Complex Numbers Problem Set #1

Walter Johnson Mathematics Team

May 2021

Problems are ordered from easiest to hardest difficulty, with high probability. None of the problems require a calculator, calculus, analysis, or an abacus. If you have any questions, just ask!

1

If $f(z) = \frac{z+1}{z-1}$, then find $f^{1991}(2+i)$

2

Let $\arg(z)$ be the angle that complex number z makes with the positive real axis. Compute

$$\arg(2+i) + \arg(3+i)$$

3

Define a sequence of complex numbers by $z_1 = 0$, $z_{n+1} = z_n^2 + i$ for $n \geq 1$. How far away from the origin is z_{111}?

4

A function f is defined by $f(z) = i\overline{z}$, where $i = \sqrt{-1}$ and $\overline{z}$ is the complex conjugate of z. How many values of z satisfy both $|z| = 5$ and $f(z) = z$?

5

For a non-zero complex number z, let $f(z) = 1/\overline{z}$. Let $\omega = f(z)$. As z varies along the line

$$(1+2i)z - (1-2i)\overline{z} = i$$

what curve does ω trace?

6

A function f is defined on the complex numbers by $f(z) = (a+bi)z$, where a and b are positive numbers. This function has the property that the image of each point in the complex plane is equidistant from that point and the origin. Given that $|a+bi| = 8$, find the value of b^2.

7

Let

$$z = \frac{1+i}{\sqrt{2}}$$

Evaluate

$$\left(z^{1^2} + z^{2^2} + \cdots + z^{12^2}\right) \cdot \left(\frac{1}{z^{1^2}} + \frac{1}{z^{2^2}} + \cdots + \frac{1}{z^{12^2}}\right)$$

8

Find the number of ordered pairs of real numbers (a, b) such that $(a+bi)^{2002} = a - bi$.

9

For how many positive integers n less than or equal to 1000 is

$$(\sin t + i \cos t)^n = \sin nt + i \cos nt$$

true for all real t?

10

Let z be a complex number satisfying

$$z^2 + z + 1 = 0$$

Compute

$$\left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right)^2 + \cdots + \left(z^{45} + \frac{1}{z^{45}}\right)^2$$

11

Let $\xi = \cos(\frac{2\pi}{7}) + i\sin(\frac{2\pi}{7})$ be a seventh root of unity. Compute the value of

$$(2\xi+\xi^2)(2\xi^2+\xi^4)(2\xi^3+\xi^6)(2\xi^4+\xi^8)(2\xi^5+\xi^{10})(2\xi^6+\xi^{12})$$

12

There exists a degree 3 polynomial f in four complex variables such that the four complex numbers z_1, z_2, z_3, z_4 form a parallelogram on the complex plane if and only if $f(z_1, z_2, z_3, z_4) = 0$. Find

$$\frac{f(1,6,1,8)}{f(0,3,3,9)}$$

13

Complex numbers a, b and c are the zeros of a polynomial $P(z) = z^3 + qz + r$, and $|a|^2 + |b|^2 + |c|^2 = 250$. The points corresponding to a, b and c in the complex plane are the vertices of a right triangle with hypotenuse h. Find h^2.

14

Let $P(z) = z^8 + (4\sqrt{3}+6)z^4 - (4\sqrt{3}+7)$. What is the minimum perimeter amoung all the 8-sided polygons in the complex plane whose vertices are precisely the zeros of $P(z)$?

15

Find the probability that when 12 elements of this set are randomly chosen and multiplied, their product is -1

$$\{\sqrt{2}i, -\sqrt{2}i, \frac{1}{\sqrt{8}}(1+i), \frac{1}{\sqrt{8}}(1-i), \frac{1}{\sqrt{8}}(-1-i), \frac{1}{\sqrt{8}}(-1+i)\}$$

16

Let complex numbers z_1, z_2, z_3 be in geometric progression. The average of z_1, z_2, z_3 is 10, and the average of the squares of z_1, z_2, z_3 is $20i$. Compute z_2

17

Let z be a non-real complex number with $z^{23} = 1$. Compute

$$\sum_{k=0}^{22} \frac{1}{1 + z^k + z^{2k}}$$