MATH 442

Homework 1

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1. Prove that $2^{2n} \geq n^4$ for all $n \geq 4$.

Base Case:
Let $S(n)$ be the statement that $2^{2n} \geq n^4$ for all $n \geq 4$.
$S(4) = 2^8 \geq 4^4 \rightarrow 256 \geq 256$ and this statement is true.

Induction Step:
Suppose that $S(n)$ is true, prove that $S(n+1)$ is true ie. $2^{2(n+1)} \geq (n+1)^4$ for all $n \geq 4$.

$2^{2(n+1)} = 2^{2n+2} = 4 * 2^{2n} \geq 4 * n^4 \geq (n+1)^4 = n^4 + 4 * n^3 + 6 * n^2 + 4 * n + 1$

The first inequality is true from the induction hypothesis, and the second inequality is true for $n \geq 4$. Hence, we can conclude that the statement $2^{2n} \geq n^4$ is true for all $n \geq 4$

2. Prove that $x - y$ divides $x^n - y^n$ for all $n \geq 1$.

Base Case:
Let $S(n)$ be the statement that $x - y$ divides $x^n - y^n$ for all $n \geq 1$.
$S(1) = x - y$ divides $x^1 - y^1 \rightarrow \frac{x^1 - y^1}{x - y} = 1$

Induction Step:
Suppose that $S(n)$ is true for some $k \mid \ k \in \mathbb{N}$ for $1 \leq k \leq n$, ie $x - y$ divides $x^k - y^k$. Prove that $S(n+1)$ is true ie. $x - y$ divides $x^{n+1} - y^{n+1}$ for all $n \geq 1$.

$x^{n+1} - y^{n+1}$
$= x * (x^n) - y * (y^n) \rightarrow (x + y - y) * x^n - (y + x - x) * y^n$
$= (x + y)(x^n - y^n) - yx^n + xy^n$
$= (x + y)(x^n - y^n) + xy(x^{n-1} - y^{n-1})$

For the case where $n = 1$, $x^{n-1} = y^{n-1} = 0$ and the second term goes away, leaving us only with the first term.
From the induction step, $x - y$ divides both terms and thus divides the summation of the two terms. Hence, $x - y$ divides $x^{n+1} - y^{n+1}$

3. Prove that for every odd number $n \geq 1$, we have that 9 divides $4^n + 5^n$.

Base Case:
Let $S(n)$ be the statement that for every odd number $n \geq 1$, we have that 9 divides $4^n + 5^n$.
$S(1) = 9$ divides $4^1 + 5^1 = 9$.

Induction Step:
Suppose that $S(n)$ is true for $k \in \mathbb{N}$ for $1 \leq k \leq n$, ie 9 divides $4^n + 5^n$. Prove that $S(n+1)$ is true ie 9 divides $4^{n+1} + 5^{n+1}$

$4^{n+1} + 5^{n+1}$
$= (4 + 5 - 5) * 4^n + (5 + 4 - 4) * 5^n$
$= (9)4^n + (9)5^n - (5)4^n - (4)5^n$
$= (9)(4^n + 5^n) - (5 * 4)4^{n-1} - (4 * 5)5^{n-1}$
$= (9)4^n + (9)5^n - (5 * 4)(4^{n-1} + 5^{n-1})$

From the induction step, we see that 9 divides both terms and thus divides the summation of the two terms. We can say this for the second term as we can reduce it to a form of $(z)(4^{n-3} + 5^{n-3}$ and further more for $n - 5$ as well. According to the inductive hypothesis, there exists a number k between 1 and n that $S(k)$ is true. Hence, 9 divides $4^{n+1} + 5^{n+1}$

4. Prove that for every positive integer n, one of the numbers $n, n+1, n+2, \ldots, 2n$ is the square of an integer.

Base Case:
Let $S(n)$ be the statement that for every positive integer n, one of the numbers $n, n+1, n+2, \ldots, 2n$ is the square of an integer.
$S(1) =$ the set from 1 to 2. In this case, 1 is the square of 1.

Induction Step:
Suppose that $S(n)$ is true, prove for $S(n+1)$ ie for every positive integer $n+1$, one of the numbers $n+1, n+2, \ldots, 2(n+1)$ is the square of an integer.

If n is not a square, then by the induction hypothesis, an integer between $n+1$ and $2n$ must be a square of an integer and thus, a number between $n+1$ and $2(n+1)$ is the square of an integer.

If n is a square, then it must be of the form $n = x^2$ where $x \in \mathbb{Z}$. Using this, we want to show that $w = (x+1)^2$ lies in between $n+1$ and $2(n+1)$ inclusively.

Expanding $w = (x+1)^2 = x^2 + 2x + 1 \rightarrow n + 2\sqrt{n} + 1$. From this, we can see that w is clearly greater than $n+1$ and for all $n \in \mathbb{Z}$, $n + 1 \geq 2\sqrt{n}$. ($(n+1)$ came from the fact that $2(n+1) - (n+1) = n+1$).

Hence, for every positive integer n, one of the numbers $n, n+1, n+2, \ldots, 2n$ is the square of an integer.

5. A *composition* of a natural number n is an ordered list of positive integers whose sum is n. Let $c(n)$ be the number of compositions of n. Conjecture and then prove a formula for $c(n)$ for all $n \geq 1$.

Conjecture:
The formula for $c(n)$ for a natural number n is 2^{n-1}.

Base Case:
Let $S(n)$ be the statement that for any natural number n, the number of compositions is defined by $c(n) = 2^{n-1}$ where compositions of $n+1$ can be constructed by taking the compositions of n and:
A. adding 1 to the last number of each ordered list
B. appending 1 to each ordered list.

$S(1) = 2^0 = 1$ and for 1, there is only 1 composition, $\{1\}$.

Further checking: $S(2) = 2^(1) = 2$ and for 2, there are two compositions, $\{2\}$ and $\{1, 1\}$. $\{2\}$ can be acquired from adding 1 to $\{1\}$ and $\{1, 1\}$ can be obtained by appending 1 to $\{1\}$.

Induction Step:
Suppose that $S(n)$ is true, prove for $S(n + 1)$ ie for any natural number $n + 1$, the number of compositions is defined by $c(n + 1) = 2^n$ and that the way compositions are constructed as stated in the base case.

For $n + 1$, $c(n + 1) = 2^n = 2 * 2^{n-1}$, which is $2 * c(n)$. To see that this is true, we have to consider how we construct the different compositions for $n + 1$ from the compositions of n.

As stated above, the compositions of $n+1$ are obtained from the composition of n by adding 1 to the last number of each ordered list or by appending 1 to each ordered list. By taking all compositions of $n + 1$, we see that each ordered list (composition) has to either end in a 1 or an integer $k \mid k > 1$.

If the composition ended in a 1, then we know it followed rule B. If the composition ended in a $k > 1$, we know that it followed rule A. As both rules are followed, $n + 1$ has exactly twice the amount of compositions as n

Hence, for any natural number n, the number of compositions is defined by $c(n) = 2^{n-1}$.