MATH 442

Homework 6

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1. Answer:

Proof. This will be proved both ways.
For $\rightarrow$
Suppose that G_1 and G_2 are homeomorphic. Consider H_1, the maximal reduction of G_1, is obtained by removing vertices of degree 2 until vertices of degree 2 no longer exist in the graph. The same process can be repeated to obtain H_2 from G_2.

Since G_1 and G_2 are homeomorphic, then H_2 can be obtained from G_1 and H_1 can be obtained from G_2. Thus, H_1 and H_2 must be isomorphic.

For $\leftarrow$
Suppose that H_1 and H_2 are the maximal reductions of G_1 and G_2 respectively. Consider that H_1 is obtained by removing vertices of degree 2 until vertices of degree 2 no longer exist in the graph. Now, since H_1 and H_2 are isomorphic, H_2 can also be obtained from G_1 following the same process with some slight modifications.

Since H_2 is the maximal reduction of G_2, then we can reconstruct G_2 from H_2. As shown above, H_2 can be obtained from G_1 as well, implying that G_1 and G_2 are homeomorphic. □

2. Answer:

Proof. The first graph is planar. Figure 1 is a planar drawing of it.

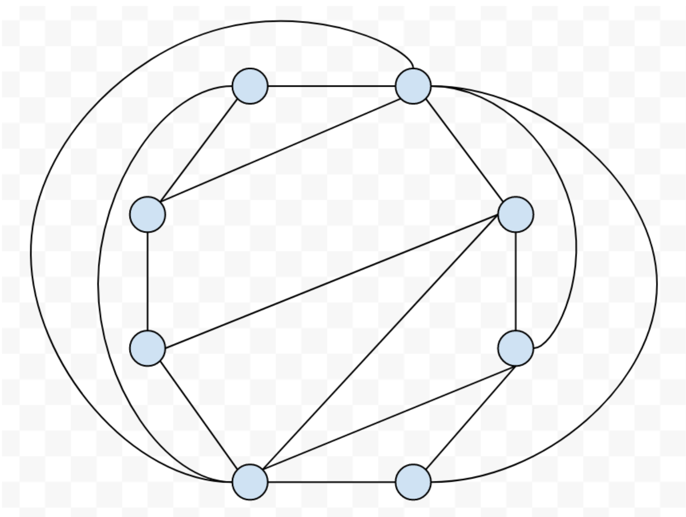

Figure 1: Graphs G needed for L(G) to be K_3 and K_4

Unfortunately I could not come up with a subgraph of 2 such that it is homeomorphic to either K_5 or $K_{3,3}$. $\square$

3. Answer:

Proof. Suppose that G_1 and G_2 are homeomorphic.

Consider H_1, the maximal reduction of G_1, which is homeomorphic to G_2, as G_1 is homeomorphic to G_2. This relationship can be said to be true for H_2 as well. Thus, H_1 and H_2 are isomorphic, meaning that they have the same number of vertices and edges (m_3 and n_3).

Consider the steps to obtain H_1. We started from either G_1 or G_2 and removed vertices of degree 2 until vertices of degree 2 no longer exist in the graph. In this process, the number $m_i - n_i$ would remain unchanged as we lose one vertex and one edge.

Thus, if G_1 and G_2 are homeomorphic, we have $m_1 - n_1 = m_2 - n_2$. $\square$

4. Answer:

Proof. Suppose a graph G is a simple planar graph with 24 edges and 8 faces and each vertex has degree of at least 3.

Following Euler's formula that $v - e + f = 2$, we see that with the above configuration, $v = 2 - f + e = 18$. However, given that the graph has exactly 24 edges and that each vertex has degree of at least 3, the G has maximum 8 vertices. Thus, this graph cannot exist. $\square$

5. Answer:

Proof. Suppose that G is a regular graph where each vertex has degree $k > 0$. Therefore, every vertex in G has k edges.

As each edge will be connected to two vertices, we have to consider both ends. At the first end of the edge, it will be "meeting" $k - 1$ edges. At the second end of the edge, it will still be "meeting" $k - 1$ edges as G is regular.

Thus, each vertex in $L(G)$ should have a degree of $2k - 2$ and therefore is a regular graph with degree $2k - 2$. $\square$

6. Answer:

Proof. Consider a graph G, then the edges in $L(G)$ can be described as a pair, (e_1, e_2), which also uniquely describes a vertex in G.

Consider the ith vertex v_i, which has degree d_i. Then for v_i, there are $\binom{d_i}{2}$ edges of $L(G)$. Thus the total number of edges in $L(G)$ is

$\sum_{i=1}^{n} \binom{d_i}{2} = \sum_{i=1}^{n} \frac{d_i(d_i-1)}{2}$. $\square$

7. Answer:

Proof. To find a graph G such that $L(G) = K_n$ where $n \geq 3$, we can first observe a few requirements.

$L(G)$ would have n vertices, $\frac{n(n-1)}{2}$ edges, and each vertex is of degree $n-1$ (regular). This would mean that G has exactly n edges.

Consider a graph G with $n+1$ vertices. We place one vertex in the middle and arrange the other n vertices to surround the central vertex. By doing so, we can connect every other vertex to the central vertex with one edge each, totalling n edges. The resulting graph always has exactly one (central) vertex that has n edges "meeting" and thus $L(G)$ will be K_n, where every vertex is connected to every other vertex.

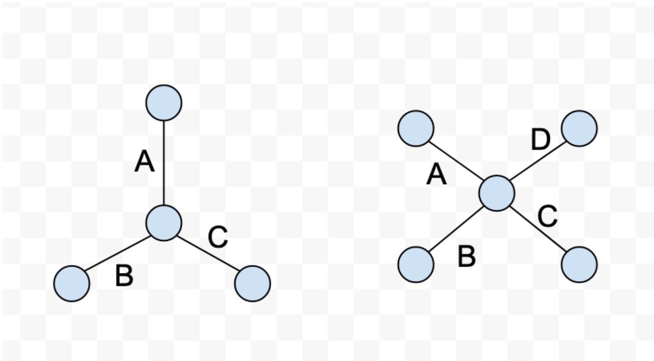

Figure 2: Graphs G needed for L(G) to be K_3 and K_4

$\square$