Pricing Apple Stock Options with Derman and Kani Implied Volatility Trees

The main goal of this project was to build a Derman & Kani type implied arbitrage volatility tree. The project was carried out using the following important details:

- The underlying stock was chosen to be Apple Inc. The main reason for this selection was due to its respective options being highly liquid which was important for this analysis.
- The reference date selected was Tuesday the 13^{th} of March 2018.
- The tree is modelled using a Δt of one month. This criteria was selected mainly due to the large selection of available strike prices and also as it's quite short-term, it is interesting to see how the analysis is reflected over the coming months.
- The analysis includes modelling the tree in both the European and American setting, along with **discrete** dividends, with a schedule provided below.

The structure of this report is in line with the coherent approach to the analysis. Firstly the standard Cox-Ross-Rubinstein (CRR) tree was developed to provide a benchmark. This was followed by the flexible tree built in the European framework which led into building the same tree within the American framework. Before delving into outlining the methods, a few tools were needed to proceed with the aforementioned trees.

Bi-Linear Interpolation

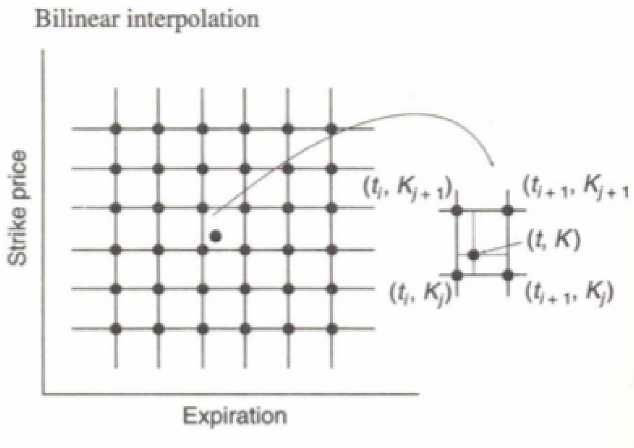

Figure 1: Visual display of notation for the Bi-Linear Interpolation.

This form of interpolation is used to calculate the option prices and implied volatilities. This is an important tool as you most certainly will always find the option needed to price will have an 'awkward strike', that's not available in the market. The following equations show as an example the equation needed to find an implied volatility with the index notation matching the Figure 1 above.

$$a = \frac{t = t_i}{t_{i+1} - t_i}$$
$$b = \frac{K - K_j}{K_{j+1} - K_j}$$

Combining these with,

$$\sigma_{t,K}^{imp} = (1-a)(1-b)\sigma_{i,j}^{imp} + a(1-b)\sigma_{i+1,j}^{imp} + ab\sigma_{i+1,j+1}^{imp} + (1-a)b\sigma_{i,j+1}^{imp} \ ,$$

will yield the implied volatility for the required 'awkward strike' option. To try and reduce both computation and human error, a number of VBA functions were created throughout the analysis. One of which is `PriceIntirr`, which is much like the VLOOKUP function in excel, it will go and find

the required strikes and correct maturities to interpolate between and apply the above formula.
It must be noted that there was a form of linear interpolation required for the rates, for example the 5-month rate between the 3-month and 6-month rates was achieved by,

$$r_t = \frac{[\frac{t-t_1}{t_2-t_1}r(t_2) + \frac{t_2-t_1}{t_2-t_1}r(t_1)]}{t_2 - t_1}$$

Dividends - All trees were modelled using dividends, with data being gathered through Bloomberg which provided the implied dividends.

March	April	May	June	July	August

As the reference date is March and Apple has a quarterly dividends structure, the highlighted months of May and August have a discrete dividend of \$0.74. In all trees the same approach in dealing with this known level of dividends took place. Once the dividend schedule was drawn out which is clearly visible in the excel file, the S_0 spot price was adjusted by the present value of all future dividend payments to,

$$S_0 = S_0 - \sum_{i=0}^{N} e^{-r_i \Delta t} D_i \,.$$

Then the nodes with dividend payments are adjusted by raising the nodes (May) by the present value of all dividend payments **after** that time-step.

Greeks
After each tree type is formed, the hedge parameters are computed using the below formula. All hedge parameters measure some rate of change of some variable with respect to another variable. Firstly, the delta is the rate of change of the option value with respect to the price, expressed as,

$$Delta(\Delta) = \frac{C_u - C_d}{S_u - S_d} \,.$$

The Γ is the rate of change of the Δ with respect to the stock price, expressed as,

$$Gamma(\Gamma) = \frac{\frac{C_{uu}-C_{ud}}{S_{uu}-S_{dd}} - \frac{C_{ud}-C_{dd}}{S_{ud}-S_{dd}}}{S_u - S_d} \,.$$

The θ is the rate of change of the option price with respect to time, expressed as,

$$Theta(\theta) = \frac{C_{ud} - C_0}{2\Delta t}$$

CRR Tree
The CRR approach builds up the ratio of the tree with

$$u = e^{\sigma\sqrt{\Delta t}} \,,$$

where $\Delta t = \frac{1}{12}$ for one period and $\sigma = 23.79\%$. As we are centring at the forward, this implies

$$ud = e^{2r\Delta t} \,.$$

where r is held constant. Therefore to ensure a recombining tree,

$$d = \frac{e^{2r\Delta t}}{u} \,.$$

The expected value equation states,

$$pu + (1-p)d = e^{u\Delta t}$$

Rearranging this equations lets us calculate p,

$$p = \frac{e^{r\Delta t} - d}{u - d}$$

which is the up transition probability which allows the tree to be built by the correct expected value at each node and has approximately a local volatility of $\sigma\sqrt{\Delta t}$, which can be defined as,

$$\sigma_{local} = \frac{1}{\sqrt{t_1 - t_0}} \sqrt{p(1-p)} \log\left(\frac{S_u}{S_d}\right)$$

In the *CRR Tree Forward* excel sheet, the CRR is displayed with and without dividends. Firstly with regards to some of the inputs for the above equations, The Apple price on the 13^{th} march was $S_0 = 179.97$, the rate was taken as the 0×5 spot rate on the 13^{th} of March and the implied volatility was the 5-month implied volatility for a put option, both of which where held constant throughout the tree. The tree set-up is relatively straight forward, we just have to take care of the dividends, with the May time-step shown below.

$$S_{20} = S_{10}d + PV_{div}, \quad S_{21} = S_{10}u + PV_{div} = S_{11}d + PV_{div}, \quad S_{22} = S_{11}u + PV_{div}.$$

The second equation holds as the CRR is a recombining tree. We have to be careful when computing the value of the stocks and ensure they recombine, we calculate all the nodes based off S_{00} and adding the dividends where necessary. By this we mean that S_{22} will be computed as $S_{00}u^2 + PV_{div}$ and $S_{32} = S_{00}du^2$. This will ensure the tree is recombining at all nodes.
Once we have computed the stock prices for all nodes, we can move on to compute the value of the option at each node. As the option is American, we will have to compute the intrinsic value and determine the maximum between the value and intrinsic value. The intrinsic value is simply calculated as $(K - S)^+$ as we chose a put option to construct the tree. If the payoff value is greater than the value of the option, then exercise the option early, if they are equal then it does not matter whether you hold or exercise the option, otherwise keep hold of it all of which is visible in the Excel worksheet.

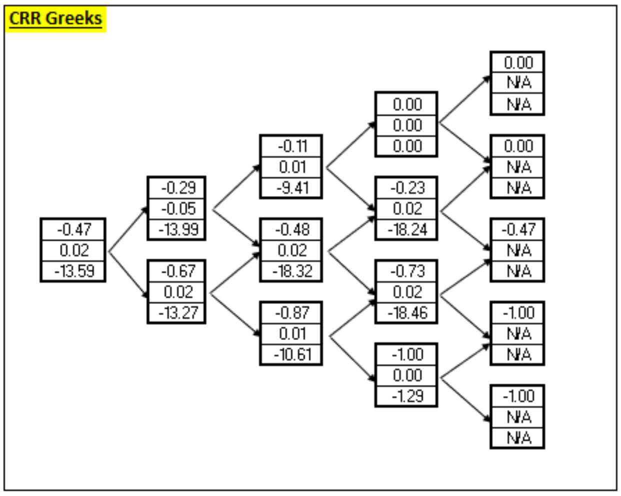

Figure 2: CRR Tree Greeks. Each box contains the Delta, Gamma and Theta of that node.

It is visible from the Figure 2 above the hedge parameters from the CRR model. As the tree was modelled using a put option, it is correct that the Δ tends to -1 below and 0 above the tree. Within the spreadsheet, it can be seen although the input volatility is 23.79%, the actual local volatility is 23.13%. Although this is almost exact, as you increase the time steps, will local volatility will tend to the input implied volatility.

Derman & Kani Implied Volatility Tree

Now that the CRR has been developed as a benchmark, the main goal of this report was to build implied volatility tree models using the Derman & Kani (1994) approach. Implied volatility trees allow hedge parameters to be computed to analyse that option market while also been used for pricing and hedging. This approach will provide prices and hedge ratios that will reflect the entire smile and not just a single volatility.

The main idea behind the Derman & Kani method is to use call options above the initial center line of t_i and put options below it. As you move through the tree it is vital that each option at any time step, is priced with a K value from the previous time-step stock price. This implies that the options are out of the money which is important, as if we used in the money, they are not sensitive to changes in volatility, which defeats the purpose of trying to capture the markets implied volatility.

Another important feature of this method is the use of option valuation using Arrow-Debreu prices, which by definition, says that the Arrow-Debreu price of any node is the probability of reaching that node discounted by a factor of $e^{-r(t_n - t_0)}$, where $(t_n - t_0)$ represents the time between today and the time where the node is. Of course, one may think that when the number of time steps is large, this may require intensive computational effort, but fortunately we only want to know the Arrow-Debreu price at a given time step from the Arrow-Debreu prices at the previous time step. Given that the Arrow-Debreu price of a node is the present value of the probability of reaching that node, we obtain

$$\lambda_{i+1,j+1} = e^{-r\Delta t} p_{i,j} \lambda_{i,j} \ , \tag{1}$$

where p_n is the risk-neutral transition probability. Note, there are two ways of reaching node$_{i+1,j}$, which will yield,

$$\lambda_{i+1,j} = e^{-r\Delta t}(p_{i,j-1}\lambda_{i,j-1} + (1 - p_{i,j})\lambda_{i,j}) \ . \tag{2}$$

As can be seen within the Excel attachment, the Derman & Kani approach has been applied in both the European and American setting. In both methods many of the steps in structuring the tree are exactly the same, but the main difference is calculating the value of each node as due to the early exercise component in an American option, this must be included within the value of the node. To handle this, a false position method was applied and is discussed further down in the report.

In both trees, as previously mentioned the present value of all the dividend payments is subtracted from the initial $S_0 = \$179.97$

To start, an ATM option is priced with maturity in one time-step ($\Delta t = 1/12$) and K=S. For the European, this can be computed from Black-Scholes (BS) after the bilinear interpolation is done to find the implied volatility.

- **Node Value (v)** - For now we will just describe the steps for the value of a node in the European case. To start with, the value of the node at $v_{0,0}$, can be simply calculated using the P_{eu} which was just calculated and dividing by the Arrow-Debreu price, which at the first node is just one. This makes sense as you are currently standing at node$_{0,0}$, therefore the probability of you standing there is one. This formula will hold for all outer nodes on the tree when $j = 1$ for the call side and $j = 0$ on the put side of the tree. For inner nodes, we must take into account that the value of that node will be affect from the outer node, such that $n_{4,1}$ is affected by $n_{4,0}$. To deal with this, we used the following equations

$$\textbf{Below Center} \ - \ v^{put}_{i-1,j} = \frac{P_{eu}(t_i, S_{i-1,j}) - \sum}{\lambda_{i-1,j}} \ , \tag{3}$$

$$\text{where} \ \sum = \sum_{k=0}^{j-1} \lambda_{i-1,k}(e^{-r\Delta t} S_{i-1,j} - S_{i-1,k} + e^{-r_i \Delta t} D_i \ . \tag{4}$$

$$\textbf{Above Center} \ - \ v^{call}_{i-1,j} = \frac{C_{eu}(t_i, S_{i-1,j}) - \sum}{\lambda_{i-1,j}} \ , \tag{5}$$

$$\text{where} \ \sum = \sum_{k=j+1}^{i} \lambda_{i-1,k}(S_{i-1,k} - e^{-r_i \Delta t} D_i - e^{-r_i \Delta t} S_{i-1,j}) \ . \tag{6}$$

It's important to note that $D_i = 0$ for time-steps where there is no dividend payment at that time-step.

- **Stock Price** $(S_{i,j})$ - For nodes at the center, we can calculate the up probability which will allows us to calculate the stock price of nodes up and below the center, where

$$u = \frac{v^{put}_{i-1,j} + K}{e^{-r\Delta t}K - v^{put}_{i-1,j}} .$$

As we are centring at the forward,

$$S_u = S_{i-1,j}u$$
$$S_d = \frac{S_{i-1,j}e^{-r\Delta t}}{u} .$$

For all other nodes that are not situated on the centre, we can use the following equations when we are moving up (call) and down (put) the tree respectively.

$$S^{call}_{i,j} = \frac{v^{put}_{i-1,j}S_{i,j+1} + (S_{i-1,j} - e^{-r\Delta t}S_{i,j+1})S_{i-1,j}}{v^{put}_{i-1,j} + S_{i-1,j} - e^{-r\Delta t}S_{i,j+1}}$$

$$S^{put}_{i,j+1} = \frac{v^{call}_{i-1,j}S_{i,j} + (e^{-r\Delta t}S_{i,j} - S_{i-1,j})S_{i-1,j}}{v^{call}_{i-1,j} + e^{-r\Delta t}S_{i,j} - S_{i-1,j}}$$

- **Probability** (p_n) - Once we attain the stock prices at the next step, we were able to attain the transition probability through the following equation for both European and American,

$$p_{i-1,j} = \frac{e^{r\Delta t}S_{i-1,j} - D_i - S_{i,j}}{S_{i,j+1} - S_{i,j}}$$

where $D_i = 0$ if there is no dividend at that nodes time-step. In our case D_i will have a value at the May and August time-step.

- **Lambda** (λ) - After we obtain the probability, we are able to calculate the λ using equation 1 and 2 defined above for calculating the lambda out outer and inner nodes respectively, both of which are valid for the European and American trees.

The above description of formula and steps was carried out in both the European tree with the American differing slightly in the value calculation which will be discussed. In developing the implied volatility flexible trees, it is most likely one will encounter a bad probability which is essentially a violation of the forward condition. Figure 2 below gives a good understanding of the issue which leads to a negative probability which is obviously not correct since the probability must be between zero and one. The technique used to solve this issue is to change the local volatility by ensuring that the spacing is the same as the closest node previous. Although examples are provided below, the solution is mathematically expressed as:

$$\textbf{Below Center} \quad - \quad S'_{i,j+1} = \frac{S_{i,j+2}S_{i-1,j}}{S_{i-1,j+1}}$$
$$\textbf{Above Center} \quad - \quad S'_{i,j+1} = \frac{S_{i,j}S_{i-1,j+1}}{S_{i-1,j}}$$

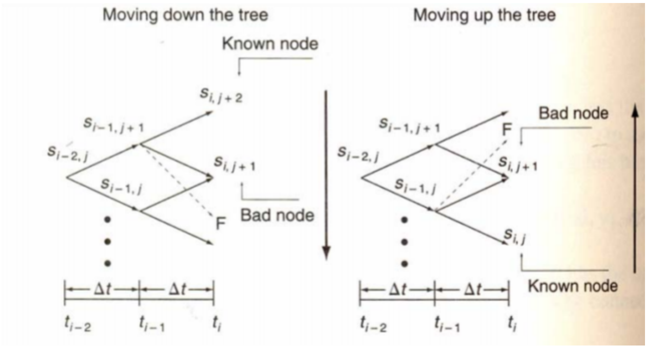

Figure 3: Visual description of a bad probability

Looking at the European tree we can see somewhat surprisingly there is no bad probabilities which gives a first impression of some calibration or estimation error. But when we look at the options being priced we can see an implied volatility (Displayed in separate tree) that is quite constant which may explain how the European tree is so well behaved per se.

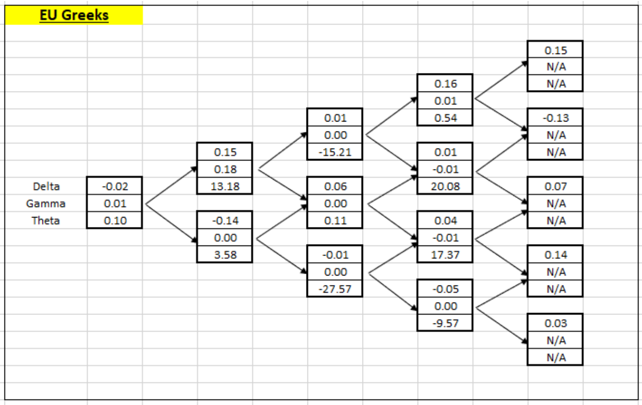

Figure 4: Derman European Greeks

It can be seen the wide range in Theta within Figure 4 but it must be noted that in flexible trees, and even when centered at the forward, therefore the Theta can become an unreliable estimate of the actual theta. Will to is the case for the American tree further down.
To extend the European tree for the American tree, we have to take into account the early exercise component that an American option contains, which means computing the true value of the node will not satisfy Equations 3-6 defined above. To help determine the correct value v, the **False Position Method** can be used to compute the value of the node for the American case. The reason we can't implement Equations 3-6, is that the value will be higher in the American option as the market price already contains an extra premium. Therefore the first step is to compute the Σ for the European option as given in Equations 4 and 6 and then set v_0 as Equations 3 and 5 whereby the P_{eu} is replaced by the market price of the American option P_{am}. v_0 and $v_1 = 0$ will be our first guesses for the value of the node. With these two values, we can obtain $\mathrm{P}(v_1)$ and $\mathrm{P}(v_0)$ using $\mathrm{P}(v_i) = v_i\lambda + \Sigma_{AM} - \mathrm{P}$ where

$$\Sigma_{AM} = \sum_{k=0}^{j-1} \lambda_{i-1,k} \max\left[(e^{-r_i \Delta t} S_{i-1,j} - S_{i-1,k} + e^{-r_i \Delta t} D_i),\ \ S_{i-1,j} - S_{i-1,k}\right].$$

Ideally we want $\mathrm{P}(v_i)$ to be as close as possible to zero so that the value of the node matches that of the market. This is most likely not the case with just the first two guesses, although this iteration method converges very quickly, therefore we mostly likely only require two iterations. We obtain

$$v_2 = \frac{v_1 m + \mathrm{P}(v_1)}{m}, \quad m = \frac{\mathrm{P}(v_1) - \mathrm{P}(v_0)}{v_1 - v_0}$$

which is our next guess for the correct value of the node. We can then determine $\mathrm{P}(v_2)$. There are only three possibilities for $\mathrm{P}(v_2)$: it is either greater, less than or equal to zero. If it is equal to zero, we stop and set $v = v_2$, otherwise we move on to the next iteration. Note that these formulas were used for a put option. For a call option the iterative procedure remains the same but the two Σ are given as

$$\Sigma_{eu} = \sum_{k=0}^{j-1} \lambda_{i-1,k}(S_{i-1,k} - e^{-r_i \Delta t} S_{i-1,j} + e^{-r_i \Delta t} D_i),$$
$$\Sigma_{am} = \sum_{k=0}^{j-1} \lambda_{i-1,k} \max\big[(S_{i-1,j} - e^{-r_i \Delta t} S_{i-1,k} + e^{-r_i \Delta t} D_i), \;\; S_{i-1,k} - S_{i-1,j}\big].$$

The *Derman American* tab on the excel file show the results of this false position method (FP) applied to the value of the inner nodes. The VBA `FPGroup7` function was created to follow the above outlined steps for the FP. We can see from the large number of bad probabilities that we faced, which was in sharp contrast to that of the European tree. When we look at the Greeks in comparison with the European, we see similar values with low Gamma's. One point to note is some excessively high theta's, particularly below the center, but as mentioned these may be inaccurate.

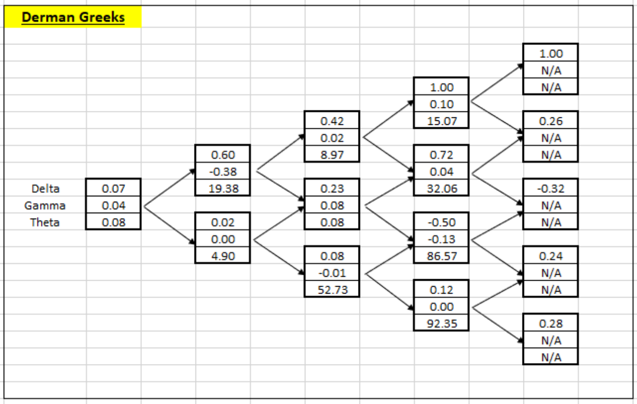

Figure 5: Derman American Greeks

To conclude, it can been seen how the Derman & Kani is an extremely useful approach in order to attain prices and hedging parameters that traders can use in a butterfly strategy for example. It was seen how the American had a lot more variation within the results, one suggestion may be instead of using interpolation of prices, to use the Bloomberg model for example, as there can be error such as reporting error in the historical data used.

APPENDIX A

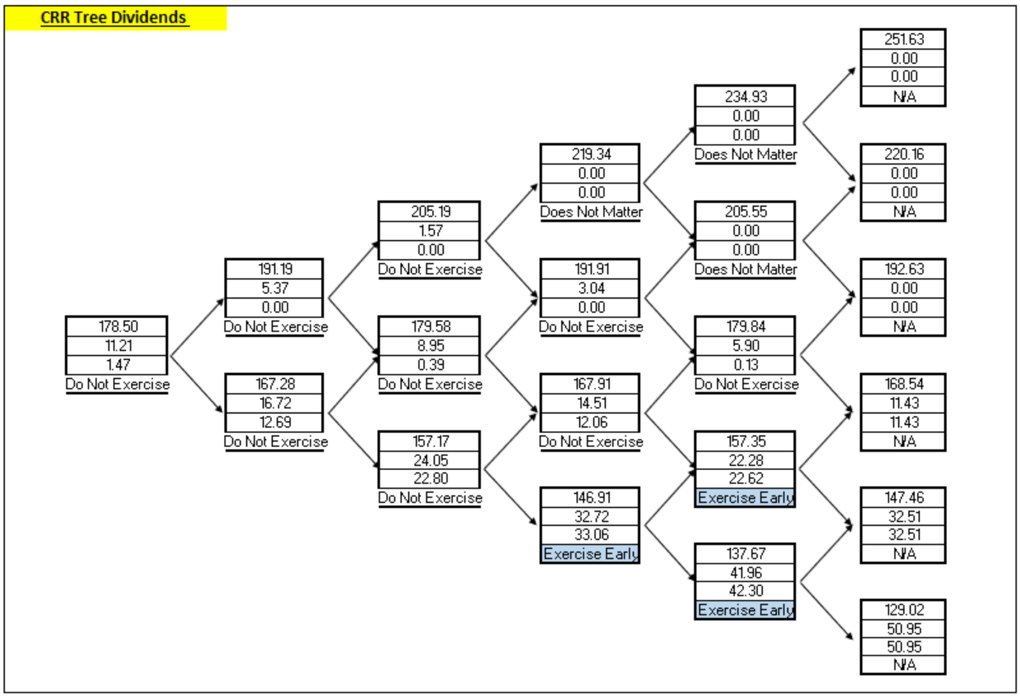

Figure 6: CRR Tree

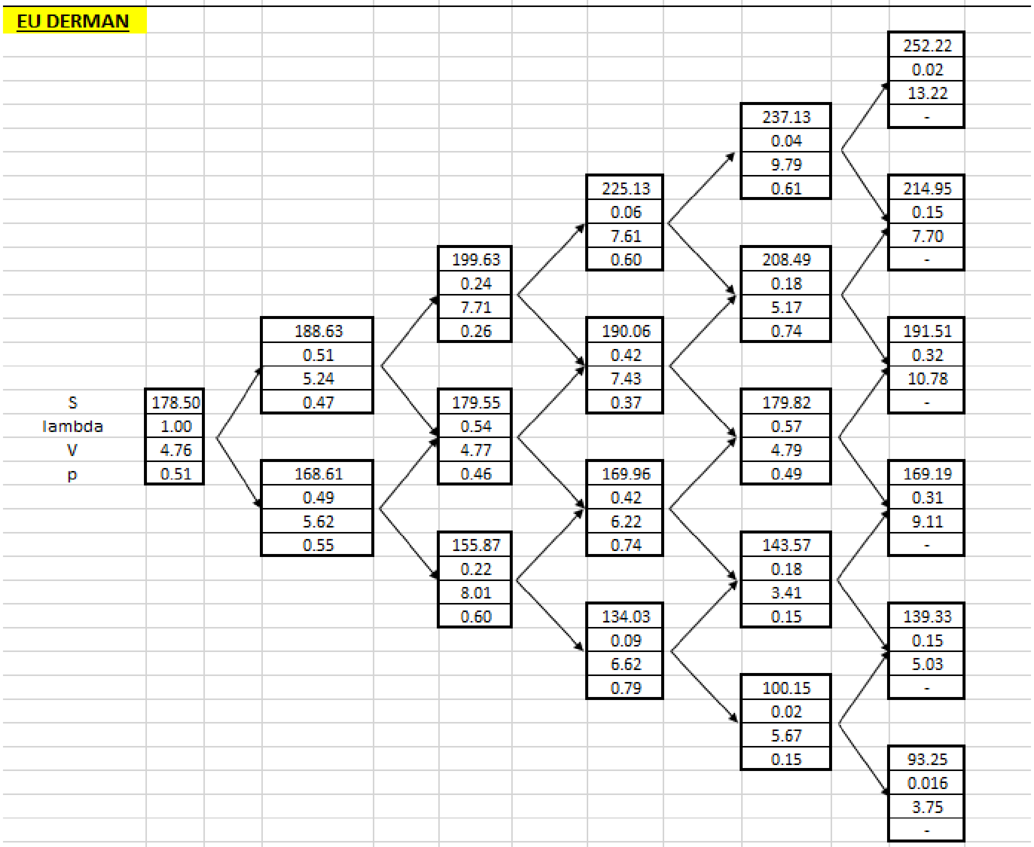

Figure 7: European Tree

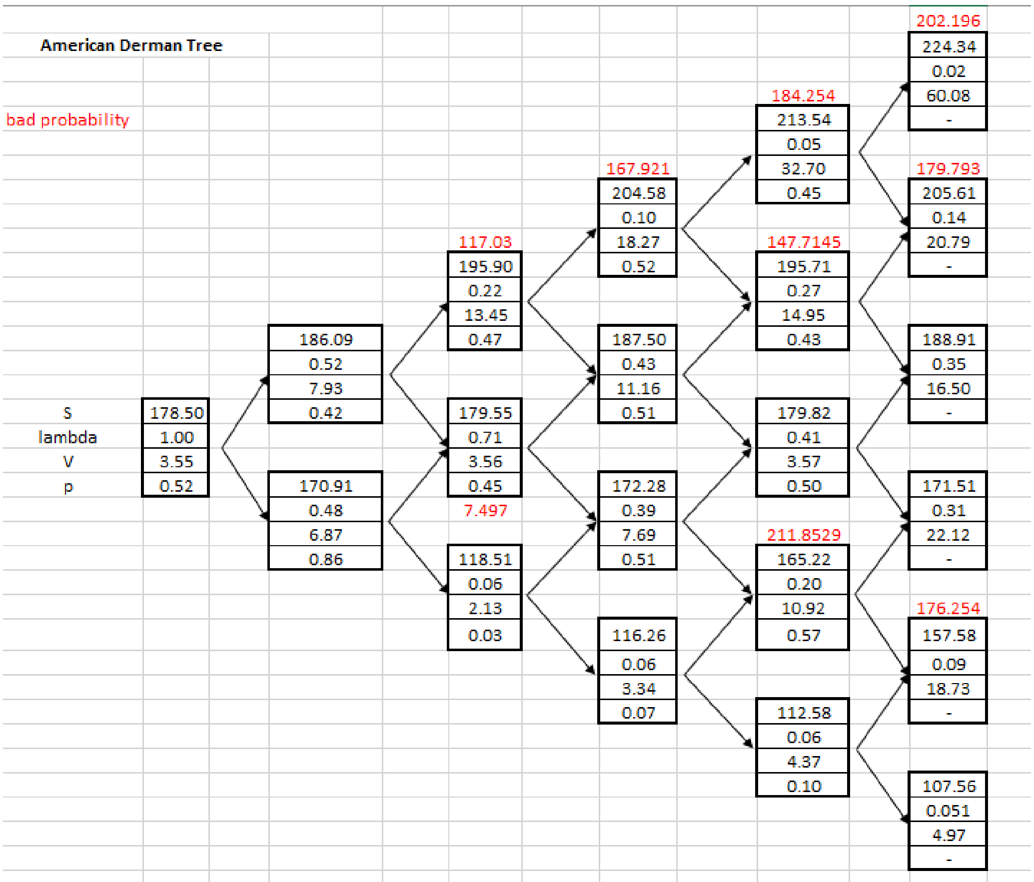

Figure 8: American Tree

BBG		**US Dollar Swaps (30/360, S/A) Curve (REF)**					
Tenor		Spot	1 Mo (P)	2 Mo (P)	3 Mo (P)	4 Mo (P)	5 Mo (P)
1 Mo	1	2.19303	2.1205	2.39004	2.18744	2.37111	2.40999
2 Mo	2	2.15991	2.20096	2.38415	2.29262	2.3948	2.3909
3 Mo	3	2.17171	2.23981	2.39647	2.32148	2.38818	2.42352
4 Mo	4	2.21177	2.26677	2.40496	2.34083	2.40949	2.44623
5 Mo	5	2.25184	2.29372	2.41346	2.36018	2.4308	2.46894
6 Mo	6	2.2919	2.32068	2.42195	2.37953	2.45211	2.49165
9 Mo		2.33843	2.38057	2.4577	2.43424	2.49053	2.51793
1 Yr		2.38908	2.4309	2.52036	2.47206	2.54578	2.58239
2 Yr		2.5809	2.60938	2.66629	2.63652	2.68495	2.70825
3 Yr		2.6881	2.70912	2.74992	2.72899	2.76342	2.77955
4 Yr		2.7439	2.76065	2.79226	2.77587	2.80301	2.81531
5 Yr		2.7786	2.79249	2.81907	2.80533	2.82812	2.8385
7 Yr		2.82975	2.84067	2.86194	2.85099	2.86939	2.87785
9 Yr		2.8716	2.88103	2.89916	2.88996	2.90577	2.91315
10 Yr		2.89121	2.90036	2.91767	2.90875	2.92421	2.93128
12 Yr		2.9273	2.93489	2.94957	2.94205	2.95493	2.96086
15 Yr		2.9599	2.96616	2.97812	2.97203	2.98258	2.9874
20 Yr		2.9829	2.98751	2.99647	2.99194	2.9997	3.00324
30 Yr		2.9611	2.96424	2.97041	2.96733	2.97255	2.9749
50 Yr		2.89006	2.89123	2.89369	2.89253	2.89419	2.89488

Figure 9: Forward Rates