Convex Optimization

Chapter 3: Convex Function

Notes by Renxiong Liu

We summarize some basic results for convex function in this notes.

3.1 Basic definition

We first introduce some definitions related with convex function.

Definition 3.1 (Convex function). A function $f : \mathbb{R}^n \to \mathbb{R}$ is called a convex function if

- $\mathbf{dom}(f)$ is a convex set.
- For all $\alpha \in [0, 1]$ and $x, y \in \mathbf{dom}(f)$, it holds that

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y) \tag{3.1}$$

Remark. if instead for all $\alpha \in [0, 1]$, $x, y \in \mathbf{dom}(f)$ with $x \neq y$, it holds that

$$f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha)f(y) \tag{3.2}$$

then f is a strictly convex function.

Proposition 3.1. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if for any $x \in \boldsymbol{dom}(f)$, function $g(t) = f(x + tv)$ is convex.*

3.2 First-order condition

In this section, we assume function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable for any $x \in \mathbf{dom}(f)$. The following two propositions establish the sufficient and necessary conditions to justify the convexity of a function.

Proposition 3.2. *A differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if for any $x, y \in \boldsymbol{dom}(f)$, it holds that*

$$f(y) \geq f(x) + \nabla f(x)^\top (y - x) \tag{3.3}$$

Similarly, $f : \mathbb{R}^n \to \mathbb{R}$ is strictly convex if and only if for any $x, y \in \boldsymbol{dom}(f)$ with $x \neq y$, it holds that

$$f(y) > f(x) + \nabla f(x)^\top (y - x) \tag{3.4}$$

A quick results based on Proposition 3.2 is,

Proposition 3.3. *A differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if for any $x, y \in$ $\boldsymbol{dom}(f)$, it holds that*

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 0 \tag{3.5}$$

Similarly, $f : \mathbb{R}^n \to \mathbb{R}$ is strictly convex if and only if for any $x, y \in \boldsymbol{dom}(f)$ with $x \neq y$, it holds that

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle > 0 \tag{3.6}$$

3.3 Second order condition

Throughout this section, we will assume function $f : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable for any $x \in \mathbf{dom}(f)$, meaning that $\nabla^2 f(x)$ exists for any $x \in \mathbf{dom}(f)$.

Proposition 3.4. *A twice differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if for any $x \in \boldsymbol{dom}(f)$, it holds that*

$$\nabla^2 f(x) \succeq 0 \tag{3.7}$$

Remark. A strictly convex function does not necessarily satisfy $\nabla^2 f(x) \succ 0$. However, if f satisfies $\nabla^2 f(x) \succ 0$, then we must have that f is a strictly convex function.

3.4 Epigraph method

In this section, we formally introduce the notion of epigraph, with which we can justify whether a function is convex or not. Toward this end, we first define the α-sublevel set of function f to be

$$C_\alpha = \{x \in \mathbb{R}^n \mid f(x) \leq \alpha\} \tag{3.8}$$

It's easy to see that when f is convex, then for any $\alpha \in \mathbb{R}$, C_α is a convex set. The converse is not true. For example $f(x) = -e^x$ satisfies for any $\alpha \in \mathbb{R}$, C_α is a convex set, but f is concave.

Now we trun to the points of epigraph. The epigraph of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\mathrm{epi}(f) = \{(x, \alpha) \mid f(x) \leq \alpha\} \tag{3.9}$$

A useful tool to justify if function f is convex is via the epigraph. More specifically, we have the following proposition.

Proposition 3.5. *function is convex if and only if its epigraph is a convex set.*

A direct corollary for a differentiable function f is: if $(x, \alpha) \in \mathrm{epi}(f)$, then it holds that

$$\alpha \geq f(x) \geq f(y) + \nabla f(y)^\top (x - y) \tag{3.10}$$

3.5 Operations that preserve convexity

In this section, we introduce some operations that will preseve the convexity of original functions. This will be quite useful to help us justify the convexity of some complicated functions.

Proposition 3.6 (Nonnegative weighted sums). *Let $f_1, f_2, \ldots, f_n$ be a list of convex functions, then for any $\alpha_1, \alpha_2, \ldots, \alpha_n$, we have that $\sum_{k=1}^n \alpha_k f_k$ is also convex over $\cap_{k=1}^n \boldsymbol{dom} f_k$.*

Remark. As an extension, let $f(x, y)$ be convex in x for any y, then we have that $g(x) = \int_S w(y) f(x, y) dy$ is also a convex function.

Proposition 3.7 (Composition with an affine mapping). *let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. Then for any $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, function $g(x) = f(Ax + b)$ is convex.*

Proposition 3.8 (Pointwise supremum). *let $f_\alpha : \mathbb{R}^n \to \mathbb{R}$ be a set of convex functions. Then $g(x) = \sup_\alpha f_\alpha(x)$ is convex.*

Remark. As a corollary, if for each $y \in \mathcal{A}$, $f(x, y)$ is convex in x, then $g(x) = \sup_{y \in \mathcal{A}} f(x, y)$ is convex.

Remark 2. Another corollary is, we can always represent a convex function as the pointwise supremum of any affine function. That is for any convex function f, we have that

$$f(x) = \sup\{g(x) \mid g \text{ if affine, } g(z) \le f(z) \text{ for any } z \in \mathbf{dom}(f)\} \tag{3.11}$$

Proposition 3.9 (Minimization). *Let $f : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$ be a convex function, C be a convex set. Then $g(x) = \inf_{y \in C} f(x, y)$ is convex.*

3.6 Conjugate function

For any $f : \mathbb{R}^n \to \mathbb{R}$, we define the conjugate function of f to be

$$f^\star(y) = \sup_x \langle y, x\rangle - f(x) \tag{3.12}$$

It's easy to see that $f^\star$ is always a convex function. Moreover, by definition, Fenchels inequality holds:

$$f(x) + f^\star(y) \ge \langle x, y\rangle \tag{3.13}$$

Another important property of conjugate function is given as follows.

Proposition 3.10. *If f is convex and closed (i.e. $\boldsymbol{epi}(f)$ is a closed set), then $f^{\star\star} = f$.*