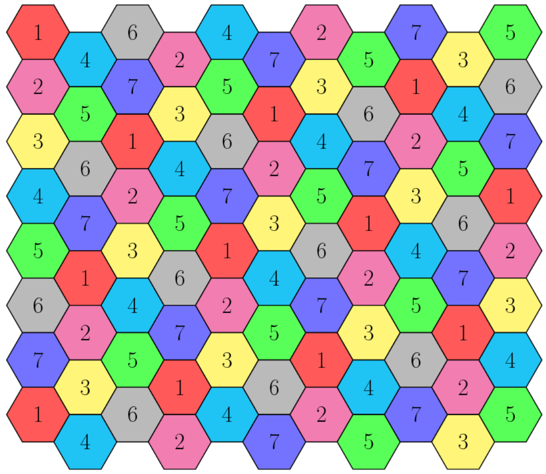
Math 442 Homework 7 Solutions

1. Let $k \geq 2$. It suffices to find a graph G that is k-colourable, and a minor G' of G that is not k colourable. Indeed, for each $k \geq 2$, we will find a graph G that is 2-colourable (and thus k-colourable) that has K_{k+1} as a minor. Let G be the graph obtained by taking K_{k+1} and inserting an additional vertex on each edge, i.e. if v_i and v_j are vertices of K_{k+1} with edge e between them, then delete edge e, add a vertex v_{ij}, and add an edge from v_i to v_{ij} and from v_j to v_{ij}. The resulting graph G is bipartite: the original vertices $v_1, \ldots, v_{k+1}$ are only adjacent to the newly inserted vertices $\{v_{ij}\}$, and vice-versa. Of course K_{k+1} is a minor of G, since we can contract each of the newly inserted vertices v_{ij} along one of their edges.

2. Let H be a regular hexagon with diameter 1 (i.e. the distance between two opposite vertices is 1). Let $H' \subset \mathbb{R}^2$ consist of the interior of the hexagon H, plus the top line segment of H. Observe that translates of H' tile the plane in a "honeycomb" fashion, i.e. we can express the plane $\mathbb{R}^2$ as an (infinite) union of translated copies of H', so that every point in $\mathbb{R}^2$ is contained in precisely one of these copies.
Given a tiling of the plane by copies of H', we will colour these copies of H' with 7 colours so that any two copies of H' of the same colour are separated by at least two copies of H'. As a basic building block, consider the following arrangement of 7 copies of H' coloured with 7 colours:

Since the diameter of each hexagon is one, and H' only contains the top part of the boundary of the hexagon, every pair of points of the same colour in the above image have distance less than 1.
We will now tile the plane with copies of the above object. This results in the following tiling[1].

[1]This image was generated by Cranston and Rabern in their paper "The fractional chromatic number of the plane," *Combinatorica* (37:837–861, 2017, which is on a related topic

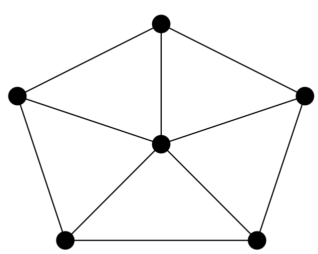


Note that in the above image, any pair of hexagons of the same colour are separated by at least two hexagons. As a result, any pair of hexagons of the same colour have distance greater than one.
With this colouring of the plane, no pair of points with distance 1 have the same colour—if two points have the same colour then either they are in the same copy of H' (and thus have distance < 1), or are in different copies of H' (and thus have distance > 1). This establishes that G is 7-colourable.

3. We will do a strong induction on the number of vertices n.

Base case: $n = 1$. One vertex is clearly 4-colourable.

Induction step: Assume true for every graph with $1 \leq n < m$ vertices, whose vertices have degree ≤ 3. Take a graph with m vertices and delete a vertex and edges joined to it. By induction colour the remaining graph with 4 colours. Insert the vertex again, and since it is adjacent to no more than 3 vertices there is a colour free to colour it with, and the graph is 4-colourable.



4. Using deletion-contraction the chromatic polynomial for the graph on the left is $k(k-1)(k-2)(k^2-3k+3)$ and on the right is $k(k-1)^3(k-2)$.

5. We will do a strong induction on the number of copies of K_n, N.

Base case: $N = 1$. Since all the vertices in K_n are adjacent we have k choices of colour for the first vertex, $k-1$ for the second, $\ldots$, $k-(n-1)$ for the n-th vertex. Hence

$$P_{Wd(n,1)}(k) = P_{K_n}(k) = k(k-1)\cdots(k-(n-1)) = k\prod_{i=1}^{n-1}(k-i)^1.$$

Induction step: Now assume that the result is true for $1 < N < m$ and consider $Wd(n,m)$. Delete one copy of K_n from $Wd(n,m)$ to form $Wd(n,m-1)$. Then by induction $P_{Wd(n,m-1)}(k) = k\prod_{i=1}^{n-1}(k-i)^{m-1}$. Inserting the one remaining copy of K_n we can colour the remaining vertices in $(k-1)(k-2)\cdots(k-(n-1))$ colours and hence

$$P_{Wd(n,m)}(k) = k\prod_{i=1}^{n-1}(k-i)^m.$$