinite states** easily.

Generating Functions

Let's look at an example.
How many ordered pairs of integers (a, b, c) satisfy

$$3a + b + c = 24?$$

Surprisingly, you **cannot** solve this with Stars and Bars (*think deeply about the bijection!*) But we can solve this with Generating functions.

Generating Functions

$$3a + b + c = 24$$

We see the generating function for b is

$$G_b = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

and the generating function for c is

$$G_c = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

because there is one way for each of these to be 0, 1, 2, etc.

Generating Functions

$$3a + b + c = 24$$

We know that the generating function for $3a$ is

$$G_{3a} = 1 + x^3 + x^6 + x^9 + \cdots = \frac{1}{1 - x^3}$$

Because $3a$ can be either 0, 3, 6, etc. And there is only one way for it to be each of these.

Generating Functions

With this, the generating function for $3a + b + c$ is equivalent to

$$G_{3a}G_bG_c = \frac{1}{(1-x^3)(1-x)^2}$$

And with this, we see the coefficient is 117:

$$\frac{1}{(1-x^3)(1-x)^2} = 1 + 2x + 3x^2 + 5x^3 + \cdots + 108x^{23} + 117x^{24} + \cdots$$

Generating Functions

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If this is a bit confusing, especially as to how we determine the expansion for the function, we'll talk about that in a minute.

Negative Binomial Expansion

So if you're wondering how exactly you're supposed to expand a function like

$$\frac{1}{1-x} \text{ or } \frac{x^3}{(1-x)^5}$$

We'll briefly discuss this in the next few slides.

Negative Binomial Expansion

We won't derive these here, but the formula is as such:

$$\frac{1}{(1-x)^n} = \binom{n+0-1}{0} + \binom{n+1-1}{1}x + \binom{n+2-1}{2}x^2 + \cdots$$

And thus:

$$\boxed{\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k}$$

and

$$\boxed{\frac{1-x^k}{1-x} = 1 + x + \cdots + x^{k-1}}$$

And with this, *if* there is a problem which can be solved with generating functions, these formulas should suffice for whatever you need to do with Generating Functions.

Questions?

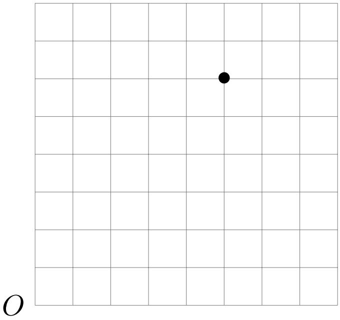
Block Walking

Block Walking is the analysis of one-directional choices made on a grid.

Imagine a bug starting at the origin. This bug can only move up or right.

Block Walking

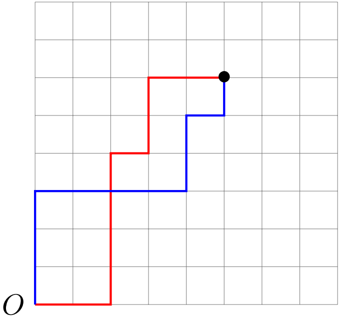
Let's say the bug's final destination is $(5, 6)$:



How many ways are there for the bug to get to this location?

Block Walking

Here are a couple possible paths.

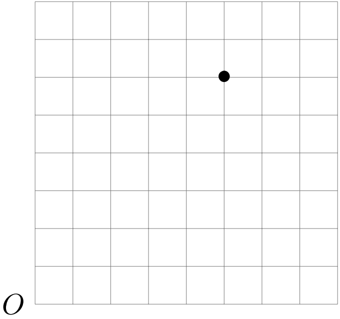


Block Walking

Turns out, there are exactly

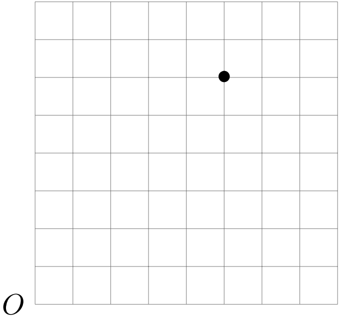
$$\binom{5+6}{6} \text{ or, equivalently } \binom{5+6}{5} \text{ or } \binom{11}{5}$$

Ways the bug can get from the origin to $(5, 6)$.



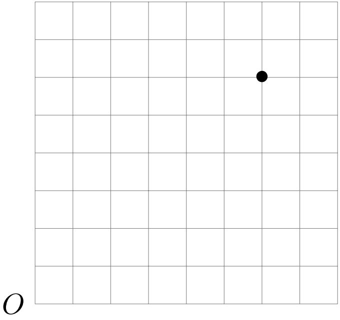
Block Walking

This is because regardless of the path taken, the bug will make exactly $5+6=11$ moves every time, and exactly 5 of them will be to the left, and 6 will be up. We then just choose which 5 moves will be to the left, and which won't, which gives us $\binom{11}{5}$



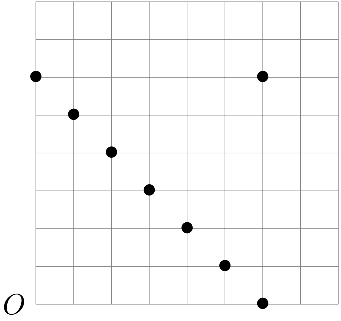
Block Walking

Let's say the destination point is $(6,6)$. We know the amount of ways to get to this location is $\binom{12}{6} = 924$, but there is another way to find how many ways we can get there.



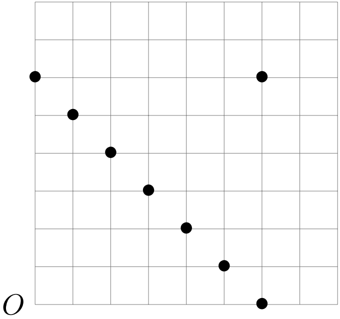
Block Walking

Notice, every possible path to the destination *must* go through one of these intermediate points, and can only go through one of them.



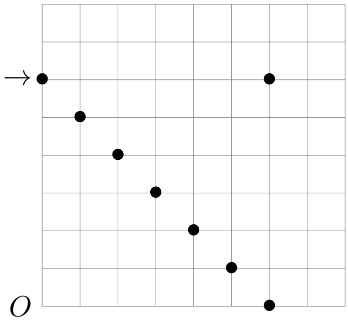
Block Walking

For each of these intermediate points, how many ways can the bug get there? How many ways can it get there to the final destination?



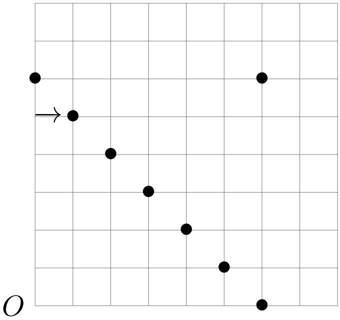
Block Walking

For the first intermediate point, there are $\binom{6}{0}$ way to get there, and $\binom{6}{0}$ way to get from there to the destination. Thus, the total number of paths through this point is $\binom{6}{0}^2$



Block Walking

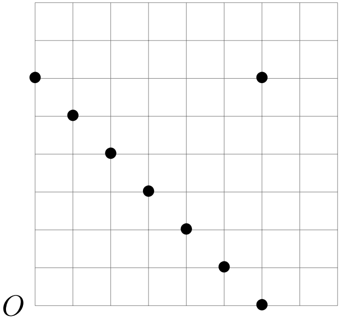
For the second intermediate point, there are $\binom{6}{1}$ way to get there, and $\binom{6}{1}$ way to get from there to the destination. Thus, the total number of paths through this point is $\binom{6}{1}^2$



Block Walking

Since the total number of ways to reach the destination is the sum of all of the ways to get to the destination through each point, we have the identity:

$$\binom{2 \cdot 6}{6} = \sum_{k=0}^{6} \binom{6}{k}^2$$



Block Walking

Since the total number of ways to reach the destination is the sum of all of the ways to get to the destination through each point, we have the identity:

$$\binom{2 \cdot 6}{6} = \sum_{k=0}^{6} \binom{6}{k}^2$$

Notice though, when we were block walking, there was nothing special about the number 6. This generalizes into the mind-bending identity:

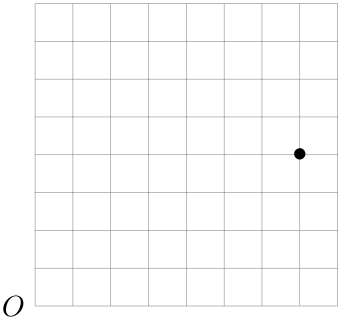
$$\boxed{\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2}$$

This is referred to as the **Hat Stacking Identity**

Questions?

Block Walking

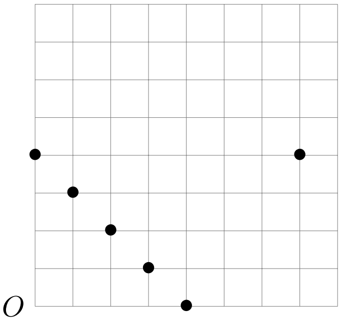
There's another incredibly important identity which comes from Block Walking. It is a generalization of the identity we just proved, but with variable dimensions. Let's say the destination is now $(7, 4)$:



We know that there are $\binom{11}{4} = 330$ ways to traverse this path.

Block Walking

But let's break the path up again based on intermediate coordinates:



Block Walking

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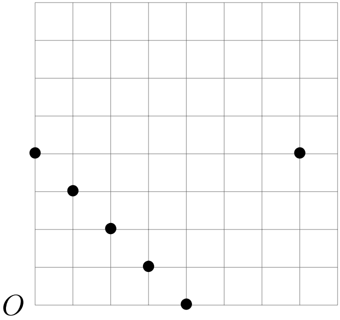
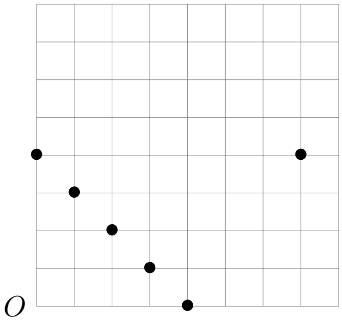


Let's do the same analysis for some of these points.

Block Walking

Looking through this lens, we see the number of ways to get to the destination is:

$$\binom{4}{0}\binom{7}{0}+\binom{4}{1}\binom{7}{1}+\binom{4}{2}\binom{7}{2}+\binom{4}{3}\binom{7}{3}+\binom{4}{4}\binom{7}{4}=330$$



This is very powerful as well, because there is nothing special about our location $(4, 7)$!

Block Walking

We generalize this to:

$$\binom{n+m}{n} = \binom{n+m}{m} = \sum_{k=0}^{n} \binom{n}{k}\binom{m}{k}$$

This can be generalized further with grids of higher dimension. Although this is not Vandermonde's Identity, it is a unique case of it.

Questions?

States

States is a way for us to analyse situations where certain *objects* can be in different *states*.
This is a vague definition, but we'll jump right in with an example.