# *Review!:* *UMD Competition, #16*

*A square $ABCD$ has side length $|AB| = 1$. Points $E$ and $F$ are taken on the sides $BC$ and $CD$ such that $AEF$ is an equilateral triangle. What is the area of $AEF$?*

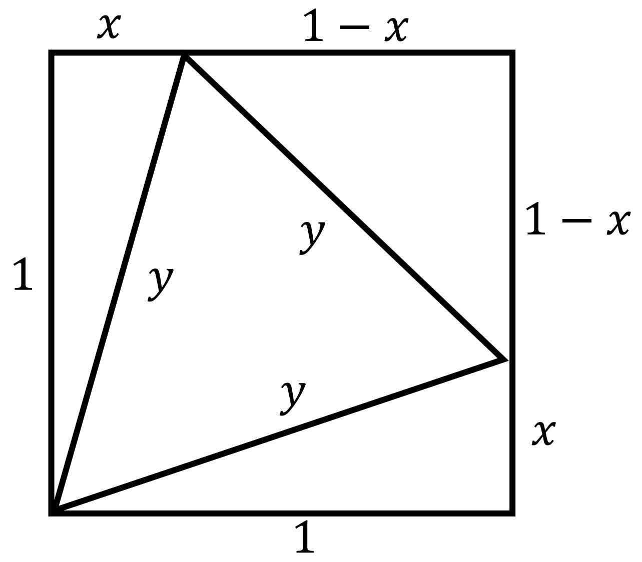
*Two circles, $C_1$ and $C_2$ are tangent on the same side as line $\ell$ at $A$ and $B$. $\overline{AB} = 20$. Their radii are $1$ and $16$. A third circle, $\omega$ is tangent to all three.*

*What is the sum of all possible radii of this third circle $\omega$?*

# Review!: UMD Competition, #16

*A square $ABCD$ has side length $|AB| = 1$. Points $E$ and $F$ are taken on the sides $BC$ and $CD$ such that $AEF$ is an equilateral triangle. What is the area of $AEF$?*

We see that we can split this square into 4 triangles, with two right triangles having side lengths of $x$ and 1 and another having side lengths $1 - x$ and $1 - x$.




With the Pythagorean theorem we solve for $x$

# *Review!:* UMD Competition, #16

*A square $ABCD$ has side length $|AB| = 1$. Points $E$ and $F$ are taken on the sides $BC$ and $CD$ such that $AEF$ is an equilateral triangle. What is the area of $AEF$?*

We see that

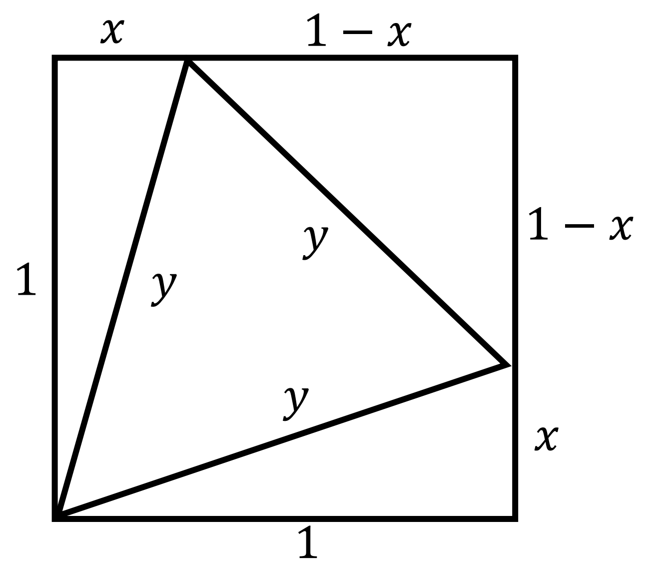
$1^2 + x^2 = y = 2(1-x)^2$

$1 + x^2 = 2 - 4x + 2x^2$

$x^2 - 4x - 1 = 0$

$x = 2 - \sqrt{3}$

Now we solve for $y$. We find this equals $\sqrt{6} - \sqrt{2}$, and from there we use $y, y\sqrt{3}, {}^{y}/_{2}$ to find the area of an equilateral triangle.

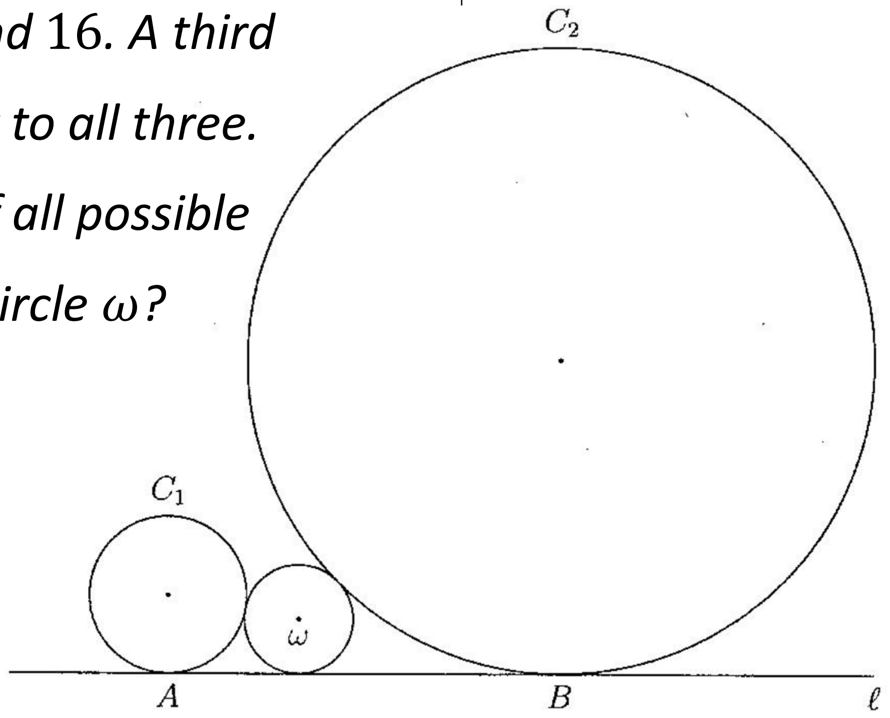


# *Review!:* *UMD Competition, #20*

*Two circles, $C_1$ and $C_2$ are tangent on the same side as line $\ell$ at $A$ and $B$.* $\overline{AB} = 20$.

*Their radii are $1$ and $16$. A third circle, $\omega$ is tangent to all three. What is the sum of all possible Radii of this third circle $\omega$?*

We approach this problem realizing our diagram gives us an excess of information. We start by annotating our given diagram.



# *Review!:* *UMD Competition, #20*

*Two circles, $C_1$ and $C_2$ are tangent on the same side as line $\ell$ at $A$ and $B$.* $\overline{AB} = 20$.

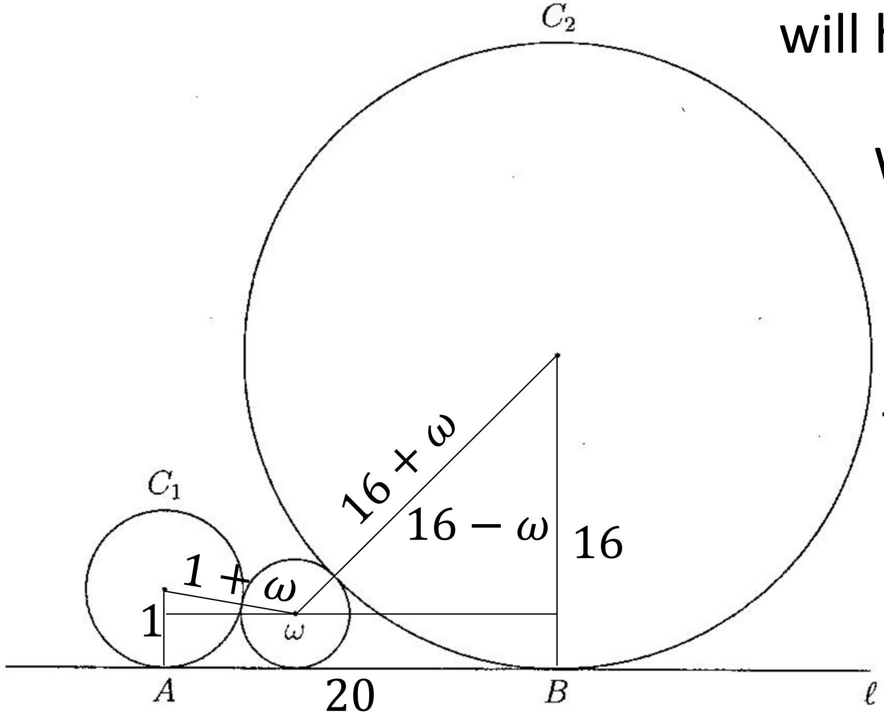
*Their radii are* $1$ *and* $16$. *A third circle, ω is tangent to all three.*

*What is the sum of all possible*

*Radii of this third circle $\omega$?*

We will add a particular line, intersecting $\omega$ and parallel to $\overline{AB}$, and equal in length to $\overline{AB}$. This line will help justify our algebra.

We see that we can use the Pythagorean Theorem to solve for $\omega$ in this instance.



# Review!: UMD Competition, #20

*Two circles, $C_1$ and $C_2$ are tangent on the same side as line $\ell$ at $A$ and $B$.* $\overline{AB} = 20$.

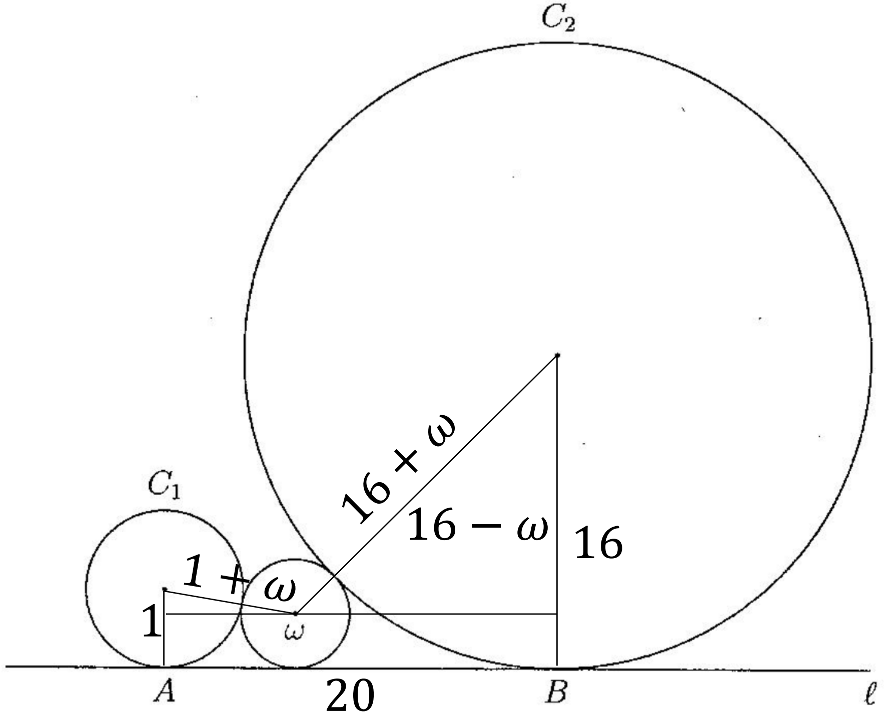
*Their radii are* $1$ *and* $16$. *A third circle,* $\omega$ *is tangent to all three. What is the sum of all possible Radii of this third circle* $\omega$?

$$20 = \sqrt{(1+\omega)^2 - (1-\omega)^2} + \sqrt{(16+\omega)^2 - (16-\omega)^2}$$

$$20 = \sqrt{4\omega} + \sqrt{64\omega} = 2\sqrt{\omega} + 8\sqrt{\omega} = 10\sqrt{\omega}$$

$$w = 4$$

Now for our second solution.



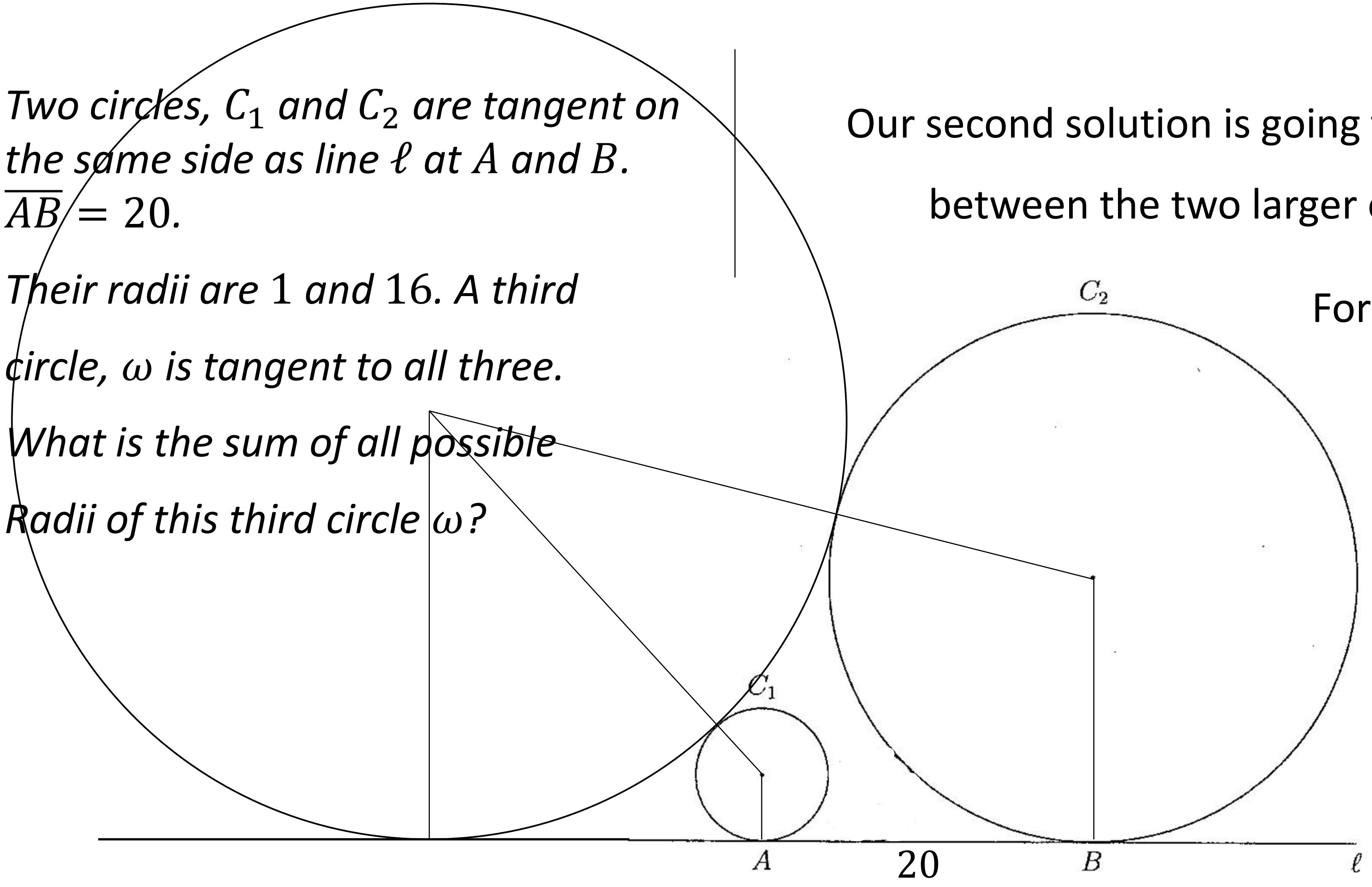
# Review!: UMD Competition, #20

*Two circles, $C_1$ and $C_2$ are tangent on the same side as line $\ell$ at $A$ and $B$. $\overline{AB} = 20$.*

*Their radii are $1$ and $16$. A third circle, $\omega$ is tangent to all three. What is the sum of all possible Radii of this third circle $\omega$?*

Our second solution is going to be if the circle is not in between the two larger ones, but exterior to one.

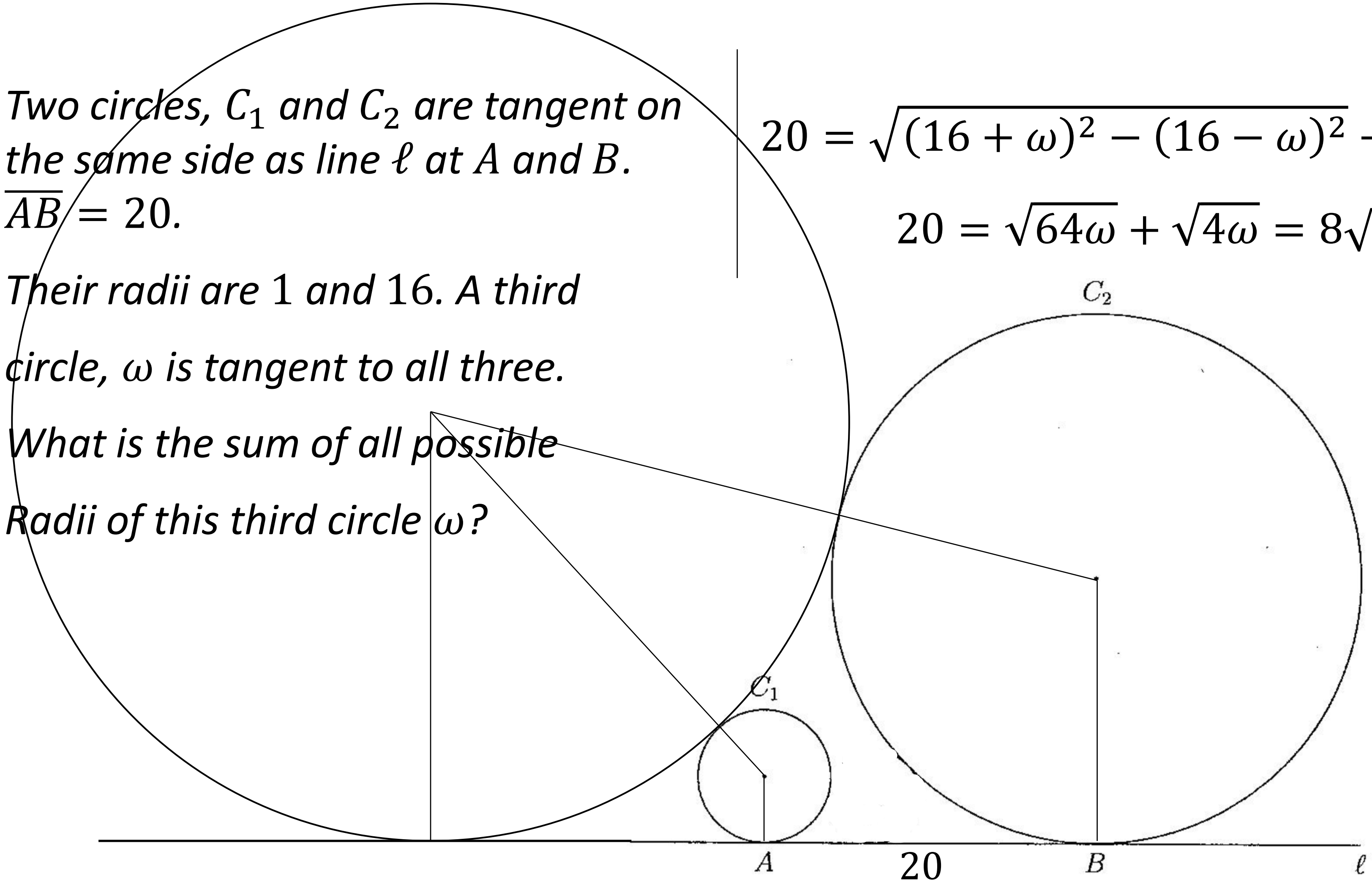
For this, we just subtract the larger value from the smaller one two acquire the radii.



# *Review!:* *UMD Competition, #20*

*Two circles, $C_1$ and $C_2$ are tangent on the same side as line $\ell$ at $A$ and $B$. $\overline{AB} = 20$.*

*Their radii are 1 and 16. A third circle, ω is tangent to all three. What is the sum of all possible Radii of this third circle $\omega$?*




$$20 = \sqrt{(16+\omega)^2 - (16-\omega)^2} - \sqrt{(1+\omega)^2 - (1-\omega)^2}$$

$$20 = \sqrt{64\omega} + \sqrt{4\omega} = 8\sqrt{\omega} - 2\sqrt{\omega} = 6\sqrt{\omega}$$

$$\sqrt{\omega} = {}^{20}/_{6} = {}^{10}/_{3}$$

$$\omega = {}^{100}/_{9}$$

The sum of the solutions is $4 + {}^{100}/_{9} = \frac{136}{9}$

# *Review!:* *UMD Competition, #16*

*For how many angles $x$ for $0 \leq x \leq 2\pi$*

*do we have*

$\sin x = \cos 3x$*?*

*Given*

$$\log(6!) = a\log(2) + b\log(3) + c\log(5)$$

*What is*

$a + b + c$*?*

# Review!: UMD Competition, #14

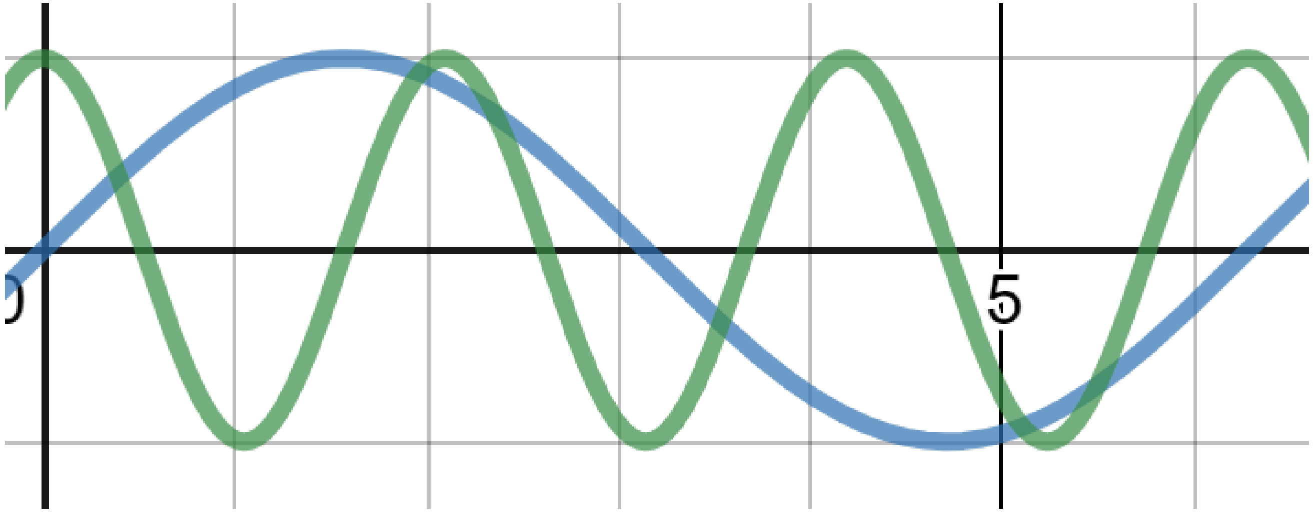
*For how many angles $x$ for*

$$0 \leq x \leq 2\pi$$

*do we have*

$$\sin x = \cos 3x?$$

We can solve this graphically.



# *Review!:* *UMD Competition, #4*

*Given*

$\log(6!)$

$= a\log(2) + b\log(3)$

$+ c\log(5)$

*What is*

$a + b + c?$

We know, based on the laws of logarithms,

$a\log(2) = \log(2^a)$ and so now we find

$$\log(6!) = \log(2^a) + \log\left(3^b\right) + \log(5^c)$$

And the additive property of logarithms we see

$$\log(6!) = \log\left(2^a 3^b 5^c\right)$$

And now we see

$$6! = 2^a 3^b 5^c$$

# *Review!:* *UMD Competition, #4*

*Given*

$\log(6!)$

$= a\log(2) + b\log(3)$

$+ c\log(5)$

*What is*

$a + b + c?$

$6! = 2^a 3^b 5^c$

$6! = 2^4 3^2 5^1$

$a + b + c = 4 + 2 + 1 = 7$

# Guided Discussion: Back into Number Theory

Slide Components

Problems

Walter Johnson Math Team

## *Guided Discussion: Looking into things*

One key concept of Number Theory which is useful in higher mathematics is the concept of a $unit\ mod\ m$. A unit is a number, $a$, such that there exists another number $n$ which satisfies

$$a^n \equiv 1\ mod\ m$$

These numbers are important! And not all numbers can satisfy this for different mods either.

The relation between these is clear, however.

$$3^1 \equiv 3, 3^2 \equiv 9, 3^3 \equiv 3\ mod\ 12$$

$$3^1 \equiv 3\ mod\ 11$$
$$3^2 \equiv 9\ mod\ 11$$
$$3^3 \equiv 5\ mod\ 11$$
$$3^4 \equiv 5 * 3 \equiv 15 \equiv 4\ mod\ 11$$
$$3^5 \equiv 4 * 3 \equiv 12 \equiv 1\ mod\ 11$$

This works because 3 and 11 are relatively prime, but 3 and 12 are not.

## *Guided Discussion: Food for Thought*

We're going to investigate some Phenomena that occur with polynomials in $\mathbb{Z}/m\mathbb{Z}$.

How many solutions are there to the polynomial

$$x^2 - 2x - 15 = 0?$$

Yeah, well not in $\mathbb{Z}/m\mathbb{Z}$

$$x^2 - 2x - 15 \equiv 0 \; mod \; 21$$

Now we can easily factor

$$(x - 5)(x + 3) \equiv 0 \; mod \; 21$$

And find solutions $x \equiv 5$ and $x \equiv -3 \equiv 18$, but if we notice that $15 \equiv 99 \; mod \; 21$, this polynomial is equal to

$$x^2 - 2x - 99 \equiv x^2 - 2x - 15 \equiv 0 \; mod \; 21$$

*Guided Discussion:*
*Food for Thought*

We’re going to investigate some Phenomena that occur with polynomials in $\mathbb{Z}/m\mathbb{Z}$.

How many solutions are there to the polynomial

$$x^2 - 2x - 15 = 0?$$

Yeah, well not in $\mathbb{Z}/m\mathbb{Z}$

$$x^2 - 2x - 99 \equiv x^2 - 2x - 15 \equiv 0\ mod\ 21$$

And we see that again, we can easily factor

$$(x - 11)(x + 9) \equiv 0\ mod\ 21$$

And we find solutions $x \equiv 11$ and $x \equiv -9 \equiv 12$. The polynomial has 4 distinct solutions.

This can be checked by setting the factors of 0 equal to our factors in the polynomial. Namely, the factors 6 and 14 of 21

# *Guided Discussion: Food for Thought*

We're going to investigate some Phenomena that occur with polynomials in $\mathbb{Z}/m\mathbb{Z}$.

How many solutions are there to the polynomial

$$x^2 - 2x - 15 = 0?$$

Yeah, well not in $\mathbb{Z}/m\mathbb{Z}$

$$x^2 - 2x - 99 \equiv x^2 - 2x - 15 \equiv 0 \; mod \; 21$$

This can be checked by setting the factors of 0 equal to our factors in the polynomial. Namely, the factors 6 and 14 of 21

We check $x - 5 \equiv 6$ and $x + 3 \equiv 14$, and indeed 11 is a solution to both. Many such cases should be checked to ensure all solutions are found.

# Problems: AIME, UMD HS Math Competition

*How many integers $m$ exist such that*

$$m = \frac{4 * 10^n - 1}{13}$$

*Where $n$ is an integer less than 50?*

*Suppose $n_0, n_1, n_2$ … is a sequence of integers satisfying all the following:*

- $0 \le n_k \le 123$ *for every* $k \ge 0$
- *For every* $k \ge 0$, $n_{k+1}$ *is the remainder when* $2n_k + 1$ *is divided by* $124$
- $n_1 \ne n_0$

*What is the least possible value of $\ell > 0$ for which $n_\ell = n_0$?*

# *Problems:* *AIME, UMD HS Math Competition*

*How many integers $m$ exist such that*

$$m = \frac{4 * 10^n - 1}{13}$$

*Where $n$ is an integer less than 50?*

Well we start off my looking at this, realizing that $m$ is indeed only an integer if $4 * 10^n - 1$ is divisible by 13, and thus we get the congruency

$$4 * 10^n - 1 \equiv 0 \; mod \; 13$$

Which can be re-written

$$4 * 10^n \equiv 1 \; mod \; 13$$

# *Problems:* *AIME, UMD HS Math Competition*

*How many integers $m$ exist such that*

$$m = \frac{4 * 10^n - 1}{13}$$

*Where $n$ is an integer less than 50?*

$$4 * 10^n \equiv 1 \; mod \; 13$$

Solving for $n$, we should start looking for $4^{-1} \; mod \; 13$, or 4 inverse $mod \; 13$.

After trial and error, we find that

$$1 = 4 * 4^{-1} = 4 * 10 \equiv 1 \; mod \; 13$$

And so we multiply both sides by $10 \equiv 4^{-1}$

$$10^n \equiv 10 \; mod \; 13$$

# *Problems:* *AIME, UMD HS Math Competition*

*How many integers $m$ exist such that*

$$m = \frac{4 * 10^n - 1}{13}$$

*Where $n$ is an integer less than 50?*

$$10^n \equiv 10 \; mod \; 13$$

Well, now we have one solution. When $n = 1$ we see the congruency holds. But what about other numbers $n$?

If we multiply both sides by 1 we would get another solution, so let’s find the number

$$10^x \equiv 1 \; mod \; 13$$

Trial and error shows us that $x = 6$

# *Problems:* *AIME, UMD HS Math Competition*

*How many integers $m$ exist such that*

$$m = \frac{4 * 10^n - 1}{13}$$

*Where $n$ is an integer less than 50?*

Now we see that our solutions extend to all numbers in the form of

$$10^{1+6j} \equiv 1 \: mod \: 13$$

Where all integers $j$ satisfy. So how many numbers are there such that $50 \geq n = 1 + 6j$?

1, 7, 13, 19, 25, 31, 37, 43, 49

9 such integers exist

# *Problems:* *AIME, UMD HS Math Competition*

*Suppose $n_0, n_1, n_2$ … is a sequence of integers satisfying all the following:*

- $0 \leq n_k \leq 123$ *for every* $k \geq 0$
- *For every $k \geq 0$, $n_{k+1}$ is the remainder when $2n_k + 1$ is divided by 124*
- $n_1 \neq n_0$

*What is the least possible value of $\ell > 0$ for which $n_\ell = n_0$?*

What a perfect application for modular arithmetic. We see clearly that

$$n_1 \equiv 2n_0 + 1 \; mod \; 124$$

$$n_2 \equiv 2(2n_0 + 1) + 1 \equiv 2^2 n_0 + 2 + 1 \; mod \; 124$$

$$n_3 \equiv 2^3 n_0 + 2^2 + 2 + 1 \; mod \; 124$$

Now we see that

$$n_\ell \equiv 2^\ell n_0 + 2^\ell - 1 \; mod \; 124$$

# *Problems:* *AIME, UMD HS Math Competition*

*Suppose $n_0, n_1, n_2 \dots$ is a sequence of integers satisfying all the following:*

- $0 \leq n_k \leq 123$ *for every* $k \geq 0$
- *For every* $k \geq 0$, $n_{k+1}$ *is the remainder when* $2n_k + 1$ *is divided by* $124$
- $n_1 \neq n_0$

*What is the least possible value of* $\ell > 0$ *for which* $n_\ell = n_0$?

$$n_\ell \equiv 2^\ell n_0 + 2^\ell - 1 \; mod \; 124$$

Now, given the equality we're trying to solve for, we see

$$n_\ell \equiv n_0 \equiv 2^\ell n_0 + 2^\ell - 1 \; mod \; 124$$

And thus

$$0 \equiv 2^\ell n_0 + 2^\ell - n_0 - 1 \; mod \; 124$$

Which means

$$0 \equiv (2^\ell - 1)(n + 1) \; mod \; 124$$

$$124 \,|(2^\ell - 1)(n + 1)$$

# *Problems:* AIME, UMD HS Math Competition

*Suppose* $n_0, n_1, n_2 \ldots$ *is a sequence of integers satisfying all the following:*

- $0 \leq n_k \leq 123$ *for every* $k \geq 0$
- *For every* $k \geq 0$, $n_{k+1}$ *is the remainder when* $2n_k + 1$ *is divided by* $124$
- $n_1 \neq n_0$

*What is the least possible value of* $\ell > 0$ *for which* $n_\ell = n_0$?

$$0 \equiv (2^\ell - 1)(n + 1) \ mod\ 124$$

$$124 \,|(2^\ell - 1)(n + 1)$$

Since $124 = 4 * 31$,

$$4|(n + 1)$$

And thus

$$n_0 + 1 \equiv 4 \ or\ 0 \ mod\ 124$$

# *Problems:* AIME, UMD HS Math Competition

*Suppose $n_0, n_1, n_2$ … is a sequence of integers satisfying all the following:*

- $0 \leq n_k \leq 123$ *for every* $k \geq 0$
- *For every* $k \geq 0$, $n_{k+1}$ *is the remainder when* $2n_k + 1$ *is divided by* 124
- $n_1 \neq n_0$

*What is the least possible value of* $\ell > 0$ *for which* $n_\ell = n_0$*?*

$$n_0 + 1 \equiv 4 \text{ } or \text{ } 0 \text{ } mod \text{ } 124$$

The case of 0 is impossible, since this would imply $n_0 \equiv -1$, which would mean

$$n_1 \equiv 2(-1) + 1 \equiv -1 \equiv n_0$$

Which was not allowed. Therefore,

$$n_0 + 1 \equiv 4 \text{ } mod \text{ } 124$$

Which means $31 | 2^\ell - 1$, meaning $\ell \geq 5$. The smallest is thus

$$\ell = 5$$