Warm Up! Number Theo. And Mod. Arith.

What is the tens and units digit of

7^{1942}?

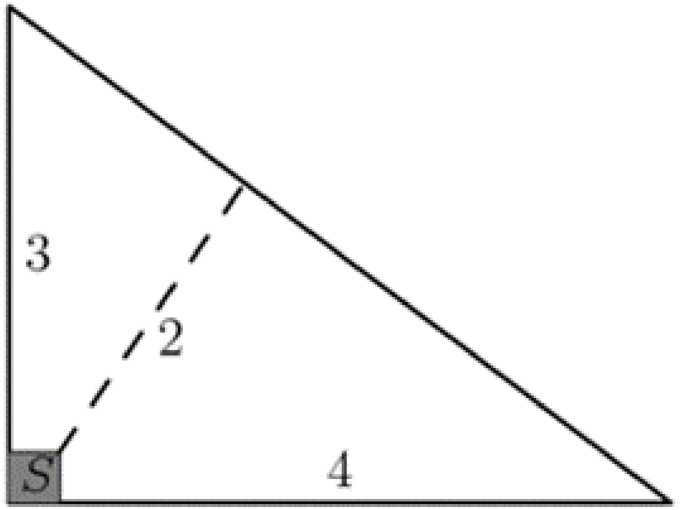
Suppose the real number x satisfies

$$\sqrt{49-x^2}-\sqrt{25-x^2}=3,$$

What is the value of

$$\sqrt{49-x^2}+\sqrt{25-x^2}?$$

The shortest distance from S to the hypotenuse is 2. What fraction of the following triangle's area is unshaded?



Warm Up! *Complex Numbers GD*

What is the tens and units digit of 7^{1942}*?*

We see the repetition modulo 100

$$7^1 \equiv 7 \mod 100$$
$$7^2 \equiv 49 \mod 100$$
$$7^3 \equiv 43 \mod 100$$
$$7^4 \equiv 1 \mod 100$$
$$7^5 \equiv 7 \mod 100$$

And so we see that the successive powers of 7 will repeat after every fourth term.

Warm Up! *Complex Numbers GD*

What is the tens and units digit of 7^{1942}?

And so we see that the successive powers of 7 will repeat after every fourth term.

With this we just find that

$$1942 \equiv 2 \; mod \; 4$$

And we have that

$$7^{1942} \equiv 7^2 \equiv 49 \; mod \; 100$$

Guided Discussion: Complex Numbers

Introduction, Parts, Forms, Euler's Identity, Roots of Unity

AMC, AIME

Walter Johnson Math Team

Guided Discussion: Introduction

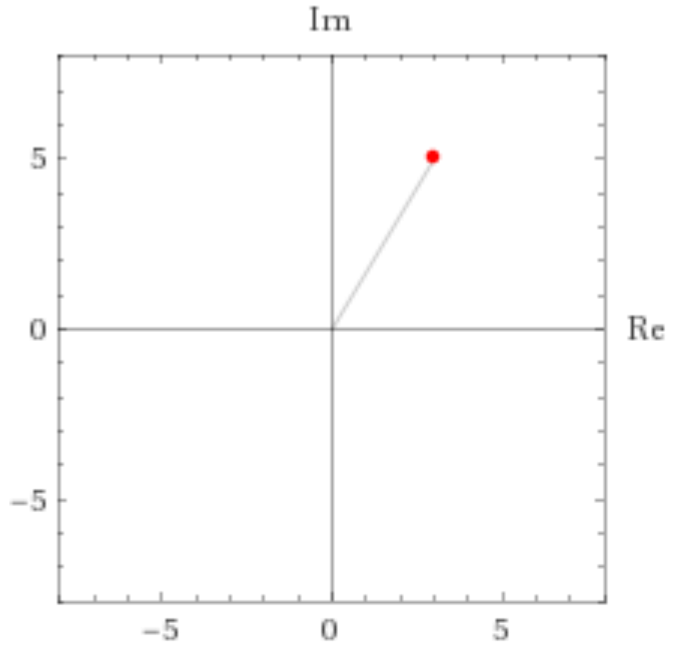
Numbers in the form $a + bi$ where $i = \sqrt{-1}$, $a, b \in \mathbb{R}$. Typically denoted z and $z \in \mathbb{C}$ (The set of complex numbers. Also a field with subring $\mathbb{R}$)

These numbers exist on the complex plane, that which has an axis representing the real component of the number, denoted $\Re(z)$ or $Re(z)$, and the imaginary component, denoted $\Im(z)$ or $Im(z)$. Note, $\Im(z) = b$ in the standard form of an imaginary number, and thus $\Im(z)$ denotes a real number.

The argument, $\arg(z)$, of a complex number is the angle θ that the point makes with the horizontal line of the reals.

$$z = 3 + 5i$$

$$\arg(z) = \arctan(\frac{5}{3})$$



Guided Discussion: Introduction

The complex conjugate, denoted $\bar{z}$ is equivalent to $a - bi$.

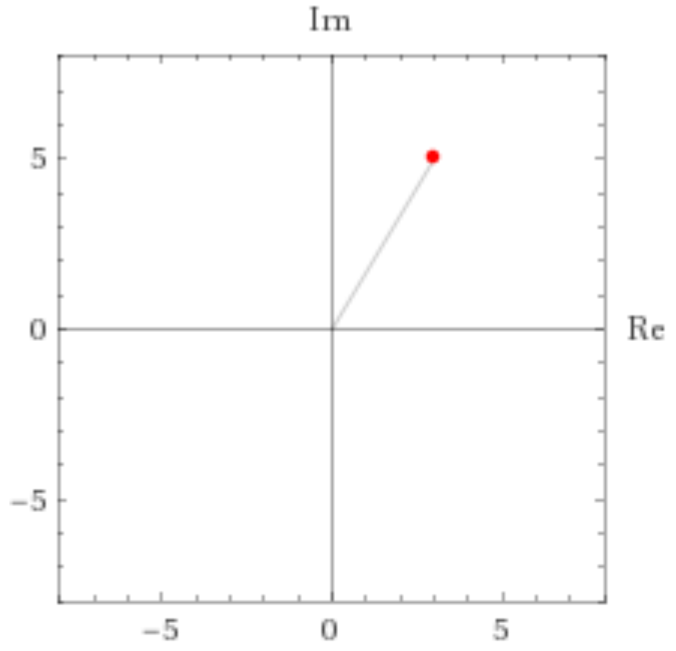
This is important as the product of a complex number and it's conjugate is always a real number, $z * \bar{z} \in \mathbb{R}$.

The magnitude ($|z|$) of a complex number denotes the distance from the origin.

$$z = 3 + 5i$$

$$\bar{z} = 3 - 5i$$

$$|z| = \sqrt{3^2 + 5^2} = \sqrt{34}$$



Guided Discussion : Parts

One obvious point, but one that can go unnoticed, is that if two complex numbers, z and w are equal, their real parts are equal, as well as their imaginary parts.

This comes into play a lot when solving for complex numbers, an example equation (to the right)

Looks elusive at first, but setting coefficients equal to each other, the problem becomes simple algebra.

$$\frac{a + 164i}{a + 164i + n} = 4i$$

Guided Discussion : Parts

One obvious point, but one that can go unnoticed, is that if two complex numbers, z and w are equal, their real parts are equal, as well as their imaginary parts.

This comes into play a lot when solving for complex numbers, an example

$$\frac{a+164i}{a+164i+n} = 4i$$

Looks elusive at first, but setting coefficients equal to each other, the problem becomes simple algebra.

$$\frac{a+164i}{a+164i+n} = 4i$$

$$a + 164i = 4ai - 656 + 4ni$$

And by parts we have

$$a = -656$$

And

$$164i = 4ai + 4ni$$

$$41 = a + n$$

$$n = 41 - (-656)$$

$$n = 697$$

Guided Discussion : Exp Function

Introducing the Exp function is just groundwork to understanding Euler's form of a complex number.

A lot of people do not understand that the notation of e^{ix} stems from the exp function itself.

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Now what happens when we plug some numbers into this function?

$$\exp(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots = e$$

$$\exp(2) = 1 + 2 + \frac{4}{2} + \frac{8}{6} + \cdots = e^2$$

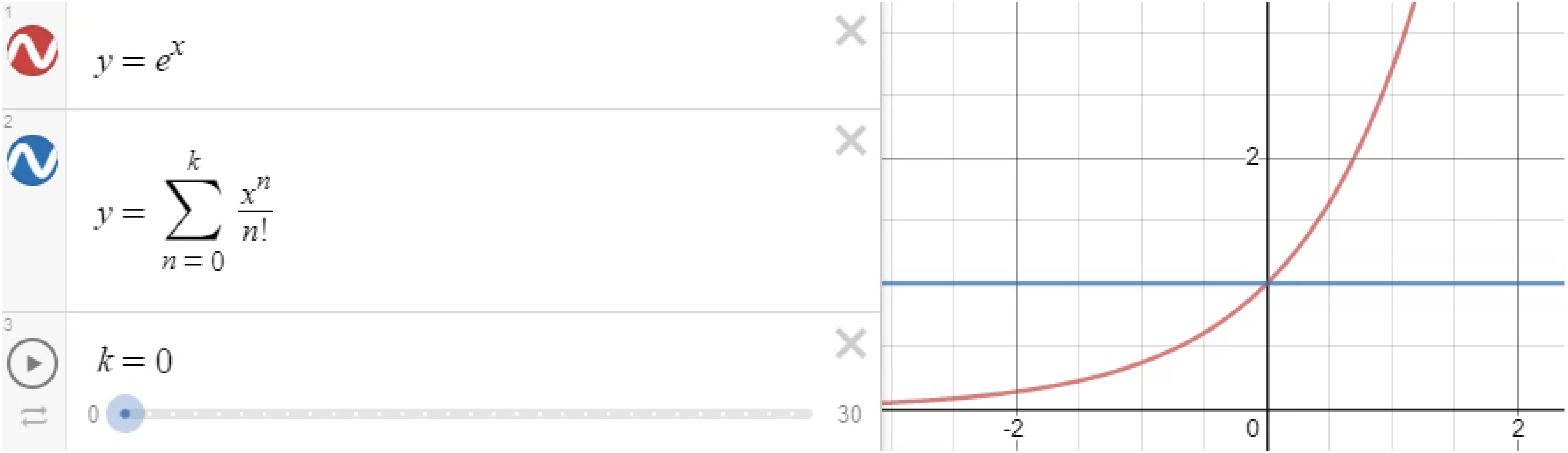
So now we use the abbreviation

$$\exp(x) = e^x$$

Guided Discussion: Euler's Identity

$$\exp(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

This makes sense, as we can see this infinite sum of polynomials does eventually converge to e^x



Guided Discussion: Euler's Identity

$$e^{ix} = 1 + ix - \frac{x^2}{2} - \frac{ix^3}{6} + \cdots = \left(1 - \frac{x^2}{2} + \cdots\right) + \left(ix - \frac{ix^3}{6} + \cdots\right)$$

$$e^{ix} = \sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n}}{(2n)!}\right) + i\left(\sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)\right)$$

Now, looking to plug ix into this function, we find we can split the function into an imaginary and a complex part.

Notice how every even exponent of i makes the term real? It just occasionally switches the sign.

Guided Discussion: Euler's Identity

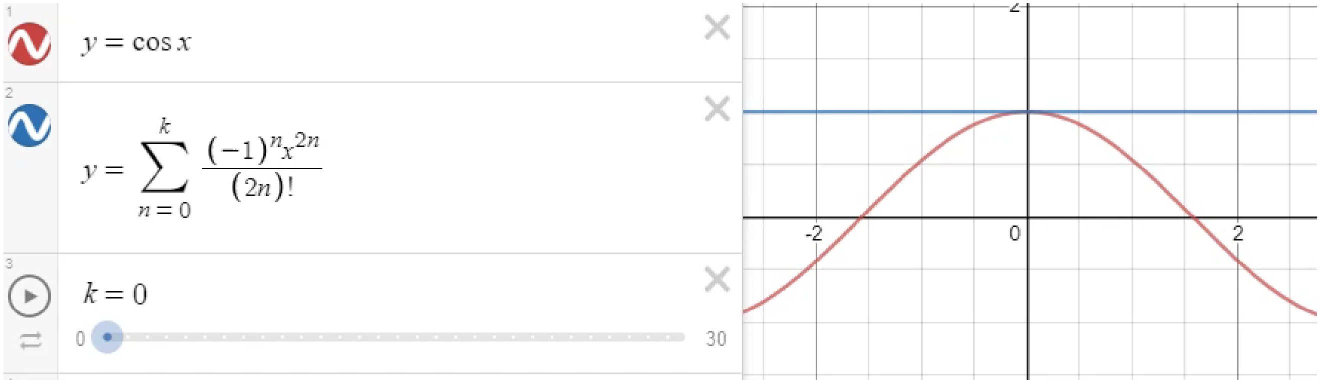
$$e^{ix} = \sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n}}{(2n)!}\right) + i\left(\sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)\right)$$

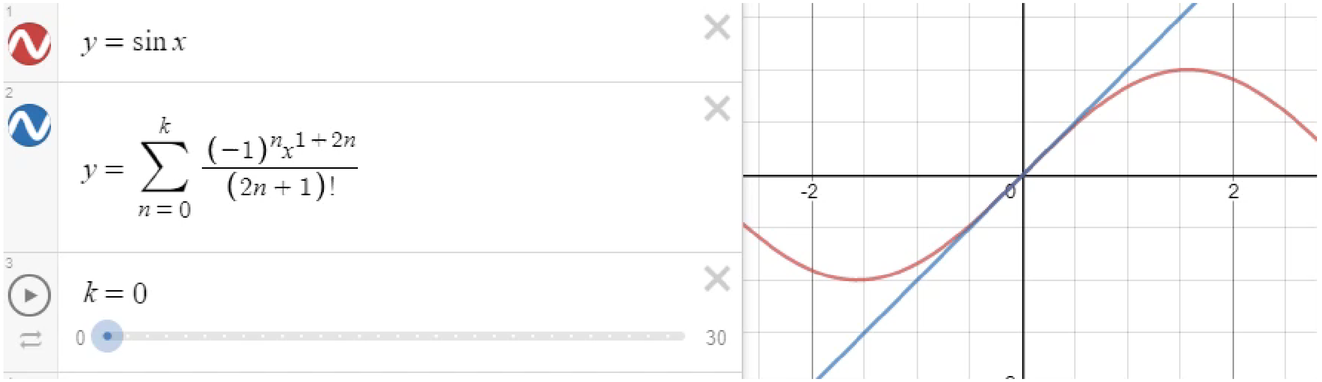
Now, looking at each of these infinite sums independently, we see they converge to two familiar functions:

$$\cos x = \sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n}}{(2n)!}\right)$$

$$\sin x = \sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)$$

$$e^{ix} = \cos x + i \sin x = \text{cis } x$$





And thus, we find the identity to be true, as $\cos \pi = -1$ and $i \sin \pi = 0$, so

$$e^{i\pi} = -1$$

Guided Discussion: Euler's Identity

Now, looking at each of these infinite sums independently, we see they converge to two familiar functions:

$$\cos x = \sum_{n=0}^{\infty} \left(\frac{(-1)^n x^{2n}}{(2n)!} \right)$$

$$\sin x = \sum_{n=0}^{\infty} \left(\frac{(-1)^n x^{2n+1}}{(2n+1)!} \right)$$

And thus, we find the identity to be true, as $\cos \pi = -1$ and $i \sin \pi = 0$, so

$$e^{i\pi} = -1$$

$$e^{ix} = \sum_{n=0}^{\infty} \left(\frac{(-1)^n x^{2n}}{(2n)!} \right) + i \left(\sum_{n=0}^{\infty} \left(\frac{(-1)^n x^{2n+1}}{(2n+1)!} \right) \right)$$

$$e^{ix} = \cos x + i \sin x = \operatorname{cis} x$$

This is commonly known as Euler's Formula (one of many Euler's formulas).

Also represented in cis *notation on occasion.*

Guided Discussion: Forms

There are three primary forms of which complex numbers are represented. The first, more standard form $z = a + bi$

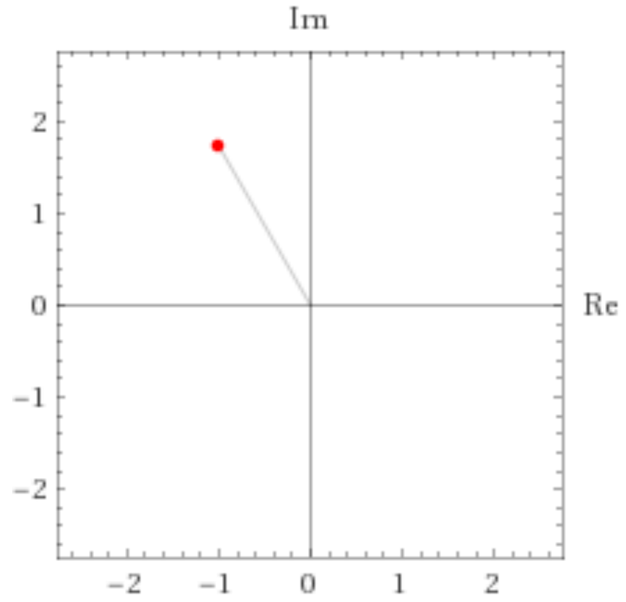
The trigonometric form, $z = r(\cos(\theta) + i\sin(\theta))$, where r is the magnitude of the complex number and θ is the argument of the number.

This last form is simply an abbreviation for what the last few slides went over, denoted $z = re^{i\theta}$

$$z = i\sqrt{3} - 1$$

$$z = \cos\left(\frac{2\pi}{3}\right) + i * \sin\left(\frac{2\pi}{3}\right)$$

$$z = 2e^{(2i\pi)/3}$$



Guided Discussion: Euler's Identity

Because cosine and sine waves are periodic in 2π, we can just add any multiple of 2π to our equality where x suffices.

$$e^{ix} = \cos x + i \sin x$$

$$e^{ix+2\pi n} = e^{ix} = \cos x + i \sin x$$

$$\cos x + i \sin x = \cos(x + 2\pi m) + i \sin(x + 2\pi p)$$

$$e^{ix+2\pi n} = \cos(x + 2\pi m) + i \sin(x + 2\pi p)$$

Guided Discussion: Roots of Unity

Now that we have a clear representation of a complex number $z = re^{i\theta}$, we can explore the complex roots of 1.

What is $(-1)^2$? So what is $e^{2i\pi}$? Now we have

$$1 = e^{2i\pi}$$

And

$$\sqrt[n]{1} = e^{(2i\pi)\backslash\mathrm{n}}$$

Which is important, as it gives n roots for 1 in the complex plane, which form a regular n-gon in the complex plane which inscribes the unit circle (e^{ix}) in the complex plane.

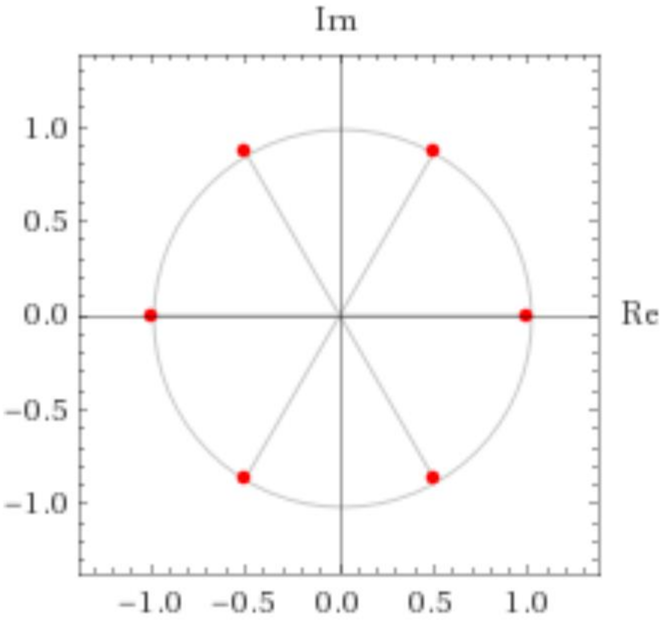
$$\sqrt[6]{1}$$

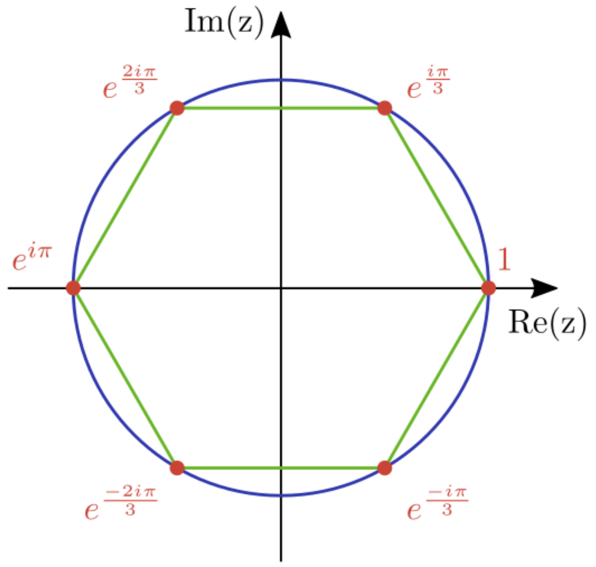
$$1 = 1^6$$

$$1 = (-1)^6$$

$$1 = (e^{2\pi/6})^6$$
$$\Downarrow$$
$$1 = (\cos(2\pi/6) + i\sin(2\pi/6))^6$$
$$\Downarrow$$
$$1 = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^6$$

And so on for the rest of the corresponding points of the regular hexagon.





Guided Discussion: Roots of Unity

Now that we have a clear representation of a complex number $z = re^{i\theta}$, we can explore the complex roots of 1.

What is $(-1)^2$? So what is $e^{2i\pi}$? Now we have

$$1 = e^{2i\pi}$$

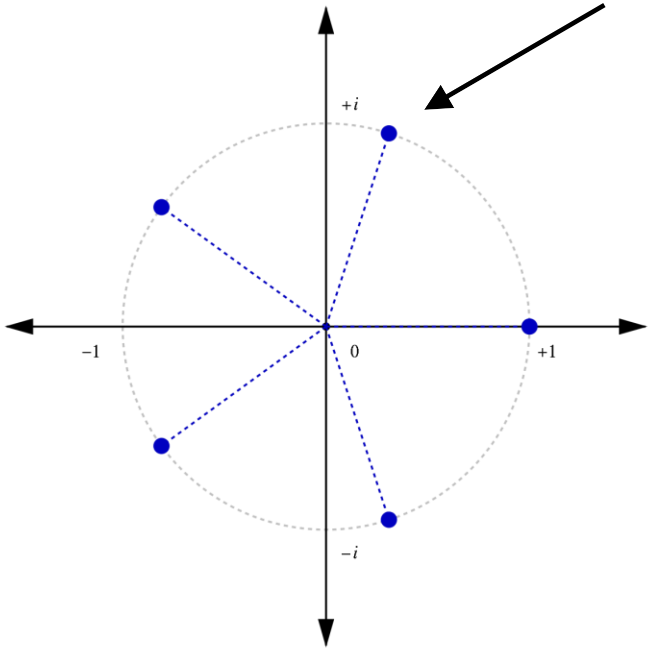
And

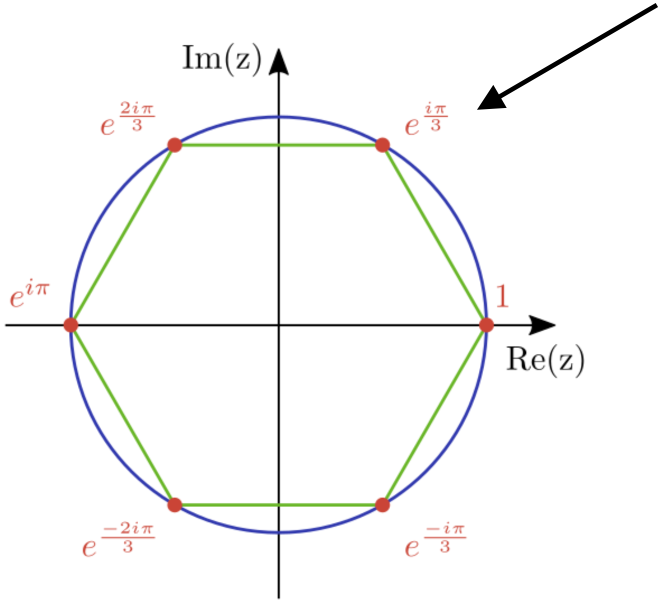
$$\sqrt[n]{1} = e^{(2i\pi)\backslash \mathrm{n}}$$

Which is important, as it gives n roots for 1 in the complex plane, which form a regular n-gon in the complex plane which inscribes the unit circle (e^{ix}) in the complex plane.

The “principle” n-th root of unity is the first root such that all the other roots can be expressed as an exponent of this root.

This comes immediately after 1 in the roots.





Guided Discussion: Roots of Unity

What's more, the product of these points map to each other $mod\ n$. If we list the points starting at 0 and going to $n-1$ counterclockwise at $1+0i$, we have a set $\mathbb{Z}/m\mathbb{Z}$ closed under addition.

For the following example of a pentagon

$$0+2 \equiv 2\ mod\ 5$$
$$1+3 \equiv 4\ mod\ 5$$
$$2+3 \equiv 0\ mod\ 5$$
$$4+4 \equiv 3\ mod\ 5$$

This works intuitively with the exponential form of the complex numbers, where the magnitude is 1.

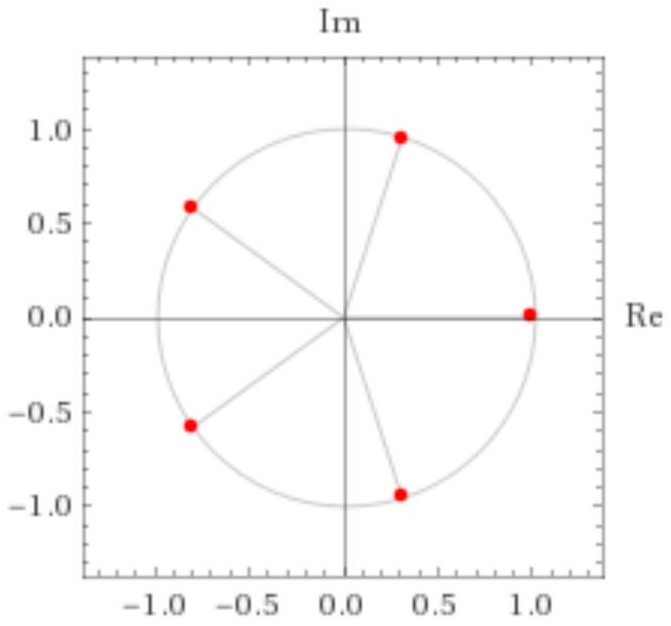
***Apply concepts from Modular Arithmetic*

$$e^{0} * e^{(2i\pi/5)*2} = e^{(2i\pi/5)*2}$$

$$e^{(2i\pi/5)} * e^{(2i\pi/5)*3} = e^{(2i\pi/5)*4}$$

$$e^{(2i\pi/5)*2} * e^{(2i\pi/5)*3} = 1$$

$$e^{(2i\pi/5)*4} * e^{(2i\pi/5)*4}$$
$$= e^{(2i\pi/5)*3}$$



Problems: *Complex Numbers*

The solutions to

$z^2 = 4 + 4\sqrt{15}i$ and

$z^2 = 2 + 2\sqrt{3}i$

form points of a quadrilateral in the complex plane.

What is the area of this quadrilateral?

If a, b, c are integers which satisfy $c =$

$(a + bi)^3 - 107i$,

Find c

How many numbers are both a 74^{th} root of unity as well as a 111^{th} root of unity?

What is the sum of all these numbers?

Let z be a complex number such that

$z + {}^1/_z = \sqrt{3}$. Find $z^{2000} + {}^1/_{z^{2000}}$

Let $\omega_0, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5$ be the 6th roots of unity. Compute

$(2 + \omega_0) \times \cdots \times (2 + \omega_5)$

Warm Up! Complex Numbers GD

If a, b, c *are integers which*

satisfy $c = (a+bi)^3 - 107i$,

Find c

$$c + 107i = (a+bi)^3$$
$$= a^3 + 3a^2bi - 3ab^2 - b^3i$$

$$c + 107i = \left(a^3 - 3ab^2\right) + \left(3a^2b - b^3\right)i$$

$$c = a^3 - 3ab^2$$

$$107 = 3a^2b - b^3$$

Warm Up! *Complex Numbers GD*

If a, b, c are integers which

satisfy $c = (a + bi)^3 - 107i$,

Find c

Now some algebra is required.

$$107 = b(3a^2 - b^2)$$

As 107 is a prime number, we know b has to be either 1 or 107. If $b = 107$, then

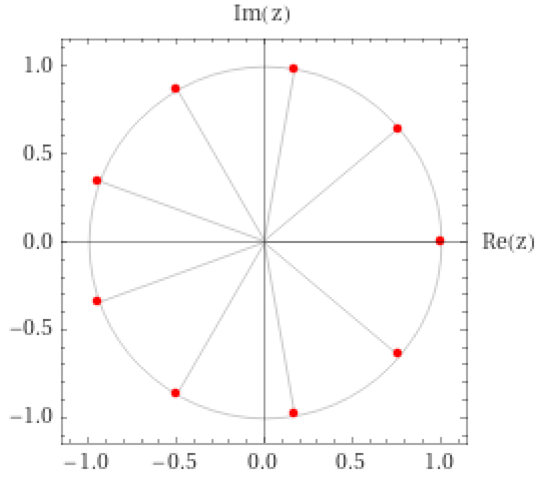
$$3a^2 = 107^2 + 1$$

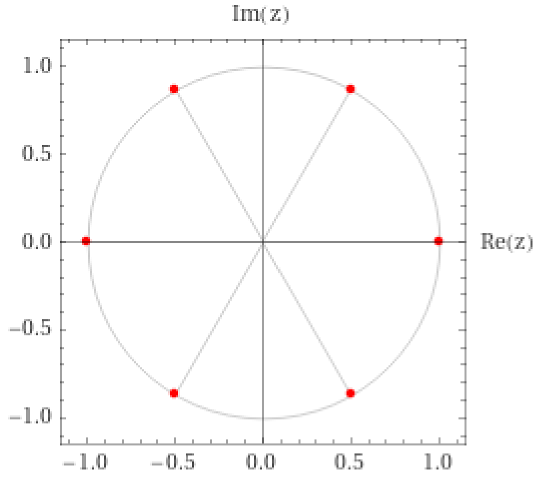
And as $107^2 + 1$ is not divisible by 3, this is a contradiction. So $b = 1$ and $a = \sqrt{108/3} = \sqrt{36} = 6$, making $c = 6^3 - 3(6) = 198$

Warm Up! *Complex Numbers GD*

How many numbers are both a 74^{th} root of unity as well as a 111^{th} root of unity?

Let's look at an easier case. The first plot are the 9th roots of unity. The second are the 6th roots of unity. Which points do these have in common?





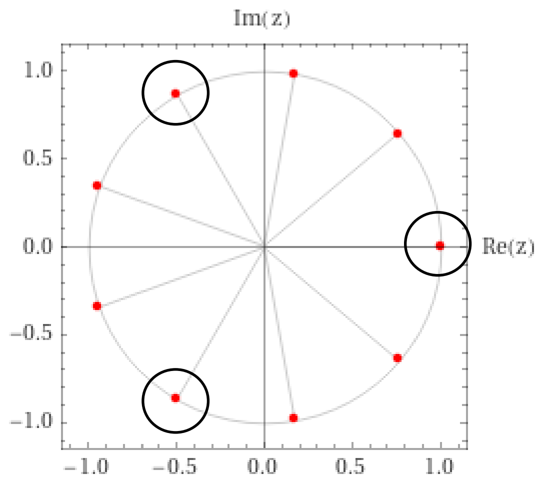
Warm Up! Complex Numbers GD

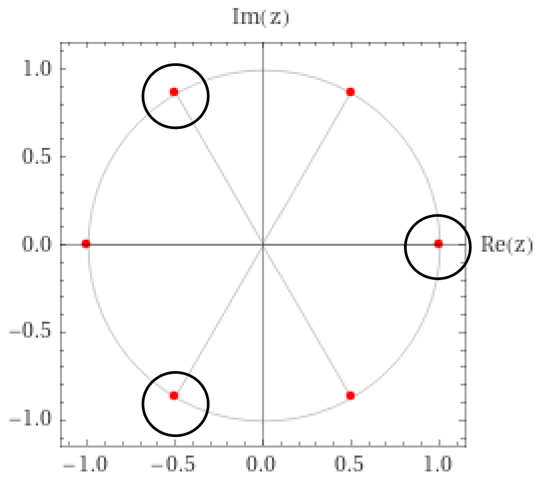
How many numbers are both a 74^{th} root of unity as well as a 111^{th} root of unity?

Let's look at an easier case. The first plot are the 9th roots of unity. The second are the 6th roots of unity. Which points do these have in common?

Only the points which are also the 3rd roots of unity.

This is because the only shared roots of unity of m- and n-th roots are those for which are factors of both m and n.





Warm Up! Complex Numbers GD

How many numbers are both a 74^{th} root of unity as well as a 111^{th} root of unity?

Now we can apply this to our problem; what is the greatest common divisor (gcd) of 74 and 111?

Warm Up! Complex Numbers GD

How many numbers are both a 74^{th} root of unity as well as a 111^{th} root of unity?

Now we can apply this to our problem; what is the greatest common divisor (gcd) of 74 and 111?

37*2 = 74, and 37*3 = 111

So we know we would have 37 roots shared between these two.

Warm Up! Complex Numbers GD

How many numbers are both a 74^{th} root of unity as well as a 111^{th} root of unity?

What is the sum of all these numbers?

Now, we have that these numbers are all in the form of

$$z^{37} = 1$$

This gives us a polynomial

$$z^{37} - 1 = 0$$

Using Vieta's formulas, what is the sum of all of these solutions?

Warm Up! *Complex Numbers GD*

How many numbers are both a 74^{th} root of unity as well as a 111^{th} root of unity?

What is the sum of all these numbers?

$$z^{37} - 1 = 0$$

As we see, Vieta's formulas give us the sum of all these roots to be 0.

Warm Up! *Complex Numbers GD*

Let z be a complex number such that

$z + {}^{1}/_{z} = \sqrt{3}$. Find $z^{2000} + {}^{1}/_{z^{2000}}$

We find that $z^2 - z\sqrt{3} + 1 = 0$

With the quadratic formula, we find that the solutions to this are

$$\frac{\sqrt{3} \pm i}{2}$$

Giving these a quick plot on the complex plane, we see this includes a 12th root of unity.

Warm Up! Complex Numbers GD

Let z be a complex number such that

$z + {}^{1}/_{z} = \sqrt{3}$. Find $z^{2000} + {}^{1}/_{z^{2000}}$

This means $z^{12} = 1$, and thus

$$z^{2000} = z^{2000 \, mod \, 12}$$

And we find that $2000 \equiv 8 \; mod \; 12$

And thus we are trying to find

$$z^8 + \frac{1}{z^8}$$

Warm Up! Complex Numbers GD

Let z be a complex number such that

$z + {}^{1}/_{z} = \sqrt{3}$. Find $z^{2000} + {}^{1}/_{z^{2000}}$

$$z^8 + \frac{1}{z^8}$$

We find this to be

$$\frac{-1-\sqrt{3}i}{2} + \frac{2}{-1-\sqrt{3}i}$$

Warm Up! *Complex Numbers GD*

Let z be a complex number such that

$z + {}^{1}/_{z} = \sqrt{3}$. Find $z^{2000} + {}^{1}/_{z^{2000}}$

$$\frac{-1-\sqrt{3}i}{2} + \frac{2}{-1-\sqrt{3}i}$$

$$\frac{\left(-1-\sqrt{3}i\right)\left(-1-\sqrt{3}i\right)+4}{-2-2\sqrt{3}i}$$

$$\frac{1+2\sqrt{3}i-3+4}{-2-2\sqrt{3}i}$$

Warm Up! *Complex Numbers GD*

Let z be a complex number such that

$z + {}^{1}/_{z} = \sqrt{3}$. Find $z^{2000} + {}^{1}/_{z^{2000}}$

$$\frac{1 + 2\sqrt{3}i - 3 + 4}{-2 - 2\sqrt{3}i}$$

$$\frac{2 + 2\sqrt{3}i}{-2 - 2\sqrt{3}i} = -1$$

Warm Up! Complex Numbers GD

Let $\omega_0, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5$ be the 6th roots of unity. Compute

$$(2+\omega_0) \times \cdots \times (2+\omega_5)$$

We see that ω_n are the roots of our polynomial

$$x^6 = 1$$

Or

$$x^6 - 1 = 0$$

This polynomial can we rewritten as

$$(x-\omega_0) \times \cdots \times (x-\omega_5) = 0$$

Warm Up! *Complex Numbers GD*

Let $\omega_0, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5$ be the 6^{th} roots of unity. Compute

$$(2+\omega_0)\times\cdots\times(2+\omega_5)$$

$$x^6 - 1 = (x-\omega_0)\times\cdots\times(x-\omega_5) = 0$$

And so we have that

$$(-2)^6 - 1 = (-2-\omega_0)\times\cdots\times(-2-\omega_5)$$

Which equals the initial product we were searching for. Thus the value is equal to

$$(-2)^6 - 1 = 63$$

Guided Discussion: 2018 AMC 12A #22

The solutions to

$z^2 = 4 + 4\sqrt{15}i$ and

$z^2 = 2 + 2\sqrt{3}i$

form points of a quadrilateral in the complex plane.

What is the area of this quadrilateral?

Guided Discussion: 2018 AMC 12A #22

The solutions to

$z^2 = 4 + 4\sqrt{15}i$ and

$z^2 = 2 + 2\sqrt{3}i$

form points of a quadrilateral in the complex plane.

What is the area of this quadrilateral?

$$z^2 = 4 + 4\sqrt{15}i$$

$$(a + bi)^2 = 4 + 4\sqrt{15}i$$

$$a^2 + 2abi - b^2 = 4 + 4\sqrt{15}i$$

And thus

$$a^2 - b^2 = 4$$

And

$$2ab = 4\sqrt{15}$$

$$ab = \sqrt{60}$$

$$a^2b^2 = 60$$

$$a^2 - b^2 = 4$$

Guided Discussion: 2018 AMC 12A #22

The solutions to

$$z^2 = 4 + 4\sqrt{15}i \text{ and}$$

$$z^2 = 2 + 2\sqrt{3}i$$

form points of a quadrilateral in the complex plane.

What is the area of this quadrilateral?

$$a^2b^2 = 60$$

$$a^2 - b^2 = 4$$

$$a^2 = 10 \text{ and } b^2 = 6$$

So we have our first 2 solutions, and for the next two

$$(a + bi)^2 = a^2 + 2abi - b^2 = 2 + 2\sqrt{3}i$$

$$a^2 - b^2 = 2$$

$$2ab = 2\sqrt{3}$$

$$a^2 = 3 \text{ and } b = 1$$

From there the solution is finding the area. (We will not go into)

Guided Discussion: 1985 AIME #3

If a, b, c *are integers which satisfy* $c = (a + bi)^3 - 107i$,

Find c

Guided Discussion: 1985 AIME #3

If a, b, c are integers which satisfy $c = (a + bi)^3 - 107i$,

Find c

$$c + 107i = (a + bi)^3 = a^3 + 3a^2bi - 3ab^2 - b^3i$$

$$c + 107i = \left(a^3 - 3ab^2\right) + \left(3a^2b - b^3\right)i$$

$$c = a^3 - 3ab^2$$

$$107 = 3a^2b - b^3$$

Now some algebra is required.

$$107 = b(3a^2 - b^2)$$

As 107 is a prime number, we know b has to be either 1 or 107. If $b = 107$, then

$$3a^2 = 107^2 + 1$$

And as $107^2 + 1$ is not divisible by 3, this is a contradiction. So $b = 1$ and $a = \sqrt{108/3} = \sqrt{36} = 6$, making $c = 6^3 - 3(6) = 198$

Guided Discussion: 1984 AIME #8

The equation

$$z^6 + z^3 + 1 = 0$$

has complex roots with argument θ between $90°$ and $180°$ in the complex plane.

Determine the degree measure of θ.

***Remember* $(z^3 - 1)(z^6 + z^3 + 1) = z^9 - 1$

***Remember this introduces extraneous solutions*

Guided Discussion: 1984 AIME #8

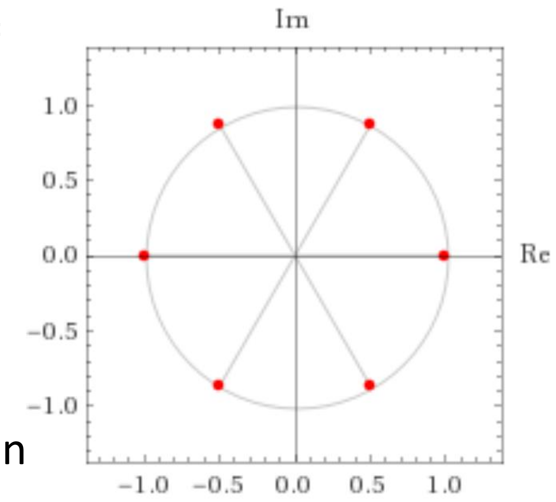
The equation $z^6 + z^3 + 1 = 0$ has complex roots with argument θ between $90°$ and $180°$ in the complex plane. Determine the degree measure of θ.

***Remember* $(z^3 - 1)(z^6 + z^3 + 1) = z^9 - 1 = 0$

***Remember this introduces extraneous solutions*

***Think about the fixed degree measures which are possible for n^{th} roots of unity.*

$z^9 - 1 = 0$ can be re-written $z^9 = 1$, $z = \sqrt[9]{1}$ and so we find the roots of this have degree measure multiple of $360°m/9$ or $40m°$ for m an integer.



There are two solutions for this, where $40m = 120$ and 160.

These, however, are the solutions to $z^9 - 1 = 0$, and so one of them is extraneous in $z^6 + z^3 + 1 = 0$.

We consider if $120° = \frac{2}{3}\pi$ is extraneous by plugging in $e^{i2\pi*1/3}$

$$(e^{i2\pi*1/3})^6 + (e^{i2\pi*1/3})^3 + 1 = 0$$

$$e^{i4\pi} + e^{i2\pi} + 1 = 3 \neq 0$$

And thus $120°$ is the extraneous solution and $160°$ is the solution.