Convex Optimization

Chapter 3: Convex Set

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3.1 Basic Definitions

We first introduce a list of notions that will be discussed in convex analysis.

Definition 3.1 (Convex set). A set A is called a convex set, if for any $x, y \in A$ and $\alpha \in [0, 1]$, we have that $\alpha x + (1-\alpha)y \in A$.

Definition 3.2 (Convex combination). A vector constructed as $\sum_{k=1}^{n} \alpha_k x_k$ with $\sum_{k=1}^{n} \alpha_k = 1$ and $\alpha_k > 0$ is called a convex combination.

Definition 3.3 (Affine set). A set A is called an affine set, if for any $x, y \in A$ and $\alpha \in \mathbb{R}$, we have that $\alpha x + (1-\alpha)y \in A$.

Definition 3.4 (Affine combination). A vector constructed as $\sum_{k=1}^{n} \alpha_k x_k$ with $\sum_{k=1}^{n} \alpha_k = 1$ is called an affine combination.

Definition 3.5 (Convex hull). Given an arbitrary set A, the convex hull of A is the smallest convex set that contains A.

Definition 3.6 (Cone). A set A is called a cone if for any $x \in A$ and $\alpha > 0$, we have that $\alpha x \in A$.

Definition 3.7 (Convex cone). A set is a convex cone if it is a cone and convex.

Definition 3.8 (Conic combination). A vector constructed as $\sum_{k=1}^{n} \alpha_k x_k$ with $\alpha_k \geq 0$ is called a conic combination.

Definition 3.9 (Conic hull). Given an arbitrary set A, the conic hull of A is the smallest convex cone that contains A.

Next, we see some examples of definitions above.

Example 3.1. Hyperplanes and half spaces: $\{x \mid a^\top x + b = 0\}$ and $\{x \mid a^\top x + b > 0\}$.

Example 3.2. Euclidean balls and ellipsoids: $\{x \mid \|x - x_c\|_2 \leq r\}$ and $\{x \mid (x - c_c)^\top P^{-1}(x - x_c) \leq r^2\}$.

Example 3.3. Polyhedrons:

$$\{x \mid a_i^\top x \leq b_i, \text{ for } i = 1, 2 \ldots, m \;, c_j^\top x = d_j \text{ for } j = 1, 2 \ldots, n\} \tag{3.1}$$

3.2 Operations that preserve convexity

- Intersection of sets over index set: $\bigcap_\alpha C_\alpha$ is a convex set if C_α is a convex set for $\alpha \in \mathcal{I}$.
- Cartesian product of sets: if C_i is a convex for $i = 1, 2 \ldots, n$, then

$$C_1 \times C_2 \ldots \times C_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in C_i, i = 1, 2 \ldots, n\} \tag{3.2}$$

  is a convex set.

- Sum of two sets: if X, Y are convex, then we have that

$$Z = \alpha X + \beta Y := \{z = \alpha x + \beta y \mid x \in X, y \in Y\} \tag{3.3}$$

  is a convex set.

- **Affine maps**: let $f : \mathbb{R}^n \to \mathbb{R}^m$ be an affine map, $C_1 \subseteq \mathbb{R}^n$ and $C_2 \subseteq \mathbb{R}^m$ be two convex sets, then $f(C_1)$ and $f^{-1}(C_2)$ are both convex sets.
- **Projection**: given perspective function $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$ with $p(x,t) = \frac{x}{t}$, $C_1 \subseteq \mathbb{R}^{n+1}$ and $C_2 \subseteq \mathbb{R}^n$ are convex sets, then both $p(C_1)$ and $p(C_2)$ are convex set.

3.3 Separation theorems

Given two convex set, we have the following separation hyperplane theorem.

Theorem 3.1 (separation hyperplane theorem)**.** *Let $C, D \subseteq \mathbb{R}^n$ be two convex set with $C \cap D = \emptyset$. Then there must exist $a \neq 0$ and $b \in \mathbb{R}$, such that*

$$a^\top x + b \leq 0, \forall x \in C, \text{ and } a^\top x + b \geq 0, \forall x \in D \tag{3.4}$$

we call hyperplane $\{x \mid a^\top x + b = 0\}$ the separating hyperplane for C and D.

Note the separating hyperplane is not necessary unique, the weak inequality above is all necessary. For strict separation, a useful theorem is given below.

Theorem 3.2. *Let C be a closed convex set, $x_0 \in \mathbb{R}^n$ and $x_0 \notin C$. Then there exist $a \neq 0$ and $b \in \mathbb{R}$, such that*

$$a^\top x + b < 0, \forall x \in C, \text{ and } a^\top x_0 + b > 0 \tag{3.5}$$