

# Algorithmic Learning Theory

## Spring 2017

### Lecture 2

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1. Review Bayes Theory(Lecture 1)
2. Random Variable and Distribution
  - (a) Random variable
    - i. DRV, discrete random variable
    - ii. CRV, continuous random variable
  - (b) Distribution function
    - i. CDF, cumulative distribution function
    - ii. pdf or pmf, probability density(Mass) function
  - (c) Discrete distribution
    - i. Discrete uniform distribution
    - ii. Beunoulli's distribution
    - iii. Binomial distribution
  - (d) Continuous distribution
    - i. Continuous uniform distribution
    - ii. Normal distribution
3. Multivariate Distributions
  - (a) Random vector
  - (b) Discrete multivariate distribution
  - (c) Binormal distribution
  - (d) Marginal distribution
  - (e) Conditional distribution
4. Bayes Classification

## 1 Review Bayes Theory(Lecture 1)

See notes in "Lecture 1".

## 2 Random Variable and Distribution

### 2.1 Random Variable

#### 2.1.1 Discrete Random Variable(D.R.V)

$x \rightarrow t_1, t_2, \dots, t_n$

#### 2.1.2 Continuous Random Variable(C.R.V)

$x \rightarrow [a, b]$  a range of value.

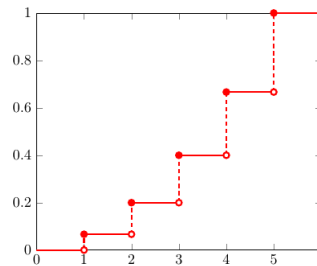
### 2.2 Distribution Function

#### 2.2.1 CDF:

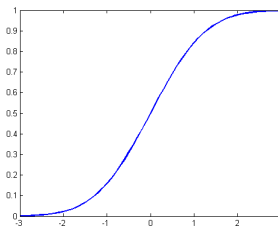
Cumulative Distribution Function

$F_x(t) = P_r[x \leq t]$ , probability can only increase

1. For Discrete:



2. For Continuous:

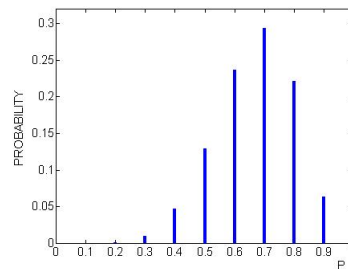


### 2.2.2 pdf or pmf:

Probability Density(Mass) Function

1. For Discrete:

$$x : F_x(t) = P_r[x = t]$$

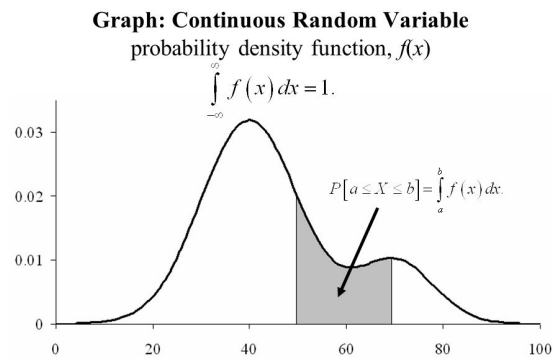


2. For Continuous:

$$f_x(t) = \frac{d}{dt} F_x(t), F_x(t) = \int_{-\infty}^t f_x(t) dt$$

$$i \ f_x(t) \geq 0$$

$$ii \ \int_{-\infty}^{\infty} f_x(t) dt = 1$$

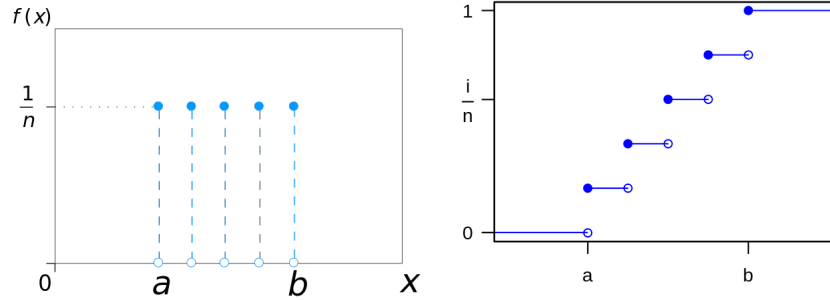


## 2.3 Discrete Distribution

### 2.3.1 Discrete Uniform Distribution

$$x : 1, 2, 3, \dots, k$$

$$pdf : u_x(t) = \begin{cases} \frac{1}{n}, & \text{if } t = 1, 2, \dots, n \\ 0, & \text{otherwise} \end{cases}$$



### 2.3.2 Bernoulli Distribution

$$\text{pdf: } f_x(t) = \begin{cases} p, & x = 1 \\ 1 - p, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{CDF: } F_x(t) = \begin{cases} 0, & x \leq 0 \\ 1 - p, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

### 2.3.3 Binomial Distribution

numbers of 0's in independent Bernoulli trial with  $P[0] = p$

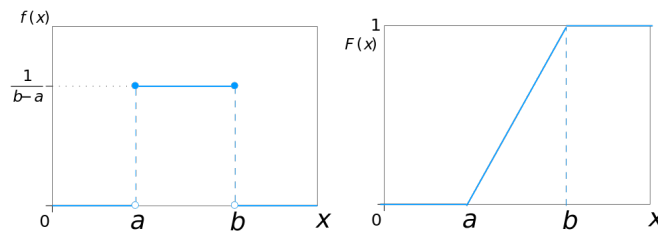
$$\text{pdf: } b(t|p, n) = \binom{n}{t} p^t (1-p)^{n-t}, \quad \binom{n}{t} = \frac{n!}{t!(n-t)!}$$

$$\text{CDF: } B(t|p, n) = \sum_{n=0}^t b(t|p, n)$$

## 2.4 Continuous distribution

### 2.4.1 Continuous uniform distribution

$$u(t|a, b) = \begin{cases} \frac{1}{b-a}, & a \leq t \leq b \\ 0, & \text{otherwise} \end{cases}$$



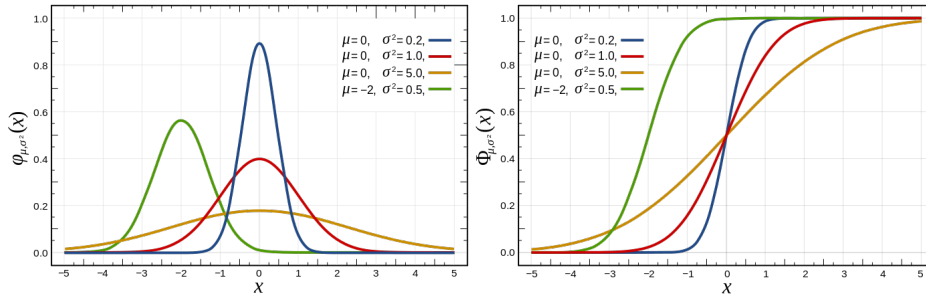
### 2.4.2 Normal distribution

mean =  $\mu$  and std. =  $\sigma$

$$\text{pdf: } f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\text{CDF: } \frac{1}{2} [1 + \text{erf}(\frac{x-\mu}{\sigma\sqrt{2}})]$$

$$\text{erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^x e^{-t^2} dt$$



## 3 Multivariate Distributions

### 3.1 Random Vector

$X_N = (x_1, x_2, \dots, x_N)$  can be continuous or discrete.

$$\text{CDF: } F_x(t_1, t_2, \dots, t_n) = P_r[x_1 \leq t_1, x_2 \leq t_2, \dots, x_n \leq t_n]$$

$$\text{pdf: } \begin{cases} \frac{\partial}{\partial t_1 \partial t_2 \partial t_3 \dots \partial t_n} F(t_1, t_2, \dots, t_n) = f_x(t_1, \dots, t_n), & \text{all continuous} \\ P_r[x_1 = t_1, x_2 = t_2, \dots, x_n = t_n] = f_x(t_1, \dots, t_n), & \text{all discrete} \end{cases}$$

both are joint distribution R.V.  $x_1, \dots, x_n$

### 3.2 Discrete Multivariate Distribution

$$Y \rightarrow 1, 2, \dots, r; \quad P[Y = r_1] = P_u; \quad \sum P_u = 1$$

repeat  $n$  times,  $x_k$  = number of times  $Y = k$  occurs

$$\underline{x} = (x_1, \dots, x_n)$$

$$\text{pdf: } f_x(x_1, x_2, \dots, x_n) = P[x_1 = t_1, x_2 = t_2, \dots, x_n = t_n] \binom{n}{t_1, t_2, \dots, t_n} p_1^{t_1} p_2^{t_2} \dots p_n^{t_n}$$

### 3.3 Multinomial Distribution

P.D.F:  $f_{\underline{x}}(t_1, t_2, \dots, t_r) = P_r[x_1 = t_1, x_2 = t_2, \dots, x_n = t_n] = \binom{n}{t_1, t_2, \dots, t_r} p_1^{t_1} p_2^{t_2} \dots p_r^{t_r}$

$\binom{n}{t_1, t_2, \dots, t_r} = \frac{n!}{t_1! t_2! \dots t_r!}$   
 $\underline{x} = (x_1, x_2)$  both continuous,  $\underline{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$

$\sigma_1^2 \rightarrow x_1, \sigma_2^2 \rightarrow x_2, \sigma_{12} \rightarrow x_1 x_2$

Covariance Matrix:  $\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$

$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \dots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \dots & \dots & \sigma_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{1m} & \dots & \dots & \dots & \sigma_{1n} \end{bmatrix}$

$\phi(t_1, t_2 | \begin{pmatrix} \mu \\ \mu \end{pmatrix}, \Sigma) = \frac{1}{\sqrt{2\pi \text{Det}(\Sigma)}} \exp[(t - \underline{\mu})^T \Sigma^{-1} (t - \underline{\mu})], t = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$

		$x_2$			
		1	2	3	$f_{x_2}(t)$
joint p.d.f. $f_{(x_1, x_2)}(t_1, t_2)$	$x_1$	0.1	0.4	0.2	0.7
	0	0.2	0.05	0.05	0.3
	1	0.3	0.45	0.45	

(1):  $\frac{0.1}{0.7} = \frac{1}{7}$

$f_{x_1|x_2=0}$  in (2):  $\frac{0.4}{0.7} = \frac{4}{7}$

$f_{x_1|x_2=0}$  in (3):  $\frac{0.2}{0.7} = \frac{2}{7}$

### 3.4 Marginal Distribution

$\underline{x} = (x_1, x_2, \dots, x_n) \rightarrow f_{\underline{x}}(t_1, \dots, t_n)$

p.d.f.:  $(x_1, x_2, \dots, x_n) = \bar{x}$

$f_{\bar{x}}(t_1, t_2, \dots, t_n) = \int_{t_{k+1}, \dots, t_n} f_{\underline{x}}(t_1, \dots, t_n) dt_{k+1}, \dots, t_n = \sum_{t_{k+1}} \sum_{t_{k+2}} \dots \sum_{t_n} f_{\underline{x}}(t_1, \dots, t_n)$

$P_r[x = 0] = P_r[x_2 = 0, x_1 = 1] + P_r[x_2 = 0, x_1 = 2] + P_r[x_2 = 0, x_1 = 3]$

$P_r[x = 1] = P_r[x_2 = 1, x_1 = 1] + P_r[x_2 = 1, x_1 = 2] + P_r[x_2 = 1, x_1 = 3]$

$\phi(t_1, t_2 | \underline{\mu}, \Sigma)$

$\phi(t_1) = \int_{-\infty}^{\infty} \phi(t_1, t_2 | \dots) dt_2$

Find p.d.f.  $f(t_1, t_2)$

Example:  $x_1 \rightarrow \text{Height}, x_2 \rightarrow \text{Gender} \begin{cases} 0 & \text{male} \\ 1 & \text{female} \end{cases}$

for  $x_2 = 0 \rightarrow \text{Height} \sim N(69, 4.5) \Leftrightarrow f_{x_1|x_2}(t_1|t_2 = 0)\phi(t_1|\mu = 69, \sigma = 4.5)$

for  $x_2 = 1 \rightarrow \text{Height} \sim N(65, 4.2) \Leftrightarrow f_{x_1|x_2}(t_1|t_2 = 1)\phi(t_1|\mu = 65, \sigma = 4.2)$

marginal distribution of height for people

$f_{x_1}(t_1) = f_{x_1|x_2}(t_1|t_2 = 0) * f_{x_2}(0) + f_{x_1|x_2}(t_1|t_2 = 1) * f_{x_2}(1) = \phi(t_1|69, 4.5) *$

$0.5 + \phi(t_2|65, 4.2) * 0.5 = \phi(t_1|\frac{69+65}{2}, \sqrt{\frac{4.5^2+4.2^2}{2}})$

### 3.5 Conditional Distribution in 2 Variables

$$\begin{aligned}x &= (x_1, x_2) \rightarrow \text{joint } f(t_1, t_2) \\f_{x_1|x_2}(t_1|t_2) &= \frac{f_x(t_1, t_2)}{f_{x_2}(t_2)} \\f_x(t_1, t_2) &= f_{x_1|x_2}(t_1|t_2) * f_{x_2}(t_2) \\f_{x_2|x_1}(t_2|t_1) &= \frac{f_x(t_1, t_2)}{f_{x_1}(t_1)} \\f_x(t_1, t_2) &= f_{x_2|x_1}(t_2|t_1) * f_{x_1}(t_1)\end{aligned}$$

## 4 Bayes

### 4.1 Bayes Formula

$$\begin{aligned}f_{x_1|x_2}(t_1|t_2) &= \frac{(f_{x_2|x_1}(t_2|t_1) * f_{x_1}(t_1))}{f_{x_2}(t_2)} \\ \text{Discrete } x_1 : f_{x_2}(t_2) &= \sum_{t_1} f_{x_2|x_1}(t_2|t_1) * f_{x_1}(t_1) \\ \text{Continuous } x_1 : f_{x_2}(t_2) &= \int_{-\infty}^{\infty} f_{x_2|x_1}(t_2|t_1) * f_{x_1}(t_1) dt \\ \text{Example: A person has height } 6'7'' & \\ f(x_2 = 0|x_1 = 6'7'') &= \frac{f_{x_1|x_2}(6'7''|x_2=0) * f_{x_2}(0)}{f_{x_1}(6'7'')} \\ f(x_2 = 1|x_1 = 6'7'') &= \frac{f_{x_1|x_2}(6'7''|x_2=1) * f_{x_2}(1)}{f_{x_1}(6'7'')}\end{aligned}$$

Example: In a box  $\frac{1}{4}$  of coins are fake,  $\frac{3}{4}$  of coins are real

The probability to get fake:  $P_r[\text{head}] = \frac{1}{3}, P_r[\text{tail}] = \frac{2}{3}$

The probability to get real:  $P_r[\text{head}] = \frac{1}{2}, P_r[\text{tail}] = \frac{1}{2}$

Take a random coin selected,  $n = 20$  times,  $t = 7$  heads, what is  $P_r[\text{real}]$ ? what is  $P_r[\text{false}]$ ?

$x_1$  = number of heads in  $n = 20$  trials

$$x_2 = \begin{cases} 0, \text{fake} & f_{x_2}(0) = \frac{1}{4} \\ 1, \text{real} & f_{x_2}(1) = \frac{3}{4} \end{cases}$$

$$\begin{aligned}f(t_2 = 0|x_1 = 7, n = 20) &= \frac{f_{x_1|x_2}(7|\text{fake}, n=20) * f_{x_2}(0)}{f_{x_1}(7)} = \binom{20}{7} \left(\frac{1}{3}\right)^7 \left(\frac{2}{3}\right)^{13} * 0.25 = 0.45 \\ f(t_2 = 1|x_1 = 7, n = 20) &= \frac{f_{x_1|x_2}(7|\text{real}, n=20) * f_{x_2}(1)}{f_{x_1}(7)} = \binom{20}{7} \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^{13} * 0.75 = 0.55\end{aligned}$$

### 4.2 Bayes Classification

$$\text{Loss}(\hat{f}, x|f), y = f(x), y = \hat{f}(x)$$

$$\text{Minimum } E_x : \text{Loss}(\hat{f}|f)$$

$$\text{Misclassification Rate: } \text{Loss}(\hat{f}, x|f) : \begin{cases} 0 & f_x = \hat{f}(x) \\ 1 & f_x \neq \hat{f}(x) \end{cases}$$

$$\text{Risk} = E_x \text{ Loss}(\hat{f}, x|f)$$

$$\text{Probability of Misclassification: } E_x = \sum_{t_i} t_i f_x(t_i) = \int_0^1 t \int_x(t) dt$$

$$x = \begin{cases} p & 0 \\ 1-p & 1 \end{cases}$$

### 4.3 Bayes Classification Rule

Binary choose  $k$   $P_r[y = k, x]$

$$k = \text{category} = \text{argmax} \frac{P_0[x|k]P_r[k]}{P_r[x]} \propto P_0[x|k]P_r[k]$$

### 4.4 Classification Cost Matrix

$C_{ij}$  = cost of classification  $a$  + number of classification of  $i$  + number of classification of  $j$

$$f(x) = j, E_x(\text{loss}(j = \hat{j}, x|i) = \sum_{i=1}^k P_r[i|x]C_{ij} = \sum_{i=1}^k \frac{f(x|i)P_r[i]C_{ij}}{f_x} \propto \sum_{i=1}^k f(x|i)P_r[i]C_{ij}$$

### 4.5 Modify Bayes Rule (Uneven Cost)

$$c = \text{avgmin} \sum_{i=1}^k f(x|i)P_r[i]C_{ij}$$

$$\text{Example: Real and Fake: } C = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$$

$$\text{real: } P_r[x_2 = 1|x_1 = 7] * C_{11} + P_r[x_2 = 0|x_1 = 7] * C_{12}$$

$$\text{fake: } P_r[x_2 = 1|x_1 = 7] * C_{21} + P_r[x_2 = 0|x_1 = 7] * C_{22}$$

$$\min : \begin{cases} 0.55 * 0 + 0.45 * 1 = 0.45 \\ 0.55 * 4 + 0.45 * 0 = 1.8 \end{cases}$$