Math 442 Homework 8 Solutions

1. We will first prove the coefficient of k^n is 1 AND the coefficient of k^{n+r} for $r > 0$ is 0. This will be a strong induction on the number of edges E.

Base case: $E = 0$. $P_G(k) = k^n$ from the notes.

Induction step: Assume the result is true for up to and including $m-1$ edges. Then for a graph G with m edges, by deletion-contraction on some $e \in E(G)$ we have

$$P_G(k) = P_{G-e}(k) - P_{G/e}(k).$$

By induction the coefficient of k^n is 1 in $P_{G-e}(k)$ and 0 in $P_{G/e}(k)$ and the coefficient of k^{n+r} for $r > 0$ is 0, and the result follows.

We now prove the coefficient of k^{n-1} is $-|E(G)|$ by strong induction on the number of edges E.

Base case: $E = 0$. $P_G(k) = k^n$ from the notes and the coefficient of k^{n-1} is $0 = -|E(G)|$.

Induction step: Assume the result is true for up to and including $m-1$ edges. Then for a graph G with m edges, by deletion-contraction on some edge $e \in E(G)$ we have

$$P_G(k) = P_{G-e}(k) - P_{G/e}(k).$$

By induction the coefficient of k^{n-1} in $P_{G-e}(k)$ is $-(m-1)$ and in $P_{G/e}(k)$ is 1 by the first part of this question. Hence the coefficient of k^{n-1} in $P_G(k)$ is $-(m-1)-1 = -m = -|E(G)|$ as desired.

2. We will do a weak induction on the number of components C.

Base case: $C = 1$. $P_G(k)$ is the chromatic polynomial of the one component.

Induction step: Assume the result is true for a graph with m components. Then for $m+1$ components, in components $\mathcal{C}_1, \ldots, \mathcal{C}_m$ we know that since no vertex in them is connected to any vertex in $\mathcal{C}_{m+1}$, then $\mathcal{C}_{m+1}$ can be coloured independently from the other components in $P_{\mathcal{C}_{m+1}}(k)$ ways. Hence

$$P_G(k) = P_{\mathcal{C}_1,\ldots,\mathcal{C}_m}(k)P_{\mathcal{C}_{m+1}}(k) = P_{\mathcal{C}_1}(k)\cdots P_{\mathcal{C}_m}(k)P_{\mathcal{C}_{m+1}}(k)$$

by the induction hypothesis, and the result follows.

3. First, if $G = K_n$ then G is n colourable since we can assign a different colour to each vertex. On the other hand, by the pigeonhole principle we must have that any colouring of G using fewer than n colours must assign the same colour to two vertices, which is forbidden since every pair of vertices are adjacent.

Conversely, if $G \neq K_n$ then it contains a vertex v such that $deg(v) < n-1$. Consider $G' = G - v$. Since G' has $n-1$ vertices we can colour it with $n-1$ colours. Reinsert v into G' to recover G. Then since $deg(v) < n-1$ there must be at least one of the $n-1$ colours, c, not used to colour a vertex adjacent to v. Colour v with colour c. Hence $\chi(G) \leq n-1$, so $\chi(G) \neq n$.

4. We will do a strong induction on the number of edges E.

Base case: Let $E = 0$. Then $G = N_n$ for $n \geq 1$ and we proved in lecture that $P_G(k) = k^n$, whose one term alternates in sign.

Induction step: Now assume that the result is true for $0 < E < m$. Now given a simple graph with $E = m$ edges and n vertices, by deletion-contraction we get

$$P_G(k) = P_{G-e}(k) - P_{G/e}(k)$$

where by induction giving us that the coefficients alternate in sign, and knowing that the coefficient of the leading term of the chromatic polynomial is 1 by Question 1 we also know that

$$P_{G-e}(k) = k^n - a_{n-1}k^{n-1} + a_{n-2}k^{n-2} - \cdots + (-1)^n a_0$$
$$P_{G/e}(k) = k^{n-1} - b_{n-2}k^{n-2} + b_{n-3}k^{n-3} - \cdots + (-1)^{n-1} b_0$$

for $a_i, b_i \geq 0$ for all i. Hence

$$P_G(k) = k^n - (a_{n-1} + 1)k^{n-1} + (a_{n-2} + b_{n-2})k^{n-2} - \cdots + (-1)^n (a_0 + b_0)$$

and the result follows by induction.

5. At the point where we delete and reinsert a vertex of degree 5 in the five colour theorem we rely on the fact that K_5 is not planar. However, when we adjust the proof to delete and reinsert a vertex of degree 4 in proving the four colour theorem we would need that K_4 is not planar, but this is not true.

6. Let $\chi(G) = k$ and the deletion of any vertex v yields a graph with a smaller chromatic number, i.e. $\chi(G - v) = m \leq k - 1$. If there exists a vertex $\tilde{v} \in V(G)$ of degree less than $k-1$ delete it. Then colour the remaining graph in $k-1$ colours. Reinsert $\tilde{v}$. As it is adjacent to at most $k-2$ vertices there is at least one of the $k-1$ colours we could colour it, so $\chi(G) \leq k-1$. Since this is not true, the degree of every vertex must be at least $k-1$.