Complex Numbers

Review,
Complex Plane and Rectangular Form,
Euler's Form and Manipulations,
Roots of Unity and Vietta's

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Review

Complex Numbers, $\mathbb{C}$ are formed by extending the Real numbers with a number i such that

$$\boxed{i^2 = -1}$$

or alternately

$$i = \sqrt{-1}$$

This forms two dimensional numbers which have a real part, a and an "imaginary part" b:

$$\boxed{z = a + bi}$$

Review

The **Complex Conjugate** of a number is defined as

$$\boxed{\overline{z} = a - bi} \text{ for } z = a + bi$$

The product of a complex number and it's conjugate is always a real number:

$$\boxed{z\overline{z} \in \mathbb{R}}$$

Review

The **Extended Complex Numbers** is an extension of the set of complex numbers:

$$\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$$

Which will mostly be relevant in complex analysis, not so much in our competitive problems.

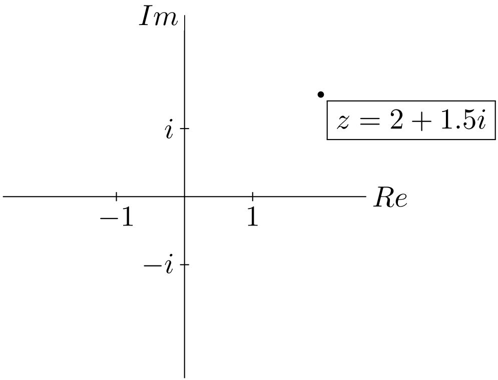
Order

We see that the imaginary unit has an exponential **Order** to it:

$$
\begin{aligned}
i^1 &= i \\
i^2 &= -1 \\
i^3 &= -i \\
i^4 &= 1 \\
i^5 &= i && = i^1 \\
i^6 &= -1 && = i^2 \\
i^7 &= -1 && = i^3 \\
&\ldots
\end{aligned}
$$

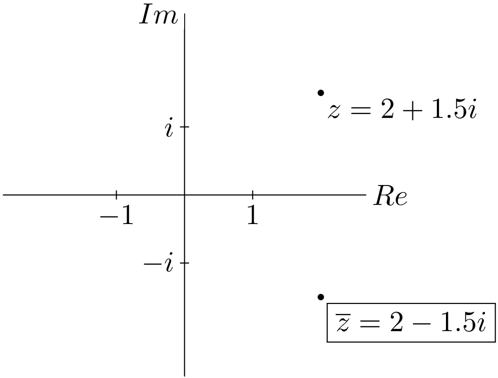
Complex Plane

What makes complex numbers so relevant is their two degrees of freedom. Because there are two dimensions to a complex number, we can plot the complex numbers on the **Complex Plane**:



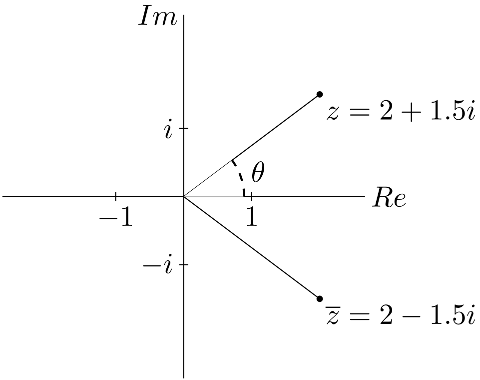
Complex Plane

We can plot the complex conjugate of a number as well.



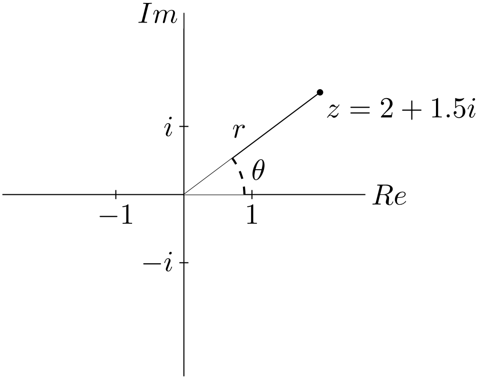
Complex Plane

The **Argument** of a complex number is the angle θ that the number makes with the positive real axis. Notice that the angle the conjugate makes with the positive real axis is $-\theta$.



Complex Plane

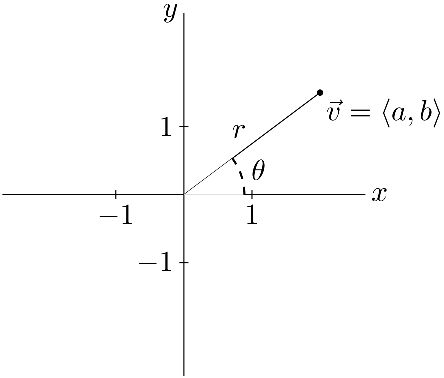
The **Magnitude** r of a complex number is it's distance from the origin. Otherwise notated as the **Absolute Value** of a complex number, $|z| = r$



Questions?

Review

When we have two dimensional vectors, we can represent them with **Cartesian Coordinates** and we can represent them with **Polar Coordinates**.



Review

We can *always* convert between the two via a reparameterization:

$$\vec{v} = \langle a, b \rangle = \langle r, \angle\theta \rangle$$

And we can go between the two:

$$\boxed{r = a^2 + b^2}$$

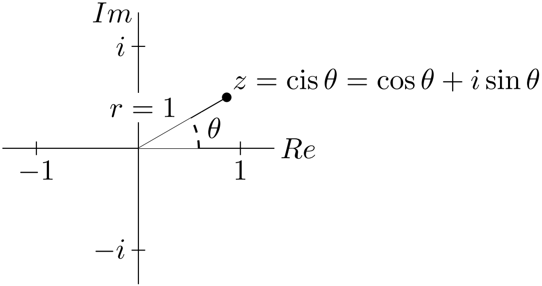
$$\boxed{\theta = \arctan\left(\frac{b}{a}\right)}$$

CIS

CIS is an abbreviation for

$$\boxed{\operatorname{cis}\theta = \cos\theta + i\sin\theta}$$

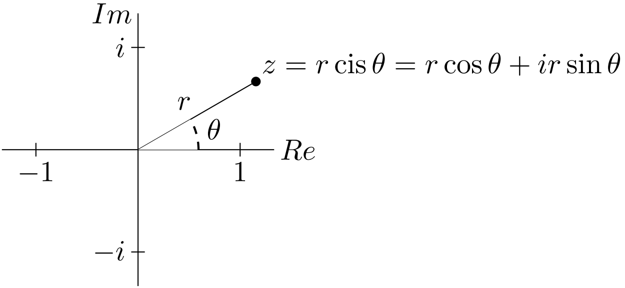
Which places a cartesian representation of an angle in the complex plane.



CIS

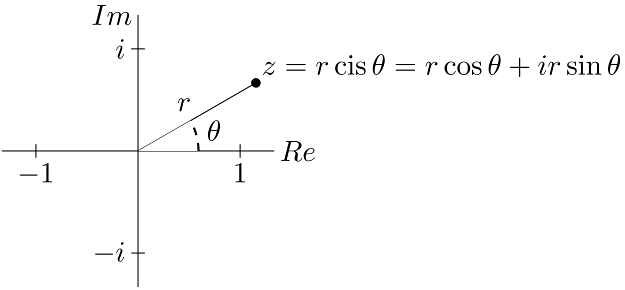
As a matter of fact, we can represent any and all complex numbers as an argument multiplied by a real number for it's radius:

$$\boxed{z = r\operatorname{cis}\theta = r\cos\theta + ir\sin\theta}$$



CIS

All we're doing here is saying that some complex number z can be represented as a cartesian coordinates, where the radius is the hypotenuse and thus the a and b is just the sides of the triangle, found with trig functions of θ



De Moivre's Theorem

Although we're not going to prove the theorem itself, **De Moivre's Theorem** is the most important theorem in our study of complex numbers, and provides the foundation of a key representation of complex numbers. De Moivre's Theorem states that

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)}$$

or

$$\boxed{(\operatorname{cis}\theta)^n = \operatorname{cis}(n\theta)}$$

Which is wild, and not in any way intuitive!
You really don't even have to remember this theorem for the most part, but you *must* know it is true.

Euler's Formula

Euler's Formula is a bit of an extension on De Moivre's Theorem, but is just as unintuitive. First, let's look at Euler's Formula:

$$\boxed{\cos\theta + i\sin\theta = e^{i\theta}}$$

Which is absolutely wild!

This you **must** remember. This will become second nature to you with some experience with complex numbers.

Euler's Formula

$$\boxed{\cos\theta + i\sin\theta = e^{i\theta}}$$

Now, we know what you're wondering.

Why is this true? Why is e raised to an imaginary power equal to the sum of $\cos\theta$ and $i\sin\theta$?

Euler's Formula

$$\boxed{\cos\theta + i\sin\theta = e^{i\theta}}$$

Now, we know what you're wondering.
Why is this true? Why is e raised to an imaginary power equal to the sum of $\cos\theta$ and $i\sin\theta$?
Long story short, **Don't worry about it**. It's a beautiful derivation, but for today's purposes, will only confuse you. You really only need to be concerned with the fact that it is true, not why.

Euler's Formula

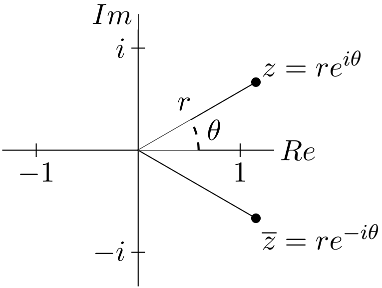
What this means is we *can* represent every number in two ways; rectangular form where $z = a + bi$, but also as $r\cos\theta + ir\sin\theta$ and thus also as $re^{i\theta}$:

$$\boxed{z = a + bi = r\cos\theta + ir\sin\theta = re^{i\theta}}$$

The fact that we can reparameterize the variable z from cartesian coordinates to polar coordinates is an incredibly important attribute of complex numbers, which a lot of problems rely on.

Euler's Form

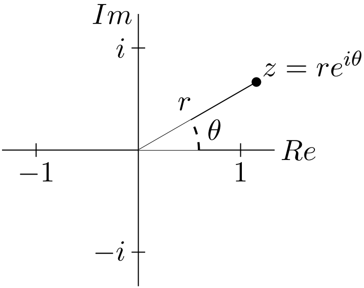
Let's get a better look at what this means on the complex plane.



Notice, the complex conjugate is the same but with a negative angle $-\theta$ instead.

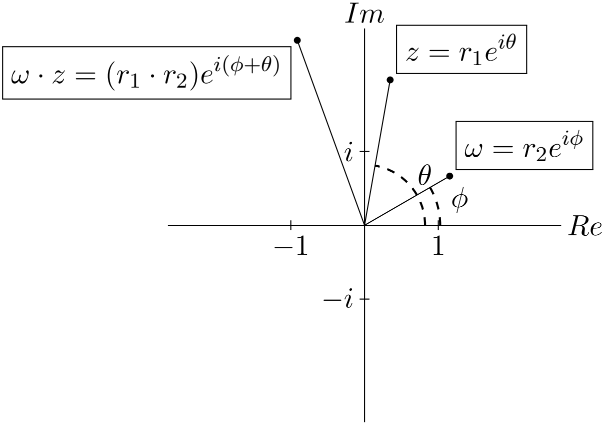
Euler's Form

Now let's get a better sense of what Euler's form really means in context of multiplying complex numbers.



Euler's Form

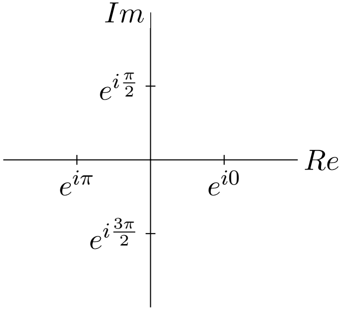
When we multiply two complex numbers, it's just like multiplying exponents. Take $z = r_1 e^{i\theta}$ and $\omega = r_2 e^{i\phi}$. We see that when we multiply them, the angles add in the exponents; $\phi + \theta$, and the radii multiply; $r_1 \cdot r_2$:



Euler's Form

Let's look at some special cases of Euler's Form:

$$\begin{aligned} \theta &= 0 & e^{i0} &= 1 \\ \theta &= \frac{\pi}{2} & e^{i\frac{\pi}{2}} &= i \\ \theta &= \pi & e^{i\pi} &= -1 \\ \theta &= \frac{3\pi}{2} & e^{i\frac{3\pi}{2}} &= -i \end{aligned}$$



Euler's Form

Let's look at an example problem.

A function f is defined by $f(z) = i\overline{z}$.
How many values of z satisfy both
$|z| = 5$ and $f(z) = z$?

Hmm... How do we go about this?

Euler's Form

A function f is defined by $f(z) = i\overline{z}$.
How many values of z satisfy both
$|z| = 5$ and $f(z) = z$?

First thing we should do is state exactly what the problem is asking:

$$f(z) = \boxed{i\overline{z} = z}$$

Now this is something we can work with. Let's put z in Euler's Form:

$$\boxed{z = re^{i\theta}}$$

Euler's Form

A function f is defined by $f(z) = i\overline{z}$. How many values of z satisfy both $|z| = 5$ and $f(z) = z$?

$$i\overline{z} = z$$

$$z = re^{i\theta}$$

Even better, the problem gives us $r = 5$, so now we have $z = 5e^{i\theta}$, and we can plug this into our equation:

$$i5e^{-i\theta} = 5e^{i\theta}$$

and simplifying we get

$$ie^{-i\theta} = e^{i\theta}$$

Ah-ha!

Euler's Form

A function f is defined by $f(z) = i\overline{z}$. How many values of z satisfy both $|z| = 5$ and $f(z) = z$?

$$i\overline{z} = z$$

$$ie^{-i\theta} = e^{i\theta}$$

Since we know $i = e^{i\frac{\pi}{2}}$, we can plug this in:

$$ie^{-i\theta} = e^{i\frac{\pi}{2}}e^{-i\theta} = e^{i\frac{\pi}{2}-i\theta} = e^{i\theta}$$

And now we can just look at the exponents!

$$i\frac{\pi}{2} - i\theta = i\theta$$

Euler's Form

A function f is defined by $f(z) = i\overline{z}$.
How many values of z satisfy both
$|z| = 5$ and $f(z) = z$?

$$i\frac{\pi}{2} - i\theta = i\theta$$

And with this, we find

$$\frac{\pi}{2} = 2\theta$$

There are two solutions to this equation,

$$\boxed{\theta = \frac{\pi}{4}} \text{ and } \boxed{\theta = \frac{-3\pi}{4}} \text{ with } r = 5$$

Questions?

Roots of Unity

The n-th **Roots of Unity** are simply the complex solutions to the equation

$$x^n - 1$$

Which, by setting equal to 0 and solving, we find

$$x^n - 1 = 0$$

and

$$x = \sqrt[n]{1}$$

With our knowledge of complex numbers, we're going to be able to find exact complex solutions to this.

Roots of Unity

The n-th **Roots of Unity** are simply the complex solutions to the equation

$$x^n - 1$$

$$x = \sqrt[n]{1}$$

We don't really know where to go from here. But what can we substitute in for 1 in order to find an exact value?

Roots of Unity

The n-th **Roots of Unity** are simply the complex solutions to the equation

$$x^n - 1$$

$$x = \sqrt[n]{1}$$

We know that $e^{2\pi i} = 1$, so we can substitute that in for 1 in the equation:

$$x = \sqrt[n]{e^{2\pi i}}$$

which gives us:

$$x = e^{\frac{2\pi i}{n}}$$

Roots of Unity

$$x^n - 1$$

$$x = \sqrt[n]{1}$$

What's even better is that not only is $e^{2\pi i} = 1$, but when we raise this to an integer power k, it's still equal to 1, $e^{2\pi i \times k} = 1$, so we can substitute this in as well:

$$\boxed{x = e^{\frac{2\pi \times k}{n}}}$$

With given values for k and n, we have determined values for x.

Roots of Unity

$$x^n - 1$$

$$x = \sqrt[n]{1}$$

$$\boxed{x = e^{\frac{2\pi \times k}{n}}}$$

What happens if we plot these on the complex plane?

Roots of Unity

$$x^n - 1$$

$$x = \sqrt[n]{1}$$

$$\boxed{x = e^{\frac{2\pi \times k}{n}}}$$

What happens if we plot these on the complex plane? Because of Euler's Formula and De Moivre's Formulas:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

We can intuit that we would have a regular n-gon on the complex plane.
Let's look at an example of this.

Roots of Unity

An example of this are the 3rd roots of unity:

$$x^3 - 1 = 0$$

and

$$x = e^{\frac{2\pi i \times 0}{3}}, e^{\frac{2\pi i \times 1}{3}}, e^{\frac{2\pi i \times 2}{3}}$$

and we can plot these:

Vietta's Formula

Now that we know the nature of the solutions to the equations $x^n - 1$, we can also factor them into their linear factors:

$$x^n - 1 = (x-1)(x - e^{\frac{2\pi}{n}}) \cdots (x - e^{\frac{2\pi \times (n-1)}{n}})$$

This can be an important thing to remember regarding complex functions.

Questions?