Université Paris Dauphine - PSL

Stochastic Calculus

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Chapter 1

Brownian Motion

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Justify that $B_t - B_s$ is independent of B_1 for all $1 \leqslant s \leqslant t$ and determine its law.

Exercise 1.2

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Exercise 1.3

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Exercise 1.4

Let $\eta = \int_0^2 B_s ds$. Compute the conditional expectation $\mathbb{E}\left[B_1 \,|\, \eta\right]$.

Exercise 1.5

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Exercise 1.6

Let $T := \inf\{t \geqslant 0 \mid |B_t| = 1\}$.

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Exercise 1.7

Justify that almost surely, $(B_t)_{t\geqslant 0}$ is not monotone on any interval.

Exercise 1.8

1. Show that for all $a > 0$, $(\frac{1}{\sqrt{a}} B_{at})_{t\geqslant 0}$ is a Brownian motion.

2. Show that for all $a > 0$, $(B_a - B_{a-t})_{0\leqslant t\leqslant a}$ is a Brownian motion.

3. Define $X_0 = 0$ and $X_t = tB_{1/t}$ for all $t > 0$. Show that $(X_t)_{t\geqslant 0}$ is a Brownian motion.

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Exercise 1.9

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1. Show that $\int_0^1 \frac{B_s}{s} ds$ is well-defined a.s.

2. Let $\beta_t := B_t - \int_0^1 \frac{B_s}{s} ds$. Show that $(\beta_t)_{t\geqslant 0}$ is a Brownian motion.

Exercise 1.13 (Brownian bridge)

We define the *Brownian bridge* as the process $Z_t := B_t - tB_1$ for all $0 \leqslant t \leqslant 1$.

1. Show that Z is a Gaussian process independent of B_1.

2. Prove that Z has the same law as the process Y defined by

$$Y_t = \begin{cases} (1-t)B_{\frac{t}{1-t}} & \text{if } 0 \leqslant t < 1, \\ 0 & \text{if } t = 1. \end{cases}$$

Exercise 1.14

Let T be a random variable distributed according to the exponential distribution of mean 1. What is the law of B_T ?

Exercise 1.15

In this exercise, $\mathbf{B}$ is a d-dimensional ($d \in \mathbb{N}$) standard Brownian motion, that is $\mathbf{B}_t = (B_t^1, \ldots, B_t^d)$, where B^i's are independent standard Brownian motions. Let $U \in \mathbb{R}^{d\times d}$ be an orthogonal matrix. Prove that the process $(\mathbf{W}_t)_{t\geqslant 0} = (U\mathbf{B}_t)_{t\geqslant 0}$ is a d-dimensional standard Brownian motion.

Exercise 1.16

Let τ be a stopping time that is almost surely finite. Show that the process $(B_{t+\tau} - B_\tau)_{t\geqslant 0}$ is a Brownian motion independent of $\mathscr{F}_\tau$.

Exercise 1.17

Prove that almost surely, $\sup_{0 \leqslant t \leqslant s} B_t > 0$ for all $s > 0$ and $\sup_{t \geqslant 0} B_t = \infty$.

Exercise 1.18

1. Show that for all $a \geqslant 0$, $\tau_a = a^2 \tau_1$ in distribution.

2. Let $0 \leqslant a \leqslant b < \infty$. Justify that $\tau_b - \tau_a$ has the same distribution as τ_{b-a} and is independent of $\mathscr{F}_{\tau_a}$.

Chapter 2

Stopping times and Martingales

For each $x \in \mathbb{R}$, we define the stopping time $\tau_x := \inf\{t \geqslant 0 : B_t = x\}$.

Exercise 2.1

1. Show that $(B_t^2 - t)_{t \geqslant 0}$ is a martingale.

2. Construct a martingale from $(B_t^3)_{t \geqslant 0}$. Same question with B_t^4.

3. Prove that $(e^{\lambda B_t - \frac{\lambda^2}{2} t})_{t \geqslant 0}$ is a martingale for all $\lambda \in \mathbb{R}$.

Solution

1. To show that $(B_t^2 - t)$ is a martingale, we compute the conditional expectation:

$$\mathbb{E}\left[B_t^2 \,\middle|\, \mathscr{F}_s\right] = \mathbb{E}\left[(B_t - B_s + B_s)^2 \,\middle|\, \mathscr{F}_s\right] = B_s^2 + \mathbb{E}\left[(B_t - B_s)^2\right] = B_s^2 + (t - s),$$

since $B_t - B_s$ is independent of $\mathscr{F}_s$ and normally distributed with mean zero and variance $t - s$. Therefore, we have

$$\mathbb{E}\left[B_t^2 - t \,\middle|\, \mathscr{F}_s\right] = B_s^2 - s,$$

confirming that $(B_t^2 - t)_{t \geqslant 0}$ is indeed a martingale.

2. For all $t > s$, we expand B_t^3 using $B_t = B_s + (B_t - B_s)$:

$$B_t^3 = (B_s + (B_t - B_s))^3 = B_s^3 + 3B_s^2(B_t - B_s) + 3B_s(B_t - B_s)^2 + (B_t - B_s)^3.$$

Taking the conditional expectation given $\mathscr{F}_s$, we find:

$$\mathbb{E}\left[B_t^3 \,\middle|\, \mathscr{F}_s\right] = B_s^3 + 3B_s\mathbb{E}\left[(B_t - B_s)^2\right] = B_s^3 + 3B_s(t - s).$$

Therefore, the process $(B_t^3 - 3B_s t)_{t \geqslant 0}$ is a martingale.

Similarly, for B_t^4, we expand:

$$B_t^4 = (B_s + (B_t - B_s))^4 = B_s^4 + 4B_s^3(B_t - B_s) + 6B_s^2(B_t - B_s)^2 + 4B_s(B_t - B_s)^3 + (B_t - B_s)^4.$$

Taking the conditional expectation, we obtain:

$$\mathbb{E}\left[B_t^4 \,\middle|\, \mathscr{F}_s\right] = B_s^4 + 6B_s^2(t - s) + 3(t - s)^2.$$

Thus, $(B_t^4 - 6B_t^2 t + 3t^2)_{t \geqslant 0}$ is a martingale.

3. Let $\lambda \in \mathbb{R}$. We compute the conditional expectation:

$$\begin{aligned}\mathbb{E}\left[e^{\lambda B_t-\frac{\lambda^2}{2}t}\,\middle|\,\mathscr{F}_s\right] &= e^{\lambda B_s-\frac{\lambda^2}{2}s}\mathbb{E}\left[e^{\lambda(B_t-B_s)-\frac{\lambda^2}{2}(t-s)}\,\middle|\,\mathscr{F}_s\right]\\ &= e^{\lambda B_s-\frac{\lambda^2}{2}s}\mathbb{E}\left[e^{\lambda(B_t-B_s)-\frac{\lambda^2}{2}(t-s)}\right]\\ &= e^{\lambda B_s-\frac{\lambda^2}{2}s},\end{aligned}$$

since $B_t - B_s$ is independent of $\mathscr{F}_s$ and $\mathbb{E}\left[e^{\lambda(B_t-B_s)-\frac{\lambda^2}{2}(t-s)}\right] = 1$ due to the properties of the normal distribution. This confirms that $(e^{\lambda B_t-\frac{\lambda^2}{2}t})_{t\geqslant 0}$ is a martingale.

Exercise 2.2

Let $a > 0$ and define $T_a^\star := \inf\{t \geqslant 0 : |B_t| = a\}$.

1. Using the martingale $(B_t^2 - t)_{t\geqslant 0}$ compute the expectation of $T_a^\star$.

2. Using a well-chosen martingale, compute the variance of $T_a^\star$ and its Laplace transform.

3. Compute the Laplace transform of τ_a and find that it has the law as $(a/B_1)^2$. What is the value of $\mathbb{E}[\tau_a]$?

Solution

1. Since $T_a^\star$ is almost surely finite, the stopped martingale $(B^2_{t\wedge T_a^\star} - t\wedge T_a^\star)$ converges almost surely to $B^2_{T_a^\star} - T_a^\star = a^2 - T_a^\star$. Moreover, $B^2_{t\wedge T_a^\star}$ is bounded by a^2, ensuring L^1 convergence by the dominated convergence theorem. In the other hand, $t\wedge T_a^\star$ converge in L^1 to $T_a^\star$ by monotone convergence. Therefore,

$$0 = \mathbb{E}\left[B^2_{t\wedge T_a^\star} - t\wedge T_a^\star\right] \xrightarrow[t\to\infty]{} a^2 - \mathbb{E}\left[T_a^\star\right].$$

Hence, $\mathbb{E}\left[T_a^\star\right] = a^2$.

2. To compute the variance, we consider the martingale $(B_t^4 - 6B_t^2 t + 3t^2)_{t\geqslant 0}$. Applying similar reasoning as before, we have:

$$\mathbb{E}\left[B^4_{t\wedge T_a^\star} - 6B^2_{t\wedge T_a^\star}(t\wedge T_a^\star) + 3(t\wedge T_a^\star)^2\right] = 0.$$

Taking the limit as $t\to\infty$ and using $B^2_{T_a^\star} = a^2$, we get:

$$a^4 - 6a^2\mathbb{E}\left[T_a^\star\right] + 3\mathbb{E}\left[(T_a^\star)^2\right] = 0.$$

Substituting $\mathbb{E}\left[T_a^\star\right] = a^2$, we solve for $\mathbb{E}\left[(T_a^\star)^2\right]$:

$$a^4 - 6a^4 + 3\mathbb{E}\left[(T_a^\star)^2\right] = 0 \quad\Longrightarrow\quad \mathbb{E}\left[(T_a^\star)^2\right] = \frac{5}{3}a^4.$$

Therefore, the variance is:

$$\mathrm{Var}\left(T_a^\star\right) = \mathbb{E}\left[(T_a^\star)^2\right] - \left(\mathbb{E}\left[T_a^\star\right]\right)^2 = \frac{5}{3}a^4 - a^4 = \frac{2}{3}a^4.$$

For the Laplace transform, consider the martingale for $\lambda > 0$ (check it):

$$M_t = \exp\left(\sqrt{2\lambda}B_t - \lambda t\right) + \exp\left(-\sqrt{2\lambda}B_t - \lambda t\right).$$

The stopped martingale $M^{T_a^\star}$ is dominated in L^1 by $2e^{\sqrt{2\lambda}a}$, and thus uniformly integrable. Therefore, $\mathbb{E}\left[M_{T_a^\star}\right] = 2$. Evaluating $M_{T_a^\star}$, we have:

$$\begin{aligned} 2 = \mathbb{E}\left[M_{T_a^\star}\right] &= \mathbb{E}\left[e^{-\lambda T_a^\star}(e^{\sqrt{2\lambda}a} + e^{-\sqrt{2\lambda}a})\mathbb{1}_{\{B_{T_a^\star}=a\}}\right] + \mathbb{E}\left[e^{-\lambda T_a^\star}(e^{-\sqrt{2\lambda}a} + e^{\sqrt{2\lambda}a})\mathbb{1}_{\{B_{T_a^\star}=-a\}}\right] \\ &= 2\cosh(\sqrt{2\lambda}a)\mathbb{E}\left[X\right] \end{aligned}$$

Hence $\mathbb{E}\left[e^{-\lambda T_a^\star}\right] = \frac{1}{\cosh(\sqrt{2\lambda}a)}$.

3. To compute the Laplace transform of τ_a, we utilize the martingale $M_t = e^{\sqrt{2\lambda}B_t - \lambda t}$. At the stopping time τ_a, we have $B_{\tau_a} = a$, so:

$$M_{\tau_a} = e^{\sqrt{2\lambda}a - \lambda\tau_a}.$$

Since M_t is a martingale, $\mathbb{E}\left[M_{\tau_a}\right] = M_0 = 1$. Therefore,

$$\mathbb{E}\left[e^{-\lambda\tau_a}\right] = e^{-\sqrt{2\lambda}a}.$$

Exercise 2.3

1. Let M be a continuous martingale such that $M_0 = x \geqslant 0$. Suppose that $M_t \geqslant 0$ for all $t \geqslant 0$ and that $M_t \to 0$ as $t \to \infty$, a.s. Show that, for all $y > x$,

$$\mathbb{P}\left(\sup_{t\geqslant 0} M_t \geqslant y\right) = \frac{x}{y}.$$

2. Deduce the law of

$$\sup_{t\leqslant T_0} B_t,$$

when B is a Brownian motion started at $x > 0$ and $T_0 = \inf\{t \geqslant 0 : B_t = 0\}$.

3. Suppose now that is started at 0, and let $\mu > 0$. Using a well-chosen exponential martingale, prove that

$$\sup_{t\geqslant 0}(B_t - \mu t)$$

follows the exponantial distribution of parameter 2μ.

Exercise 2.4

Let $a < 0 < b$ and set $T = \tau_a \wedge \tau_b$.

1. Prove that, for all $\lambda > 0$,

$$\mathbb{E}\left[\exp\left(-\lambda T\right)\right] = \frac{\cosh\left(\frac{b+a}{2}\sqrt{2\lambda}\right)}{\cosh\left(\frac{b-a}{2}\sqrt{2\lambda}\right)}.$$

(*Hint*: introduce the martingale

$$M_t = \exp\left(\sqrt{2\lambda}(B_t - \alpha) - \lambda t\right) + \exp\left(-\sqrt{2\lambda}(B_t - \alpha) - \lambda t\right),$$

with a well-chosen α.)

2. Similarly, prove that, for all $\lambda > 0$,

$$\mathbb{E}\left[\exp(-\lambda T)\mathbb{1}_{\{T=\tau_a\}}\right] = \frac{\sinh\left(b\sqrt{2\lambda}\right)}{\sinh\left((b-a)\sqrt{2\lambda}\right)}.$$

3. With the help of question 2, give an expression of $\mathbb{P}(\tau_a < \tau_b)$.

Exercise 2.5

Let $M = (M_t)_{0\leqslant t\leqslant 1}$ be a sub-martingale. Let $(\mathscr{G}_s)_{s\geqslant 0}$ be a sub-filtration of $(\mathscr{F}_s)_{s\geqslant 0}$. Prove that $N_t = \mathbb{E}[M_t \mid \mathscr{G}_t]$ is a $(\mathscr{G}_s)$-sub-martingale.

Solution

First note that N is clearly adapted and integrable. We prove the martingale property, let $0 \leqslant s \leqslant t$:

$$\begin{aligned}
\mathbb{E}[N_t \mid \mathscr{G}_s] &= \mathbb{E}[\mathbb{E}[M_t \mid \mathscr{G}_t] \mid \mathscr{G}_s] \\
&= \mathbb{E}[M_t \mid \mathscr{G}_s] \\
&= \mathbb{E}[\mathbb{E}[M_t \mid \mathscr{F}_s] \mid \mathscr{G}_s] \\
&\geqslant \mathbb{E}[M_s \mid \mathscr{G}_s] = N_s.
\end{aligned}$$

Exercise 2.6

Let $\sigma \leqslant \tau$ be two bounded stopping times. Show that

$$\mathbb{E}\left[(B_\tau - B_\sigma)^2\right] = \mathbb{E}\left[B_\tau^2\right] - \mathbb{E}\left[B_\sigma^2\right] = \mathbb{E}[\tau - \sigma].$$

Exercise 2.7

Let $M = (M_t)_{0\leqslant t\leqslant 1}$ be a sub-martingale such that $\mathbb{E}[M_0] = \mathbb{E}[M_1]$. Prove that M is a martingale.

Solution

Since M is a sub-martingale we have for all $t \in [0,1]$, $\mathbb{E}[M_0] \leqslant \mathbb{E}[M_t] \leqslant \mathbb{E}[M_1]$, hence $\mathbb{E}[M_0] = \mathbb{E}[M_t]$ for all t. Let $0 \leqslant s \leqslant t \leqslant 1$, M is a sub-martingale so that $\mathbb{E}[M_t - M_s \mid \mathscr{F}_s]$ is non negative and of expectation 0, therefore $\mathbb{E}[M_t - M_s \mid \mathscr{F}_s] = 0$ and M is a martingale.

Exercise 2.8

Let M be a càd martingale. Let $t \geqslant 0$. Prove that $M_{t+\varepsilon} \xrightarrow[\varepsilon\to 0]{L^1} M_t$.

Solution

The càd property give us $M_{t+\varepsilon} \xrightarrow[\varepsilon\to 0]{\text{a.s.}} M_t$. For every $n \geqslant 1$ we have

$$\mathbb{E}\left[M_{t+1} \,\middle|\, \mathscr{F}_{t+\frac{1}{n}}\right] = M_{t+\frac{1}{n}}.$$

Hence the sequence $(M_{t+1/n})_{n\geqslant 1}$ is uniformly integrable and $M_{t+1/n} \xrightarrow[n\to\infty]{L^1} M_t$.

Exercise 2.9

Let M be a local continuous martingale such that $M_0 = 0$ a.s.

1. Let $a > 0$ and let $\sigma_a := \inf\{t \geqslant 0 : \langle M, M\rangle_t \geqslant a^2\}$. Show that

$$\mathbb{P}\left(\sup_{s\in[0,\sigma_a]}|M_s| > a\right) \leqslant \frac{1}{a^2}\mathbb{E}\left[a^2 \wedge \langle M, M\rangle_\infty\right].$$

2. Show that $\mathbb{P}\left(\sup_{t\geqslant 0}|M_t| > a\right) \leqslant \mathbb{P}(\langle M, M\rangle_\infty \geqslant a^2) + a^{-2}\mathbb{E}\left[a^2 \wedge \langle M, M\rangle_\infty\right]$.

3. Show that $\mathbb{E}\left[\sup_{t\geqslant 0}|M_t|\right] \leqslant 3\mathbb{E}\left[\sqrt{\langle M, M\rangle_\infty}\right]$.

4. Show that if $\mathbb{E}\left[\sqrt{\langle M, M\rangle_\infty}\right] < \infty$, then M is a uniformly integrable martingale.

5. Show that if $\mathbb{E}\left[\sqrt{\langle M, M\rangle_t}\right] < \infty$ for every $t \geqslant 0$, then M is a integrable martingale.

Solution

1. We observe that $\mathbb{P}\left(\sup_{s\in[0,\sigma_a]}|M_s| > a\right) = \mathbb{P}\left(\sup_{t\geqslant 0}|M_{t\wedge\sigma_a}| > a\right)$ and since $\mathbb{E}\left[\langle M\rangle_{t\wedge\sigma_a}\right] \leqslant a^2$, M^{σ_a} and $(M^{\sigma_a})^2 - \langle M^{\sigma_a}\rangle$ are uniformly integrable martingales. Therefore we can apply Doob maximal inequality:

$$a^2\mathbb{P}\left(\sup_{t\geqslant 0}|M_{t\wedge\sigma_a}| > a\right) \leqslant \sup_{t\geqslant 0}\mathbb{E}\left[M_{t\wedge\sigma_a}^2\right] \leqslant \mathbb{E}\left[\langle M\rangle_{\sigma_a}\right] \leqslant \mathbb{E}\left[a^2 \wedge \langle M\rangle_\infty\right].$$

Indeed, $\langle M\rangle_{\sigma_a} = a^2\mathbb{1}_{\sigma_a<\infty} + \langle M\rangle_\infty(1 - \mathbb{1}_{\sigma_a<\infty}) \leqslant a^2$ and $a^2\mathbb{1}_{\sigma_a<\infty} + \langle M\rangle_\infty(1 - \mathbb{1}_{\sigma_a<\infty}) \leqslant \langle M\rangle_\infty$.

2. We have

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\geqslant 0}|M_t| > a\right) &= \mathbb{P}\left(\sup_{t\geqslant 0}|M_t| > a, \sigma_a < \infty\right) + \mathbb{P}\left(\sup_{t\geqslant 0}|M_t| > a, \sigma_a = \infty\right)\\
&\leqslant \mathbb{P}(\sigma_a < \infty) + \mathbb{P}\left(\sup_{t\in[0,\sigma_a]}|M_t| > a\right).
\end{aligned}$$

3. By integrating with respect to a we have

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\geqslant 0}|M_t|\right] &\leqslant \int_0^\infty \mathbb{P}(\langle M, M\rangle_\infty \geqslant a^2) + a^{-2}\mathbb{E}\left[a^2 \wedge \langle M, M\rangle_\infty\right] da\\
&\leqslant \int_0^\infty \mathbb{P}(\sqrt{\langle M, M\rangle_\infty} \geqslant a)da + \mathbb{E}\left[\int_0^\infty 1 \wedge \frac{\langle M, M\rangle_\infty}{a^2} da\right]\\
&\leqslant \mathbb{E}\left[\sqrt{\langle M, M\rangle_\infty}\right] + \mathbb{E}\left[\int_0^{\sqrt{\langle M,M\rangle_\infty}} da\right] + \mathbb{E}\left[\langle M, M\rangle_\infty \int_{\sqrt{\langle M,M\rangle_\infty}}^\infty a^{-2}da\right]\\
&\leqslant 3\mathbb{E}\left[\sqrt{\langle M, M\rangle_\infty}\right].
\end{aligned}$$

4. Using the previous question we have that $\sup_{t\geqslant 0}|M_t| = C < \infty$ which implies that the local martingale M is dominated in L^1 and thus a uniformly integrable martingale.

5. By the same reasoning we that for every $T > 0$, the process $(M_t)_{t\in[0,T]}$ is dominated in L^1, that is a martingale on $[0, T]$ for every $T > 0$ and thus a martingale on $\mathbb{R}_+$.

Chapter 3

Stochastic Integration

Exercise 3.1

Let M be a local martingale. Show that the family $\{M_\tau,\ \tau < \infty\}$ is uniformly integrable if, and only if, M is a uniformly integrable martingale.

Solution

Assume first that M is a uniformly integrable martingale, therefore for all stopping time τ we have that $M_\tau = \mathbb{E}\left[M_\infty \,|\, \mathscr{F}_\tau\right]$ hence $\{M_\tau,\ \tau < \infty\}$ is uniformly integrable.

Assume now that the family $\{M_\tau,\ \tau < \infty\}$ is uniformly integrable, in particular the family $\{M_t\}_{t \geqslant 0}$ is uniformly integrable, we have to prove the martingale property. Let τ_n be a localizing sequence, for every $0 \leqslant s \leqslant t$ and every $A \in \mathscr{F}_{s \wedge \tau_n} \subseteq \mathscr{F}_s$, we know that

$$\mathbb{E}\left[M_{s \wedge \tau_n} \mathbb{1}_A\right] = \mathbb{E}\left[M_{t \wedge \tau_n} \mathbb{1}_A\right].$$

By assumption, we have that $(M_{x \wedge \tau_n})_{x \geqslant 0}$ is uniformly integrable, so we have L^1 converge in both terms, we have

$$\mathbb{E}\left[M_s \mathbb{1}_A\right] = \mathbb{E}\left[M_t \mathbb{1}_A\right]$$

for all $A \in \mathscr{F}_s$ and M is a uniformly integrable martingale.

Exercise 3.2

Let M be a bounded local martingale. Show that $\langle M \rangle_\infty < \infty$ a.s.

Solution

Clearly, M is a bounded martingale. Hence $\lim_{t \to \infty} M_t^2 = M_\infty^2$ in L^1 and $\lim_{t \to \infty} \mathbb{E}\left[\langle M \rangle_t\right] = \mathbb{E}\left[\langle M \rangle_\infty\right]$ by monotone convergence. We obtain

$$0 = \mathbb{E}\left[M_t^2 - \langle M \rangle_t\right] \xrightarrow[t \to \infty]{} \mathbb{E}\left[M_\infty^2 - \langle M \rangle_\infty\right].$$

This implies that $\mathbb{E}\left[\langle M \rangle_\infty\right] = \mathbb{E}\left[M_\infty^2\right] < \infty$.

Exercise 3.3

Let B be a standard Brownian motion with $B_0 = x > 0$. Set $T = \inf\{t \geqslant 0 : B_t = 0\}$. Let $f : \mathbb{R}_+ \to \mathbb{R}$ compactly supported. Compute $\mathbb{E}\left[\int_0^T f(B_s)ds\right]$.

Exercise 3.4

Let $\alpha : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function. With the help of the stochastic integration by parts formula, check that the process

$$X_t := \int_0^t \exp\left(\int_s^t \alpha(u)du\right) dB_s,$$

satisfies the stochastic differential equation $dX_t = \alpha(t)X_t dt + dB_t$.

Solution

First, we have for every $t \geqslant 0$

$$\begin{aligned} X_t &= \int_0^t \exp\left(\int_0^t \alpha(u)du - \int_0^s \alpha(u)du\right) dB_s \\ &= \exp\left(\int_0^t \alpha(s)ds\right) \int_0^t \exp\left(-\int_0^s \alpha(u)du\right) dB_s \\ &= e^{Y_t} \int_0^t e^{-Y_s} dB_s \end{aligned}$$

Where $Y_t = \int_0^t \alpha(s)ds$. Now using integration by part we obtain

$$\begin{aligned} dX_t &= d\left(e^{Y_t}\right) \int_0^t e^{-Y_s} dB_s + e^{Y_t} d\left(\int_0^t e^{-Y_s} dB_s\right) \\ &= \alpha(t)e^{Y_t}dt \int_0^t e^{-Y_s} dB_s + e^{Y_t}e^{-Y_t}dB_t. \end{aligned}$$

Hence, $dX_t = \alpha(t)X_t dt + dB_t$.

Exercise 3.5

Let $f : \mathbb{R} \to \mathbb{R}$ be continuously twice-differentiable. Show that the process

$$X_t := f(B_t) - \frac{1}{2}\int_0^t f''(B_s)ds,$$

is a continuous local martingale. Give a sufficient condition for X to be a martingale.

Exercise 3.6

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be a (deterministic) function in $L^2_{\text{loc}}(\mathbb{R}_+)$ (i.e. $\int_0^t \varphi^2(s)ds < \infty$ for every $t \geqslant 0$) and $Z^\varphi = (Z_t^\varphi)_{t \geqslant 0}$ the associated Doléans-Dade exponential process. Check that Z^φ is a martingale.

Solution

Since, $\varphi \in L^2_{\text{loc}}$, $\int_0^t \varphi(s)dB_s$ is well-defined and is a centered gaussian random variable with variance $\int_0^t \varphi(s)^2 ds < \infty$, this implies that $Z_t^\varphi \in L^1$ for every $t \geqslant 0$. Let $0 \leqslant s \leqslant t$ we have

$$\mathbb{E}\left[e^{\int_0^t \varphi(u)dB_u - \frac{1}{2}\int_0^t \varphi(u)^2 du} \,\middle|\, \mathscr{F}_s \right] = \mathbb{E}\left[e^{\int_0^t \varphi(u)dB_u - \int_0^s \varphi(u)dB_u} \,\middle|\, \mathscr{F}_s \right] e^{\int_0^s \varphi(u)dB_u - \frac{1}{2}\int_0^t \varphi(u)^2 du}.$$

Recall that, $\int_0^t \varphi(u)dB_u - \int_0^s \varphi(u)dB_u = \int_s^t \varphi(u)dB_u$ is independent of $\mathscr{F}_s$ and is a centered gaussian variable with variance $\int_s^t \varphi(s)^2 ds$, we thus obtain

$$\begin{aligned}
\mathbb{E}\left[e^{\int_0^t \varphi(u)dB_u - \frac{1}{2}\int_0^t \varphi(u)^2 du} \,\middle|\, \mathscr{F}_s \right] &= \mathbb{E}\left[e^{\int_s^t \varphi(u)dB_u} \right] e^{\int_0^s \varphi(u)dB_u - \frac{1}{2}\int_0^t \varphi(u)^2 du} \\
&= e^{\frac{1}{2}\int_s^t \varphi(s)^2 ds} \cdot e^{\int_0^s \varphi(u)dB_u - \frac{1}{2}\int_0^t \varphi(u)^2 du} \\
&= e^{\int_0^s \varphi(u)dB_u - \frac{1}{2}\int_0^s \varphi(u)^2 du}.
\end{aligned}$$

Hence the result.

Exercise 3.7

Find a progressive process $X = (X_t)_{t \geqslant 0}$ such that the process $Z = (Z_t)_{t \geqslant 0}$ defined by $Z_t = \exp(X_t - B_t^2)$ is a martingale.

Solution

Assume that the process X has the form

$$X_t = \int_0^t b_s ds + \int_0^t \sigma_s dB_s,$$

and $X_0 = 0$, then by Itô's formula we have

$$\begin{aligned}
Z_t &= 1 + \int_0^t Z_s dX_s - 2\int_0^t Z_s B_s dB_s + \frac{1}{2}\int_0^t Z_s d\langle X, X\rangle_s - 2\int_0^t Z_s B_s d\langle X, B\rangle_s + \int_0^t Z_s(2B_s^2 - 1)ds \\
&= 1 + \int_0^t Z_s(\sigma_s - 2B_s)dB_s + \int_0^t Z_s\left(\frac{1}{2}\sigma_s^2 - 2B_s\sigma_s + 2B_s^2 - 1 + b_s\right)ds.
\end{aligned}$$

Then, by taking $X_t = \int_0^t (1 - B_s^2/2)ds + \int_0^t B_s dB_s$, $Z_t = 1 - \int_0^t Z_s B_s dB_s$ is a local martingale. Using the fact that $B_t^2 - t = 2\int_0^t B_s dB_s$, we have

$$Z_t = e^{-\int_0^t B_s dB_s - \frac{1}{2}\int_0^t B_s^2 ds}.$$

Exercise 3.8

Let X and Y be two $(\mathscr{F}_t)$ independent Brownian motions and let H be a progressive process. We set

$$\begin{aligned}
\beta_t &= \int_0^t \cos(H_s)dX_s - \int_0^t \sin(H_s)dY_s, \\
\gamma_t &= \int_0^t \sin(H_s)dX_s + \int_0^t \cos(H_s)dY_s.
\end{aligned}$$

Show that β and γ are independent $(\mathscr{F}_t)$ Brownian motions.

Exercise 3.9

Let B be a Brownian motion. Show that $\int_0^t \mathbb{1}_{\{B_s=0\}} dB_s = 0$.

Exercise 3.10

1. Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function. Let $f : \mathbb{R}_+ \to (0, \infty)$ be a $\mathscr{C}^2$ function such that $f'' = 2gf$ on $\mathbb{R}_+$ and $f(0) = 1$, $f'(1) = 0$. We set

$$u(t) := \frac{f'(t)}{2f(t)}, \quad t \geqslant 0.$$

Show that $u' + 2u^2 = g$ on $\mathbb{R}_+$.

2. Let β a $(\mathscr{F}_t)$ standard Brownian motion. Let $x_0 \geqslant 0$, $a \geqslant 0$ and X an adapted, continous and non negative process such that

$$X_t = x_0 + 2\int_0^t \sqrt{X_s} d\beta_s + at.$$

Show that $u(t)X_t - \int_0^t g(s)X_s ds = u(0)X_0 + \int_0^t u(s)dX_s - 2\int_0^t u(s)^2 X_s ds$, $t \geqslant 0$.

3. Set $M_t := u(0)x_0 + 2\int_0^t u(s)\sqrt{X_s} d\beta_s$, $t \geqslant 0$. Show that

$$f(t)^{-a/2} \exp\left(u(t)X_t - \int_0^t g(s)X_s ds\right) = \mathscr{E}(M)_t.$$

4. Show that f is non increasing on $[0, 1]$ and show that

$$\mathbb{E}\left[\exp\left(-\int_0^1 g(s)X_s ds\right)\right] = f(1)^{a/2} e^{x_0 f'(0)/2}.$$

5. Show that

$$\mathbb{E}\left[\exp\left(-\frac{\theta^2}{2}\int_0^1 X_s ds\right)\right] = \frac{1}{\cosh(\theta)^{a/2}} \exp\left(-\frac{x_0}{2}\theta\tanh(\theta)\right), \quad \forall\theta \in \mathbb{R}.$$

6. Let B be a standard Brownian motion. For every $x \in \mathbb{R}$, show that,

$$\mathbb{E}\left[\exp\left(-\frac{\theta^2}{2}\int_0^1 (B_s + x)^2 ds\right)\right] = \frac{1}{\cosh(\theta)^{1/2}} \exp\left(-\frac{x^2}{2}\theta\tanh(\theta)\right), \quad \forall\theta \in \mathbb{R}.$$

7. Let B and $\tilde{B}$ be two independent standard Brownian motions. For every $t > 0$ show that $\inf\{s \geqslant 0 : |B_s| = t\} = \int_0^t B_s^2 ds + \int_0^t \tilde{B}_s^2 ds$ in law.

Solution

4. Use definition of f, that is, we have domination in L^1 for every $t \in [0, 1]$ $(X \geqslant 0, u \leqslant 0)$.

5. Take $f(t) = \frac{\cosh(\theta(t-1))}{\cosh(\theta)}$.

Exercise 3.11

Let B be a Brownian motion and let $S_t := \sup_{s\in[0,t]} B_s$. We set $X_t := S_t - B_t$.

1. Show that $\int_0^t \mathbb{1}_{\{X_u \neq 0\}} dS_u = 0$.

2. Show that $Y_t := X_t^2 - t$ is a martingale.

3. Let $\tau := \inf\{t \geqslant 0 : X_t = 1\}$. Compute $\mathbb{E}[\tau]$.

Chapter 4

Stochastic differential equations

Exercise 4.1

Let $M_t := \frac{1}{2}(B_t^2 - t)$, $t \geqslant 0$.

1. Justify that M is a martingale and express M as a stochastic integral.
2. Show that for all $b \geqslant 0$, the exponential local martingale $\mathscr{E}(-bM)$ is a martingale. For all $T > 0$, justify that $\mathbb{Q} := \mathscr{E}(-bM)_T \cdot \mathbb{P}$ defines a probability measure.
3. Determine the SDE satisfied by $(B_t)_{t\in[0,T]}$ on $\mathbb{Q}$. Deduce the distribution of B_t, $t \in [0,T]$ on $\mathbb{Q}$.
4. Deduce that for all $a, b \geqslant 0$,
$$\mathbb{E}\left[\exp\left\{-aB_t^2 - \frac{b^2}{2}\int_0^t B_s^2 ds\right\}\right] = \left(\frac{b}{b\cosh(bt) + 2a\sinh(bt)}\right)^{1/2}.$$
5. Using that for all $\alpha, \beta > 0$ and $s \geqslant 0$,
$$\frac{1}{\Gamma(\alpha)}\int_0^\infty \beta^\alpha x^{\alpha-1} e^{-(\beta+s)x} dx = \left(\frac{\beta}{s+\beta}\right)^\alpha,$$
compute
$$\mathbb{E}\left[\exp\left\{-\frac{b^2}{2}\int_0^t B_s^2 ds\right\}\middle| B_t = y\right], \quad b > 0,\ y \in \mathbb{R}.$$

Solution

1. By Itô's formula we have that $B_t^2 = 2\int_0^t B_s dB_s + t$, hence $M_t = \int_0^t B_s dB_s$ is a local martingale. Moreover, $\mathbb{E}\left[\langle M\rangle_t\right] = \mathbb{E}\left[\int_0^t B_s^2 ds\right] = \int_0^t \mathbb{E}\left[B_s^2\right] ds = \frac{t^2}{2} < \infty$, hence M is a martingale.
2. For every $T \geqslant 0$, $t \in [0,T]$, we have $\mathscr{E}(-bM)_t = e^{-\frac{b}{2}B_t^2 + \frac{bt}{2} - \frac{b^2}{2}\langle M\rangle_t} \leqslant e^{\frac{bT}{2}}$. Hence, $\mathscr{E}(-bM)$ is a martingale, $\mathbb{E}\left[\mathscr{E}(-bM)_T\right] = 1$ and thus $\mathbb{Q}$ define a probability measure.
3. By Girsanov theorem we have that $\beta_t := B_t + b\langle B, M\rangle_t$ is a Brownian motion under $\mathbb{Q}$, i.e. B solve
$$dB_t = d\beta_t - bB_t dt.$$

We recognize the SDE satisfied by the Ornstein-Uhlenbeck process and thus,

$$B_t = \int_0^t e^{-b(t-s)} d\beta_s.$$

Therefore, B is a centered gaussian process with variance $\frac{1-e^{-2bt}}{2b}$.

4. For every $a, b \geqslant 0$ we have

$$\begin{aligned}
\mathbb{E}\left[\exp\left\{-aB_t^2 - \frac{b^2}{2}\int_0^t B_s^2 ds\right\}\right] &= \mathbb{E}\left[e^{-aB_t^2 + b(B_t^2 - t)}\mathscr{E}(-bM)_t\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[e^{(\frac{b}{2}-a)B_t^2 - \frac{b}{2}t}\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[e^{(\frac{b}{2}-a)(\frac{1-e^{-2bt}}{2b})Z - \frac{b}{2}t}\right]
\end{aligned}$$

Where $Z \sim \mathcal{X}^2(1)$, and the expectation is well defined if $(\frac{b}{2}-a)(\frac{1-e^{-2bt}}{2b}) < \frac{1}{2}$, indeed

$$\left(\frac{b}{2}-a\right)\left(\frac{1-e^{-2bt}}{2b}\right) \leqslant \frac{b-2a}{4b} < \frac{1}{2}.$$

Therefore,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[e^{(\frac{b}{2}-a)(\frac{1-e^{-2bt}}{2b})Z - \frac{b}{2}t}\right] &= \frac{e^{-\frac{bt}{2}}}{\sqrt{1-(\frac{b}{2}-a)(\frac{1-e^{-2bt}}{b})}} \\
&= \frac{1}{\sqrt{e^{bt}(1-(\frac{b}{2}-a)(\frac{1-e^{-2bt}}{b}))}} \\
&= \left(\frac{b}{be^{bt}-(\frac{b}{2}-a)(e^{bt}-e^{-bt})}\right)^{1/2} \\
&= \left(\frac{b}{b\cosh(bt)+2a\sinh(bt)}\right)^{1/2}.
\end{aligned}$$

Exercise 4.2

1. Justify that for all $T < 1$ and $x \in \mathbb{R}$, there exists almost surely a solution to the SDE:

$$X_t^x = x + B_t - \int_0^t \frac{X_s^x}{1-s} ds, \quad t \in [0,T].$$

2. By applying the Itô formula to $(\frac{X_t^0}{1-t})_{t\in[0,T]}$, find an explicit formula for $(X_t^0)_{t\in[0,T]}$.

3. Show that $X_t^x = X_t^0 + x(1-t)$, $t \in [0,T]$ and determine its distribution.

4. Show that X^x can be extented to a continuous process on $[0,1]$.

5. What does $(X_t^x)_{t\in[0,1]}$ represent ?

Solution

1. The coefficients $\sigma(t,x)=1$ and $b(t,x)=-\frac{x}{1-t}$ are Lipschitz in space and bounded in time, thus there exists a unique solution.

2. By Itô formula we have

$$\frac{X_t^0}{1-t}=\int_0^t \frac{1}{1-s}dX_s^0+\int_0^t \frac{X_s^0}{(1-s)^2}ds=\int_0^t \frac{1}{1-s}dB_s.$$

Hence, $X_t^0=\int_0^t \frac{1-t}{1-s}dB_s$ and $X_t^0\sim\mathcal{N}\left(0,\int_0^t(\frac{1-t}{1-s})^2ds\right)=\mathcal{N}(0,t(1-t))$ for every $t\in[0,T]$.

3. Let $t\in[0,T]$,

$$x+B_t-\int_0^t \frac{X_t^0+x(1-t)}{1-s}ds=B_t-\int_0^t \frac{X_t^0}{1-s}ds+x-\int_0^t xds=X_t^0+x(1-t).$$

$X_t^0+x(1-t)$ solve the SDE and by uniqueness we have that $X_t^x=X_t^0+x(1-t)$. Moreover $X_t^x\sim\mathcal{N}(x(1-t),t(1-t))$.

4. It suffices to show that X^0 extend to a continous process on $[0,1]$. We know that $\frac{X_t^0}{t-1}$ is a centered gaussian process with covariance $\min\left(\frac{t}{1-t},\frac{s}{1-s}\right)$, thus $X_t^0=(1-t)B_{\frac{t}{1-t}}$ as processes on $[0,T]$. And by time inversion we have that

$$\lim_{t\to1}(1-t)B_{\frac{t}{1-t}}=\lim_{t\to\infty}\frac{B_t}{t}=0.$$

5. X^x is a Brownian bridge between x and 0.

Exercise 4.3

Let $b\in\mathbb{R}$, $a>0$ and set $X_t=B_t-bt$. Let $T=\inf\{t\geqslant0:B_t=a\}$.

1. Find a probability measure $\mathbb{Q}$ on $\mathscr{F}_\infty$ such that $(X_t)_{t\geqslant0}$ is a Brownian motion.

2. Deduce the value of $\mathbb{P}(T\leqslant t)$ and the distribution of $Z=\sup_{t\geqslant0}X_t$ under $\mathbb{P}$.

Exercise 4.4

1. Let $X=(X_t)_{t\geqslant0}$ solution to $E(\sigma,b)$ with value in an open set $D\subset\mathbb{R}^d$. Let $\lambda\in\mathbb{R}$. Let $f:D\to\mathbb{R}$ be twice continuously differentiable and such that $\mathscr{L}f=\lambda f$, where

$$(\mathscr{L}f)(x):=b(x)^\top\nabla f(x)+\frac{1}{2}\mathrm{tr}\left((\sigma\sigma^\top)(x)D^2f(x)\right).$$

Show that $(f(X_t)e^{-\lambda t})_{t\geqslant0}$ is a continuous local martingale.

2. Let $B=(B^1,B^2,B^3)$ be a Brownian motion with value in $\mathbb{R}^3$, and $B_0:=a\in\mathbb{R}^3\setminus\{0\}$. Let $X=\|B\|^2$. Show that X solves some SDE $E(\sigma,b)$ and give the coefficients σ and b.

3. We now assume that $\lambda\geqslant0$. Show that $2tf''(t)+3f'(t)=\lambda f(t)$, $t>0$, for $f(t)=\frac{\sinh(\sqrt{2\lambda t})}{\sqrt{2\lambda t}}$.

4. Let $x>\|a\|^2$, and let $T_x=\inf\{t\geqslant0:X_t=x\}$. Show that for every $\lambda\geqslant0$, we have

$$\mathbb{E}\left[e^{-\lambda T_x}\right]=\frac{\sinh(\sqrt{2\lambda\|a\|^2})}{\sqrt{2\lambda\|a\|^2}}\frac{\sqrt{2\lambda x}}{\sinh(\sqrt{2\lambda x})}.$$

Exercise 4.5

1. Let H, Z and X be continuous semimartingales, such that $X_t = H_t + \int_0^t X_s dZ_s$. Express X as a function of H and Z.

Hint: start with $H \equiv 1$ and use a variation of the constant method as in classical ODE.

2. Solve $X_t = x + B_t - \beta \int_0^t X_s ds$, where $x \in \mathbb{R}$ and $\beta \geqslant 0$ are constants. The process X is called Ornstein-Uhlenbeck process.

Solution

1. By setting $H \equiv 1$, we have that $X = \mathscr{E}(Z)$, so we need to find a process Y such that $X_t = Y_t \mathscr{E}(Z)_t$. First by Itô formula we have

$$\begin{aligned}\frac{1}{\mathscr{E}(Z)_t} &= 1 - \int_0^t \frac{1}{\mathscr{E}(Z)_s^2} d\mathscr{E}(Z)_s + \int_0^t \frac{1}{\mathscr{E}(Z)_s^3} d\langle \mathscr{E}(Z)\rangle_s \\ &= 1 - \int_0^t \frac{1}{\mathscr{E}(Z)_s} dZ_s + \int_0^t \frac{1}{\mathscr{E}(Z)_s} d\langle Z\rangle_s.\end{aligned}$$

Therefore, using again Itô formula, we have

$$\begin{aligned}dY_t = d\left(\frac{X_t}{\mathscr{E}(Z)_t}\right) &= X_t d\left(\frac{1}{\mathscr{E}(Z)_t}\right) + \frac{dX_t}{\mathscr{E}(Z)_t} + d\left\langle X, \frac{1}{\mathscr{E}(Z)}\right\rangle_t \\ &= X_t\left(-\frac{1}{\mathscr{E}(Z)_t} dZ_t + \frac{1}{\mathscr{E}(Z)_t} d\langle Z\rangle_t\right) + \frac{1}{\mathscr{E}(Z)_t}(dH_t + X_t dZ_t) + d\left\langle X, \frac{1}{\mathscr{E}(Z)}\right\rangle_t \\ &= \frac{1}{\mathscr{E}(Z)_t}(dH_t + X_t d\langle Z\rangle_t) + d\left\langle X, \frac{1}{\mathscr{E}(Z)}\right\rangle_t \\ &= \frac{1}{\mathscr{E}(Z)_t}(dH_t + X_t d\langle Z\rangle_t) - \frac{1}{\mathscr{E}(Z)_t} d\langle H, Z\rangle_t - \frac{X_t}{\mathscr{E}(Z)_t} d\langle Z\rangle_t \\ &= \frac{dH_t - d\langle H, Z\rangle_t}{\mathscr{E}(Z)_t}.\end{aligned}$$

We deduce that

$$X_t = \mathscr{E}(Z)_t \left(H_0 + \int_0^t \frac{1}{\mathscr{E}(Z)_s} dH_s - \int_0^t \frac{1}{\mathscr{E}(Z)_t} d\langle H, Z\rangle_s\right).$$

2. Taking $H_t = x + B_t$ and $Z_t = -\beta t$ in the previous question, we obtain

$$X_t = e^{-\beta t}\left(x + \int_0^t e^{\beta s} dB_s\right).$$

Chapter 5

Exams

5.1 M2MO - 2024

Exercise 1

1. We know that $B_t^2 - t = 2\int_0^t B_s dB_s$ and that $\mathscr{E}(L)$ is a local martingale where $L_t = -2\int_0^t B_s dB_s$. Thus we have

$$\mathscr{E}(L)_t = e^{-2\int_0^t B_s dB_s - \frac{4}{2}\int_0^t B_s^2 ds} = e^{-B_t^2 - t - 2\int_0^t B_s^2 ds}.$$

We deduce that $a = -2$.

2. The sum of two independent gaussian variables is still gaussian, thus $(\frac{B_t+W_t}{\sqrt{2}})_{t\geqslant 0}$ is a gaussian process, $\mathbb{E}\left[\frac{B_t+W_t}{\sqrt{2}}\right] = 0$ and for every $t, s \geqslant 0$

$$\operatorname{Cov}\left(\frac{B_s + W_s}{\sqrt{2}}, \frac{B_t + W_t}{\sqrt{2}}\right) = \frac{1}{2}\left(\operatorname{Cov}(B_s, B_t) + \operatorname{Cov}(W_s, W_t)\right) = s \wedge t.$$

We deduce that $(\frac{B_t+W_t}{\sqrt{2}})_{t\geqslant 0}$ is a Brownian motion and therefore for every $t \geqslant 0$, $\left\langle \frac{B+W}{\sqrt{2}} \right\rangle_t = t$. On the other hand we have

$$\left\langle \frac{B+W}{\sqrt{2}} \right\rangle_t = \frac{1}{2}\left(\langle B\rangle_t + 2\langle B, W\rangle_t + \langle W\rangle_t\right) = t + \langle B, W\rangle_t.$$

Hence, for every $t \geqslant 0$, $\langle B, W\rangle_t = 0$.

3. Fix $t \geqslant 0$, by Girsanov's theorem, it suffice to find a process L such that for every $s \in [0, t]$, $B_s + s^2 = B_s - \langle B, L\rangle_s$ and $(\mathscr{E}(L)_s)_{s\in[0,t]}$ is a martingale. Take $L_s = -2\int_0^s u dB_u$ and since

$$\mathbb{E}\left[e^{\frac{1}{2}\langle L\rangle_t}\right] = e^{2\int_0^t s^2 ds} < \infty,$$

by Novikov's criterion, $(\mathscr{E}(L)_s)_{s\in[0,t]}$ is a martingale, let $d\mathbb{Q} = \mathscr{E}(L)_t d\mathbb{P}$ then $B_s + s^2$ is a Brownian motion under $\mathbb{Q}$.

Exercise 2

1. Set $X_n = B_n - B_{n-1}$, $n \geqslant 1$ be an i.i.d. sequence of integrable random variables, then by the law of large numbers:

$$\lim_{n\to\infty} \frac{B_n}{n} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_n = \mathbb{E}\left[X_1\right] = 0.$$

2. Using basic properties of Brownian motion we have

$$\begin{aligned}\mathbb{P}(L_n > n^{2/3}) = \mathbb{P}\left(\sup_{t\in[0,1]} |B_t| > n^{2/3}\right) &\leqslant 2\mathbb{P}\left(\sup_{t\in[0,1]} B_t > n^{2/3}\right) \\ &\leqslant 2\mathbb{P}(|B_1| > n^{2/3}) \leqslant 2\mathbb{P}(B_1^2 > n^{4/3}) \leqslant 2\mathbb{P}\left(B_1^2 > \frac{1}{4}n^{4/3}\right).\end{aligned}$$

(I do not see why $\frac{1}{4}$ plays an important role in this exercise...)

3. By Markov's inequality, we have $\mathbb{P}(B_1^2 > \frac{n^{3/4}}{4}) \leqslant \frac{4}{n^{3/4}}\mathbb{E}\left[B_1^2\right] = \frac{4}{n^{3/4}}$. By the previous question we obtain that

$$\mathbb{P}\left(\frac{L_n}{n} > \frac{1}{n^{1/3}}\right) \leqslant \frac{8}{n^{4/3}}.$$

This prove first that $\frac{L_n}{n}$ converges toward 0 in probability and since $\frac{8}{n^{4/3}}$ is summable, by Borel Cantelli, it converges also almost surely toward 0 as $n \to \infty$.

4. For every $t > 0$, there exists $n \in [1, t]$ such that $t \in [n, n+1]$ and

$$\frac{|B_t|}{t} \leqslant \frac{|B_t - B_n|}{t} + \frac{|B_n|}{t} \leqslant \frac{L_n}{t} + \frac{|B_n|}{t} \leqslant \frac{L_n}{n} + \frac{|B_n|}{n},$$

taking $t \to \infty$ then $n \to \infty$ and by using previous questions we obtain $\lim_{t\to\infty} \frac{B_t}{t} = 0$ almost surely.

5. The process M is adapted (to the natural filtration of B), for every $t \geqslant 0$, $\mathbb{E}\left[|M_t|\right] \leqslant \mathbb{E}\left[e^{B_t}\right] = e^{\frac{t}{2}} < \infty$ and for every $0 \leqslant s \leqslant t$, using that $B_t - B_s$ is independent of $\mathscr{F}_s$, we have

$$\begin{aligned}\mathbb{E}\left[e^{B_t - \frac{t}{2}} \,\middle|\, \mathscr{F}_s\right] = \mathbb{E}\left[e^{B_t - \frac{t}{2}} \,\middle|\, \mathscr{F}_s\right] &= \mathbb{E}\left[e^{B_t - B_s} \,\middle|\, \mathscr{F}_s\right] e^{B_s - \frac{t}{2}} \\ &= \mathbb{E}\left[e^{B_t - B_s}\right] e^{B_s - \frac{t}{2}} = e^{\frac{t-s}{2}} e^{B_s - \frac{t}{2}} = M_s.\end{aligned}$$

6. Using question 4. we have

$$M_\infty = \lim_{t\to\infty} e^{B_t - \frac{t}{2}} = \lim_{t\to\infty} e^{t\left(\frac{B_t}{t} - \frac{1}{2}\right)} = 0.$$

7. If M is a uniformly integrable martingale then $1 = \mathbb{E}\left[M_0\right] = \mathbb{E}\left[M_\infty\right] = 0$. Contradiction, M is not a uniformly integrable martingale.

8. Again, if $\sup_{t\geqslant 0} M_t$ is integrable, this will imply that M is uniformly integrable, since M would be dominated in L^1 by $\sup_{t\geqslant 0} M_t$, hence $\sup_{t\geqslant 0} M_t$ is not integrable.

Exercise 3

1. Cleary, ψ_n is even. Set $g_n(x) = \int_0^x \rho_n(u)du$, since ρ_n is continuous, g_n is $\mathscr{C}^1$ and therefore ψ_n is $\mathscr{C}^2$. We have that

$$\psi_n'(x) = \frac{x}{|x|}g_n(|x|) \quad \text{and} \quad \psi_n''(x) = \rho_n(|x|).$$

(for $\mathscr{C}^2$ proof, one has that $\psi_n''(x) = \rho_n(|x|)$ is a continuous function)
And observe that g_n is non decreasing, if $x \leqslant a_n$, $g_n(x) = 0$ and if $x \geqslant a_{n-1}$, $g_n(x) = 1$. Thus $|\psi_n'(x)| = g_n(|x|) \leqslant 1$.

2. For all $y \geqslant 0$, since $\lim_{n\to\infty} a_n = 0$

$$\exists N \geqslant n,\ \forall n \geqslant N,\ a_{n-1} \leqslant y.$$

We deduce that for every $y \geqslant 0$

$$\lim_{n\to\infty} \int_0^y \rho_n(u)du = \lim_{n\to\infty} \int_{a_n}^{a_{n-1}} \rho_n(u)du = 1.$$

Since g_n is non decreasing, by monotone convergence theorem,

$$\lim_{n\to\infty} \psi_n(x) = \int_0^{|x|} \lim_{n\to\infty} \int_0^y \rho_n(u)du = |x|,$$

ψ_n is non decreasing, because g_n it is.

3. For every $n \geqslant 1$ and $t \geqslant 0$,

$$\begin{aligned}\mathbb{E}\left[\int_0^t \psi_n''(\Delta_s)(\sigma(X_s) - \sigma(Y_s))^2 ds\right] &\leqslant \mathbb{E}\left[\int_0^t \rho_n(|\Delta_s|)h(|\Delta_s|)^2 ds\right] \\ &\leqslant \mathbb{E}\left[\int_0^t \frac{2}{n} ds\right] = \frac{2t}{n}.\end{aligned}$$

4. We apply Itô's formula to the function $(x, y) \mapsto \psi_n(x - y)$:

$$\begin{aligned}d\psi_n(\Delta_t) &= \psi_n'(\Delta_t)dX_t - \psi_n'(\Delta_t)dY_t + \frac{1}{2}\psi_n''(\Delta_t)d\left\langle X\right\rangle_t + \frac{1}{2}\psi_n''(\Delta_t)d\left\langle Y\right\rangle_t - \psi_n''(\Delta_t)d\left\langle X, Y\right\rangle_t \\ &= \psi_n'(\Delta_t)(dX_t - dY_t) + \frac{1}{2}\psi_n''(\Delta_t)(\sigma(X_t) - \sigma(Y_t))^2 dt \\ &= \psi_n'(\Delta_t)(b(X_t) - b(Y_t))dt + \psi_n'(\Delta_t)(\sigma(X_t) + \sigma(Y_t))dB_t + \frac{1}{2}\psi_n''(\Delta_t)(\sigma(X_t) - \sigma(Y_t))^2 dt.\end{aligned}$$

The assumptions (1) makes $\psi_n'(\Delta_t)(\sigma(X_t) + \sigma(Y_t))dB_t$ be a true martingale and thus have a expectation of 0. Using question 1., 3. and the Lipschitz condition on b we obtain

$$\begin{aligned}\mathbb{E}\left[\psi_n(\Delta_t)\right] &\leqslant \mathbb{E}\left[\int_0^t K|\Delta_s|ds\right] + \frac{1}{2}\mathbb{E}\left[\int_0^t \psi_n''(\Delta_s)(\sigma(X_s) - \sigma(Y_s))^2 ds\right] \\ &\leqslant K\int_0^t \mathbb{E}\left[|\Delta_s|\right] ds + \frac{t}{n}.\end{aligned}$$

5. By question 2., we can apply monotone convergence theorem, and passing to the limit

$$\mathbb{E}\left[|\Delta_t|\right] \leqslant K\int_0^t \mathbb{E}\left[|\Delta_s|\right] ds,$$

we conclude with Grönwall lemma, that gives us $\mathbb{E}\left[|\Delta_t|\right] = 0$ which imply that for every $t \geqslant 0$, $|X_t - Y_t| = 0$ almost surely. Paths continuity of X and Y ensure that almost surely for every $t \geqslant 0$, $|X_t - Y_t| = 0$ and we have path uniqueness of the solution of the SDE.

Exercise 4

1. If $x = 0$, we observe that $X_t \equiv 0$ is a solution and by uniqueness, this is the only one.

2. Fix $t \geqslant 0$, we apply the Itô's formula to the function $(s, x) \mapsto \exp\left(-\frac{\lambda x}{1+\frac{1}{2}(t-s)}\right)$:

$$\begin{aligned} dM_s &= -\frac{\lambda}{1+\frac{\lambda}{2}(t-s)} M_s dX_s + \frac{\lambda^2}{2} \frac{X_s}{(1+\frac{\lambda}{2}(t-s))^2} M_s ds + \frac{1}{2} \frac{\lambda^2}{1+\frac{\lambda}{2}(t-s)} d\langle X \rangle_s \\ &= -\frac{\lambda}{1+\frac{\lambda}{2}(t-s)} M_s \sqrt{X_s} dB_s. \end{aligned}$$

Therefore, $(M_s)_{s\in[0,t]}$ is a local martingale, moreover $M_s \leqslant 1$ for every $s \in [0,t]$, thus, this is a martingale on $[0,t]$.

3. Obviously,

$$e^{-\frac{\lambda x}{1+\frac{\lambda t}{2}}} = \mathbb{E}[M_0] = \mathbb{E}[M_t] = \mathbb{E}\left[e^{-\lambda X_t}\right].$$

4. Since $e^{-\lambda X_t}$ converges almost surely to $\mathbb{1}_{X_t=0}$ as $\lambda \to \infty$, by dominated convergence theorem we obtain the following equalities

$$e^{-\frac{2x}{t}} = \lim_{\lambda\to\infty} \exp\left(-\frac{\lambda x}{1+\frac{\lambda t}{2}}\right) = \lim_{\lambda\to\infty} \mathbb{E}\left[e^{-\lambda X_t}\right] = \mathbb{P}(X_t = 0),$$

As $t \to \infty$, the probability $\mathbb{P}(X_t = 0)$ approaches 1, indicating that the process X_t converges to zero almost surely over time. Thus, we deduce that $X_t \to 0$ almost surely as t tends to infinity.

5.

$$\begin{aligned} dZ_t &= dX_t + dY_t \\ &= \sqrt{X_t} dB_t + \sqrt{Y_t} dW_t \\ &= \sqrt{X_t + Y_t}\left(\sqrt{\frac{X_t}{X_t+Y_t}} dB_t + \sqrt{\frac{Y_t}{X_t+Y_t}} dW_t\right). \end{aligned}$$

Take $\beta_t = \sqrt{\frac{X_t}{X_t+Y_t}} dB_t + \sqrt{\frac{Y_t}{X_t+Y_t}} dW_t$ which is a local martingale starting at 0 and for every $t \geqslant 0$ since B and W are independent Brownian motion, we have $\langle B, W\rangle_t = 0$ by **Exercise 1** and

$$\begin{aligned} \langle \beta \rangle_t &= \left\langle \int_0^\cdot \frac{X_s}{X_s+Y_s} dB_s \right\rangle_t + \left\langle \int_0^\cdot \frac{Y_s}{X_s+Y_s} dW_s \right\rangle_t \\ &= \int_0^t \frac{X_s}{X_s+Y_s} ds + \int_0^t \frac{Y_s}{X_s+Y_s} ds = t. \end{aligned}$$

By Lévy's caracterization, β is a Brownian motion and $dZ_t = \sqrt{Z_t} d\beta_t$.