Algorithmic Learning Theory
Spring 2017
Lecture 2

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2. Random Variable and Distribution
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      ii. CRV, continuous random variable
   (b) Distribution function
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      ii. pdf or pmf, probability density(Mass) function
   (c) Discrete distribution
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3. Multivariate Distributions
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1 Review Bayes Theory(Lecture 1)

See notes in "Lecture 1".

2 Random Variable and Distribution

2.1 Random Variable

2.1.1 Discrete Random Variable(D.R.V)

$x \rightarrow t_1, t_2, \cdots, t_n$

2.1.2 Continuous Random Variable(C.R.V)

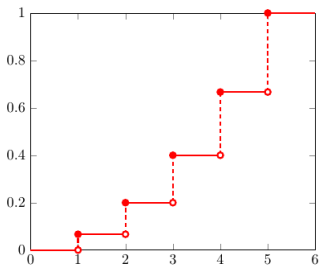
$x \rightarrow [a, b]$ a range of value.

2.2 Distribution Function

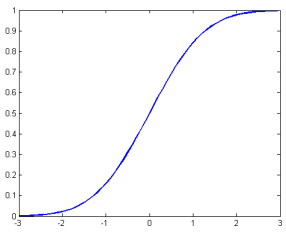
2.2.1 CDF:

Cumulative Distribution Function
$F_x(t) = P[x \leq t]$, probability can only increase

1. For Discrete:



2. For Continuous:

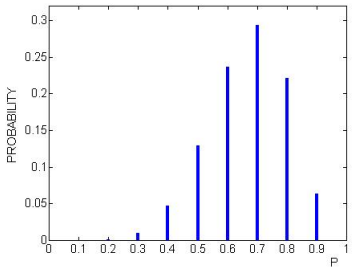


2.2.2 pdf or pmf:

Probability Density(Mass) Function

1. For Discrete:

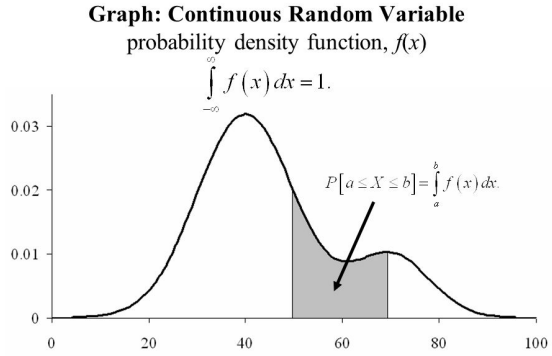
$$x : F_x(t) = P[x = t]$$



2. For Continuous:

$$f_x(t) = \frac{d}{dt}F_x(t),\ F_x(t) = \int_{-\infty}^{t} f_x(t)dt$$

i $f_x(t) \geq 0$
ii $\int_{-\infty}^{\infty} f_x(t)dt = 1$

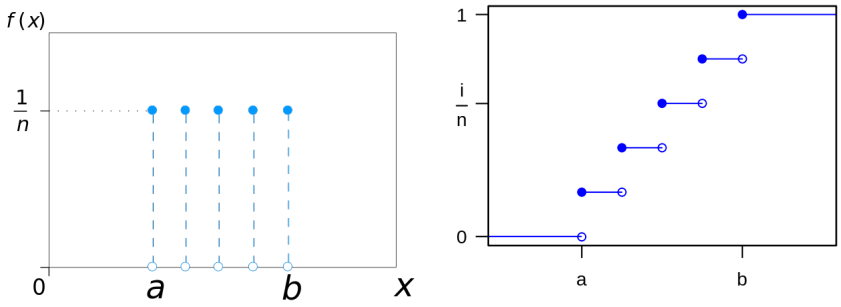


2.3 Discrete Distribution

2.3.1 Discrete Uniform Distribution

$$x : 1, 2, 3, \cdots, k$$

$$\text{pdf} : u_x(t) = \begin{cases} \frac{1}{n}, & \text{if } t = 1, 2, \cdots, n \\ 0, & \text{otherwise} \end{cases}$$



2.3.2 Bernoulli Distribution

$$\text{pdf}: f_x(t) = \begin{cases} p, & x = 1 \\ 1-p, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{CDF}: F_x(t) = \begin{cases} 0, & x \leqslant 0 \\ 1-p, & 0 \leqslant x < 1 \\ 0, & x \geqslant 1 \end{cases}$$

2.3.3 Binomial Distribution

numbers of 0's in independent Bernoulli trial with $P[0] = p$

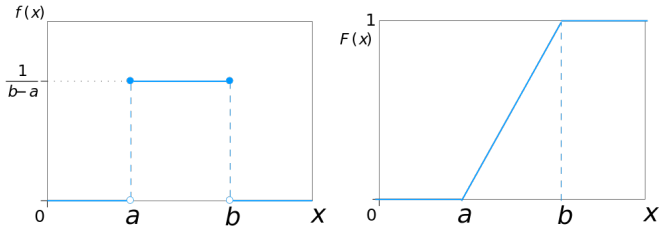
$$\text{pdf: } b(t|p,n) = \binom{n}{t} p^t (1-p)^{n-t}, \quad \binom{n}{t} = \frac{n!}{t!(n-t)!}$$

$$\text{CDF: } B(t|p,n) = \sum_{n=0}^{t} b(t|p,n)$$

2.4 Continuous distribution

2.4.1 Continuous uniform distribution

$$u(t|a,b) = \begin{cases} \frac{1}{b-a}, & a \leqslant t \leqslant b \\ 0, & \text{otherwise} \end{cases}$$



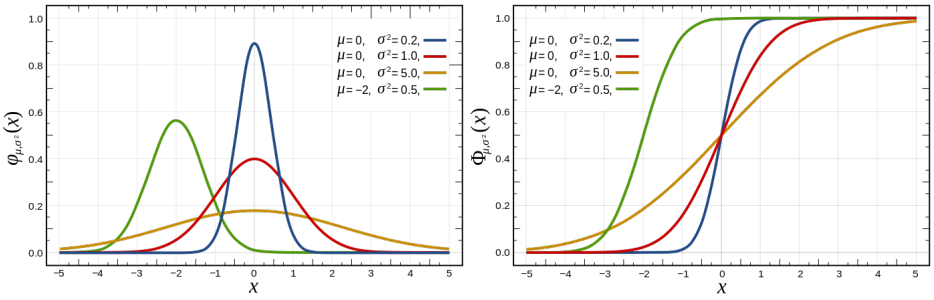
2.4.2 Normal distribution

mean= μ and std.= σ

$$\text{pdf: } f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\text{CDF: } \frac{1}{2}[1+\text{erf}(\frac{x-\mu}{\sigma\sqrt{2}})]$$

$$\text{erf}(x) = \frac{1}{\sqrt{\pi}}\int_{-x}^{x} e^{-t^2}\,dt$$



3 Multivariate Distributions

3.1 Random Vector

$X_N = (x_1, x_2, \cdots x_N)$ can be continuous or discrete.

$$\text{CDF: } F_x(t_1,t_2,\cdots t_n) = P[x_1 \leqslant t_1, x_2 \leqslant t_2, \cdots, x_n \leqslant t_n]$$

$$\text{pdf: } \begin{cases} \frac{\partial}{\partial t_1 \partial t_2 \partial t_3 \cdots \partial t_n} F(t_1,t_2,\cdots t_n) = f_x(t_1,\cdots,t_n), & \text{all continuous} \\ P[x_1 = t_1, x_2 = t_2, \cdots, x_n = t_n] = f_x(t_1,\cdots,t_n), & \text{all discrete} \end{cases}$$

both are joint distribution R.V. $x_1, \cdots, x_n$

3.2 Discrete Multivariate Distribution

$$Y \to 1,2,\cdots,r; \quad P[Y=r_1] = P_u; \quad \sum P_u = 1$$

repeat n times, x_k= number of times $Y = k$ occurs

$$\underset{\sim}{x} = (x_1,\cdots,x_n)$$

$$\text{pdf: } f_x(x_1,x_2,\cdots,x_n) = P[x_1=t_1,x_2=t_2,\cdots,x_n=t_n]\binom{n}{t_1,t_2,\cdots,t_n} {p_1}^{t_1}{p_2}^{t_2}\cdots{p_n}^{t_n}$$

$$\binom{n}{t_1,t_2,\cdots,t_r} = \frac{n!}{t_1!t_2!\cdots t_n!}$$

3.3 Binormal distribution

$$\underset{\sim}{x} = (x_1, x_2), \text{both continuous}$$

$$\underset{\sim}{\mu} = \begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}, \sigma_1^2 \to x_1, \sigma_2^2 \to x_2, \sigma_{21} \to x_1, x_2$$

$$\text{Covariance Matrix: } \Sigma = \begin{pmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{pmatrix}$$

$$\phi(t_1, t_2|\underset{\sim}{\mu}, \Sigma) = \frac{1}{\sqrt{2\pi \cdot Det(\Sigma)}} exp[(t - \underset{\sim}{\mu})^T \Sigma^{-1}(t - \underset{\sim}{\mu}), t = \begin{pmatrix}t_1\\ t_2\end{pmatrix}$$

$$\phi(t_1) = \int_{-\infty}^{\infty} \phi(t_1, t_2|\cdots)dt_2$$

Joint pdf: $f_{(x1,x2)}(t_1, t_2)$
In a similar way, for multi-normal distribution,

$$\Sigma = \begin{bmatrix}\sigma_1^2 & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1n}\\ \sigma_{12} & \sigma_2^2 & \cdots & \cdots & \sigma_{2n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \sigma_{n1} & \cdots & \cdots & \cdots & \sigma_n^n\end{bmatrix}$$

3.4 Marginal Distribution

$$\underset{\sim}{x} = (x_1, x_2, \cdots, x_n) \to f_{x_\sim}(t_1, \cdots, t_n)$$

To calculate pdf,

$$f_{x_\sim}(t_1, t_2, \cdots, t_n) = \int_{t_{k+1}, \cdots, t_n}^{\infty} f_{x_\sim}(t_1, \cdots, t_n)dt_{k+1}, \cdots, dt_n = \sum_{t_{k+1}} \sum_{t_{k+2}} \cdots \sum_{t_n} f_{x_\sim}(t_1, \cdots, t_n)$$

Joint pdf: $f(t_1, t_2)$

3.5 Conditional Distribution (2 Vars)

$$\underset{\sim}{x} = (x_1, x_2) \to \text{ joint:} f(t_1, t_2)$$

Conditional distribution means: $f_{x_1|x_2}(t_1|t_2 = a)$, a is a given constant, x_2 is given, fixed.

$$f_{x_1|x_2}(t_1|t_2) = \frac{f_{x_\sim}(t_1, t_2)}{f_{x_2}(t_2)}$$

By defination, $f_{x_\sim}(t_1, t_2)$ is joint distribution, $f_{x_2}(t_2)$ is marginal distribution. And, we have,

$$f_{x_\sim}(t_1, t_2) = f_{x_1|x_2}(t_1|t_2) f_{x_2}(t_2)$$

$$f_{x_2|x_1}(t_2|t_1) = \frac{f_{x_\sim}(t_1,t_2)}{f_{x_1}(t_1)}$$

$$f_{x_\sim}(t_1,t_2) = f_{x_2|x_1}(t_2|t_1)f_{x_1}(t_1)$$

So,

$$f_{x_1|x_2}(t_1|t_2) = \frac{f_{x_2|x_1}(t_2|t_1)f_{x_1}(t_1)}{f_{x_2}(t_2)} \rightarrow \text{Bayes Rule}$$

e.g.

x_2 \ x_1	1	2	3	$f_{x_2}(t)$
0	0.1	0.4	0.2	0.7
1	0.2	0.05	0.05	0.3
$f_{x_1}(t)$	0.3	0.45	0.45	

$$P[x_2=0] = P[x_2=0|x_1=1] + P[x_2=0|x_1=2] + P[x_2=0|x_1=3] = 0.7$$

$$P[x_2=1] = 1 - P[x_2=0] = 0.3$$

4 Bayes Classification

$$f_{x_1|x_2}(t_1|t_2) = \frac{f_{x_2|x_1}(t_2|t_1)f_{x_1}(t_1)}{f_{x_2}(t_2)} \rightarrow \text{Bayes Rule}$$

For discrete: $f_{x_2}(t_2) = \sum_{t_1} f_{x_2|x_1}(t_2|t_1)f_{x_1}(t_1)$

For continuous: $f_{x_2}(t_2) = \int_{-\infty}^{\infty} f_{x_2|x_1}(t_2|t_1)f_{y_1}(t_1)dt$

e.g. Height Supposed that: $x_1 \rightarrow$ Height, $x_2 \rightarrow$ Gender, $\begin{cases} 0 & \text{male} \\ 1 & \text{female} \end{cases}$

For $x_2 = 0 \rightarrow$ Height $\sim N(69, 4.5) \Leftrightarrow f_{x_1|x_2}(t_1|t_2=0)\phi(t_1|\phi=69, \sigma=4.5)$

For $x_2 = 1 \rightarrow$ Height $\sim N(65, 4.2) \Leftrightarrow f_{x_1|x_2}(t_1|t_2=1)\phi(t_1|\phi=65, \sigma=4.2)$

Marginal distribution of height for people:

$$\begin{aligned} f_{x_1}(t_1) &= f_{x_1|x_2}(t_1|t_2=0) * f_{x_2}(0) + f_{x_1|x_2}(t_1|t_2=1) * f_{x_2}(1) \\ &= \phi(t_1|69, 4.5) \cdot 0.5 + \phi(t_2|65, 4.2) * 0.5 \\ &= \phi(t|\frac{69+65}{2}, \sqrt{\frac{4.5^2+4.2^2}{2}}) \end{aligned}$$

A person has height 6'7", caculate the probability of each gender.

$$f(x_2=0|x_1=67) = \frac{f_{x_1|x_2}(67|x_2=0)f_{x_2}(0)}{f_{x_1}(67)} = \frac{\phi(67|69,4.5)\cdot 0.5}{f_{x_1}(67)}$$

$$f(x_2 = 1|x_1 = 67) = \frac{f_{x_1|x_2}(67|x_2 = 1)f_{x_2}(1)}{f_{x_1}(67)} = \frac{\phi(67|65, 4.2) \cdot 0.5}{f_{x_1}(67)}$$

$$\text{loss}(\hat{f}, \underset{\sim}{x}|f), \rightarrow \text{Minimum}E_x : \text{Loss}(\hat{f}|f)$$

Misclassification Rate: $\text{loss}(\hat{f}, \underset{\sim}{x}|f) : \begin{cases} 0, & f_x = \hat{f}(x) \\ 1, & f_x \neq \hat{f}(x) \end{cases}$, Risk $= E_x[\text{loss}(\hat{f}, \underset{\sim}{x}|f)]$

Probability of Misclassification: $E_x = \begin{cases} \sum_{t_i} t_i f_x(t_i), & \text{for discrete} \\ \int_0^1 t \int_x(t)dt, & \text{for continuous} \end{cases}$

Bayes Classification Rule for Binary: choose k that $P[y = k|\underset{\sim}{x}]$,

$$k = \underset{k}{\text{argmax}} \frac{P[x|k]P[k]}{P[x]} \propto P[x|k]P[k]$$

For cost matrix C, assumed that

$$C_{ij} = \text{the cost of classification } i\text{by wrong classified in } j$$

$$E_x(\text{loss}(j = \hat{f}(x)|i) = \sum_{i=1}^{k} P[i|x]C_{ij} = \sum_{i=1}^{k} \frac{f(x|i)P[i]C_{ij}}{f_x} \cdot C_{ij}$$

$$\propto \sum_{i=1}^{k} f(x|i)P[i]C_{ij}$$

Modified Bayes Rule: $c = \underset{j}{\text{argmin}} \sum_{i=1}^{k} f(x|i)P_r[i]C_{ij}$

e.g. Coins In a box, $\frac{1}{4}$ of coins are fake, $\frac{3}{4}$ of coins are real.

$$\text{For fake: } P[\text{head}] = \frac{1}{3}, P[\text{tail}] = \frac{2}{3}$$

$$\text{For real: } P[\text{head}] = \frac{1}{2}, P[\text{tail}] = \frac{1}{2}$$

Take a random coin selected, n= 20 times, t= 7 heads, what is $P[\text{real}]$ and $P[\text{false}]$?

x_1 = number of heads in n= 20 trials.

$$x_2 = \begin{cases} 0, & \text{fake} \\ 1, & \text{real} \end{cases}$$

So, $f_{x_2}(0) = \frac{1}{4}, f_{x_2}(1) = \frac{3}{4}$

$$f_{x_2|x_1}(t_2 = 0|x_1 = 7, n = 20) = \frac{f_{x_1|x_2}(7|\text{fake}, n = 20)f_{x_2}(0)}{f_{x_1}(7)} = \frac{\binom{20}{7}(\frac{1}{3})^7(\frac{2}{3})^{13} * 0.25}{f_{x_1}(7)} = \frac{0.45}{f_{x_1}(7)}$$

$$f_{x_2|x_1}(t_2 = 1|x_1 = 7, n = 20) = \frac{f_{x_1|x_2}(7|real, n = 20)f_{x_2}(1)}{f_{x_1}(7)} = \frac{\binom{20}{7}(\frac{1}{2})^7(\frac{1}{2})^{13} * 0.75}{f_{x_1}(7)} = \frac{0.55}{f_{x_1}(7)}$$

$$f_{x_1}(7) = f_{x_1|x_2}(7|fake)f_{x_2}(0) + f_{x_1|x_2}(7|real)f_{x_2}(1)$$

Cost matrix for real and fake: $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$

For real:$P[x_2 = 1|x_1 = 7]C_{11} + P[x_2 = 0|x_1 = 7]C_{12}$

For fake:$P[x_2 = 1|x_1 = 7]C_{21} + P[x_2 = 0|x_1 = 7]C_{22}$