Assignment 1
Author: Yuan Qu **Date:** 01/31/2017

Question 1

a.

This defines an independence system $(E, \mathcal{F}), \mathcal{F} \subseteq 2^E$

First, prove that $\emptyset \in \mathcal{F}$.
Obviously, we have $\emptyset \subseteq E = \{1, 2, ..., \text{n}\}$, and $a(\emptyset) = \sum_{j \in \emptyset} = 0 \leqslant b \in \mathbb{R}_+$
So, $\emptyset \in \mathcal{F}$.

Second, prove that $X \subseteq Y \in \mathcal{F} \Rightarrow X \in \mathcal{F}$

$$Y \in \mathcal{F} \Rightarrow a(Y) = \sum_{j \in Y} a_j \leqslant b$$

$$X \subseteq Y \Rightarrow a(X) = \sum_{j \in X} a_j = \sum_{i \in Y} a_i - \sum_{j \in Y \setminus X} a_j \leqslant \sum_{j \in Y} a_j \leqslant b$$

So, this difines an independence system $(E, \mathcal{F}), \mathcal{F} \subseteq 2^E$

b.

According to the defination, we have

$$\text{n} = 6 \Rightarrow S \subseteq E = \{1, 2, 3, 4, 5, 6\}$$

$$a = (1, 1, 1, 4, 4, 5), \text{b} = 8 \Rightarrow a(S) = \sum_{j \in S} a_j \leqslant 8$$

According to the defination of *rank*, we have

$$r(X) = \max_{F \in \mathcal{F}} |F \cap X| = \max_{B \in \mathcal{B}_X} |B|, \text{ and } \rho(X) = \min_{B \in \mathcal{B}_X} |B|$$

We can find that

$$\mathcal{B}_X = \big\{\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 3, 6\}, \{4, 5\}\big\}$$

So that we have $r(X) = 4, \rho(X) = 2$

c.

$$O\big(r(X)\big) = O(nlogn)$$

As the greedy algorithm, order the a from small to large, use the *Best-in Greedy* to select numbers from the smallest. So the complexity is $O(nlogn)$

d.

$$O\big(\rho(X)\big) = O(n^2)$$

Need to find all the subset of S So the complexity is $O(n^2)$

e.

As *b.* mentioned, $B_3 = \{1,2,3,6\}, B_4 = \{4,5\}$. If we use *Best-in Greedy*, we may lose the B_4, because there is a solution when the first choice is 6.

Question 2

a.

We have the job set E like following

Job	1	2	3	4	5	6	7
Due	3	2	4	1	4	4	6
Profit	2	3	4	3	3	6	7

The purpose is to find a independent subset $S \subseteq E$
The initialization of *Best-in Greedy* is sorting the jobs, as following

Job	7	6	3	5	4	2	1
Due	6	4	4	4	1	2	3
Profit	7	6	4	3	3	3	2

Then, as *Best-in Greedy*, the process would be

Date	1	2	3	4
Job	7	6	3	5
Due	6	4	4	4
Profit	7	6	4	3

The total profit would be: $7 + 6 + 4 + 3 = 20$

b.

Suppose A and B are two independent subsets of E, so we have $A, B \subseteq E$, and $A, B \in \mathcal{F}$. Construct A and B as:

$$A = \{a_1, a_2, ..., a_\mathrm{k}\}, B = \{b_1, b_2, ..., b_\mathrm{m}\}, a_\mathrm{i} \text{ and } b_\mathrm{j} \text{ means the job in } A, B$$

Suppose that $\mathrm{k} < \mathrm{m}$, so $|A| < |B|$.

If all the due of a_i is less or equal than k, it's easy to find a b_j whose due is larger than k, because of $\mathrm{k} < \mathrm{m}$. So, we can add b_j to A as $a_{\mathrm{k}+1}$ and all the jobs can be processed,

$$b_\mathrm{j} \in B \setminus A, A \cup \{b_\mathrm{j}\} \in \mathcal{F}$$

If $\exists a_i, d_{a_i} > \mathrm{k}$, we can find a $b_j \in B \setminus A$ whose due is larger than i, because of $\mathrm{k} - \mathrm{i} < \mathrm{m} - \mathrm{i}$. So, we can put a_i as $(\mathrm{k}+1)$th process and set b_j as ith process, all the jobs can be processed,

$$b_j \in B \setminus A, A \cup \{b_j\} \in \mathcal{F}$$

So, $(E, \mathcal{F})$ is a matroid.

Question 3

a.

$\big(V(G), \mathcal{F}\big)$ is a matroid.

First, prove that $\emptyset \in \mathcal{F}$.
Obviously, we have $\emptyset \subseteq V(G)$, and there is no edge because of no vertex.
So, $\emptyset \in \mathcal{F}$.

Second, prove that $X \subseteq Y \in \mathcal{F} \Rightarrow X \in \mathcal{F}$
If Y is independent, which means no edge inside the Y, obviously all the subsets of Y contain no edge inside.
So, this difines an independence system $\big(V(G), \mathcal{F}\big), \mathcal{F} \subseteq 2^{V(G)}$

Third, to prove that the independence system $\big(V(G), \mathcal{F}\big)$ is a matroid, according to the *lemma 2*, only need to prove **for each $X \subseteq E$, all bases of X are of the same size.**
Supposed $|V(G)| = \mathrm{n}$.
Apparently, no matter which vertices are chosen, for n is *odd*, the size of all the bases of X is equal to $\frac{n-1}{2}$, and for n is *even*, the size of all the bases of X is equal to $\frac{n}{2}$.

So, $\big(V(G), \mathcal{F}\big)$ is a matroid.

b.

While $n = 5$, according to *a.*, so that the size of bases $r(S) = \frac{n-1}{2} = 2$.
Therefore, the defining inequalities for $P_{V(G),\mathcal{F}}$:

$$S_1 = \{x_1, x_3\} \quad x(S_1) = x_1 + x_3 \leqslant r(S) = 2$$

$$S_2 = \{x_2, x_4\} \quad x(S_2) = x_2 + x_4 \leqslant r(S) = 2$$

$$S_3 = \{x_3, x_5\} \quad x(S_3) = x_3 + x_5 \leqslant r(S) = 2$$

$$S_4 = \{x_4, x_1\} \quad x(S_4) = x_4 + x_1 \leqslant r(S) = 2$$

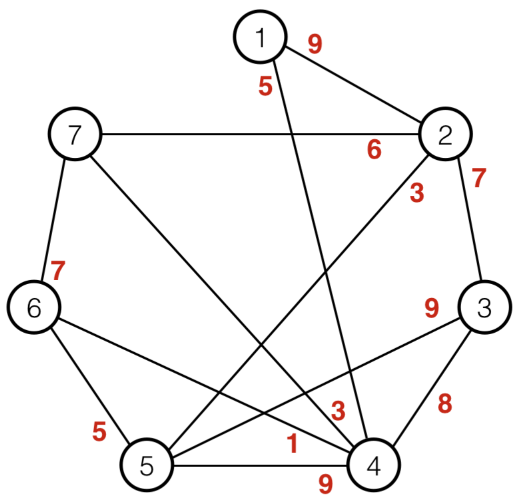
$$S_5 = \{x_5, x_2\} \quad x(S_5) = x_5 + x_2 \leqslant r(S) = 2$$

c.

Yes. According to the *Edmonds(1970)* The output is the linear, especially binary in this case, combination of the characteristic vectors, which are integral. So the ploytope $P_{V(G),\mathcal{F}}$ is integral.

Question 4

As the defination, the graph is following.



Use *Kruskal's Algorithm* to find the maximum-weight spanning tree.

Initialize Sort E by weight from largest to smallest.

$E = \big\{(1,2),(3,5),(4,5),(3,4),(2,3),(6,7),(2,7),(1,4),(5,6),(2,5),(4,7),(4,6)\big\}$

Set $T = \emptyset$

1. $T \cup \{(1,2)\}$, cycle free, $T = T \cup \{(1,2)\}$
2. $T \cup \{(3,5)\}$, cycle free, $T = T \cup \{(3,5)\}$
3. $T \cup \{(4,5)\}$, cycle free, $T = T \cup \{(4,5)\}$
4. $T \cup \{(3,4)\}$, cycle!

5. $T \cup \{(2,3)\}$, cycle free, $T = T \cup \{(2,3)\}$
6. $T \cup \{(6,7)\}$, cycle free, $T = T \cup \{(6,7)\}$
7. $T \cup \{(2,7)\}$, cycle free, $T = T \cup \{(2,7)\}$
8. $T \cup \{(1,4)\}$, cycle! No alone vertex, means rest edges make cycle, algorithm end.

So that the $T = \{(1,2),(3,5),(4,5),(2,3),(6,7),(2,7)\}$

The weight of the maximum-weight spanning tree is

$$w_{1,2} + w_{3,5} + w_{4,5} + w_{2,3} + w_{6,7} + w_{2,7} = 47$$

Question 5

a.

$$|B| = \frac{15 \times 3 + 18 \times 5 + 12 \times 15}{1 + 2 + 4} = 45$$

b.

The maximum cardinality matching in this graph is 45.

If graph $G = (A \cup B, E)$ has a *Perfect Matching*, which has $\nu(G) = |A|$, the maximum cardinality matching will be *Perfect Matching*.

Now prove that in this case $G = (A \cup B, E)$ has a *Perfect Matching*.

According to *Theorem 3 (Frobeiniu(1971), Hall(1935))*, only need to prove,

$$|S| \leqslant |N(S)| \quad \text{for all} \quad S \subseteq A$$

Proof by contradiction. If $\exists |S| > |N(S)|, S \subseteq A$. Suppose that $\mathrm{k} = |S|, \mathrm{m} = |N(S)|$, i.e. k > m.

Suppose that

$$k_1 + k_2 + k_3 = k > m$$

in which k_1 refers the number of vertices of degree 3, k_2 refers the number of vertices of degree 5, k_3 refers the number of vertices of degree 15.

According to the defination of B, we could get $m \geqslant 3k_1$, $m \geqslant \frac{5}{2}k_2$, $m \geqslant \frac{15}{4}k_3$, so we have

$$m = (\frac{1}{3} + \frac{2}{5} + \frac{4}{15})m \geqslant k_1 + k_2 + k_3 = k > m$$

Which makes $m > m$, **contradiction**.

So graph $G = (A \cup B, E)$ has a *Perfect Matching*, and $\nu(G) = |A|$.

Question 6

Question 7

Question 8

Question 9

Question 10

$$\mathcal{B} + \mathcal{F}$$