Example Problems

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1 Number Theory

Find n such that

$$133^5 + 110^5 + 84^5 + 27^5 = n^5$$

2 Polynomials and Algebra

Let a, b and c be the roots of

$$f(x) = -x^3 - 4x^2 + 16x - 3$$

Find $a^2 + b^2 + c^2$

3 Complex Numbers

Let $\xi = \cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)$ be a seventh root of unity. Compute the value of

$$(2\xi + \xi^2)(2\xi^2 + \xi^4)(2\xi^3 + \xi^6)(2\xi^4 + \xi^8)(2\xi^5 + \xi^{10})(2\xi^6 + \xi^{12})$$

4 Geometry

Two circles, C_1 and C_2 are tangent on the same side as line l at A and B. $\overline{AB} = 20$. Their radii are 1 and 16. A third circle, ω is tangent to all three. What is the sum of all possible radii of this third circle, ω?

5 Series and Recursion

Let $ABCD$ be a unit square. Let Q_1 be the midpoint of $\overline{CD}$. For $i = 1, 2, \cdots$, let P_i be the intersection of $\overline{AQ_i}$ and $\overline{BD}$, and let Q_{i+1} be the foot of the perpendicular from P_i to $\overline{CD}$. What is

$$\sum_{i=1}^{\infty} [\triangle DQ_iP_i]$$

Where $[DQ_iP_i]$ denotes the area of that triangle?

6 Combinatorics

Suppose 10 points are drawn on a plane such that exactly 4 of the points are collinear and amoung the remaining points, no three points are collinear. How many distinct lines can be drawn by connecting any 2 among these 10 points?

7 Miscellaneous

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \cdots, b_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$. Show that

$$\frac{a_1^2}{a_1 + b_1} + \frac{a_2^2}{a_2 + b_2} + \cdots + \frac{a_n^2}{a_n + a_n} \geq \frac{a_1 + a_2 + \cdots + a_n}{2}$$