Assignment 1
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Question 1

a.

First, prove that $\emptyset \in S$.
Obviously, we have $\emptyset \in E = \{1, 2, ..., \text{n}\}$, and $a(\emptyset) = \sum_{j \in \emptyset} = 0 \leqslant b \in \mathbb{R}_+$
So, $\emptyset \in S$.
Second, prove that $X \subseteq Y \in S \Rightarrow X \in S$

$$Y \in S \Rightarrow a(Y) = \sum_{j \in Y} a_j \leqslant b$$

$$X \subseteq Y \Rightarrow a(X) = \sum_{j \in X} a_j = \sum_{i \in Y} a_i - \sum_{j \in Y \setminus X} a_j \leqslant \sum_{j \in Y} a_j \leqslant b$$

So, this difines an independence system $\mathcal{F} \subseteq 2^E$

b.

According to the defination, we have

$$\text{n} = 6 \Rightarrow S \subseteq E = \{1, 2, 3, 4, 5, 6\}$$

$$a = (1, 1, 1, 4, 4, 5), \text{b} = 8 \Rightarrow a(S) = \sum_{j \in S} a_j \leqslant 8$$

According to the defination of *rank*, we have

$$r(X) = \underset{F \in \mathcal{F}}{max} |F \cap X| = \underset{B \in \mathcal{B}_X}{max} |B|, \text{ and } \rho(X) = \underset{B \in \mathcal{B}_X}{min} |B|$$

We can find that

$$\mathcal{B}_X = \Big\{\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 3, 6\}, \{4, 5\}\Big\}$$

So that we have $r(X) = 4, \rho(X) = 2$

c.

$$O\big(r(X)\big) = O(nlogn)$$

As the greedy algorithm, order the a from small to large, use the *Best-in Greedy* to select numbers from the smallest. So the complexity is $O(nlogn)$

d.

$$O\big(\rho(X)\big) = O(n^2)$$

Need to find all the subset of S So the complexity is $O(n^2)$

e.

As *b.* mentioned, $B_3 = \{1, 2, 3, 6\}, B_4 = \{4, 5\}$. If we use *Best-in Greedy*, we may lose the B_4, because there is a solution when the first choice is 6.

Question 2

a.

We have the job set E like following

Job	1	2	3	4	5	6	7
Due	3	2	4	1	4	4	6
Profit	2	3	4	3	3	6	7

The purpose is to find a independent subset $S \subseteq E$
The initialization of *Best-in Greedy* is sorting the jobs, as following

Job	7	6	3	5	4	2	1
Due	6	4	4	4	1	2	3
Profit	7	6	4	3	3	3	2

Then, as *Best-in Greedy*, the process would be

Date	1	2	3	4
Job	7	6	3	5
Due	6	4	4	4
Profit	7	6	4	3

The total profit would be: $7 + 6 + 4 + 3 = 20$

b.

Suppose A and B are two independent subsets of E, so we have $A, B \subseteq E$, and $A, B \in \mathcal{F}$. Construct A and B as:

$$A = \{a_1, a_2, ..., a_k\}, B = \{b_1, b_2, ..., b_m\}, a_i \text{ and } b_j \text{ means the job in } A, B$$

Suppose that k < m, so $|A| < |B|$.

If all the due of a_i is less or equal than k, it's easy to find a b_j whose due is larger than k, because of k < m. So, we can add b_j to A as a_{k+1} and all the jobs can be processed,

$$b_j \in B \setminus A, A \cup \{b_j\} \in \mathcal{F}$$

If $\exists a_i, d_{a_i} > \mathrm{k}$, we can find a $b_j \in B \setminus A$ whose due is larger than i, because of $\mathrm{k} - \mathrm{i} < \mathrm{m} - \mathrm{i}$. So, we can put a_i as $(\mathrm{k} + 1)$th process and set b_j as ith process, all the jobs can be processed,

$$b_j \in B \setminus A, A \cup \{b_j\} \in \mathcal{F}$$

So, $(E, \mathcal{F})$ is a matroid.

Question 3

Question 4

Question 5

Question 6

Question 7

Question 8

Question 9

Question 10

$$\mathcal{B} + \mathcal{F}$$