Algebra Problem Set #1

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May 2021

Problems are ordered from easiest to hardest difficulty, with high probability. None of the problems require a calculator, calculus, analysis, or an abacus. If you have any questions, just ask!

1

Find the largest value of x for which

$$x^2 + y^2 = x + y$$

has a solution, if x and y are real.

2

For positive real numbers $x \neq 1$ simplify

$$\frac{1}{\log_3 x} + \frac{1}{\log_4 x} + \frac{1}{\log_5 x}$$

3

Simplify

$$\sqrt{3 + 2\sqrt{2}}$$

4

Given there is only one solution for x in the equation

$$4x^2 + ax + 8x + 9 = 0$$

Find the sum of all possible values a.

5

For what value of x does

$$\log_{\sqrt{2}} \sqrt{x} + \log_2 x + \log_4 \left(x^2\right) + \log_8 \left(x^3\right) + \log_{16} \left(x^4\right) = 40?$$

6

Find all solutions to

$$x^{\log x} = \frac{x^3}{100}$$

7

Let $f(x) = x^2(1-x)^2$. What is the value of the sum

$$\sum_{n=1}^{2018} \left((-1)^n \times f\left(\frac{n}{2019}\right) \right)$$

8

What is the minimum value of the product

$$\prod_{i=1}^{6} \frac{a_i - a_{i+1}}{a_{i+1} - a_{i+3}}$$

given that $(a_1, a_2, a_3, a_4, a_5, a_6)$ is a permutation of $(1, 2, 3, 4, 5, 6)$? (note, $a_7 = a_1, a_8 = a_2 \cdots$)

9

All the roots of polynomial

$$z^6 - 10z^5 + Az^4 + Bz^3 + Cz^2 + Dz + 16$$

are positive integers. What is the value of B?

10

Compute

$$\sum_{k=2}^{\infty} \frac{k-3}{k(k^2-1)}$$

11

Define

$$P(x) = \left(x - 1^2\right)\left(x - 2^2\right)\left(x - 3^2\right) \cdots \left(x - 100^2\right)$$

How many integers n are there such that $P(n) \leq 0$?

12

If $60^a = 3$ and $60^b = 5$, then find

$$12^{\left(\frac{1-a-b}{2-2b}\right)}$$

13

A sequence of numbers is defined recursively by $a_1 = 1$, $a_2 = \frac{3}{7}$, and

$$a_n = \frac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}}$$

for all $n \geq 3$. Then a_{2019} can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p+q$?

14

The equations $x^3+5x^2+px+q=0$ and $x^3+7x^2+px+r=0$ have two roots in common. If the third root of each equation is represented as x_1 and x_2, respectively, compute the ordered pair (x_1, x_2).

15

Let $a+ar_1+ar_1^2+ar_1^3+\cdots$ and $a+ar_2+ar_2^2+ar_2^3+\cdots$ be two different infinite geometric series of positive numbers with the same first term. The sum of the first series is r_1, and the sum of the second series is r_2. What is r_1+r_2?

16

Solve

$$\sqrt[3]{3x-5} + \sqrt[3]{2x-4} = \sqrt[3]{5x-9}$$

17

The function f defined by $f(x) = \frac{ax+b}{cx+d}$ where a, b, c and d are nonzero real numbers, has the properties $f(19) = 19$, $f(97) = 97$, and $f(f(x)) = x$ for all values except $\frac{-d}{c}$. Find the unique number that is not in the range of f.

18

Let $f(x)$ be a function satisfying the equation $f(f(x)) = 3x+1$. If we are told that $f(0) = 223$, then what must be the value of $f(4)$?

19

Let $g(x)$ be a polynomial with leading coefficient 1, whose three roots are the reciprocals of the three roots of $f(x) = x^3+ax^2+bx+c$, where $1 < a < b < c$. What is $g(1)$ in terms of a, b and c?

20

Consider the polynomial

$$P(x) = \prod_{k=0}^{10}(x^{2^k}+2^k) = (x+1)(x^2+2)\cdots(x^{1024}+1024)$$

The coefficient of x^{2012} is equal to 2^a. What is a?

21

Let the sequence $\{a_i\}_{i=0}^{\infty}$ be defined by $a_0 = \frac{1}{2}$ and $a_n = 1+(a_{n-1}-1)^2$. Find the product

$$\prod_{i=0}^{\infty} a_i = a_0 a_1 a_2 \ldots$$

22

Let $a_1 = 3$, $a_2 = 8$ and $a_n = \sum_{k=1}^{n-1} a_k$ for $n > 2$. The value of $\sum_{n=1}^{\infty} \frac{1}{a_n}$ can be written as a common fraction $\frac{p}{q}$. Compute $p+q$.

23

Let a and b be real numbers, and let r, s, and t be the roots of $f(x) = x^3+ax^2+bx-1$. Also, $g(x) = x^3+mx^2+nx+p$ has roots r^2, s^2, and t^2. If $g(-1) = -5$, find the maximum possible value of b.

24

Compute

$$\sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\cdots\sum_{a_7=0}^{\infty}\frac{a_1+a_2+\cdots+a_7}{3^{a_1+a_2+\cdots+a_7}}$$

25

For distinct complex numbers $z_1, z_2, \ldots, z_{673}$, the polynomial

$$(x-z_1)^3(x-z_2)^3\cdots(x-z_{673})^3$$

can be expressed as $x^{2019}+20x^{2018}+19x^{2017}+g(x)$, where $g(x)$ is a polynomial with complex coefficients and with degree at most 2016. The value of

$$\left|\sum_{1\leq j<k\leq 673} z_j z_k\right|$$

can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.