Inequalities

Review,
Triangle, AM GM and Power Mean Inequalities,
Cauchy Schwartz Inequalities,
Schur's and Jenson's Inequalities

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What are Inequalities?

Inequalities state that some expression is less than or equal to another expression. Inequalities are often used to do one of two things:

- **Prove Other Inequalities**. You may be asked to prove that a certain inequality or statement is true. Given some universally applicable and widely known inequalities, you can prove this new one.
- Find the **Minimum or Maximum** of an expression. Given certain universally accepted inequalities, you can find that some expression can be manipulated to fit into an inequality, and thus you can find the minimum or maximum of an expression that way.

Triangle Inequality

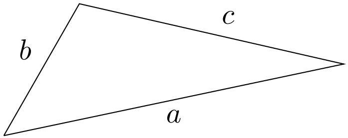
The **Triangle Inequality** is essentially the statement that three values a, b, and c can be the sides of a triangle with positive area. This implies three inequalities:

$$a + b > c$$
$$b + c > a$$
$$c + a > b$$

Any set of variables that don't satisfy this cannot be the sides of a triangle. This might seem niche, but it can play a significant role if necessary.

Ravi Substitution

Another interesting conclusion we can draw from this, is when given arbitrary real numbers a, b, and c that satisfy the triangle inequality, we can construct a triangle with them:



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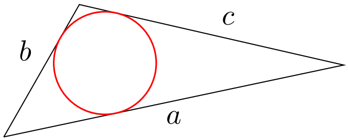
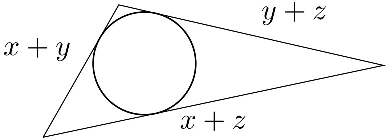


And with this triangle, we can also inscribe a circle.

Ravi Substitution



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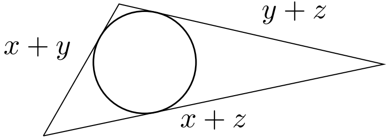
With our geometry knowledge, we can find that the intersection of b and the circle split the segment into $x+y=b$. Segment a is split into $x+z=a$, and c is split into $z+y=c$.

Ravi Substitution



And with this triangle, we can also inscribe a circle.
With our geometry knowledge, we can find that the intersection of b and the circle split the segment into $x + y = b$. Segment a is split into $x + z = a$, and c is split into $z + y = c$.
This allows us to reparameterize *any* three variables which satisfy the Trianle Inequality. We call this substitution the **Ravi Substitution**.

Mixed Terms

In the practice problem sets, you'll be looking at some problems like these

$$a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$$

$$a^4 + b^4 + c^4 \geq a^2bc + b^2ca + c^2ab$$

Notice anything?

Mixed Terms

$$a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$$

$$a^4 + b^4 + c^4 \geq a^2bc + b^2ca + c^2ab$$

The terms on the left and the right hand side both have the same degree.
Furthermore, the terms on the right tend to be more **"mixed"**.
This is a good intuition! And typically when we're faced with an inequality problem to be solved with AM-GM, this will occur. So keep this in mind!

AM GM Inequality

The **Arithmetic Mean-Geometric Mean Inequality** is a fascinating inequality that states the Arithmetic Mean of a set of positive real numbers is always greater than the Geometric mean of these real numbers.

$$\boxed{\frac{r_1 + r_2 + \cdots + r_n}{n} \geq \sqrt[n]{r_1 r_2 \cdots r_n}}$$

This is fascinating because it's always true! Take any real numbers, they satisfy this. Always.

AM GM Inequality

The **Arithmetic Mean-Geometric Mean Inequality** is a fascinating inequality that states the Arithmetic Mean of a set of positive real numbers r_i is always greater than the Geometric mean of these real numbers.

$$\boxed{\frac{r_1 + r_2 + \cdots + r_n}{n} \geq \sqrt[n]{r_1 r_2 \cdots r_n}}$$

Another way we represent this is

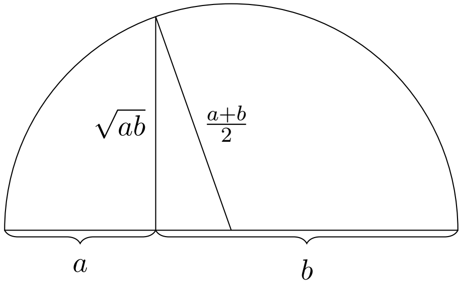
$$\boxed{\frac{\sum_{i=0}^{n} r_i}{n} \geq \sqrt[n]{\prod_{i=0}^{n} r_i}}$$

AM GM Inequality

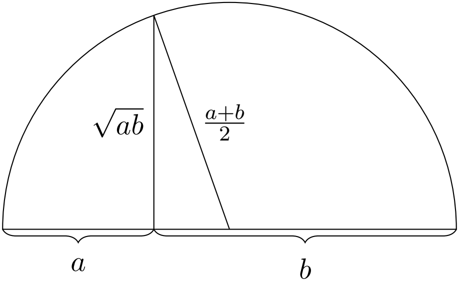
We're going to prove this with two numbers:

$$\frac{a+b}{2} \geq \sqrt{ab}$$

This is called a proof by picture. The diameter is $a+b$, the radius is the arithmetic mean, and the foot from the intersection to the diameter is the geometric mean.



AM GM Inequality



As you can clearly see, the arithmetic mean is always greater than or equal to the geometric mean.

AM GM Inequality

Now we know the AM GM Inequality, but what would this look like in a problem?
Let's try an example.

Find the maximum of the expression

$$\sqrt{2a^3b + 2b^3a} \text{ given } a + b = \sqrt{7}$$

AM GM Inequality

Find the maximum of the expression

$$\sqrt{2a^3b + 2b^3a} \text{ given } a + b = \sqrt{7}$$

Oh boy... Where do we start?

AM GM Inequality

Find the maximum of the expression

$$\sqrt{2a^3b + 2b^3a} \text{ given } a + b = \sqrt{7}$$

Ok, well our goal is to use AM GM to solve this problem. We see that the expression we're trying to solve is already under a radical. Recall, the AM GM Inequality states the Arithmetic Mean is always less than the Geometric Mean of two numbers, meaning the arithmetic mean is the maximum of the expression of the geometric mean.

AM GM Inequality

Find the maximum of the expression

$\sqrt{2a^3b + 2b^3a}$ given $a + b = \sqrt{7}$

Ok, well our goal is to use AM GM to solve this problem. We see that the expression we're trying to solve is already under a radical. Recall, the AM GM Inequality states the Arithmetic Mean is always less than the Geometric Mean of two numbers, meaning the arithmetic mean is the maximum of the expression of the geometric mean. Let's try to manipulate this expression.

AM GM Inequality

Find the maximum of the expression

$$\sqrt{2a^3b + 2b^3a} \text{ given } a + b = \sqrt{7}$$

We see that we can factor this expression:

$$\sqrt{2a^3b + 2b^3a} = \sqrt{(2ab)(a^2 + b^2)}$$

Hm, where should we go next?

AM GM Inequality

Find the maximum of the expression

$$\sqrt{2a^3b + 2b^3a} \text{ given } a + b = \sqrt{7}$$

We see that we can factor this expression:

$$\sqrt{2a^3b + 2b^3a} = \sqrt{(2ab)(a^2 + b^2)}$$

Let $x = 2ab$ and $y = a^2 + b^2$. Then we see this expression equals

$$\sqrt{(2ab)(a^2 + b^2)} = \boxed{\sqrt{xy}}$$

Ah-ha! This left hand side is the Geometric Mean of x and y! What can we do now?

AM GM Inequality

Find the maximum of the expression

$\sqrt{2a^3b + 2b^3a}$ given $a + b = \sqrt{7}$

$$x = 2ab \text{ and } y = a^2 + b^2$$

$$\sqrt{(2ab)(a^2 + b^2)} = \boxed{\sqrt{xy}}$$

AM GM tells us that this is always less than the Arithmetic Mean of x and y:

$$\boxed{\frac{x + y}{2} \geq \sqrt{xy}}$$

AM GM Inequality

Find the maximum of the expression

$\sqrt{2a^3b + 2b^3a}$ given $a + b = \sqrt{7}$

$$x = 2ab \text{ and } y = a^2 + b^2$$

$$\sqrt{(2ab)(a^2 + b^2)} = \sqrt{xy}$$

With this, we can substitute a and b back in place of x and y:

$$\frac{(2ab) + (a^2 + b^2)}{2} = \frac{x + y}{2} \geq \sqrt{xy} = \sqrt{(2ab)(a^2 + b^2)}$$

AM GM Inequality

$$\boxed{\text{Find the maximum of the expression } \sqrt{2a^3b+2b^3a} \text{ given } a+b=\sqrt{7}}$$

With this, we can substitute a and b back in place of x and y:

$$\frac{(2ab)+(a^2+b^2)}{2}=\frac{x+y}{2}\geq\sqrt{xy}=\sqrt{(2ab)(a^2+b^2)}$$

Now we have something we can work with:

$$\frac{2ab+a^2+b^2}{2}\geq\sqrt{2a^3b+2b^3a}$$

Hmm... What can we do now?

AM GM Inequality

Find the maximum of the expression

$$\sqrt{2a^3b + 2b^3a} \text{ given } a + b = \sqrt{7}$$

Now we have something we can work with:

$$\frac{2ab + a^2 + b^2}{2} \geq \sqrt{2a^3b + 2b^3a}$$

We see that this is just an expanded binomial on the left:

$$\frac{2ab + a^2 + b^2}{2} = \frac{(a+b)^2}{2} \geq \sqrt{2a^3b + 2b^3a}$$

AM GM Inequality

Find the maximum of the expression

$\sqrt{2a^3b + 2b^3a}$ given $a + b = \sqrt{7}$

$$\frac{(a+b)^2}{2} \geq \sqrt{2a^3b + 2b^3a}$$

And substituting in $a + b = \sqrt{7}$, which was given, we have:

$$\boxed{\frac{7}{2} \geq \sqrt{2a^3b + 2b^3a}}$$

Power Mean Inequality

The **Power Mean Inequalities** define an inequality on *all* of the Power Means. It states that if $m > n$, then the m^{th} power mean is greater than or equal to the n^{th} power mean.
What is a power mean?

Power Mean Inequality

The **Power Mean Inequalities** define an inequality on *all* of the Power Means. It states that if $m > n$, then the m^{th} power mean is greater than or equal to the n^{th} power mean.
A power mean for some integer n and some set of positive integers r_i is this:

$$\boxed{\sqrt[n]{\frac{1}{k}\sum^{k} r_i^n} \text{ or alternatively } \sqrt[n]{\frac{r_0^n + r_1^n + \cdots + r_k^n}{k}}}$$

Power Mean Inequality

A power mean for some integer n and some set of positive integers r_i is this:

$$\sqrt[n]{\frac{1}{k}\sum^{k} r_i^n}$$

So, for example, the 2^{nd} Power Mean (or **Quadratic Mean**) is

$$\boxed{\sqrt[2]{\frac{1}{k}\sum^{k} r_i^2} = \sqrt[2]{\frac{r_0^2 + r_1^2 + \cdots + r_k^2}{k}}}$$

and the 3^{rd} is

$$\sqrt[3]{\frac{1}{k}\sum^{k} r_i^3} = \sqrt[3]{\frac{r_0^3 + r_1^3 + \cdots + r_k^3}{k}}$$

Power Mean Inequality

A power mean for some integer n and some set of positive integers r_i is this:

$$\sqrt[n]{\frac{1}{k}\sum^{k} r_i^n}$$

So, for example, the 2^{nd} Power Mean (or **Quadratic Mean**) is

$$\boxed{\sqrt[2]{\frac{1}{k}\sum^{k} r_i^2} = \sqrt[2]{\frac{r_0^2 + r_1^2 + \cdots + r_k^2}{k}}}$$

What happens if we go to negative numbers?

Power Mean Inequality

The power mean for -1 is also known as the harmonic mean:

$$\boxed{\frac{k}{\sum^k \frac{1}{r_i}} = \frac{k}{\frac{1}{r_0} + \frac{1}{r_1} + \cdots + \frac{1}{r_k}}}$$

And there are subsequent others which follow the same rule.
What about when $n = 0$?

Power Mean Inequality

When $n = 0$, we substitute the **Geometric Mean** in:

$$\boxed{\sqrt[k]{\prod^{k} r_i} = \sqrt[k]{r_0 r_1 \cdots r_k}}$$

Power Mean Inequality

Where does the *Inequality* come into place?
Essentially, given any set of positive real numbers r_i, the power means satisfy an inequality which orders them. As n gets greater, the n^{th} Power Mean gets greater as well. And thus, the means are ordered as such:

$$\boxed{\cdots \geq \sqrt[2]{\frac{1}{k}\sum^{k} r_i^2} \geq \frac{1}{k}\sum^{k} r_i \geq \sqrt[k]{\prod^{k} r_i} \geq \frac{k}{\sum^{k} \frac{1}{r_i}} \geq \cdots}$$

Seen here are the 2^{nd}, 1^{st}, 0^{th}, and -1^{st} Power Means. Otherwise known as the Quadratic, Arithmetic, Geometric, and Harmonic Means.

Questions?

Review

The **Dot Product** of two vectors is a measure of how alike the vectors are in direction (more precisely, how linearly dependent they are). If $\vec{u}$ and $\vec{v}$ are three dimensional vectors, the following defines the dot product:

$$\boxed{\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = |\vec{u}||\vec{v}| \cos\theta}$$

Notice, the dot product is a scalar!

Review

The **Dot Product** of two vectors is a measure of how alike the vectors are in direction (more precisely, how linearly dependent they are). What is important to recall is that if $\vec{u}$ and $\vec{v}$ are higher dimensional vectors, the dot product is still defined. Let them be n-dimensional vectors:

$$\boxed{\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i}$$

Cauchy Schwartz Inequality

By far the most in-depth of all the inequalities, the **Cauchy Schwartz Inequality** has variants and generalizations from multi-dimensional linear algebra to multi-variable integration. We're primarily going to be exposed to the linear algebra explanation.

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Cauchy Schwartz Inequality

Cauchy Schwartz says that the two vectors that are most alike, and thus have the highest possible dot product, is a vector and itself. And thus, the inequality:

$$\boxed{(\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v}) \geq (\vec{u} \cdot \vec{v})^2}$$

Or, with the definition of a dot product:

$$\boxed{\left(\sum_{i=0}^{n} u_i^2\right)\left(\sum_{i=0}^{n} v_i^2\right) \geq \left(\sum_{i=0}^{n} u_i v_i\right)^2}$$

Holder's Inequality

We're able to generalize Cauchy Schwartz with multiple variables, which turns into **Holder's Inequality**:

$$\boxed{\prod_{i=0}^{n}(a_{i_0} + a_{i_1} + \cdots a_{i_k})^{\lambda_i} \geq \sum_{i=0}^{k} a_i^{\lambda_1} b_i^{\lambda_2} \cdots c_i^{\lambda_n}} \text{ for } \sum_{i=0}^{n} \lambda_i = 1$$

Which is just wild to muse at.
If this makes absolutely no sense to you, that's completely ok.

Cauchy Schwartz Integral Inequality

Note! This slide is only for Calculus students. If you haven't been exposed to much Calculus, avert your eyes!

This can be extended to the Cauchy Schwartz Integral Inequality:

$$\left(\int_{-\infty}^{\infty} |x(t)|^2\right)\left(\int_{-\infty}^{\infty} |y(t)|^2\right) \geq \left|\int_{-\infty}^{\infty} x(t)y(t)dt\right|^2$$

Schur's Inequalities

Schur's Inequality states that for $a, b, c \geq 0$:

$$\boxed{a(a-b)(a-c) + b(b-a)(b-c) + c(c-a)(c-b) \geq 0}$$

Which is actually quite intuitive when we arbitrarily choose an ordering for the variables.
Notice, that this is a symmetric polynomial.

Schur's Inequalities

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$$a(a-b)(a-c) + b(b-a)(b-c) + c(c-a)(c-b) \geq 0$$

Because this is a symmetric polynomial, we can arbitrarily choose an ordering for any of the variables. Let's say $a < c < b$.

Schur's Inequalities

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$$a(a-b)(a-c) + b(b-a)(b-c) + c(c-a)(c-b) \geq 0$$

Because this is a symmetric polynomial, we can arbitrarily choose an ordering for any of the variables. WLOG $a < b < c$. We see the term to the left is positive, so we can just focus on what is left. We can factor out $c - b$, which is negative:

$$a(a-b)(a-c) + (c-b)\left(c(c-a) - b(b-a)\right) \geq 0$$

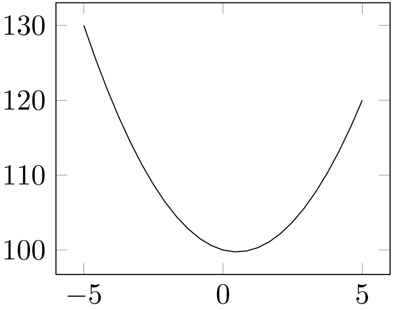
And thus $c - b$ is negative, and $c(c-a)$ is less than $b(b-a)$, so what's in the parenthesis is negative too. Thus, the whole thing is positive.

Jenson's Inequality

Jenson's Inequality is one of the most important inequalities in all of modern algebra, because it is so general that it can be used to prove a variety of other inequalities.
The inequality is best understood visually.

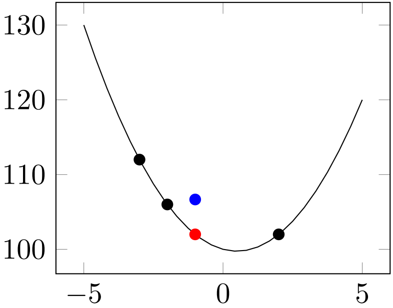
Jenson's Inequality

Consider an arbitrary concave function, $f(x)$:



Take a bunch of x values, $x_0, x_1 \cdots x_n$. Jenson's Inequality says that f evaluated at the average of those values is strictly less than or equal to the average of f evaluated at each point.

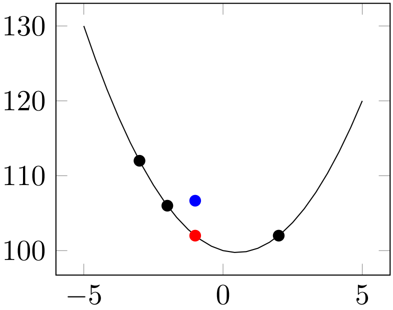
Jenson's Inequality



The red dot represents f evaluated at the average of the x inputs. The blue dot represents the average of the points on the function at each x value.

Jenson's Inequality

For any and all functions which are concave up, the inequality holds.

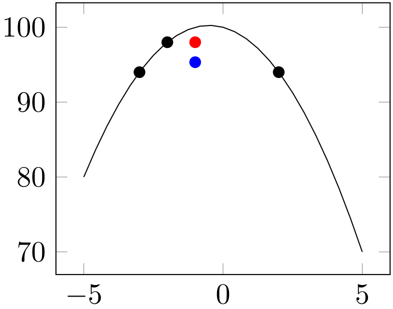


$$\boxed{\frac{1}{n}\sum_{i=0}^{n} f(x_i) \geq f\left(\frac{1}{n}\sum_{i=0}^{n} x_i\right)}$$

What if a function is concave down?

Jenson's Inequality

If a function is concave down, the $\geq$ sign flips to a $\leq$ sign:



$$\boxed{\frac{1}{n}\sum_{i=0}^{n} f(x_i) \leq f\left(\frac{1}{n}\sum_{i=0}^{n} x_i\right)}$$

Let's look at an example of this in practice.

Jenson's Inequality

Prove that $\forall n \in \mathbb{N}$

$$\sum_{i=1}^{n} \sqrt{i^2+1} \geq \frac{n}{2}\sqrt{n^2+2n+5}$$

Wow... How do we go about doing this?

Jenson's Inequality

Prove that $\forall n \in \mathbb{N}$

$$\sum_{i=1}^{n} \sqrt{i^2 + 1} \geq \frac{n}{2}\sqrt{n^2 + 2n + 5}$$

Well if we're going to prove it using Jenson's Inequality, we know we have to find some function $f(x)$ which will work well. How about we figure that out once it comes to us later.

Jenson's Inequality

Prove that $\forall n \in \mathbb{N}$

$$\sum_{i=1}^{n} \sqrt{i^2+1} \geq \frac{n}{2}\sqrt{n^2+2n+5}$$

First, we need the left hand side to be an average, so let's divide both sides by n:

$$\frac{1}{n}\sum_{i=1}^{n} \sqrt{i^2+1} \geq \frac{1}{2}\sqrt{n^2+2n+5}$$

Now we can tell that $f(x) = \sqrt{x^2+1}$. But what about the right hand side?

Jenson's Inequality

Prove that $\forall n \in \mathbb{N}$

$$\sum_{i=1}^{n} \sqrt{i^2+1} \geq \frac{n}{2}\sqrt{n^2+2n+5}$$

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Now we can tell that $f(x) = \sqrt{x^2+1}$. In order for the right hand side to work with Jenson's, it needs to be the average of all of the points that the function is evaluated at. What are these points?

Jenson's Inequality

Prove that $\forall n \in \mathbb{N}$

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Now we can tell that $f(x) = \sqrt{x^2+1}$. In order for the right hand side to work with Jenson's, it needs to be the average of all of the points that the function is evaluated at. We see that these points are all of the integers from 1 to n. Thus, the average of these integers is:

$$\frac{1+2+\cdots+n}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

Jenson's Inequality

Prove that $\forall n \in \mathbb{N}$

$$\sum_{i=1}^{n} \sqrt{i^2+1} \geq \frac{n}{2}\sqrt{n^2+2n+5}$$

$$\frac{1}{n}\sum_{i=1}^{n} \sqrt{i^2+1} \geq \frac{1}{2}\sqrt{n^2+2n+5} \text{ and } f(x) = \sqrt{x^2+1}$$

Ah-ha! Now we can evaluate this function at this average:

$$f\left(\left(\frac{n+1}{2}\right)^2+1\right) = \sqrt{\frac{n^2+2n+1}{4}+1}$$

Jenson's Inequality

$$\boxed{\text{Prove that } \forall n \in \mathbb{N} \quad \sum_{i=1}^{n} \sqrt{i^2+1} \geq \frac{n}{2}\sqrt{n^2+2n+5}}$$

$$\frac{1}{n}\sum_{i=1}^{n} \sqrt{i^2+1} \geq \boxed{\frac{1}{2}\sqrt{n^2+2n+5}} \text{ and } f(x) = \sqrt{x^2+1}$$

Ah-ha! Now we can evaluate this function at this average:

$$\sqrt{\frac{n^2+2n+1}{4}+1} = \sqrt{\frac{n^2+2n+5}{4}} = \boxed{\frac{1}{2}\sqrt{n^2+2n+5}}$$

Jenson's Inequality

Prove that $\forall n \in \mathbb{N}$

$$\sum_{i=1}^{n} \sqrt{i^2+1} \geq \frac{n}{2}\sqrt{n^2+2n+5}$$

Now we have shown that

$$\frac{1}{n}\sum_{i=1}^{n} \sqrt{i^2+1} \geq \frac{1}{2}\sqrt{n^2+2n+5}$$

Is Jenson's Inequality for $f(x) = \sqrt{x^2+1}$. What is the final condition we need for Jenson's to work?

Jenson's Inequality

Prove that $\forall n \in \mathbb{N}$

$$\sum_{i=1}^{n} \sqrt{i^2+1} \geq \frac{n}{2}\sqrt{n^2+2n+5}$$

$$\frac{1}{n}\sum_{i=1}^{n} \sqrt{i^2+1} \geq \frac{1}{2}\sqrt{n^2+2n+5}$$

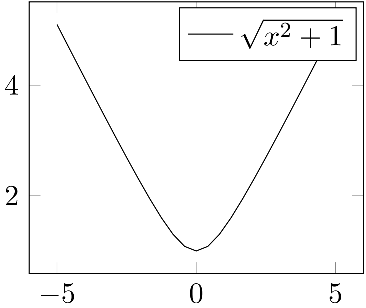
In order for Jenson's to work, we have to show that $f(x)$ is concave up!

Jenson's Inequality

Prove that $\forall n \in \mathbb{N}$

$$\sum_{i=1}^{n} \sqrt{i^2+1} \geq \frac{n}{2}\sqrt{n^2+2n+5}$$

A simple plot shows us that it is indeed:



Questions?