Mathematical Finance M551
Section 9.1 Lecture Notes
Indiana University East

In this section, we provide an example of when a no-arbitrage cost for an option may not be unique.

Example: Consider a stock with intial price 100. After one period, the stock may either be $200, 100$, or 50. So we now allow for the possibity that the stock price stays the same (notice the difference with the model given in 5.1). Suppose we wish to price a call option with strike price 150 (i.e. we have the option to buy stock for 150 after 1 period. Assume that interest rate is 0 for simplicity.

$$\begin{array}{ccccc} & & 100 & & \\ & \swarrow & \downarrow & \searrow & \\ 200 & & 100 & & 50 \end{array}$$

By the arbitrage theorem, we know arbitrage will not be present povidd there exist probabilities P_{50}, P_{100}, P_{200} such that the expected gain when purchasing a stock or option is 0. That is

$$E[G_s] = 0 \quad \text{and} \quad E[G_c] = 0.$$

Where G_s denotes the gain at time 1 from buying the stock and G_c " " buying the option.

Hence,

$$G_s = \begin{cases} 1000 & if \quad S(1) = 200 \\ 0 & if \quad S(1) = 100 \\ -50 & if \quad S(1) = 50 \end{cases}$$

Which implies,

$$E[G_s] = 100P_{200} + 0P_{100} + (-50)P_{50}$$

If we let C be the price of the option,

then

$$G_s = \begin{cases} 50 - C & if \quad S(1) = 200 \\ -C & if \quad S(1) = 100 \quad \text{or } S(1) = 50 \end{cases}$$

So,

$$E[G_c] = (50 - C)P_200 - CP_{50} - CP_{100} = 50P_{200} - C$$

For no arbitrage, we must have
$0 = E[G_s]$ and $0 = E[G_c]$, thus

$$E[G_s] = E[G_c]$$

This implies

$$P_{200} = \frac{1}{2}P_{50} \quad \text{and} \quad C = 50P_{200}$$

but $P_{200} = \frac{1}{2}P_{50} \implies P_{200} \leq \frac{1}{3}$ (Since $P_{50} + P_{100} + P_{200} = 1$). It follows that $0 \leq C \leq \frac{50}{3}$.

So the noarbitrage price C in this case can be any value in the interval $[0, \frac{5)}{3}]$, hence it is not unique!