MATH 312

Assignment 1

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Combinatorics

(1) We can expand $\binom{n}{k+1}$ into $\frac{n!}{(k+1)!(n-k-1)!}$ and $\binom{n}{k}$ into $\frac{n!}{(n-k)!}$
Then, the problem becomes:
$= \frac{n!}{(k+1)!(n-k-1)!} + \frac{n!}{k!(n-k)!}$
$= \frac{n!(n-k)}{(k+1)!(n-k-1)!(n-k)} + \frac{n!(k+1)}{k!(n-k)!(k+1)}$
$= \frac{n!n-n!k+n!k+n!}{(k+1)!(n-k)!}$
$= \frac{(n+1)!}{(k+1)!(n-k)!}$
$= \binom{n+1}{k+1}$

(2) Claim: $\sum_{j=0}^{n}\binom{j}{0} = \binom{n+1}{1}$ Proof: For any j, $\binom{j}{0} = 1$.
As j starts at 0, the summation is the sum of $n+1$ 1s, or $n+1$.
Looking at the right hand side, $\binom{n+1}{1} = n+1$.
Thus, $LHS = RHS$ or $n+1 = n+1$

Induction

(3) Claim: For $n \in \mathbb{Z}_{\geq 1}, \exists n, n+1, n+2|$ one of the three is divisible by 3.
Proof:
   (a) Base Case: n $= 1 \to 1, 2, 3.$ $3|3$.
   (b) Assume that the Claim is true for $n = k \to$ prove for $n = k+1$
   (c) For $k+1, k+2, k+3$:
   If $3|k \to$ Claim is true as $3|k+3$
   If $3|k+1 \to$ Claim is true for $k+1$
   If $3|k+2 \to$ Claim is true for $k+2$

(4) Claim: Fix $k \geq 0$, for all $n \geq 0$, $\sum_{j=0}^{n}\binom{j}{k} = \binom{n+1}{k+1}$. Proof:
   (a) Base Case: n $= 1$
   LHS $= \binom{0}{k} + \binom{1}{k}$.
   RHS $= \binom{2}{k+1}$
   (b) Assume that the Claim is true for $n = m \to$ prove for $n = m+1$
   (c) $\sum_{j=0}^{n+1}\binom{j}{k} = \binom{n+2}{k+1}$
   LHS $= \sum_{j=0}^{n+1}\binom{j}{k}$
   $= \sum_{j=0}^{n}\binom{j}{k} + \binom{n+1}{k}$
   B $= \binom{n+1}{k+1} + \binom{n+1}{k}$ via the Induction Hypothesis.

   From Question 1, $\binom{n}{k+1} + \binom{n}{k} = \binom{n+1}{k+1}$
   OR: $\binom{n+1}{k+1} + \binom{n+1}{k} = \binom{n+2}{k+1}$
   The LHS has the form $\binom{n+1}{k+1} + \binom{n+1}{k}$, which is equal to $\binom{n+2}{k+1}$.
   This is exactly the RHS. Thus, we have proved this Claim.

(5) Question 5
(6) asdf asdf

Factorials and Primes

(7) Claim: For $2 \leq j \leq n$, show that $n! + j$ is not a prime number. Proof: $n!$ by itself is not a prime number as it can be factored by any number $< n$. Consider that $2 \leq j \leq n$:

(8) The only two consecutive prime numbers are 2 and 3. Thus, $p = 2$.