Inequalities Problem Set #1

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Problems are ordered from easiest to hardest difficulty, with high probability. None of the problems require a calculator, calculus, analysis, or an abacus. If you have any questions, just ask!

1

Let x be a positive real number. Find the minimum value of

$$8x^5 + 10x^{-4}$$

2

For $a, b, c > 0$ prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$

3

For $a, b, c > 0$ prove that $a^4 + b^4 + c^4 \geq a^2bc + b^2ca + c^2ab$

4

For $a, b, c > 0$ prove that

$$a^5 + b^5 + c^5 \geq a^3bc + b^3ca + c^3ab \geq abc(ab + bc + ca)$$

5

Let $a, b, c > 0$ with $a + b + c = 1$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 3 + 2 \cdot \frac{a^3 + b^3 + c^3}{abc}$$

6

If $abcd = 1$ for $a, b, c, d > 0$, Prove that

$$a^4b + b^4c + c^4d + d^4a \geq a + b + c + d$$

7

Let a, b, c be positive reals. Prove that

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c$$

8

If $a, b, c \geq 0$ prove that

$$\sqrt{3(a + b + c)} \geq \sqrt{a} + \sqrt{b} + \sqrt{c}$$

9

Show that for

$$f(x) = \frac{(x + k)^2}{x^2 + 1}$$

that

$$f(x) \leq k^2 + 1$$

10

Let $a, b, c > 0$. Prove that

$$3(a + b + c) \geq 8\sqrt[3]{abc} + \sqrt[3]{\frac{a^3 + b^3 + c^3}{3}}$$

11

For positive real numbers a, b, c, d find the minimum value of the expression

$$(a + b + c + d)\left(\frac{25}{a} + \frac{36}{b} + \frac{81}{c} + \frac{144}{d}\right)$$

12

For $a, b, c > 0$ prove

$$\frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \geq \frac{3}{2}$$

13

Let $r_1, r_2, r_3 \cdots, r_n$ be n real numbers each greater than zero. Prove that for any real number $x > 0$,

$$(x+r_1)(x+r_2)\cdots(x+r_n) \leq \left(x + \frac{r_1 + r_2 + \cdots + r_n}{n}\right)^n$$

14

Prove for any positive reals a, b that

$$\frac{a+b}{2} - \sqrt{ab} \geq \sqrt{\frac{a^2+b^2}{2}} - \frac{a+b}{2}$$

15

The positive real numbers w, x, y, z satisfy $w + 2x + 3y + 4z = 5$. What is the minimum possible value of

$$w^2 + \frac{1}{2}x^2 + \frac{1}{3}y^2 + \frac{1}{4}z^2$$

16

Prove that for positive reals a, b, c summing to 1, we have

$$\frac{1}{a+b} + \frac{16}{c} + \frac{81}{a+b+c} \geq 98$$

17

Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$. Show that

$$\frac{a_1^2}{a_1+b_2} + \frac{a_2^2}{a_2+b_2} + \cdots + \frac{a_n^2}{a_n+b_n} \geq \frac{a_1+a_2+\cdots+a_n}{2}$$

18

For $a, b, c > 0$ prove that

$$\frac{a^3+3b^3}{5a+b} + \frac{b^3+3c^3}{5b+c} + \frac{c^3+3a^3}{5c+a} \geq \frac{2}{3}(a^2+b^2+c^2)$$