Guided Discussion: Topology II

Lhuilier's Exceptions,

Walter Johnson Math Team

Guided Discussion: Lhuiler’s Exceptions

***Regular Polygons** have equal side lengths and equal angles.*

*A **Polyhedron** is a loosely defined object in geometry. It is a convex 3-dimensional object in geometry. If needed to fill with water, would only have to be filled once.*

*A **Regular Polyhedron** is a polyhedron which has congruent regular polygonal faces.*

***Euler's Formula:** $\boldsymbol{V-E+F=2}$ for polyhedrons.*

You may be shocked to hear, but Euler’s Formula does indeed have exceptions.

Initially, mathematician Lhuiler developed some of the first exceptions to this rule, but later, more exceptions were found, specifically by mathematician Louis Poinsot.

This branched off to the study of objects which appeared locally as surfaces.

Guided Discussion:

The start of this investigation in Topology looks at surfaces.

Surfaces are any object that look locally like a plane.

A sphere, disk, torus, and cylinder all exhibit these properties.

This brings about the idea of intrinsic and extrinsic dimension of a surface.

***Intrinsic Dimension** – the dimensionality of a topological object locally.*

***Extrinsic Dimension** – the dimensionality of a topological object globally.*

Guided Discussion:

Intrinsic Dimension *– the dimensionality of a topological object locally.*

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Surface *– Actually a Compact Surface, for which the surface is bounded and contains all its curves. The 2-dimensional plane is not one of these.*

This brings about the idea of intrinsic and extrinsic dimension of a surface.

An ant (very very small ant) would tell you that the surface of a donut (very large donut (torus)) is that of a plane. This ant would say the same about the surface of the earth. They are ***intrinsically*** *the same with respect to their dimensionality.*

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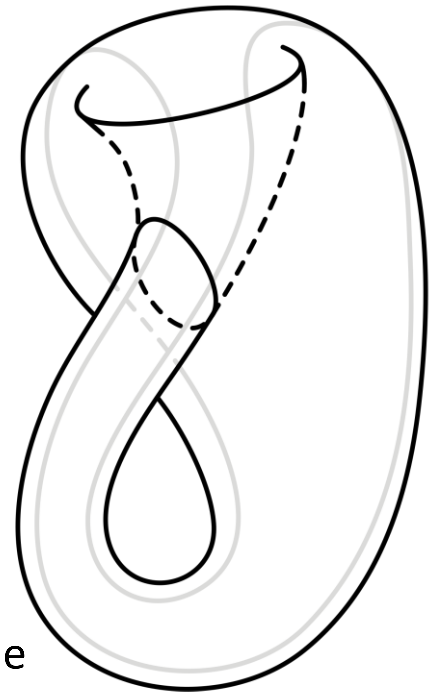
However, in order to construct a disk, or a piece of paper, we must have a space for it to exist in – an enveloping space. For a torus and a sphere, this is a three-dimensional space. For a piece of paper, this is a two-dimensional space.

For a Klein bottle, this 4 dimensional.



The Klein Bottle

Guided Discussion:

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Now, what can we do with this knowledge of surfaces having both intrinsic and extrinsic dimension?

We can construct them with intricate polygons enclosing the same intrinsic dimensioned-surface.

Felix Klein invented a way of constructing these, best imagined to be made of "pliable rubber material"

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Felix Klein invented a way of constructing these, best imagined to be made of "pliable rubber material"

This shows the construction of a Torus through rubber sheet topology. Arrows designate how the sides should connect.

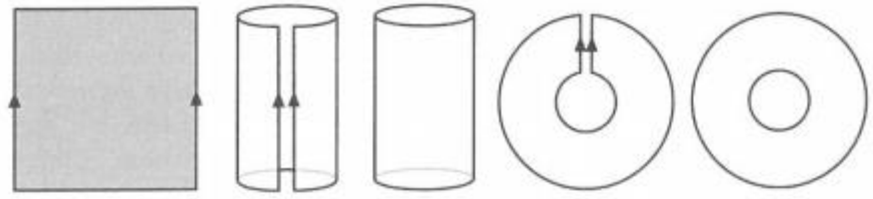

Figure 16.2. A cylinder or annulus.

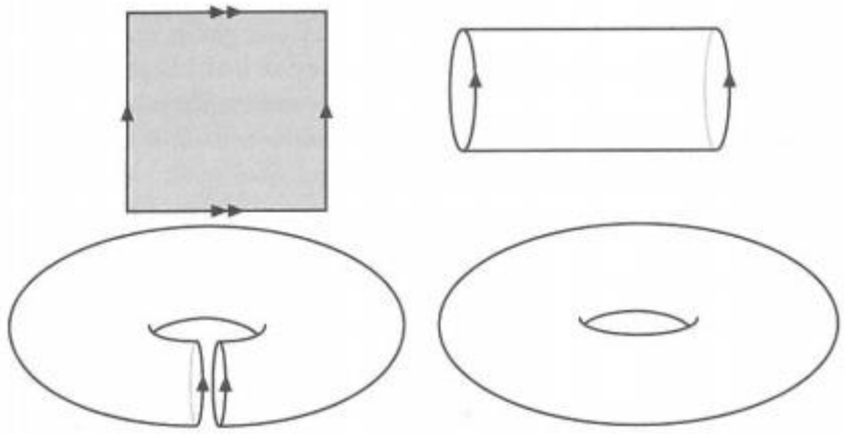

Figure 16.3. Making a torus from a square.

Guided Discussion:

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Rubber Sheet Topology *– The construction of a topological manifold with an intrinsic dimension of 2 but variable extrinsic dimension.*

Felix Klein invented a way of constructing these, best imagined to be made of "pliable rubber material"

This shows the construction of a Double-Torus through rubber sheet topology. Arrows designate how the sides should connect.

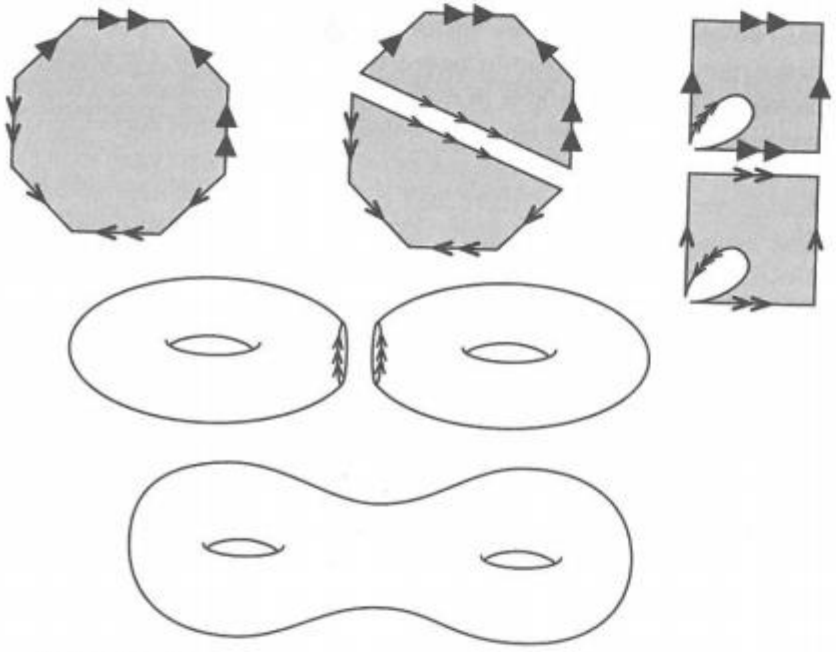

Figure 16.5. A double torus.

Guided Discussion:

These "surfaces" are topological manifolds. Specifically, they are intrinsically two dimensional manifolds.

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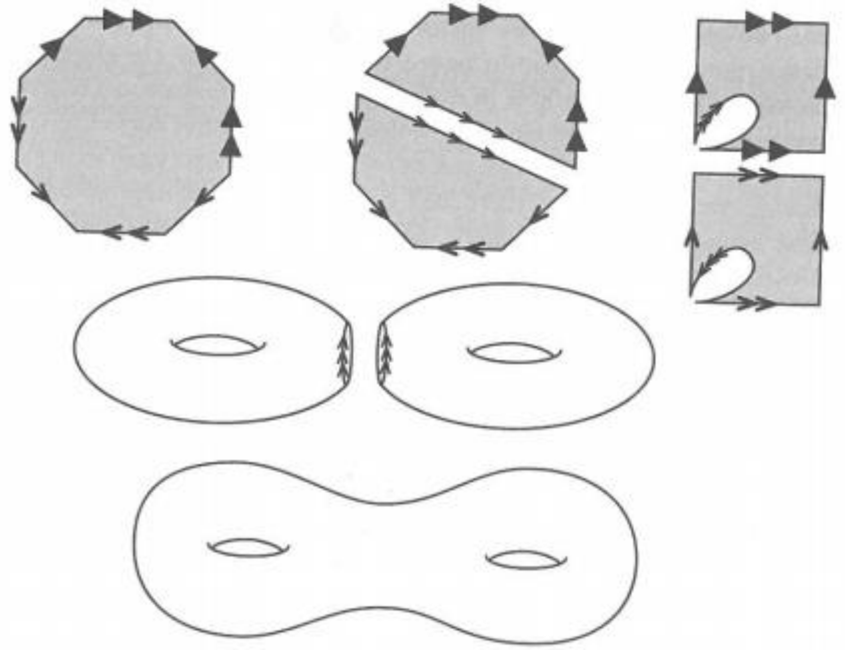

Figure 16.5. A double torus.

Guided Discussion:

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The famous Mobius Strip – the one-sided band, can be constructed this way.

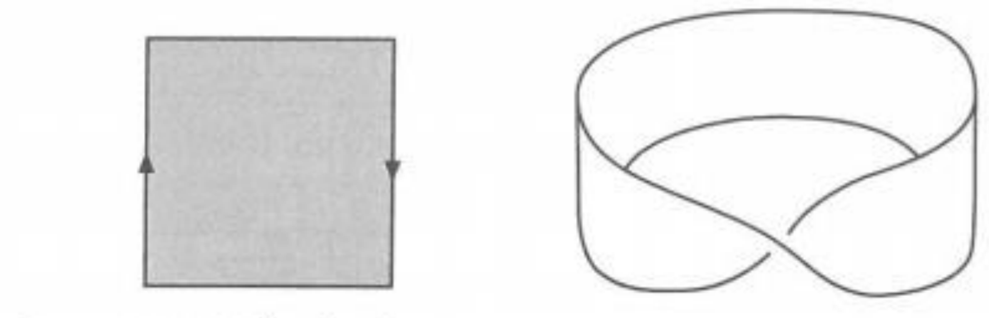

Figure 16.6. A Möbius band.

This is most famously portrayed in the artistic mathematician M.C. Escher, who's art delves deep into mathematics.
As one can see, a single ant can traverse the entire mobius strip without ever crossing the edge.

Guided Discussion: *Orientability*

This one-sidedness property was classified by Möbius as ***orientability.***

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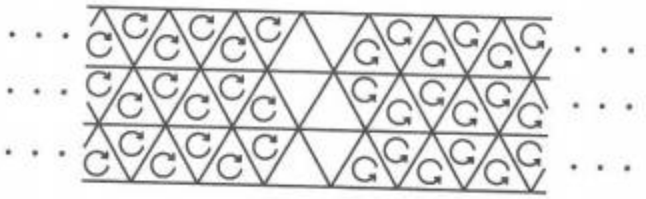

Figure 16.9. A traingulation of a Möbius band cannot be oriented.

An object, such as these oriented circles, is called an “indicatrix”

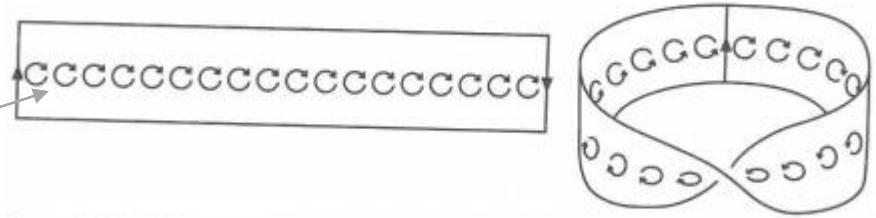

Figure 16.10. The Möbius band is not orientable.

*Note!! This is not the same as only having one side!! Mathematicians differentiate those two. How this is determined in those photos above, is if a circle, for instance, with a certain orientation is chosen, if moving this circle around the surface and back to its original position one can *guarantee* that it’s orientation is the same, then it is orientable.*

Guided Discussion: Orientability

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Why don’t mathematicians just classify orientability as the property for a surface to have one or two sides?

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Why don't mathematicians just classify orientability as the property for a surface to have one or two sides?

Because this loses meaning in higher extrinsic dimension.

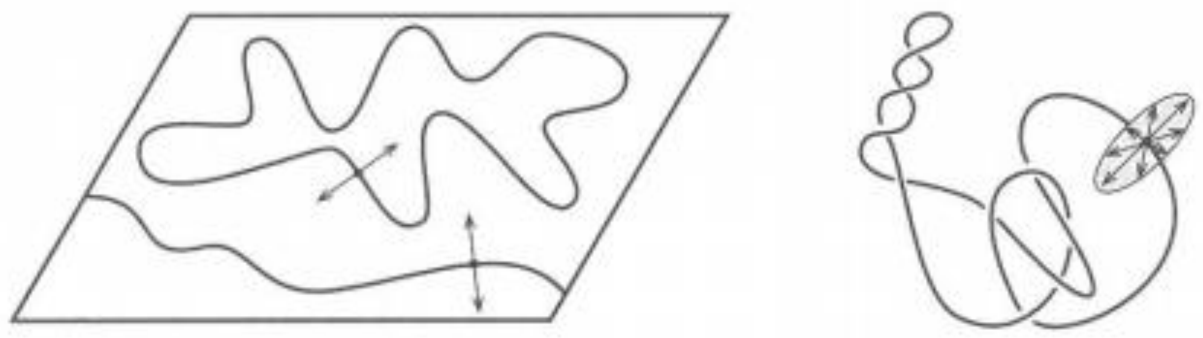

Figure 16.11. A curve in the plane is two sided, but one in 3-dimensional space has no sides.

Both curves are intrinsically one-dimensional, but their extrinsic dimensionality is different, for one, it is 2 dimensional, the other, 3.

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There are also many indicatrices, for example a normal vector, or a coordinate axis serve as some.

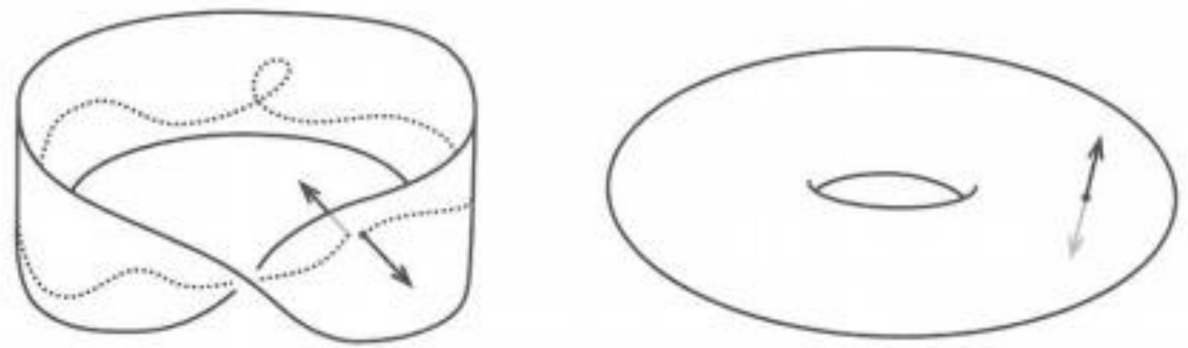

Figure 16.12. In 3-dimensional space the Möbius band is one-sided and the torus is two-sided.

Guided Discussion: Extrinsic Dimensionality

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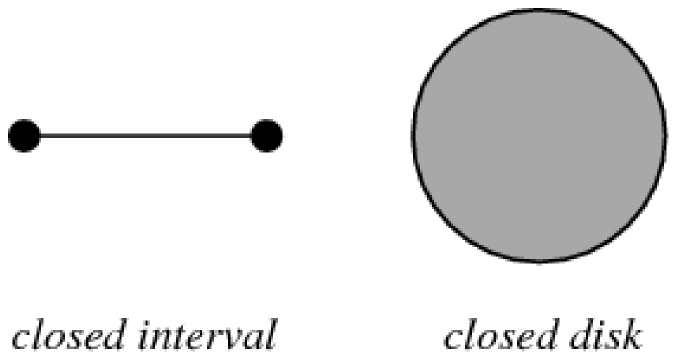
Now we've seen some intrinsically 2 dimensional surfaces which are extrinsically 1 and 2 dimensional (the line, the disk)



And we can imagine how these can exist in 1 or 2 dimensional extrinsic space.

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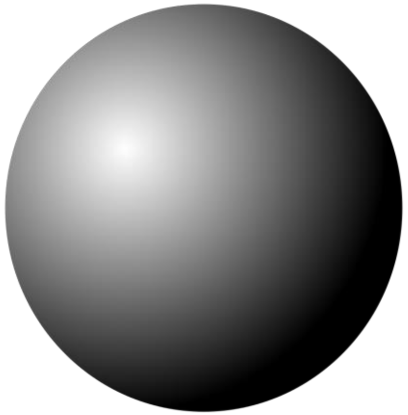
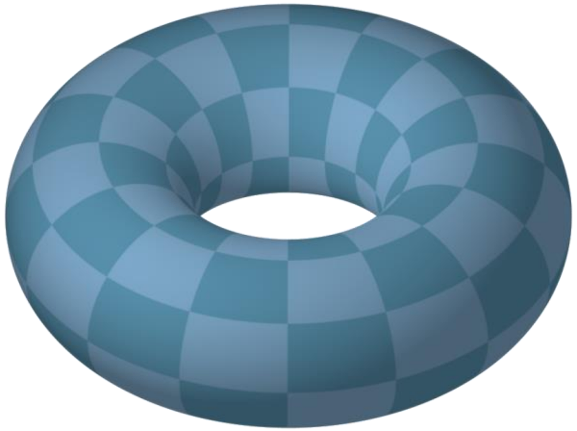
We've even looked at how intrinsically 2 dimensional surfaces can be extrinsically 3 dimensional, such as the sphere and the torus.





But what about higher dimensions?

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The Klein Bottle is the best example we have of this, which is intrinsically a two dimensional but extrinsically four dimensional!

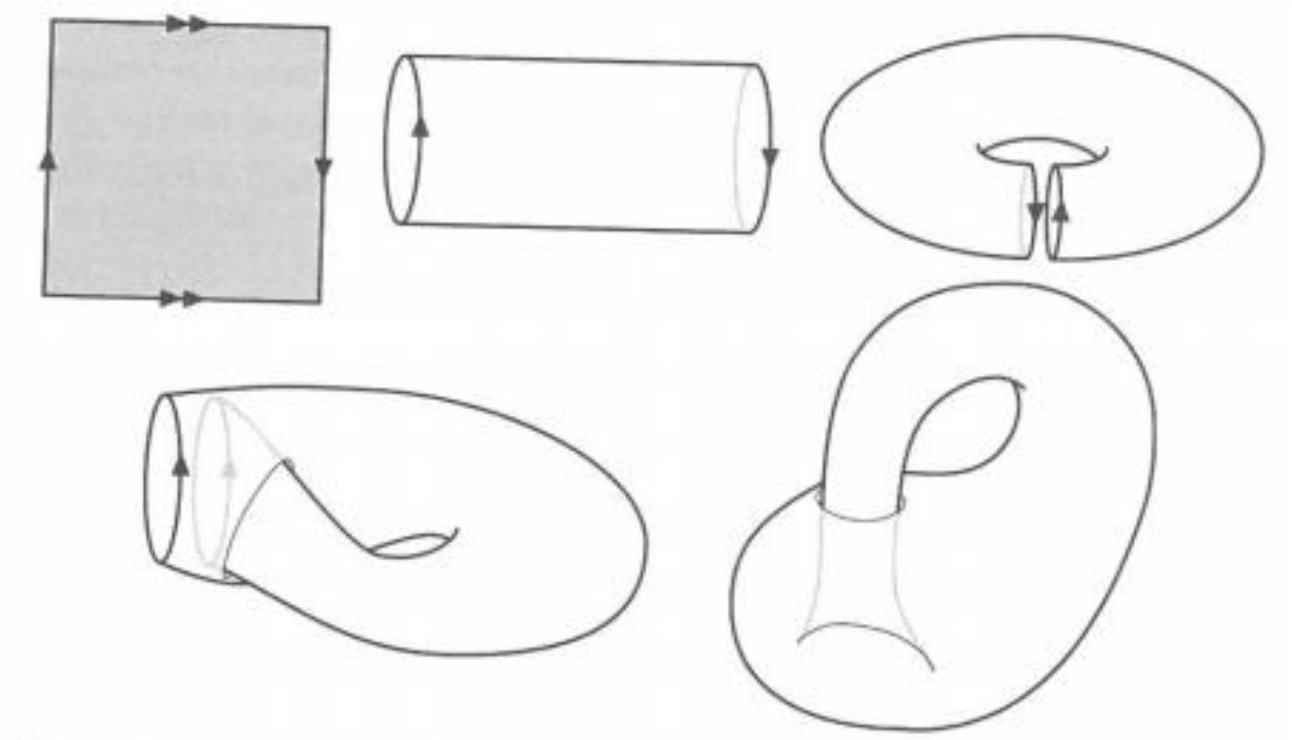

Figure 16.13. A Klein bottle.

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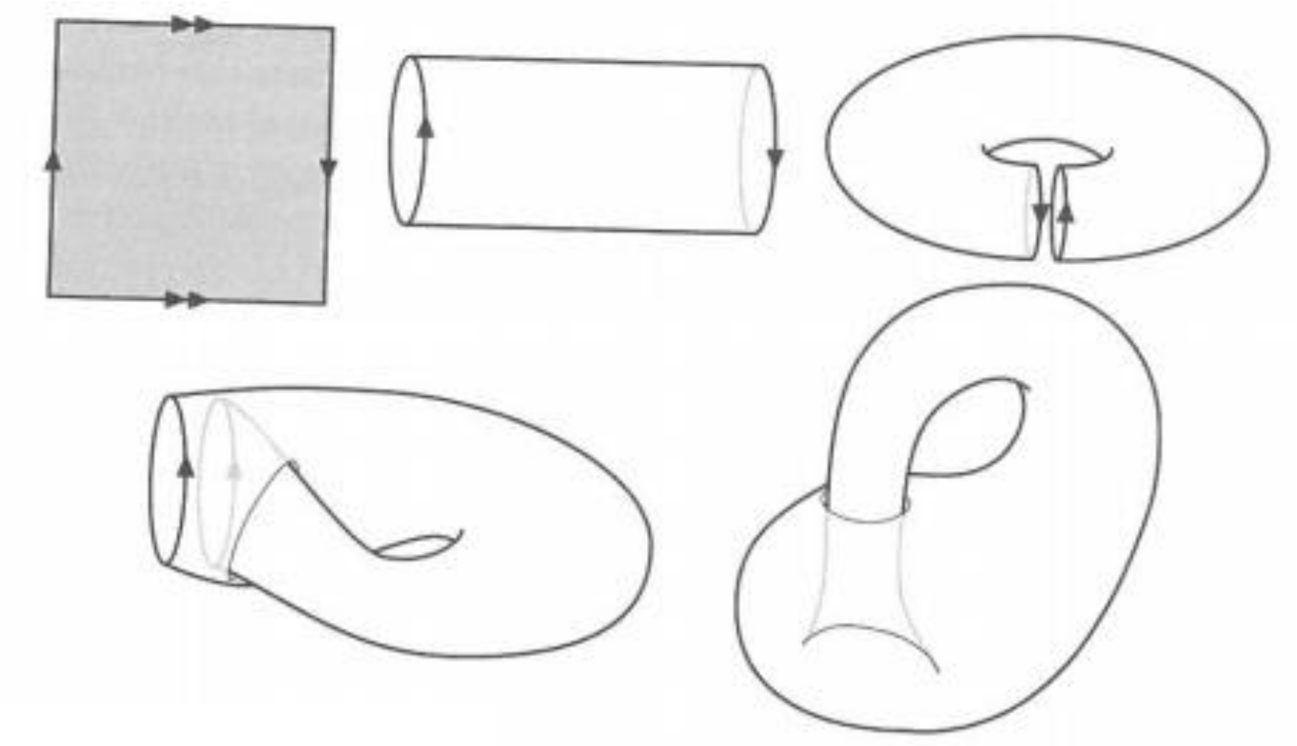

Is this orientable? Does it have boundaries? Why can it not exist in 3 dimensions?

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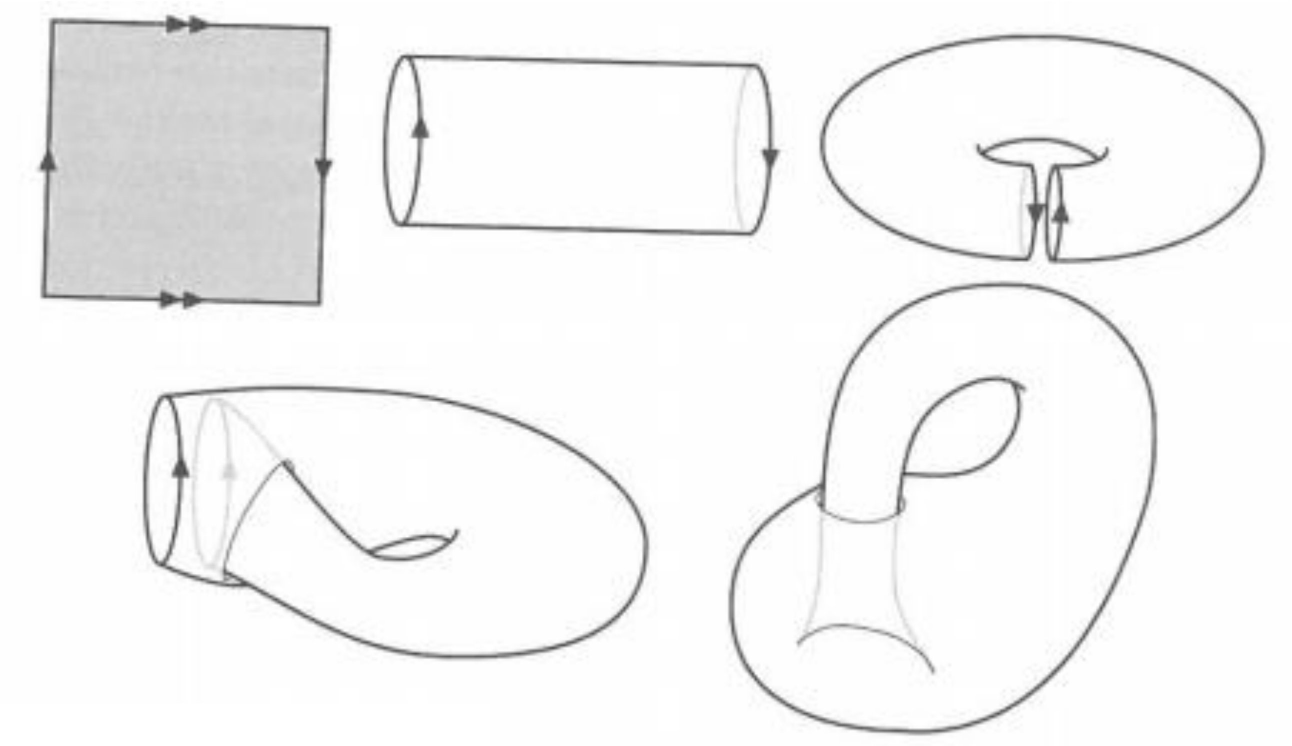

In order to stop the bottle from intersecting itself, the surface must take a brief detour in the 4th dimension.

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This concept of a surface taking a detour in a higher dimension to avoid intersecting itself is more than necessary in a variety of situations.

Figure 16.14. With a slight detour in the third dimension, we can allow two lines to pass without crossing.

Guided Discussion: Composition of Manifolds

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As you can see to the left, a torus cut through it's main loop forms a handle for a sphere with two disks removed. This forms a new torus.

Topological manifolds of intrinsic dimension of 2 (surfaces) also can be created by construction of other topological objects with the removal of disks:

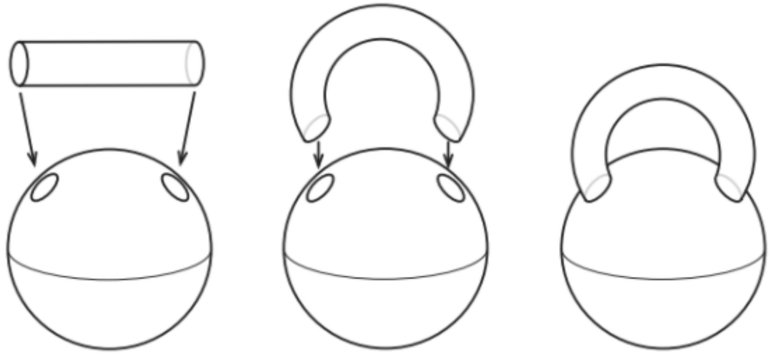

Figure 16.16. A sphere with a handle (a torus).

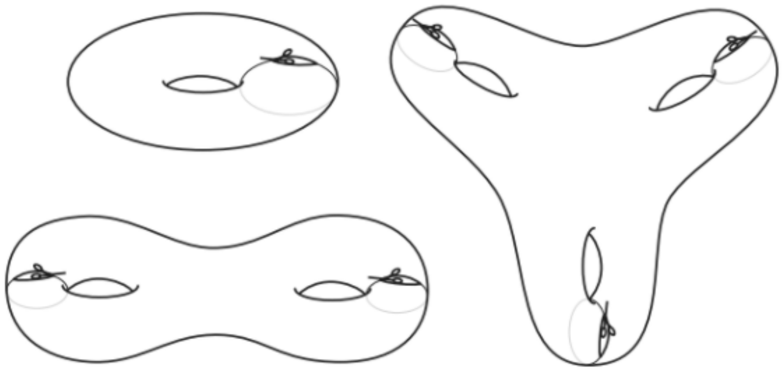

Figure 16.17. Surfaces of genus 1, 2, and 3.

Guided Discussion: Invariants

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As you can see to the left, a torus cut through it's main loop forms a handle for a sphere with two disks removed. This forms a new torus.

Now we're starting to get a grip on what a topological manifold is.

But how do Topologists study these? (without application)

Guided Discussion: Invariants

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A ***Topological Invariant*** *is a characteristic, usually a number, which uniquely distinguishes topological manifolds from each other. A few invariants:*

- *Euler Characteristic*
- *Genus*
- *Orientability*

Now we're starting to get a grip on what a topological manifold is.

But how do Topologists study these?

They primarily look at what are called ***Topological Invariants****, which are properties of topological manifolds which, if different between two topological manifolds, then the manifolds are definitively different.*

Euler's Characteristic Is an Invariant.

Guided Discussion: *Invariants*

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***Euler's Characteristic is an invariant**.*

Genus is an invariant.

But there is one which we shall discuss later…

Meanwhile, let's look at some examples of topologically identical surfaces.

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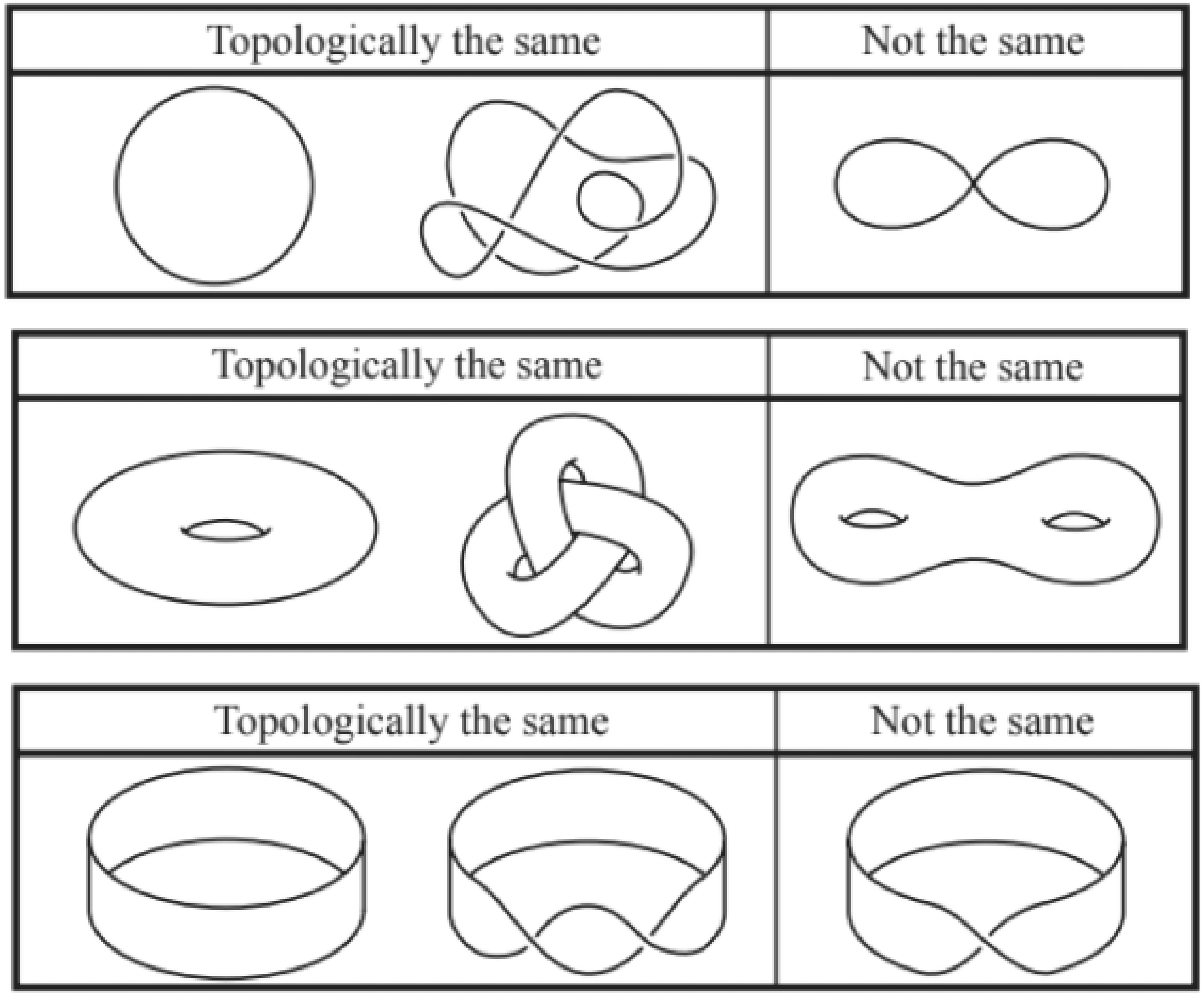
Some topological manifolds which are equivalent:



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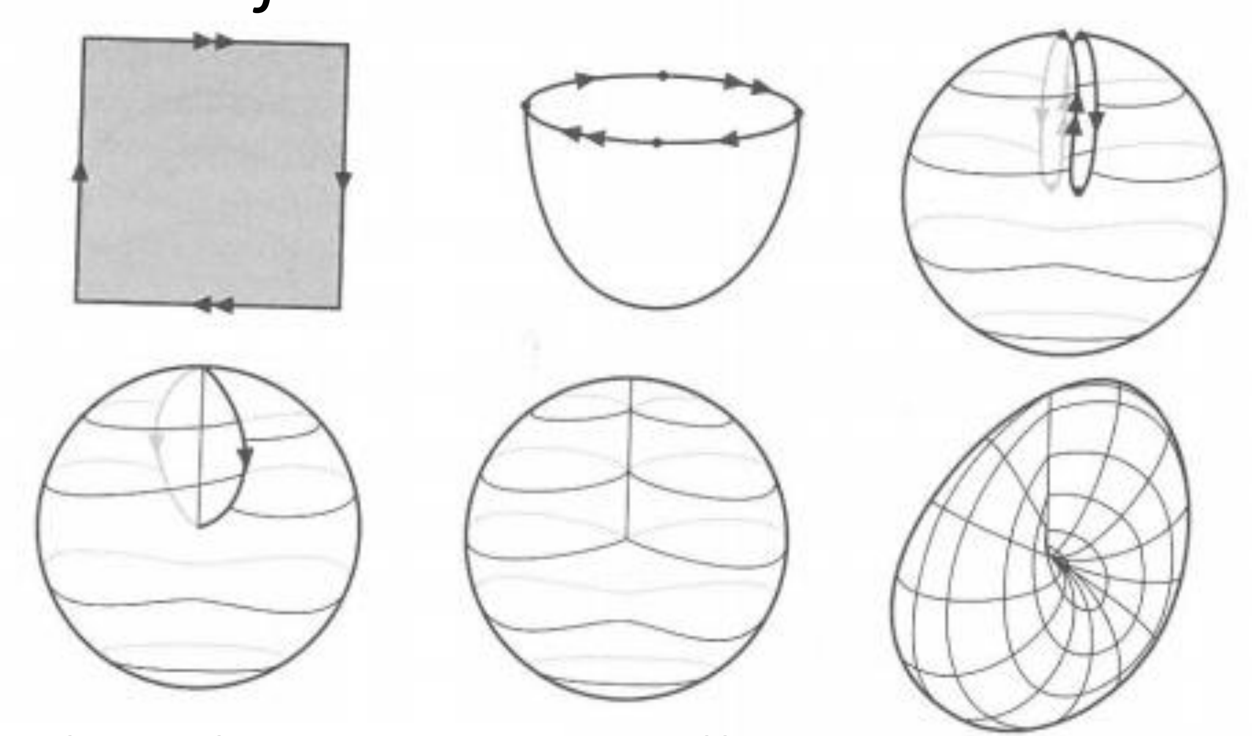
Let's look at a funky new object.

The Projective Plane.



This object was initially discovered in projective geometry (a geometric system for which any two lines meet at a single point (even those which are parallel))