Mathematical Finance M551
Section 8.5
Indiana University East

To determine the no-arbitrage price of a call using the Black-Scholes formula, the parameters s, t, k, r and σ are needed. Of these parameters, $s, t, k,$ and r are known. The value of σ must be estimated.

In this seciton, we will investigate two techniques for estimating σ: the historical approach, and the standdard approuch.

Suppose $X_1, X_2, ..., X_n$ are independent random variables each with the same probability distribution (we call such random variables *i.i.d.*: independent, identically distributed.

Suppose each X_i has mean μ and variance σ^2. Then $\bar{X}$ defined by

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

is a usual estimator of the mean. In fact, $\bar{X}$ is call an 'unbiased' estimator of μ_0 because $E[\bar{X}] = \mu_0$(prove this).

Further, the random varibale S^2 defined by

$$S^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$$

is an unbiased estimator of the variance σ_0^2, i.e. $E[S^2] = \sigma_0^2$.

Hence, if $C_0, C_1, ..., C_n$ represent closing prices of a stock over $n+1$ days then, defining

$$X_i = \log(\frac{C_i}{C_{i-1}})$$

gives as collection of n *i.i.d.* random varibles which, under the assumption that the daily stock prices follow a G.B.M., are normal with mean μt and variance $t\sigma^2$. But $t = 1$ ($C_i - C_{i-1}$) is one day).

So, the variance is given by σ_0^2, which can be estimated by S^2. This means $\sigma_0 \approx \sqrt{S^2}$.

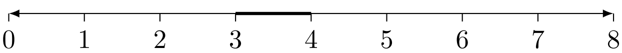
Thus, the volatility parameter can be estimated by

$$\sigma_0 \approx \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x}^2}{n-1})}$$

Now suppose we wish to estimate σ (the volatility parameter) over an interval of time t of historical data.

So we assume the present time is t and that we have historical price data $S(y)$ for our stock, where $0 \leq y \leq t$.

Partition the interval $[0, t]$ by fixing n and letting $L = \frac{t}{n}$.



$$\frac{t}{n} = 1$$

Define the random variables $X_1, X_2, ..., X_n$ by

$$X_1 = \log(\frac{S(\ell)}{S(0)})$$

$$X_2 = \log(\frac{S(2\ell)}{S(\ell)})$$

$$X_3 = \log(\frac{S(3\ell)}{S(2\ell)})$$

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$$X_n = \log(\frac{S(n\ell)}{S(n-\ell)\ell})$$

Assuming that the price evolution $S(y)$ follows a G.B.M. with parameters μ and σ, it follows that the X_i's are i.i.d. normal r.v's with mean $\ell\mu$ and variance $\ell\sigma^2$.

We will assume $t = 1$ trading year, or 252 days, and that ℓ represents 1 day. Hence $\ell = \frac{1}{252}$.

Using $X_i = \log(\frac{C_i}{C_{i-1}})$ where $C_0, C_1, ..., C_n$ are $n+1$ successive closing prices, we know the X_i's are independent and

$$X_i \sim N(\frac{1}{252}\mu, \frac{1}{252}\sigma^2)$$

for each i.
where increments here are days as proportion of 1 year(252 days)

Hence,

$$\frac{1}{252}\sigma^2 \approx S^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$$

So that the volatiltity parameter can be approxameted by

$$\sigma \approx S\sqrt{252}$$