

Question 1

a.

First, prove that $\emptyset \in S$.

Obviously, we have $\emptyset \in E = \{1, 2, \dots, n\}$, and $a(\emptyset) = \sum_{j \in \emptyset} a_j = 0 \leq b \in \mathbb{R}_+$

So, $\emptyset \in S$.

Second, prove that $X \subseteq Y \in S \Rightarrow X \in S$

$$Y \in S \Rightarrow a(Y) = \sum_{j \in Y} a_j \leq b$$

$$X \subseteq Y \Rightarrow a(X) = \sum_{j \in X} a_j = \sum_{i \in Y} a_i - \sum_{j \in Y \setminus X} a_j \leq \sum_{j \in Y} a_j \leq b$$

So, this defines an independence system $\mathcal{F} \subseteq 2^E$

b.

According to the definition, we have

$$n = 6 \Rightarrow S \subseteq E = \{1, 2, 3, 4, 5, 6\}$$

$$a = (1, 1, 1, 4, 4, 5), b = 8 \Rightarrow a(S) = \sum_{j \in S} a_j \leq 8$$

According to the definition of *rank*, we have

$$r(X) = \max_{F \in \mathcal{F}} |F \cap X| = \max_{B \in \mathcal{B}_X} |B|, \text{ and } \rho(X) = \min_{B \in \mathcal{B}_X} |B|$$

We can find that

$$\mathcal{B}_X = \{\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 3, 6\}, \{4, 5\}\}$$

So that we have $r(X) = 4, \rho(X) = 2$

c.

$$O(r(X)) = O(n \log n)$$

Question 2

Question 3

Question 4

Question 5

Question 6

Question 7

Question 8

Question 9

Question 10

$$\mathcal{B} + \mathcal{F}$$