Algorithmic Learning Theory
Spring 2017
Lecture 2

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1. Review Bayes Theory(Lecture 1)
2. Random Variable and Distribution
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      i. DRV, discrete random variable
      ii. CRV, continuous random variable
   (b) Distribution function
      i. CDF, cumulative distribution function
      ii. pdf or pmf, probability density(Mass) function
   (c) Discrete distribution
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3. Multivariate Distributions
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4. Bayes Classification

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1 Review Bayes Theory(Lecture 1)

See notes in "Lecture 1".

2 Random Variable and Distribution

2.1 Random Variable

2.1.1 Discrete Random Variable(D.R.V)

$x \to t_1, t_2, \cdots, t_n$

2.1.2 Continuous Random Variable(C.R.V)

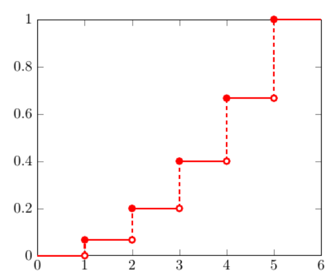
$x \to [a, b]$ a range of value.

2.2 Distribution Function

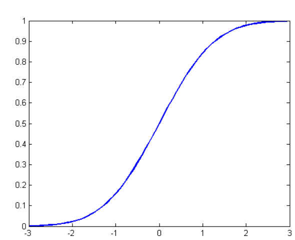
2.2.1 CDF:

Cumulative Distribution Function
$F_x(t) = P[x \leq t]$, probability can only increase

1. For Discrete:



2. For Continuous:

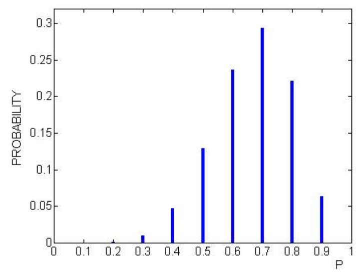


2.2.2 pdf or pmf:

Probability Density(Mass) Function

1. For Discrete:

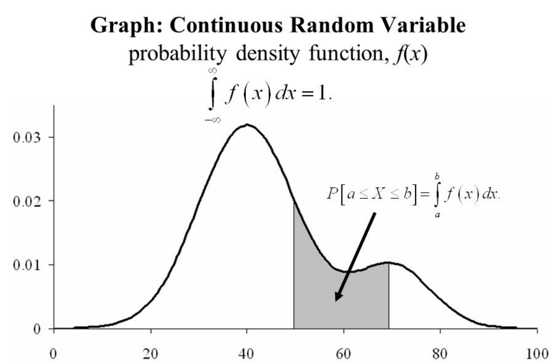
$$x : F_x(t) = P[x = t]$$



2. For Continuous:

$$f_x(t) = \frac{d}{dt}F_x(t), \; F_x(t) = \int_{-\infty}^{t} f_x(t)dt$$

i $f_x(t) \geq 0$
ii $\int_{-\infty}^{\infty} f_x(t)dt = 1$

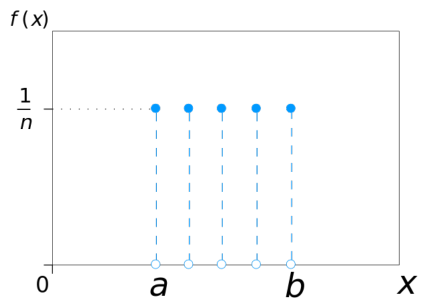


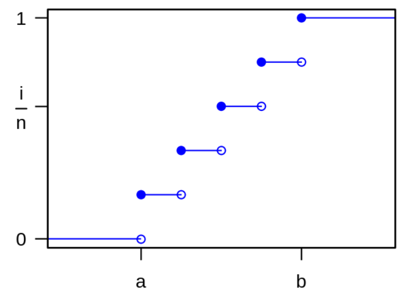
2.3 Discrete Distribution

2.3.1 Discrete Uniform Distribution

$$x : 1, 2, 3, \cdots, k$$

$$\mathrm{pdf} : u_x(t) = \begin{cases} \frac{1}{n}, & \text{if } t = 1, 2, \cdots, n \\ 0, & \text{otherwise} \end{cases}$$





2.3.2 Bernoulli Distribution

$$\text{pdf}: f_x(t) = \begin{cases} p, & x = 1 \\ 1-p, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{CDF}: F_x(t) = \begin{cases} 0, & x \leqslant 0 \\ 1-p, & 0 \leqslant x < 1 \\ 0, & x \geqslant 1 \end{cases}$$

2.3.3 Binomial Distribution

numbers of 0's in independent Bernoulli trial with $P[0] = p$

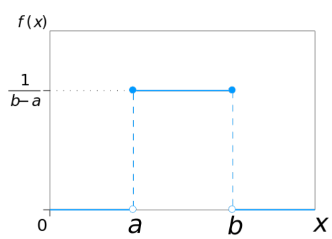
$$\text{pdf: } b(t|p,n) = \binom{n}{t} p^t (1-p)^{n-t}, \ \binom{n}{t} = \frac{n!}{t!(n-t)!}$$

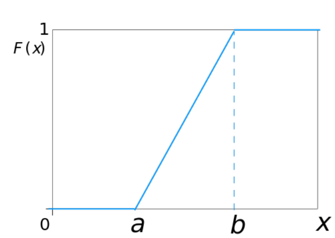
$$\text{CDF: } B(t|p,n) = \sum_{n=0}^{t} b(t|p,n)$$

2.4 Continuous distribution

2.4.1 Continuous uniform distribution

$$u(t|a,b) = \begin{cases} \frac{1}{b-a}, & a \leqslant t \leqslant b \\ 0, & \text{otherwise} \end{cases}$$





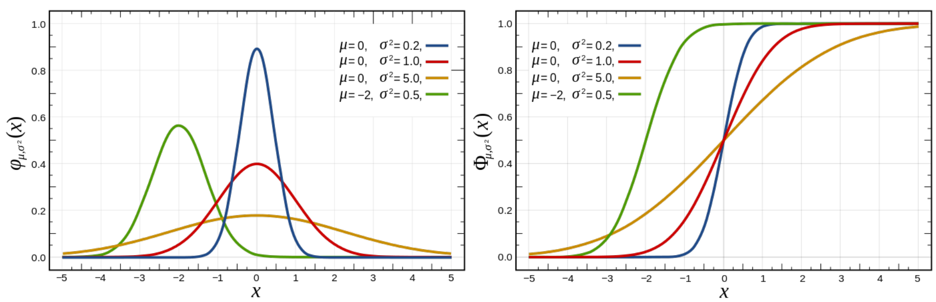
2.4.2 Normal distribution

mean$= \mu$ and std.$= \sigma$

$$\text{pdf: } f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\text{CDF: } \frac{1}{2}[1 + \mathrm{erf}(\frac{x-\mu}{\sigma\sqrt{2}})]$$

$$\mathrm{erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} e^{-t^2} dt$$



3 Multivariate Distributions

3.1 Random Vector

$X_N = (x_1, x_2, \cdots x_N)$ can be continuous or discrete.

$$\text{CDF: } F_x(t_1, t_2, \cdots t_n) = P[x_1 \leqslant t_1, x_2 \leqslant t_2, \cdots, x_n \leqslant t_n]$$

$$\text{pdf: } \begin{cases} \frac{\partial}{\partial t_1 \partial t_2 \partial t_3 \cdots \partial t_n} F(t_1, t_2, \cdots t_n) = f_x(t_1, \cdots, t_n), & \text{all continuous} \\ P[x_1 = t_1, x_2 = t_2, \cdots, x_n = t_n] = f_x(t_1, \cdots, t_n), & \text{all discrete} \end{cases}$$

both are joint distribution R.V. $x_1, \cdots, x_n$

3.2 Discrete Multivariate Distribution

$$Y \to 1, 2, \cdots, r; \quad P[Y = r_1] = P_u; \quad \sum P_u = 1$$

repeat n times, x_k= number of times $Y = k$ occurs

$$\underset{\sim}{x} = (x_1, \cdots, x_n)$$

$$\text{pdf: } f_x(x_1, x_2, \cdots, x_n) = P[x_1 = t_1, x_2 = t_2, \cdots, x_n = t_n] \binom{n}{t_1, t_2, \cdots, t_n} {p_1}^{t_1} {p_2}^{t_2} \cdots {p_n}^{t_n}$$

$$\binom{n}{t_1, t_2, \cdots, t_r} = \frac{n!}{t_1! t_2! \cdots t_n!}$$

3.3 Binormal distribution

$$\underset{\sim}{x} = (x_1, x_2), \text{both continuous}$$

$$\underset{\sim}{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \sigma_1^2 \to x_1, \sigma_2^2 \to x_2, \sigma_{21} \to x_1, x_2$$

$$\text{Covariance Matrix: } \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

$$\phi(t_1, t_2 | \underset{\sim}{\mu}, \Sigma) = \frac{1}{\sqrt{2\pi \cdot \mathrm{Det}(\Sigma)}} \exp[(t - \underset{\sim}{\mu})^T \Sigma^{-1} (t - \underset{\sim}{\mu}), t = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

$$\phi(t_1) = \int_{-\infty}^{\infty} \phi(t_1, t_2 | \cdots) dt_2$$

Joint pdf: $f_{(x1,x2)}(t_1, t_2)$
In a similar way, for multi-normal distribution,

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \cdots & \cdots & \sigma_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \cdots & \cdots & \sigma_n^n \end{bmatrix}$$

3.4 Marginal Distribution

$$\underset{\sim}{x} = (x_1, x_2, \cdots, x_n) \to f_{\underset{\sim}{x}}(t_1, \cdots, t_n)$$

To calculate pdf,

$$f_{\underset{\sim}{x}}(t_1, t_2, \cdots, t_n) = \int_{t_{k+1}, \cdots, t_n}^{\infty} f_{\underset{\sim}{x}}(t_1, \cdots, t_n) dt_{k+1}, \cdots, dt_n = \sum_{t_{k+1}} \sum_{t_{k+2}} \cdots \sum_{t_n} f_{\underset{\sim}{x}}(t_1, \cdots, t_n)$$

Joint pdf: $f(t_1, t_2)$

3.5 Conditional Distribution (2 Vars)

$$\underset{\sim}{x} = (x_1, x_2) \to \text{ joint:} f(t_1, t_2)$$

Conditional distribution means: $f_{x_1|x_2}(t_1|t_2 = a)$, a is a given constant, x_2 is given, fixed.

$$f_{x_1|x_2}(t_1|t_2) = \frac{f_{\underset{\sim}{x}}(t_1, t_2)}{f_{x_2}(t_2)}$$

By defination, $f_{\underset{\sim}{x}}(t_1, t_2)$ is joint distribution, $f_{x_2}(t_2)$ is marginal distribution. And, we have,

$$f_{\underset{\sim}{x}}(t_1, t_2) = f_{x_1|x_2}(t_1|t_2) f_{x_2}(t_2)$$

$$f_{x_2|x_1}(t_2|t_1) = \frac{f_{\underset{\sim}{x}}(t_1, t_2)}{f_{x_1}(t_1)}$$

$$f_{\underset{\sim}{x}}(t_1, t_2) = f_{x_2|x_1}(t_2|t_1) f_{x_1}(t_1)$$

So,

$$f_{x_1|x_2}(t_1|t_2) = \frac{f_{x_2|x_1}(t_2|t_1) f_{x_1}(t_1)}{f_{x_2}(t_2)} \longrightarrow \text{Bayes Rule}$$

e.g.

x_2 \ x_1	1	2	3	$f_{x_2}(t)$
0	0.1	0.4	0.2	0.7
1	0.2	0.05	0.05	0.3
$f_{x_1}(t)$	0.3	0.45	0.45	

$$P[x_2 = 0] = P[x_2 = 0|x_1 = 1] + P[x_2 = 0|x_1 = 2] + P[x_2 = 0|x_1 = 3] = 0.7$$

$$P[x_2 = 1] = 1 - P[x_2 = 0] = 0.3$$

4 Bayes Classification

$$f_{x_1|x_2}(t_1|t_2) = \frac{f_{x_2|x_1}(t_2|t_1) f_{x_1}(t_1)}{f_{x_2}(t_2)} \longrightarrow \text{Bayes Rule}$$

For discrete: $f_{x_2}(t_2) = \sum_{t_1} f_{x_2|x_1}(t_2|t_1) f_{x_1}(t_1)$

For continuous: $f_{x_2}(t_2) = \int_{-\infty}^{\infty} f_{x_2|x_1}(t_2|t_1) f_{y_1}(t_1) dt$

e.g. Height Supposed that: $x_1 \longrightarrow$ Height, $x_2 \longrightarrow$ Gender, $\begin{cases} 0 & \text{male} \\ 1 & \text{female} \end{cases}$

For $x_2 = 0 \longrightarrow$ Height $\sim N(69, 4.5) \Leftrightarrow f_{x_1|x_2}(t_1|t_2 = 0)\phi(t_1|\phi = 69, \sigma = 4.5)$

For $x_2 = 1 \longrightarrow$ Height $\sim N(65, 4.2) \Leftrightarrow f_{x_1|x_2}(t_1|t_2 = 1)\phi(t_1|\phi = 65, \sigma = 4.2)$

Marginal distribution of height for people:

$$f_{x_1}(t_1) = f_{x_1|x_2}(t_1|t_2 = 0) * f_{x_2}(0) + f_{x_1|x_2}(t_1|t_2 = 1) * f_{x_2}(1)$$
$$= \phi(t_1|69, 4.5) \cdot 0.5 + \phi(t_2|65, 4.2) * 0.5$$
$$= \phi(t|\frac{69 + 65}{2}, \sqrt{\frac{4.5^2 + 4.2^2}{2}})$$

A person has height 6'7", caculate the probability of each gender.

$$f(x_2 = 0|x_1 = 67) = \frac{f_{x_1|x_2}(67|x_2 = 0) f_{x_2}(0)}{f_{x_1}(67)} = \frac{\phi(67|69, 4.5) \cdot 0.5}{f_{x_1}(67)}$$

$$f(x_2 = 1|x_1 = 67) = \frac{f_{x_1|x_2}(67|x_2 = 1)f_{x_2}(1)}{f_{x_1}(67)} = \frac{\phi(67|65, 4.2) \cdot 0.5}{f_{x_1}(67)}$$

$$\text{Loss}(\hat{f}, \underset{\sim}{x}|f), \rightarrow \text{Minimum} E_x : Loss(\hat{f}|f)$$

Misclassification Rate: Loss$(\hat{f}, \underset{\sim}{x}|f) : \begin{cases} 0, & f_x = \hat{f}(x) \\ 1, & f_x \neq \hat{f}(x) \end{cases}$, Risk $= E_x[Loss(\hat{f}, \underset{\sim}{x}|f)]$

Probability of Misclassification: $E_x = \begin{cases} \sum_{t_i} t_i f_x(t_i), & \text{for discrete} \\ \int_0^1 t \int_x (t)dt, & \text{for continuous} \end{cases}$

Bayes Classification Rule for Binary: choose k that $P[y = k|\underset{\sim}{x}]$,

$$k = \underset{k}{\text{argmax}} \frac{P[x|k]P[k]}{P[x]} \propto P[x|k]P[k]$$

For cost matrix C, assumed that

$$C_{ij} = \text{the cost of classification } i \text{by wrong classified in } j$$

$$E_x(loss(j = \hat{f}(x)|i) = \sum_{i=1}^{k} P[i|x]C_{ij} = \sum_{i=1}^{k} \frac{f(x|i)P[i]C_{ij}}{f_x} \cdot C_{ij}$$

$$\propto \sum_{i=1}^{k} f(x|i)P[i]C_{ij}$$

us

Example: In a box $\frac{1}{4}$ of coins are fake, $\frac{3}{4}$ of coins are real
The probability to get fake: $P_r[head] = \frac{1}{3}, P_r[tail] = \frac{2}{3}$
The probability to get real: $P_r[head] = \frac{1}{2}, P_r[tail] = \frac{2}{2}$
Take a random coin selected, n= 20 times, t= 7 heads, what is $P_r[real]$? what is $P_r[false]$?
x_1= number of heads in n= 20 trials
$x_2 = \begin{cases} 0, \text{fake} & f_{x_2}(0) = \frac{1}{4} \\ 1, \text{real} & f_{x_2}(1) = \frac{3}{4} \end{cases}$
$f(t_2 = 0|x_1 = 7, n = 20) = \frac{f_{x_1|x_2}(7|fake, n=20) * f_{x_2}(0)}{f_{x_1}(7)} = \binom{20}{7} \frac{1}{3}^7 \frac{2}{3}^1 3 * 0.25 = 0.45$
$f(t_2 = 1|x_1 = 7, n = 20) = \frac{f_{x_1|x_2}(7|real, n=20) * f_{x_2}(1)}{f_{x_1}(7)} \binom{20}{7} \frac{1}{2}^7 \frac{1}{2}^1 3 * 0.75 = 0.55$

4.1 Bayes Classfication

4.2 Modify Bayes Rule(Uneven Cost)

$c = avgmin \sum_{i=1}^{k} f(x|i)P_r[i]C_{ij}$
Example: Real and Fake: $C = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$

real:$P_r[x_2 = 1|x_1 = 7] * C_{11} + P_r[x_2 = 0|x_1 = 7] * C_{12}$
fake:$P_r[x_2 = 1|x_1 = 7] * C_{21} + P_r[x_2 = 0|x_1 = 7] * C_{22}$

$$min : \begin{cases} 0.55 * 0 + 0.45 * 1 = 0.45 \\ 0.55 * 4 + 0.45 * 0 = 1.8 \end{cases}$$