Math 442 Homework 6

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1. Let G and H be graphs. H is called a maximal reduction of G if H is homeomorphic to G, and H has no vertices of degree two. Prove that two graphs G_1 and G_2 are homeomorphic if and only if their maximal reductions are isomorphic.

Proof. We will begin by showing that if G_1 and G_2 are homeomorphic, then their maximal reductions H_1 and H_2 are isomorphic.

If G_1 and G_2 are homeomorphic, then we are guaranteed that for G_1, we can add vertices to the middle of edges, or delete vertices of degree 2 and join the 2 edges such we create a new graph G_1' that equal to G_2. Since G_1' and G_2 are the same graph, they have the same maximal reduction H (as maximal reductions are unique), which is isomorphic to itself.

Now we will show that if 2 maximal reductions H_1 and H_2 are isomorphic, then G_1 and G_2 are homeomorphic.

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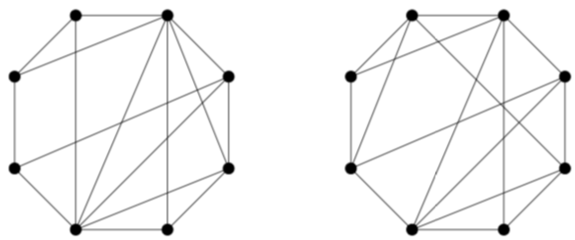
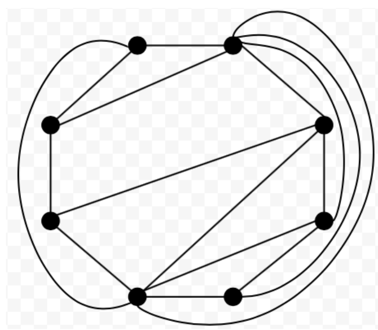
□

2. Which of the two graphs below is planar? For the one that is give a planar embedding. For the one that isn't find a sub-graph homeomorphic to K_5 or $K_{3,3}$.



Proof. Graph 1 in the above diagram is planar. The planar embedding for this graph is:



Couldn't find a sub graph homeomorphic to K_5 or $K_{3,3}$ for the second graph :($\square$

3. Let G_1 and G_2 be two homeomorphic graphs. Let G_1 have n_1 vertices and m_1 edges, and let G_2 have n_2 vertices and m_2 edges. Show that $m_1 - n_1 = m_2 - n_2$.

Proof. Since G_1 and G_2 are homeomorphic graphs, they will have maximal reductions H_1 and H_2 that are isomorphic to each other. Therefore, H_1 and H_2 will both have n vertices and m edges. Now, we've defined a homeomorphic transformations as either adding a vertex in-between an edge, or by deleting vertices of degree 2 and joining their edges together. We can build G_1 and G_2 from H_1 and H_2 by doing these operations. We will evaluate the effect of each of these operations:

- Case 1: Adding vertices to the middle of edges. This operation takes a single edge, and adds 1 additional vertex and 1 additional edge. Thus we increase n and m by 1.
- Case 2: Deleting vertices of degree 2, and connecting the corresponding edges. This operation removes exactly 1 vertex and leaves behind a single edge joining the neighbours of the removed edge. Therefore, this operation decreases n and m by 1.

Since H_1 and H_2 both have n vertices and m edges, m - n for both maximal reductions is equal. Since we can build up G_1 and G_2 with homeomorphic operations which either increase or decrease the b both number of vertices/edges by 1, we can guarantee that the difference in edges and vertices ($m_1 - n_1$ or $m_2 - n_2$) will always be equal, since they started at the same maximal reduction.

□

4. An equivalent definition of a *polyhedral graph* is that it is a simple connected planar graph where every vertex has degree at least 3.

Prove that no polyhedral graph with exactly 24 edges and 8 faces can exist.

Proof. We will prove this by contradiction, suppose there exists a polyhedral graph G with 24 edges and 8 faces. Then by Euler's theorem ($v - e + f = 2$), G must have 18 vertices. Now, using the definition of a polyhedral graph, each vertex in G must have degree at least 3. Thus the number of edges e in G must be:

$$e \geq \frac{18 * 3}{2} = 27$$

This is a contraction as we proposed that a polyhedral graph with 24 edges and 8 faces can exist, but by Euler's theorem, this graph requires at minimum 27 edges to be polyhedral. Therefore we have proven that no polyhedral graph with exactly 24 edges and 8 faces can exist. □

5. The *line graph* $L(G)$ of a simple graph G is the graph whose vertices are in one-to-one correspondence with the *edges* of G, and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G meet at a vertex. Prove that if a simple graph G is regular of degree $k > 0$, then $L(G)$ is regular of degree $2k - 2$.

Proof. We will prove this directly. Suppose graph G is a regular graph with v vertices such that all vertices in G have degree $k > 0$. Therefore, each edge in G will have a head and tail which connect to a unique vertex that has $k - 1$ other edges connected to it. Using our definition of $L(G)$, we can conclude that each vertex in $L(G)$ must be adjacent to $2 * (k - 1) = 2k - 2$ other vertices. □

6. Given a simple graph G with vertices $v_1, \ldots, v_n$, prove that the number of edges in $L(G)$ is

$$\sum_{i=1}^{n} \frac{d_i(d_i - 1)}{2}$$

where d_i is the degree of vertex v_i for $1 \leq i \leq n$.

Proof. We will prove this directly. Take some v_i in G and it will have d_i edges connected to it. Therefore d_i edges connected to v_i correspond to vertices $u_1, ..., u_{d_i}$ in $L(G)$. Each of these vertices $u_j \in u_1, ..., u_{d_i}$ will have their degree increased by $d_i - 1$ as there are $d_i - 1$ other edges connected to v_i. Therefore a vertex v in G with degree d_i, will increase the sum of the all the degrees in $L(G)$ by $d_i(d_i - 1)$.

Thus the sum of all the degrees in $L(G)$ can be calculated by evaluating all vertices in G. For the sum of all the degrees in $L(G)$ we have:

$$\sum_{i=1}^{n} d_i(d_i - 1)$$

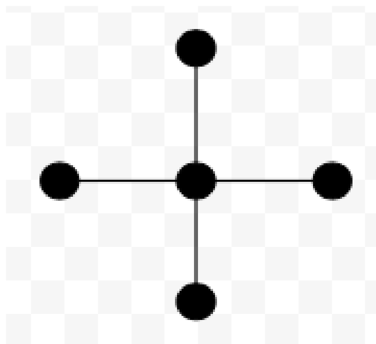
Therefore the number of edges in $L(G)$ must be:

$$\sum_{i=1}^{n} \frac{d_i(d_i - 1)}{2}$$

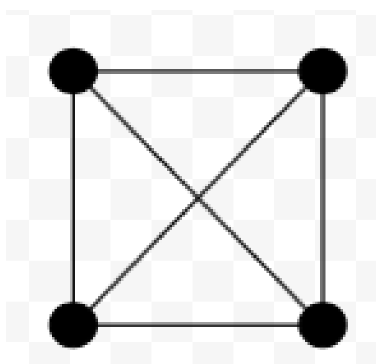
□

7. For every $n \geq 3$ find a graph G whose line graph $L(G) = K_n$. Explain your answer.

Proof. For every $n \geq 3$, we can create a graph G that has $n + 1$ vertices with n edges in a star formation. This graph G will have 1 center vertex of degree n and n outer vertices with degree 1, connected to the center vertex via an edge. This "star" graph for 5 vertices is shown below:



The graph G with $n + 1$ vertices will always have a line graph equal to K_n as each one of the n edges in G shares the center vertex with every other edge in G. The line graph for the star graph of 5 vertices is shown below and is the K_4 graph:



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