Math 442 Homework 3 Solutions

1. Recall that the adjacency matrix of G is given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Note that

$$A^2 = A \cdot A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

First, we will prove by induction that for each positive integer k,

$$A^{2k-1} = \begin{bmatrix} 0 & 2^{2k-2} & 2^{2k-2} & 0 \\ 2^{2k-2} & 0 & 0 & 2^{2k-2} \\ 2^{2k-2} & 0 & 0 & 2^{2k-2} \\ 0 & 2^{2k-2} & 2^{2k-2} & 0 \end{bmatrix}$$

The base case $k = 1$ follows from the expression we have above for $A = A^1$. For the inductions step, suppose that the result has already been proved for some value of k. Then

$$\begin{aligned} A^{2(k+1)-1} &= A^{2k-1}A^2 \\ &= \begin{bmatrix} 0 & 2^{2k-2} & 2^{2k-2} & 0 \\ 2^{2k-2} & 0 & 0 & 2^{2k-2} \\ 2^{2k-2} & 0 & 0 & 2^{2k-2} \\ 0 & 2^{2k-2} & 2^{2k-2} & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2^{2(k+1)-2} & 2^{2(k+1)-2} & 0 \\ 2^{2(k+1)-2} & 0 & 0 & 2^{2(k+1)-2} \\ 2^{2(k+1)-2} & 0 & 0 & 2^{2(k+1)-2} \\ 0 & 2^{2k-2} & 2^{2k-2} & 0 \end{bmatrix}, \end{aligned}$$

which completes the induction step.
Next, we will prove by induction that for each positive integer k,

$$A^{2k} = \begin{bmatrix} 2^{2k-1} & 0 & 0 & 2^{2k-1} \\ 0 & 2^{2k-1} & 2^{2k-1} & 0 \\ 0 & 2^{2k-1} & 2^{2k-1} & 0 \\ 2^{2k-1} & 0 & 0 & 2^{2k-1} \end{bmatrix}.$$

The base case $k = 1$ follows from our computation for A^2 above. For the induction step, suppose the result has been proved for some value of k. Then

$$\begin{aligned}
A^{2(k+1)} &= A^{2k}A^2 \\
&= \begin{bmatrix} 2^{2k-1} & 0 & 0 & 2^{2k-1} \\ 0 & 2^{2k-1} & 2^{2k-1} & 0 \\ 0 & 2^{2k-1} & 2^{2k-1} & 0 \\ 2^{2k-1} & 0 & 0 & 2^{2k-1} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 2^{2(k+1)-1} & 0 & 0 & 2^{2(k+1)-1} \\ 0 & 2^{2(k+1)-1} & 2^{2(k+1)-1} & 0 \\ 0 & 2^{2(k+1)-1} & 2^{2(k+1)-1} & 0 \\ 2^{2(k+1)-1} & 0 & 0 & 2^{2(k+1)-1}, \end{bmatrix}
\end{aligned}$$

which closes the induction. We conclude that the number of walks of length k from from A to B is 2^{k-1} if k is odd, and 0 if k is even.

2. Each edge has 2 ends, and each end contributes 1 to the total sum of the degrees. Hence if there are e edges in the graph the total sum of the degrees is $2e$, which is even.

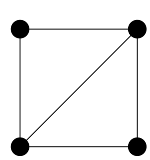
Let G be a graph with total sum of degrees $2e$. If we subtract the degrees of every vertex with even degree we are left with an even number of degrees $2t$. This must be the sum of the degrees of every vertex with odd degree. The only way this can happen is if the number of vertices of odd degree is even.

If we draw a graph with a vertex for each person at the party, and an edge between them if they have shaken hands, by the previous part of this question, if the vertex representing me has degree 5 then another vertex has odd degree, since the number of vertices of odd degree must be even and ≥ 1.

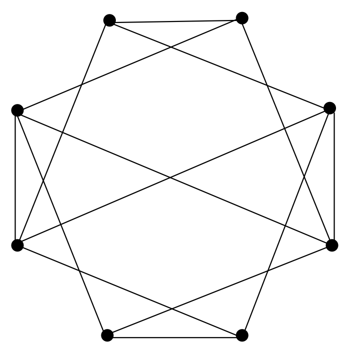
3. Yes, it is possible. Let our graph have $v = 2m$ vertices. Counting degrees, the degree contribution from the m vertices of degree a is ma and the degree contribution from the m vertices of degree $a+1$ is $m(a+1)$. We know by the previous question that the sum of the degrees of all vertices is even, say $2k$. Hence we know that

$$2k = ma + m(a+1) = 2ma + m \Rightarrow m = 2(k - ma)$$

so m must be even, say 2ℓ. Hence $v = 2m = 4\ell$, and a necessary condition is that the number of vertices must be a multiple of 4.

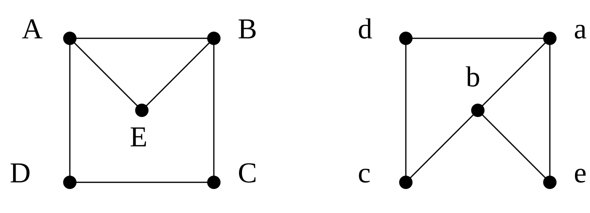


4. If an n vertex graph is self-complementary then we know its number of edges, e, is half that in the complete graph K_n, so $e = \frac{1}{4}n(n-1)$. However, e must be an integer and so n or $n-1$ is a multiple of 4, or $n = 4k$ or $4k+1$ for $k \in \mathbb{Z}$.



5. We know that if two graphs are isomorphic then there exists an invariant, such as the number of edges, which differs. So to prove our statement let us check the number of edges in G and $\overline{G}$, both of which have n vertices. When $n = 2$ if $E(G) = 0$ then $E(\overline{G}) = 1$, and if $E(G) = 1$ then $E(\overline{G}) = 0$. When $n = 3$ if $E(G) = 0$ then $E(\overline{G}) = 3$, if $E(G) = 1$ then $E(\overline{G}) = 2$, if $E(G) = 2$ then $E(\overline{G}) = 1$, and if $E(G) = 3$ then $E(\overline{G}) = 0$. In each case of G, we have that $E(G) \neq E(\overline{G})$, and hence G is not self-complementary.

6. Labelling the graphs from left to right, top to bottom, looking at vertices we see: 3 of them (1, 2, 6) have 2 of degree 3, and 3 of degree 2; 1 (3) has 1 of degree 4, 1 of degree 3, 2 of degree 2, and 1 of degree 1; 1 (4) has 3 of degree 3, 1 of degree 2, and 1 of degree 1; 1 (5) has 1 of degree 4, and 4 of degree 2. Hence, although they all have 5 vertices and 6 edges, by degrees only 1, 2, 6 could be isomorphic. 1 and 6 are not isomorphic to 2 since the former have the 2 degree 3 vertices adjacent, and the latter doesn't. An isomorphism between 1 and 6 is



where A maps to a etc.

7. The number of vertices of Q_k is 2^k

Proof By definition, each vertex corresponds to a sequence of k symbols $(a_1, \ldots, a_k)$ where each $a_i = 0, 1$. So we have 2 choices for a_1, 2 choices for a_2, ... , 2 choices for a_k. That is 2^k choices in total, and hence 2^k vertices. $\square$

The number of edges of Q_k is $k2^{k-1}$

Proof Consider a vertex v corresponding to the sequence of 0s and 1s $(a_1, \ldots, a_k)$. Then v is connected to k other vertices each of which differ from v in exactly one of $a_1, \ldots, a_k$. Now note that the total number of vertices by above is 2^k, so the total number of edges coming out of some vertex is $k2^k$. However, we have now counted each edge twice, and hence the number of edges is $k2^k/2 = k2^{k-1}$. $\square$