Dynamics of Third Order Beverton Holt Difference Equation

M. R. S. Kulenović and R. Sullivan
Department of Mathematics
University of Rhode Island,

September 10, 2024

Abstract

We consider third order difference equation

$$y_{n+1} = a_1 \frac{y_n^2}{1 + y_n^2} + a_2 \frac{y_{n-1}^2}{1 + y_{n-1}^2} + a_3 \frac{y_{n-2}^2}{1 + y_{n-2}^2}$$

as discrete time model of population dynamics with three generation involved. We obtain some global dynamics results for this equation. We describe the parts of the basins of attraction of three equilibrium points that this equation equation admits. (2nd and 1st)

The **sigmoid Beverton Holt** equation with three generations is the following equation

$$y_{n+1} = a_1 \frac{y_n^2}{1+y_n^2} + a_2 \frac{y_{n-1}^2}{1+y_{n-1}^2} + a_3 \frac{y_{n-2}^2}{1+y_{n-2}^2}, \quad n = 0, 1, \ldots, \tag{1}$$

where $a_i \in [0, \infty), i = 1, 2, 3$ and $|a_1| + |a_2| + |a_3| > 0$.

The **sigmoid Beverton Holt** equation with three generations is the following equation

$$y_{n+1} = a_1 \frac{y_n^2}{1+y_n^2} + a_2 \frac{y_{n-1}^2}{1+y_{n-1}^2} + a_3 \frac{y_{n-2}^2}{1+y_{n-2}^2}, \quad n = 0, 1, \ldots, \quad (1)$$

where $a_i \in [0, \infty), i = 1, 2, 3$ and $|a_1| + |a_2| + |a_3| > 0$. Special case $a_3 = 0$ was considered by Bilgin, Kulenovic, and Pilav

Example 1

For instance, the equation

$$y_{n+1} = 20\frac{y_n^2}{1+y_n^2} + \frac{y_{n-1}^2}{1+y_{n-1}^2} + \frac{y_{n-2}^2}{1+y_{n-2}^2}, n = 0, 1, \ldots$$

has three equilibrium points: $0, \frac{1}{11+2\sqrt{30}}, 11+2\sqrt{30}$. The characteristic equation at $\frac{1}{11+2\sqrt{30}}$ is

$$242m^3 - \frac{\sqrt{30}m^2}{1331} - \frac{1}{242}\left(11+2\sqrt{30}\right)m - \frac{\sqrt{3}}{121} - \frac{1}{22} = 0$$

which has characteristic roots $m_1 = 1.88792$ outside of the unit circle, and $m_{2,3} = -0.0367531 \pm 0.216108i$ inside the unit circle. So the middle equilibrium is a saddle point with stable space of dimension 2.

Example 2

The equation:

$$y_{n+1} = \frac{y_n^2}{1+y_n^2} + 20\frac{y_{n-1}^2}{1+y_{n-1}^2} + \frac{y_{n-2}^2}{1+y_{n-2}^2}, \quad n = 0, 1, \dots, \qquad (2)$$

has three equilibrium points: $0, \frac{1}{11+2\sqrt{30}}, 11+2\sqrt{30}$. The characteristic equation at $\frac{1}{11+2\sqrt{30}}$ has the form

$$\frac{22^2}{22+4\sqrt{30}}m^3 - m^2 - 20m - 1 = 0$$

with characteristic roots:
$m_1 = -1.27576, m_2 = 1.41668, m_3 = -0.0501957$ showing that this equilibrium solution is a saddle, with stable space of dimension 1.

Example 3

The equation:

$$y_{n+1} = \frac{y_n^2}{1+y_n^2} + 20\frac{y_{n-1}^2}{1+y_{n-1}^2} + 70\frac{y_{n-2}^2}{1+y_{n-2}^2}, \quad n = 0, 1, \ldots, \quad (3)$$

has three equilibrium points: $0, \frac{2}{91+2\sqrt{8277}}, \frac{91+2\sqrt{8277}}{2}$. The characteristic equation at $\frac{2}{91+2\sqrt{8277}}$ has the form

$$\frac{92^2}{91+2\sqrt{8277}}m^3 - m^2 - 20m - 70 = 0$$

with characteristic roots:
$m_{1,2} = -0.633469 \pm 0.890047i, m_3 = 1.28891$ showing that this equilibrium solution is a source.

Preliminary Results

Negative Feedback Theorem:

Let I be an open interval of real numbers and $\bar{x}$ be an equilibrium point of difference equation

$$y_{n+1} = f(y_n, y_{n-1}, y_{n-2}), \quad n = 0, 1, \ldots \tag{4}$$

Assume that f satisfies the following:

1. f is increasing in each of its arguments.
2. f satisfies the negative feedback condition:

$$(u - \overline{x})(f(u, u, u) - u) < 0$$

for all $u \in I - \{\overline{x}\}$.

Then the equilibrium point $\bar{y}$ is a global attractor for Equation (4).

Preliminary Results

Global Convergence in an Interval

For a nonempty set $R \subset \mathbb{R}^n$ and $\preceq$ a partial order on $\mathbb{R}^n$, let $T : R \to R$ be an order preserving map, and let $a, b \in R$ be such that $a \prec b$ and $[\![a, b]\!] \subset R$. If $a \preceq T(a)$ and $T(b) \preceq b$, then $[\![a, b]\!]$ is an invariant set and

i. There exists a fixed point of T in $[\![a, b]\!]$.
ii. If T is strongly order preserving, then there exists a fixed point in $[\![a, b]\!]$ which is stable relative to $[\![a, b]\!]$.
iii. If there is only one fixed point in $[\![a, b]\!]$, then it is a global attractor in $[\![a, b]\!]$ and therefore asymptotically stable relative to $[\![a, b]\!]$.

Preliminary Results

The following result is a consequence of the Trichotomy Theorem of Dancer and Hess,

Preliminary Results

The following result is a consequence of the Trichotomy Theorem of Dancer and Hess,

Corollary

If the nonnegative cone of a partial ordering $\preceq$ is a generalized quadrant in $\mathbb{R}^n$, and if T has no fixed points in $[\![u_1, u_2]\!]$ other than u_1 and u_2, then the interior of $[\![u_1, u_2]\!]$ is either a subset of the basin of attraction of u_1 or a subset of the basin of attraction of u_2.

Preliminary Results

Theorem: Invariant Surfaces (Kulenovic, Marcotte, Merino)

[15] For $\mathrm{z} = (z_1, \ldots, z_{n-1}, z_n)$ in $\bar{\mathbb{R}}^n$ with $n \geq 2$, we denote with $\mathrm{z}^\pi = (z_1, \ldots, z_{n-1})$ the projection of z onto $\bar{\mathbb{R}}^{n-1}$, and for $S \subset \bar{\mathbb{R}}^n$, set $S^\pi := \{\mathrm{z}^\pi : \mathrm{z} \in S\}$. Let $\mathrm{p}, \mathrm{q} \in \bar{\mathbb{R}}^n$ be such that $\mathrm{p} <_\sigma \mathrm{q}$. With $R := \mathrm{int}[\mathrm{p}, \mathrm{q}]_\sigma$, let $T : R \to R$ be a strongly monotone map. Set $B(\mathrm{p}) := \{\, \mathrm{x} \in R \,:\, T^m(\mathrm{x}) \to \mathrm{p} \,\}$. Define the function $\phi : R^\pi \to [p_n, q_n]$ by

$$\phi(x_1, \ldots, x_{n-1}) := q_n \qquad \text{if } (x_1, \ldots, x_{n-1}, q_n) \notin B(\mathrm{p}) \tag{5}$$

and

$$\phi(x_1, \ldots, x_{n-1}) := \sup \{t \in [p_n, q_n] : T^m(x_1, \ldots, x_{n-1}, t) \to \mathrm{p}\} \text{ if } (x_1, \ldots, x_{n-1}, q_n) \in B(\mathrm{p}). \tag{6}$$

If $\emptyset \neq B(\mathrm{p}) \neq R$, then

i. The function ϕ is continuous.

ii. $\mathrm{graph}(\phi) \cap R = \partial B(p) \cap R$.

iii. $\mathrm{graph}(\phi) \cap R$ consists of non-comparable points.

iv. If T is differentiable on R and such that the n-th column of $T'(z)$ has positive entries for $z \in R$, then ϕ is Lipschitz on $(\mathrm{graph}(\phi) \cap R)^\pi$.

Preliminary Results

Theorem

Assume q_0, q_1, q_2 are real numbers such that

$$|q_0| + |q_1| + |q_2| < 1.$$

Then all roots of $\lambda^3 + q_0\lambda^2 + q_1\lambda + q_2 = 0$ lie in the unit circle.

The following result from [14] will be used to establish local stability.

Theorem

A necessary and sufficient condition for all roots of

$$\lambda^3 + q_0\lambda^2 + q_1\lambda + q_2 = 0$$

to be in the unit disk is the following:

$$|q_2 + q_0| < 1 + q_1, \text{ and } |q_2 - 3q_0| < 3 - q_1, \text{ and } q_0^2 + q_1 - q_0q_2 < 1$$

Local Dynamics

1. If
$$a_1 + a_2 + a_3 < 2 \tag{7}$$
then $\bar{y} = 0$ is the unique equilibrium. Also, $f_x(0,0,0) = f_y(0,0,0) = f_z(0,0,0) = 0$, so the equilibrium is locally asymptotically stable.

Local Dynamics

1. If
$$a_1 + a_2 + a_3 < 2 \tag{7}$$
then $\bar{y} = 0$ is the unique equilibrium. Also, $f_x(0,0,0) = f_y(0,0,0) = f_z(0,0,0) = 0$, so the equilibrium is locally asymptotically stable.
2. If $a_1 + a_2 + a_3 = 2$, then there are two equilibrium points, $\bar{y}_0 = 0$ and $\bar{y}_1 = 1$. Then $\bar{y}_0$ is locally asymptotically stable, and $\bar{y}_1$ is non-hyperbolic equilibrium point.

Local Dynamics

1. If
$$a_1 + a_2 + a_3 < 2 \tag{7}$$
then $\bar{y} = 0$ is the unique equilibrium. Also, $f_x(0,0,0) = f_y(0,0,0) = f_z(0,0,0) = 0$, so the equilibrium is locally asymptotically stable.
2. If $a_1 + a_2 + a_3 = 2$, then there are two equilibrium points, $\bar{y}_0 = 0$ and $\bar{y}_1 = 1$. Then $\bar{y}_0$ is locally asymptotically stable, and $\bar{y}_1$ is non-hyperbolic equilibrium point.
3. If
$$a_1 + a_2 + a_3 > 2 \tag{8}$$
then there are three equilibrium points, $\bar{y}_0 = 0$, $\bar{y}_+ = \frac{1}{2}(a_1 + a_2 + a_3 + \sqrt{(a_1 + a_2 + a_3)^2 - 4})$, and $\bar{y}_- = \frac{1}{2}(a_1 + a_2 + a_3 - \sqrt{(a_1 + a_2 + a_3)^2 - 4})$.

Local Dynamics

Theorem: Local Dynamics

Assume $a_1 + a_2 + a_3 > 2$. The equilibrium solutions 0 and $\bar{y}_+$ are always locally asymptotically stable. The equilibrium solution $\bar{y}_-$ is always unstable.

Local Dynamics

Theorem: Local Dynamics

Assume $a_1 + a_2 + a_3 > 2$. The equilibrium solutions 0 and $\bar{y}_+$ are always locally asymptotically stable. The equilibrium solution $\bar{y}_-$ is always unstable.

Proof: The characteristic equation for $\bar{y}_-$ is

$$\lambda^3 - q_0\lambda^2 - q_1\lambda - q_0 = 0$$

where $q_0 = f_x(\bar{y}_-, \bar{y}_-, \bar{y}_-), q_1 = f_y(\bar{y}_-, \bar{y}_-, \bar{y}_-)$, and $q_2 = f_z(\bar{y}_-, \bar{y}_-, \bar{y}_-)$. Since

$$q_2 + q_0 = \frac{(a_3 + a_1)(a_1 + a_2 + a_3 + \sqrt{(a_1 + a_2 + a_3)^2 - 4}}{(a_1 + a_2 + a_3)^2} > 0$$

we have $|-q_2 - q_0| = q_0 + q_2$.

Local Dynamics

Theorem: Local Dynamics

Assume $a_1 + a_2 + a_3 > 2$. The equilibrium solutions 0 and $\bar{y}_+$ are always locally asymptotically stable. The equilibrium solution $\bar{y}_-$ is always unstable.

Proof: The characteristic equation for $\bar{y}_-$ is

$$\lambda^3 - q_0\lambda^2 - q_1\lambda - q_0 = 0$$

where $q_0 = f_x(\bar{y}_-, \bar{y}_-, \bar{y}_-), q_1 = f_y(\bar{y}_-, \bar{y}_-, \bar{y}_-)$, and $q_2 = f_z(\bar{y}_-, \bar{y}_-, \bar{y}_-)$. Since

$$q_2 + q_0 = \frac{(a_3 + a_1)(a_1 + a_2 + a_3 + \sqrt{(a_1 + a_2 + a_3)^2 - 4}}{(a_1 + a_2 + a_3)^2} > 0$$

we have $| - q_2 - q_0| = q_0 + q_2$.
Hence, if $\bar{y}_-$ is stable, then $q_0 + q_2 < 1 - q_1 \implies q_0 + q_1 + q_2 < 1$, So

$$\frac{(a_1 + a_2 + a_3)(a_1 + a_2 + a_3 + \sqrt{(a_1 + a_2 + a_3)^2 - 4}}{(a_1 + a_2 + a_3)^2} < 1,$$

which means

$$1 + \frac{\sqrt{(a_1 + a_2 + a_3)^2 - 4}}{a_1 + a_2 + a_3} < 1,$$

To show $\bar{y}_+$ is always locally asymptotically stable we will show that the sum of the absoulte values of the coefficients of the characteristic equation is less than 1.

To show $\bar{y}_+$ is always locally asymptotically stable we will show that the sum of the absoulte values of the coefficients of the characteristic equation is less than 1.

The characteristic polynomial for $\bar{y}_+$ is

$$\lambda^3 - p_0\lambda^2 - p_1\lambda - p_2 = 0$$

where $p_0 = f_x(\bar{y}_+, \bar{y}_+, \bar{y}_+)$, $p_1 = f_y(\bar{y}_+, \bar{y}_+, \bar{y}_+)$, and $p_2 = f_z(\bar{y}_+, \bar{y}_+, \bar{y}_+)$

$$|p_0| + |p_1| + |p_2| = \frac{4}{(a_1 + a_2 + a_3)(a_1 + a_2 + a_3 + \sqrt{(a_1 + a_2 + a_3)^2 - 4}}$$

Since $a_1 + a_2 + a_3 > 2$,

$$0 < \frac{4}{(a_1 + a_2 + a_3)(a_1 + a_2 + a_3 + \sqrt{(a_1 + a_2 + a_3)^2 - 4})} < \frac{4}{(2)(2 + \sqrt{2^2 - 4})} = 1$$

Hence $\bar{y}_+$ is always locally asymptotically stable.

Global Results

Theorem

If $a_1 + a_2 + a_3 < 2$, *Then the zero equilibrium is globally asymptotically stable.*

Global Results

Theorem

If $a_1 + a_2 + a_3 < 2$, Then the zero equilibrium is globally asymptotically stable.

Proof Since f is nondecreasing in all arguments, the interval $[0, a_1 + a_2 + a_3]$ is an invariant and attracting interval and 0 is the unique equilibrium. Now an application of Theorem A.0.1 from [14] completes the proof.

Global Results

The global attractivity result of Theorem 4 can be extended to the case of non-autonomous difference equation:

$$y_{n+1} = a_1(n)\frac{y_n^2}{1+y_n^2} + a_2(n)\frac{y_{n-1}^2}{1+y_{n-1}^2} + a_3(n)\frac{y_{n-2}^2}{1+y_{n-2}^2}, \quad n = 0, 1, \ldots, \tag{9}$$

where the coefficients $a_i(n)$ are bounded sequences $|a_i(n)| \leq A_i, i = 1, 2, 3.$

Global Results

Theorem

Consider Equation (10) subject to the condition

$$A_1 + A_2 + A_3 < 2, \;\; where \; |a_i(n)| \leq A_i, i = 1, 2, 3. \tag{10}$$

Then the zero equilibrium is globally asymptotically stable.

Global Results

Theorem

Consider Equation (10) *subject to the condition*

$$A_1 + A_2 + A_3 < 2, \; \textit{where} \; |a_i(n)| \leq A_i, i = 1, 2, 3. \tag{10}$$

Then the zero equilibrium is globally asymptotically stable.

Proof follows from Corollary 1 in (12) (Janowski, Kulenovic)

Global Dynamics

Remark: We can extend Theorem 5 to the difference equation

$$y_{n+1} = \sum_{i=0}^{k} a_{i+1}(n) \frac{y_{n-i}^2}{1 + y_{n-i}^2}, \quad n = 0, 1, \ldots. \tag{11}$$

Hence if

$$\sum_{i=0}^{k} A_{i+1} < 1, \text{ where } |a_i(n)| \le A_i, i = 1, \ldots, k,$$

then the zero equilibrium of Equation (11) is globally asymptotically stable.

General Result

Theorem: Global Asymptotic Stability

Consider Equation

$$y_{n+1} = f(y_n, y_{n-1}, y_{n-2}), \quad n = 0, 1, \ldots \tag{12}$$

subject to the condition that $f : I^3 \to I$, where I is an interval and f is continuous and non-decreasing in all its arguments. Assume that Equation (12) has three equilibrium points $\bar{y}_0 < \bar{y}_1 < \bar{y}_2$, where $\bar{y}_0, \bar{y}_2$ are locally asymptotically stable and $\bar{y}_1$ is unstable. Then $\bar{y}_0$ and $\bar{y}_2$ are globally asymptotically stable within the sets $[\bar{y}_0, \bar{y}_1)^3$ and $(\bar{y}_1, \infty)^3$ respectively.

Extension to kth Order Equation

The previous result can be extended to the general k-th order difference equation

$$y_{n+1} = f(y_n, y_{n-1}, \dots, y_{n-k+1}), \quad n = 0, 1, \dots \tag{13}$$

subject to the condition that $f : I^k \to I$, where I is an interval, is continuous and non-decreasing in all its arguments. Assume that Equation (13) has m equilibrium points $\bar{y}_1 < \bar{y}_2 \dots < \bar{y}_m$, where $\bar{y}_{2k-1}, k = 1, 2, \dots$ are locally asymptotically stable and $\bar{y}_{2k}, k = 1, 2, \dots$ are unstable. Then $\bar{y}_{2k-1}, k = 1, 2, \dots$ are globally asymptotically stable within the sets $[\bar{y}_{2k-1}, \bar{y}_{2k})^3$.

Figure

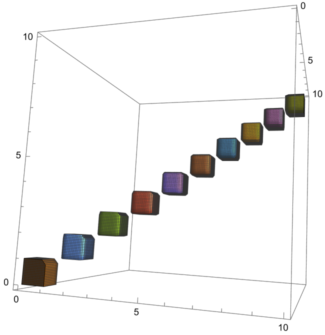

Figure: Representing a Possible (Substantial) Basin of Attraction

Application to Equation (1)

Corollary

Consider Equation (1) *subject to the condition* $a_1 + a_2 + a_3 > 2$. *Then the zero equilibrium is globally asymptotically stable with the basin of attraction that contains the set* $[0, \bar{y}_-)^3$ *and* $\bar{y}_+$ *is globally asymptotically stable with the basin of attraction that contains the set* $(\bar{y}_-, \infty)^3$.

Numerical Simulations

Since each Equilibrium solution has a substantial basin of attraction, we now make an attempt to find the complete Basins of Attraction for 0 and $\bar{y}_+$

Numerical Simulations

Since each Equilibrium solution has a substantial basin of attraction, we now make an attempt to find the complete Basins of Attraction for 0 and $\bar{y_+}$ Numerous simulations show that Equation (1) can have period-two or period-three solutions.

Example

For example, choosing $a_1 = 0.02$, $a_2 = 3$, and $a_3 = 0.02$ for Equation (1) with initial conditions $y_0 = \frac{3}{2} - \frac{1}{2}\sqrt{5}$, $y_1 = \frac{3}{2} + \frac{1}{2}\sqrt{5}$, and $y_2 = 0$, gives the following sequence,

0.381966, 2.61803, 0, 2.62058, 0.0349114, 2.61871, 0.0385646, 2.61826, 0.0393637, 2.61815, 0.039549, 2.61813, 0.0395925, 2.61812, 0.0396028, 2.61812, 0.0396052, 2.61812, 0.0396057, 2.61812, 0.0396059, 2.61812, 0.0396059, 2.61812, 0.0396059, 2.61812, 0.0396059

Example

For example, choosing $a_1 = 0.02$, $a_2 = 3$, and $a_3 = 0.02$ for Equation (1) with initial conditions $y_0 = \frac{3}{2} - \frac{1}{2}\sqrt{5}$, $y_1 = \frac{3}{2} + \frac{1}{2}\sqrt{5}$, and $y_2 = 0$, gives the following sequence,

$$0.381966, 2.61803, 0, 2.62058, 0.0349114, 2.61871, 0.0385646, 2.61826, 0.0393637, 2.61815, 0.039549, 2.61813, 0.0395925, 2.61812, 0.0396028, 2.61812, 0.0396052, 2.61812, 0.0396057, 2.61812, 0.0396059, 2.61812, 0.0396059, 2.61812, 0.0396059, 2.61812, 0.0396059$$

Taking $a_1 = 0.14$, $a_2 = .121$, and $a_3 = 10.0$ for Equation (1) with initial conditions $y_0 = 0.0012379503869046587$, $y_1 = 0.10102004113206411$, and $y_2 = 0.0014350532543266266$, gives the period-three solution $\{0.00143505, 0.00123795, 0.10102\}$.

Special Case

The special case of Equation (1)

$$y_{n+1} = a_3 \frac{y_{n-2}^2}{1 + y_{n-2}^2} \tag{14}$$

has eight period three solutions $(0, 0, \bar{x}_+), (0, \bar{x}_+, \bar{x}_+)$ and

$$\begin{aligned} &P_1 = (0, 0, \bar{x}_-), P_2 = (0, \bar{x}_-, \bar{x}_-), P_3 = (\bar{x}_+, \bar{x}_+, \bar{x}_-), \\ &P_4 = (\bar{x}_+, \bar{x}_-, \bar{x}_-), P_5 = (0, \bar{x}_-, \bar{x}_+), P_6 = (0, \bar{x}_+, \bar{x}_-) \end{aligned} \tag{15}$$

where $\bar{x}_-$ and $\bar{x}_+$ are smaller and larger positive equilibrium solution of Equation (14). The local stability character of these period three solutions follow from local stability character of three equilibrium points $0, \bar{x}_-, \bar{x}_+$ and so $(0, 0, \bar{x}_+)$ and $(0, \bar{x}_+, \bar{x}_+)$ are locally stable while the the remaining six period three points are saddle points.

These points do not belong to the basins of attraction of the major attractors $0(0,0,0)$ and $E_+(\bar{x}_+,\bar{x}_+,\bar{x}_+)$. Every solution $\{x_n\}$ of Equation (14) is clearly bounded and breaks into three subsequences $\{x_{3k}\},\{x_{3k+1}\}$, and $\{x_{3k+2}\}$ which are eventually monotonic and so are convergent. Thus every solution of Equation (14) converges to a period-three solution or to an equilibrium. Thus, the equilibrium solutions $E_0(0,0,0)$ and $E_+(\bar{x}_+,\bar{x}_+,\bar{x}_+)$ have substantial basins of attractions $\mathcal{B}(E_0)$ and $\mathcal{B}(E_+)$, respectively as well as two period three solutions $E_{3+}(0,0,\bar{x}_+)$ and $E_{3-}(\bar{x}_+,\bar{x}_+,0)$ with the basins of attractions $\mathcal{B}(E_{3+})$ and $\mathcal{B}(E_{3-})$, respectively. Thus we can formulate the result

Theorem

The boundary $\partial B(p)$ of the four basins of attractions for Equation (14) is a continuous manifold consisting of non-comparable points and contain six period period three points given with (15) as well as their stable maniflods $\mathcal{W}(P_i), i = 1, \ldots, 6$ and the equilibrium solution $E_-(\bar{x}_-, \bar{x}_-, \bar{x}_-)$. In other words

$$\partial B(p) = \{E_-(\bar{x}_-, \bar{x}_-, \bar{x}_-)\} \cup \bigcup_{i=1}^{6} \mathcal{W}(P_i).$$

Figures

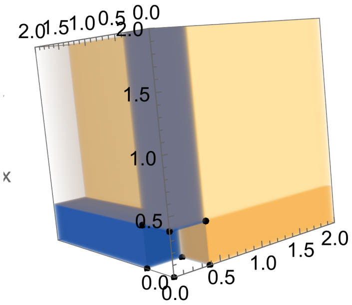

Figure: Basins of Attraction of Period 3 Points of Equation (16) with $a_3 = 3$

Figures

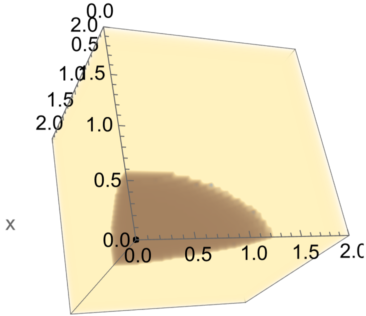

Figure: Basins of Attraction of Equilibrium Solutions of Equation (1) with $a_1 = 0,\ a_2 = 2,\ a_3 = 1$

Global Result

Theorem

The boundary $\partial B(p)$ of the basin of attraction for Equation (1) is a continuous manifold consisting of non-comparable points which contains all periodic solutions and the equilibrium solution $E_{-}(\bar{x}_{-}, \bar{x}_{-}, \bar{x}_{-})$ and is Lipschitz surface.

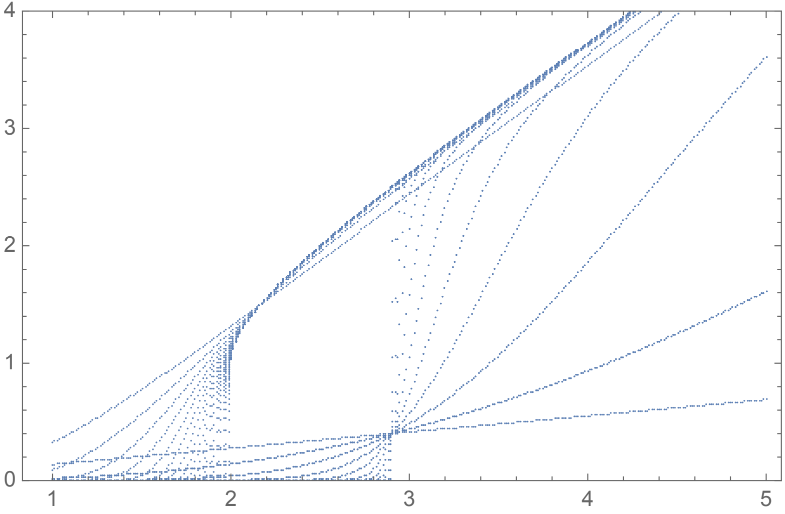

Figure: Bifurcation Plot of Equation (16)

- L. Allen, *An Introduction to Mathematical Biology*, Pearson, Boston, 2006.
- A. Bilgin and M. R. S. Kulenović, Global asymptotic stability for discrete single species population models. *Discrete Dyn. Nat. Soc.* 2017, Art. ID 5963594, 15 pp.
- A. Bilgin, M. R. S. Kulenović and E. Pilav, Basins of Attraction of Period-Two Solutions of Monotone Difference Equations, *Advances in Difference Equations*, (2016), 2016: 25p.
- E. Camouzis and G. Ladas, *Dynamics of third-order rational difference equations with open problems and conjectures.* Advances in Discrete Mathematics and Applications, 5. Chapman & Hall/CRC, Boca Raton, FL, 2008.
- A. Cima, A. Gasull and V. Manosa, Asymptotic Stability for Block Triangular Maps, *Sarajevo J. Math.*, 2022,

- J. M. Cushing, Difference equations as models of evolutionary population dynamics. *J. Biol. Dyn.* 13 (2019), no. 1, 103–127.
- J. M. Cushing, A Darwinian Ricker equation. *Progress on difference equations and discrete dynamical systems*, 231–243, Springer Proc. Math. Stat., 341, Springer, Cham, [2020]
- S. Elaydi, *An Introduction to Difference Equations*, 3rd ed., Springer-Verlag, New York, 2005.
- J. K. Hale and H. Kocak, *Dynamics and bifurcations*. Texts in Applied Mathematics, 3. Springer-Verlag, New York, 1991.
- P. Hess, *Periodic-parabolic boundary value problems and positivity.* Pitman Research Notes in Mathematics Series, 247. Longman Scientific and Technical, Harlow; copublished in the United States with John Wiley and Sons, Inc., New York, 1991.

- W. Jamieson and O. Merino, Asymptotic behavior results for solutions to some nonlinear difference equations. *J. Math. Anal. Appl.* 430 (2015), no. 2, 614–632.
- E. J. Janowski and M. R. S. Kulenović, Attractivity and global stability for linearizable difference equations, *Comput. Math. Appl.* 57 (2009), no. 9, 1592–1607.
- E. J. Janowski and M. R. S. Kulenović, Global Behavior of Certain Nonautonomous Linearizable Three Term Difference Equations, *Mathematics*, 2018; 4(1):20,
- M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations, with Open Problems and Conjectures*, Chapman& Hall/CRC Press, 2001.
- M. R. S. Kulenović, J. Marcotte and O. Merino, Cones for Coordinate-wise Monotone Functions and Dynamics of Monotone Maps, (to appear).

- M. R. S. Kulenović and O. Merino, *Discrete Dynamical Systems and Difference Equations with Mathematica*, Chapman and Hall/CRC, Boca Raton, London, 2002.
- M. R. S. Kulenović and O. Merino, A global attractivity result for maps with invariant boxes. *Discrete Contin. Dyn. Syst. Ser. B* 6(2006), 97–110.
- M. R. S. Kulenović and O. Merino, Global Bifurcation for Competitive Systems in the Plane, *Discrete Contin. Dyn. Syst. B*, 12 (2009), 133–149.
- M. R. S. Kulenović and Z. Nurkanović, Global behavior of a three-dimensional linear fractional system of difference equations, *J. Math. Anal. Appl.*, 310(2005), 673-689.
- Y. Kuznetsov, *Elements of Applied Bifurcation Theory*, Springer, NewYork, 1998.

H. R. Thieme, *Mathematics in population biology*. Princeton Series in Theoretical and Computational Biology. Princeton University Press, Princeton, NJ, 2003. xx+543 pp.

S. Wiggins, *Introduction to applied nonlinear dynamical systems and chaos*. Second edition. Texts in Applied Mathematics, 2. Springer-Verlag, New York, 2003.