Week 7 Lecture Notes

A. Background and Motivation

The Plan, Data and Analysis Phases of PPDAC involved the use of specialized tools and techniques to inter-relate these three phases.

> *def:* **Statistical Inference** is the process of drawing conclusions from sample data.

Key Learning Outcomes

- concepts of basic probability theory (condition probability), independent and mutually exclusive events, discrete and continuous random variables
- prosecutor's fallacy, base rate fallacy and conjunction fallacy
- statistical inference using
  - confidence intervals
  - hypothesis tests

The Analysis Phase of PPDAC

- use of Summary Statistics (Chp 1)
- analysis of categorical variables using rates, rules of rates etc. (Chp 2)
- analysis of numerical variables using the correlation coefficient, visualizations and five-point summary stats (Chp 3)

B. Introduction to Probability

Uncertainty

- use `"chance"` to mean that something is indefinite or not certain to occur
- when comparing likelihood of occurrence, we use terms like `more or less likely` $\implies$ common for everyday use but not precise when we deal with data @ a deeper level (need some concrete way of defining uncertainty)

> *def:* Probability is a mathematical means to reason about uncertainty.

Classic Problem and Motivation

- determine the possible outcomes and "likelihood" of the side where a coin lands after two tosses

$$Set(Outcomes) = sample\_space = \{HH, HT, TH, TT\}$$

> *def:* A Probability Experiment is any procedure that can be **infinitely repeated** and has a well-defined, **precise set of outcomes**.

- probability experiment must be repeatable

> *def:* A **sample space** is the collection of *all possible outcomes* of a probability experiment *(see above)*

- an event is a group of element(s) / sub-collection of the sample space

$$Event_1 = HT$$
$$Event_2 = HT, TH, TT$$

> For a probability experiment with an associated sample space, the *probability of an event* of the sample space is the total probability that the outcome of the experiment is an element of the event.

$$Event_n \subset \mathcal{E}$$

$$\{2, 4, 5\} \subset Set(Dice\,rolls)$$

Probability of an Event, `P(E)`

The probability of Event E, denoted $P(E)$ takes on a numerical value between *zero and one* inclusive $\implies 0 \leq P(E) \leq 1$.

- note that $P(E)$ should be the **long-run proportion** of observing E with many repetitions (i.e. N, which is also called the sample space).
- $P(E)$ is estimated using the following formula

$$P(E) = \frac{count(E)}{N}$$

- $P(E) = 0$ *iff* E is an impossible event
- $P(E) = 1$ *iff* E is a certain event

Important Notes

1. The estimate of $P(E)$ obtained from N repetitions is likely to be **different**, if we repeat the experiment (and get estimate of $P(E)$) another N times.
2. Such estimates of $P(E)$ get more accurate and closer to the actual true $P(E)$ value as N becomes larger.

Rules of Probabilities

- Virtually impossible to verify the true probability of an event for a given probability experiment
- $\therefore$ sufficient to just provide estimates as if it were the true probability
- need to ensure that some rules are observed

1. The probability of each event E, denoted $P(E)$, is a number between 0 and 1 inclusive.
2. If we denote the entire sample space by S, then $P(S)$ is 1.
3. If E and F are mutually exclusive events (can't happen simultaneously), then $P(E \cup F) = P(E) + P(F)$

We will stick to assumption that the sample space contains only a finite number of outcomes.

Uniform Probability

> *def:* **Uniform probability** is the way of assigning probabilities to outcomes such that **equal probability** is assigned to *every outcome* in the *finite sample space*. Thus, if the sample space contains a total of N different outcomes, then the probability assigned to each outcome is $\frac{1}{N}$.

- probability of selected any particular unit in this case if $\frac{1}{N}$
- the sampling frame is **exactly the sample space** of the uniform probability experiment

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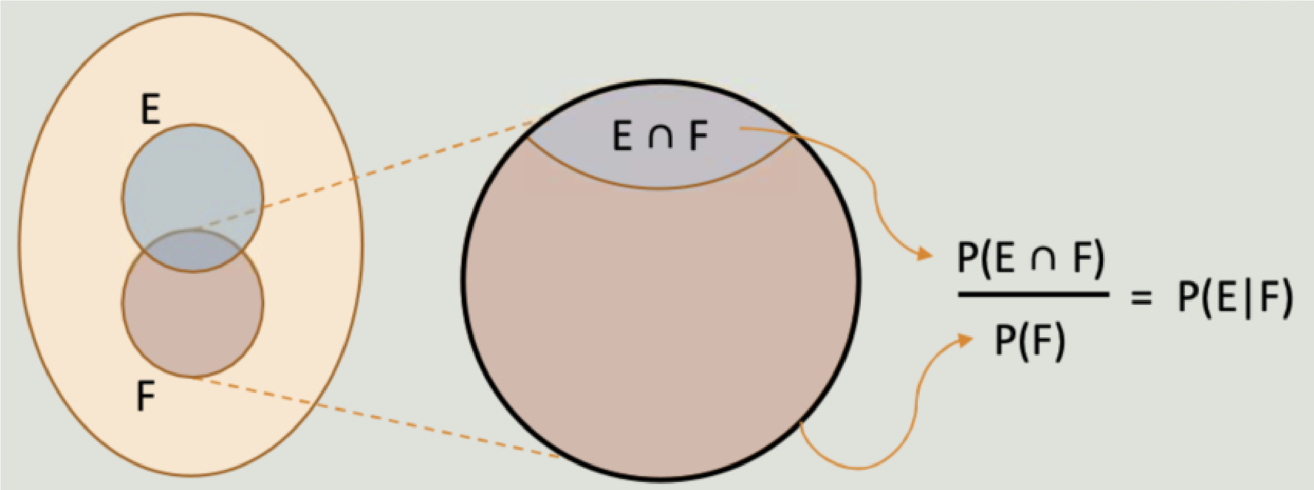
C. Prosecutor's Fallacy

- expert witness in dual-murder case (of children) commented that the chance of two children from an affluent family dying from Sudden Infant Death Syndrome is 1 in 73 million.
  - $\implies$ later on the Royal Statistics Society decided to issue a public statement of how this probability calculated was misrepresented
  - expert witness assumed that first child's death and second child's death are independent of each other
    - used the formula $P(First\,infant\,death \mid Clark\,innocent) \times P(Second\,infant\,death \mid Clark\,innocent)$ when they might in fact be related
- occurs when we assume that $P(A \mid B) = P(B \mid A)$ ❌ $\implies$ "all cows have four legs" does not imply "all four legged animals are cows"

Conditional Probability

> *def:* **Conditional Probability** is the probability of one even given another event, where both events are of the same sample space



- Probability of Event E given F, $P(E\,|F)$
  - Computation is based on restricting the focus of the given event F as the **restricted sample space** (rather than the entire sample space N)
  - Events F and E *may or may not have overlaps* which are denoted as $E \cap F$

$$P(E\,|F) = \frac{P(E \cap F)}{P(F)}$$

Cases where Conditional Probability is zero

1. If $P(E \cap F)$ is zero, it indicates that E and F **do not happen simultaneously**. In this case, $P(E\,|F)$ is also zero.
2. If event F itself cannot occur (i.e. $P(F) = 0$) $\implies P(E\,|F) = 0$, provided E and F are in the same sample space.

Conditional Probabilities as rates

Rates and Conditional probability are related by the equation:

$$P(A\,|B) = \frac{P(A \cap B)}{P(B)} = rate(A\,|B)$$

Independent Events

> *def:* The probability of an Event A *is the same* as the probability of A given B. So the fact that B has not occurred does not affect the probability of A occurring.

$$P(A) = P(A\,|B)$$

Conditional probability with A and B being independent from each other:

$$P(A\,|B) = \frac{P(A \cap B)}{P(B)}, P(A) = \frac{P(A \cap B)}{P(B)} \implies P(A) \times P(B) = P(A \cap B)$$

Recall: We say that two variables are **not related** if $rate(A) = rate(A\,|B)$.

- Thus, A and B are independent events whenever they are **not associated** with one another.

Conditional Independence

- Two events A & B are conditionally independent given an Event C with $P(C) > 0$) if:

$$P(A \cap B \mid C) = P(A \mid C) \times P(B \mid C)$$

- using the "distributive" technique for expansion

D. Conjunction Fallacy

> *def:* The ***law of total probability*** states that if E, F and G are events from the same sample space S such that:
> 1. E and F are mutually exclusive (i.e. $E \cap F = \emptyset$)
> 2. $E \cup F = S$
> Then, $P(G) = P(G|E) \times P(E) + P(G|F) \times P(F)$

> *def:* **Conjunction fallacy** occurs when one believes that the chances of two things happening together is higher than the chance of one of the two things happening alone.

i.e. is incorrectly assuming that:

$$P(A \cap B) > P(A) \; or \; P(B \cap A) > P(B)$$

when in actuality:

$$P(A \cap B) \leq P(A) \; or \; P(B \cap A) \leq P(B)$$

E. Base Rate Fallacy

> *def:* The **base rate fallacy** is a decision-making error in which the information about the rate of the occurrence of some trait in a population, called the base rate information, is ignored or not given appropriate weight.

Example case study: Accuracy of ART Test Kits

1. Scenario 1: Infected Individual's ART result is positive
    1. True Positive Rate / **Sensitivity of the Test**: $P(Tests\ Positive | Individual\ is\ infected)$
2. Scenario 2: Infected Individual's ART result is negative
3. Scenario 3: Healthy Individual's ART result is positive
4. Scenario 4: Healthy Individual's ART result is negative
    1. True Negative Rate / **Specificity of the Test**: $P(Tests\ Negative | Individual\ is\ not\ infected)$

Results from scenario 1 and 2 are not helpful for the average layman (don't actually have the means to afford testing options which 100% confirm diagnosis), thus more useful to find out:

$$P(\text{Individual is infected} \mid \text{Tests Positive})$$

Apart from the **sensitivity (TP rate)** and **specificity (TN rate)**, it is also important to consider the **base rate of infection** of the population.

- Base rate: is the basic probability of a particular event without any conditional probability

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F. Random variables

> *def:* A **random variable** is a **numerical** variable with probabilities assigned to *each of the possible* numerical values taken by the numerical variable (i.e. rolling a dice.

- applies to both discrete and continuous random variables
- The mode of a discrete random variable is the value of x that attains the highest y-value.

Discrete Random variables

Example of probabilities involved when rolling an **unequal six-sided / biased die** *might* be as follows:

$$P(Y=1) = \frac{1}{3}$$

$$P(Y=2) = \frac{1}{3}$$

$$P(Y=3) = \frac{1}{12}$$

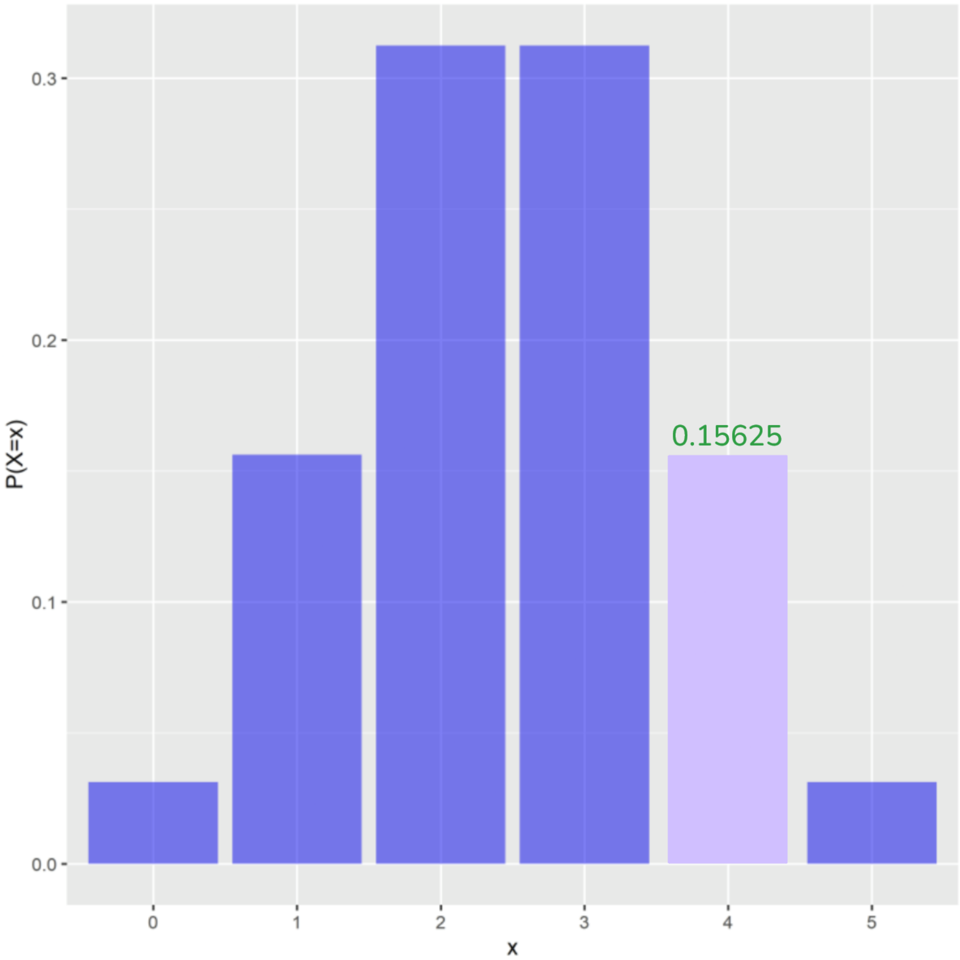
$$\cdots$$

$$P(Y=6) = \frac{1}{12}$$

$$\therefore \sum_{i=1}^{6} P(Y=i) = 1$$

Example 2 (Tutorial 4 Qn 1)

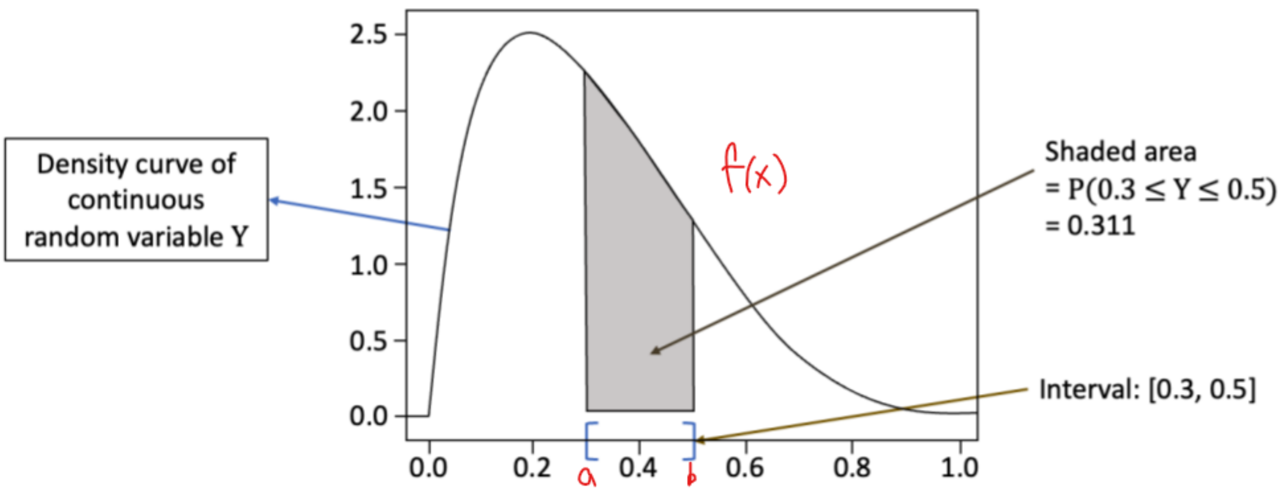
$X = x$	0	1	2	3	4	5
$P(X = x)$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$



Continuous Random variables

- continuous random is defined over an interval of values and is represented by the area under the **density curve**
- use integration to obtain (not in scope of GEA)

$$P(a \leq Y \leq b) = \int_a^b f(x)dx \ , \ on \ [a, b]$$



2.5
2.0
1.5
1.0
0.5
0.0
0.0
0.2
0.4
0.6
0.8
1.0
a
b
f(x)
Density curve of continuous random variable Y
Shaded area
= P(0.3 ≤ Y ≤ 0.5)
= 0.311
Interval: [0.3, 0.5]