Guided Discussion: Intro To Topology

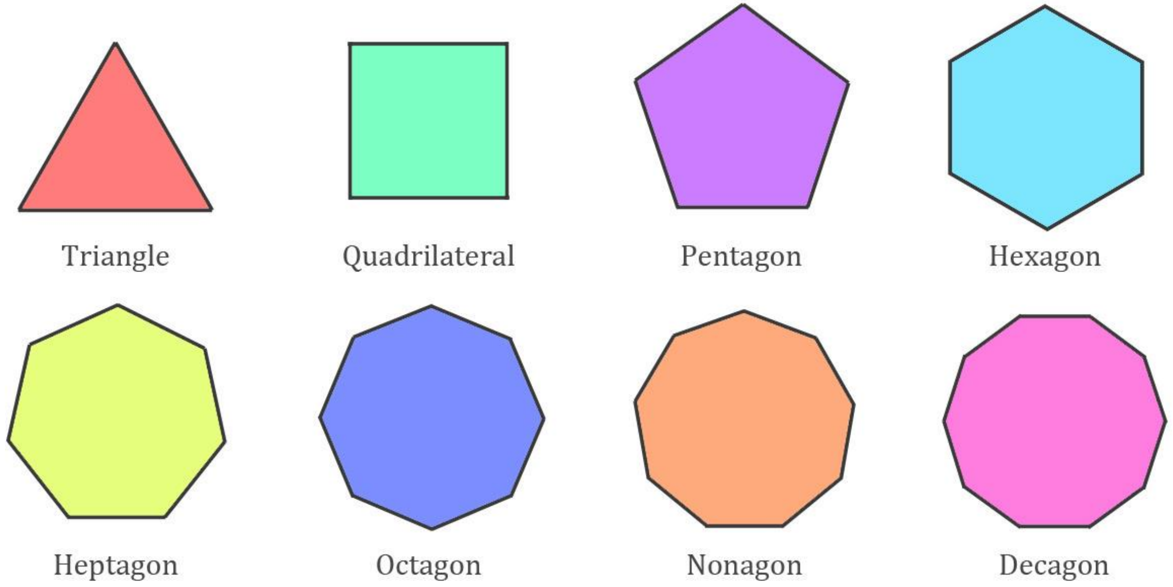
No Knot Theory! That's for sure

Walter Johnson Math Team

Guided Discussion: Geometry

I apologize, there's still some geometry that is interesting before topology comes into play! But it leads right into it.

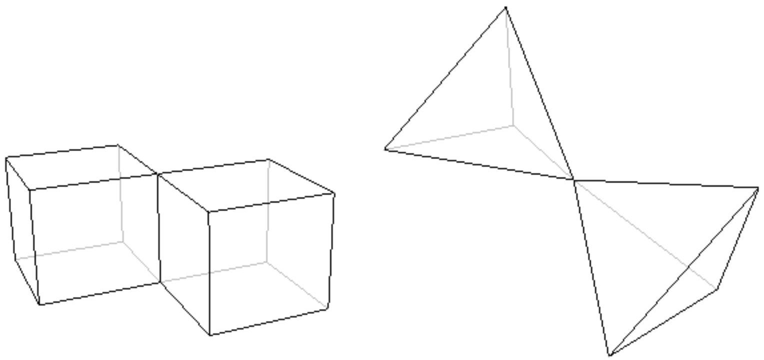
- *We know what regular polygons are*
- *Equal length sides, equal angles.*



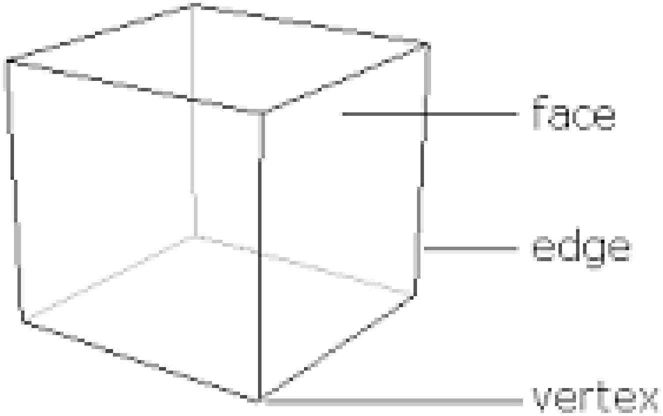
Guided Discussion: Geometry

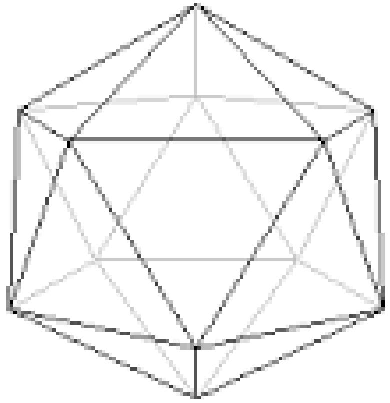
Regular Polygons *have equal side lengths and equal angles.*

Not *Polyhedrons:*



Polyhedrons:





- *A polyhedron is a loosely defined object in geometry. It is a 3 dimensional solid which is typically convex.*
- *Also defined so that if one were to fill it with water, it would only have to be filled once.*

Guided Discussion: *Geometry*

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A ***Polyhedron*** *is a loosely defined object in geometry. It is a convex 3-dimensional object in geometry. If needed to fill with water, would only have to be filled once. No holes.*

- *The definition for a polyhedron was more implied historically, with rigorous definitions hard to pinpoint because exceptions were often found.*
- *But certain attributes of what mathematicians typically thought of polyhedrons because loosely associated with them over time.*

Guided Discussion: Geometry

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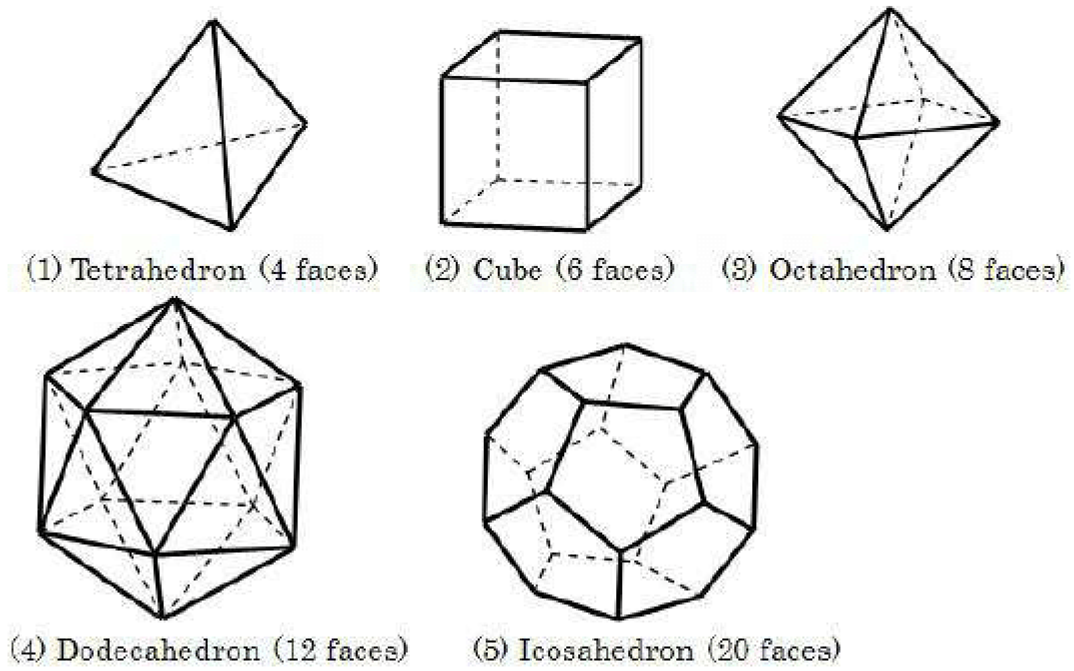
Tetrahedron *has 4 equilateral triangles as faces*

Icosahedron *has 20 regular pentagons as faces*

- *A* ***Regular Polyhedron*** *is a polyhedron which has faces which are all congruent regular polygons.*
- *There are* ***only 5***



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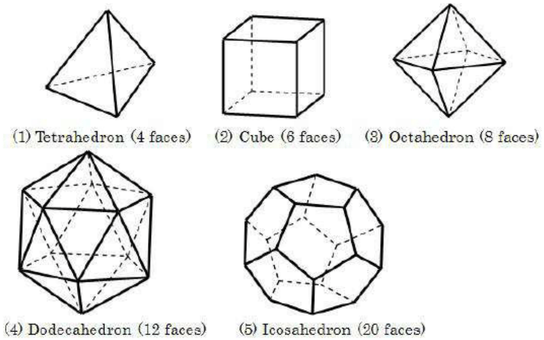

- *Historically, these objects were heavily studied, through famous mathematicians such as Pythagoras, Plato, Kepler, Euclid, and Euler.*
- *These objects were idolized by mathematicians, making a variety of false theories about the universe stemming from these objects.*

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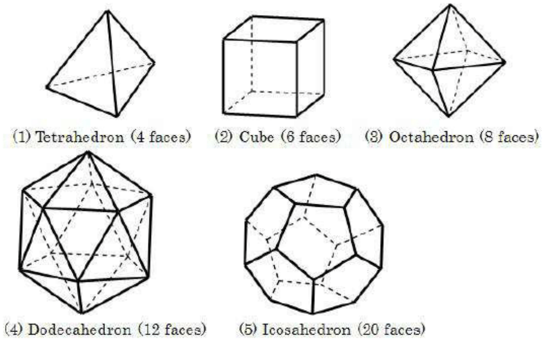
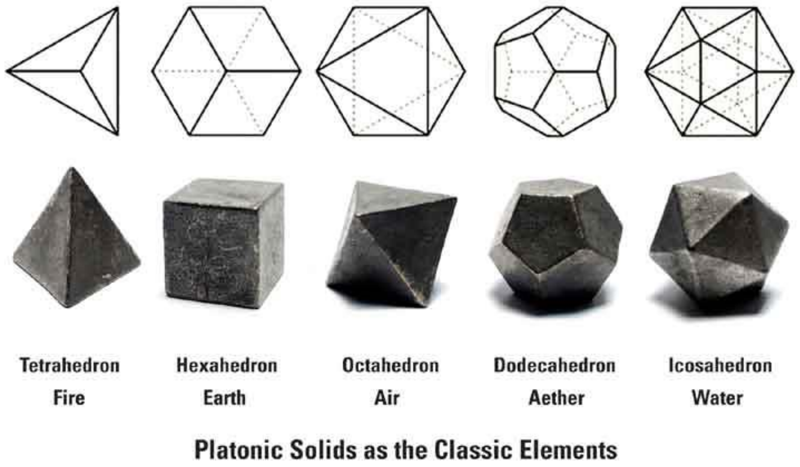
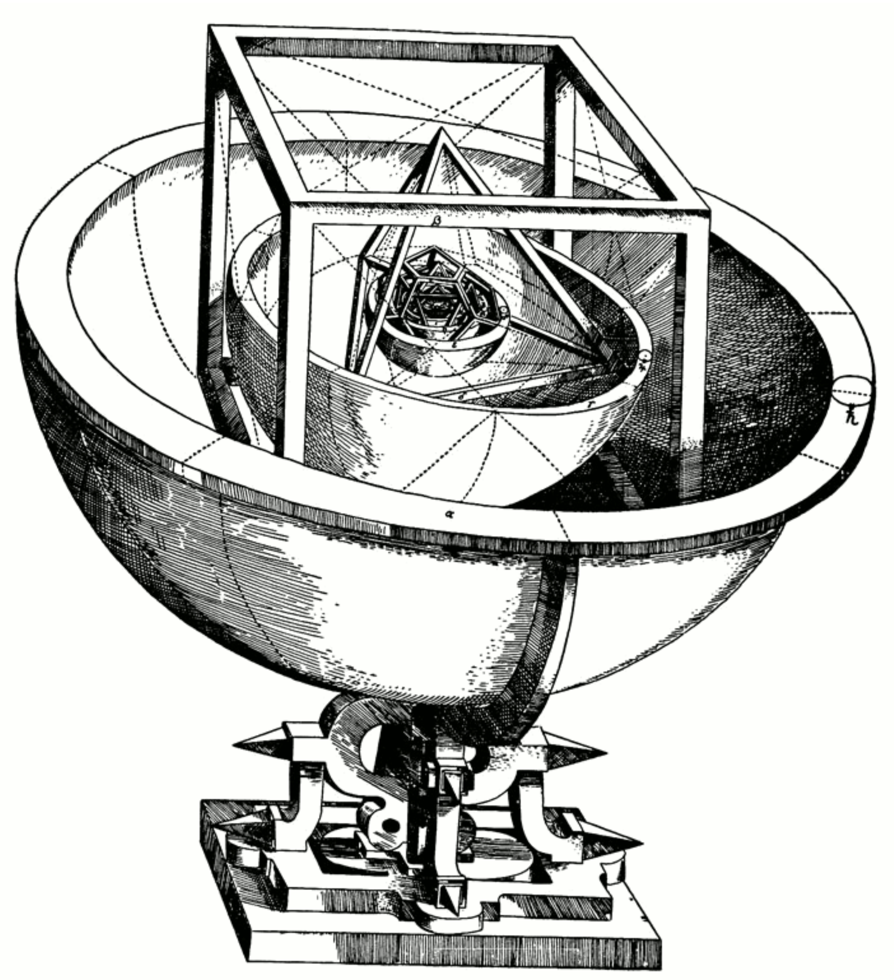


- *Plato believed the "elements" were constructed of these 4 regular polyhedrons, with the universe as the dodecahedron.*
- *This was widely accepted until the "Skeptical Chemist" in 1661 and the modern theory of chemistry.*



Guided Discussion: *Geometry*



(4) Dodecahedron (12 faces) (5) Icosahedron (20 faces)

- *Kepler constructed a diagram of the universe revolving around these regular polyhedrons*
- *He tried to justify them with actual measurements of the planet's paths in the sky but eventually gave up pushing this theory.*

Guided Discussion: Geometry

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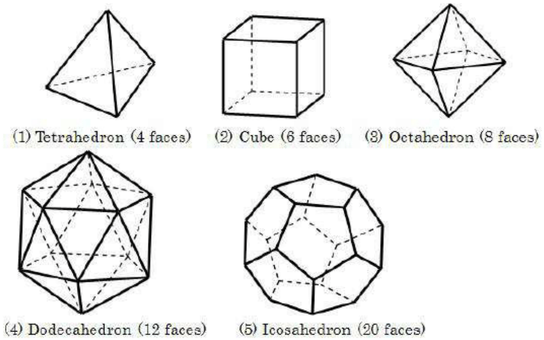
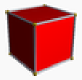
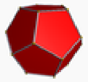
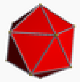


- *But real phenomena occurred with regular polyhedrons as well*
- *What do we notice about the number of vertices, edges, and faces of these figures?*

Name	Image	Vertices V	Edges E	Faces F
Tetrahedron		4	6	4
Hexahedron or cube		8	12	6
Octahedron		6	12	8
Dodecahedron		20	30	12
Icosahedron		12	30	20

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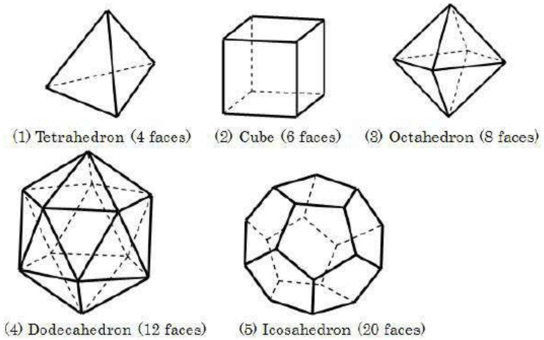

- *But real phenomena occurred with regular polyhedrons as well*
- *What do we notice about the number of vertices, edges, and faces of these figures?*
- *There is symmetry!*

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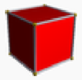
Hexahedron or cube		8	12	6
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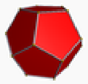
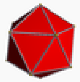
Dodecahedron		20	30	12
Icosahedron		12	30	20

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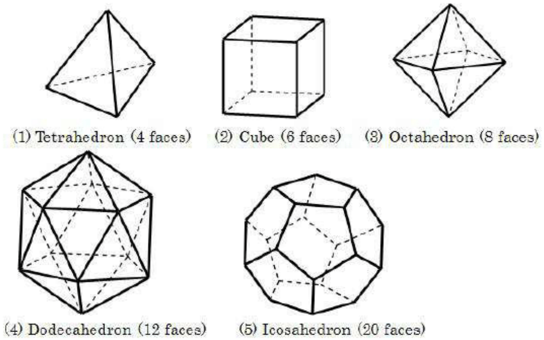

- *This symmetry explains why the regular polyhedrons can fit inside their pairs.*

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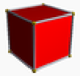
Hexahedron or cube		8	12	6
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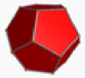
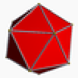
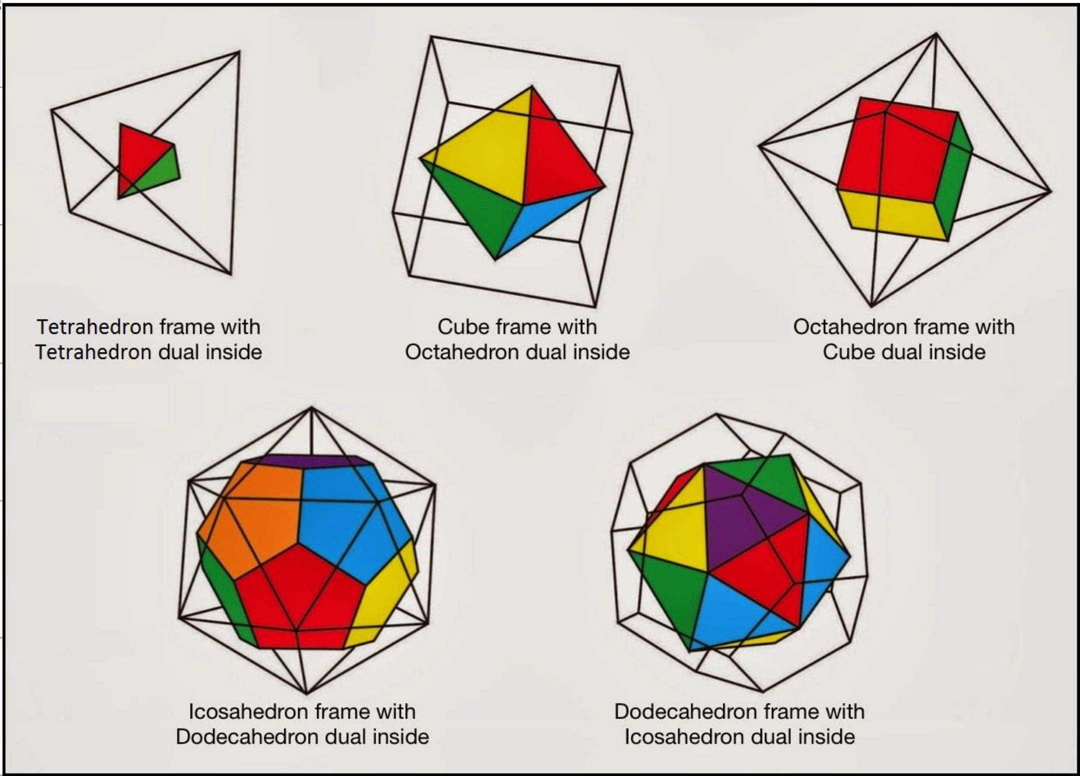
Dodecahedron		20	30	12
Icosahedron		12	30	20



Guided Discussion: *Euler's Number*

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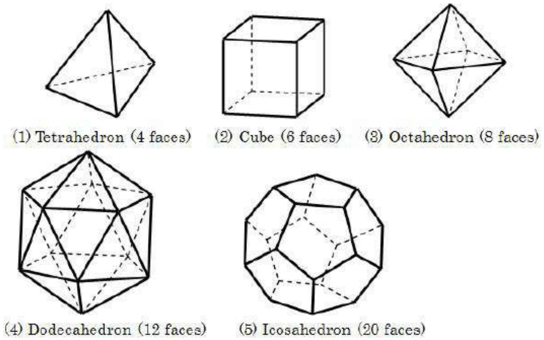

- *Euler made a major discovery with polyhedrons, by looking at them in a context no one had ever done before, by acknowledging that they had edges.*
- *With this new eye of looking at polyhedrons, he came to the now widely known Euler's Formula*

Guided Discussion:
Euler's Number

$$V - E + F = 2$$

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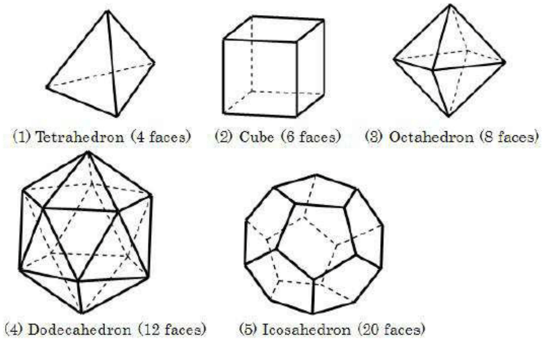
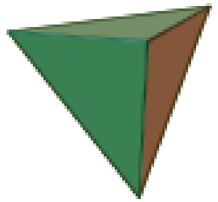
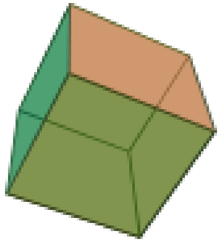
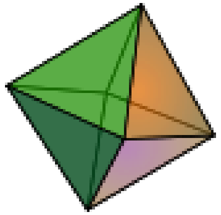
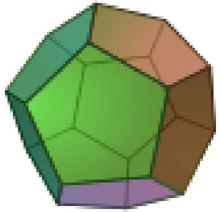
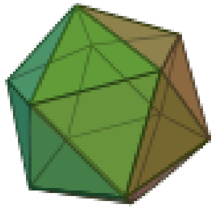


- *Probably not the formula you were expecting, right?*
- *This relates the number of vertices, edges, and faces of any polyhedron.*
- *It's fascinating. And crazy.*
- *We're now going to count some*

Guided Discussion: *Euler's Number*

Tetrahedron V=4, E=6, F=4	Cube or hexahedron V=8, E=12, F=6	Octahedron V=6, E=12, F=8	Dodecahedron V=12, E=30, F=20	Icosahedron V=20, E=30, F=12
				

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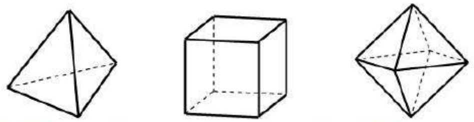
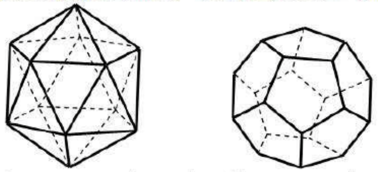


(1) Tetrahedron (4 faces) (2) Cube (6 faces) (3) Octahedron (8 faces)



(4) Dodecahedron (12 faces) (5) Icosahedron (20 faces)

$$V - E + F = 2$$

- *Keep in mind, just because we are testing them for the regular polyhedrons, this formula works for __all__ polyhedrons*
- *We can also imagine this as a projection onto the 2d plane, and a description from there*

Guided Discussion: Euler's Formula

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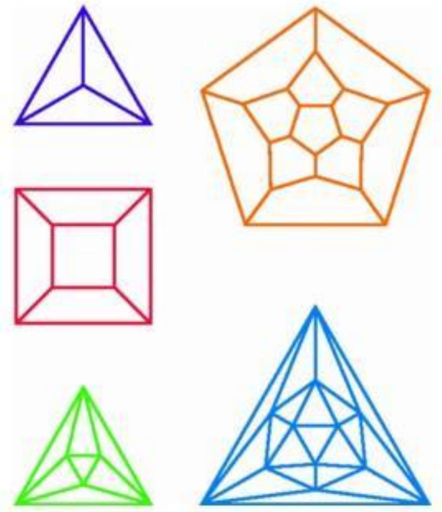
Euler's Formula: $\boldsymbol{V - E + F = 2}$ *for polyhedrons.*

$$V - E + F = 2$$

- *We can also imagine this as a projection onto the 2d plane, and a description from there*
- *Counting the outside as a face, we see this formula holds.*



Guided Discussion: Euler’s Formula

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***Euler's Formula:** $V - E + F = 2$ for polyhedrons.*

- *The number 2 in the Euler formula only holds for Polyhedrons, which do not have holes.*
- *This includes objects such as the sphere.*
- *Although we cannot count points and edges and faces on a sphere, we can say that the sphere approximates and has the same topology as a polyhedron, and thus has the same **Euler Number***

Guided Discussion: Euler's Formula

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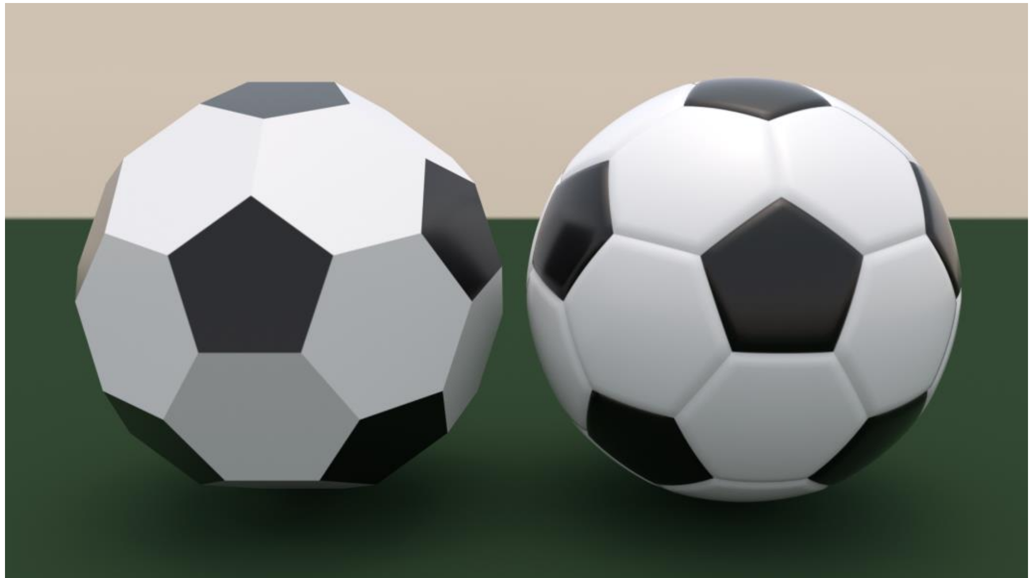

Guided Discussion: Topology

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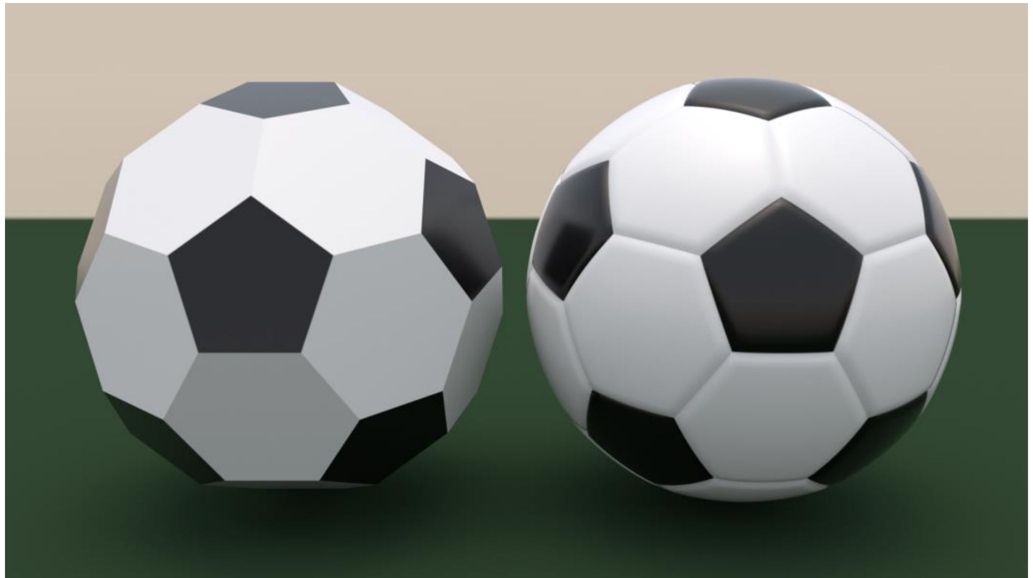
***Euler's Formula:** $V - E + F = 2$ for polyhedrons.*

- *Don't take this idea of deformation for granted!*
- *Remember, Topology relies on the idea of smooth deformations. But not all deformations are smooth.*



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Euler's Formula: $\boldsymbol{V - E + F = 2}$ *for polyhedrons.*

- *So we see the* ***Euler Number*** *for all objects topologically equivalent to the sphere have an Euler Number of 2. Meaning*

$$V - E + F = 2$$

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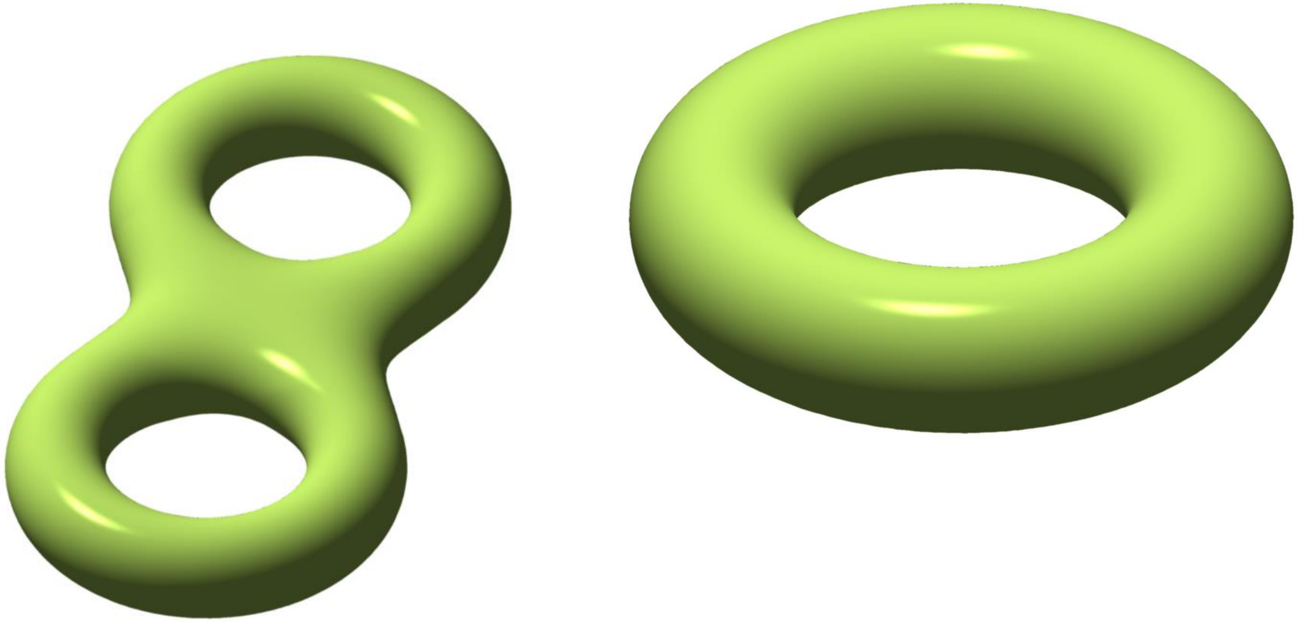
***Euler's Formula:** $V - E + F = 2$ for polyhedrons.*

- *Well what about objects which do have holes?*
- *What about 3 dimensional objects that aren't spheres?*
- *How do we find their Euler Number?*



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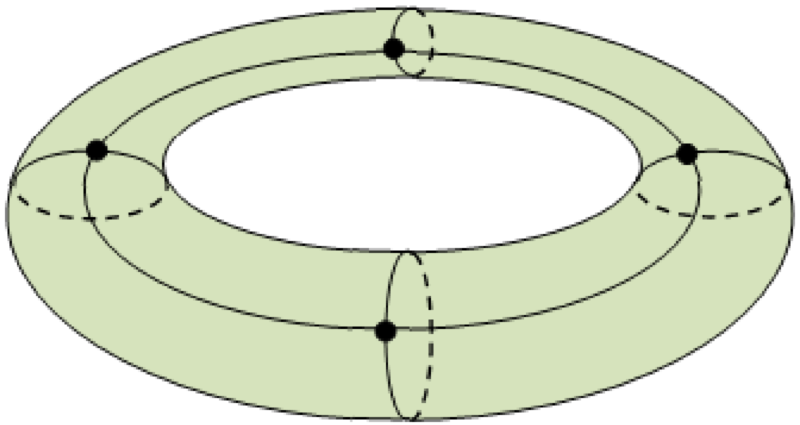
Euler's Number: $V - E + F$

- *We can find rigid objects which are topologically equivalent.*
- *Here we see a torus partitioned into faces, edges, and points.*
- *Counting the relation, what is this figure's Euler Number?*



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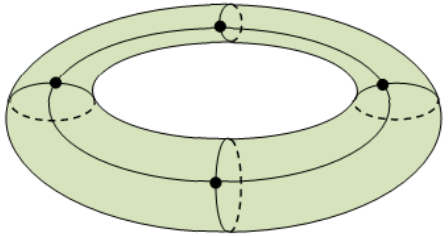
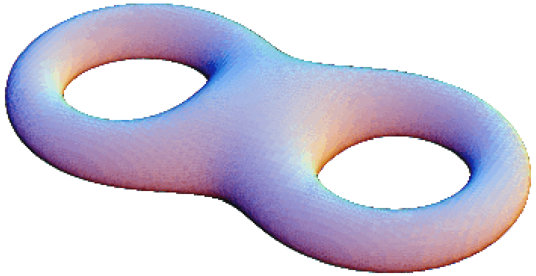
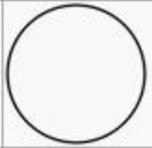


- *2*
- *What do you think the Euler Number will be for this figure?*



Guided Discussion: Topology

Name	Image	Euler characteristic
Interval		1
Circle		0
Disk		1
Sphere		2
Torus (Product of two circles)		0
Double torus		−2
Triple torus		−4

- *What do you think the Euler Number will be for this figure?*
- ***-2***
- *We see this relation continues for objects with successive numbers of holes.*

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Euler's Formula: $\boldsymbol{V - E + F = 2}$ *for polyhedrons.*

- *Scroll back a bit…*
- *We are saying there are only 5 regular polyhedrons. How can we prove that?*

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- *Were going to do that with Euler's Theorem.*

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- *Given Euler's Formula, we look at what would define a regular polyhedron. So we start with*

$$V - E + F = 2$$

- *Because we're dealing with a regular polyhedron, we're looking at the fact that each face has the same number of edges. Call this* n

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Guided Discussion: Topology

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A ***Polyhedron*** *is a loosely defined object in geometry. It is a convex 3-dimensional object in geometry. If needed to fill with water, would only have to be filled once.*

A ***Regular Polyhedron*** *is a polyhedron which has congruent regular polygonal faces.*

Euler's Formula: $\boldsymbol{V - E + F = 2}$ *for polyhedrons.*

- *Because we're dealing with a regular polyhedron, we're looking at the fact that each face has the same number of edges. Call this* n
- *We see that* $n \geq 3$
- *We also see that the number of edges meeting at a vertex, let's call this* m*, must also be* $m \geq 3$
- *And sense each edge is shared by two faces on the polyhedron, we have*

$$E = \frac{1}{2}Fn$$

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- *And sense each face contributes* n *vertices, and each vertex meet at* m *faces, we have*

$$V = \frac{Fn}{m}$$

Plugging these into Euler's formula we find the following

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- *Plugging these into Euler's formula we find the following*

$$V - E + F = 2$$

$$\frac{Fn}{m} - \frac{Fn}{2} + F = 2$$

$$F\left(\frac{n}{m} - \frac{n}{2} + 1\right) = 2$$

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$$F\left(\frac{n}{m} - \frac{n}{2} + 1\right) = 2$$

$$F\left(\frac{2n - mn + 2m}{2m}\right) = 2$$

$$F = \frac{4m}{2n - mn + 2m}$$

And we know that $4m$ *and* F *have to be positive*

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$$F = \frac{4m}{2n - mn + 2m}$$

And we know that $4m$ and F must be positive, so

$$2n - mn + 2m > 0$$

And we see the only satisfying pairs of integers are

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$$2n - mn + 2m > 0$$

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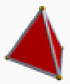
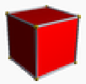
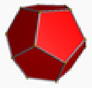
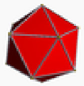
Name	Image	Vertices V	Edges E	Faces F
Tetrahedron		4	6	4
Hexahedron or cube		8	12	6
Octahedron		6	12	8
Dodecahedron		20	30	12
Icosahedron		12	30	20