# Math 442 Homework 5

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**1.** Prove that every tree with at least two vertices has a vertex of degree one. You may not use the fact that $v = e + 1$.

*Proof.* This statement will be proven by induction:

- Base Case (2 vertices): The simplest and only tree we can create with 2 vertices is a tree where both vertices are connected by a single edge. This tree has 2 vertices with degree 1. Thus the base case satisfies this claim.
- Inductive step: Assume we have a tree with $n$ vertices and there exists a vertex $u$ with degree 1. If we were to add an additional vertex $v$ to this tree, we are only able to connect it to the existing tree with a single edge, as a $2^{nd}$ edge would create a cycle, and therefore we would no longer have a tree. At this point we have 2 cases:
  - Case 1: We connect $v$ to a vertex in the existing tree whose degree is larger than 1. In this case, we will be guaranteed the tree has at least 2 vertices of degree 1, our new vertex $v$, and $u$.
  - Case 2: We connect $v$ to $u$. In this case $u$ will no longer be of degree 1, but $v$ will now be of degree 1.
- Therefore by adding a new vertex $v$ with a single edge, we have shown the inductive step to satisfy the claim.

Since the base case, and the inductive steps satisfy the original claim, we can conclude that this statement is true. $\square$

**2.** Let $G$ be a (simple) planar graph that contains no cycle of length $\leq k$. Prove that

$$e \leq \frac{k+1}{k-1}v.$$

*Proof.* If $G$ is a simple planar graph with no cycles of length $\leq k$, then all faces $f$ in $G$ must be bounded by at least $k+1$ edges. Additionally, since edges are the boundaries between 2 faces (or the same one), we can observe that:

$$2e \geq (k+1)f$$

Using Euler's theorem for planar graphs ($v - e + f = 2$):

$$\begin{align} 2e &\geq (k+1)(2-v+e) \tag{1} \\ 2e &\geq 2(k+1) - v(k+1) + e(k+1) \tag{2} \\ 2e - ek - e &\geq 2(k+1) - v(k+1) \tag{3} \\ e(k-1) &\leq v(k+1) - 2(k+1) \tag{4} \\ e &\leq v\frac{k+1}{k-1} - 2\frac{k+1}{k-1} \tag{5} \end{align}$$

Which is a stronger inequality than the one presented in the original question. Thus we have proven the original statement. □

**3.** Prove that if a graph has no closed paths of odd length then it is bipartite.

*Proof.* This statement essentially claims that if a graph G has only closed paths of even length, or no closed paths at all, then the G is bipartite. We can now evaluate this claim by attempting to colour each alternative vertex in this graph red or blue such that each red vertex is only adjacent to a blue vertex.

- Case 1: G contains no closed paths at all. Then G must be a tree. We can pick any starting vertex $v$ in this tree and colour it red. Then we can colour all its neighbours blue. We can continue this pattern such that all vertices at odd length away from $v$ are blue and all vertices an even length away from $v$ are red. We are guaranteed that this colouring will work as without loss of generality, no red vertex will be adjacent to any other red vertex as G contains no cycles.

- Case 2: G contains a closed path of even length. We can use a similar approach as in case 1, and choose any vertex as a starting point. Suppose our starting vertex is red, and we follow this even length closed cycle. Then in this cycle (closed path) the final vertex before returning to the starting vertex must be blue as it was an odd number of steps away. Therefore, even cycles will obey our colouring scheme. If G contains extra vertices that are not part of even cycles, they must follow case 1, as there are no odd cycles.

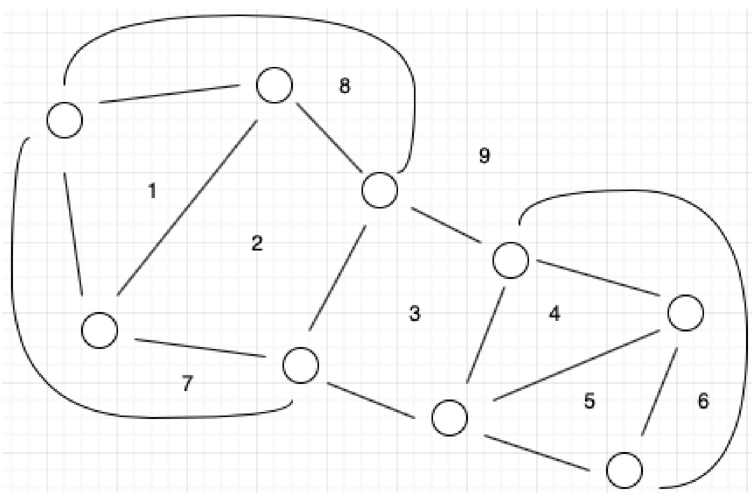
We have shown that it is possible to colour a graph with no closed paths of odd lengths in an alternative red or blue sequence. This is the definition of a bipartite graph, and therefore the statement has been proven. □

**4.** If a simple connected planar graph consists of 5 vertices of degree 4, and 4 vertices of degree 3, then how many faces does it have? Give an example of such a graph.

*Proof.* If there are 5 vertices of degree 4 and 4 vertices of degree 3 then the sum of all the degrees in the planar graph is 32. Therefore there must be 16 edges. Using Euler's Theorem for planar graphs our number of faces $f$ is:

$$f = 2 - v + e = 2 - 9 + 16 = 9$$

An example of a graph with these characteristics is:




□

**5.** For exactly which values of $k$ is $Q_k$ planar? Prove your answer.

*Proof.* In $Q_k$, each vertex is labelled as a sequence $S = a_1, a_2, ..., a_k$ of length k. We are able to travel from one vertex to another by changing $a \in S$ to be 1 or 0. Suppose we create a cycle of minimum length, then we would start at vertex $v$, and modify $a_i \in S$ to get to vertex $v'$. At this point, we would be able to complete this cycle with minimal length by first modifying $a_j \in S, a_j \neq a_i$ to get to vertex $v''$. Now we can undo our changes to $a_i$ and move to a vertex $v'''$ that is adjacent to $v$ by undoing changes to $a_j$. Thus, the minimum length cycle in $Q_k$ is of length 4.

Now this means that each face is bounded by at least 4 edges, which are boundaries between 2 faces (or 2 sides of the same face). Then we get that:

$$4f \leq 2e$$

Using Euler's theorem we can conclude that for simple planar graphs with at least 4 vertices:

$$2v - 4 \geq e$$

Using the formula's we've proven from past homeworks for the number of vertices and edges in $Q_k$ we get that:

$$2^{k+1} - 4 \geq k2^{k-1}$$
$$4 - \frac{4}{2^{k-1}} \geq k$$

Now this inequality only holds for $k = 2, 3$ and at $k \geq 4$, the right hand side of the inequality grows much faster, and therefore the inequality will no longer hold true. To prove that $Q_1$ is also planar, it is very easy to see that it is a graph with 2 vertices (0 and 1) and a single edge joining them; this is obviously planar.

Therefore, using Euler's theorem, we have proven that $Q_k$ is planar for $k = 1, 2, 3$. □

**6.** Prove that every simple connected planar graph with fewer than 12 vertices contains a vertex of degree at most 4.

*Proof.* We will prove this by contradiction.
Suppose we take a planar graph G with $v < 12$ vertices and every vertex in G has at least degree 5. Then the number of edges in G must be at least $\lceil \frac{5v}{2} \rceil$ edges. Using the lemma $e \leq 3v - 6$ from Euler's theorem for simple planar graphs with more than 2 vertices, we get that for graph G:

$$\left\lceil \frac{5v}{2} \right\rceil \leq 3v - 6 \tag{6}$$
$$5v \leq 6v - 12 \tag{7}$$
$$12 \leq v \tag{8}$$

But graph G can have at most 11 vertices, and so by contraction, we have proven the original statement. □

**7.** Let $G$ be a simple graph with at least 11 vertices. Prove that $G$ and $\overline{G}$ are not both planar.

*Proof.* Take a graph G with 11 vertices and assume that it is planar. Then by Euler's formula the number of edges it can have and still be planar is:

$$e \leq 33 - 6 = 27$$

Now $\overline{G}$ must have the number of edges in a complete graph of 11 vertices minus 27. We get that $\overline{G}$ has 28 vertices but then $\overline{G}$ cannot be planar since we already showed that for a graph with 11 vertices, it can have at most 27 edges whilst still being planar.

As the number of vertices in G grows, we get that the number of edges in $\overline{G}$,
$|\overline{E}| = (\frac{v(v-1)}{2} - (3v - 6))$ grows much faster than the number of vertices in $\overline{G}$ and therefore this statement is also true for all graphs with more than 11 vertices. □