Math 442 Homework 5 Solutions

1. Let G be a tree, and let P be a path in G of maximal length, i.e. sequence $v_1, e_1, v_2, e_2, \ldots, e_{k-1}, v_k$ of alternating vertices and edges so that e_i is incident to v_i and v_{i+1}; no vertex is repeated, and no edge is repeated. We have that both v_1 and v_k must have degree one. Indeed, if v_1 did not have degree 1, then there is a vertex v' adjacent to v_1 with $v' \neq v_2$. Since G is a tree, v' is distinct from $v_2, \ldots, v_k$, so the path $v', e', v_1, e_1, \ldots, v_k$ is a path in G of length $k+1$; this contradicts the fact that P is a path of maximal length. Similarly, v_k must have degree one.
We conclude that G contains at least *two* vertices of degree one.

2. First, if G is a tree, then $e < v$, so certainly the result is true.
If G is not a tree, then we will follow a similar proof strategy to the one used in lecture. Let $\tilde{G}$ be a plane drawing of G. Let f be the number of faces of this plane drawing. X be the number of (face, edge-side) pairs, where the edge-side is adjacent to the face. Since each edge has two sides, we have $X = 2e$. On the other hand, since G contains no cycles of length $\leq k$, each face is adjacent to at least $k+1$ edge sides (this is clearly the case for each bounded face. For the unbounded face, the result follows since G is not a tree). Thus we have $X \geq (k+1)f$, and hence $(k+1)f \leq 2e$.
Thus by Euler's formula $f = 2 - v + e$, we have $(k+1)(2 - v + e) \leq 2e$, or $(k-1)e \leq (k+1)v - 2(k+1) \leq (k+1)v$. Re-arranging, we obtain $e \leq \frac{k+1}{k-1}v$, as desired.

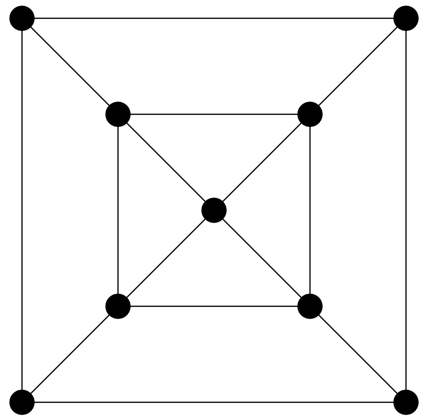
3. We will do an induction on the number of edges E.

Base case: $E = 0$. If there are no edges then the graph is bipartite as we can colour every vertex either black or white.

Induction step: Assume every graph with $0 < k < m$ edges with no closed paths of odd length is bipartite. Now consider a graph G with no closed paths of odd length and m edges, and delete one of its edges e whose end points are u and v. Note that no closed paths of odd length are created so by induction $G - e$ is bipartite.

If e disconnects a component of the graph into two, then by induction each component is bipartite, and we can colour the vertices such that u and v are different colours. If e does not disconnect a component, then since e completes a cycle of even length in G (by hypothesis) there must be a path of odd length in $G - e$ between u and v. Thus in the bipartite colouring of $G - e$ we have that u and v are different colours. Hence in both cases G is bipartite.

4. Euler's Theorem gives $v - e + f = 2$ so $9 - \frac{20+12}{2} + f = 2$ and $f = 9$.



5. Q_k **is planar for** $k \leq 3$ **and** Q_k **is not planar for** $k \geq 4$**.**

For $k = 1, 2, 3$ we can easily draw Q_k (do it) and see they are planar. For $k = 4$ note that since Q_k contains no triangles we have that if Q_4 was planar then it would satisfy $e \leq 2v - 4$. However since $e = 32, v = 16$ using our formula from HW2 Q7 this is not satisfied and hence Q_4 is not planar.

For $k > 4$ consider a subgraph of Q_k consisting of the set vertices whose last $k - 4$ digits are identical. Then this subgraph is isomorphic to Q_4 by the isomorphism ϕ : $(a_1, a_2, a_3, a_4, \dots) \mapsto (a_1, a_2, a_3, a_4)$, and so is not planar. Hence since a subgraph of Q_k is not planar, then Q_k is not planar.

6. We prove the contrapositive, namely if every vertex has degree ≥ 5 then the number of vertices $v \geq 12$. Assume that every vertex has degree at least 5. Then $2e =$ (total sum of degrees of the vertices) $\geq 5v$. So $2e \geq 5v$. Since our graph is simple, connected and planar it satisfies the useful inequality $e \leq 3v - 6$. Hence putting these inequalities together we get

$$\frac{5}{2}v \leq e \leq 3v - 6$$

so $5v \leq 6v - 12$ and $v \geq 12$.

7. Let G have n vertices. We know that one of G or $\overline{G}$ will have at least half the number of edges of the complete graph K_n so by Proposition 1 at least $\frac{n(n-1)}{4}$ edges. Also we will not have planarity if $e > 3v - 6$: this is true if the graph is connected by the useful inequality, and it is true if the graph is not connected by adding together the useful inequality for each connected component. Also $e > 3v - 6$ is trivially true when the number of vertices is 1 or 2. Therefore we will not have planarity if

$$\begin{aligned} \frac{n(n-1)}{4} &> 3n - 6 \\ \Rightarrow 24 &> 13n - n^2 \end{aligned}$$

or $n \geq 12$. For $n = 11$ one of G or $\overline{G}$ will have at least half the number of edges of the complete graph K_n so by Proposition 1 at least 28 edges. But $3n - 6 = 27 < 28 \leq e$ so one of G or $\overline{G}$ is not planar.