Guided Discussion: Proof Strategies

Algebraic Proof, Proof by Induction, Strong Induction, Proof by Contradiction, Pigeonhole Principle

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Guided Discussion: Algebraic Proof

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Prove that

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$$f_n + f_{n+1} + f_{n+2} = 2f_{n+2}$$

$$f_{n+2} + f_{n+2} = 2f_{n+2}$$

The proof technically goes in reverse, starting with the last line shown here.

Guided Discussion: Proof by Induction

Induction is a technique used to solve general-case proof problems. There are generally two steps to these kinds of problems:

To prove that P is true for all n

- Base case for some n
- An algebraic proof that if P is true for n, it must be true for $n+1$
- Thus P is proved for all $\geq n$

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A practice problem:

Prove that

$$n! > 2^n$$

For

$$n \geq 4$$

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- *Base Case:* $4! \geq 2^n$
- *Induction Step:*

$$(n+1)! = n!\,(n+1)$$
$$> 2^n(n+1) > 2^n * 2 = 2^{n+1}$$

And it is proved.

Guided Discussion: Proof by Induction

Strong induction is a slightly different kind of induction.

Instead of proving a statement given a base case n, we prove that it is true given base cases $k < i < n$, where i are the indices which are given to be true.

Other than that, it is the same general argument.

We use strong induction over induction when we are unable to prove by induction, or wish to make a stronger claim.

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We have that

$$f_{n+1} = f_n + f_{n-1}$$

$$f_n + f_{n-1} < 2^n + 2^{n-1}$$

$$2^n + 2^{n-1} < 2^n + 2^n = 2^{n+1}$$

And it is proved.

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Take this example:

Prove

$$\sqrt{2}$$

Is irrational

We will prove by contradiction

If $\sqrt{2}$ was rational, then we would have some n, m such that

$$\sqrt{2} = {}^{n}/_{m}$$

Such that n and m are reduced and relatively prime.

This would mean

$$2m^2 = n^2$$

Because n^2 is divisible by 2, then n^2 is divisible by 4.

We're not done yet

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Because n^2 is divisible by 2, then n^2 is divisible by 4. This implies there is some k such that $n^2 = 4k$, which implies

$$2m^2 = 4k$$

$$m^2 = 2k$$

And thus m is even. But if both n and m are both divisible by 2, then ${}^{n}/_{m}$ is not reduced, a contradiction.

Guided Discussion: Pigeonhole Principle

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The pigeonhole principle states that if there are $n + 1$ pigeons in n pigeonholes, that at least 1 pigeonhole contains at least 2 pigeons.

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We look at an example:

Suppose S is a set of $n + 1$ integers. Prove that there exist distinct $a, b \in S$ such that $a - b$ is a multiple of n

*Don't be intimidated by this practice problem, it's tough, but look at some clear indicators of the kind of *arithmetic* you will be doing.*

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Let's look at S modulo n. We have that there are $n+1$ elements, fit into n different congruency classes (mod n), so by the pigeonhole principle, there must exist some $a, b \in S$ such that

$$a \equiv b \bmod m$$

Which implies

$$a - b \equiv 0 \bmod m$$

Which is our statement.