Application of Double Exponential Jump Diffusion for Option Pricing

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Abstract

When pricing for options, an essential issue is the accuracy in estimating the price of the underlying security. Observed that the best-known Black-Scholes model is unable to explain several economic facts when modeling stock price dynamics, the resulting option price under this model is not conclusive. This report focus on pricing European options using an advanced model and implement it using C++. In particular, we reproduce the asymmetric leptokurtic feature of stock return and simulate the price for European options by double exponential jump diffusion (DEJD) model. We also put an effort in parameter estimation through maximum likelihood estimation (MLE) using SPY adjusted close price.
(Option Pricing; double exponential jump diffusion; asymmetric leptokurtic; MLE)

1. Introduction

It is generally recognized that the basic stock price dynamics follows a continuous sample-path process, namely the Black-Scholes-Merton Model, where the log return for the stock price follows a Geometric Brownian Motion (Merton, 1976). Despite varies alternative models that were designed afterwards, the Black-Scholes model is still the fundamentals of all and it reflects the real prices of stocks to a certain degree. However, there are several limitations for this method. Besides the regular assumptions concerning transaction costs and dividend issues, Black-Scholes model does not meet some important economic facts such as the asymmetric leptokurtic feature for stock return and the volatility smile for option pricing. The first fact concerning stock return is basically a result of human behavior. As there are both overreaction and underreaction to the outside news in the market, the assumption that the return is a random walk by Black-Scholes model underestimates the probability of extreme values and produces heavy tails on both sides. While with a significant overreaction to bad news, the left tail is heavier than the right. The reason is investors fear to lose, so the price is pushed down in a more rapid speed under the market impact and the probability of extreme low prices increases. Therefore, the real stock's return resembles higher peak and heavier left tail compared with normal distribution (an assumption of Black-Scholes model), known as the asymmetric leptokurtic feature. The second fact regarding option pricing is another major issue that Black-Scholes model assumptions conflicts with reality. Different from stocks, we measure the value of options in terms of their implied volatility. The more volatile the stock's return is, the more likely the option will pay off, and the more valuable the option should be. The implied volatility is thus a measure by a certain model from an option price to the future volatility the stock must have to insure the fairness of the option price, as stated by Emanuel Derman in his book 'My Life as a Quant' (Derman, 2004). People then notice that out-of-the-money options have higher implied volatility than at-the money options with same maturity dates, that is, the volatility is highly skewed and convex as the strike price changes, namely the volatility 'smile', or "smirk" in an equity market. However, the Black-Scholes model couldn't accommodate for this convexity for it assumes volatility to be constant over time and along different strike prices.

In hope of explaining the facts stated above, Robert C. Merton came up with the idea of adding 'jumps' to the original Black-Scholes model as a representation to incoming news. Merton (1976) derived the normal jump diffusion (NJD) model where underlying stock returns are generated by a mixture of both continuous process and jump processes. The only assumption that Merton's model differs from Black-Scholes model is the non-continuous stock price dynamics. The source of discontinuity is the 'Poisson-driven' process. Compared to the option price of Black-Scholes model, the theoretical price of deeply out-of-the-money option obtained from DEJD model will be higher while the deeply in-the-money option will be lower because of the existence of the jump, which is in accordance with the feature resembled from real option prices. This normal jump diffusion model can explain the heavy tail, high peak and show volatility smile to a certain level. But there is no simulation provided in the paper to justify these features. In addition, Merton (1973) also pointed out that this method is valid when the interest rate dynamics is stochastic.

However, the stock's return provided by NJD model is still symmetric and it does not give analytical solutions to path dependent options. To solve these problems, Kou (2002) introduced the double exponential jump-diffusion (DEJD) model with jump sizes double exponentially distributed. The validity of this model is backed up by the research of Ramezani and Zeng (1998) and Burger and Kliaris (2013) who tested historical data on both models and observed that the double exponential jump-diffusion model fits stock data better than the normal jump diffusion model. Further study of Cai and Kou (2011) illustrate that the jump size can be modified so that the model can approximate any distribution generated from real stock returns. In addition, Kou and Wang (2004) developed analytic approximation for finite-horizon American option. The pricing method for path dependent options such as barrier option and lookback option is given by Kou and Wang (2004) and Fuh, C. D., Luo, S. F., & Yen, J. F. (2013) under the scheme of DEJD model.

One important issue in validating the DEJD model is the estimation of the parameter values based on empirical observation. Plenty of methods have been proposed for parameter estimation problem of traditional Affine Jump Diffusion Models. Aït-Sahalia and Hansen (2004) surveyed a range of different estimation frameworks, including the generalized method of moments, the simulated moment estimation, and Markov Chain Monte Carlo (MCMC) methods.

Frame and Ramezani (2014) applied the MCMC method in the DEJD model. To employ the model in practice, numerical techniques based on the MCMC methods are proposed. One Bayesian methods ensure global convergence assuming the properties of Markov Chain are adequately satisfied. However, the result of this method is contingent on the prior assumptions (Kostrzewski, 2014).

A more general method is to apply the Maximum Likelihood (MLE) framework. Sorensen (1991) proved that, for large samples, MLE is the best estimation choice for its consistency, asymptotically normality and asymptotically efficiency (Bates, 2003b; Aït-Sahalia 2002). Ramezani and Zeng (2007) proved the Double Exponential Jump Diffusion Model (DEJD) is a specification of a Pareto-Beta Jump Diffusion (PBJD) model.

Our report is organized as follows, in chapter 2 we first lay out a detailed description of the DEJD model then goes on to explain the simulation method for European option prices under the model and closed this chapter with the MLE approach for parameter estimation. Chapter 3 contains all simulation outputs done with C++ language that the model features, including the asymmetric leptokurtic feature and a comparison among option prices obtained from DEJD model and BS model. We conclude our results with analysis for outputs and limitations for the DEJD model in chapter 4. Relevant C++ Codes for simulation are attached in appendix.

2. The DEJD Model

In this section, we first review theories for the stock price dynamics, which is a combination of basic BS model and an added 'jump' term, then go on to the pricing of options using Monte Carlo simulation. For a brief illustration of the rationale of the DEJD model, we only consider European option pricing in this report. All parameters involved are estimated using maximum likelihood ratio referenced from Ramezani & Zeng (2007).

2.1 Stock Price Dynamics

The formula for asset price dynamics of double exponential jump diffusion model is described as follows:

$$\frac{dS(t)}{S(t-)} = \mu dt + \sigma dW(t) + d\left(\sum_{i=1}^{N(t)} (V_i - 1)\right) \tag{1}$$

where $S(t)$ is the asset price, μ is the drift term, σ is the volatility of the asset, $W(t)$ is a standard Brownian motion, $N(t)$ is a Poisson process with rate λ representing the average times of jumps per unit time interval, $\{V_i\}$ is a sequence of independent identically distributed nonnegative random variables measuring the size of each jumps where $\Upsilon = \log(V)$ has an asymmetric double exponential distribution with density

$$f_Y(y) = p \cdot \eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q \cdot \eta_2 e^{\eta_2 y} 1_{\{y<0\}}, \eta_1 > 1, \eta_2 > 0 \tag{2}$$

where $p + q = 1$, $p > 0$, $q > 0$, which are probabilities of upward jumps and downward jumps respectively. $\frac{1}{\eta_1}, \frac{1}{\eta_2}$ measure the expectation of the size of the upward jumps and downward jumps respectively, normally $\frac{1}{\eta_1} < \frac{1}{\eta_2}$ to represent the overreaction of people behavior to bad news.

The assumption made in this dynamic is that all parts that involve randomness, that is $\mathrm{W(t)}$, $\mathrm{N(t)}$, $\{V_i\}$ should be independent, drift term, volatility and interest rate are constant, and Brownian motions and jumps are one dimensional.

To simulate stock and option price using Monte Carlo method, the stock dynamics under risk neutral measure is needed. By choosing a risk-neutral measure P^*, the dynamics of stock price becomes:

$$\frac{dS(t)}{S(t-)} = (r - \lambda^* \xi^*)dt + \sigma dW^*(t) + d\left(\sum_{i=1}^{N^*(t)} (V_i^* - 1)\right) \tag{3}$$

Note that the stock price still follows the DEJD model but with different coefficients. Specifically, both the coefficients for drift term and jump term changed under this risk neutral measure P^* while that for the diffusion term remain unchanged, which is similar with the change of measure for BS model. The jump sizes V_i^* are still independent identically distributed with their log values following a new asymmetric double exponential distribution. All random parts are still random.

Solving the stochastic differential equation (3), we obtain the return process $X(t) = \log(\frac{S(t)}{S(0)})$:

$$X(t) = \left(r - \frac{1}{2}\sigma^2 - \lambda^* \xi^*\right) t + \sigma W^*(t) + \sum_{i=1}^{N^*(t)} (\Upsilon_i^*), \qquad X(0) = 0 \tag{4}$$

where $\xi^* := E^*[V^*] - 1 = \frac{p^* \eta_1^*}{\eta_1^* - 1} + \frac{q^* \eta_2^*}{\eta_2^* + 1} - 1$, * terms meaning the term is under risk neutral measure.

Under the structure of DEJD model, the asymmetric leptokurtic feature is explained through the term $\sum_{i=1}^{N^*(t)} (\Upsilon_i^*)$ which is compound Poisson process with jump sizes double exponentially distributed while the other parts in the formula forms a Brownian motion as the Black-Scholes model.

As the difference between DEJD model and BS model is the jump term, the main issue involved in simulating stock prices is then to generate double exponential random variables and Poisson random variables, here we apply inverse transform method.

1) Generating double exponential random variables:

Step 1: from the probability density function of double exponential distribution given by formula (2), integrate it to get cumulative density function

$$F_Y(y) = \begin{cases} 1 - p^* e^{-\eta_1^* y}, & y \geq 0 \\ (1 - p^*) e^{\eta_2^* y}, & y < 0 \end{cases};$$

Step 2: compute its inverse function

$$y = F_\Upsilon^{-1}(u) = \begin{cases} -\frac{1}{\eta_1^*} ln(1 - u), & u < p^* \\ \frac{1}{\eta_2^*} ln(u), & u \geq p^* \end{cases};$$

Step 3: generate independent variable U_1 from uniform distribution $\mathrm{U}(0, 1)$;

Step 4: if $U_1 < p^*$, plug U_1 into F_Υ^{-1} and we get $y = -\frac{1}{\eta_1^*} ln\left(\frac{1 - U_1}{p^*}\right)$;

else, $y = \frac{1}{\eta_2^*} ln\left(\frac{U_1}{1 - p^*}\right)$.

Thus $\Upsilon^* = y$ is the double exponential random variable we want.

2) Generating Poisson random variables: (referenced from Knuth on Wikipedia)

Step 1: let $L = exp(-\lambda)$ be some small number approximating 0 and let $k = 0, p = 1$;

Step 2: generate independent variable U_1 from uniform distribution $\mathrm{U}(0, 1)$ and let $k = k + 1, p = p * U_1$;

Step 3: if $p \geq L$, turn to Step 2; else, return $\mathrm{k} - 1$.

2.2 European Option Pricing

As the closed form option pricing formula given by Kou (2002) is complicated and the result using Monte Carlo simulation is approximately the same as that calculated from the formula, we will use Monte Carlo approach to simulate paths for stock return $X(T)$ at maturity using formula (4) and further get the option price $\phi^c(T)$ at maturity. which independence structure among random parts is assumed, we can simply simulate all the components and add them up in the dynamic formula to get the simulated return in a single simulation.

Take European call option for example, the option price formula at maturity is given by:

$$\phi_i^c(T) = \max(X_i(T) - K, 0), \qquad i = 1, \dots, n \tag{5}$$

n is the number of simulation paths. Since the option price is simulated under risk neutral measure, the initial value of option is the discounted expected value of the option at maturity given by:

$$\phi^c(0) = e^{-rT} E^{P^*}[\phi_i^c(T)] \tag{6}$$

where r is the interest rate, $\phi^c(0)$ is the value of the call option at initial time.

The above expectation is approximated by the mean of terminal values $\phi_i^c(T)$ simulated from n independent paths:

$$E^{P^*}[\phi^c(T)] \approx \frac{1}{n}\sum_{i=1}^{n} \phi_i^c(T) \tag{7}$$

2.3 Parameter Estimation

Parameter estimation by MLE is a challenging task for DEJD or PBJD models, because of the well-known complication associated with the likelihood function of mixture distributions. The MLE optimization requires evaluation of infinite summations and integration of improper integrals. In practice, both the summations and the integrals are truncated using an appropriate convergence criteria. Moreover, the likelihood function of the mixture distribution may explode unless the parameter space is "appropriately" restricted. At last, during the MLE process, the traditional gradient calculations (by way of numerical first and second derivatives in a multivariate Newton's method framework) will suffer from model complexity.

To solve these issues in MLE methods, piecewise Gaussian quadrature was employed to compute the integrals in Ramezani and Zeng's paper. Also, the volatility parameter is bounded away from zero to prevent likelihood function exploding. Powell method is then applied to avoid the gradient complexity. These techniques helped to meet the conditions described in previous studies by Kiefer (1978), Hamilton (1994), and Honoré (1998).

In this report, we apply a modified method based on the methods in Ramezani and Zeng (2007). The steps are as follows.

Let $D = \{S(0), S(1), S(2), \dots, S(M)\}$ denote the stock price at time $k = 0,1,2,\dots,M$. The one period rate of return $r_i = lnS(i) - lnS(i-1)$ is independent identically distributed. The unconditional density of 1 period returns $f(r)$ is:

$$\begin{aligned} f(r) = e^{-(\lambda_u+\lambda_d)} f_{0,0}(r) + \mathrm{e}^{-\lambda_u} \sum_{n=1}^{\infty} P(n,\lambda_d) f_{0,n}(r) \\ + \mathrm{e}^{-\lambda_d} \sum_{m=1}^{\infty} P(m,\lambda_u) f_{m,0}(r) \\ + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} P(n,\lambda_d) P(m,\lambda_u) f_{m,n}(r) \end{aligned} \tag{8}$$

Where $P(n,\lambda) = \frac{e^{-\lambda}\lambda^n}{n!}$ and $f_{m,n}(r)$ $(n \geq 0 \text{ and } m \geq 0)$ is the conditional density for one period returns, conditional on given numbers of up and down jumps (m, n).

The log-likelihood given M equally spaced returns observations is:

$$L(D; \lambda_u, \lambda_d, \eta_u, \eta_d, \mu, \sigma^2) = \sum_{i=1}^{M} ln\big(f(r_i)\big) \tag{9}$$

The unconditional density $f(r)$ is a mixture density (i.e., a Poisson weighted sum of four conditional densities). Our aim is then maximizing $L(D; \lambda_u, \lambda_d, \eta_u, \eta_d, \mu, \sigma^2)$ in terms of all 6 parameters $(\lambda_u, \lambda_d, \eta_u, \eta_d, \mu, \sigma^2)$, which in turn demands simulation for $f(r_i)$. From formula (8) we observe that the main difficulty in simulating $f(r)$ is the infinite summation of m and n and that for $f_{m,n}(r)$. Detailed formulae and derivations for a faster numerical computation for $f_{m,n}(r)$ $(n \geq 0$ and $m \geq 0)$ is shown in the appendix.

For maximizing $L(D; \lambda_u, \lambda_d, \eta_u, \eta_d, \mu, \sigma^2)$, we use an optimization method proposed by Powell (2009) called BOBYQA, which is a modification of POWELL by himself and does not need to calculate derivatives as well. Powell described it as a method that seeks the least value of a function of many variables, by applying a trust region method that forms quadratic models by interpolation. There is usually some freedom in the interpolation conditions, which is taken up by minimizing the Frobenius norm of the change to the second derivative of the model, beginning with the zero matrix. The values of the variables are constrained by upper and lower bounds.

3. Data and Results

3.1 Asymmetric Leptokurtic Feature

We simulate sample returns using DEJD model and the parameters we use were quoted from Kou (2002), which is reasonable for US stocks. Below is a comparison between the returns simulated from DEJD model and the probability density function of a normal distribution.

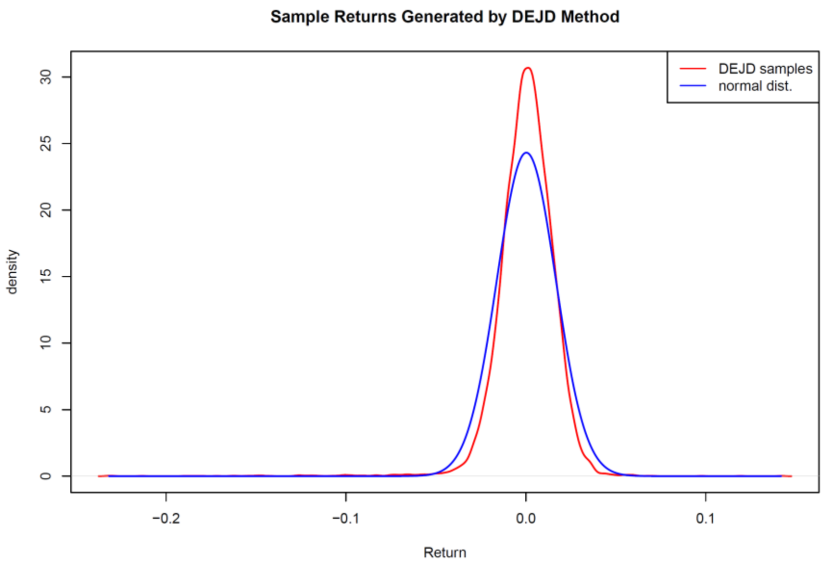


Figure 1. Sample returns generated from DEJD model and the pdf of normal distribution. Parameters for DEJD model are taken as: $\Delta t = 1$ day $= 1/250, \sigma = 8\times10^{-4}, \mu = 6\times10^{-4}, \lambda = 0.04, p = 0.30, \eta_1 = 50, \eta_2 = 25$ (red line). Parameters for normal distribution are sample mean and sample standard deviation calculated from the returns generated by DEJD model.

In the figure 1 shown above, the leptokurtic feature is quite apparent. The sample has an obvious higher peak than the theoretical stock return for Black Scholes model, the corresponding kurtosis is 27.54218, which is a quite persuading result. The skewness is -2.6136, which means the DEJD return has a left-skewness, indicating a higher probability of huge loss. Thus, generally the result is consistent with the DEJD model theory.

3.2 Option Prices

In the following paragraphs, we will use a new set of parameters estimated by MLE estimation we proposed in section 2.3. First we give our estimated results in Table 1 for parameters in DEJD model, the testing period is from 01/01/2015 to 04/01/2016. Here 10 hours of computational time is needed. Those parameters for BS model is estimated from sample mean and standard deviation.

Table 1. Parameters estimated from S&P500 daily return, 01/01/2015 – 04/01/2016.

	DEJD	**BS**
$\hat{\boldsymbol{\eta}}_1$	213.216	-
$\hat{\boldsymbol{\eta}}_2$	313.242	-
$\hat{\boldsymbol{r}}$	5.45E-06	2.47E-05
$\hat{\boldsymbol{\lambda}}$	1.24551	-
$\hat{\boldsymbol{p}}$	0.217218	-
$\hat{\boldsymbol{\sigma}}$	0.000225	0.005633

The value of $\hat{\boldsymbol{\lambda}}$ indicate that on average jumps (including both upward and downward jumps) occur a little more than once per day. $\hat{\boldsymbol{p}} < 0.5$ signals a higher probability of downward jump when a jump is observed. $\hat{\boldsymbol{\sigma}}_{DEJD} < \hat{\boldsymbol{\sigma}}_{BS}$ is because part of the probability is explained by the jump term. However, we notice that $\hat{\boldsymbol{\eta}}_1 < \hat{\boldsymbol{\eta}}_2$, which means the upward jump size is bigger than the downward jump size. It is not consistent to the theory. One possible reason is, the result of MLE method is not accurate enough. On the other hand, here is a trade-off between accuracy and running time. Since the method is already time-consuming, to let the algorithm finish in a tolerable time, we accept this result.

Now we can give our European option prices in Table 2 and Table 3 using equations (4) - (7) for DEJD model based on the parameters (except λ) given above. We set $T = 250$ to simulate one-year payoff.

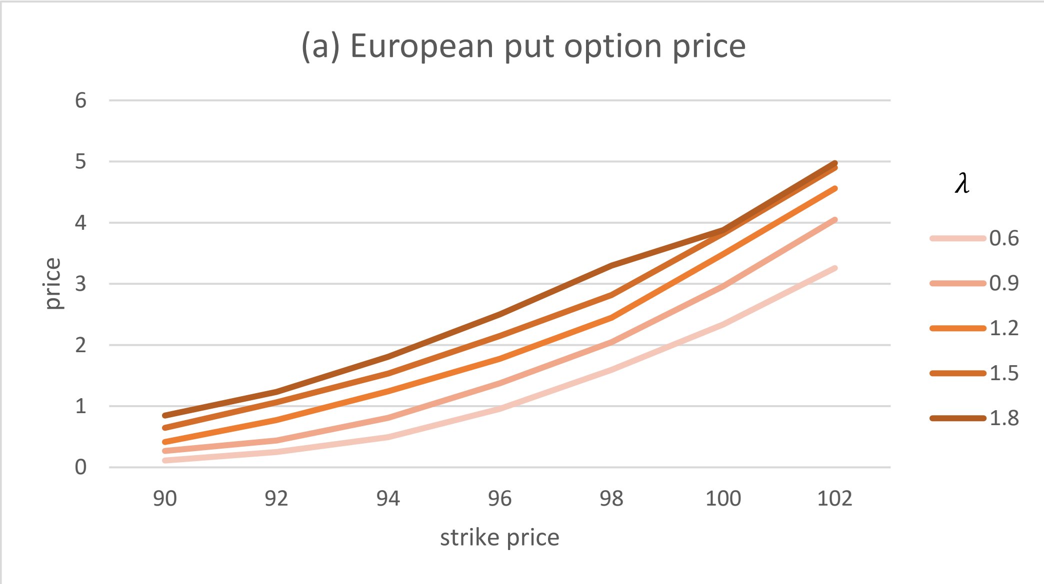
Table 2. European put prices by DEJD model

T	K	lambda 0.6	0.9	1.2	1.5	1.8
250	80	0.0011	0.0013	0.0141	0.0207	0.0576
	90	0.1142	0.2496	0.4900	0.6016	0.9967
	100	2.3818	2.9570	3.2719	3.7220	3.9045
	110	9.8796	10.1966	10.6106	10.7795	11.0328
	120	19.5625	19.3581	19.8515	19.6859	20.1251
500	80	0.0094	0.0876	0.1343	0.2377	0.3897
	90	0.3650	0.8430	1.1623	1.5068	2.0732
	100	3.3533	4.2734	5.0381	5.4814	6.1884
	110	10.3082	11.0168	11.2425	11.8311	12.2708
	120	19.5317	19.7731	20.2222	19.8647	20.4891
750	80	0.0727	0.2016	0.4327	0.6106	0.8516
	90	0.8544	1.3610	1.8792	2.4608	2.9143
	100	3.8781	4.8718	6.0467	6.5820	6.9760
	110	10.5358	11.2264	12.5304	12.8357	13.4219
	120	19.7555	19.9945	20.2699	20.3677	21.3990

Table 3. European call prices by DEJD model

T	K	lambda 0.6	0.9	1.2	1.5	1.8
250	80	19.7612	19.7707	19.6690	19.8329	20.3584
	90	10.0460	10.2957	10.4929	10.9018	10.3836
	100	2.5882	2.9493	3.5081	3.8865	4.6041
	110	0.2228	0.3666	0.6065	0.9335	1.0665
	120	0.0010	0.0268	0.0731	0.1636	0.1827
500	80	19.9499	20.0109	19.8933	20.0898	20.6215
	90	10.7935	10.9147	11.0791	11.8135	12.4814
	100	3.5990	4.2418	4.6171	5.3434	5.8933
	110	0.6189	1.2732	1.6207	2.2158	2.4910
	120	0.0641	0.2067	0.4398	0.7088	0.9795
750	80	19.9760	20.7402	21.0157	20.8685	21.2783
	90	11.1471	11.5989	12.1119	12.9037	13.6080
	100	4.5011	5.7530	5.9444	6.4585	7.7548
	110	1.2732	2.1600	2.5915	3.1456	3.8434
	120	0.2821	0.6142	0.8298	1.4453	1.8328

The corresponding graph for $T = 250$ is showing below:



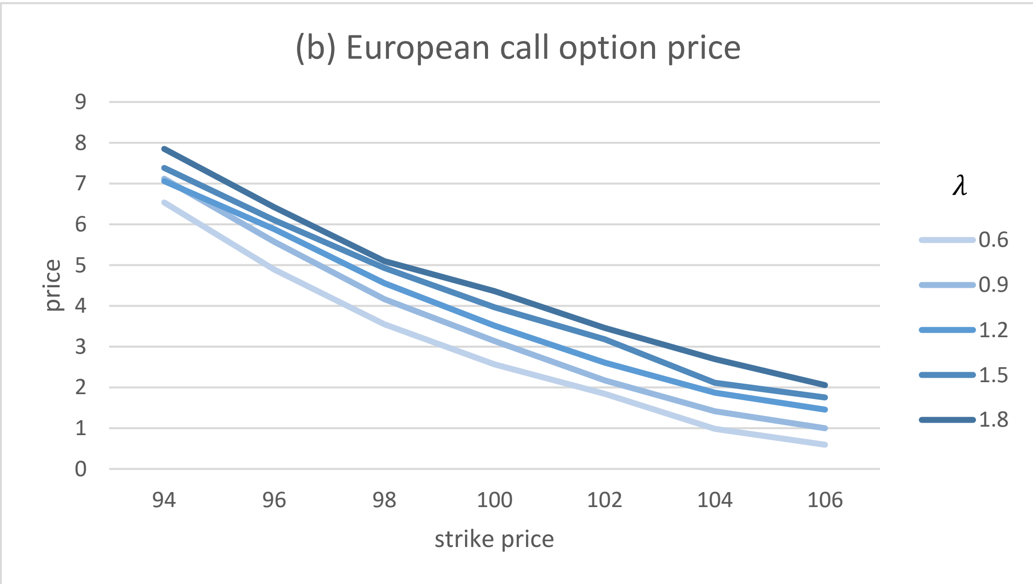
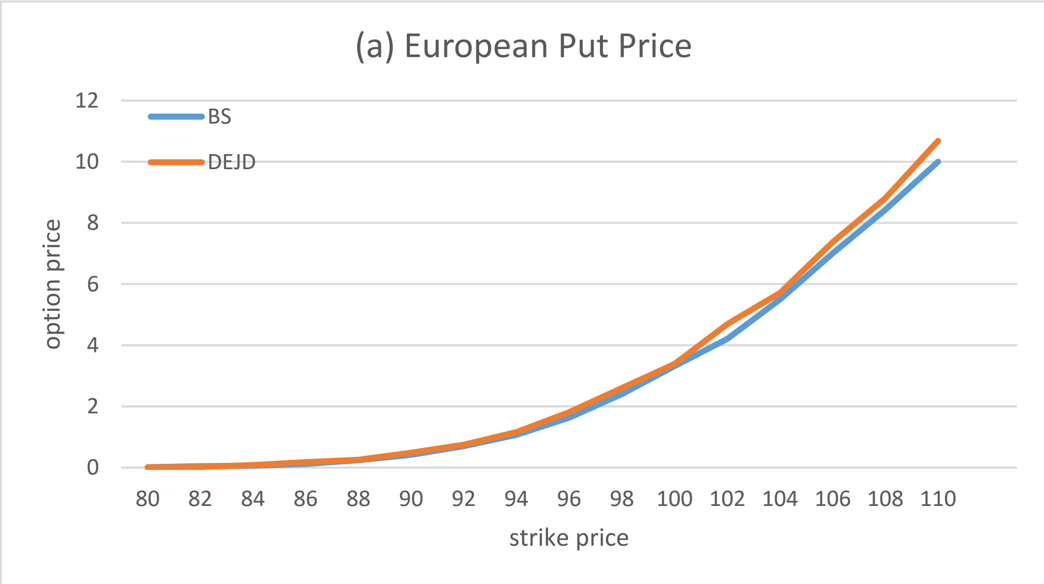


Figure 2. (a) European put prices for DEJD model vs strike prices. (b) European call prices for DEJD model vs strike prices

Figure 2 shows that the price for European put option is positively correlated to the strike price while the price for European call option is negatively correlated to the strike price, which is consistent with the result from Black Scholes model.

On the other hand, as λ increases, prices of both European puts and calls increase. This is a natural result since a higher λ means more jumps which leads to a higher volatility and thus a higher price.

We then compare the price obtained from DEJD model and that from Black Scholes model in Figure 3. Here λ is taken to be the estimated value in Table 1.



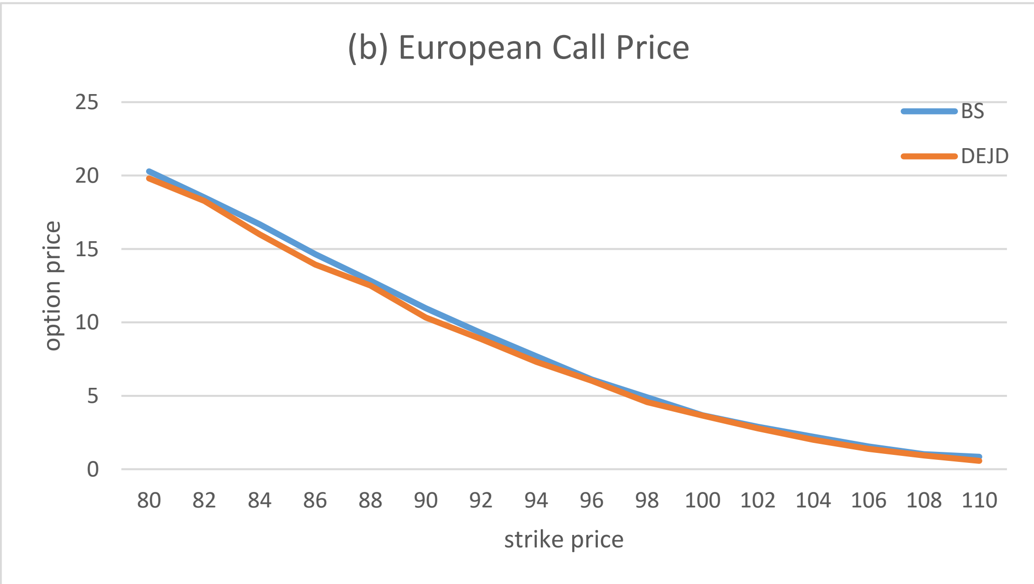


Figure 3. (a) European put prices for DEJD model vs Black Scholes model. (b) European call prices for DEJD model vs Black Scholes model

Figure 3 shows that the European option prices obtained from DEJD model are very much consistent with what we get from Black Scholes model. This means an adding with the jump term in the pricing formula does not affect the price of European options.

4. Conclusion

The report presents an empirical analysis of the DEJD model as well as parameter estimation using maximum likelihood estimation. We found that option prices obtained from DEJD model are similar to option prices obtained from Black Scholes model. Therefore, despite of jumps observed in the underlying asset price process, adding jump term in the pricing formula have little effects on option pricing result. On the other hand, although the random walk assumption does not fit the stock market reality where the asymmetric leptokurtic feature is quite common, Black-Scholes model is still an acceptable method in pricing European call and put options.

Meanwhile, the DEJD model can explain the asymmetric leptokurtic feature of stock returns with the jump term added in its pricing formula, where the size of the jumps follows a double exponential distribution and the number of jumps follows a Poisson process.

There are also several drawbacks for the DEJD model. Similar to normal jump diffusion model, under the double exponential jump-diffusion model, one cannot hedge without risk due to the non-linear function of the stock price derived. Only when required expected return on option is known or assume CAPM is valid can the option pricing formula be derived as the paper of Merton (1976) suggests. Another essential limitation is that there may be certain level of dependence in stock return while the model itself assumes independent increment.

A number of further directions for empirical simulations regarding the DEJD model can be extended as follows. As a start, several papers mentioned the effectiveness of the DEJD model in pricing path dependent options such as barrier options and lookback options and we can try the empirical analysis for these options. Also, there can be other modifications for the MLE method to improve the computational time and accuracy.

Appendix

As is stated in the appendix by Ramezani and Zeng (2007), the formulae for $f_{m,n}(r)$ $(n \geq 0, m \geq 0)$ is shown below:

$$f_{r(s)|0,0}(r) = \frac{1}{\sqrt{2\pi s}\sigma} e^{-\frac{1}{2\sigma^2 s}(r-t-\mu s+0.5\sigma^2 s)^2} \tag{A1}$$

$$f_{r(s)|0,n}(r) = \frac{\eta_d^n}{(n-1)!\sqrt{2\pi s}\sigma} \int_{-\infty}^{0} (-x)^{n-1} \mathrm{e}^{\eta_d x - \frac{1}{2\sigma^2 s}(r-x-\mu s+0.5\sigma^2 s)^2} dx \tag{A2}$$

$$f_{r(s)|m,0}(r) = \frac{\eta_u^m}{(m-1)!\sqrt{2\pi s}\sigma} \int_{0}^{\infty} (x)^{m-1} \mathrm{e}^{-\eta_u x - \frac{1}{2\sigma^2 s}(r-x-\mu s+0.5\sigma^2 s)^2} dx \tag{A3}$$

$$\begin{aligned} f_{r(s)|m,n}(r) = \frac{\eta_u^m \eta_d^n}{(m-1)!(n-1)!\sqrt{2\pi s}\sigma} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{0\wedge t} (-x)^{n-1}(t-x)^{m-1} e^{(\eta_u+\eta_d)x} dx \right) \\ \times e^{-\eta_u t} e^{-\frac{1}{2\sigma^2 s}(r-t-\mu s+0.5\sigma^2 s)^2} dt \end{aligned} \tag{A2}$$

Equation (A1) is computationally straight forward. To simplify the numerical computation for the above equations (A2) to (A4), we try to change the improper integral into finite integrals by doing the following steps:

For equation (A2), define

$$g_{0,n}(x) = (-x)^{n-1} \exp\left(\eta_d x - \frac{1}{2\sigma^2 s}(r-x-\mu s+0.5\sigma^2 s)^2\right)$$

Then

$$\begin{aligned} f_{r(s)|0,n}(r) &= \frac{\eta_d^n}{(n-1)!\sqrt{2\pi s}\sigma} \int_{-\infty}^{0} g_{0,n}(x)dx \\ &= \frac{\eta_d^n}{(n-1)!\sqrt{2\pi s}\sigma} \left\{ \int_{-1}^{0} g_{0,n}(x)dx + \int_{-\infty}^{-1} g_{0,n}(x)dx \right\} \end{aligned}$$

To solve $\int_{-\infty}^{-1} g_{0,n}(x)dx$, define

$$h_{0,n}(x) = \frac{1}{x^2} g_{0,n}\left(\frac{1}{x}\right)$$

Then

$$\begin{aligned} h_{0,n}(x) &= \frac{1}{x^2}(-x)^{-(n-1)} \exp\left(\frac{\eta_d}{x} - \frac{1}{2\sigma^2 s}\left(r - \frac{1}{x} - \mu s + 0.5\sigma^2 s\right)^2\right) \\ &= (-x)^{-(n+1)} \exp\left(\frac{\eta_d}{x} - \frac{1}{2\sigma^2 s}\left(r - \frac{1}{x} - \mu s + 0.5\sigma^2 s\right)^2\right) \end{aligned}$$

Thus

$$\int_{-\infty}^{-1} g_{0,n}(x)dx = \int_{-\infty}^{-1} g_{0,n}(x)(-x^2)d\left(\frac{1}{x}\right) = \int_{0}^{-1} -\frac{1}{y^2} g_{0,n}\left(\frac{1}{y}\right) dy = \int_{-1}^{0} h_{0,n}(y)dy$$

and

$$f_{r(s)|0,n}(r) = \frac{\eta_d^n}{(n-1)!\sqrt{2\pi s}\sigma} \left\{ \int_{-1}^{0} g_{0,n}(x)dx + \int_{-1}^{0} h_{0,n}(y)dy \right\} \tag{B}$$

Similarly, for equation (A3), define

$$g_{m,0}(x) = x^{m-1} \exp\left(-\eta_u x - \frac{1}{2\sigma^2 s}(r - x - \mu s + 0.5\sigma^2 s)^2\right)$$

$$h_{m,0}(x) = x^{-(m+1)} \exp\left(-\frac{\eta_u}{x} - \frac{1}{2\sigma^2 s}\left(r - \frac{1}{x} - \mu s + 0.5\sigma^2 s\right)^2\right)$$

Then

$$f_{r(s)|m,0}(r) = \frac{\eta_u^m}{(m-1)!\sqrt{2\pi s}\sigma}\left\{\int_0^1 g_{m,0}(x)dx + \int_0^1 h_{m,0}(y)dy\right\} \quad \text{(C)}$$

As for equation (A4), define

$$I_{t_1}(x) = (-x)^{-n-1}\left(\frac{1}{t} - \frac{1}{x}\right)^{m-1} \exp\left\{\frac{\eta_u + \eta_d}{x}\right\}$$

$$I_{t_2}(x) = (-x)^{-n-1}\left(t - \frac{1}{x}\right)^{m-1} \exp\left\{\frac{\eta_u + \eta_d}{x}\right\}$$

$$I_{t_3}(x) = (-x)^{n-1}(t - x)^{m-1} \exp\{(\eta_u + \eta_d)x\}$$

$$I_{t_4}(x) = (-x)^{n-1}\left(\frac{1}{t} - x\right)^{m-1} \exp\{(\eta_u + \eta_d)x\}$$

$$g_{m,n}(t) = \exp\left\{-\eta_u t - \frac{1}{2\sigma^2 s}(r - t - \mu s + 0.5\sigma^2 s)^2\right\}$$

$$h_{m,n}(t) = \frac{1}{t^2}\exp\left\{-\frac{\eta_u}{t} - \frac{1}{2\sigma^2 s}\left(r - \frac{1}{t} - \mu s + 0.5\sigma^2 s\right)^2\right\}$$

Then

$$\begin{aligned} f_{r(s)|m,n}(r) = \frac{\eta_u^m \eta_d^n}{(m-1)!\,(n-1)!\sqrt{2\pi s}\sigma}\Bigg[&\int_{-1}^{0}\left(\int_t^0 I_{t_1}(x)dx\right)h_{m,n}(t)dt \\ &+ \int_{-1}^{0}\left(\int_{\frac{1}{t}}^0 I_{t_2}(x)dx\right)g_{m,n}(t)dt \\ &+ \int_0^1\left(\int_{-1}^0 I_{t_2}(x)dx + \int_{-1}^0 I_{t_3}(x)dx\right)g_{m,n}(t)dt \\ &+ \int_0^1\left(\int_{-1}^0 I_{t_1}(x)dx + \int_{-1}^0 I_{t_4}(x)dx\right)h_{m,n}(t)dt\Bigg] \end{aligned} \quad \text{(D)}$$

Up till now, all improper integrals in the formula of $f_{m,n}(r)$ $(n \geq 0$ and $m \geq 0)$ are transformed into finite integrals (B) to (D) and we are still left with the infinite summation of $m \geq 1$ and $n \geq 1$. To solve this, we use a similar approach in Ramezani and Zeng (2007). We set an estimation error $\varepsilon = 10^{-3}$, calculate interval increment $\mathrm{X}_m = P(m, \lambda_u) f_{m,0}(r), m = 1,2, \ldots$ and the summation $\mathrm{S}_{i+1} \triangleq \sum_{m=1}^{i+1} \mathrm{X}_m$ up till now, check whether $2|\mathrm{X}_{i+1}| < \varepsilon \times (\mathrm{S}_{i+1} + \mathrm{S}_i)$, if no, then let $i = i + 1$ and calculate the increment for the next interval; else, return S_{i+1}.

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