Number Theory and Modular Arithmetic Homework

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1 Review

1.1 Modular Arithmetic, Modular Sets, Congruency Classes

We have that a set modulo m contains the congruency classes $0, \cdots, m-1$

$$\mathbb{Z}/m\mathbb{Z} = \{0, \cdots, m-1\}$$

for example

$$\mathbb{Z}/5\mathbb{Z} = \{0, 1, 2, 3, 4\}$$

These conguency classes contain all numbers which are congruent to the designated number mod m. For example, the congruency class 2 mod 5 contains

$$\{\cdots, -8, -3, 2, 7, 12, \cdots\}$$

This is synonymous with our congruency statements written as

$$-3 \equiv 2 \pmod 5$$

and

$$2 \equiv 12 \pmod 5$$

Notice any multiple of m is congruent to 0 mod m.
There is a reason Modular Arithmetic is called Modular Arithmetic. That

is, arithmetic operations apply under congruency. Notice a result:

$$(-1)^2 \equiv 1 \pmod{m} \tag{1}$$

$$(am-1)^2 \equiv a^2m^2 - 2am + 1 \equiv 1 \pmod{m} \tag{2}$$

This is just an example of how arithmetic under a modulus gives us interesting results.

A **unit** mod m is a number a such that there exists an n such that

$$a^n \equiv 1 \pmod{m}$$

An example of a number which is **not** a unit mod 12 is 3, as we see here:

$$3^1 \equiv 3 \pmod{12} \tag{3}$$

$$3^2 \equiv 9 \pmod{12} \tag{4}$$

$$3^3 \equiv 3 \pmod{12} \tag{5}$$

So we see there will never be a power of 3 which will equal 1. However, let's look at 3 under mod 11

$$3^1 \equiv 3 \pmod{11} \tag{6}$$

$$3^2 \equiv 9 \pmod{11} \tag{7}$$

$$3^3 \equiv 5 \pmod{11} \tag{8}$$

$$3^4 \equiv 4 \pmod{11} \tag{9}$$

$$3^5 \equiv 1 \pmod{11} \tag{10}$$

So we see that 3 is a unit mod 11. Furthermore,

$$3^{5k+b} \equiv 3^{5k} \cdot 5^b \equiv 3^b \pmod{11}$$

The reason why 3 is a unit mod 11 but not 12? As you may have guessed, it is because 3 is relatively prime to 11 but not 12.

1.2 Fermat's Little Theorem, Prime Numbers

We restate Fermat's Little Theorem, but do not prove it below.
Fermat's Little Theorem states that if p is a prime modulo, then

$$a^{p-1} \equiv 1 \pmod{p}$$

Or that any number a, raised to the $p-1$ power, is congruent to 1, mod p. This also has implications involving the set $\mathbb{Z}/p\mathbb{Z}$;

$$\{0, 1, 2, \ldots, p-1\}$$

As when this set is multiplied by a, the remaining set

$$\{0, 1a, 2a, \ldots, (p-1)\,a\}$$

is simply a reordering of the original set. Thus

$$\{0, 1, 2, \ldots, p-1\} = \{0, 1a, 2a, \ldots, (p-1)\,a\}$$

This is extremely important.

2 Introductory Problems

You should be reminded that none of these problems will be straightforward. Nothing here will be plug-and-chug formulas.

2.1

Is $2^{21} - 2$ divisible by 11?

2.2

How many digits are in the number

$$(4^5)(5^{13})$$

2.3

What is the remainder when

$$3^0 + 3^1 + 3^2 + \cdots + 3^{99}$$

is divided by 13?

2.4

Consider the sequence $a_1, a_2, a_3 \dots$ such that $a_1 = 3$, $a_2 = 7$, and

$$a_{n+1} = a_n - a_{n-1}$$

For all $n > 2$. What is a_{2019}?

2.5

What is the tens digit in the sum

$$7! + 8! + 9! + \cdots + 2016!$$

2.6

What is the units digit of

$$7^{7^7}$$

2.7

Let

$$k = 2008^2 + 2^{2008}$$

Find the units digit of k^2

2.8

Let p be the smallest prime number with 2019 digits. What is the remainder when p^2 is divided by 12?

3 Further Application

3.1

In base 10, the number 2013 ends in the digit 3. In base 9, the same number is 2676_9 and ends in the digit 6. For how many positive integers b does the base-b representation of 2013 end in 3?

3.2

How many integers m exist such that

$$m = \frac{4 * 10^n - 1}{13}$$

Where n is an integer less than 50?

3.3

Find the least positive integer such that when its leftmost digit is deleted, the resulting integer is 1/29 of the original integer.

3.4

Find the remainder when the number

$$1^{40} 2^{39} 3^{38} \cdots 39^2 40^1$$

is divided by 41

3.5

Find the remainder when

$$1^{20} + 2^{20} + \cdots + 101^{20}$$

is divided by 101.

3.6

One of Euler's conjectures was disproved in the 1960s by three mathematicians when they showed there was a positive integer such that

$$133^5 + 110^5 + 84^5 + 27^5 = n^5$$

Find the value of n

3.7

If

$$f(x) = x^{x^{x^{x}}}$$

Find the last two digits of

$$f(17) + f(18) + f(19) + f(20)$$