$$\tilde{\beta} = \mathcal{I}_0^3(b)\beta u_{\rm IM}^b \sqrt{\Phi_b - \frac{1}{2}\vec{p_b}^{\intercal} P^{-1}\vec{p_b}} \tag{1}$$

For $\tilde{\beta}$ to have a real value for all $b \neq 0, 3$, we must ensure that the quantity under the square root is positive. The only way for this quantity to be negative is if the integral,

$$\int_{p_0}^{1} p^{-5}\left((p^b - 1) - \frac{b}{3}(p^3 - 1)\right)^2 dp, \tag{2}$$

is negative. This is only possible if $p_0 > 1$. Thus, we must ensure that $p_0 \leq 1$, or

$$f_{\rm min} \leq f_{\rm IM}. \tag{3}$$

We note that $f_{\rm IM}$ can be written as

$$f_{\rm IM} = \frac{[M f_{\rm IM}]c^3}{GM}, \tag{4}$$

where the quantity $[M f_{\rm IM}]$ is a constant. Substituting this into Eqn. 3 and solving for M, we find

$$M \leq \frac{[M f_{\rm IM}]c^3}{G f_{\rm min}}. \tag{5}$$

Thus, to ensure that $\tilde{\beta}$ always has a real value, we must place an upper bound on the total system mass. To quantify how large this upper bound is, we can choose some common values for the quantities in Eqn. 5, such as $[M f_{\rm IM}] = 0.018$ and $f_{\rm min} = 20$ Hz. This gives us a maximum total mass of 182 $\mathrm{M}_{\odot}$.