

## Question 1

a.

This defines an independence system  $(E, \mathcal{F}), \mathcal{F} \subseteq 2^E$

First, prove that  $\emptyset \in \mathcal{F}$ .

Obviously, we have  $\emptyset \subseteq E = \{1, 2, \dots, n\}$ , and  $a(\emptyset) = \sum_{j \in \emptyset} a_j = 0 \leq b \in \mathbb{R}_+$   
So,  $\emptyset \in \mathcal{F}$ .

Second, prove that  $X \subseteq Y \in \mathcal{F} \Rightarrow X \in \mathcal{F}$

$$Y \in \mathcal{F} \Rightarrow a(Y) = \sum_{j \in Y} a_j \leq b$$

$$X \subseteq Y \Rightarrow a(X) = \sum_{j \in X} a_j = \sum_{i \in Y} a_i - \sum_{j \in Y \setminus X} a_j \leq \sum_{j \in Y} a_j \leq b$$

So, this defines an independence system  $(E, \mathcal{F}), \mathcal{F} \subseteq 2^E$

b.

According to the definition, we have

$$n = 6 \Rightarrow S \subseteq E = \{1, 2, 3, 4, 5, 6\}$$

$$a = (1, 1, 1, 4, 4, 5), b = 8 \Rightarrow a(S) = \sum_{j \in S} a_j \leq 8$$

According to the definition of *rank*, we have

$$r(X) = \max_{F \in \mathcal{F}} |F \cap X| = \max_{B \in \mathcal{B}_X} |B|, \text{ and } \rho(X) = \min_{B \in \mathcal{B}_X} |B|$$

We can find that

$$\mathcal{B}_X = \{\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 3, 6\}, \{4, 5\}\}$$

So that we have  $r(X) = 4, \rho(X) = 2$

c.

$$O(r(X)) = O(n \log n)$$

As the greedy algorithm, order the  $a$  from small to large, use the *Best-in Greedy* to select numbers from the smallest. So the complexity is  $O(n \log n)$

d.

$$O(\rho(X)) = O(n^2)$$

Need to find all the subset of  $S$  So the complexity is  $O(n^2)$

e.

As  $b$ . mentioned,  $B_3 = \{1, 2, 3, 6\}$ ,  $B_4 = \{4, 5\}$ . If we use *Best-in Greedy*, we may lose the  $B_4$ , because there is a solution when the first choice is 6.

## Question 2

a.

We have the job set  $E$  like following

Job	1	2	3	4	5	6	7
Due	3	2	4	1	4	4	6
Profit	2	3	4	3	3	6	7

The purpose is to find a independent subset  $S \subseteq E$

The initialization of *Best-in Greedy* is sorting the jobs, as following

Job	7	6	3	5	4	2	1
Due	6	4	4	4	1	2	3
Profit	7	6	4	3	3	3	2

Then, as *Best-in Greedy*, the process would be

Date	1	2	3	4
Job	7	6	3	5
Due	6	4	4	4
Profit	7	6	4	3

The total profit would be:  $7 + 6 + 4 + 3 = 20$

b.

Suppose  $A$  and  $B$  are two independent subsets of  $E$ , so we have  $A, B \subseteq E$ , and  $A, B \in \mathcal{F}$ . Construct  $A$  and  $B$  as:

$$A = \{a_1, a_2, \dots, a_k\}, B = \{b_1, b_2, \dots, b_m\}, a_i \text{ and } b_j \text{ means the job in } A, B$$

Suppose that  $k < m$ , so  $|A| < |B|$ .

If all the due of  $a_i$  is less or equal than  $k$ , it's easy to find a  $b_j$  whose due is larger than  $k$ , because of  $k < m$ . So, we can add  $b_j$  to  $A$  as  $a_{k+1}$  and all the jobs can be processed,

$$b_j \in B \setminus A, A \cup \{b_j\} \in \mathcal{F}$$

If  $\exists a_i, d_{a_i} > k$ , we can find a  $b_j \in B \setminus A$  whose due is larger than  $i$ , because of  $k - i < m - i$ . So, we can put  $a_i$  as  $(k + 1)$ th process and set  $b_j$  as  $i$ th process, all the jobs can be processed,

$$b_j \in B \setminus A, A \cup \{b_j\} \in \mathcal{F}$$

So,  $(E, \mathcal{F})$  is a matroid.

### Question 3

a.

$(V(G), \mathcal{F})$  is a matroid.

First, prove that  $\emptyset \in \mathcal{F}$ .

Obviously, we have  $\emptyset \subseteq V(G)$ , and there is no edge because of no vertex.

So,  $\emptyset \in \mathcal{F}$ .

Second, prove that  $X \subseteq Y \in \mathcal{F} \Rightarrow X \in \mathcal{F}$

If  $Y$  is independent, which means no edge inside the  $Y$ , obviously all the subsets of  $Y$  contain no edge inside.

So, this defines an independence system  $(V(G), \mathcal{F}), \mathcal{F} \subseteq 2^{V(G)}$

Third, to prove that the independence system  $(V(G), \mathcal{F})$  is a matroid, according to the *lemma 2*, only need to prove **for each  $X \subseteq E$ , all bases of  $X$  are of the same size.**

Supposed  $|V(G)| = n$ .

Apparently, no matter which vertices are chosen, for  $n$  is *odd*, the size of all the bases of  $X$  is equal to  $\frac{n-1}{2}$ , and for  $n$  is *even*, the size of all the bases of  $X$  is equal to  $\frac{n}{2}$ .

So,  $(V(G), \mathcal{F})$  is a matroid.

b.

While  $n = 5$ , according to a., so that the size of bases  $r(S) = \frac{n-1}{2} = 2$ .

Therefore, the defining inequalities for  $P_{V(G), \mathcal{F}}$ :

$$S_1 = \{x_1, x_3\} \quad x(S_1) = x_1 + x_3 \leq r(S) = 2$$

$$S_2 = \{x_2, x_4\} \quad x(S_2) = x_2 + x_4 \leq r(S) = 2$$

$$S_3 = \{x_3, x_5\} \quad x(S_3) = x_3 + x_5 \leq r(S) = 2$$

$$S_4 = \{x_4, x_1\} \quad x(S_4) = x_4 + x_1 \leq r(S) = 2$$

$$S_5 = \{x_5, x_2\} \quad x(S_5) = x_5 + x_2 \leq r(S) = 2$$

c.

Yes. According to the *Edmonds(1970)* The output is the linear, especially binary in this case, combination of the characteristic vectors, which are integral. So the polytope  $P_{V(G), \mathcal{F}}$  is integral.

Question 4

Question 5

Question 6

Question 7

Question 8

Question 9

Question 10

$$\mathcal{B} + \mathcal{F}$$