Algebra

Logarithms and Algebraic Manipulation,
Symmetric Polynomials, Vietta's and Simon's Formulas,
Conjugate Root Theorem, Nested Radicals and Series,
Functional Equations and Newton Sums

Walter Johnson Mathematics Team

Review

We know that you all are already masters of Algebra, so we're not going to review the basics.
But there are a few things to keep in mind that go easily forgotten:

- **Don't Be Afraid** to just manipulate expressions like crazy to get what you need from them. Most times, they will bend to your will.
- **Difference Of Squares** can pop up in wild and unexpected situations, so keep an open mind about it!
- **Use What You Already Know**. Substitute equations into each other to make the expressions give you what you want.

Review

A **Polynomial** is an incredibly important function which we will dig through extensively this year. You **must** remember that a polynomial **can** be factored into linear equations, and can always be expanded. This is important because sometimes just the knowledge of the function being factor-able introduces a insight into a problem.

Review

The sum of the first n counting numbers:

$$\boxed{\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}}$$

These are called the **Triangular Numbers**, and they are the sums of the first n numbers. We'll look more into them in Combinatorics.

Review

Complex and **Irrational Conjugates**:

$$\boxed{\overline{a+bi} = a - bi}$$

and

$$\boxed{\overline{a+\sqrt{b}} = a - \sqrt{b}}$$

When a complex number and its conjugate are multiplied, they make a real number.
When irrational conjugates are multiplied, they make a rational number.

Logarithms

A recap of the basics of Logarithms:

$$\boxed{a^b = c} \text{ means } \boxed{\log_a(c) = b}$$

The exponent rules:

$$\boxed{\log_{a^x} b^y = \frac{y}{x} \log_a b}$$

The change of base rules:

$$\boxed{\log_a b = \frac{\log_c b}{\log_c a}}$$

Which should be either familiar or comfortable for you.

Logarithms

Some more uncommon identities include the **Chain Rule** for logarithms, which follows from eliminating the denominator from the change of base rule:

$$\boxed{\log_a(b)\log_b(c) = \log_a c}$$

Which can be chained to multiple variables:

$$\boxed{\sum_{i=0}^{n-1} \log_{x_i} x_{i+1} = \log_{x_0} x_1 \log_{x_1} x_2 \cdots \log_{x_{n-1}} x_n = \log_{x_0} x_n}$$

and **Reciprocal Rule**:

$$\boxed{\log_a b = \frac{1}{\log_b a}}$$

Change of Variables

One important way to handle manipulating equations is to utilize **changing variables**. This is much like **reparameterization**. We'll show some examples.
Let's say we have variables a, b, c such that $abc = 1$ and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = a + b + c$$

Change of Variables

Let's say we have variables a, b, c such that $abc = 1$ and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = a + b + c$$

If we reparameterize these variables to be:

$$a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}, \text{ so } abc = \frac{xyz}{yzx} = 1$$

Plugging into our original inequality, we find

$$\frac{y}{x} + \frac{z}{y} + \frac{x}{z} = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$$

and

$$y^2z + z^2x + x^2y = x^2z + y^2x + z^2y$$

Which, may prove more useful than the original in certain scenarios.

Symmetric Polynomials

Before we get into Vietta's Formulas, there's a special type of multivariable polynomial which is important for us to know about. These are called **Symmetric Polynomials**. What makes a certain polynomial symmetric?

> A polynomial is symmetric if interchanging any of its variables doesn't change the polynomial.

We'll look at some examples on the next slide.

Symmetric Polynomials

For example, the following polynomials are symmetric:

$$f(x, y, z) = x^2yx + y^2xz + z^2xy$$

$$g(a, b, c) = (a - bc)(b - ac)(c - ab)$$

$$h(x, y) = x^2 - y + y^2 - x - xy$$

If you swap any of the variables, the same polynomial holds.
Are there any special kinds of Symmetric Polynomials?

Symmetric Polynomials

Within any set of variables $(a_0, a_1, \cdots, a_n)$, there are the **Elementary Symmetric Polynomials**. Let's say we just have three variables a, b, and c. Our elementary symmetric polynomials are

$$e_0 = 1$$

$$e_1 = a + b + c$$

$$e_2 = ab + ac + bc$$

$$e_3 = abc$$

We can generalize this for a set of any variables.

Symmetric Polynomials

For any set of variables $a_0, a_1, \cdots, a_n$ we have the elementary symmetric polynomials are:

$$\boxed{e_0 = 1}$$

$$\boxed{e_1 = \sum_{0 \le i \le n}^{n} a_i}$$

$$\boxed{e_2 = \sum_{0 \le i < k \le n}^{n} a_i a_k}$$

$$\boxed{e_3 = \sum_{0 \le i < k < j \le n}^{n} a_i a_k a_j}$$

$$\ldots$$

Symmetric Polynomials

Essentially, we have that

The n^{th} elementary symmetric polynomial is equal to the sum of the n-tuples of the variables.

Symmetric Polynomials

Why are Elementary Symmetric Polynomials important?
They're important because of the **Fundamental Theorem of Symmetric Polynomials**, which says

> Every Symmetric Polynomial in n variables can be factored into the n Elementary Symmetric Polynomials

This is a big deal!
Let's look at one of our previous examples to see if it's true.

Symmetric Polynomials

$$f(x, y, z) = x^2yz + y^2xz + z^2xy$$

Let's factor this:

$$f(x, y, z) = (xyz)(x + y + z) = \boxed{e_3e_1}$$

Symmetric Polynomials

What about this one:

$$g(a,b,c) = (a-bc)(b-ac)(c-ab)$$

First let's expand it:

$$g(a,b,c) = \boxed{abc - a^2b^2c^2} - a^2b^2 - a^2c^2 - b^2c^2 + \boxed{a^3bc + ab^3c + abc^3}$$

$$g(a,b,c) = \boxed{e_3 - e_3^2} - a^2b^2 - a^2c^2 - b^2c^2 + \boxed{e_3(e_1^2 - 2e_2)}$$

What can we do with the middle terms?

Symmetric Polynomials

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$$g(a,b,c) = \boxed{e_3 - e_3^2} - a^2b^2 - a^2c^2 - b^2c^2 + \boxed{e_3(e_1^2 - 2e_2)}$$

We can tell that

$$a^2b^2 + a^2c^2 + b^2c^2 = (ab+bc+ac)^2 - 2(a^2bc + ab^2c + abc^2)$$

$$a^2b^2 + a^2c^2 + b^2c^2 = \boxed{e_2^2} - 2(abc)(a+b+c) = \boxed{e_2^2 - 2e_3e_1}$$

So we have

$$g(a,b,c) = \boxed{e_3 - e_3^2 - e_2^2 + 2e_1e_3 + e_3e_1^2 - 2e_3e_2}$$

Symmetric Polynomials

Just to recap, the reason why Symmetric Polynomials are important to identify is because they can always be **factored** into their **elementary symmetric polynomials**.

Vietta's Formulas

Vietta's Formulas are perhaps the most important formulas in algebraic competitive math. What do they do?

> Vietta's Formulas relate the coefficients of a polynomial to the roots of that polynomial.

Which can be incredibly powerful in a host of situations.

Vietta's Formulas

Note, we will primarily look at monic polynomials. The same applies for other polynomials as well.

We'll begin with Vietta's for 2-degree polynomials:

$$x^2 + bx + c$$

is a monic quadratic polynomial. It can be factored:

$$(x - r_1)(x - r_2)$$

Which can be expanded:

$$x^2 - (r_1 + r_2)x + r_1 r_2$$

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Which can be expanded:

$$x^2 - (r_1 + r_2)x + r_1 r_2$$

Notice, that this allows us to find the coefficients of the quadratic in terms of the roots, which are the elementary symmetric polynomials of the roots:

$$-b = r_1 + r_2 = e_1 \text{ and } c = r_1 r_2 = e_2$$

Vietta's Formulas

$$-b = r_1 + r_2 = e_1 \text{ and } c = r_1 r_2 = e_2$$

What's special about this is that it can be used for *any* quadratic! This relates the roots to the coefficients, and vice versa.

This is not unique to quadratics either, as well see in the next few slides.

Vietta's Formulas

Let's look at a monic third degree polynomial:

$$x^3 + bx^2 + cx + d$$

It can be factored into its roots

$$(x - r_1)(x - r_2)(x - r_3)$$

And expanded to

$$x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3$$

$$e_0x^3 - e_1x^2 + e_2x - e_3$$

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Notice the alternating signs!

Vietta's Formulas

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$$(x - r_1)(x - r_2)(x - r_3)$$

And expanded to

$$x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3$$

Now we have

$$-b = r_1 + r_2 + r_3 \ , \ c = r_1r_2 + r_1r_3 + r_2r_3 \ , \ -d = r_1r_2 + r_1r_3 + r_2r_3$$

Vietta's Formulas

$$x^3 + bx^2 + cx + d$$

$$(x - r_1)(x - r_2)(x - r_3)$$

$$x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3$$

$$e_0x^3 - e_1x^2 + e_2x - e_3$$

This actually works for *all* polynomials!

Vietta's Formulas

This gives a general formula for relating the coefficients of a polynomial to its roots:

> The n^{th} coefficient of a polynomial is the sum of the n-tuples of the roots of the polynomial, with alternating signs.

Where an n-tuple is the product of n numbers.

Vietta's Formulas

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> The n^{th} coefficient of a polynomial is the sum of the n-tuples of the roots of the polynomial, with alternating signs.

Where an n-tuple is the product of n numbers.

$$x^n + bx^{n-1} + cx^{n-2} + dx^{n-3} \cdots$$

$$\boxed{x^n - \left(\sum_{i}^{n} r_i\right) x^{n-1} + \left(\sum_{i,k}^{n} r_i r_k\right) x^{n-2} - \cdots}$$

Vietta's Formulas

Which can be re-written in consideration of elementary symmetric polynomial as:

The n^{th} coefficient of a polynomial is the n^{th} elementary symmetric polynomial.

And thus every monic polynomial looks like this:

$$\boxed{e_0x^n - e_1x^{n-1} + e_2x^{n-2} \cdots}$$

For non-monic polynomials, we just multiply everything by the constant.

Vietta's Formulas

Let's look at an example of this in action:

Let α, β, and γ be the three roots of the polynomial

$$x^3 + 20x^2 - 13x - 14$$

And let $p(x)$ be the monic cubic polynomial whose roots are $\alpha + \beta$, $\alpha + \gamma$, and $\beta + \gamma$. Compute $p(0)$.

Hmm... Where do we go from here?

Vietta's Formulas

Let α, β, and γ be the three roots of the polynomial $x^3 + 20x^2 - 13x - 14$, and let $p(x)$ be the monic cubic polynomial whose roots are $\alpha + \beta$, $\alpha + \gamma$, and $\beta + \gamma$. Compute $p(0)$.

Well, we know that

$$p(x) = x^3 + ax^2 + bx + c$$

for some a, b, and c. Well, then we know that

$$p(0) = c$$

in this case. $p(0)$ is just the constant term of $p(x)$!

Vietta's Formulas

Let α, β, and γ be the three roots of the polynomial $x^3 + 20x^2 - 13x - 14$, and let $p(x)$ be the monic cubic polynomial whose roots are $\alpha + \beta$, $\alpha + \gamma$, and $\beta + \gamma$. Compute $p(0)$.

So now we're just trying to find the constant term of $p(x)$. Let's use Vietta's Formulas:

$$\boxed{p(x) = x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3}$$

And so Vietta's tell us that the constant term c is equal the negative of the product of the roots:

$$\boxed{-c = r_1r_2r_3 = e_3}$$

Vietta's Formulas

Let α, β, and γ be the three roots of the polynomial $x^3 + 20x^2 - 13x - 14$, and let $p(x)$ be the monic cubic polynomial whose roots are $\alpha + \beta$, $\alpha + \gamma$, and $\beta + \gamma$. Compute $p(0)$.

$$\boxed{-c = r_1 r_2 r_3 = e_3}$$

And the problem tells us that the roots of $p(x)$ are $\alpha + \beta$, $\alpha + \gamma$, and $\beta + \gamma$:

$$c = -r_1 r_2 r_3 = -(\alpha + \beta)(\alpha + \gamma)(\beta + \gamma)$$

$$-c = (\alpha+\beta)(\alpha+\gamma)(\beta+\gamma) = 2\alpha\beta\gamma+\alpha^2\beta+\alpha^2\gamma+\alpha\beta^2+\alpha\gamma^2+\beta^2\gamma+\beta\gamma^2$$

This is going to be some trouble!

Vietta's Formulas

Let α, β, and γ be the three roots of the polynomial $x^3 + 20x^2 - 13x - 14$, and let $p(x)$ be the monic cubic polynomial whose roots are $\alpha + \beta$, $\alpha + \gamma$, and $\beta + \gamma$. Compute $p(0)$.

$$-c = 2\alpha\beta\gamma + \alpha^2\beta + \alpha^2\gamma + \alpha\beta^2 + \alpha\gamma^2 + \beta^2\gamma + \beta\gamma^2$$

Luckily, we can use Vietta's on the given polynomial to help us a bit. We know that $e_3 = \alpha\beta\gamma = 14$ using Veitta's on the given polynomial, giving us:

$$-c = 2 \times 14 + \alpha^2\beta + \alpha^2\gamma + \alpha\beta^2 + \alpha\gamma^2 + \beta^2\gamma + \beta\gamma^2$$

Vietta's Formulas

Let α, β, and γ be the three roots of the polynomial $x^3 + 20x^2 - 13x - 14$, and let $p(x)$ be the monic cubic polynomial whose roots are $\alpha + \beta$, $\alpha + \gamma$, and $\beta + \gamma$. Compute $p(0)$.

Now let's try to factor a bit:

$$-c = 28 + \alpha\beta\gamma\left(\frac{\alpha}{\gamma} + \frac{\alpha}{\beta} + \frac{\beta}{\gamma} + \frac{\gamma}{\beta} + \frac{\beta}{\alpha} + \frac{\gamma}{\alpha}\right)$$

I'm going to add in $\frac{\alpha}{\alpha} + \frac{\beta}{\beta} + \frac{\gamma}{\gamma} - 3$, which is 0, into this:

$$-c = 28 + \alpha\beta\gamma\left(\frac{\alpha}{\alpha} + \frac{\alpha}{\beta} + \frac{\alpha}{\gamma} + \frac{\beta}{\alpha} + \frac{\beta}{\beta} + \frac{\beta}{\gamma} + \frac{\gamma}{\alpha} + \frac{\gamma}{\beta} + \frac{\gamma}{\gamma} - 3\right)$$

Vietta's Formulas

Let α, β, and γ be the three roots of the polynomial $x^3 + 20x^2 - 13x - 14$, and let $p(x)$ be the monic cubic polynomial whose roots are $\alpha + \beta$, $\alpha + \gamma$, and $\beta + \gamma$. Compute $p(0)$.

$$-c = 28 + \alpha\beta\gamma\left(\frac{\alpha}{\alpha} + \frac{\alpha}{\beta} + \frac{\alpha}{\gamma} + \frac{\beta}{\alpha} + \frac{\beta}{\beta} + \frac{\beta}{\gamma} + \frac{\gamma}{\alpha} + \frac{\gamma}{\beta} + \frac{\gamma}{\gamma} - 3\right)$$

We can factor out the $e_3 = \alpha\beta\gamma$:

$$-c = 28 + \alpha\beta\gamma\left(3\alpha\beta\gamma\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) - 3\right)$$

Hmm... What can we do now?

Vietta's Formulas

Let α, β, and γ be the three roots of the polynomial $x^3 + 20x^2 - 13x - 14$, and let $p(x)$ be the monic cubic polynomial whose roots are $\alpha + \beta$, $\alpha + \gamma$, and $\beta + \gamma$. Compute $p(0)$.

$$-c = 28 + \alpha\beta\gamma\left(3\alpha\beta\gamma\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) - 3\right)$$

Putting this fraction under a common denominator reveals something!

$$-c = 28 + \alpha\beta\gamma\left(3\alpha\beta\gamma\left(\frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}\right) - 3\right)$$

Now everything is an elementary symmetric polynomial!

Vietta's Formulas

Let α, β, and γ be the three roots of the polynomial $x^3 + 20x^2 - 13x - 14$, and let $p(x)$ be the monic cubic polynomial whose roots are $\alpha + \beta$, $\alpha + \gamma$, and $\beta + \gamma$. Compute $p(0)$.

$$-c = 28 + \alpha\beta\gamma \left(3\alpha\beta\gamma \left(\frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma} \right) - 3 \right)$$

We can rewrite this with the elementary symmetric polynomials:

$$-c = 28 + e_3 \left(3e_3 \left(\frac{e_2}{e_3} \right) - 3 \right) = 28 + 3e_3e_2 - 3e_3$$

Vietta's Formulas

Let α, β, and γ be the three roots of the polynomial $x^3 + 20x^2 - 13x - 14$, and let $p(x)$ be the monic cubic polynomial whose roots are $\alpha + \beta$, $\alpha + \gamma$, and $\beta + \gamma$. Compute $p(0)$.

And with this, Vietta's gets us the rest of the way:

$$c = 3e_3 - 3e_3e_2 - 28 = 3(14) - 3(14)(-13) - 28$$

$$\boxed{p(0) = c = 560}$$

Simon's Favorite Factoring Trick

Simon's Favorite Factoring Trick, or just Simon's for short, is an intuitive but often overseen method of factoring a non-linear Diophantine Equation.
Non-linear Diophantine Equations have a host of solutions, and Simon's lets us find them.

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A simple non-linear Diophantine Equation is below:

$$xy + 2x + 5y = 8$$

How do we factor this?

Simon's Favorite Factoring Trick

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A simple non-linear Diophantine Equation is below:

$$xy + 2x + 5y = 8$$

We factor this as such:

$$(x + 5)(y + 2) - 10 = 8$$

$$(x + 5)(y + 2) = 18$$

And now we can find a host of solutions.

Simon's Favorite Factoring Trick

$$xy + 2x + 5y = 8$$

We factor this as such:

$$(x + 5)(y + 2) - 10 = 8$$

$$(x + 5)(y + 2) = 18$$

And now we can find a host of solutions by setting the factors of 18 equal to the terms on the left:

$$(x+5)(y+2) = 18 = 18\times 1 = 9\times 2 = 6\times 3 = 3\times 6 = 2\times 9 = 1\times 18$$

And by plugging in each of these factorizations into the left, we find the following solutions:

$$(13, -1), (4, 0), (1, 1), (-2, 4), (-3, 7), (-4, 16)$$

Simon's Favorite Factoring Trick

The generalized version of this is:

$$\boxed{Kxy + Nx + My = J}$$

which is factored:

$$\boxed{K\left(x + \frac{M}{K}\right)\left(y + \frac{N}{K}\right) = J + \frac{MN}{K}}$$

Which is not as scary as it looks, you can derive this yourself easily.
You don't need to memorize this, just remember that you can always deal with this kind of non-linear Diophantine Equation with just Simon's.

Conjugate Root Theorem

Conjugate Root Theorem allows us to recognize if there are certain roots in a polynomial based on it's coefficients.
This is primarily applied when we have a polynomial with **rational coefficients**, but **complex** or **irrational roots**.

Conjugate Root Theorem

Essentially, the statement is that for a polynomial with **rational coefficients**, it can be factored into its linear factors:

$$P(x) = (x-a)(x-b)(x-c)\cdots$$

if b is irrational or complex, then we know that $(x-\bar{b})$ must also be a factor of P, so b and its conjugate $\bar{b}$ are both roots of P.

Conjugate Root Theorem

This makes sense because it means that

$$P(x) = (x - a)(x - b)(x - c) \cdots (x - \bar{b}) \cdots$$

and

$$P(x) = (x - a)\left(x^2 - b\bar{b}\right)(x - c) \cdots$$

Since $b\bar{b}$ is always real, we know this polynomial, once expanded, has real coefficients.
The theorem holds because in order to have real coefficients, the polynomial *has* to have this conjugate pair as roots.

Questions?

Nested Radicals

There are a variety of ways that radicals can be nested in each other, the first being infinitely:

$$\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-\cdots}}}}$$

and otherwise:

$$\sqrt{2-\sqrt{3}}$$

Nested Radicals

There are a variety of ways that radicals can be nested in each other, the first being infinitely:

$$\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-\cdots}}}} = 1$$

and otherwise:

$$\sqrt{2-\sqrt{3}} = \sqrt{\frac{3}{2}} - \frac{\sqrt{2}}{2}$$

Both of which we can evaluate!
This is called **Denesting**.

Infinite Nesting

For the case of infinite nested radicals, our case is actually simpler:

$$\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-\cdots}}}}$$

We can just let the expression equal x:

$$x=\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-\cdots}}}}$$

and then plug in x:

$$x=\sqrt{2-x}$$

which gives us:

$$x^2+x-2=0$$

Infinite Nesting

This can be applied for infinite fractions and infinite exponents as well:

$$e^{x^{e^{x^{e^{\cdots}}}}}$$

and

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}$$

Reminder, these are very basic examples! Sometimes they get a bit more tricky with special changes as the nesting continues, but most times they can still be solved systematically.

Finite Nested Radicals

What about finite nested radicals?

$$\sqrt{2-\sqrt{3}}$$

Not all of these can be solved, but most times they can be.

Finite Nested Radicals

What about finite nested radicals?

$$\sqrt{2-\sqrt{3}}$$

Not all of these can be solved, but most times they can be. There is a certain family of finite nested radicals that we're going to focus on, and that is those which can be expressed in this form:

$$\sqrt{a+\sqrt{b}}$$

These can **always** be solved. We'll see how.

Finite Nested Radicals

Let's look at our example

$$\sqrt{2-\sqrt{3}}$$

First step, let's make a guess!
It will probably be a binomial, and will probably have a $\sqrt{3}$ term in there.

$$\sqrt{2-\sqrt{3}} = x + y\sqrt{3}$$

Now this is something we can work with. What should we do next?

Finite Nested Radicals

$$\sqrt{2-\sqrt{3}}$$

$$\sqrt{2-\sqrt{3}} = x + y\sqrt{3}$$

Now we can square both sides:

$$2-\sqrt{3} = x^2 + 3y^2 + 2\sqrt{3}xy$$

Ah-ha! What do we do now?

Finite Nested Radicals

$$\sqrt{2-\sqrt{3}}$$

$$\sqrt{2-\sqrt{3}} = x + y\sqrt{3}$$

$$2-\sqrt{3} = x^2 + 3y^2 + 2\sqrt{3}xy$$

We can predict that the parts with a factor of $\sqrt{3}$ are equal, which gives us two equations:

$$-\sqrt{3} = 2\sqrt{3}xy$$

and

$$2 = x^2 + 3y^2$$

Squaring this first one gives us

$$1 = 4x^2y^2$$

Ah-ha! What's our next step?

Finite Nested Radicals

$$\sqrt{2-\sqrt{3}} = x + y\sqrt{3}$$
$$2 = x^2 + 3y^2$$
$$1 = 4x^2y^2$$

If we add the equations together and substitute $a = x^2$ and $b = y^2$, we get

$$3 = x^2 + 3y^2 + 4x^2y^2 = a + 3b + 4ab$$

$$3 = a + 3b + 4ab$$

What does this look like?

Finite Nested Radicals

This is a non-linear diophantine equation that we can solve with Simon's! After using Simon's, we solve the problem and have the variables.

$$\sqrt{2-\sqrt{3}} = \pm\left(\sqrt{\frac{3}{2}} - \frac{\sqrt{2}}{2}\right)$$

This was a very involved process, but most times it can be streamlined and it gets much faster. You rarely need to work this hard, but it is good to know that **all** expressions of that form **can** be simplified using Simon's.

Series and Products

Series are just sums of a bunch of numbers.
Infinite Series pop of quite frequently, and we'll analyse two ways to handle them:

- **Telescoping Series** are series for which terms will cancel each other out.

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots = 1$$

- **Geometric Series** are infinite series where the terms have a common ratio. We'll look at more examples later.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

Products

Although these are technically not series, sometimes there will be very very long products that need to be evaluated. These are solved in a similar manner as the series.

Telescoping Series

Telescoping Series have terms which cancel each other out. Consider this series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots$$

It is not intuitive how to evaluate this, but let's look at it further:

$$\frac{1}{n^2+n} = \frac{1}{n(n+1)}$$

Telescoping Series

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We can use **Partial Fraction Decomposition**:

$$\frac{1}{n^2+n} = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

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and now we see:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots$$

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Ah-ha! Now we see that

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \cdots$$

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And we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} = 1$$

Not all telescoping series will be exactly like this, but a lot of them involve either Partial Fraction Decomposition or some other form of splitting up into multiple terms.

Geometric Series

Geometric Series have a common ratio, so they are all of this form:

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots$$

And there is a very plain and simple solution:

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots = \frac{a}{1 - r}$$

Why does this work?

Geometric Series

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots = \frac{a}{1-r}$$

We can see this works because if we multiply both sides by the denominator:

$$(1-r)\left(a + ar + ar^2 + ar^3 + ar^4 + \cdots\right) = a$$

all of the terms would cancel out except for the a.

Geometric Series

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots = \frac{a}{1-r}$$

We call this the r-form of an infinite geometric series:

$$\boxed{\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}}$$

Geometric Series

The other form is the p-form, which is the same thing! Just represented a bit differently:

$$\frac{a}{p} + \frac{a}{p^2} + \frac{a}{p^3} + \frac{a}{p^4} + \cdots$$

Geometric Series

The other form is the p-form, which is the same thing! Just represented a bit differently:

$$\frac{a}{p} + \frac{a}{p^2} + \frac{a}{p^3} + \frac{a}{p^4} + \cdots$$

Once again, we have a plain and simple solution to this:

$$\frac{a}{p} + \frac{a}{p^2} + \frac{a}{p^3} + \frac{a}{p^4} + \cdots = \frac{a}{p-1}$$

Geometric Series

$$\frac{a}{p} + \frac{a}{p^2} + \frac{a}{p^3} + \frac{a}{p^4} + \cdots = \frac{a}{p-1}$$

As we see, if we multiply both sides by $p-1$, we get a telescoping series:

$$(p-1)\left(\frac{a}{p} + \frac{a}{p^2} + \frac{a}{p^3} + \frac{a}{p^4} + \cdots\right) = a$$

Geometric Series

This gives us our p-form of an infinite geometric series:

$$\boxed{\sum_{n=1}^{\infty} \frac{a}{p^n} = \frac{a}{p-1}}$$

Questions?

Functional Equations

Functional Equations are equations where we're primarily working with *functions*, not variables or expressions.
Here are examples of functional equations:

$$f(x + f(y)) = x + y$$

$$f(f(f(xy))) + f(f(x) + f(y)) = y + x + 2$$

There are a variety of ways to handle these kinds of equations.

Functional Equations

There are a variety of key techniques we use to solve these sort of problems, including:

- **Testing cases**, such as $f(x) = ax + b$ and making predictions from there.
- **Testing 0**, which involves either seeing what happens when the variables are zero, $x = 0$ or when $f(x) = 0$.
- **Testing** fff, which we will talk about soon.
- **Cauchy's Equation**, which is in the form $f(x + y) = f(x) + f(y)$.

Testing 0

This should be the first thing that comes to mind, as it can sometimes completely break a problem. Let's look at an example:

$$f(x + f(y)) = x + y$$

What can eliminate with just testing $x = 0$?

Testing 0

$$f(x + f(y)) = x + y$$

We have

$$\boxed{f(0 + f(y)) = 0 + y}$$

and thus

$$f(f(y)) = y$$

which is a major step in the right direction. With a little more work, we see the solutions to f are

$$\begin{aligned} f(y) &= y \\ &= c - y, \forall c \in \mathbb{R} \\ &= \frac{c}{y}, \forall c \in \mathbb{R} \backslash \{0\} \end{aligned}$$

and a few others.

The fff Trick

Let's say we have the equation

$$f(f(x)) = x + 2$$

Then we can find $f(f(f(x)))$ in a variety of ways. We can wrap both sides in f:

$$f(f(f(x))) = f(x + 2)$$

And we can substitute $x = f(x)$ in:

$$f(f(f(x))) = f(x) + 2$$

Which both serve to help us.

The fff Trick

$$f(f(x)) = x + 2$$

$$f(f(f(x))) = f(x + 2)$$

$$f(f(f(x))) = f(x) + 2$$

These give us the equation

$$\boxed{f(x) + 2 = f(f(f(x))) = f(x + 2)}$$

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And with these, we can test out a linear solution, $f(x) = ax + b$.

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And with these, we can test out a linear solution, $f(x) = ax + b$:

$$\boxed{ax + b + 2 = ax + 2a + b}$$

And we can solve from there:

$$b + 2 = 2a + b \implies a = 1$$

The fff Trick

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$$f(x) + 2 = f(x+2)$$

$$ax + b + 2 = ax + 2a + b$$

$$b + 2 = 2a + b \implies a = 1$$

Which gives us:

$$\boxed{f(x) = x + b \text{ for } b \in \mathbb{R}}$$

Of course, we're not done until we check that this solution works, or find a new restriction on b. We also don't know if there are any other non-linear equations that work. We would have to go back and check those as well.

Functional Equations

One primary way we handle functional equations is via **Substitution**. For example, let's try to solve this functional equation:

$$f(x) + f(x + y) = y + 2$$

Functional Equations

$$f(x) + f(x + y) = y + 2$$

Hmm... what should we do first?

$$f(x) + f(x+y) = y + 2$$

Let's let $y = 0$. We get

$$f(x) + f(x+0) = 0 + 2$$

and

$$2f(x) = 2$$

and

$$f(x) = 1$$

Which is a fairly disappointing function. But regardless, we solved it.

Newton Sums

Newton Sums are a direct application of Vietta's Formulas and symmetric polynomials. Let's say we have the polynomial

$$x^3 - 4x^2 + 5x + 13$$

with roots r_1, r_2, and r_3, and we want to find

$$r_1^3 + r_2^3 + r_3^3$$

We solve this using Newton Sums.

Newton Sums

Generally, any problem which asks us to find the k-th powers of n numbers is solved with Newton Sums, which generally looks like this:

$$\boxed{P_k = x_1^k + x_2^k + x_3^k + \cdots + x_n^k}$$

Which we notate P_k. We'll look at how to solve this in a later slide.

Newton Sums

Generally, any problem which asks us to find the k-th powers of n numbers is solved with Newton Sums, which generally looks like this:

$$P_k = x_1^k + x_2^k + x_3^k + \cdots + x_n^k$$

The key is to notice that this is indeed a symmetric polynomial in n variables, but a particularly unique kind of symmetric polynomial. Because it is a symmetric polynomial, we know that we *can* represent it exclusively with the other elementary symmetric polynomials.

For the rest of the presentation, we will exclusively be using symmetric polynomial notation, e_n.

Newton Sums

Let's look at the simple case with three variables and $k = 1$:

$$P_1 = x + y + z$$

Well we know that this is just the elementary symmetric polynomial for 3 variables:

$$P_1 = x + y + z = e_1$$

Newton Sums

Now let's consider $k = 2$:

$$P_2 = x^2 + y^2 + z^2$$

Well we know the easiest ways to get the squared terms is as such:

$$e_1 \times e_1 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$$

But as we can see, this gives $2e_2$ left over. So we subtract this and we solve:

$$P_2 = x^2 + y^2 + z^2 = e_1^2 - 2e_2$$

Newton Sums

We can note that since $P_1 = e_1$, this is actually

$$P_2 = x^2 + y^2 + z^2 = e_1^2 - 2e_2 = e_1 P_1 - 2e_2$$

Newton Sums

Now let's consider $k = 3$:

$$P_3 = x^3 + y^3 + z^3$$

We can find that this is indeed actually

$$P_3 = x^3 + y^3 + z^3 = e_1 P_2 - e_2 P_1 + 3e_3$$

Just observing some patterns.

Newton Sums

Now let's consider $k = 4$. We have

$$P_4 = x^4 + y^4 + z^4$$

and this is

$$P_4 = e_1 P_3 - e_2 P_2 + e_3 P_1$$

Newton Sums

In general, for 3 variables, we have

$$P_i = x^i + y^i + z^i = e_1 P_{i-1} - e_2 P_{i-2} + e_3 P_{i-3}$$

We'll see this pattern turn up again.

Newton Sums

In general, for n variables, the i-th Newton Sum is given by

$$P_i = \sum_{k=0}^{n} x_k^i = e_1 P_{i-1} + e_2 P_{i-2} + \cdots + e_n P_{i-n} = \sum_{k=1}^{n} e_k P_{i-k}$$

Or more concisely:

$$\boxed{P_i = \sum_{k=1}^{n} e_k P_{i-k}}$$

Questions?