# Algorithmic Learning Theory
# Spring 2017
# Lecture 2

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1/25/2017

1. Review Bayes Theory(Lecture 1)
2. Random Variable and Distribution
   (a) Random variable
      i. DRV, discrete random variable
      ii. CRV, continuous random variable
   (b) Distribution function
      i. CDF, cumulative distribution function
      ii. pdf or pmf, probability density(Mass) function
   (c) Discrete distribution
      i. Discrete uniform distribution
      ii. Beunoulli's distribution
      iii. Binomial distribution
   (d) Continuous distribution
      i. Continuous uniform distribution
      ii. Normal distribution
3. Multivariate Distributions
   (a) Random vector
   (b) Discrete multivariate distribution
   (c) Binormal distribution
   (d) Marginal distribution
   (e) Conditional distribution
4. Bayes Classification

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# 1 Review Bayes Theory(Lecture 1)

See notes in "Lecture 1".

# 2 Random Variable and Distribution

## 2.1 Random Variable

### 2.1.1 Discrete Random Variable(D.R.V)

$x \to t_1, t_2, \cdots, t_n$

### 2.1.2 Continuous Random Variable(C.R.V)

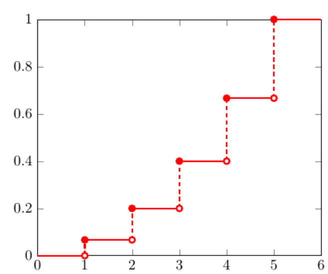
$x \to [a, b]$ a range of value.

## 2.2 Distribution Function

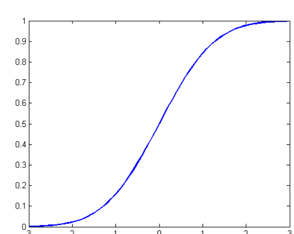
### 2.2.1 CDF:

Cumulative Distribution Function
$F_x(t) = P_r[x \leq t]$, probability can only increase

1. For Discrete:




2. For Continuous:

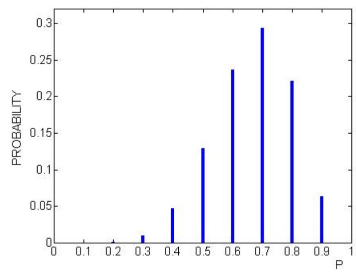


### 2.2.2 pdf or pmf:

Probability Density(Mass) Function

1. For Discrete:

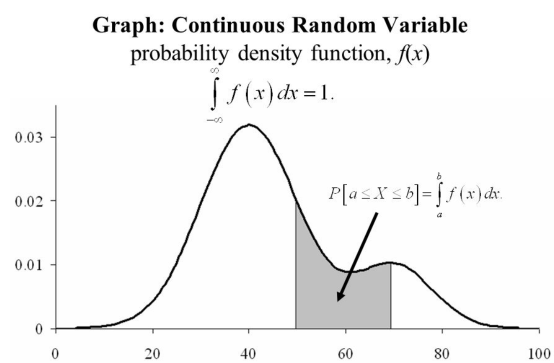
$$x : F_x(t) = P_r[x = t]$$




2. For Continuous:

$$f_x(t) = \frac{d}{dt}F_x(t),\ F_x(t) = \int_{-\infty}^{t} f_x(t)dt$$

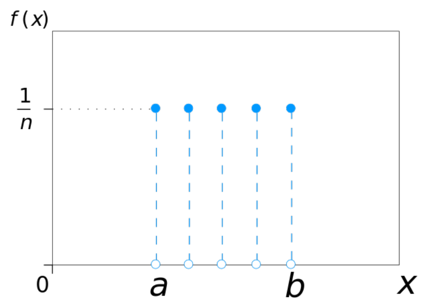
i $f_x(t) \geq 0$
ii $\int_{-\infty}^{\infty} f_x(t)dt = 1$

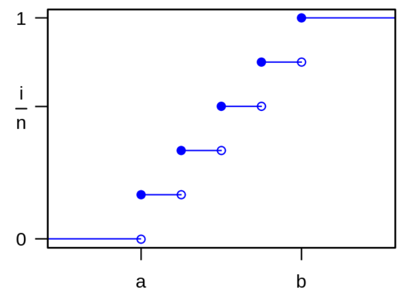



## 2.3 Discrete Distribution

### 2.3.1 Discrete Uniform Distribution

$$x : 1, 2, 3, \cdots, k$$
$$\text{pdf} : u_x(t) = \begin{cases} \frac{1}{n}, & \text{if } t = 1, 2, \cdots, n \\ 0, & \text{otherwise} \end{cases}$$




### 2.3.2 Bernoulli Distribution

$$\text{pdf} : f_x(t) = \begin{cases} p, & x = 1 \\ 1-p, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{CDF} : F_x(t) = \begin{cases} 0, & x \leqslant 0 \\ 1-p, & 0 \leqslant x < 1 \\ 0, & x \geqslant 1 \end{cases}$$

### 2.3.3 Binomial Distribution

numbers of 0's in independent Bernoulli trial with $P[0] = p$

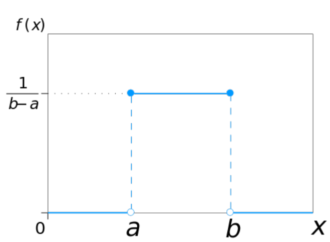
$$\text{pdf: } b(t|p,n) = \binom{n}{t} p^t (1-p)^{n-t}, \ \binom{n}{t} = \frac{n!}{t!(n-t)!}$$

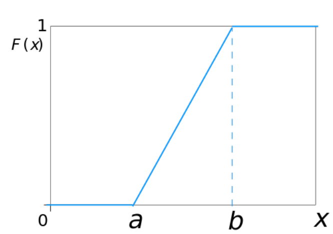
$$\text{CDF: } B(t|p,n) = \sum_{n=0}^{t} b(t|p,n)$$

## 2.4 Continuous distribution

### 2.4.1 Continuous uniform distribution

$$u(t|a,b) = \begin{cases} \frac{1}{b-a}, & a \leqslant t \leqslant b \\ 0, & \text{otherwise} \end{cases}$$



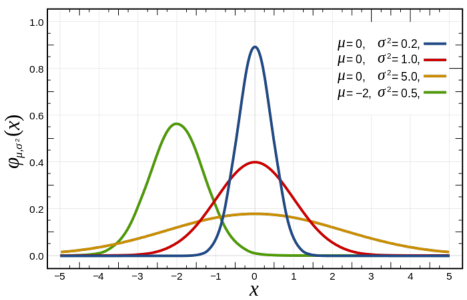
### 2.4.2 Normal distribution

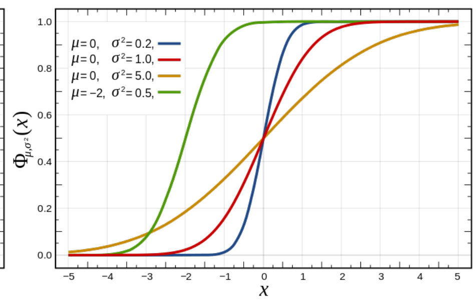
mean= $\mu$ and std.= $\sigma$

$$\text{pdf: } f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\text{CDF: } \frac{1}{2}[1+\mathrm{erf}(\frac{x-\mu}{\sigma\sqrt{2}})]$$

$$\mathrm{erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} e^{-t^2} dt$$




# 3 Multivariate Distributions

## 3.1 Random Vector

$X_N = (x_1, x_2, \cdots x_N)$ can be continuous or discrete.

$$\text{CDF: } F_x(t_1, t_2, \cdots t_n) = P_r[x_1 \leqslant t_1, x_2 \leqslant t_2, \cdots, x_n \leqslant t_n]$$

$$\text{pdf: } \begin{cases} \frac{\partial}{\partial t_1 \partial t_2 \partial t_3 \cdots \partial t_n} F(t_1, t_2, \cdots t_n) = f_x(t_1, \cdots, t_n), & \text{all continuous} \\ P_r[x_1 = t_1, x_2 = t_2, \cdots, x_n = t_n] = f_x(t_1, \cdots, t_n), & \text{all discrete} \end{cases}$$

both are joint distribution R.V. $x_1, \cdots, x_n$

## 3.2 Discrete Multivariate Distribution

$$Y \to 1, 2, \cdots, r; \quad P[Y = r_1] = P_u; \quad \sum P_u = 1$$

repeat $n$ times, $x_k$= number of times $Y = k$ occurs

$$\underset{\sim}{x} = (x_1, \cdots, x_n)$$

$$\text{pdf: } f_x(x_1, x_2, \cdots, x_n) = P[x_1 = t_1, x_2 = t_2, \cdots, x_n = t_n] \binom{n}{t_1, t_2, \cdots, t_n} {p_1}^{t_1} {p_2}^{t_2} \cdots {p_n}^{t_n}$$

$$\binom{n}{t_1, t_2, \cdots, t_r} = \frac{n!}{t_1! t_2! \cdots t_n!}$$

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## 3.3 Binormal distribution

$$\underset{\sim}{x} = (x_1, x_2), \text{both continuous}$$

$$\underset{\sim}{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \sigma_1^2 \rightarrow x_1, \sigma_2^2 \rightarrow x_1, \sigma_{12} \rightarrow x_1 x_2$$

$$\text{Covariance Matrix: } \sum = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

## 3.4 Multinomial Distribution

The rest

$$\sum = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \cdots & \cdots & \sigma_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{1m} & \cdots & \cdots & \cdots & \sigma_{1n} \end{bmatrix}$$

$\phi(t_1, t_2 | \binom{\mu}{n}, \sum) = \frac{1}{\sqrt{2}\pi \mathrm{Det}(\sum)} \exp[(t - \underset{\sim}{\mu})^T \sum^- 1(t - \underset{\sim}{\mu}), t = \binom{t_1}{t_2}$

joint p.d.f. $f_{(x1,x2)}(t_1, t_2)$

| $x_1$ \ $x_2$ | 1 | 2 | 3 | $f_{x_2}(t)$ |
|---|---|---|---|---|
| 0 | 0.1 | 0.4 | 0.2 | 0.7 |
| 1 | 0.2 | 0.05 | 0.05 | 0.3 |
| $f_x(t)$ | 0.3 | 0.45 | 0.45 | |

$f_{x_1|x_2=0}$ in

(1): $\frac{0.1}{0.7} = \frac{1}{7}$
$f_{x_1|x_2=0}$ in (2): $\frac{0.4}{0.7} = \frac{4}{7}$
$f_{x_1|x_2=0}$ in (3): $\frac{0.2}{0.7} = \frac{2}{7}$

## 3.5 Marginal Distribution

$\underset{\sim}{x} = (x_1, x_2, \cdots, x_n) \rightarrow f_x(t_1, \cdots, t_n)$
p.d.f.: $(x_1, x_2, \cdots, x_n) = \underset{\sim}{\overline{x}}$
$f_{\overline{x}} = (t_1, t_2, \cdots, t_n) = \int_{t_{k+1}, \cdots, t_n}^{\infty} f_x(t_1, \cdots, t_n) dt_{k+1}, \cdots, t_n = \sum_{t_{k+1}} \sum_{t_{k+2}} \cdots \sum_{t_n} f_x(t_1, \cdots, t_n)$
$P_r[x = 0] = P_r[x_2 = 0, x_1 = 1] + P_r[x_2 = 0, x_1 = 2] + P_r[x_2 = 0, x_1 = 3]$
$P_r[x = 1] = P_r[x_2 = 1, x_1 = 1] + P_r[x_2 = 1, x_1 = 2] + P_r[x_2 = 1, x_1 = 3]$
$\phi(t_1, t_2 | \underset{\sim}{\mu}, \sum)$
$\phi(t_1) = \int_{-\infty}^{\infty} \phi(t_1, t_2 | \cdots) dt_2$
Find p.d.f.$f(t_1, t_2)$

Example: $x_1 \rightarrow$ Height, $x_2 \rightarrow$ Gender $\begin{cases} 0 & \text{male} \\ 1 & \text{female} \end{cases}$

for $x_2 = 0 \rightarrow$ Height $\sim N(69, 4.5) \Leftrightarrow f_{x_1|x_2}(t_1|t_2 = 0)\phi(t_1|\phi = 69, \sigma = 4.5)$
for $x_2 = 1 \rightarrow$ Height $\sim N(65, 4.2) \Leftrightarrow f_{x_1|x_2}(t_1|t_2 = 1)\phi(t_1|\phi = 65, \sigma = 4.2$
marginal distribution of height for people

$f_{x_1}(t_1) = f_{x_1|x_2}(t_1|t_2 = 0) * f_{x_2}(0) + f_{x_1|x_2}(t_1|t_2 = 1) * f_{x_2}(1) = \phi(t_1|69, 4.5) * 0.5 + \phi(t_2|65, 4.2) * 0.5 = \phi(t_1|\frac{69+65}{2}, \sqrt{\frac{4.5^2+4.2^2}{2}})$

### 3.6 Conditional Distribution in 2 Variables

$x = (x_1, x_2) \to$ joint $f(t_1, t_2)$
$f_{x_1|x_2}(t_1|t_2) = \frac{f_x(t_1,t_2)}{f_x(t_2)}$
$f_x(t_1, t_2) = f_{x_1|x_2}(t_1|t_2) * f_{x_2}(t_2)$
$f_{x_2|x_1}(t_2|t_1) = \frac{f_x(t_1,t_2)}{f_{x_1}(t_1)}$
$f_x(t_1, t_2) = f_{x_2|x_1}(t_2|t_1) * f_{x_1}(t_1)$

# 4 Bayes

### 4.1 Bayes Formula

$f_{x_1|x_2}(t_1|t_2) = \frac{(f_{x_2|x_1}(t_2|t_1)*f_{x_1}(t_1))}{f_{x_2}(t_2)}$
Discrete $x_1 : f_{x_2}(t_2) = \sum_{t_1} f_{x_2|x_1}(t_2|t_1) * f_{x_1}(t_1)$
Continuous $x_1 : f_{x_2}(t_2) = \int_{-\infty} \infty f_{x_2|x_1}(t_2|t_1) * f_{y_1}(t_1)dt$
Example: A person has height 6?7"
$f(x_2 = 0|x_1 = 6'7") = \frac{f_{x_1|x_2}(6'7"|x_2=0)*f_{x_2}(0)}{f_{x_1}(67)}$
$f(x_2 = 1|x_1 = 6'7") = \frac{f_{x_1|x_2}(6'7"|x_2=1)*f_{x_2}(0)}{f_{x_1}(67)}$
Example: In a box $\frac{1}{4}$ of coins are fake, $\frac{3}{4}$ of coins are real
The probability to get fake: $P_r[head] = \frac{1}{3}, P_r[tail] = \frac{2}{3}$
The probability to get real: $P_r[head] = \frac{1}{2}, P_r[tail] = \frac{2}{2}$
Take a random coin selected, n= 20 times, t= 7 heads, what is $P_r[real]$? what is $P_r[false]$?
$x_1$= number of heads in n= 20 trials

$$x_2 = \begin{cases} 0, \text{fake} & f_{x_2}(0) = \frac{1}{4} \\ 1, \text{real} & f_{x_2}(1) = \frac{3}{4} \end{cases}$$

$f(t_2 = 0|x_1 = 7, n = 20) = \frac{f_{x_1|x_2}(7|fake,n=20)*f_{x_2}(0)}{f_{x_1}(7)} = \binom{20}{7}\frac{1}{3}^7\frac{2}{3}^13 * 0.25 = 0.45$
$f(t_2 = 1|x_1 = 7, n = 20) = \frac{f_{x_1|x_2}(7|real,n=20)*f_{x_2}(1)}{f_{x_1}(7)}\binom{20}{7}\frac{1}{2}^7\frac{1}{2}^13 * 0.75 = 0.55$

### 4.2 Bayes Classfication

Loss$(\hat{f}, \underset{\sim}{x}|f), y = f(x), y = \hat{f}(x)$

Minimum $E_x : Loss(\hat{f}|f)$

Misclassification Rate: Loss$(\hat{f}, \underset{\sim}{x}|f) : \begin{cases} 0 & f_x = \hat{f}(x) \\ 1 & f_x \neq \hat{f}(x) \end{cases}$

Risk = $E_x$ Loss$(\hat{f}, \underset{\sim}{x}|f)$

Probability of Misclassification: $E_x = \sum_{t_i} t_i f_x(t_i) = \int_0^1 t \int_x (t) dt$

$x = \begin{cases} p & 0 \\ 1-p & 1 \end{cases}$

## 4.3 Bayes Classification Rule

Binary choose k $P_r[y = k, \underset{\sim}{x}]$

k= category= avemax$\frac{P_0[x|k]P_r[k]}{P_r[x]} \propto P_0[x|k]P_r[k]$

## 4.4 Classification Cost Matrix

$C_{ij}$= cost of classification a+ number of classification of i + number of classification of j

$f(x) = j$, $E_x(\text{loss}(j = \hat{j}, x|i) = \sum_{i=1}^{k} P_r[i|x]C_{ij} = \sum_{i=1}^{k} \frac{f(x|i)P_r[i]C_{ij}}{f_x} \propto \sum_{i=1}^{k} f(x|i)P_r[i]C_{ij}$

## 4.5 Modify Bayes Rule(Uneven Cost)

$c = avgmin \sum_{i=1}^{k} f(x|i)P_r[i]C_{ij}$

Example: Real and Fake: $C = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$

real:$P_r[x_2 = 1|x_1 = 7] * C_{11} + P_r[x_2 = 0|x_1 = 7] * C_{12}$

fake:$P_r[x_2 = 1|x_1 = 7] * C_{21} + P_r[x_2 = 0|x_1 = 7] * C_{22}$

$min : \begin{cases} 0.55 * 0 + 0.45 * 1 = 0.45 \\ 0.55 * 4 + 0.45 * 0 = 1.8 \end{cases}$