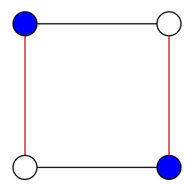
Math 442 Homework 9 Solutions

1.



2. Recall that in Q_k a vertex labelled by $a_1 \ldots a_k$ where $a_i = 0, 1$ for all $1 \leq i \leq k$ is adjacent to k vertices since there are k choices for where a sequence can differ from $a_1 \ldots a_k$ in just one place.

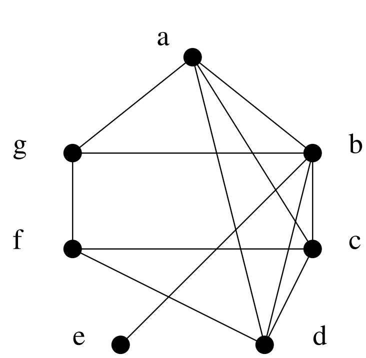
We can colour the edges of Q_k with k colours in the following way. If two adjacent vertices have their labels differing in position i where $1 \leq i \leq k$ then colour the edge between them colour i. Note that by construction every edge incident at a vertex is thus a different colour, and hence we have an edge colouring of Q_k with k colours.

Consequently, $\chi'(Q_k) \leq k$. However, since k edges meet at a vertex, $\chi'(Q_k) \geq k$. Therefore $\chi'(Q_k) = k$.

3. We have no loops as the graph is simple, so it is edge colourable. Assume it is edge colourable in $d \geq 2$ colours for our d regular graph.

We know that since G is regular $vd = 2e$, so $v/2 = e/d$ so e/d is not an integer since v is odd. On the other hand since each vertex has d edges incident to it, coloured d colours, this means there are the same number of edges of each colour, say p. So $pd = e$ and e/d is an integer. We have a contradictiction. Thus by Vizing's Theorem, the chromatic index is $d + 1$.

4. Drawing a graph with a vertex for each lecture, and an edge between them if they must not coincide we get the following.



Notice there is a K_4 subgraph so we need at least 4 colours. If we colour a and e colour 1, b and f colour 2, c and g colour 3 and d colour 4 we need at most 4 colours, so 4 periods are needed.
Alternatively, using deletion-contraction, we can compute the chromatic polynomial to be $k(k-1)^2(k-2)(k-3)(k^2-5k+8)$ again giving that 4 colours and hence periods are needed.

5. If we have a tree then since it is a simple connected graph there is exactly one path between any two vertices. Connecting two of them them creates exactly two paths between them, that is, one cycle, as since we had a tree there were none before.
Conversely, if adding one edge creates exactly one cycle in a simple connected graph then this means we started with a simple connected graph with no cycles, that is, a tree.

6. We will show that: If v is the number of vertices in T then

$$v = \frac{2}{2-a}.$$

Proof. Let v be the number of vertices in T, and e be the number of edges. Since $v = e + 1$, or $e = v - 1$. We also know that $2e = $ (sum of degrees) $= av$. Hence $2(v-1) = av$ and rearranging gives $v = \frac{2}{2-a}$. $\square$

7. We will do a proof by strong induction on the number of edges E.

Base case: If $E = 1$ then T is K_2 and T has 2 leaves and 0 non-leaf vertices.

Induction step: Assume the result is true for all trees with fewer than $m-1$ edges and consider a tree T with $m-1$ edges, L leaves and $m-L$ non-leaves (since $v = e+1$,T has m vertices in total). Delete a leaf ℓ. If deleting ℓ results in no vertices of degree 2 then by induction

$$L - 1 > m - L \Rightarrow L > m - L.$$

If no such ℓ exists, that is, deleting every leaf results in a vertex of degree 2, then this means that in T that some vertex v is adjacent to exactly 2 leaves, or $T = K_{1,3}$. If $T = K_{1,3}$ then the result follows. If not then delete both leaves v is adjacent to, resulting in v becoming a leaf. Then by induction

$$L - 2 + 1 > m - L - 1 \Rightarrow L > m - L.$$

Hence the result follows by induction.