Complex Numbers Homework

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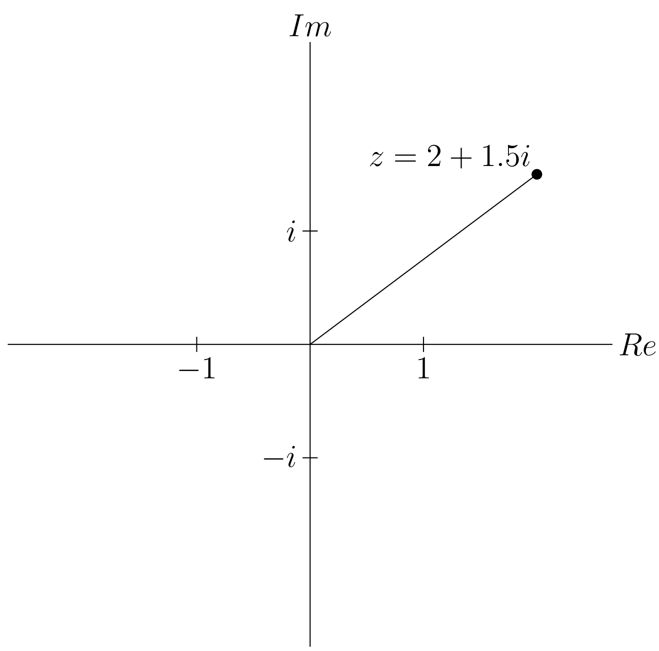
1 Review

1.1 Cartesian Complex Numbers

The Complex numbers are represented on a cartesian coordinate plane.
As you may remember, I very much dislike representing complex numbers in cartesian coordinates. However, for simple complex number problems, it suffices on occasion.



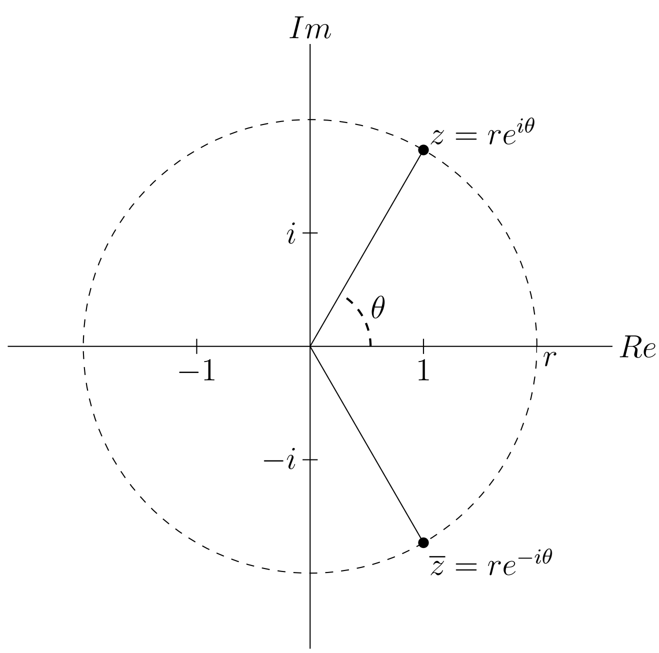
The primary thing to remember is that if two complex numbers are equal, the imaginary components and the real components are equal.

1.2 Euler's Form

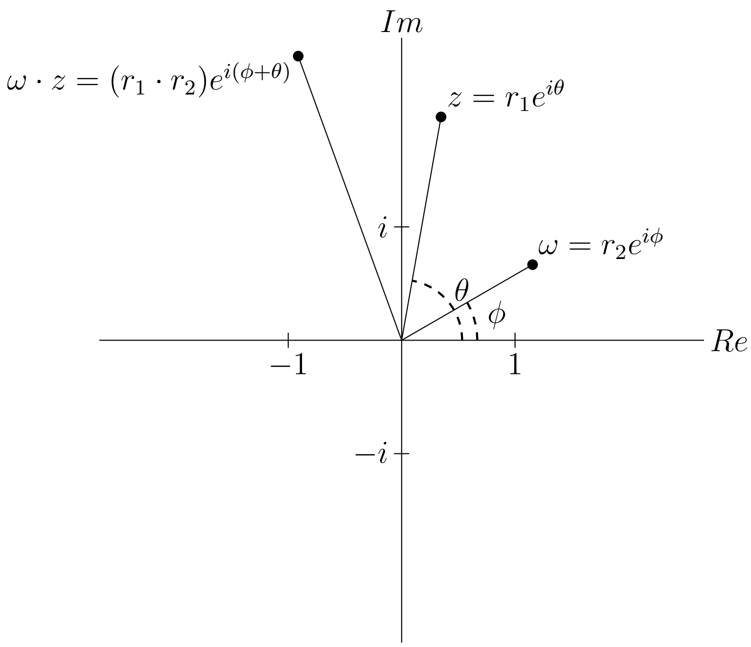
Euler's form of a complex number should be the first thing you should do when approaching any moderately difficult complex number problem.
Euler's form of a complex number restates the complex number in terms of it's radius r from the origin, and the angle θ it makes from the positive real axis.

$$z = re^{i\theta}$$

This opens a variety of identities and relationships for us to utilize.



Observe that the conjugate of a complex number $re^{i\theta}$ is simply $re^{-i\theta}$. Notice that multiplication of two complex numbers results in a product of their magnitudes and addition of their angles.



1.3 Roots of Unity

The n-th roots of unity are the solutions to the equation

$$z^n - 1 = 0 \tag{1}$$
$$z^n = 1 \tag{2}$$
$$z = \sqrt[n]{1} \tag{3}$$

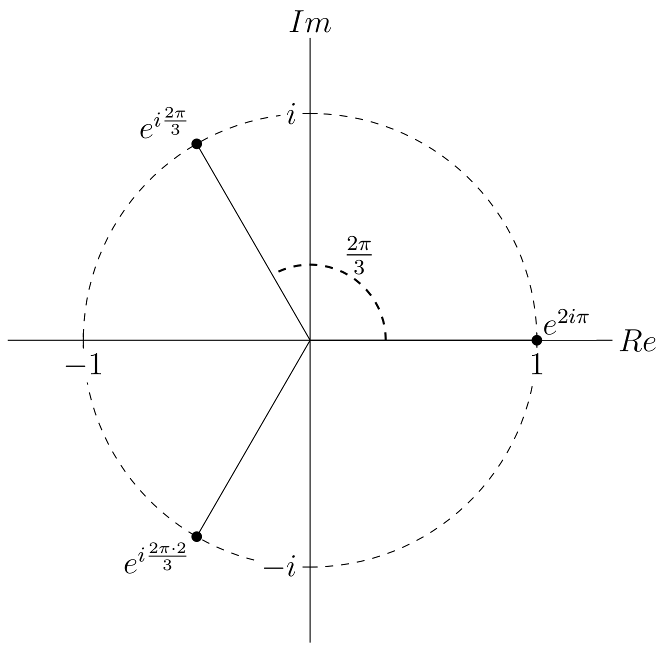
Which then has mathematical significance as

$$1 = e^{2i\pi \times k}$$

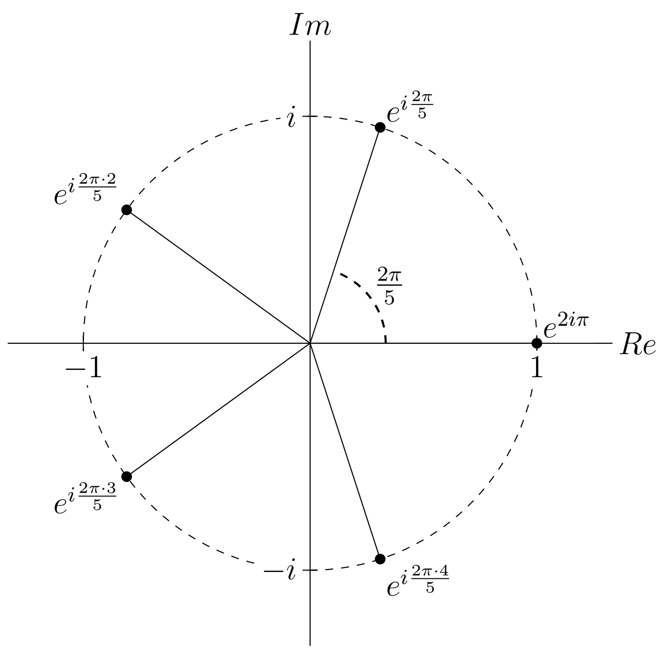
for an integer k, which gives us

$$\sqrt[n]{1} = e^{i\frac{2\pi \times k}{n}}$$

On the complex plane, these form an n-gon.
An example of this are the 3-rd roots of unity, as a triangle on the complex plane:



Another example of the 5-th roots of unity



Going back to our initial equation for the n^{th} roots of unity,

$$z^n - 1 = 0$$

We know the roots of this are the roots of unity. Given that and Vieta's Formulas, we see that the sum of the 1-tuples of the roots of unity are equal to 0, that the sum of the 2-tuples of the roots of unity are equal to 0, and so forth. But the n-tuple of the n roots of unity (equivalent to the product of all of the roots of unity) equals either 1 or -1, depending on the parity of n. It should also be noted that the equation can be factored into

$$(x - \omega_1)(x - \omega_2)(x - \omega_3) \cdots (x - 1) = 0$$

With ω_i being the i-th root of unity, and 1 being the last root of unity.
The principle root of unity is where $k = 1$, or

$$e^{i\frac{2\pi}{n}}$$

This is important because when raised to successive integer powers, generates the other ROU. Note that if n is prime, all non-trivial ROU generate each other with successive powers of themselves.

2 Introductory Problems

2.1

If a, b, c are integers which satisfy

$$c = (a + bi)^3 - 107i$$

Find c

2.2

Let $\arg(z)$ be the angle that complex number z makes with the positive real axis. Compute $\arg(2 + i)$+$\arg(3 + i)$.

2.3

Find all poles and roots to the following

$$\frac{e^z}{(z^4 - 1)(z^2 - 1)}$$

2.4

Let z be a complex number such that

$$z + \frac{1}{z} = \sqrt{3}$$

Find

$$z^{2000} + \frac{1}{z^{2000}}$$

2.5

What is the sum of all the 37^{th} roots of unity?

2.6

A function f is defined by $f(z) = i\overline{z}$, where $i = \sqrt{-1}$ and $\overline{z}$ is the complex conjugate of z. How many values of z satisfy both $|z| = 5$ and $f(z) = z$?

2.7

How many numbers are both a 74^{th} root of unity as well as a 111^{th} root of unity?

2.8

Compute

$$\sin^2 4^\circ + \sin^2 8^\circ + \sin^2 12^\circ + \cdots + \sin^2 176^\circ$$

2.9

Let

$$z = \frac{1+i}{\sqrt{2}}$$

Evaluate

$$\left(z^{1^2} + z^{2^2} + z^{3^2} + \cdots + z^{12^2}\right) \cdot \left(\frac{1}{z^{1^2}} + \frac{1}{z^{2^2}} + \frac{1}{z^{3^2}} + \cdots + \frac{1}{z^{12^2}}\right)$$

3 Further Application

3.1

Find the number of ordered pairs of real numbers (a, b) such that

$$(a+bi)^{2002} = a - bi$$

3.2

Let

$$P(z) = z^8 + (4\sqrt{3}+6)z^4 - (4\sqrt{3}+7)$$

What is the minimum perimeter among all the 8-sided polygons in the complex plane whos vertices are precisely the zeros of $P(z)$?

3.3

Let $\xi = \cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)$ be a seventh root of unity. Compute the value of

$$(2\xi+\xi^2)(2\xi^2+\xi^4)(2\xi^3+\xi^6)(2\xi^4+\xi^8)(2\xi^5+\xi^{10})(2\xi^6+\xi^{12})$$

3.4

Let complex numbers z_1, z_2, z_3 be in geometric progression. The arithmetic average of z_1, z_2, z_3 is 10, and the average of the squares of z_1, z_2, z_3 is $20i$. Compute z_2