Warm Up! AIME #1, Complex Numbers GD

Consider the integer $N = 9 + 99 + \cdots$ *with the final term having 321 digits.*

Find the sum of the digits of N.

The number 100000 *is factored as a product of two positive integers* m *and* n. *Suppose that neither* m *or* n *has a* 0 *in their base-10 expansion.*

What is $m + n$?

The curves $x^2 + y^2 = a^2$ *and* $y = x^2 - a$ *intersect at exactly 3 points. Find the set of all points* a *which satisfy this.*

Given that $\log_2(\log_2 x)$, $\log_4(\log_4 x)$ *and* $\log_{16}(\log_{16} x)$ *form an arithmetic progression in that order, find* x

Warm Up! *AIME #1*

Consider the integer $N = 9 + 99 + \cdots$ *with the final term having 321 digits.*

Find the sum of the digits of N.

Note that $9 = 10 - 1$ *and* $99 = 100 - 1$

Now N has become $\underbrace{11111 \cdots 11110}_{322\ digits} - 321$ and thus our number is now $N = \underbrace{11111 \cdots 10789}_{322}$ and our sum becomes

$$1 * 318 + 7 + 8 + 9 = 342$$

Warm Up! *UMD HS Math Competition*

The number 100000 *is factored as a product of two positive integers* m *and* n*. Suppose that neither* m *or* n *has a* 0 *in their base-10 expansion.*

What is $m + n$*?*

We can see that $100000 = 10^5$. Now the prime factorization of this $2^5 * 5^5$. Our values n and m are going to be complimentary combinations of these 2s and 5s, but given that neither can have a 0, we know that 10 cannot be a factor of either.

Warm Up! UMD HS Math Competition

The number 100000 is factored as a product of two positive integers m and n. Suppose that neither m or n has a 0 in their base-10 expansion.

What is $m + n$?

Which means if either n or m has a 2 or a 5 it cannot have the other, because then 10 becomes a factor of it. Thus, our numbers are $2^5 = 32$ and $5^5 = 3125$, and their sum

$$3125 + 32 = 3157$$

Warm Up! 10A #21

The curves $x^2 + y^2 = a^2$ and

$y = x^2 - a$ intersect at exactly

3 points.

Find the set of all points a which satisfy this.

Substituting $y = x^2 - a$ into $x^2 + y^2 = a^2$ we get

$$x^2 + \left(x^2 - a\right)^2 = a^2$$

With this we get $x^2 + x^4 - 2ax^2 = 0$

And $x^2\left(x^2 - 2a + 1\right) = 0$

Thus with 4 factors, two being 0, the final solutions

are $\pm\sqrt{2a - 1}$ which means

$$a \geq {}^{1}/_{2}$$

Warm Up! *ARML*

Given that $\log_2(\log_2 x)$, $\log_4(\log_4 x)$ *and* $\log_{16}(\log_{16} x)$ *form an arithmetic progression in that order, find* x

We can see that

$$\log_2(\log_2 x) + \log_{16}(\log_{16} x) = 2\log_4(\log_4 x)$$

Let y be $\log_2 x$, so

$$\log_{16} x = \frac{\log_2 x}{\log_2 16} = \frac{y}{4}$$

and

$$\log_4 x = \frac{y}{2}$$

Warm Up! *ARML*

Given that $\log_2(\log_2 x)$, $\log_4(\log_4 x)$ *and* $\log_{16}(\log_{16} x)$ *form an arithmetic progression in that order, find* x

Expanding out we find

$$\log_2(\log_2 x) + \log_{16}(\log_{16} x) = 2\log_4(\log_4 x)$$

$$\log_2(y) + \log_{16}({}^{y}/_{4}) = 2\log_4({}^{y}/_{2})$$

$$\log_2(y) + \log_{16}(y) - \log_{16} 4 = 2\log_4(y) - 2\log_4 2$$

Now let $z = \log_2 y$, we find

$$z + \frac{z}{4} + \frac{1}{2} = 2\frac{z}{2}$$

Interest Concept Sheets: Infinite Series

Key Concepts: Rearrangement, Taylor Series

Walter Johnson Math Team

Concept Sheets: Rearrangement

Jumping right in, we'll consider the series to our left:

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \cdots$$

Which for interest's sake, is equal to $\ln 2 \approx 0.693$

Now consider this being multiplied by $^{1}/_{2}$

$$\frac{1}{2} S = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} \cdots$$

Concept Sheets: Rearrangement

Now consider this being multiplied by $^1/_2$

$$\frac{1}{2}S = +\frac{1}{2} \qquad -\frac{1}{4} \qquad +\frac{1}{6}\dots$$

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7}\dots$$

Now this gets interesting, as adding the two together we get $\frac{3}{2}S$.

$$\frac{3}{2}S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9}\dots$$

And we can see that this is the same series as S, but in a different order.

Concept Sheets: Rearrangement

$$\frac{1}{2}S = +\frac{1}{2} \qquad -\frac{1}{4} \qquad +\frac{1}{6}\ldots$$

And we can see that this is the same series as S, but in a different order.

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7}\ldots$$

What this confirms is the commutativity of addition does not hold for infinite terms.

$$\frac{3}{2}S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9}\cdots$$

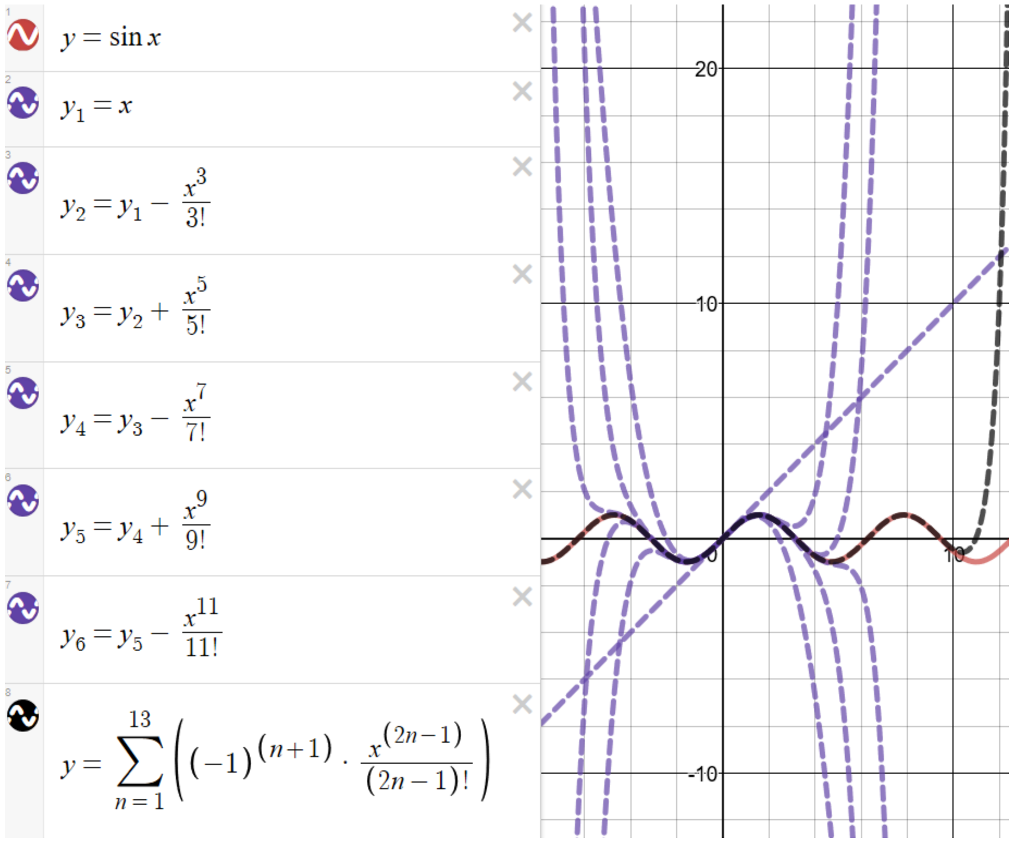
Concept Sheets: Taylor Series

A Taylor Series is an infinite sum of polynomials which sums to equal a function.

This is typically intended to represent a trigonometric function.

And so there are Taylor Series representations of many types of trigonometric functions.

$$\sin x = \sum_{n=1}^{\infty}\left(\frac{(-1)^n x^{2n-1}}{(2n-1)!}\right)$$



Concept Sheets: *Taylor Series*

And so there are Taylor Series representations of many types of trigonometric functions, but the one in our interest is the arctan x or $\tan^{-1} x$ function.

This has its importance rooted in $\arctan 1 = \frac{\pi}{4}$

$$\tan^{-1} x = \sum_{n=1}^{\infty} \left(\frac{-1^{n-1} x^{2n-1}}{2n-1} \right)$$

$$\frac{\pi}{4} = \tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots$$

Competitive Concept Sheets: Infinite Series

Key Concepts: Geometric Formula, Derivation

Problems: AMC 10A #18, 12A #24

Walter Johnson Math Team

Concept Sheets:
Geometric Series

Infinite geometric series are a key tool in many competitive problems, ranging in application.

There is one generic formula you will need to understand, and two forms of it.

$r \in (0,1)$

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$$

Also, it represented as

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

Concept Sheets: Geometric Series

$$r \in (0,1) \qquad \sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$$

If we substitute $^1/_p$ for r, we will find the new second important representation of the geometric series.

The different forms will be applied in different scenarios.

$$\sum_{n=1}^{\infty}\left(\frac{1}{p}\right)^n = \sum_{n=1}^{\infty}\frac{1}{p^n} = \frac{\frac{1}{p}}{1-\frac{1}{p}} = \frac{\frac{p}{p}}{p-1} = \frac{1}{p-1}$$

Concept Sheets: Geometric Series Derivation

The Geometric Series has an interesting origin in the asymptotes of exponential functions.

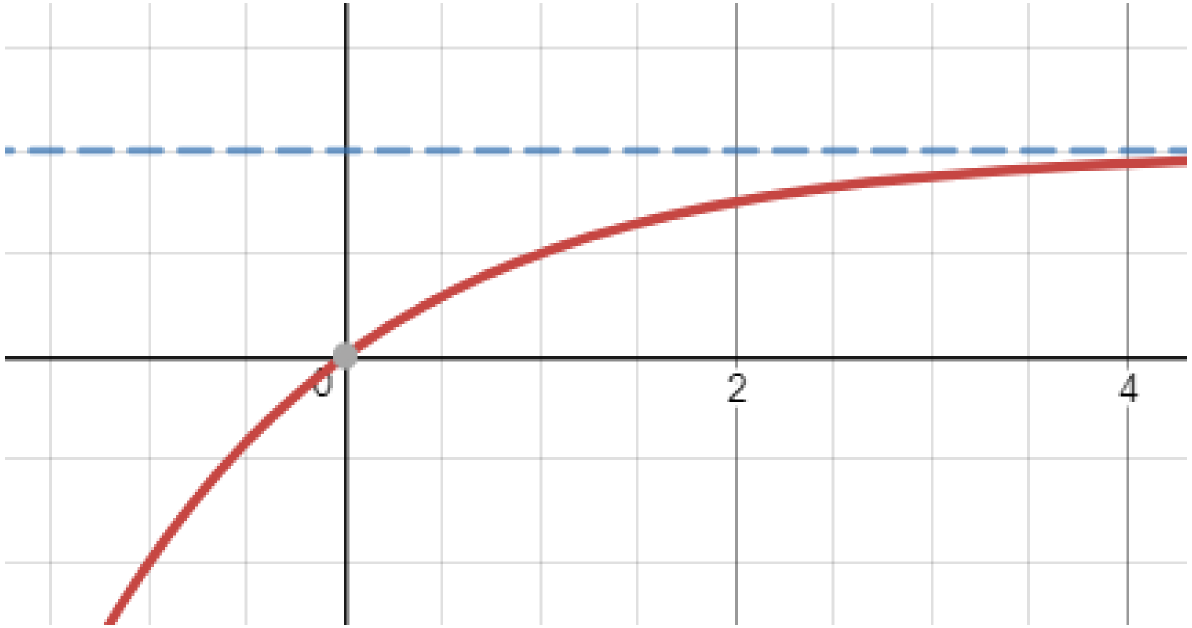
Consider $y = 1 - 2^{-x}$:

We see that

$$\lim_{x \to \infty} (1 - 2^{-x}) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$

$$y(1) = \frac{1}{2}, y(2) = \frac{3}{4}, y(3) = \frac{7}{8}$$

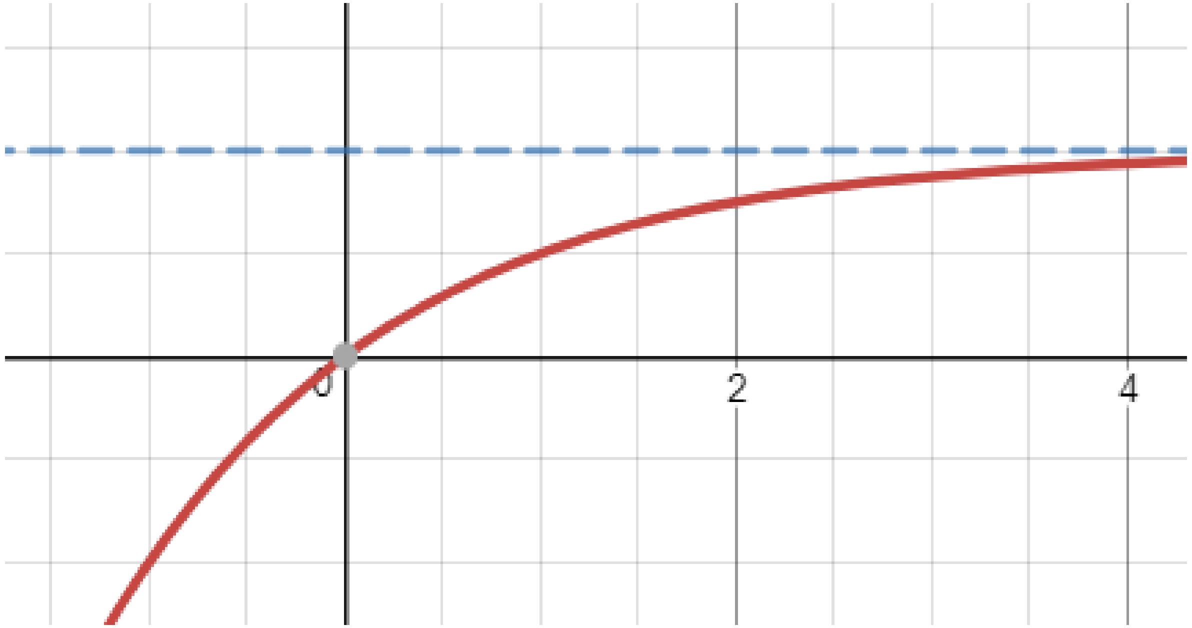
$$y(1) = \frac{1}{2}, y(2) = \frac{1}{2} + \frac{1}{4}, y(3) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$$



Concept Sheets:
Geometric Series Derivation

Fortunately this holds under all Geometric Series.

$$\lim_{x\to\infty}(1-2^{-x}) = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n} = \frac{1}{p-1}$$



Concept Sheets:

Geometric Series Derivation

Like all things in Mathematics, Geometric Series pop up in fascinating places in higher mathematics.

For example, this derivation for the arctan x derivative

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 \cdots$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} \cdots$$

$$= \int \left(1 - x^2 + x^4 - x^6 \cdots\right) dx$$

$$= \int \left(1 + (-x^2)^1 + (-x^2)^2 + (-x^2)^3 \cdots\right) dx$$

$$= \int \frac{dx}{1-(-x^2)} = \int \frac{dx}{1+x^2}$$

Problems: AIME #1, Complex Numbers GD

For some positive integer k, the repeating base-k representation of the (base-ten) fraction $\frac{7}{51}$ is $0.\overline{23}_k = 0.23232323232323\ldots_k$

What is k?

Amelia has a coin that lands on heads with probability $\frac{1}{3}$, and Blaine has a coin that lands on heads with probability $\frac{2}{5}$. Amelia and Blaine alternately toss their coins until someone lands head, and 'wins'. Amelia goes first. The probability that Amelia wins is $\frac{p}{q}$.

What is $q - p$?

Problems: *Complex Numbers GD*

For some positive integer k, the repeating base-k representation of the (base-ten) fraction $\frac{7}{51}$ is

$$0.\overline{23}_k =$$

$$0.2323232323232323\ldots_k$$

What is k?

We see that the representation of numbers in specified bases is perfect for representing them in a Geometric Series

$$0.232323\ldots = \frac{2}{k} + \frac{3}{k^2} + \frac{2}{k^3} + \frac{3}{k^4} \cdots$$

$$0.232323\ldots = \frac{2k+3}{k^2} + \frac{2k+3}{k^4} \cdots$$

Problems: Complex Numbers GD

For some positive integer k, the repeating base-k representation of the (base-ten) fraction $\frac{7}{51}$ is

$$0.\overline{23}_k =$$

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What is k?

$$0.232323\ldots = \frac{2k+3}{k^2} + \frac{2k+3}{k^4}\cdots$$

$$\sum_{n=1}^{\infty} \frac{2k+3}{k^{2n}} = \frac{7}{51}$$

$$(2k+3)\sum_{n=1}^{\infty} \frac{1}{k^{2n}} = \frac{7}{51}$$

Problems: Complex Numbers GD

For some positive integer k, the repeating base-k representation of the (base-ten) fraction $\frac{7}{51}$ is

$$0.\overline{23}_k =$$

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What is k?

$$(2k+3)\sum_{n=1}^{\infty}\frac{1}{k^{2n}}=\frac{7}{51}$$

$$\frac{2k+3}{k^2-1}=\frac{7}{51}$$

$$-7k^2+102k+160=0$$

Problems: *Complex Numbers GD*

For some positive integer k, the repeating base-k representation of the (base-ten) fraction $\frac{7}{51}$ is

$0.\overline{23}_k =$

$0.23232323232323\ldots_k$

What is k?

$$-7k^2 + 102k + 160 = 0$$

$$(k - 16)(7k + 10) = 0$$

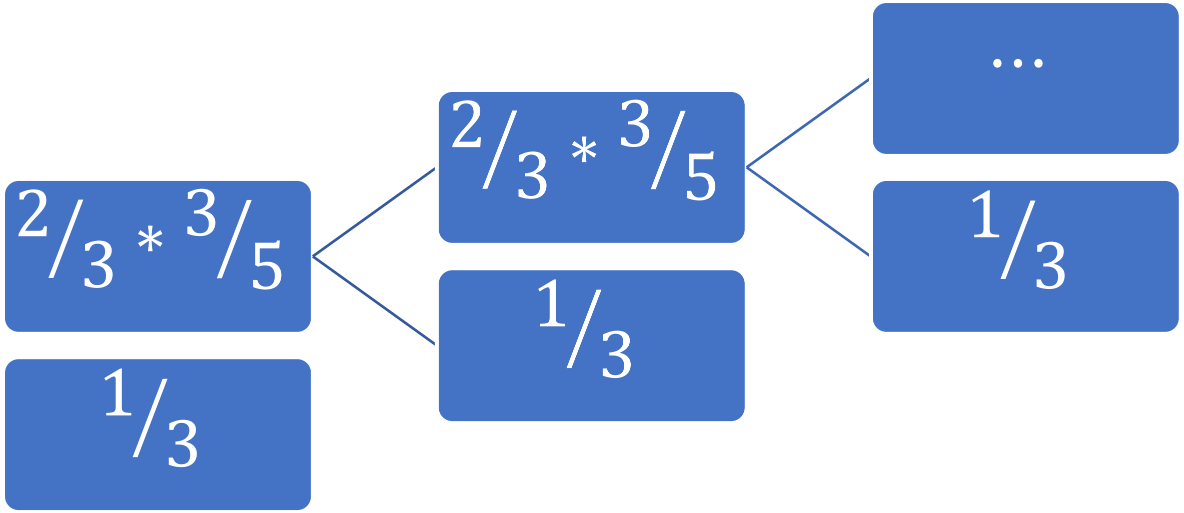
$$k = 16$$

Problems: *Complex Numbers GD*

Amelia has a coin that lands on heads with probability $\frac{1}{3}$*, and Blaine has a coin that lands on heads with probability* $\frac{2}{5}$*. Amelia and Blaine alternately toss their coins until someone lands head, and 'wins'. Amelia goes first. The probability that Amelia wins is* $\frac{p}{q}$*.*

What is $q - p$*?*

We can draw a tree to represent this probability



Problems: Complex Numbers GD

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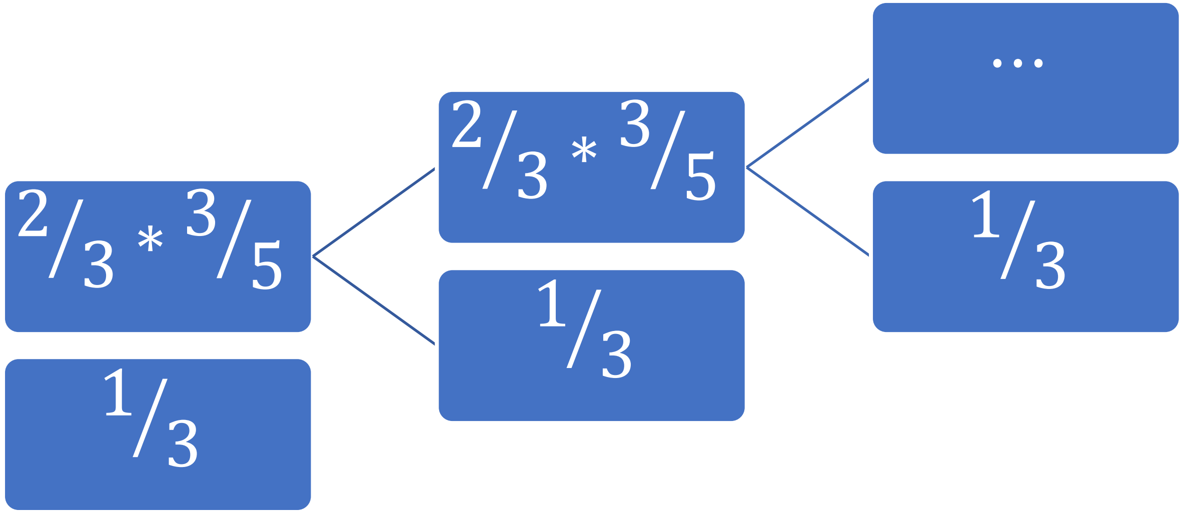

Now, we can represent this probability algebraically

$$\frac{p}{q}=\frac{1}{3}+\left(\frac{6}{15}\left(\frac{1}{3}+\frac{6}{15}\left(\frac{1}{3}+\frac{6}{15}(\cdots)\right)\right)\right)$$

Problems: *Complex Numbers GD*

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$$\frac{p}{q}=\frac{1}{3}+\left(\frac{6}{15}\left(\frac{1}{3}+\frac{6}{15}\left(\frac{1}{3}+\frac{6}{15}(\cdots)\right)\right)\right)$$

Expanding it out we get

$$\frac{p}{q}=\frac{1}{3}+\frac{6}{15}*\frac{1}{3}+\left(\frac{6}{15}\right)^2\frac{1}{3}+\left(\frac{6}{15}\right)^3\frac{1}{3}+\cdots$$

Problems: Complex Numbers GD

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What is $q - p$?

$$\frac{p}{q} = \frac{1}{3} + \frac{6}{15} * \frac{1}{3} + \left(\frac{6}{15}\right)^2 \frac{1}{3} + \left(\frac{6}{15}\right)^3 \frac{1}{3} + \cdots$$

$$\frac{p}{q} = \frac{1}{3} * \sum_{n=0}^{\infty} \left(\frac{6}{15}\right)^n = \frac{1}{3} * \frac{1}{1 - {}^{6}/_{15}}$$

$$\frac{p}{q} = \frac{1}{3} * \frac{1}{{}^{9}/_{15}} = \frac{1}{3} * \frac{15}{9} = \frac{5}{9}$$

$$q - p = 9 - 5 = 4$$