Warm Up! UMD HS Math Competition

The number 100000 is factored as a product of two positive integers m and n. Suppose that neither m or n has a 0 in their base-10 expansion.

What is $m+n$?

What is the volume of an octahedron whose vertices are the centers of the six faces on a unit cube?

Warm Up! *UMD HS Math Competition*

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What is $m + n$?

We can see that $100000 = 10^5$. Now the prime factorization of this $2^5 * 5^5$. Our values n and m are going to be complimentary combinations of these 2s and 5s, but given that neither can have a 0, we know that 10 cannot be a factor of either.

Warm Up! *UMD HS Math Competition*

The number 100000 *is factored as a product of two positive integers* m *and* n*. Suppose that neither* m *or* n *has a* 0 *in their base-10 expansion.*

What is $m + n$*?*

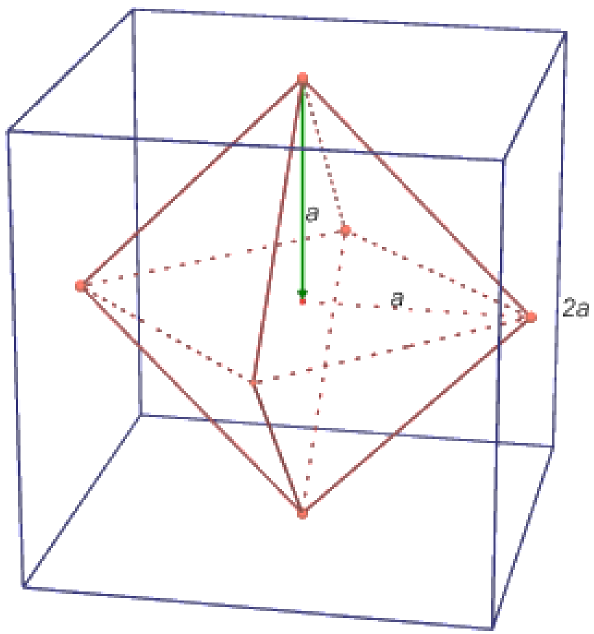
Which means if either n or m has a 2 or a 5 it cannot have the other, because then 10 becomes a factor of it. Thus, our numbers are $2^5 = 32$ and $5^5 = 3125$, and their sum

$$3125 + 32 = 3157$$

Warm Up! UMD HS Math Competition

What is the volume of an octahedron whose vertices are the centers of the six faces on a unit cube?

We can see that the area of the center cross section is equal to $^1/_2$, and thus the volume of half of the octahedron is $^1/_2 * {}^1/_2 * {}^1/_3$, and so the area of the whole figure is $^1/_6$



Warm Up! UMD HS Math Competition

A line ℓ passes through the vertex A of square $ABCD$ and has no other points in common with the square. It is known that the side of AB has length 1 and forms an angle $30°$ with ℓ. What is the shortest distance from the vertex C to the line ℓ?

Consider the sequence a_1, a_2, a_3 … such that $a_1 = 3$, $a_2 = 7$, and

$$a_{n+1} = a_n - a_{n-1}$$

For all $n > 2$

What is a_{2019}?

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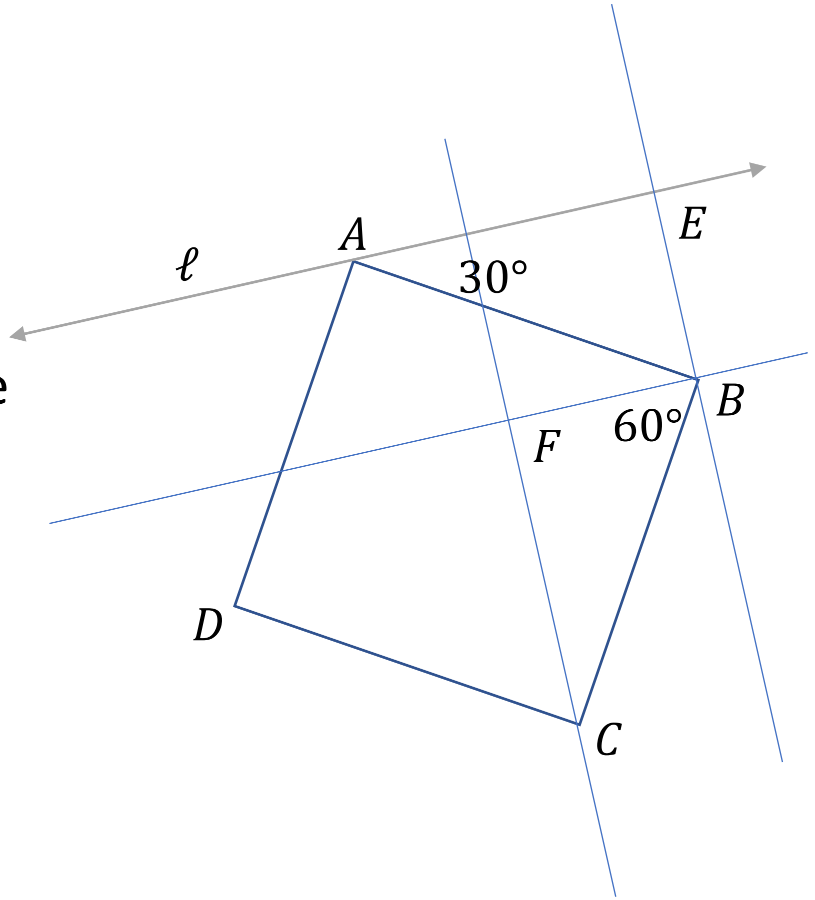
We see that this becomes simple addition of vector components, which all happen to be the sine of the designated angles.



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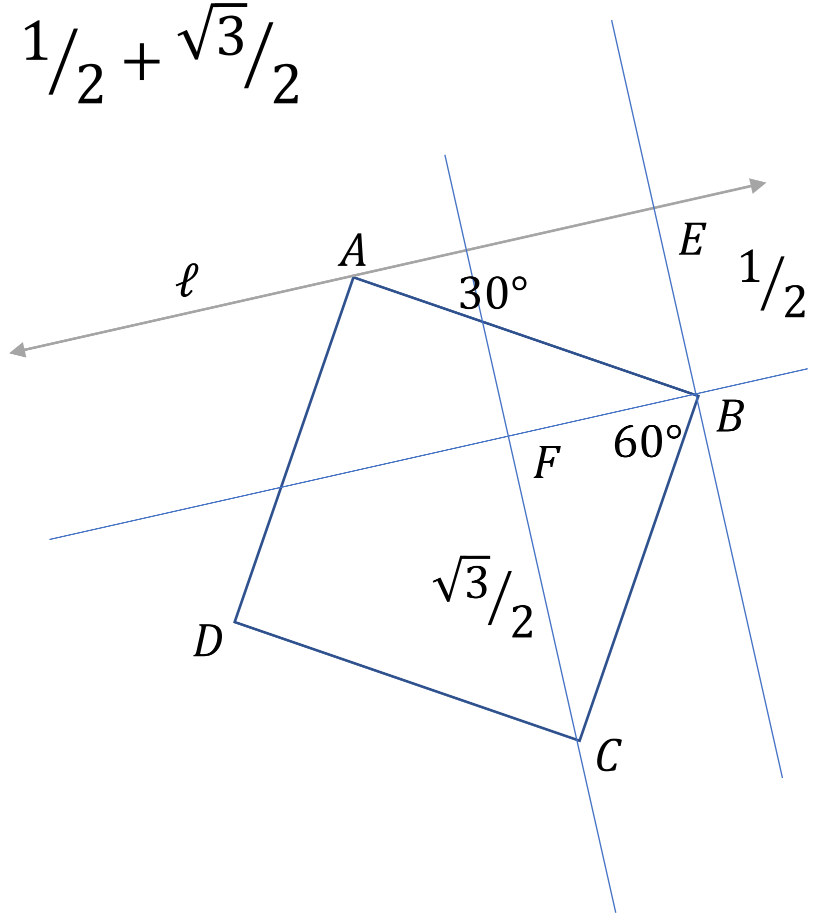
$$\sin 60° + \sin 30° = {}^{1}/_{2} + {}^{\sqrt{3}}/_{2}$$

$$= {}^{(\sqrt{3}+1)}/_{2}$$



Warm Up! *UMD HS Math Competition*

Consider the sequence

a_1, a_2, a_3 ... *such that* $a_1 = 3$,

$a_2 = 7$, *and*

$$a_{n+1} = a_n - a_{n-1}$$

For all $n > 2$

What is a_{2019}*?*

We see that the values repeat every

7[th] element, so

$$a_{2019} = a_{(2019 \, mod \, 6)}$$

And

$$2019 \equiv 3 \, mod \, 6$$

So

$$a_{2019} = a_3 = 4$$

$a_1 = 3$
$a_2 = 7$
$a_3 = 4$
$a_4 = -3$
$a_5 = -7$
$a_6 = -4$
$a_1 = a_7 = 3$
$a_2 = a_8 = 7$

Competitive Concept Sheets: Infinite Series

Key Concepts: Geometric Formula, Derivation

Problems: AMC 10A #18, 12A #24

Walter Johnson Math Team

Concept Sheets:
Geometric Series

Infinite geometric series are a key tool in many competitive problems, ranging in application.

There is one generic formula you will need to understand, and two forms of it.

$r \in (0,1)$

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$$

Also, it represented as

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

Concept Sheets: Geometric Series

$$r \in (0,1) \qquad \sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$$

If we substitute ${}^{1}/{}_{p}$ for r, we will find the new second important representation of the geometric series.

$$\sum_{n=1}^{\infty} \left(\frac{1}{p}\right)^n = \sum_{n=1}^{\infty} \frac{1}{p^n} = \frac{\frac{1}{p}}{1-\frac{1}{p}} = \frac{\frac{p}{p}}{p-1} = \frac{1}{p-1}$$

The different forms will be applied in different scenarios.

Concept Sheets: *Geometric Series Derivation*

The Geometric Series has an interesting origin in the asymptotes of exponential functions.

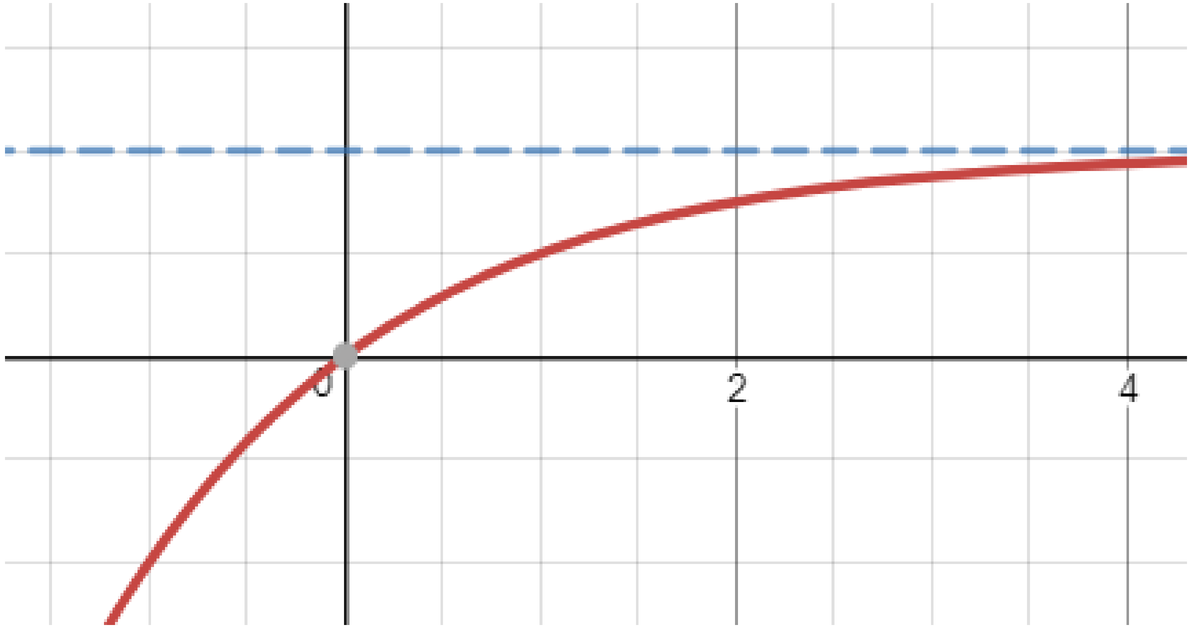
Consider $y = 1 - 2^{-x}$:

We see that

$$\lim_{x\to\infty}(1 - 2^{-x}) = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$$

$$y(1) = \frac{1}{2}, y(2) = \frac{3}{4}, y(3) = \frac{7}{8}$$

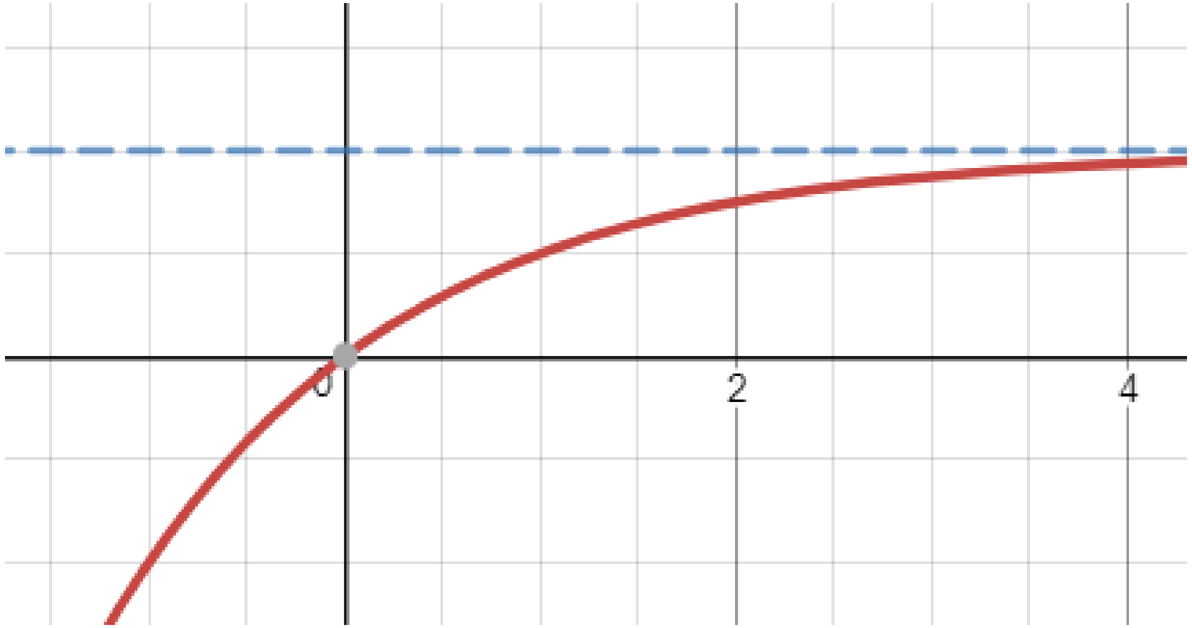
$$y(1) = \frac{1}{2}, y(2) = \frac{1}{2} + \frac{1}{4}, y(3) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$$



Concept Sheets:
Geometric Series Derivation

Fortunately this holds under all Geometric Series.

$$\lim_{x\to\infty}(1-2^{-x}) = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{p-1}$$



Concept Sheets:
Geometric Series Derivation

Like all things in Mathematics, Geometric Series pop up in fascinating places in higher mathematics.

For example, this derivation for the arctan x derivative

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 \cdots$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} \cdots$$

$$= \int \left(1 - x^2 + x^4 - x^6 \cdots\right) dx$$

$$= \int \left(1 + (-x^2)^1 + (-x^2)^2 + (-x^2)^3 \cdots\right) dx$$

$$= \int \frac{dx}{1-(-x^2)} = \int \frac{dx}{1+x^2}$$

Problems: AMC 12, Infinite Series CSs

For some positive integer k, the repeating base-k representation of the (base-ten) fraction $\frac{7}{51}$ is $0.\overline{23}_k = 0.23232323232323\ldots_k$

What is k?

Amelia has a coin that lands on heads with probability $\frac{1}{3}$, and Blaine has a coin that lands on heads with probability $\frac{2}{5}$. Amelia and Blaine alternately toss their coins until someone lands head, and 'wins'. Amelia goes first. The probability that Amelia wins is $\frac{p}{q}$.

What is $q - p$?

Problems: *AMC 12, Infinite Series CSs*

For some positive integer k, the repeating base-k representation of the (base-ten) fraction $\frac{7}{51}$ is

$$0.\overline{23}_k =$$

$$0.232323232323\ldots_k$$

What is k?

We see that the representation of numbers in specified bases is perfect for representing them in a Geometric Series

$$0.232323\ldots = \frac{2}{k} + \frac{3}{k^2} + \frac{2}{k^3} + \frac{3}{k^4} \cdots$$

$$0.232323\ldots = \frac{2k+3}{k^2} + \frac{2k+3}{k^4} \cdots$$

Problems: AMC 12, Infinite Series CSs

For some positive integer k, the repeating base-k representation of the (base-ten) fraction $\frac{7}{51}$ is

$0.\overline{23}_k =$

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What is k?

$$0.232323\ldots = \frac{2k+3}{k^2} + \frac{2k+3}{k^4}\cdots$$

$$\sum_{n=1}^{\infty} \frac{2k+3}{k^{2n}} = \frac{7}{51}$$

$$(2k+3)\sum_{n=1}^{\infty} \frac{1}{k^{2n}} = \frac{7}{51}$$

Problems: *AMC 12, Infinite Series CSs*

For some positive integer k, the repeating base-k representation of the (base-ten) fraction $\frac{7}{51}$ is

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What is k?

$$(2k+3)\sum_{n=1}^{\infty}\frac{1}{k^{2n}}=\frac{7}{51}$$

$$\frac{2k+3}{k^2-1}=\frac{7}{51}$$

$$-7k^2+102k+160=0$$

Problems: AMC 12, Infinite Series CSs

For some positive integer k, the repeating base-k representation of the (base-ten) fraction $\frac{7}{51}$ is

$$0.\overline{23}_k =$$

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What is k?

$$-7k^2 + 102k + 160 = 0$$

$$(k-16)(7k+10) = 0$$

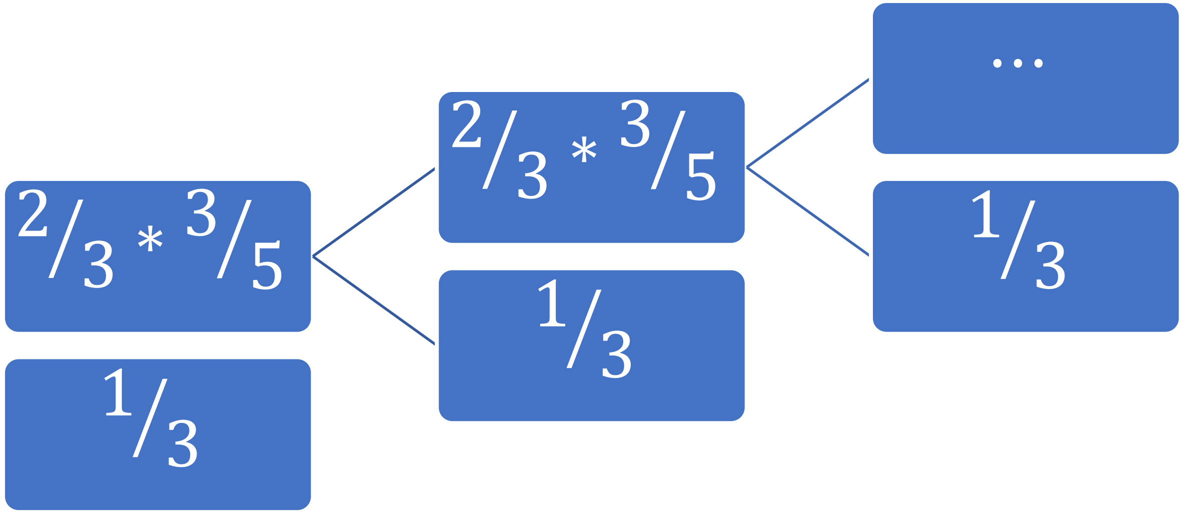
$$k = 16$$

Problems: AMC 12, Infinite Series CSs

Amelia has a coin that lands on heads with probability $\frac{1}{3}$, and Blaine has a coin that lands on heads with probability $\frac{2}{5}$. Amelia and Blaine alternately toss their coins until someone lands head, and 'wins'. Amelia goes first. The probability that Amelia wins is $\frac{p}{q}$.

What is $q - p$?

We can draw a tree to represent this probability



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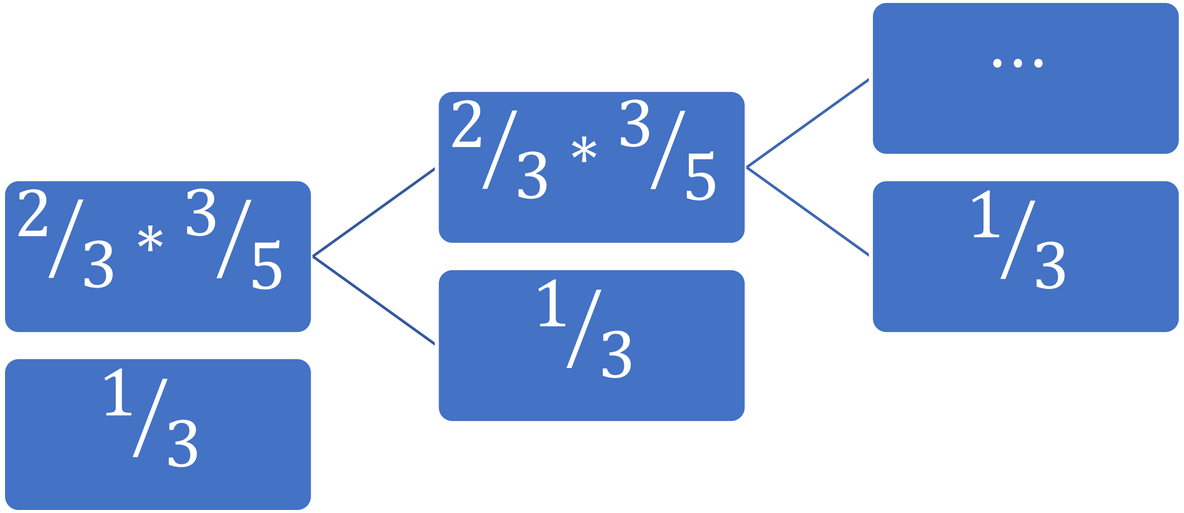

Now, we can represent this probability algebraically

$$\frac{p}{q} = \frac{1}{3} + \left(\frac{6}{15}\left(\frac{1}{3} + \frac{6}{15}\left(\frac{1}{3} + \frac{6}{15}(\cdots)\right)\right)\right)$$

Problems: *AMC 12, Infinite Series CSs*

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$$\frac{p}{q} = \frac{1}{3} + \left(\frac{6}{15}\left(\frac{1}{3} + \frac{6}{15}\left(\frac{1}{3} + \frac{6}{15}(\cdots)\right)\right)\right)$$

Expanding it out we get

$$\frac{p}{q} = \frac{1}{3} + \frac{6}{15} * \frac{1}{3} + \left(\frac{6}{15}\right)^2 \frac{1}{3} + \left(\frac{6}{15}\right)^3 \frac{1}{3} + \cdots$$

Problems: AMC 12, Infinite Series CSs

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What is $q - p$?

$$\frac{p}{q} = \frac{1}{3} + \frac{6}{15} * \frac{1}{3} + \left(\frac{6}{15}\right)^2 \frac{1}{3} + \left(\frac{6}{15}\right)^3 \frac{1}{3} + \cdots$$

$$\frac{p}{q} = \frac{1}{3} * \sum_{n=0}^{\infty} \left(\frac{6}{15}\right)^n = \frac{1}{3} * \frac{1}{1 - {}^{6}/_{15}}$$

$$\frac{p}{q} = \frac{1}{3} * \frac{1}{{}^{9}/_{15}} = \frac{1}{3} * \frac{15}{9} = \frac{5}{9}$$

$$q - p = 9 - 5 = 4$$