Complex Analysis

Walter Johnson Maths Honors Society

Requirements

In order to receive credit for this independent research project and be in good standing with the Walter Johnson Maths Honor Society, you must write a paper which does each of the following:

- Describe, briefly, what Complex Analysis is.
- Define the extended complex numbers, $\mathbb{C}^*$, and its topology.
- Define and describe functions of a complex variable.
- Derive and describe the Cauchy Riemann Equations.
- Derive the value of a closed-loop integral over a continuous complex-valued function.
- Describe why certain closed-loop contours are better for finding certain real integrals than others.
- Describe what Riemann Surfaces are.
- Complete and describe solutions to both problems.

On average, this assignment will take about 3 hours to research and write up. You may **not** work in a group or collaborate with others.

You will be assigned to groups of 6 people, each of which has completed a different independent research project. At the end of the year, you will present your findings to your group, and listen as your peers present their findings. Your presentation must briefly discuss every subject required in your paper along with 1 of the problems you solved, of your choice.

Resources

You are provided with various resources to complete your research. You are welcome to use resources that are not given here.

Complex Numbers

Complex Numbers are rarely expressed in their rectangular form, $a + bi$, and are more often expressed in terms of their radius from origin r and angle from the positive real axis, θ, with **DeMoivre's Theorem** and **Euler's Theorem**, which states a complex number z can be represented in polar form as such:

$$z = a + bi = re^{i\theta}$$

One confusing point in Complex Analysis is the misunderstanding that one complex dimension actually involves two real dimensions. This is intuitive on further thought: one complex variable z contains two real variables a and b in rectangular form or r and θ in polar form.

Cauchy Riemann Equations

- Video proof of Cauchy Riemann Equations by TheMathCoach.
- Rigorous derivation of Cauchy Riemann Equations. From Columbia Mathematics department. Start on page 5.

Contour Integration

- Video explanation of Contour Integration by TheMathCoach.
- Wolfram MathWorld page on Contour Integration.
- Wolfram MathWorld page on specific Complex Contours.

Riemann Surfaces

- Riemann Surface description by Xah Lee.
- SISSA Article on Riemann Surfaces.

Problem 1

Prove that if $f(z) = z\overline{z}$ then $f'(z)$ only exists at $z = 0$

Problem 2

Evaluate

$$\oint_C \frac{1}{z} dz$$

Where C is a contour on the complex plane from the equation $r = 2\theta$ such that $0 \leq \theta \leq 2\pi$

Note, this is not a closed contour.

Problem 3

Evaluate

$$\int_0^\infty \frac{\sin x}{x} dx$$

Given

$$\lim_{r \to 0} \int_r \frac{e^{iz} - e^{-iz}}{2z} dz = \pi$$

Where $\int_r$ is the integral over the semicircle from $+r$ to $-r$.

Note, the function $\sin x / x$ is even.

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