Geometry Homework

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12/13/20

1 Review

Competitive Geometry problems revolve around your comfort with a variety of theorems and relationships.
The primary reason you won't be successful in a geometry-based competition mathematics problem are for the following reasons:

1. You forget a relationship

2. You are unfamiliar with an appication of a relationship

So for the practice problems outlined below, don't be afraid to look at the following set of relationships so you begin feeling comfortable with them, as they will be the bedrock you need to solve any geometry problem:

1. Pythagorean Theorem

2. An Equilateral triangle has side length $2x$, and altitude $x\sqrt{3}$

3. First few pythagorean triples $3, 4, 5$ and $5, 12, 13$ and $8, 15, 17$

4. Analytic area of a triangle formula, $\frac{1}{2}ab\sin C$

You should eventually be more familiar with these than you are with your right hand. And more familiar with right angles than your right hand.
The elementary trigonometric identities include the Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The Law of Cosines (or generalized Pythagorean Theorem):

$$c^2 = a^2 + b^2 - 2ab\cos C$$

And the (more uncommon) Product-Sum Tangents Identity:

$$\tan(a) \cdot \tan(b) \cdot \tan(c) = \tan(a) + \tan(b) + \tan(c)$$

As a matter of fact, this last identity is so uncommon, I tried to find additional resources on it, and I literally could not find any that referenced it. Whole pages on advanced trigonometric identities, and not one mentioned this identity. I started thinking I was going a little crazy, insisting this identity works when it's actually nonsense, but surely enough, I plugged it into the calculator and it works for all real angles of a triangle which sum to 180. Regardless, these first four relationships and the Laws of Sines and Cosines are incredibly important, and pop up in competition math constantly. I threw the last one in for fun.
The double-angle identities are crucial as well. The Sine Double-Angle Identity:

$$\sin 2\theta = 2\sin\theta\cos\theta$$

Cosine Double-Angle Identity:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

and the tougher, less easily identifiable Tangent Double-Angle Identity:

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Consider the significance of using these with geometric constructions. The first two can be life savers in those situations, the last one has more unique applications.

1.1 More Formulas

Further relationships which are the crux of a variety of competitive geometry problems include:

1. Angle Bisector formulas

2. Midpoint Area formulas

3. Ptolemy's Theorem\Cyclic Quadrilateral

4. Shoelace formula

These first two formulas are actually going to be the crux of many geometry problems. The third is applied in much tougher scenarios, but can make or break a variety of tough geometry problems. The Shoelace formula is a theorem which will be heavily applied in other sorts of harder problems which require area to solve.

The Angle Bisector formula states that when a triangle is bisected from an angle, the ratio of two line segments formed on one side of the bisector is equal to the ratio of the two line segments formed on the other.

The Midpoint Area formula simply states that a line formed from one point and the midpoint on the opposite side of a triangle form two triangles which have equal area. This is fairly intuitive but is forgotten easily.

Ptolemy's Theorem states that for a cyclic quadrilateral (a quadrilateral inscribed in a circle) $ABCD$, that the product of the diagonals is equal to the sum of the products of the opposite sides:

$$AC \cdot BD = AB \cdot CD + AD \cdot BC$$

Which is (like many geometric relationships) unintuitive. But important to remember whenever a quadrilateral is "inscribed" in a figure.

The Shoelace formula gives the area of a polygon with vertices (x_i, y_i) as

$$A = \frac{1}{2}|(x_1y_2 + x_2y_3 + \cdots + x_ny_1) - (y_1x_2 + y_2x_3 + \cdots + y_nx_1)|$$

This is not intuitive, and the proofs for it are mind-bending. But it is an extaordinary convenience when asked to solve for areas of odd polygons, or given areas of polygons with unknowns.

2 Introductory Problems

2.1

Triangle ABC has $AB = 3$, $AC = 4$,$BC = 5$. Let D be the foot of the angle bisector from A to BC. Compute BD.

2.2

Triangle ABC has $A = (0, 0)$, $B = (30, 0)$, $C = (0, 15)$. Let D be the foot of the angle bisector from A and suppose $D = (x, y)$. Find $x + y$.

2.3

Triangle ABC has $AB = 13$, $AC = 15$, $BC = 14$. Let D be the foot of the angle bisector from A. Compute AD.

2.4

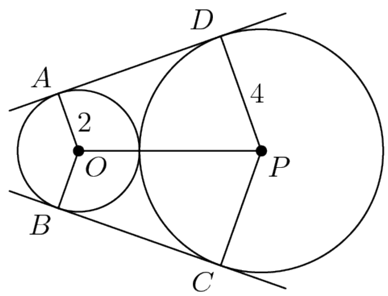
Regular hexagon $ABCDEF$ has vertices A and C at $(0, 0)$ and $(7, 1)$, respectively. What is it's area?

2.5

Let D be the point on side $\overline{BC}$ of triangle ABC for which $\overline{AD}$ is the angle bisector of $\angle A$. If $AB = AD = 6$ and $CD = 2(BD)$, then find length BD.

2.6

Circles with centers O and P have radii 2 and 4, respectively, and are externally tangent. Points A and B are on the circle centered at O, and points C and D are on the circle centered at P, such that $\overline{AD}$ and $\overline{BD}$ are common external tangents to the circles. What is the area of hexagon $AOBCPD$?



2.7

Triangle ABC is a right triangle with $\angle ACB$ as its right angle, $m\angle ABC = 60°$, and $AB = 10$. Let P be randomly chosen inside $\triangle ABC$, and extend $\overline{BP}$ to meet $\overline{AC}$ at D. What is the probability that $BD > 5\sqrt{2}$?

2.8

Select numbers a and b between 0 and 1 independently and at random, and let c be their sum. Let A, B, and C be the results when a,b and c respectively, are rounded to the nearest integer. What is the probability that $A+B = C$?

3 Further Application

3.1

What is the area of the region enclosed by the graph of the equation

$$x^2 + y^2 = |x| + |y|$$

3.2

Find all pairs (a, b) of natural numbers with $a < b$ having the property that there exists a right triangle with legs of length a and b whose hypotenuse has length $\frac{1}{3}ab - a - b$

3.3

Let C_1 and C_2 be circles defined by

$$(x-10)^2+y^2=36$$

and

$$(x+15)^2+y^2=81$$

respectively. What is the length of the shortest line segment $\overline{PQ}$ that is tangent to C_1 at P and to C_2 at Q?

3.4

Square $ABCD$ has area 36, and $\overline{AB}$ is parallel to the x-axis. Vertices A, B, and C are on the graphs of $y = \log_a x$, $y = 2\log_a x$, and $y = 3\log_a x$, respectively. What is a?

3.5

Three mutually tangent spheres of radius 1 rest on a horizontal plane. A sphere of radius 2 rests on them. What is the distance from the plane to the top of the larger sphere?

3.6

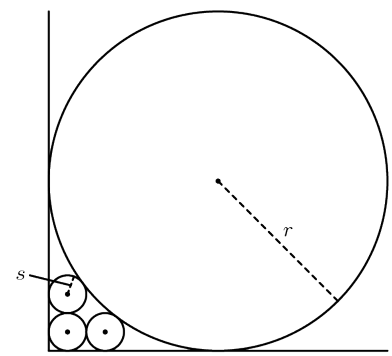
Isoceles triangle with vertex angle θ has side lengths $\sin\theta, \sqrt{\sin\theta}, \sqrt{\sin\theta}$. What is the area of this triangle?

3.7

Suppose that in quadrilateral $ABCD$ we have $m\angle ABC = m\angle ACD = 90^\circ$ and $m\angle CBD = m\angle CDB$. Label $m\angle DAC = \alpha$ and $m\angle ACB = \beta$. It follows that one of $\sin\alpha, \cos\alpha, \tan\alpha$ must always be equal to one of $\sin\beta, \cos\beta, \tan\beta$. Which two values are necessarily the same?

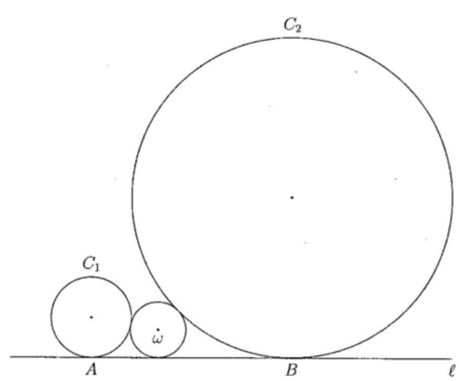
3.8

Three circles of radius s are drawn in the first quadrant of the xy-plane. The first cirlce is tangent to both axes, the second is tangent to the first circle and the x-axis, and the third is tangent to the first circle and the y-axis. A circle of radius $r > s$ is tangent to both axes and to the second and third circles. What is r/s?



3.9

Two circles, C_1 and C_2 are tangent on the same side as line l at A and B. $\overline{AB} = 20$. Their radii are 1 and 16. A third circle, ω is tangent to all three. What is the sum of all possible radii of this third circle, ω?



3.10

Suppose there are four points, A, B, C and D, along a circle for which the distances between each are integers. If the distances between adjacent points are 10, 12, 11, and 13, in that order, how many possible quadrilaterals $ABCD$ are there?