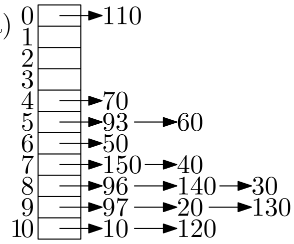
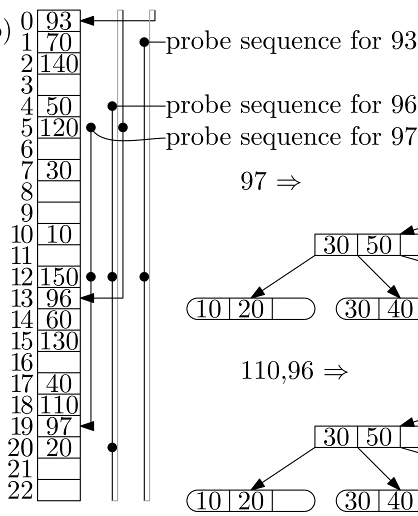
CPSC 221 — Written 3 solutions — 2 December 2016

1.

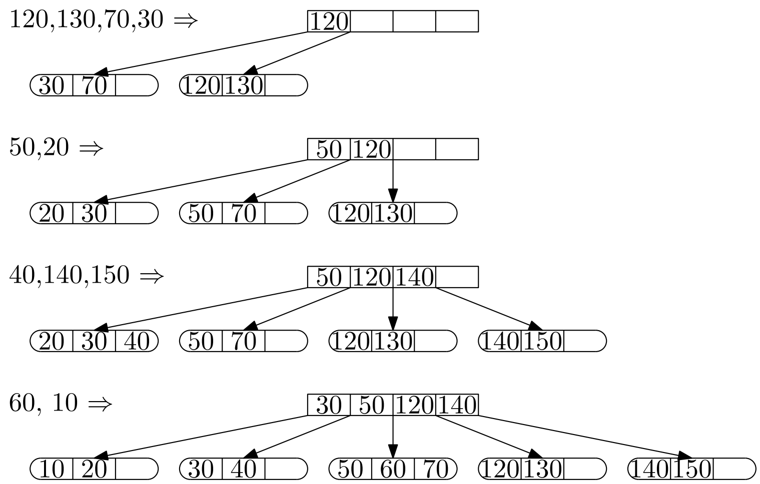
(a)

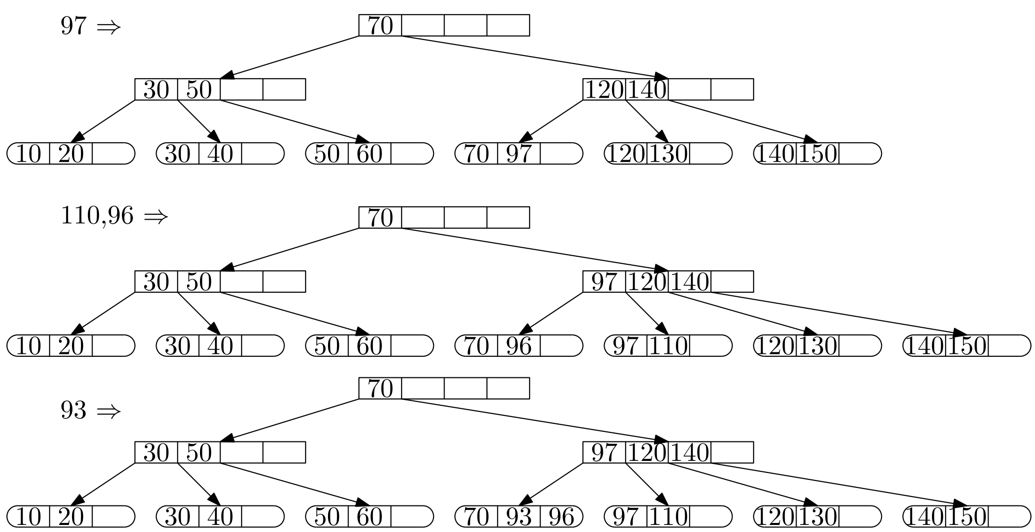


(b)

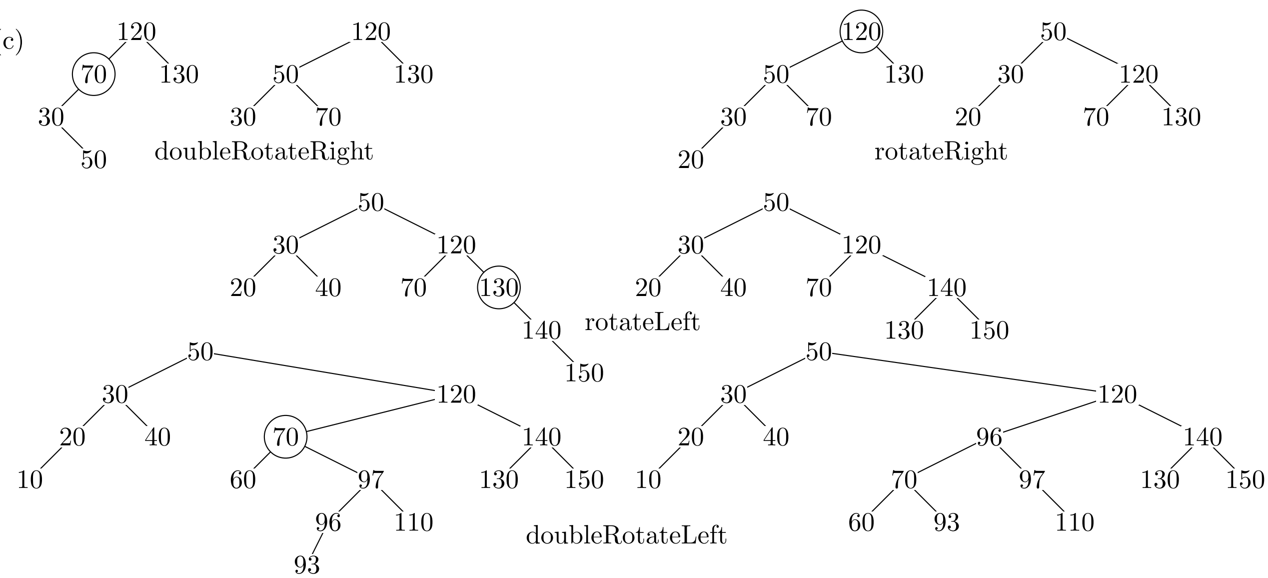


(d)





(c)



2. If a computer is not connected to any other, then the maximum number of connections at a computer is 4. Pigeons = 6 computers. Holes = number of connections at the computer, which is one of $\{0, 1, 2, 3, 4\}$. By pigeonhole principle, two computers have the same number of connections. If a computer is connected to 5 other computers, then the possible number of connections is $\{1, 2, 3, 4, 5\}$ (0 is no longer an option since one computer is connected to every other) and the pigeonhole principle applies again.

3. In an AVL tree, siblings must have height within one of each other. If the tree has height h, one of the two children of the root must have height $h-1$ and therefore the other must have height at least $h-2$. Since both of the child subtrees have the balance property, to minimize the number of nodes in the tree, one child must have the minimum number of nodes in an AVL tree of height $h-1$ and the other must have the minimum number of nodes in an AVL tree of height $h-2$. This gives us the recurrence relation for the minimum number of nodes in an AVL tree of height h as:

$$N(h) = \begin{cases} 1 & \text{if } h = 0 \\ 2 & \text{if } h = 1 \\ N(h-1) + N(h-2) + 1 & \text{otherwise.} \end{cases}$$

Claim 1. *$N(h) = F(h+3) - 1$ where $F(i)$ is the i-th Fibonacci number.*

Proof. (by induction on h)

Base Cases: For $h = 0$, $N(0) = 1$ and $F(0+3) - 1 = 2 - 1 = 1$. For $h = 1$, $N(1) = 2$ and $F(1+3) - 1 = 3 - 1 = 2$.

Inductive Hypothesis: Assume the claim is true for all $h < k$ where $k \geq 2$.

Inductive Step: For $h = k$,

$$\begin{aligned} N(h) &= N(h-1) + N(h-2) + 1 && \text{(by definition)} \\ &= F(h-1+3) - 1 + F(h-2+3) - 1 + 1 && \text{(by Ind. Hyp.)} \\ &= F(h+2) + F(h+1) - 1 \\ &= F(h+3) - 1 && \text{(by definition of } F\text{)} \end{aligned}$$

□