Intro to Probability

Homework 1

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Problem 3

Suppose it is necessary to form a committee from a set of n people.

1. How many different ways are there to select a committee of size k and a chairperson (who is one of these k people)?
2. How many different ways are there to select a chairperson and some other arbitrary set of other committee members of any size?
3. Hence show that

$$\sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}$$

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Problem 3.1

How many different ways are there to select a committee of size k and a chairperson (who is one of these k people)?

1. Choose a size k committee: $\binom{n}{k}$
2. Choose a chairperson: $\binom{k}{1}$
3. Total: $\binom{k}{1} \times \binom{n}{k} = k\binom{n}{k}$

Another way:

1. Choose a chairperson: $\binom{n}{1}$
2. Choose a size k-1 committee: $\binom{n-1}{k-1}$
3. Total: $\binom{n}{1} \times \binom{n-1}{k-1} = n\frac{(n-1)!}{(k-1)!(n-k)!} = k\frac{n!}{k!(n-k)!} = k\binom{n}{k}$

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Problem 3.2

How many different ways are there to select a chairperson and some other arbitrary set of other committee members of any size?

1. Choose a chairperson: $\binom{n}{1}$
2. Choose an any size committee: $\sum_{i=0}^{n-1} \binom{n-1}{i} = 2^{n-1}$
3. Total: $\binom{n}{1} \times 2^{n-1} = n2^{n-1}$

Problem 3.3

Hence show that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$

$\sum_{k=1}^{n} k\binom{n}{k} = \sum_{k=1}^{n} \binom{n}{k}\binom{k}{1}$ …… Choose any committee size of k, and a chairperson.

$n2^{n-1}$ …… Choose a chairperson, and any committee size of k

$$\Rightarrow \sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$$

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Problem 3.3

Hence show that $\sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}$

$$\sum_{k=1}^{n} k \binom{n}{k} = \sum_{k=1}^{n} k \cdot \frac{n!}{k!(n-k)!} = \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!}$$

$$= n \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!}$$

$$= n \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-k-1)!}$$ …… Use k to replace k-1

$$= n \sum_{k=0}^{n-1} \binom{n-1}{i} = n2^{n-1}$$ …… Binomial theory