Polynomials and Algebra Homework

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12/13/20

1 Review

A quick review of things you will easily forget includes:

1. For a quadratic to have only 1 solution, it's descriminate is equal to 0
2. Difference of squares
3. For two equal polynomials, their coefficients are equal
4. The square root of a binomial might be a binomial
5. You will need to do the work to find the solution

The last one students struggle with because they lack confidence in the theorems they are working with. I encourage you to not be afraid of getting the wrong answer. It is better than having no answer.

1.1 Vieta's Formulas

Simply stated, vieta's formulas generalize the coefficients for a monic polynomial as the product of the tuples of it's roots, with alternating signs. We will take the base case for a quadratic:

$$Q(x) = (x - r_1)(x - r_2) \tag{1}$$
$$Q(x) = x^2 - (r_1 + r_2)x + r_1 r_2 \tag{2}$$

This shows us the first coefficient is the sum of the 1-tuples of the roots, and the second coefficient is the sum of the 2-tuples of the roots.
Looking at a 3rd degree monic polynomial, we have

$$C(x) = (x - r_1)(x - r_2)(x - r_3) \tag{3}$$
$$C(x) = x^3 - (r_1 + r_2 + r_3)x^2 + (r_1 r_2 + r_1 r_3 + r_2 r_3)x - r_1 r_2 r_3 \tag{4}$$

Looking more generally, for a monic polynomial of n roots, we have

$$P(x) = (x - r_1)(x - r_2) \cdots (x - r_n) \tag{5}$$
$$P(x) = x^n - (r_1 + r_2 + \cdots + r_n)x^{n-1} + \cdots \pm (r_1 r_2 \cdots r_n) \tag{6}$$

With the kth term's coefficient as the sum of the k tuples of the roots.

1.2 Binomial Coefficients

As you should be familiar with, Pascal's Triangle is a triange starting with a 1, with each row defined as the sum of the two elements proceeding it.

$$\begin{array}{lccccccccccccc}
n=0 & & & & & & & 1 & & & & & & \\
n=1 & & & & & & 1 & & 1 & & & & & \\
n=2 & & & & & 1 & & 2 & & 1 & & & & \\
n=3 & & & & 1 & & 3 & & 3 & & 1 & & & \\
n=4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
n=5 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
n=6 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

Each row of this triangle contains the coefficients of each term in an expanded binomial. For example, in row $n = 2$, we have

$$(a + b)^2 = a^2 + 2ab + b^2 = 1 \times a^2 + 2 \times ab + 1 \times b^2$$

With the coefficients mapping to the second ($n = 2$) row of Pascal's Triangle. Try this with some binomials raised to small powers to convince yourself.

1.3 Algebraic Manipulation

The key to solving problems which are heavily based on algebraic manipulation is to remember the most important piece of information any problem gives you. This piece of information is:

1. There is a solution.

With this, and your algebraic manipulation skills, you are ready to tackle any algebraic manipulation problem. They may not be straightforward, and require a lot of work, but they are more than solvable.
The only other technique which is necessary to tackle problems, particularly case-finding problems, is Simon's Factoring Trick. This is used to find solutions to a diphantine equation. Take the following example:

$$xy + 66x - 88y = 23333$$

We can break this into

$$(x - 88)(y + 66) - (-88) \cdot (66) = 23333$$

$$(x - 88)(y + 66) = 23333 - 88 \cdot 66$$

$$(x - 88)(y + 66) = 17525$$

And from here we find our solutions (x, y)

$$(17613, -65), (3593, -61), (789, -41), (113, 635), (93, 3439), (89, 17459)$$

This generalizes to any diophantine equation

$$xy + jx + ky = a$$

With all integral coefficients. This can be formed to

$$(x + k)(y + j) = a + jk$$

This is Simon's Factoring trick.

1.4 Logarithms

Basic logarithm convienciences:

$$\log x = \log_{10} x \tag{7}$$

$$\ln x = \log_e x \tag{8}$$

$$\lg x = \log_2 x \tag{9}$$

Although the logarithm without a specified base is either base 10 or base 2 depending on the scenario, in competition math, it will always be base 10. Basic logarithm rules:

$$\log(ab) = \log(a) + \log(b) \tag{10}$$
$$\log(a^b) = b \cdot \log(a) \tag{11}$$
$$\log_b(b) = 1 \tag{12}$$

These are our basic logarithm rules which should be review. The next few rules are extensions of our Change of Base rule:

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Which, by simple manipulation, gives us the chain rule:

$$\log_c a \cdot \log_a b = \log_c b$$

And under the condition that $c = b$ we have our reciprocal rule:

$$\log_a b = \frac{1}{\log_b a}$$

Which can also be shown via the chain rule. These rules must be used dynamically, but in some cases, inspection and casework is necessary.

2 Introductory Problems

2.1

For some real numbers a and b, the equation

$$8x^3 + 4ax^2 + 2bx + a = 0$$

has three distinct positive roots. If the sum of the base-2 logarithms of the roots is 5, what is the value of a?

2.2

Both roots of the quadratic equation $x^2 - 63x + k = 0$ are prime numbers. Find the number of possible values of k.

2.3

For how many complex numbers z does

$$x^2 + xz^5 + 2x + z^5 - 5$$

have a pair of repeated roots for x?

2.4

For $i \geq 1$, let $a_{i+1} = ra_i$ where r and a_i are integers. What is the least positive integer value of r such that the quadratic equation

$$a_{i+1}x^2 + a_{i+2}x + a_i = 0$$

has exactly two distinct real solutions?

2.5

There is a real number x in the interval $0 < x < 1$ satisfying the equation

$$\sqrt{1-x} + \sqrt{1+x} = \sqrt{2.012}$$

Determine the value of x^2 as a decimal.

2.6

If m, n are integers such that

$$m^2 + 3m^2n^2 = 30n^2 + 517$$

Find $3m^2n^2$

2.7

Suppose the real number x satisfies

$$\sqrt{49 - x^2} - \sqrt{25 - x^2} = 3$$

What is the value of

$$\sqrt{49 - x^2} + \sqrt{25 - x^2}$$

3 Logarithm Problems

3.1

Compute

$$\frac{\log_{10} 8 \cdot \log_{10} 16}{\log_{10} 4 \cdot \log_{10} 2}$$

3.2

What is the value of a for which the following satisfies?

$$\frac{1}{\log_2 a} + \frac{1}{\log_3 a} + \frac{1}{\log_4 a} = 1$$

3.3

What is the closest integer to the following product?

$$\prod_{k=2}^{2020} \log_k (k+1)$$

3.4

For how many positive integers x is

$$\log_{10} (x - 40) + \log_{10} (60 - x) < 2?$$

3.5

Find the value of x for which

$$\log_{\sqrt{2}} \sqrt{x} + \log_2 x + \log_4 (x^2) + \log_8 (x^3) + \log_1 6(x^4) = 40$$

3.6

Solve the following equations for x, writing your answer in simplest form.

$$(\log_{21} 48)\, x + (\log_5 13)\, y = \log_{21} 56$$

$$(\log_{13} 3)\, x + (\log_5 21)\, y = \log_{13} 7$$

3.7

The sequence

$$\log_{12} 162, \log_{12} x, \log_{12} y, \log_{12} z, \log_{12} 1250$$

is an arithmetic progression. What is x?

3.8

If $60^a = 3$ and $60^b = 5$, then find $12^{\left(\frac{1-a-b}{2-2b}\right)}$

4 Further Application

4.1

The graph of the polynomial

$$P(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$$

has five distinct x-intercepts, one of which is at $0, 0$. Which of the five unknown coefficients cannot be zero?

4.2

For certain real numbers a, b and c the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

Has three distinct roots, and each root of $g(x)$ is also a root of

$$f(x) = x^4 + x^3 + bx^2 + 100x + c$$

What is $f(1)$?

4.3

Let a, b, and c be the roots of

$$f(x) = -x^3 - 4x^2 + 16x - 3$$

Find $a^2 + b^2 + c^2$

4.4

The sum of the zeros, the product of the zeros, and the sum of the coefficients of the function $f(x) = ax^2 + bx + c$ are equal. Their common value must also be which of the following?

1. the coefficient of x^2
2. the coefficient of x
3. the y-intercept of the graph of $y = f(x)$
4. one of the x-intercepts of the graph of $y = f(x)$
5. the mean of the x-intercepts of the graph of $y = f(x)$

4.5

Find a quadratic function $f(x) = x^2 + ax + b$ such that

$$\frac{f(f(x) + x)}{f(x)} = x^2 + 1776x + 2010$$

4.6

Find a polynomial with integer coefficients which has a root of $\sqrt{2} + \sqrt{5}$