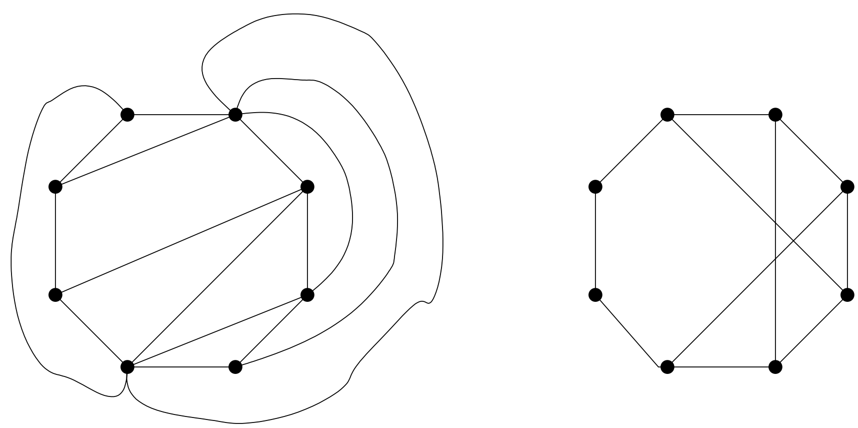
Math 442 Homework 6 Solutions

1. First, observe that if G is a graph that has no vertices of degree two, then G is a maximal reduction of itself.
Next, note that if G_1 and G_2 are graphs, neither of which contain any vertices of degree two, then G_1 and G_2 are homeomorphic if and only if they are isomorphic. Indeed, if G_1 and G_2 are ismorphic then they are clearly homeomorphic. In the order direction, if G_1 and G_2 are homeomorphic, then G_2 can be obtained from G_1 by deleting vertices of degree 2 (and "joining" the corresponding edges), or inserting vertices in the "middle" of edges. These operations either remove a vertex of degree 2 or add a vertex of degree 2. Since neither G_1 nor G_2 contain vertices of degree 2, we conclude that G_1 and G_2 must be isomorphic.
Finally, let G_1 and G_2 be graphs, and let G'_1 and G'_2 be their maximal reductions, respectively. Since G_1 is homeomorphic to G'_1, G_2 is homeomorphic to G'_2, and the property of being homeomorphic is transitive, G_1 is homeomorphic to G_2 is G'_1 and G'_2 are isomorphic (and hence homeomorphic). On the other hand, if G_1 and G_2 are homeomorphic then G'_1 and G'_2 must be homeomorphic, which implies that they are isomorphic.

2. The LHS one is, and the RHS one isn't.



3. In order to get from graph G_1 to G_2 we repeatedly either add in a vertex to an edge increasing both the number of vertices and number of edges by 1, or delete a vertex of degree 2 decreasing both the number of vertices and number of edges by 1. Hence, repeatedly, the (number of vertices)-(number of edges) stays constant, or $m_1 - n_1 = m_2 - n_2$.

4. Let us do a proof by contradiction and assume that such a polyhedral graph G exists. Let v denote the number of vertices in G, then since G is polyhedral it satisfies Euler's theorem and hence $v - 24 + 8 = 2$ and so $v = 18$.

Since every vertex has degree at least 3 we know from counting degrees that $3v \leq 2e$ since every edge has 2 ends. Hence since $v = 18$ and $e = 24$ substituting this in we have that $54 = 3.18 \leq 2.24 = 48$, a contradiction. Therefore no such polyhedral graph exists.

5. If G is regular of degree $k > 0$ then every vertex has degree k so every edge meets $k-1$ other edges at a vertex it is incident to. Every edge is incident to 2 vertices, that is has 2 ends, and hence every edge meets $2k-2$ other edges in total. Thus in $L(G)$ every vertex has degree $2k-2$ by definition, so $L(G)$ is regular of degree $2k-2$.

6. In G let e be an edge incident to v_i of degree d_i. Then in $L(G)$ the vertex corresponding to e, v_e, will be adjacent to $(d_i - 1)$ other vertices by definition of $L(G)$. Since there are d_i possibilities for e, each vertex v_i generates $d_i(d_i-1)$ edge ends in $L(G)$ and hence there are $\sum_{i=1}^{n} d_i(d_i-1)$ in $L(G)$ overall. Since every edge has 2 ends, the total number of edges in $L(G)$ is

$$\sum_{i=1}^{n} \frac{d_i(d_i-1)}{2}.$$

7. We have that the bipartite graph $K_{1,n}$ for $n \geq 3$ satisfies $L(K_{1,n}) = K_n$. This is because in $K_{1,n}$ every edge meets the $n-1$ others, and hence in $L(K_{1,n})$ every vertex is adjacent to the $n-1$ others. This is K_n by definition.