Math 442 Midterm Solutions

1. For the questions below, circle either True or False.

a) Let G and H be graphs. Suppose that G is homeomorphic to H.
(**True**) / (False): G is Eulerian if and only if H is Eulerian.

b) Let G and H be graphs. Suppose that G is homeomorphic to H.
(True) / (**False**): G is Hamiltonian if and only if H is Hamiltonian.
Note: There was a grading snafu, and this problem was graded incorrectly. This will be sorted out in lecture on March 14.

c) Let G and H be graphs. Suppose that G is a minor of H.
(True) / (**False**): G is planar if and only if H is planar.

d) In the country of Monochromaticstan every pair of cows have the same colour.
(**True**) / (False): In Monochromaticstan, all cows have the same colour.

e) (**True**) / (False): a connected 2-regular graph must remain connected if one edge is removed.

2.

a) Define a *closed path* on a graph G.

Solution. A closed path is an alternating sequence $v_1, e_1, v_2, e_2, \ldots, e_{n-1}, v_n$ of edges and vertices, so that for each $i = 1, \ldots, n-1$, the edge e_i is incident to v_i and v_{i+1}. Furthermore, each of the edges $e_1, \ldots, e_{n-1}$ are distinct; each of the vertices $v_1, \ldots, v_{n-1}$ are distinct; and $v_1 = v_n$.

It would also be acceptable to say things like a "closed trail where all of the vertices are distinct," or "a closed walk where all of the edges and vertices are distinct."

b) Define what it means for a graph to be *Hamiltonian*.

Solution. A graph is Hamiltonian if it contains a Hamiltonian cycle.

c) Define what it means for a graph to be *Eulerian*.

Solution. A graph is Hamiltonian if it contains an Eulerian circuit.
d) Define what it means for a graph to be *Planar*.

Solution. A graph is planar if the vertices of the graph can be represented by points in the plane, and the edges of the graph can be represented by curves in the plane, so that no two curves intersect. Equivalently, a graph is planar if it can be drawn in the plane with any edge crossings.

e) Give a necessary and sufficient condition for a graph to be semi-Eulerian.

Solution. A graph is semi-Eulerian if it is connected and exactly two vertices have odd degree.

3. Recall that a graph is called 2-connected if it has at least two vertices and it remains connected if one vertex is removed.

a) Prove that if a graph is Eulerian, then it is 2-connected.

Solution. The problem as stated is not correct (oops!). If a graph is Hamiltonian, then it is 2-connected. Alternately, if a graph is Eularian, then it is 2-edge-connected (i.e. it is connected, and it remains connected if an edge is removed). This problem will be discarded, but bonus points (5 marks) will be given for providing a counter-example to the original statement, or for proving either of the two (correct) statements given above.

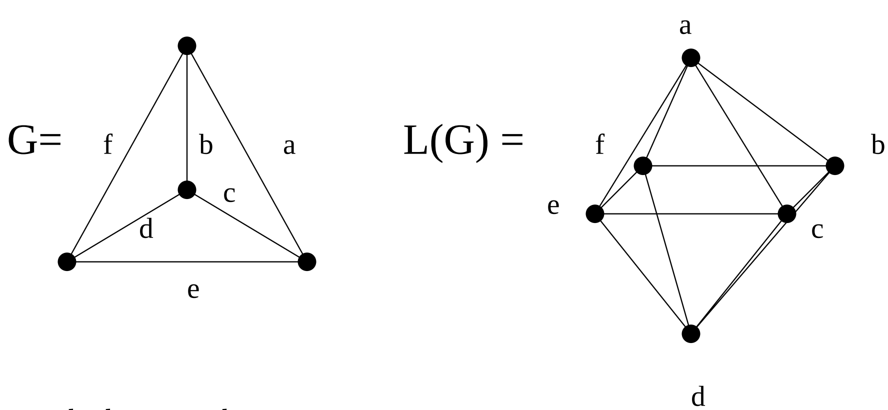
b) If a graph is semi-Eulerian, must it be 2-connected? If so, prove it. If not, provide a counter-example and show that your example is correct

Solution. No; consider the path P_4. This graph consists of 4 vertices $v_1, \ldots, v_4$ and three edges e_1, e_2, e_3, so that edge e_i is incident to vertex v_i and v_{i+1}. This graph is semi-Eulerian, since the open trail (actually path) $v_1, e_1, v_2, e_2, v_3, e_3, v_4$ visits each edge once. However, this graph is not 2-connected, since removing the vertex v_2 cuts the graph into the connected component v_1, and the connected component containing the adjacent vertices v_3 and v_4.

4. Recall that the *line graph* $L(G)$ of a simple graph G is the graph whose vertices are in one-to-one correspondence with the *edges* of G, and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G meet at a vertex.

Compute the line graph of the tetrahedron graph. Is this the graph of a platonic solid? If so, which one?

Solution. If G is the tetrahedron graph, then its line graph is given as follows:



This is the octahedron graph.

5. Let a and b be two vertices in K_4. Compute the number of walks of length 3 from a to b.

Solution. The easiest (though not only) way to solve this problem is to consider the adjacency matrix of K_4:

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

We can compute

$$A.A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix},$$

and

$$A.A.A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 7 & 7 & 7 \\ 7 & 6 & 7 & 7 \\ 7 & 7 & 6 & 7 \\ 7 & 7 & 7 & 6 \end{bmatrix}$$

Thus if a and b are two (distinct) vertices of K_4, then there are 7 walks of length 3 from a to b. (if a and b are the same vertex, then there are 6 walks, but you can receive full credit without considering this possibility).

6. Let $x_1, x_2, ..., x_6$ be a sequence of 6 distinct real numbers. Prove that there is a sub-sequence of length three that is either strictly increasing or strictly decreasing, i.e. there are integers $1 \leq i < j < k \leq 6$ so that either $x_i < x_j < x_k$, or $x_i > x_j > x_k$. Hint: think about six people at a party.

Solution. Let G be the complete graph on 6 vertices $v_1, \ldots, v_6$. For each pair of distinct integers $1 \leq i < j \leq 6$, colour the edge from v_i to v_j red if $x_i < x_j$, and colour it blue if $x_i > x_j$ (since the six numbers $x_1, \ldots, x_6$ are distinct, we must have either $x_i < x_j$ or $x_i > x_j$). By the "six people at a party" result we discussed in lecture (also known as the bound for Ramsey's number $R(3,3) = 6$), there must be three vertices of the graph v_i, v_j, v_k so that every edge between these three vertices has the same colour. Without loss of generality we can assume that $i < j < k$ (if not, we can re-index). If all edges are red then $x_i < x_j < x_k$, while if all edges are blue then $x_i > x_j > x_k$.