Warm Up!

How many real numbers x *with* $0 \leq x < 2\pi$ *satisfy*

$$\cos x + \cos 2x + \cos 3x + \cos 4x = 0?$$

(angles in radians)

Let $f(x) = x^2 + 6x + 1$ *and let* R *denote the set of points* (x, y) *in the coordinate plane such that*

$$f(x) + f(y) \leq 0 \text{ and } f(x) - f(y) \leq 0$$

What is the area of R*?*

Warm Up!

How many real numbers x *with* $0 \leq x < 2\pi$ *satisfy*

$$\cos x + \cos 2x + \cos 3x + \cos 4x = 0?$$

(angles in radians)

We know $\cos(a) + \cos(b) = 2\cos\left(\frac{b+a}{2}\right)\cos\left(\frac{b-a}{2}\right)$,

So applying this we see our equation equals

$$0 = \cos\left(\frac{5x}{2}\right)\cos\left(\frac{3x}{2}\right) + \cos\left(\frac{5x}{2}\right)\cos\left(\frac{x}{2}\right)$$

Warm Up!

How many real numbers x with

$0 \leq x < 2\pi$ satisfy

$\cos x + \cos 2x + \cos 3x +$

$\cos 4x = 0$*?*

(angles in radians)

$$0 = \cos\left(\frac{5x}{2}\right)\cos\left(\frac{3x}{2}\right) + \cos\left(\frac{5x}{2}\right)\cos\left(\frac{x}{2}\right)$$

Factoring

$$0 = \cos\left(\frac{5x}{2}\right)\left(\cos\left(\frac{3x}{2}\right) + \cos\left(\frac{x}{2}\right)\right)$$

Warm Up!

How many real numbers x *with*

$0 \leq x < 2\pi$ *satisfy*

$\cos x + \cos 2x + \cos 3x +$

$\cos 4x = 0$*?*

(angles in radians)

$$0 = \cos\left(\frac{5x}{2}\right)\left(\cos\left(\frac{3x}{2}\right) + \cos\left(\frac{x}{2}\right)\right)$$

For our first factor we have the solutions

$$x = \frac{\pi}{5}, \frac{3\pi}{5}, \frac{7\pi}{5}, \frac{9\pi}{5}$$

Now for the second factor

Warm Up!

How many real numbers x with

$0 \leq x < 2\pi$ satisfy

$\cos x + \cos 2x + \cos 3x +$

$\cos 4x = 0$*?*

(angles in radians)

Now for the second factor

$$\cos\left(\frac{3x}{2}\right) + \cos\left(\frac{x}{2}\right)$$

We see this happens for

$$\frac{3x}{2} = (2n + 1)\pi - \frac{\pi}{2}$$

Warm Up!

How many real numbers x with

$0 \leq x < 2\pi$ *satisfy*

$\cos x + \cos 2x + \cos 3x + \cos 4x = 0$*?*

(angles in radians)

$$\frac{3x}{2} = (2n+1)\pi - \frac{x}{2}$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

Thus we have 7 values for x

Warm Up!

Let $f(x) = x^2 + 6x + 1$ *and let* R *denote the set of points* (x, y) *in the coordinate plane such that*

$$f(x) + f(y) \leq 0 \textit{ and } f(x) - f(y) \leq 0$$

What is the area of R*?*

Substituting for our function we have

$$x^2 + 6x + 1 + y^2 + 6y + 1 \leq 0$$

Completing the square we have

$$(x + 3)^2 - 9 + 1 + (y + 3)^2 - 9 + 1 \leq 0$$

$$(x + 3)^2 + (y + 3)^2 \leq 16$$

Which encloses a circle of radius 4 at point $(-3, -3)$

Warm Up!

Let $f(x) = x^2 + 6x + 1$ *and let* R *denote the set of points* (x, y) *in the coordinate plane such that*

$f(x) + f(y) \leq 0$ *and* $f(x) - f(y) \leq 0$

What is the area of R*?*

Our next area is

$$x^2 + 6x + 1 - y^2 - 6y - 1 \leq 0$$

$$\left(x^2 - y^2\right) + 6(x - y) \leq 0$$

By difference of squares we can find

$$(x - y)(x + 6 + y) \leq 0$$

Warm Up!

Let $f(x) = x^2 + 6x + 1$ *and let* R *denote the set of points* (x, y) *in the coordinate plane such that*

$f(x) + f(y) \leq 0$ *and* $f(x) - f(y) \leq 0$

What is the area of R*?*

Thus either

$$x - y \geq 0, x + y + 6 \leq 0$$

Or

$$x - y \leq 0, x + y + 6 \geq 0$$

As if one factor is negative and the other is positive the outcome is negative.

Warm Up!

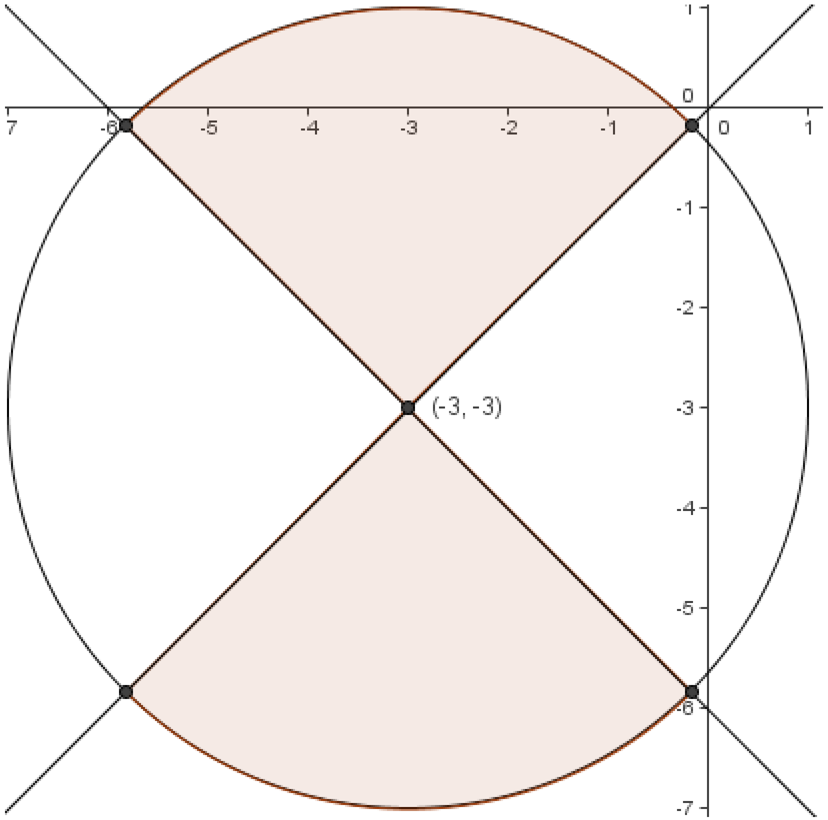
Let $f(x) = x^2 + 6x + 1$ *and let* R *denote the set of points* (x, y) *in the coordinate plane such that*

$$f(x) + f(y) \leq 0 \text{ and } f(x) - f(y) \leq 0$$

What is the area of R*?*

We see our region encloses half of our circle, so our area is

$$\frac{\pi}{2} r^2 = 8\pi$$



Warm Up!

What is the sum of all positive real solutions of x to the equation

$$2\cos(2x)\left(\cos(2x)-\cos\left(\frac{2014\pi^2}{x}\right)\right)=\cos(4x)-1?$$

In base 10, the number 2013 ends in the digit 3. In base 9, the same number is 2676_9 and ends in the digit 6. For how many positive integers b does the base-b representation of 2013 end in 3?

Warm Up!

What is the sum of all positive real solutions of x to the equation

$$2\cos(2x)\left(\cos(2x) - \cos\left(\frac{2014\pi^2}{x}\right)\right) = \cos(4x) - 1?$$

If we write $\cos 4x - 1$ as $2\cos^2 2x - 2$ and let $a = \cos 2x$ and let $b = \cos\left(\frac{2014\pi^2}{x}\right)$

And we have

$$2a(a-b) = 2a^2 - 2$$
$$ab = 1$$

Warm Up!

In base 10, the number 2013 ends in the digit 3. In base 9, the same number is 2676_9 and ends in the digit 6. For how many positive integers b does the base-b representation of 2013 end in 3?

We see immediately that $b < 3$, else it could not end in 3.

Also, any b must divide the greatest possible b, 2010

$$2013 - 2010 = 3$$

So we know that the factors of 2010 are

$$2010 = 2 * 3 * 5 * 67$$

Warm Up!

In base 10, the number 2013 ends in the digit 3. In base 9, the same number is 2676_9 and ends in the digit 6. For how many positive integers b does the base-b representation of 2013 end in 3?

$$2010 = 2 * 3 * 5 * 67$$

So for each of these factors we can or won't include, we have 2^4 possible options. But bases 1, 2, and 3 are not suitable so our number of possible bases is

$$16 - 3 = 13$$

$$13$$

Guided Discussion: AMC Season

What to expect, Approaches to Problems

Walter Johnson Math Team

Guided Discussion: Subtopics

There are a lot of types of problems. We're going to review some basic techniques to approaching all these problems to remind ourselves and get all around ready for the AMC!

Probability	Common, Hard,
Complex Numbers	Uncommon, Medium, 15-25
Functions	Common, M-Hard, 8-23
Geometry	Common, M-Hard, 10-25
Triangles	Uncommon, Hard, 10-21
Trig Equations	Rare, Hard, 15-25
Combinatorics	Rare, Hard, 13-23
Series	Uncommon, Medium, 10-20
Logarithms	Common, M-Hard,
Number Theory	Uncommon, Easy,
Word Problems	Common, Medium, 10-20

Guided Discussion: Probability

- *What techniques do we have to approach these problems?*
- *What do you already know?*

Guided Discussion: Probability

- *Infinite Series*
- *Modeling with 3-D Figures*
- *Systems of Equations*

Guided Discussion: Logarithms

For the harder problems, you'll get familiar over time with how problems are structured and what they are really asking for.

Note: *Don't be afraid if you find a problem splits up into multiple directions for different solutions!*

- *Identities*
- *Modeling with Ranges and Domains*
- *Don't be afraid of the Sigma!*

Guided Discussion: Functions

Never back away from a polynomials problem without modeling the equation and writing everything you have on paper.

- *Try to just go at problems, don't be intimidated, and go for problems.*
- *Vieta's Formulas for Monic Polynomials*
- n*-roots for* $n-1$ *degree polynomials*
- *Nested radicals (not common)*
- *Multinomial Theorem*

Guided Discussion: Functions

Never back away from a polynomials problem without modeling the equation and writing everything you have on paper.

- *Multiple root theorem (Coefficient mapping*

Guided Discussion: Counting

Easier problems will ask you how many numbers below 2020 satisfy a quantity. Don’t think in terms of splitting this up

- *Sum of n numbers*

$$\frac{n(n+1)}{2}$$

- *Combinatorics (Identities), Name some!*

Guided Discussion: Complex Numbers

- *Euler's Formula,* $e^{i\pi} = -1$
- $\operatorname{cis}\theta = \cos\theta + i\sin\theta$
- *This inscribes regular polygons in the unit circle on the complex plane*
- *If a polynomial* $f(x)$ *has real coefficients but complex solutions, it's complex solutions come in conjugates.*

Guided Discussion: Geometry

Always start just by getting a feel for a problem. Notice things if they give you a diagram, if they don't, notice that they don't.

**Typically problems will be broken up in an order for Geometry problems, making them a little easier if you do know where you're going.*

- *Power of a Point*
- *Ptolemy's Theorem (Cyclic Quadrilaterals)*
- *Brahmagupta's Formula*

$$A_{CQ} = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

- *Don't be afraid to try out anything*

Problems:

The expression

$$(x+y+z)^{2006}+(x-y-z)^2$$

Is simplified by expanding it and combining like terms. How many terms are in the simplified equation?

Let

$$S_1 = \{(x,y) \mid \log_{10}(1+x^2+y^2) \leq 1+\log_{10}(x+y)\}$$

And

$$S_2 = \{(x,y) \mid \log_{10}(2+x^2+y^2) \leq 2+\log_{10}(x+y)\}$$

What is the ratio of their areas?

Problems:

The expression

$$(x+y+z)^{2006}+(x-y-z)^{2}$$

Is simplified by expanding it and combining like terms. How many terms are in the simplified equation?

We substitute $P = y + z$ and we have

$$(x+P)^{2006}+(x-P)^{2006}$$

Problems:

For all positive integers n, let

$$a_n = \begin{cases} 11, & n \text{ is divisible by } 13 \text{ and } 14 \\ 13, & n \text{ is divisible by } 14 \text{ and } 11 \\ 14, & n \text{ is divisible by } 11 \text{ and } 13 \\ 0, & \text{otherwise} \end{cases}$$

Find $\sum_{n=1}^{2001} a_n$

Let $\{a_k\}$ be a sequence of integers such that $a_1 = 1$ and

$a_{m+n} = a_m + a_n + mn$, for all positive integers m and n.

Find a_{12}

Problems: