Pielou Equation

The Pielou Equation is

$$x_{n+1} = p\frac{x_n}{1 + x_{n-1}} \tag{1}$$

$$\begin{align}
|x_{n+1} - x_n| &= p|\frac{x_n}{1 + x_{n-1}} - \frac{x_{n-1}}{1 + x_{n-2}}| \tag{2}\\
&= p|\frac{x_n}{1 + x_{n-1}} - \frac{x_{n-1}}{1 + x_{n-1}} + \frac{x_{n-1}}{1 + x_{n-1}} - \frac{x_{n-1}}{1 + x_{n-2}}| \tag{3}\\
&\leq p|\frac{1}{1 + x_{n-1}}||x_n - x_{n-1}| + p|\frac{x_{n-1}}{(1 + x_{n-1})(1 + x_{n-2})}||x_{n-2} - x_{n+1}| \tag{4}\\
&\leq p|x_n - x_{n-1}| + p|x_{n-1} - x_{n-2}| \tag{5}
\end{align}$$

Hence (x_n) converges if $p < 1/2$

Pielou Equation with Constant Stocking

The Pielou Equation with Constant Stocking is

$$x_{n+1} = p\frac{x_n}{1 + x_{n-1}} + h \tag{6}$$

Since h is constant, $|x_{n+1} - x_n|$ is the same as (2), so (x_n) converges when $p < 1/2$.

Pielou Equation with Proportional Stocking

The Pielou Equation with proportional stocking is

$$x_{n+1} = p\frac{x_n}{1 + x_{n-1}} + hx_n \tag{7}$$

$$\begin{align}
|x_{n+1} - x_n| &= |p\frac{x_n}{1 + x_{n-1}} - p\frac{x_{n-1}}{1 + x_{n-2}} + hx_n - hx_{n-1}| \tag{8}\\
&\leq p|\frac{x_n}{1 + x_{n-1}} - \frac{x_{n-1}}{1 + x_{n-2}}| + h|x_n - x_{n-1}| \tag{9}\\
&\leq (p + h)|x_n - x_{n-1}| + p|x_{n-1} - x_{n-2}| \tag{10}
\end{align}$$

so (x_n) converges when $2p + h < 1$.

Pielou Equation with Asymptotic Stocking

Let $h_n \to h > 0$ and

$$x_{n+1} = p\frac{x_n}{1 + x_{n-1}} + h_n \tag{11}$$

Let $\varepsilon > 0$. There exists N such that $n > N$ implies $|h_n - h_{n-1}| < \varepsilon$, so for $n > N$,

$$\begin{align}
|x_{n+1} - x_n| &= |p\frac{x_n}{1 + x_{n-1}} - p\frac{x_{n-1}}{1 + x_{n-2}} + h_n - h_{n-1}| \tag{12}\\
&\leq p|x_n - x_{n-1}| + p|x_{n-1} - x_{n-2}| + \varepsilon \tag{13}
\end{align}$$

letting $\varepsilon \to 0$ implies (x_n) converges when $p < 1/2$.

Pielou Equation with Constant Harvesting

The Pielou Equation with Constant harvesting is

$$x_{n+1} = p\frac{x_n}{1 + x_{n-1}} - h \tag{14}$$

Since h is constant, $|x_{n+1} - x_n|$ is the same as (2), so (x_n) converges when $p < 1/2$.

Pielou Equation with Proportional Harvesting

The Pielou Equation with proportional harvesting is

$$x_{n+1} = p\frac{x_n}{1 + x_{n-1}} - hx_n \tag{15}$$

$$\begin{aligned}
|x_{n+1} - x_n| &= |p\frac{x_n}{1 + x_{n-1}} - p\frac{x_{n-1}}{1 + x_{n-2}} - hx_n + hx_{n-1}| &(16)\\
&\leq p|\frac{x_n}{1 + x_{n-1}} - \frac{x_{n-1}}{1 + x_{n-2}}| + h| - x_n + x_{n-1}| &(17)\\
&\leq (p + h)|x_n - x_{n-1}| + p|x_{n-1} - x_{n-2}| &(18)
\end{aligned}$$

so (x_n) converges when $2p + h < 1$.

Pielou Equation with Asymptotic Harvesting

Let $h_n \to h > 0$ and

$$x_{n+1} = p\frac{x_n}{1 + x_{n-1}} - h_n \tag{19}$$

Let $\varepsilon > 0$. There exists N such that $n > N$ implies $|h_n - h_{n-1}| < \varepsilon$, so for $n > N$,

$$\begin{aligned}
|x_{n+1} - x_n| &= |p\frac{x_n}{1 + x_{n-1}} - p\frac{x_{n-1}}{1 + x_{n-2}} - h_n + h_{n-1}| &(20)\\
&\leq p|x_n - x_{n-1}| + p|x_{n-1} - x_{n-2}| + \varepsilon &(21)
\end{aligned}$$

letting $\varepsilon \to 0$ implies (x_n) converges when $p < 1/2$.