Open Problems in Topology
J. van Mill and G.M. Reed (Editors)

Chapter 29

Problems in Knot Theory

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0. Introduction

This paper is an introduction to knot theory through a discussion of research problems. Each section (there are eleven) deals with a specific problem, or with an area in which problems exist. No attempt has been made to be either complete or particularly balanced in the composition of these problems. They reflect my combinatorial bias, and my conviction that many problems in graph theory (such as the four color problem) are really problems in the theory of knots.

In this sense the theory of knots goes beyond topology into the combinatorial structures that underpin topology. In the same sense, knot theory is also deeply related to contexts in theoretical physics, and we have touched on some of these connections, particularly in relation to the Jones polynomial and its generalizations.

Knot theory had its inception in a combinatorial exercise to list all possibilities for vortex atoms in the aether. It has always lived in the multiple worlds of combinatorics, topology and physics. This is every bit as true as it was a century ago. And the plot thickens!

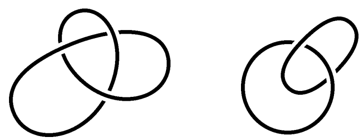
I shall let the problems speak for themselves. Earlier problems introduce information and terminology that occurs (with appropriate reference) in the later problems. In retrospect, a few fascinating classes of problems have not been touched here, so I shall mention them in this introduction. They are the problems of the understanding of frictional properties of knots (give a good mathematical model for it), understanding knotted orbits in dynamical systems, understanding physical configurations of knots and links under various conditions (tensions, fields, ...), and the applications of knot theory and differential geometry to chemistry and molecular biology.

It gives me great pleasure to thank the following people for very helpful conversations: G. Spencer-Brown, John Conway, Louis Crane, Jim Flagg, Ivan Handler, Vaughan Jones, Carol Marians, Ken Millett, Kunio Murasugi, Mario Rasetti, and Dan Sandin.

1. Reidemeister Moves, Special Moves, Concordance

For our purposes, a *knot* is a differentiable embedding of a circle into three dimensional space, and a *link* is an embedding of a collection of circles. Two links are said to be *ambient isotopic*, if there is a continuously parametrized (over the interval $[0, 1]$) family of such embeddings so that the first link corresponds to the parameter value 0, and the second link corresponds to the parameter value 1.

Any link may be projected to a plane in three-space to form a link diagram. The link diagram is a locally 4-valent planar graph with extra structure at the vertices of this graph indicating which segment of the diagram crosses over the other in the three dimensional embedding. The usual convention for this information is to indicate the undercrossing line by drawing it with a small break at the crossing. The over-crossing line goes continuously through the crossing and is seen to cleave the under-crossing line:



These diagrams can be used to formulate a purely combinatorial theory of links that is equivalent to the theory of link embeddings up to ambient isotopy in three dimensional space. The combinatorial theory is based on the Reidemeister moves (REIDEMEISTER [1948]). (See Figure 1.) These moves (along with the topological moves on the 4-valent planar graph underlying the link diagram) generate ambient isotopy for knots and links in three dimensional space. Two diagrams are related via a sequence of Reidemeister moves if and only if the link embeddings that they represent in three-space are ambient isotopic. (See BURDE and ZIESCHANG [1986] for a modern proof of this fact.)

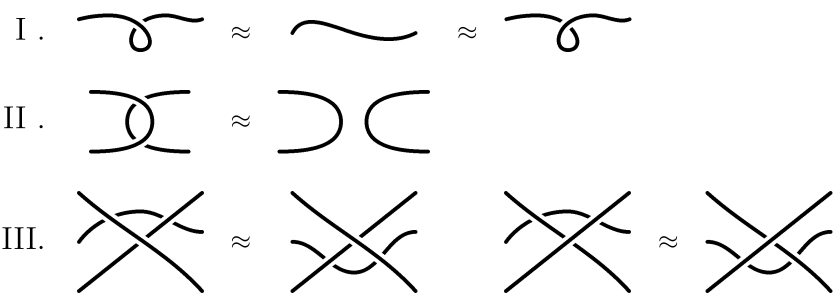
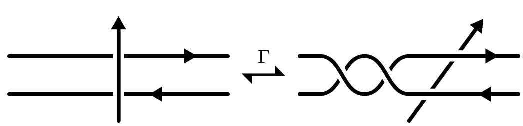


Figure 1: Reidemeister Moves

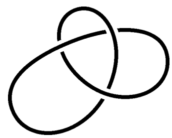
One can add extra moves to the Reidemeister moves, thereby getting larger equivalence classes and (in principle) invariants of ambient isotopy. For example, consider the following *switch move* (called the Gamma move in KAUFFMAN [1983]) on oriented diagrams:



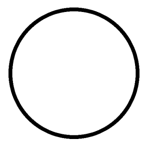
(An orientation of a diagram consists in assigning a direction of travel to each link component. This is indicated by arrows on the diagram.)

In KAUFFMAN [1983] it is shown that the equivalence relation generated by the Reidemeister moves plus the switch (call it *pass equivalence*) has exactly two equivalence classes for knots. (There is a similar statement for links but I shall not go into it here.)

The trefoil knot



represents one class, and the trivial knot



represents the other class.

Switch equivalence is interesting because *every ribbon knot is pass equivalent to the unknot.*

A *ribbon knot* is a knot that bounds a disk immersed in three space with only ribbon singularities. A *ribbon singularity* consists in a transverse intersection of two non-singular arcs from the disk: One arc is interior to the disk; one arc has its endpoint on the boundary of the disk. Examples of ribbon knots are shown in Figure 2.

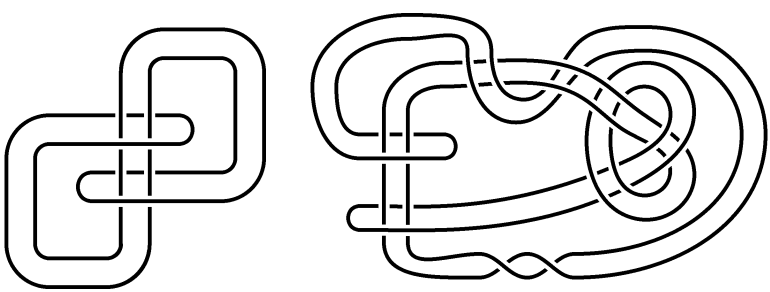


Figure 2: Ribbon Knots

In the diagram of a ribbon knot a sequence of switches can be used to remove all the ribbon singularities. Thus a ribbon knot is pass equivalent to an unknot. Since the trefoil knot is not pass equivalent to the unknot, we conclude that the trefoil is not ribbon.

The interest in the problem of ribbon knots lies in the fact that *every ribbon knot is slice.* That is, every ribbon knot bounds a smoothly embedded disk in the upper four space H^4 (if the knots and links are in the Euclidean space $\mathbb{R}^3$, then $H^4 = \mathbb{R}^+ \times \mathbb{R}^3$ where $\mathbb{R}^+ = \{r : r \geq 0,\ r \text{ a real number}\}$.) A knot that bounds a smoothly embedded disk in upper four space is called a *slice knot*

(FOX and MILNOR [1966]). We would really like to characterize slice knots. In fact, it remains an open question:

? 825. Problem 1. *Is every slice knot a ribbon knot?*

The problem of detecting slice knots is very deep, with distinct differences between the case in dimension three and the generalizations to higher dimensional knots. The most significant work on the slice problem in dimension three, definitively discriminating it from higher dimensions, is due to CASSON and GORDON [1978]. While their invariants detect significant examples of non-slice knots that are algebraically slice (from the viewpoint of the Seifert pairing), the method is in general very difficult to apply. Thus: One would like to have new and computable invariants to detect slice knots. (See Problem 2 below.)

The appropriate equivalence relation on knots and links for this matter of slice knots is the notion of *concordance*. Two knots are said to be *concordant* if there is a smooth embedding of $S^1 \times I$ into $S^3 \times I$ with one knot at one end of the embedding and the other knot at the other end. A slice knot is concordant to the unknotted circle. (The embedding of the slice disk can rise only a finite height into four-space by compactness. Locate a point of maximal height and exercise a small disk. This proceduces the concordance.)

Concordance is generated by the Reidemeister moves, in conjunction with the passage through saddle point singularities, and the passage through minima and maxima. A minimum connotes the birth of an unknotted circle, and a maximum connotes the death of an unknotted circle.

Of course, the entire history of the concordance is constrained to trace out an annulus ($S^1 \times I$) embedded in the four-space. It is this constraint that makes the subject of knot and link concordance so difficult to analyze. It is easy to construct slice knots, but very hard to recognize them!

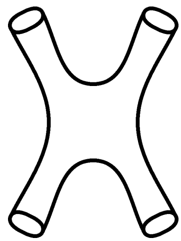
Later, we shall raise this question of slice knots and behaviour under concordance with respect to various invariants such as the Jones polynomial (JONES [1985]). The question is:

? 826. Problem 2. *Are there any new and simple invariants of concordance?*

It is possible that we are overlooking the obvious in this realm.

2. Knotted Strings?

String Theory is usually formulated in dimensions that forbid the consideration of knots. We can, however, imagine string-like particles tracing out world sheets in four dimensional spacetime. A typical string vertex will then be an embedding of a sphere with four holes in $S^3 \times I$ so that two holes are in $S^3 \times 0$ and two holes are in $S^3 \times 1$:



$$\hookrightarrow \quad S^3 \times I.$$

Just as with knot concordance (Section 1), the embedding can be quite complex, and this complexity will be indexed by the appearance of singularities in the hyperspace cross sections $S^3 \times t$ for t between zero and one. The singularities are births, deaths and saddle points. It is interesting to note that in this framework, a knot and its mirror image can interact to produce two unknots! See Figure 3.

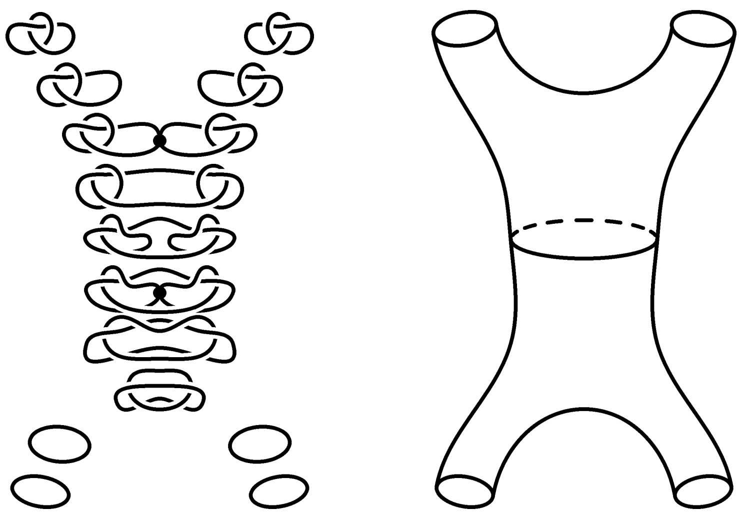


Figure 3: Interaction.

Thus, this embedded string theory contains a myriad of "particle states" corresponding to knotting and the patterns of knot concordance (Section 1).

While the physical interpretation of knotted strings is ambiguous, the mathematics of interacting knots and links is a well-defined and unexplored territory.

Problem 3. *Investigate knotted strings and four-space interactions.* **827. ?**

3. Detecting Knottedness

? **828. Problem 4.** *Does the original Jones polynomial* (JONES [1985]) *detect knottedness?*

There are many polynomial invariants of knots and links that generalize the original Jones polynomial (AKUTSU and WADATI [1988], FREYD, YETTER, HOSTE, LICKORISH, MILLETT and OCNEAUNU [1985], HO [1985], HOSTE [1986], JONES [1987, 1989], KAUFFMAN [1989a, 19∞a, 1990a], LICKORISH and MILLETT [1987], LICKORISH [1988], RESHETIKHIN [1987, 1989], TURAEV [1987], WITTEN [1989]), and the same problem can be addressed to them. Nevertheless, the problem is most charming when phrased to the original Jones polynomial. It is, at base, a combinatorial question about the structure of the bracket state model (KAUFFMAN [1987b]) that calculates this polynomial.

Recall the bracket, $[K]$. It is, at the outset a well-defined three variable polynomial for unoriented link diagrams—defined by the equations

$$1.\quad [\times] = A[\asymp] + B[)(]$$
$$2.\quad [\bigcirc \mathbf{K}] = d[\mathbf{K}], \qquad [\bigcirc] = d$$

In these equations the small diagrams stand for parts of otherwise identical larger diagrams. The second equation is to be interpreted as saying that an isolated loop (Jordan curve) contributes a factor of d to the polynomial. Since we assume that A, B and d commute, it follows easily that $\langle K\rangle$ is well-defined on unoriented diagrams. Call this polynomial the *three-variable bracket.* It is not an ambient isotopy invariant as it stands, but a specialization of the variables yields the Jones polynomial.

To be precise, one easily finds the following formula:

$$[)\!\!\times\!\!\times\!\!(] = AB[)(] + (ABd + A^2 + B^2)[\asymp].$$

Hence, if we choose $B = A^{-1}$ and $d = -A^2 - A^{-2}$, and define the *topological bracket*, $\langle K\rangle$, by the formula

$$\langle K\rangle(A) = [K](A, A^{-1}, -A^2 - A^{-2})/[\bigcirc]$$

then

$$\langle)\!\!\times\!\!\times\!\!(\rangle = \langle)(\rangle$$

achieving invariance under the second Reidemeister move. It is then easy(!) to see that this *topological bracket* is invariant under the third Reidemeister move as well. Finally, we get the formulas

$$\langle \sim\!\!\circ\!\!\sim \rangle = (-A^3)\langle \sim \rangle$$
$$\langle \sim\!\!\circ\!\!\sim \rangle = (-A^{-3})\langle \sim \rangle$$

(Thus $\langle K\rangle$ is not invariant under the first Reidemeister move. It is invariant under II and III. This is called invariance under *regular isotopy.*)

3.1. THEOREM (KAUFFMAN [1987b]). *The original Jones polynomial $V_K(t)$ is a normalized version of the special bracket. In particular,*

$$V_K(t) = f_K(t^{-\frac{1}{4}})$$

where $f_K(A) = (-A^3)^{-w(K)}\langle K\rangle$, K is oriented, $w(K)$ is the sum of the crossing signs of K, and $\langle K\rangle$ is the topological bracket evaluated on K by forgetting K's orientation. □

We can restate the question at the beginning:

Problem 4.1. *Does there exist a knot K (K is assumed to be knotted) such that $\langle K\rangle$ is a power of A?* **829. ?**

Such a knot would have extraordinary cancellations in the bracket calculation.

One way to begin to look into this problem is to consider the structure of a state summation for the bracket. That is, we can give a specific formula for the bracket as a combinatorial summation over certain configurations of the link diagram. I call these configurations "states" of the diagram—in analogy to the states of a physical system in physical mechanics. In a sense, each model for a link invariant has its own special set of states. The states for the bracket are particularly simple: let U be the four-valent plane graph underlying a given link diagram K. A (bracket) state of U is a collection of Jordan curves in the plane that is obtained by splicing each crossing of U in one of the two possible ways—as shown in Figure 4.

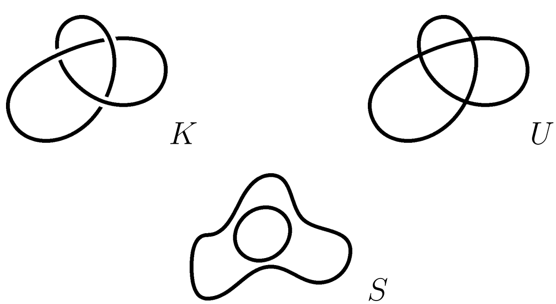
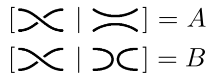
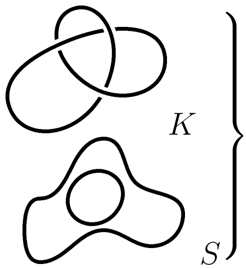


Figure 4: S is a *state* of U

For a given link diagram K, each state S of K has *vertex weights* A or B at each crossing of K. These weights depend upon the relationship of the local state configuration and the crossing in the link diagram. If C is a crossing and Q is a local state configuration, then I let $[C|Q]$ denote the vertex weight contributed by C and Q. The rules are as shown below:



If K is a link diagram and S is a state of K, then $[K|S]$ *denotes the product of the vertex weights from* K *and* S *over all the crossings of* K.



$$\Longrightarrow \quad \begin{array}{l}[K|S] = A^3 \\ \|S\| = 2\end{array}$$

We then have the specific formula

$$[K] = \sum [K|S] d^{\|S\|}$$

where the summation extends over all states of the diagram, and $\|S\|$ denotes the number of Jordan curves in the state S.

In the case of the special bracket, this summation becomes

$$\langle K \rangle = \sum A^{i(S)-j(S)}(-A^2 - A^{-2})^{\|S\|-1}.$$

Here the summation extends over all the states of the diagram, and $i(S)$ and $j(S)$ denote the number of sites in the states that receive vertex weights of A and A^{-1} respectively. From this formula, we see that the whole difficulty in understanding cancellation phenomena in the bracket is concentrated in the presence of the signs $(-1)^{\|S\|}$ in the state summation. In the cases of alternating links (KAUFFMAN [1987b], MURASUGI [1987]) and adequate links (LICKORISH [1988]) it is possible to see directly that there is no non-trivial cancellation (i.e., the polynomial $\langle K \rangle$ detects knottedness for alternating and adequate knots and links). In general, it is quite possible that there is a topologically knotted diagram K with enough cancellation to make $\langle K \rangle$ into a power of A.

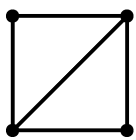
? **830. Problem 6.** *Where is this culprit* K*?*

(The culprit would answer Problem 4.1.)

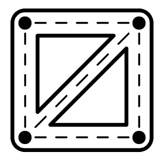
4. Knots and Four Colors

A simple, classical construction relates arbitrary planar graphs and (projected) link diagrams. This is the construction of the *medial graph* associated to any planar graph embedding. See Figure 5.

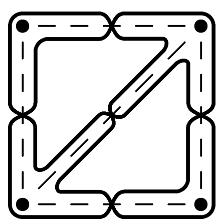
The medial graph is obtained as follows: In each region of the plane graph G (A *plane graph* is a graph that is embedded in the plane), draw a Jordan curve that describes the boundary of the region. For ease of construction, the curve should be drawn near the boundary. Now each edge of the graph will appear as shown below (with the dotted line representing the original edge, and the solid lines representing the Jordan curves).



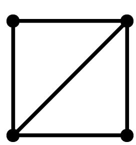
G



Once for each edge in G, replace the parallel Jordan curve segments with a crossing as shown below and in Figure 5.



The resulting locally four-valent plane graph is the medial graph, $M(G)$.



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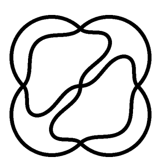
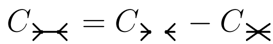


$M(G)$

Figure 5: The graph G and its medial graph $M(G)$

The upshot of this medial graph construction is that the class of locally four-valent plane graphs is sufficiently general to capture all the properties of the entire class of plane graphs. Since knots and links project to locally four-valent plane graphs, this means that *in principle, all combinatorial problems about plane graphs are problems about link diagrams.* A problem about link diagrams may or may not be a problem about the topology of links, but it is interesting and possibly very significant to see the relationship between combinatorial problems and their topological counterparts.

A first example of this correspondence is the chromatic polynomial, $C_G(q)$. This is the number of vertex colorings by q colors of the graph G such that vertices that share an edge receive distinct colors (i.e., $C_G(q)$ is the number of proper vertex colorings of G with q colors.). It is easy to see that C_G satisfies the following formulas



$$C_{\bullet G} = qC_G$$

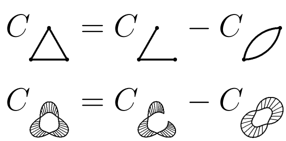
Here $\bullet G$ denotes the disjoint union of G with an isolated point, and the small diagrams indicate (in order from left to right) an edge in the graph G, the deletion of this edge, the contraction of this edge to a point. Thus the first formula states that

$$C_G = C_{G'} - C_{G''}$$

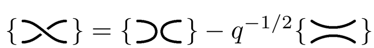
where G, G' and G" stand for the original graph, the graph with a specific edge deleted, and the graph with this edge contracted to a point, respectively.

On translating this formula to the medial graph we find

Since one must keep track of the direction of splitting in terms of the original graph it is best to work with a shaded medial, thus:



This scheme is quite convenient for working with colorings of graphs. In particular, it suggests that the chromatic polynomial is very similar to the bracket polynomial. In fact, *we can use the crossings of a knot diagram to encode the chromatic polynomial as a bracket calculation.* (See KAUFFMAN [1989d].) The result is as follows: Associate to each plane graph G an alternating link diagram $K(G)$ by taking the medial $M(G)$, and arranging a link diagram over $M(G)$ with crossings chosen to be of "A-type" for each edge of G (See Figure 6 for this convention.) Define a special bracket via



$$\{\bigcirc\mathbf{K}\} = q^{1/2}\{\mathbf{K}\}$$

Then $C_G(q) = q^{N/2}\{K(G)\}$ where N denotes the number of vertices of the original graph G.

This formula shows that the chromatic polynomial for a plane graph can be put into exactly the same framework as the Jones polynomial for a given link. Now the classical combinatorial problem about the chromatic polynomial for

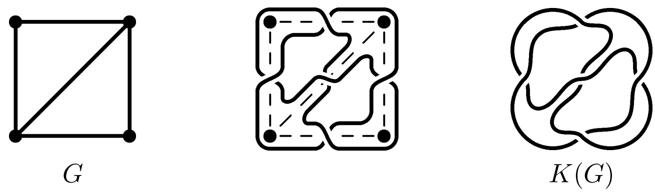


Figure 6: G and its alternating link diagram $K(G)$

plane graphs is to show that it does not vanish for $q = 4$ when G has no loop and no isthmus (See KEMPE [1879], WHITNEY [1988]). We see from this reformulation that this difficulty is very similar to the difficulty in showing that the Jones polynomial detects knottedness.

These remarks solve neither the knot detection problem nor the coloring problem, but it is significant to find that these problems share the rung in the inferno.

5. The Potts Model

The chromatic polynomial of Section 4, is a special case of the *dichromatic polynomial*, a polynomial $W_G(q, v)$ in two variables, q and v, associated with an arbitrary graph G via the formulas

$$\begin{aligned} W_{\succ\!\!-\!\!\prec} &= W_{\succ\ \prec} - vW_{\times} \\ W_{\bullet G} &= qW_G \end{aligned}$$

That is, W is a generalization of the chromatic polynomial. It specializes to the chromatic polynomial when $v = 1$.

Just as we expressed the chromatic polynomial as a bracket calculation, we can also express the dichromatic polynomial in a similar way. Generalize the special bracket of Section 4 via the rules:

$$\begin{aligned} \{\times\} &= \{\supset\subset\} + vq^{-1/2}\{\asymp\} \\ \{\bigcirc \mathbf{K}\} &= q^{1/2}\{\mathbf{K}\} \end{aligned}$$

Then one has the formula

$$W_G(q, v) = q^{N/2}\{K(G)\},$$

where N denotes the number of vertices of the graph G (G is a plane graph for this discussion.) and $K(G)$ is the alternating link diagram associated with the plane graph G via the medial construction (See Section 4).

Now it is well-known (BAXTER [1982]) that the dichromatic polynomial of a graph G can be interpreted as the partition function of a statistical mechanics model based on G. This model, known as the *Potts model* depends upon q local states at the vertices of the graph, and the variable v is related to the temperature in the model via the equation $z = \exp(\frac{1}{kT}) - 1$ (anti-ferromagnetic case) where T denotes temperature, and k is a constant (Boltzman's constant). The partition function is a summation over the physical states of the model of probability weighting for these states. The weights depend upon energy, temperature and Boltzman's constant. In this very simple model, a state σ is an assignment of values (colors, spins,...) to each vertex of the graph. The energy, $E(\sigma)$, of the state σ is then defined to be the number of coincidences of spins for pairs of vertices that are connected by an edge in the graph. The partition function is the summation

$$Z_G = \sum_{\sigma} \exp\left((\frac{1}{kT}) E(\sigma) \right)$$

where the sum extends over all states of the given graph.

The basic result is that $Z_G = W_G(q, \exp(\frac{1}{kT}) - 1)$. Hence

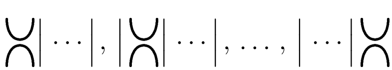
$$Z_G = q^{\frac{N}{2}} \{K(G)\}\{q, \exp(\frac{1}{kT}) - 1\}.$$

Note that this formula says that the Potts partition function at zero temperature is the chromatic polynomial.

For G a rectangular lattice in the plane it is conjectured (BAXTER [1982]) that the Potts model has a phase transition (in the limit of large lattices) for the temperature value that symmetrizes the model with respect to graph and dual graph in the plane. In terms of the special bracket link diagram representation of the model, this means that we demand that $q^{\frac{1}{2}} = \exp(\frac{1}{kT}) - 1$ since this creates the symmetry

corresponding in link diagrams to the desired duality.

Many problems about the Potts model find their corresponding formulations in terms of this special bracket for linked diagrams. In particular, it is at once obvious from the special bracket expansion for the Potts model that the Potts model can be expanded over the Temperley-Lieb algebra—with this algebra represented diagramatically via braid monoid elements of the form



There are a number of important questions about the relationship of the Temperley-Lieb algebra and other structures of the model near criticality. For example, one would hope that this approach sheds light on the relationship with the Virasoro algebra in the continuum limit of the Potts model at criticality. In general we can ask:

Problem 7. *Does the link diagrammatic approach lend insight into properties of the Potts model?* **831.** ?

We can also ask whether the concepts of statistical mechanics can be used in the topological context. For example,

Problem 8. *What does the phenomena of phase transition mean in the context of calculating link polynomials for large links?* **832.** ?

Problem 9. *Is there a way to extract topological information from the dynamical behaviour of a quasi-physical system associated with the knot?* **833.** ?

Finally, to return directly to the knot theory, one might wonder:

Problem 10. *Is the Potts partition function viewed as a function on alternating link diagrams a topological invariant of these diagrams?* **834.** ?

Instead of being nonsense, this turns out to be a deep question! It requires interpretation. The apparently correct conjecture is this:

Problem 11. *Let K be a reduced alternating diagram, then we conjecture that $[K](A, B, d)$ is an ambient isotopy invariant of K.* **835.** ?

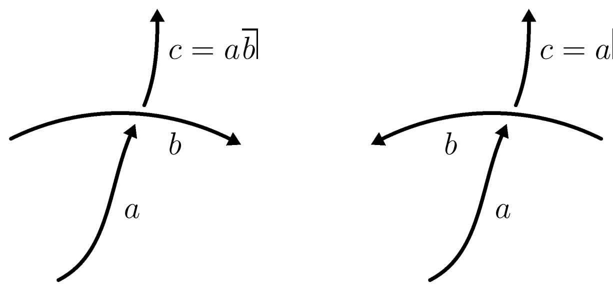
(A diagram is reduced if there are no simplifying type I moves and it is not a connected sum of two non-trivial diagrams.) This conjecture has its roots in the classical conjectures of Tait, Kirkwood and Little who suggested that two reduced alternating projections of the same link are related by a sequence of higher-order moves called *flypes*. A flype takes a tangle with two crossed input strands and two output strands, and turns the tangle by one half twist (180 degrees). This moves takes alternating projections to alternating projections. *It is easy to see that the full three-variable bracket polynomial is invariant under flyping.* Thus the Flyping Conjecture of Tait, Kirkwood and Little implies the topological invariance of the full bracket for the reduced alternating projections. This, in turn, implies the invariance of the Potts partition function for an associated reduced alternating link.

It appears that the Potts partition function contains real topological information. Perhaps eventually it will be seen that the Tait Flyping Conjecture follows from subtle properties of statistical mechanics.

6. States, Crystals and the Fundamental Group

The fundamental group of the complement of the link can be described as a special sort of state of the link diagram. In order to illustrate this point and to ask questions related to it, I shall describe a structure that simultaneously generalizes the fundamental group, the Alexander module, and the Quandle (JOYCE [1982]). We shall call this algebraic structure related to an oriented link diagram the *crystal of K*, $C(K)$. (See KAUFFMAN [1987a, 19∞b]).

The crystal is obtained by assigning an algebra element to each arc in the diagram, and writing a relation at each crossing in the form shown below:



In this formalism the mark, ⌉ (or ⌈, (there are left and right versions of the mark) is a formal operator that is handled like a root sign ($\sqrt{\ }$) in ordinary algebra. That is, the mark has the role of operator and parenthesis. It acts on the expression written within it, and it creates a parenthetical boundary for the result of the operation. The concatenation $a\overline{b|}$ is regarded as a non-commutative product of a and $\overline{b|}$. The crystal is a formal algebra that is given to be associative and (possibly) non-commutative.

Products in the crystal are built via the following rules:

(1) If a and b are in C, then $a\overline{b|}$, $a\overline{|b}$, $\overline{a|}$ and $\overline{|a}$ are in C.
(2) The labels for the arcs on the link diagram are in C.
(3) All elements of C are built via these three rules.

The crystal axioms are:

2. $x\overline{a|}\,\overline{|a} = x$ for all x and a in C.
3. $x\overline{|b}\,\overline{a|}\,\overline{b|} = x\overline{a\overline{b|}|}$ for all x, a and b in C. (and the variants motivated below.)

The axioms are labelled 2. and 3. to correspond to the Reidemeister moves 2. and 3. The diagrams in Figure 7 show this correspondence with the moves. Here we have used a modified version of the type III move (a detour) that is valid in the presence of the type two move.

By our assumption about the Crystal Axioms, the crystal acts on itself via

$$C \times C \longrightarrow C : (a, b) \longrightarrow a\overline{b|} \text{ (or } a\overline{|b}).$$

Given the associativity of the concatenation operation in the crystal, we see that Axiom 2 asserts that the *operator subset*

$$C^* = \{\, x \in C : x = \overline{a|} \text{ or } x = \overline{|a} \text{ for some } a \in C \,\}$$

II.

III.

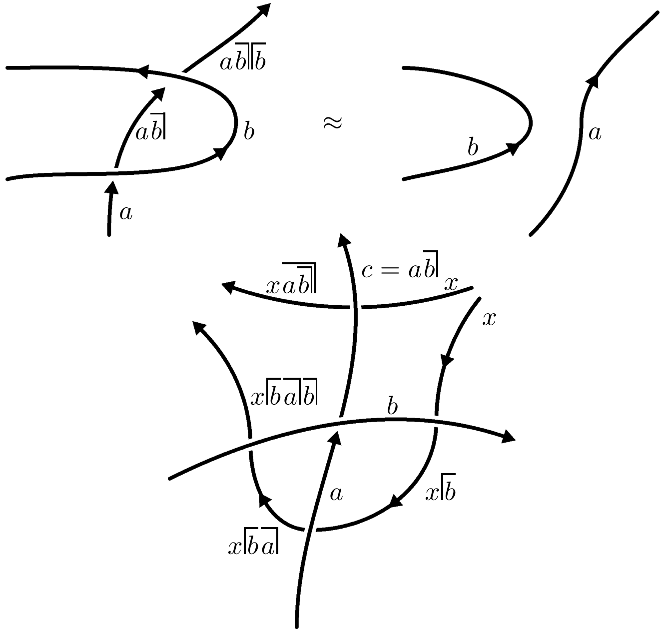
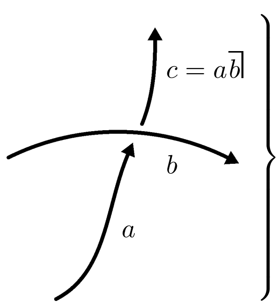


Figure 7: Crystal Axioms and Reidemeister Moves

is a group (of automorphisms of C) under the crystal multiplication. *This group is the fundamental group of the link complement.* (Compare the formalism with that of the Wirtinger presentation (Crowell and Fox [1963]) of the fundamental group.) If we wish to emphasize this group structure then we can write the axioms as:

2. $\overline{a|}\,\overline{|a} = 1, \qquad \overline{|a}\,\overline{a|} = 1,$
3. $\overline{a\overline{b|}|} = \overline{|b}\,\overline{a|}\,\overline{b|}, \quad \overline{a\overline{|b}|} = \overline{b|}\,\overline{a|}\,\overline{|b}, \quad \overline{|a\overline{b|}} = \overline{|b}\,\overline{|a}\,\overline{b|}, \quad \overline{|a\overline{|b}} = \overline{b|}\,\overline{|a}\,\overline{|b}.$

These are the operator identities, but the important point to see is that we associate one group element to each arc of the diagram, and that there is one relation for each crossing in the form shown below:



$$\Longrightarrow \qquad \begin{aligned} \alpha &= \overline{a|} \\ \beta &= \overline{b|} \\ \gamma &= \overline{c|} \\ \gamma &= \overline{a\overline{b|}|} \\ &= \overline{|b}\,\overline{a|}\,\overline{b|} \\ \gamma &= \beta^{-1}\alpha\beta \end{aligned}$$

This is the familiar Wirtinger relation for the fundamental group.

The quandle (JOYCE [1982], WINKER [1984]) is generated by *lassos*, each consisting in an arc emanating from a basepoint in the complement of the link, plus a disk whose boundary encircles the link—the interior of the disk is punctured once transversely by the link itself. One lasso acts on another to form a new lasso $a * b$ (for lassos a and b) by changing the arc of a by first travelling down b, around its disk, back to basepoint, then down the original arc of a. Since one can travel around the disk in two ways, this yields two possible operations $a * b$ and $a \overline{*} b$. These correspond formally to our abstract operations $a\overline{b\rvert}$ and $a\overline{\lvert b}$. See Figure 8.

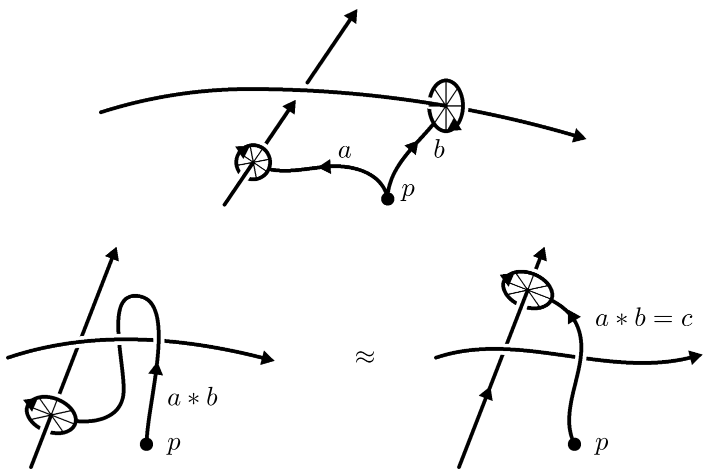
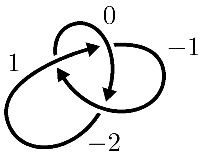


Figure 8: Lassos

The crystal contains more than the fundamental group, and in fact it classifies knots up to mirror images. (See JOYCE [1982].) I have defined the crystal so that it is a regular isotopy invariant (invariant under the second and third Reidemeister moves. It is nevertheless the case that the group $C^*(K)$ is invariant under all three Reidemeister moves. The quandle is a quotient of the crystal. We translate to the quandle by writing $a * b = a\overline{b\rvert}$. This has the effect of replacing a non-commutative algebra with operators by a non-associative algebra.

Simple representations of the crystal show its nature. For example, label the arcs of the link diagram with integers and define $a\overline{b\rvert} = a\overline{\lvert b} = 2b - a$. (This operation does not depend upon the orientation of the diagram.) Then each link diagram will have a least modulus (not equal to 1) in which the crossing equations can be solved. For example, the modulus of the trefoil is three:



$a * b = 2b - a$
modulo 3

This shows that the number three is an invariant of the trefoil knot, and it shows that we can label the arcs of a trefoil with three colors $(0, 1, 2)$ so that each crossing sees either three distinct colors or it sees only one color. The invariantce of the crystal tells us that any diagram obtained from the trefoil by Reidemeister moves can be colored in the same way (i.e., according to the same rules). In general, there is a coloring scheme corresponding to each modulus, and any knot can be colored with (sufficiently many) labels. Note that for a diagram isotopic to a given diagram there may appear different colors, since not all the colors will necessarily be used on a given diagram (even for a fixed modulus).

In any case, this modular approach to link invariants shows us a picture of a link invariant arising as a property of a special sort of "state" of the diagram (The state is a coloring of the arcs according to the crystal rules.). That property is the modulus. It is the least integer that annihilates all the state labels. The states themselves are arranged so that if the diagram is changed by a Reidemeister move, then there is a well-defined transition from the given state to a state of the new diagram.

The same picture holds for the classical Alexander polynomial. Here the crystal represents the Alexander module by the equations

$$a\overline{b|} = ta + (1 - t)b$$
$$a\overline{|b} = t^{-1}a + (1 - t^{-1})b$$

Note that when $t = -1$ we have the formalism of the modular crystal described above. The labels $a, b, \ldots$ on the arcs of the link diagram are generators of a module (hence additively commutative) over the ring $\mathbb{Z}[t, t^{-1}]$. Each crossing in the diagram gives a relation that must hold in the module. The classical Alexander polynomial is defined (up to units in $\mathbb{Z}[t, t^{-1}]$) as the generator of the annihilator ideal of the Alexander module. Once again, we have generalized states of the diagrams (labellings from the Alexander module) and a topological invariant arising from the properties of these states.

Problem 12. *A fundamental problem is to find new ways to extract significant topological information from the crystal. We would like to find a useful generalization of the crystal that would completely classify links—including the information about mirror images.* **836. ?**

I have taken the time to describe this crystalline approach to the classical invariants because it is fascinating to ask how the classical methods are related to the new methods that produce the Jones polynomial and its generalizations. At the present time there seems to be no direct relationship between the Jones polynomial and the fundamental group of the knot complement, or with a structure analogous to the Alexander module. This means that although the newer knot polynomials are very powerful, they do not have access to many classical techniques. A direct relation with the fundamental group or with the structure of the crystal would be a real breakthrough.

There is a theme in this quest that is best stated in the metaphors of mathematical physics. A given physical system has physical states. As time goes on, and as the system is changed (possibly as the system is topologically deformed) these states undergo transitions. The patterns of the state transitions reflect fundamental properties of the physics of the system. The usual method of statistical mechanics is to consider not the transitions of the states, but rather the gross average of probability weighting over all the possible states. This average is called the partition function of the system. The two points of view—transition properties and gross averages—are complementary ways of dealing with the physics of the system. They are related. For example, one hopes to extract information about phase transition from the partition function. Phase transition is a significant property of state changes in the system.

In the knot theory we have the same schism as in the physics—between the contexts of state transition and state averaging. The tension between them will produce new mathematics and new relations with the physics.

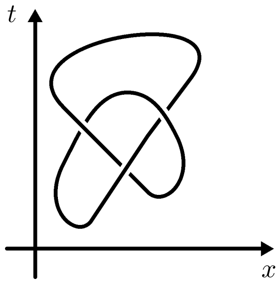
7. Vacuum-Vacuum Expectation and Quantum Group

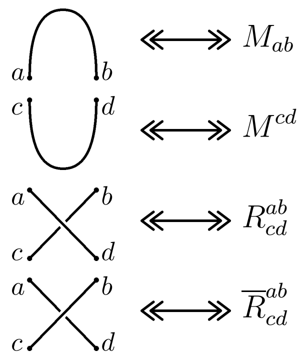
An intermediate position related to the philosophy at the end of Section 6 is the fact that a statistical mechanics model in $d+1$ dimensions of space can be construed as a quantum statistical mechanics model in d dimensions of space and 1 dimension of time. (This is called $d+1$-dimensional space-time.)

In the case of the knot invariants, this philosophy leads to the invariant viewed as a vacuum-vacuum expectation for a process occurring in $1+1$-dimensional space-time, with the link diagram in the Minkowski plane. For knots in three-space, the process occurs in a $2+1$-dimensional space-time. Here the picture is quite intuitive. One visualizes a plane moving up through three dimensional space. This is the motion through time for the flatlanders living in the plane. The flatlanders observe a complex pattern of particle creation, interaction and annihilation corresponding to the intersection of the moving plane with a link embedded in the three dimensional space.

In order to calculate the vacuum-vacuum expectation of this process, the flatlanders must know probability amplitudes for different aspects of the process—or they must have some global method of computing the amplitude. In the case of the Jones polynomial and its generalizations the global method is provided by Witten's topological quantum field theory (See Witten [1989]).

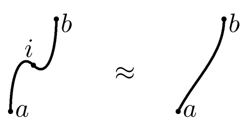
For now we shall rest content with a simpler calculation more suited to linelanders than to flatlanders The simplest version of a quantum model of this kind is obtained from the planar knot and link diagrams. There we can call attention to creations, annihilations and interactions in the form of cups, caps and crossings. Note that the crossings go over and under the plane of the diagram. This model has just a bit more than one dimension of space in its space-time.

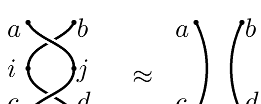


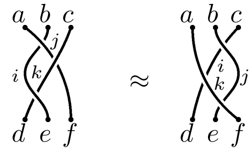


As illustrated above, I have associated each cup, cap, or crossing with a matrix whose indices denote the "spins" of the particles created or interacting, and whose value denotes a generalized quantum amplitude taking values in an (unspecified) commutative ring.

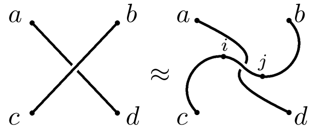
Following the principles of quantum mechanics, the amplitude is the sum (over all possible configurations of spins) of the products of the amplitudes for each configuration. In order for this amplitude to be an invariant of regular isotopy, we need matrix properties that correspond to topological moves. Thus we require

I.  $\longleftrightarrow M_{ai}M^{ib} = \delta^b_a$ (sum on i)

II.  $\longleftrightarrow R^{ab}_{ij}\overline{R}^{ij}_{cd} = \delta^a_c\delta^b_d$

III. 

$\longleftrightarrow$ Yang-Baxter Equation

IV. 

$\longleftrightarrow$ $\overline{R}^{ab}_{cd} = M_{ai} R^{ia}_{dj} M^{jb}$ Twist

(See KAUFFMAN [1990b, 1990a].)

Many link polynomials fall directly in this framework. For example, the bracket model for the Jones polynomial (See Section 3) is modelled via

$$(M_{ab}) = (M^{ab}) = M = \begin{bmatrix} 0 & \sqrt{-1}\,A \\ -\sqrt{-1}\,A^{-1} & 0 \end{bmatrix}$$

and

$$R^{ab}_{cd} = AM^{ab}M_{cd} + A^{-1}\delta^a_c\delta^b_d.$$

Now, the remarkable thing about this approach is that it is directly related to the non-commutative Hopf algebra constructions (called quantum groups) of Drinfeld and others (See DRINFELD [1986], MANIN [1988], and RESHETIKHIN [1987]). In particular, the so-called Double Construction of Drinfeld exactly parallels these extended Reidemeister moves (extended by the conditions related to creation and annihilation). (For example, the twist move corresponds to the existence of an antipode in the Hopf algebra via the use of the Drinfeld universal solution to the Yang-Baxter Equation.)

This context of link invariants as vacuum-vacuum amplitudes is a good context in which to ask the question:

? **837. Problem 13.** *Do these vacuum-vacuum amplitude invariants completely classify knots and links?*

The abstract tensor formalism of cup, cap and interaction satisfying only the properties we have listed does give a faithful translation of the regular isotopy classes of knots and links into a category of formal tensor products. In order to calculate an invariant these tensor symbols must be replaced by actual matrices.

? **838. Problem 14.** *I conjecture that for a given pair of links that are distinct, there exists a representation of the abstract tensor formalism that distinguishes them. (In fact I conjecture that there is a representation of the Drinfeld double construction, i.e., a quantum group, that distinguishes them.)*

The abstract tensor structures are related to the duality structure of conformal field theories, and to invariants of three manifolds obtained in a number of related ways (Crane [1989], Reshetikhin [1989], Witten [1989]). There is not space in this problem list to go into the details of all these constructions. However, the basic idea behind the constructions of the three-manifold invariants in the Reshetikhin-Turaev approach is to add extra conditions to the link polynomials so that they become invariants of framed links and so that they are further invariant under the Kirby moves (See Kirby [1978]). This insures that the resulting polynomials are invariants of the three manifold obtained by surgery on the framed link. The fundamental group of the three manifold is obtained as a quotient of the fundamental group of the given link complement.

Problem 15. *We now face the important question of the sensitivity of these new invariants of three-manifolds to the fundamental group of the three-manifold.* **839. ?**

If the new three manifold invariants can be non-trivial on simply connected compact three-manifolds, then there will exist a counterexample to the classical Poincaré Conjecture. The structure of these new invariants will provide a long sought after clue to the solution of this venerable conundrum. (The *Poincaré Conjecture* asserts that a compact simply connected three manifold is homeomorphic to the standard three dimensional sphere.)

8. Spin-Networks and Abstract Tensors

Another relationship between quantum networks and three dimensional spaces occurs in the Penrose theory of spin networks (Penrose [1971]). Here the formalism of spin angular momentum in quantum mechanics is made into a purely diagrammatic system. Each spin network is assigned a combinatorially computed norm. (The Penrose norm has the form of a vacuum-vacuum expectation for the whole network, but here the network is not embedded in a space-time. This bears an analogical relation with the amplitudes for knots and links that depend only upon the topology of the embedding into space-time and not upon any given choice for an arrow of time.) These norms, in turn can be used to compute probabilities of interaction between networks, or between parts of a given network.

Probabilities for interaction lead to a definition of angle between networks. The angle is regarded as well-defined if two repeated measurements yield the same result. The upshot of the Penrose work is the *Spin-Geometry Theorem* that states that *well-defined angles between subnetworks of a (large) network obey the dependency relation of angles in a three dimensional space.* In other words, the properties of three dimensional space begin to emerge from the abstract relations in the spin networks.

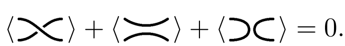
One would hope to recover distances and even space-time in this fashion. The Penrose theory obtains only angles in a fundamental way.

? **840. Problem 16.** *I conjecture* (KAUFFMAN [1990b]) *that a generalization of the spin networks to networks involving embedded knotted graphs will be able to realize the goal of a space-time spin geometry theorem.*

Here it must be understood that the embedding space of the knotted graphs is not the final space or three manifold that we aim to find. In fact, it may be possible to use a given embedding of the graph for calculating spin network norms, but that these norms will be essentially independent of the embedding (just so the Penrose spin nets are calculated through a planar immersion of the net, but they depend only on the abstract net and cyclic orders attached to the vertices).

In this vision, there will be constructions for new three dimensional manifolds, and these manifolds will carry the structure of significant topological invariants in the networks of which they are composed.

In order to bring this discussion down to earth, let me give one example of how the spin networks are already generalized by the vacuum-vacuum, amplitude models for the Jones polynomial. If we take the bracket (Section 3) at the value $A = -1$, then the bracket relation becomes



This relation is identical to the generating relation for the *Penrose binor calculus*—a translation of $SL(2,C)$ invariant tensors into diagrammatic language. The binor calculus is the underpinning of the spin networks. As A is deformed away from -1 (or from 1) the symmetry of these networks becomes the quantum group $SL(2)q$. $(A = \sqrt{\varepsilon})$

Thus the link diagrams as abstract tensor diagrams already show themselves as a generalization of the spin networks.

9. Colors Again

To come fully down to earth from Section 8, here is a spin network calculation that computes the number of edge colorings of a trivalent plane graph:

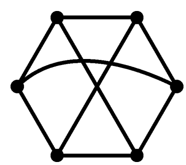
Associate to each vertex in the graph the tensor $\sqrt{-1}\epsilon_{abc}$ where ϵ_{abc} denotes the alternating symbol—that is, a, b and c run over three indices $\{0,1,2\}$; the epsilon is zero if any two indices are the same, and it is the sign of the permutation abc when the three indices are distinct. Call an assignment of indices to all the edges of the graph an em edge coloring if each vertex receives three distinct indices. For each edge coloring σ of G, let $\|\sigma\|$ denote the product of the values $\sqrt{-1}\epsilon_{abc}$ from each vertex. Thus $\|\sigma\|$ is the product of the vertex weights assigned by this tensor to the given edge coloring. Define the *norm*, $\|G\|$ of a graph G to be the sum of these products of vertex weights, summing over all edge colorings of the graph.

Then one has the following result:

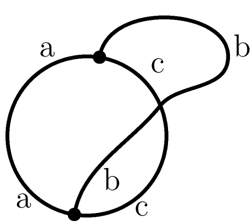
9.1. Theorem (Penrose [1971]). *If G is a trivalent plane graph, then the norm of G, $\|G\|$, is equal to the number of edge colorings of G. In fact the norm of each coloring is $+1$ if G is planar. In general, the norm for immersed graphs (with edge-crossing singularities) obeys the following equation*

with the value of a collection of (possibly overlapping) closed loops being three (3) raised to the number of loops. □

In this theorem, the norm can be evaluated for non-planar graphs by choosing a singular embedding of the graph in the plane, and then computing the norm as before. (Crossing lines may be colored differently or the same. We are actually coloring the abstract graph.) The immersion of the graph in the plane gives a specific cyclic order to the edges of each vertex, and this determines the norm computation. Of course, any graph with no colorings receives a norm zero, but non-planar graphs that have colorings can also receive norm zero. For example, the graph below has zero norm:

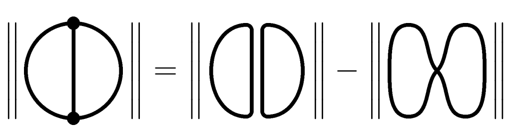


An embedding with singularities may not enumerate all the colorings with positive signs. The simplest example is



$$(\sqrt{-1}\varepsilon_{abc})(\sqrt{-1}\varepsilon_{abc}) = -(\varepsilon_{abc})^2.$$

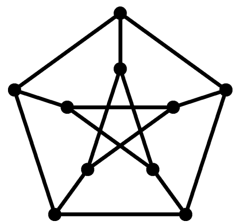
Note how the recursion formula works:



$$= 3^2 - 3 = 6.$$

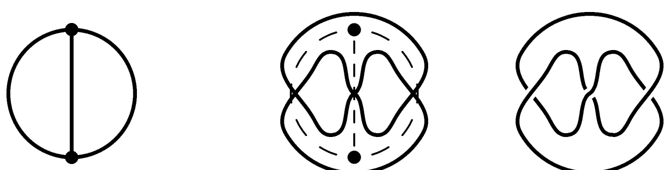
Remark: The problem of edge colorings for trivalent graphs is well-known to be equivalent to the four color problem for arbitrary plane graphs. Thus the spin network evaluation is another instance of the four color problem living in relation to a context composed of combinatorics, knot theory and mathematical physics. The relation with the knot theory could be deepened if we could add crossing tensors to the Penrose formula so that it computed the coloring number for arbitrary (not necessarily planar) trivalent graphs. This is a nice challenge for the knot diagrammatic approach.

The simplest known *snark* (a snark is a non-edge colorable trivalent graph) is the Petersen graph—shown below:



The Peterson graph is to combinatorics as the Möbius strip is to topology—a ubiquitous phenomenon that insists on turning up when least expected. Tutte has conjectured that the Petersen graph must appear in any snark.

This chromatic spin-network calculation can be reformulated in terms of link diagrams if we restrict ourselves to plane graphs. Then the medial construction comes directly into play: Take the medial construction for the trivalent graph.



G $K(G)$

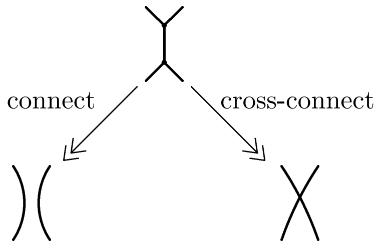
Associate link diagrammatic crossings to the crossings in the medial construction to form an alternating link (as we have done in Section 4). Define a state expansion on link diagrams via

where the value of a collection of loops (with singular crossings) is three to the number of loops. This norm computes the same coloring number as the Penrose number. (Exercise!)

Heuristics

One of the advantages of the coloring problem in relation to our concerns about state models and topology is that, while the coloring problem is very difficult, there are very strong heuristic arguments in favor of the conjecture that four colors suffice to color a plane map, (and equivalently that trivalent plane maps without loop or isthmus can be edge colored with three colors distinct at each edge.)

At an edge in a trivalent map I shall define two operations to produce smaller maps. These operations are denoted *connect* and *cross-connect* as illustrated below:

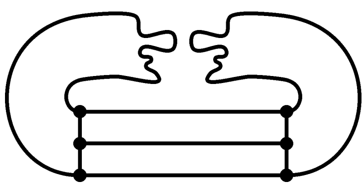


Call a trivalent map that has no edge coloring and that is minimal with respect to this property *critical*. It is obvious that if G is critical and we form H by connecting or cross-connecting an edge of G, then *the two local edges produced by the operation must receive the same color in any coloring of H.* (If they are different, then it is trivial to produce a coloring of G by using a third color on the edge deleted by the operation.) Call a pair of edges *twins* if they must receive the same color in any coloring of a graph H. Say that G *forces twins at an edge e of G* if the edges resulting from both connect and cross-connect at e are twins.

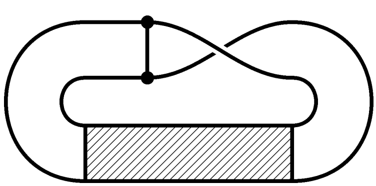
Thus

9.2. Theorem. *A critical trivalent map G forces twins at every edge of G.* □

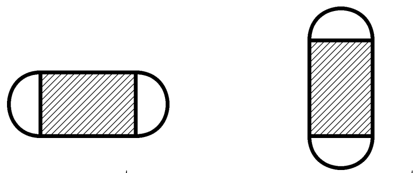
In order to design a critical map it is necessary to create maps with twins. A simple example of a pair of twins is shown below.



To give a feel for the design problem, suppose that the diagram below represents a trivalent critical map. (I have drawn it in a non-planar fashion for convenience. We are discussing the matter of design of critical maps in the abstract)

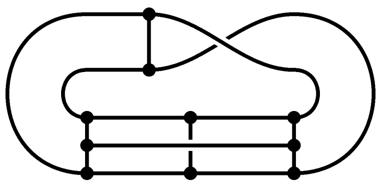


If this map is critical then the edge that is shown must force twins. The pattern of the forced pairs simplifies as shown below.



connect cross-connect

The simplest example that I can devise to create forcing from both pairs is as shown below



This graph is isomorphic to the Petersen graph.

Thus we have seen how one is lead inevitably to the Petersen graph in an attempt to design critical trivalent maps. The design side is a strong arena for investigating the coloring problem.

A similar arena exists in knot theory via the many examples that one can construct and compute, however I do not yet see the problem of designing knots that are undetectable in any similar light. The graph theory may yield clues. Time will tell.

? 841. **Problem 17.** *Can Knot Theory solve the Four Color problem and what does the truth of the four color theorem imply for three-dimensional topology?*

10. Formations

A diagrammatic approach to coloring trivalent maps clarifies some of the issues of Problem 17, and allows us to raise a central issue about map coloring. This diagrammatic technique goes as follows. Regard the three colors as red (————), blue (-----) and purple (-----). That is, regard one color (purple) as a superposition of the other two colors, and diagram red by a solid line, blue by a dotted line, and purple by a combination dotted and solid line.

With this convention, any edge three coloring of a trivalent graph has the appearance of two collections of Jordan curves in the plane. One collection consists in red curves. The other collection has only blue curves. The red curves are disjoint from one another, and the blue curves are disjoint from one another. Red and blue curves share segments corresponding to the edges that are labelled purple. Thus red curves and blue curves can interact by either sharing a segment without crossing one another (a *bounce*), or by sharing in the form of a crossing (a *cross*).

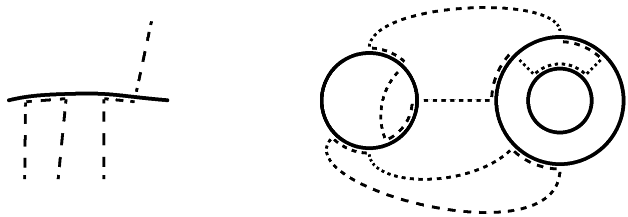


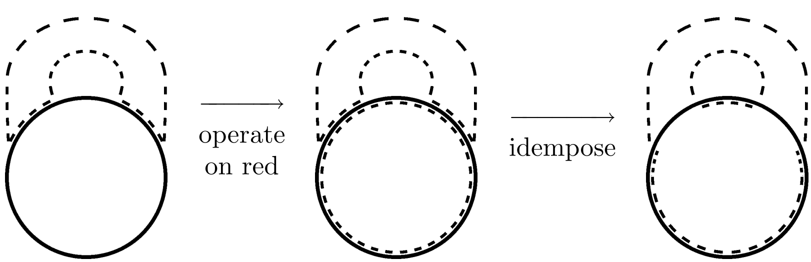
Figure 9: Bouncing and Crossing

These two forms of interaction are illustrated in Figure 9. I call a coloring shown in this form of interacting Jordan curves a *formation*. The terminology formation and the idea for this diagrammatic approach to the coloring problem is due to G. SPENCER-BROWN [1979].

The existence of edge colorings for trivalent plane maps is equivalent to the existence of formations for these maps. This point of view reveals structure. For example, we see at once that the product of imaginary values (for a given coloring) in the spin network calculation of the norm, $\|G\|$, for planar G is always equal to one. For each bounce contributes 1, while each crossing contributes -1, and the number of crossings of a collection of Jordan curves in the plane is even.

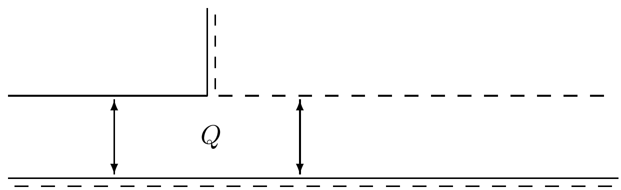
A deeper result has to do with parity. In an edge three coloring the parity of the total number of alternating color cycles (called here *cinguli*) remains unchanged under the operation of switching a pair of colors along a cingulus (a *simple operation*). One can use the language of formations to prove this result (see SPENCER BROWN [1979], KAUFFMAN [1986], and compare with TUTTE [1948]).

In the language of formations a simple operation is accomplished by drawing a curve of one color along a curve of the opposite color (red and blue are opposite, as are red/blue alternating and purple). (*Note that in a formation the red cinguli index cycles of alternating red and purple, while the blue cinguli index cycles of alternating blue and purple.*) After the curves are superimposed, *common colors are cancelled.* This cancellation is called *idemposition.* For example



While parity is preserved under simple operations, the parity necessarily changes under a *Spencer-Brown switching operation* (G. SPENCER-BROWN

[1979]) at a five-region. Spencer-Brown's operation is performed to replace one extension problem by another. We are given a configuration as shown below:



This configuration, I shall call a Q-region. It has two missing edges denoted by arrows. If these edges could be filled in to make a larger formation, the result would be a coloring of a larger map.

The problem corresponds to having a map that is all colored except for one five-sided region.

? **842.** **Problem 18.** *One wants to rearrange the colors on the given map so that the coloring can be extended over the five-sided region.*

If one can always solve this problem then the four-color theorem follows from it.

In the category of non-planar edge three-colorings it is possible for a Q-region problem to have no solution involving only simple operations.

The switching operation replaces the Q-region by another Q-region, and changes the parity in the process. In the language of formations, the switch is performed by drawing a red curve that replaces one missing edge, idemposes one edge, and travels along a blue cingulus in the original formation to complete its journey. See the example below.

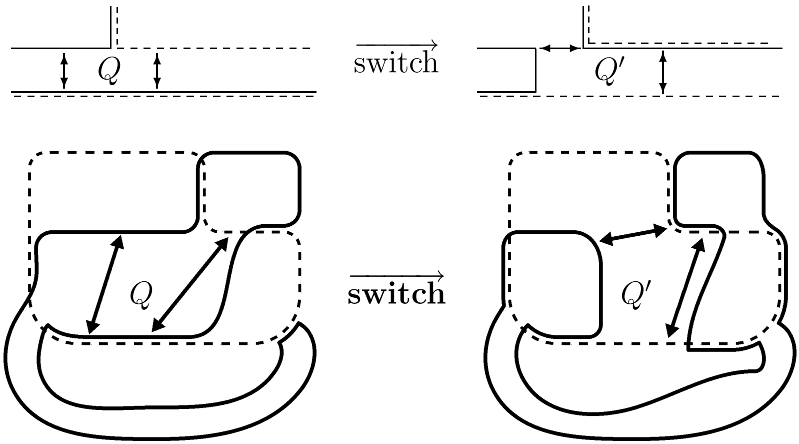
? **843.** **Problem 19.** **Switching Conjecture:** *I conjecture that a Q-region in a planar formation that is unsolvable by simple operations before the Spencer-Brown switch becomes solvable by simple operations after the switch.*

Of course this conjecture would solve the four color problem, and one might think that it is too good to be true. I encourage the reader to try it out on formations of weight three (that is with exactly one red, one blue, and one red/blue alternating cingulus). Q-regions at weight three are always unsolvable by simple operations. Switch a weight three Q-region problem and you will find higher weight (since parity changes and weight can't go down).

The switching conjecture aside, it is now possible to indicate a proof of the four color theorem that is due to G. Spencer-Brown.

Spencer-Brown's Proof:

It suffices to consider a formation with a Q-region. If the weight is larger than



three then there is an extra cingulus (red, blue or red/blue alternator) other than the three cinguli involved at the Q-region. If this extra cingulus can be used in a sequence of simple operations to solve the Q-region then we are done. If this cingulus can not be used in any such sequence, then the extra cingulus is *ineffective*, and from the point of view of the Q-region it is invisible to the problem. Hence the formation with an ineffective cingulus is structurally smaller, and hence is solved by induction. If there is no extra cingulus in the formation, then the weight is equal to three. Apply the switching operation. Now the weight is greater than three. Hence there is an extra cingulus, and the first part of the argument applies. **Q.E.D.**

Problem 20. *Understand this proof!* **844. ?**

The crux of the matter in bringing this proof to earth lies in understanding the nature of an effective cingulus. The proof is an extraordinary guide to understanding the map color problem. In G. Spencer-Brown [1979] the argument is extended to show that a formation with an extra cingulus can always be solved by complex operations.

Of course one would like to know what is the relationship among quantum physical, statistical mechanical and topological structures and these deep combinatorial matters of the coloring problem. Full understanding of the four color theorem awaits the unfolding of these relationships.

11. Mirror-Mirror

The last problem on this set is a conjecture about alternating knots that are *achiral*. A knot is achiral if it is ambient isotopic to its mirror image.

We usually take the mirror image as obtained from the original diagram by switching all the crossings. The mirror is the plane on which the diagram is drawn.

Let $G(K)$ denote the graph of the diagram K. That is, $G(K)$ is obtained from a checkerboard shading of the diagram K (unbounded region is shaded white). Each black region determines a vertex for $G(K)$. $G(K)$ has an edge for each crossing of K that is shared by shaded regions.

Let $M(K)$ denote the cycle matroid of $G(K)$ (See WELSH [1988] for the definition of the matroid.). Let $M^*(K)$ denote the dual matroid of $M(K)$.

? **845. Problem 21. Conjecture.** *K is alternating, reduced and achiral, if and only if $M(K)$ is isomorphic to $M^*(K)$ where $M(K)$ is the cycle matroid of $G(K)$ and $M^*(K)$ is its dual.*

This conjecture has its roots in the observation that for all the achiral reduced alternating knots of less than thirteen crossings, the graphs $G(K)$ and $G^*(K)$ (the planar dual) are isomorphic. One might conjecture that this is always the case, but Murasugi has pointed out that it is not so (due to flyping—compare with Section 5). The matroid formulation of the conjecture avoids this difficulty.

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