

UNIVERSITÉ PARIS DAUPHINE - PSL

Stochastic Calculus

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Chapter 1

Brownian Motion

Exercise 1.1

Justify that $B_t - B_s$ is independent of B_1 for all $1 \leq s \leq t$ and determine its law.

Exercise 1.2

Compute the conditional expectations $\mathbb{E}[B_t | \mathcal{F}_s]$ and $\mathbb{E}[B_t^2 | \mathcal{F}_s]$ for all $t \geq 1$.

Exercise 1.3

Let $\xi = \int_0^1 B_s ds$. What is the law of ξ ?

Exercise 1.4

Let $\eta = \int_0^2 B_s ds$. Compute the conditional expectation $\mathbb{E}[B_1 | \eta]$.

Exercise 1.5

Let $T := \inf \{t \geq 0 \mid B_t = 1\}$. Show that $\mathbb{P}(T < \infty) \geq \frac{1}{2}$.

Exercise 1.6

Let $T := \inf \{t \geq 0 \mid |B_t| = 1\}$.

1. Prove that $T < \infty$ a.s.
2. Show that T and $\mathbb{1}_{B_T=1}$ are independent.

Exercise 1.7

Justify that almost surely, $(B_t)_{t \geq 0}$ is not monotone on any interval.

Exercise 1.8

1. Show that for all $a > 0$, $(\frac{1}{\sqrt{a}} B_{at})_{t \geq 0}$ is a Brownian motion.
2. Show that for all $a > 0$, $(B_a - B_{a-t})_{0 \leq t \leq a}$ is a Brownian motion.

3. Define $X_0 = 0$ and $X_t = tB_{1/t}$ for all $t > 0$. Show that $(X_t)_{t \geq 0}$ is a Brownian motion.
4. Deduce that $\lim_{t \rightarrow \infty} \frac{B_t}{t} = 0$ a.s.

Exercise 1.9

Prove that $\int_0^\infty |B_s| ds = \infty$ a.s.

Exercise 1.10

Prove that for all $a > 0$, $(B_{t+a} - B_a)_{t \geq 0}$ is a Brownian motion independent of \mathcal{F}_a .

Exercise 1.11

1. Show that for all $t \geq 0$, $|B_t|$ has the same distribution as $\sqrt{t}|B_1|$.
2. Do the processes $(|B_t|)_{t \geq 0}$ and $(\sqrt{t}|B_1|)_{t \geq 0}$ have the same distribution ?

Exercise 1.12

1. Show that $\int_0^1 \frac{B_s}{s} ds$ is well-defined a.s.
2. Let $\beta_t := B_t - \int_0^1 \frac{B_s}{s} ds$. Show that $(\beta_t)_{t \geq 0}$ is a Brownian motion.

Exercise 1.13 (Brownian bridge)

We define the *Brownian bridge* as the process $Z_t := B_t - tB_1$ for all $0 \leq t \leq 1$.

1. Show that Z is a Gaussian process independent of B_1 .
2. Prove that Z has the same law as the process Y defined by

$$Y_t = \begin{cases} (1-t)B_{\frac{t}{1-t}} & \text{if } 0 \leq t < 1, \\ 0 & \text{if } t = 1. \end{cases}$$

Exercise 1.14

Let T be a random variable distributed according to the exponential distribution of mean 1. What is the law of B_T ?

Exercise 1.15

In this exercise, \mathbf{B} is a d -dimensional ($d \in \mathbb{N}$) standard Brownian motion, that is $\mathbf{B}_t = (B_t^1, \dots, B_t^d)$, where B^i 's are independent standard Brownian motions. Let $U \in \mathbb{R}^{d \times d}$ be an orthogonal matrix. Prove that the process $(\mathbf{W}_t)_{t \geq 0} = (U\mathbf{B}_t)_{t \geq 0}$ is a d -dimensional standard Brownian motion.

Exercise 1.16

Let τ be a stopping time that is almost surely finite. Show that the process $(B_{t+\tau} - B_\tau)_{t \geq 0}$ is a Brownian motion independent of \mathcal{F}_τ .

Exercise 1.17

Prove that almost surely, $\sup_{0 \leq t \leq s} B_t > 0$ for all $s > 0$ and $\sup_{t \geq 0} B_t = \infty$.

Exercise 1.18

1. Show that for all $a \geq 0$, $\tau_a = a^2 \tau_1$ in distribution.
2. Let $0 \leq a \leq b < \infty$. Justify that $\tau_b - \tau_a$ has the same distribution as τ_{b-a} and is independent of \mathcal{F}_{τ_a} .

Chapter 2

Stopping times and Martingales

For each $x \in \mathbb{R}$, we define the stopping time $\tau_x := \inf \{t \geq 0 : B_t = x\}$.

Exercise 2.1

1. Show that $(B_t^2 - t)_{t \geq 0}$ is a martingale.
2. Construct a martingale from $(B_t^3)_{t \geq 0}$. Same question with B_t^4 .
3. Prove that $(e^{\lambda B_t - \frac{\lambda^2}{2}t})_{t \geq 0}$ is a martingale for all $\lambda \in \mathbb{R}$.

Solution

1. To show that $(B_t^2 - t)$ is a martingale, we compute the conditional expectation:

$$\mathbb{E}[B_t^2 \mid \mathcal{F}_s] = \mathbb{E}[(B_t - B_s + B_s)^2 \mid \mathcal{F}_s] = B_s^2 + \mathbb{E}[(B_t - B_s)^2] = B_s^2 + (t - s),$$

since $B_t - B_s$ is independent of \mathcal{F}_s and normally distributed with mean zero and variance $t - s$. Therefore, we have

$$\mathbb{E}[B_t^2 - t \mid \mathcal{F}_s] = B_s^2 - s,$$

confirming that $(B_t^2 - t)_{t \geq 0}$ is indeed a martingale.

2. For all $t > s$, we expand B_t^3 using $B_t = B_s + (B_t - B_s)$:

$$B_t^3 = (B_s + (B_t - B_s))^3 = B_s^3 + 3B_s^2(B_t - B_s) + 3B_s(B_t - B_s)^2 + (B_t - B_s)^3.$$

Taking the conditional expectation given \mathcal{F}_s , we find:

$$\mathbb{E}[B_t^3 \mid \mathcal{F}_s] = B_s^3 + 3B_s \mathbb{E}[(B_t - B_s)^2] = B_s^3 + 3B_s(t - s).$$

Therefore, the process $(B_t^3 - 3B_s t)_{t \geq 0}$ is a martingale.

Similarly, for B_t^4 , we expand:

$$B_t^4 = (B_s + (B_t - B_s))^4 = B_s^4 + 4B_s^3(B_t - B_s) + 6B_s^2(B_t - B_s)^2 + 4B_s(B_t - B_s)^3 + (B_t - B_s)^4.$$

Taking the conditional expectation, we obtain:

$$\mathbb{E}[B_t^4 \mid \mathcal{F}_s] = B_s^4 + 6B_s^2(t - s) + 3(t - s)^2.$$

Thus, $(B_t^4 - 6B_s^2 t + 3t^2)_{t \geq 0}$ is a martingale.

3. Let $\lambda \in \mathbb{R}$. We compute the conditional expectation:

$$\begin{aligned}\mathbb{E} \left[e^{\lambda B_t - \frac{\lambda^2}{2} t} \middle| \mathcal{F}_s \right] &= e^{\lambda B_s - \frac{\lambda^2}{2} s} \mathbb{E} \left[e^{\lambda(B_t - B_s) - \frac{\lambda^2}{2}(t-s)} \middle| \mathcal{F}_s \right] \\ &= e^{\lambda B_s - \frac{\lambda^2}{2} s} \mathbb{E} \left[e^{\lambda(B_t - B_s) - \frac{\lambda^2}{2}(t-s)} \right] \\ &= e^{\lambda B_s - \frac{\lambda^2}{2} s},\end{aligned}$$

since $B_t - B_s$ is independent of \mathcal{F}_s and $\mathbb{E} \left[e^{\lambda(B_t - B_s) - \frac{\lambda^2}{2}(t-s)} \right] = 1$ due to the properties of the normal distribution. This confirms that $(e^{\lambda B_t - \frac{\lambda^2}{2} t})_{t \geq 0}$ is a martingale.

Exercise 2.2

Let $a > 0$ and define $T_a^* := \inf \{t \geq 0 : |B_t| = a\}$.

1. Using the martingale $(B_t^2 - t)_{t \geq 0}$ compute the expectation of T_a^* .
2. Using a well-chosen martingale, compute the variance of T_a^* and its Laplace transform.
3. Compute the Laplace transform of τ_a and find that it has the law as $(a/B_1)^2$. What is the value of $\mathbb{E}[\tau_a]$?

Solution

1. Since T_a^* is almost surely finite, the stopped martingale $(B_{t \wedge T_a^*}^2 - t \wedge T_a^*)$ converges almost surely to $B_{T_a^*}^2 - T_a^* = a^2 - T_a^*$. Moreover, $B_{t \wedge T_a^*}^2$ is bounded by a^2 , ensuring L^1 convergence by the dominated convergence theorem. In the other hand, $t \wedge T_a^*$ converge in L^1 to T_a^* by monotone convergence. Therefore,

$$0 = \mathbb{E} \left[B_{t \wedge T_a^*}^2 - t \wedge T_a^* \right] \xrightarrow[t \rightarrow \infty]{} a^2 - \mathbb{E} [T_a^*].$$

Hence, $\mathbb{E} [T_a^*] = a^2$.

2. To compute the variance, we consider the martingale $(B_t^4 - 6B_t^2 t + 3t^2)_{t \geq 0}$. Applying similar reasoning as before, we have:

$$\mathbb{E} \left[B_{t \wedge T_a^*}^4 - 6B_{t \wedge T_a^*}^2 (t \wedge T_a^*) + 3(t \wedge T_a^*)^2 \right] = 0.$$

Taking the limit as $t \rightarrow \infty$ and using $B_{T_a^*}^2 = a^2$, we get:

$$a^4 - 6a^2 \mathbb{E} [T_a^*] + 3\mathbb{E} [(T_a^*)^2] = 0.$$

Substituting $\mathbb{E} [T_a^*] = a^2$, we solve for $\mathbb{E} [(T_a^*)^2]$:

$$a^4 - 6a^4 + 3\mathbb{E} [(T_a^*)^2] = 0 \quad \implies \quad \mathbb{E} [(T_a^*)^2] = \frac{5}{3}a^4.$$

Therefore, the variance is:

$$\text{Var} (T_a^*) = \mathbb{E} [(T_a^*)^2] - (\mathbb{E} [T_a^*])^2 = \frac{5}{3}a^4 - a^4 = \frac{2}{3}a^4.$$

For the Laplace transform, consider the martingale for $\lambda > 0$ (check it):

$$M_t = \exp\left(\sqrt{2\lambda}B_t - \lambda t\right) + \exp\left(-\sqrt{2\lambda}B_t - \lambda t\right).$$

The stopped martingale $M^{T_a^*}$ is dominated in L^1 by $2e^{\sqrt{2\lambda}a}$, and thus uniformly integrable. Therefore, $\mathbb{E}[M_{T_a^*}] = 2$. Evaluating $M_{T_a^*}$, we have:

$$\begin{aligned} 2 = \mathbb{E}[M_{T_a^*}] &= \mathbb{E}\left[e^{-\lambda T_a^*}(e^{\sqrt{2\lambda}a} + e^{-\sqrt{2\lambda}a})\mathbb{1}_{\{B_{T_a^*}=a\}}\right] + \mathbb{E}\left[e^{-\lambda T_a^*}(e^{-\sqrt{2\lambda}a} + e^{\sqrt{2\lambda}a})\mathbb{1}_{\{B_{T_a^*}=-a\}}\right] \\ &= 2 \cosh(\sqrt{2\lambda}a)\mathbb{E}[X] \end{aligned}$$

$$\text{Hence } \mathbb{E}[e^{-\lambda T_a^*}] = \frac{1}{\cosh(\sqrt{2\lambda}a)}.$$

3. To compute the Laplace transform of τ_a , we utilize the martingale $M_t = e^{\sqrt{2\lambda}B_t - \lambda t}$. At the stopping time τ_a , we have $B_{\tau_a} = a$, so:

$$M_{\tau_a} = e^{\sqrt{2\lambda}a - \lambda\tau_a}.$$

Since M_t is a martingale, $\mathbb{E}[M_{\tau_a}] = M_0 = 1$. Therefore,

$$\mathbb{E}[e^{-\lambda\tau_a}] = e^{-\sqrt{2\lambda}a}.$$

Exercise 2.3

1. Let M be a continuous martingale such that $M_0 = x \geq 0$. Suppose that $M_t \geq 0$ for all $t \geq 0$ and that $M_t \rightarrow 0$ as $t \rightarrow \infty$, a.s. Show that, for all $y > x$,

$$\mathbb{P}\left(\sup_{t \geq 0} M_t \geq y\right) = \frac{x}{y}.$$

2. Deduce the law of

$$\sup_{t \leq T_0} B_t,$$

when B is a Brownian motion started at $x > 0$ and $T_0 = \inf\{t \geq 0 : B_t = 0\}$.

3. Suppose now that is started at 0, and let $\mu > 0$. Using a well-chosen exponential martingale, prove that

$$\sup_{t \geq 0} (B_t - \mu t)$$

follows the exponential distribution of parameter 2μ .

Exercise 2.4

Let $a < 0 < b$ and set $T = \tau_a \wedge \tau_b$.

1. Prove that, for all $\lambda > 0$,

$$\mathbb{E}[\exp(-\lambda T)] = \frac{\cosh\left(\frac{b+a}{2}\sqrt{2\lambda}\right)}{\cosh\left(\frac{b-a}{2}\sqrt{2\lambda}\right)}.$$

(Hint: introduce the martingale

$$M_t = \exp\left(\sqrt{2\lambda}(B_t - \alpha) - \lambda t\right) + \exp\left(-\sqrt{2\lambda}(B_t - \alpha) - \lambda t\right),$$

with a well-chosen α .)

2. Similarly, prove that, for all $\lambda > 0$,

$$\mathbb{E} [\exp(-\lambda T) \mathbf{1}_{\{T=\tau_a\}}] = \frac{\sinh(b\sqrt{2\lambda})}{\sinh((b-a)\sqrt{2\lambda})}.$$

3. With the help of question 2, give an expression of $\mathbb{P}(\tau_a < \tau_b)$.

Exercise 2.5

Let $M = (M_t)_{0 \leq t \leq 1}$ be a sub-martingale. Let $(\mathcal{G}_s)_{s \geq 0}$ be a sub-filtration of $(\mathcal{F}_s)_{s \geq 0}$. Prove that $N_t = \mathbb{E}[M_t | \mathcal{G}_t]$ is a (\mathcal{G}_s) -sub-martingale.

Solution

First note that N is clearly adapted and integrable. We prove the martingale property, let $0 \leq s \leq t$:

$$\begin{aligned} \mathbb{E}[N_t | \mathcal{G}_s] &= \mathbb{E}[\mathbb{E}[M_t | \mathcal{G}_t] | \mathcal{G}_s] \\ &= \mathbb{E}[M_t | \mathcal{G}_s] \\ &= \mathbb{E}[\mathbb{E}[M_t | \mathcal{F}_s] | \mathcal{G}_s] \\ &\geq \mathbb{E}[M_s | \mathcal{G}_s] = N_s. \end{aligned}$$

Exercise 2.6

Let $\sigma \leq \tau$ be two bounded stopping times. Show that

$$\mathbb{E}[(B_\tau - B_\sigma)^2] = \mathbb{E}[B_\tau^2] - \mathbb{E}[B_\sigma^2] = \mathbb{E}[\tau - \sigma].$$

Exercise 2.7

Let $M = (M_t)_{0 \leq t \leq 1}$ be a sub-martingale such that $\mathbb{E}[M_0] = \mathbb{E}[M_1]$. Prove that M is a martingale.

Solution

Since M is a sub-martingale we have for all $t \in [0, 1]$, $\mathbb{E}[M_0] \leq \mathbb{E}[M_t] \leq \mathbb{E}[M_1]$, hence $\mathbb{E}[M_0] = \mathbb{E}[M_t]$ for all t . Let $0 \leq s \leq t \leq 1$, M is a sub-martingale so that $\mathbb{E}[M_t - M_s | \mathcal{F}_s]$ is non negative and of expectation 0, therefore $\mathbb{E}[M_t - M_s | \mathcal{F}_s] = 0$ and M is a martingale.

Exercise 2.8

Let M be a càd martingale. Let $t \geq 0$. Prove that $M_{t+\varepsilon} \xrightarrow[\varepsilon \rightarrow 0]{L^1} M_t$.

Solution

The càd property give us $M_{t+\varepsilon} \xrightarrow[\varepsilon \rightarrow 0]{\text{a.s.}} M_t$. For every $n \geq 1$ we have

$$\mathbb{E}\left[M_{t+1/n} \mid \mathcal{F}_{t+\frac{1}{n}}\right] = M_{t+\frac{1}{n}}.$$

Hence the sequence $(M_{t+1/n})_{n \geq 1}$ is uniformly integrable and $M_{t+1/n} \xrightarrow[n \rightarrow \infty]{L^1} M_t$.

Exercise 2.9

Let M be a local continuous martingale such that $M_0 = 0$ a.s.

1. Let $a > 0$ and let $\sigma_a := \inf\{t \geq 0 : \langle M, M \rangle_t \geq a^2\}$. Show that

$$\mathbb{P} \left(\sup_{s \in [0, \sigma_a]} |M_s| > a \right) \leq \frac{1}{a^2} \mathbb{E} [a^2 \wedge \langle M, M \rangle_\infty].$$

2. Show that $\mathbb{P}(\sup_{t \geq 0} |M_t| > a) \leq \mathbb{P}(\langle M, M \rangle_\infty \geq a^2) + a^{-2} \mathbb{E} [a^2 \wedge \langle M, M \rangle_\infty]$.
3. Show that $\mathbb{E} [\sup_{t \geq 0} |M_t|] \leq 3 \mathbb{E} [\sqrt{\langle M, M \rangle_\infty}]$.
4. Show that if $\mathbb{E} [\sqrt{\langle M, M \rangle_\infty}] < \infty$, then M is a uniformly integrable martingale.
5. Show that if $\mathbb{E} [\sqrt{\langle M, M \rangle_t}] < \infty$ for every $t \geq 0$, then M is a integrable martingale.

Solution

1. We observe that $\mathbb{P} \left(\sup_{s \in [0, \sigma_a]} |M_s| > a \right) = \mathbb{P}(\sup_{t \geq 0} |M_{t \wedge \sigma_a}| > a)$ and since $\mathbb{E} [\langle M \rangle_{t \wedge \sigma_a}] \leq a^2$, M^{σ_a} and $(M^{\sigma_a})^2 - \langle M^{\sigma_a} \rangle$ are uniformly integrable martingales. Therefore we can apply Doob maximal inequality:

$$a^2 \mathbb{P} \left(\sup_{t \geq 0} |M_{t \wedge \sigma_a}| > a \right) \leq \sup_{t \geq 0} \mathbb{E} [M_{t \wedge \sigma_a}^2] \leq \mathbb{E} [\langle M \rangle_{\sigma_a}] \leq \mathbb{E} [a^2 \wedge \langle M \rangle_\infty].$$

Indeed, $\langle M \rangle_{\sigma_a} = a^2 \mathbf{1}_{\sigma_a < \infty} + \langle M \rangle_\infty (1 - \mathbf{1}_{\sigma_a < \infty}) \leq a^2$ and $a^2 \mathbf{1}_{\sigma_a < \infty} + \langle M \rangle_\infty (1 - \mathbf{1}_{\sigma_a < \infty}) \leq \langle M \rangle_\infty$.

2. We have

$$\begin{aligned} \mathbb{P} \left(\sup_{t \geq 0} |M_t| > a \right) &= \mathbb{P} \left(\sup_{t \geq 0} |M_t| > a, \sigma_a < \infty \right) + \mathbb{P} \left(\sup_{t \geq 0} |M_t| > a, \sigma_a = \infty \right) \\ &\leq \mathbb{P}(\sigma_a < \infty) + \mathbb{P} \left(\sup_{t \in [0, \sigma_a]} |M_t| > a \right). \end{aligned}$$

3. By integrating with respect to a we have

$$\begin{aligned} \mathbb{E} \left[\sup_{t \geq 0} |M_t| \right] &\leq \int_0^\infty \mathbb{P}(\langle M, M \rangle_\infty \geq a^2) + a^{-2} \mathbb{E} [a^2 \wedge \langle M, M \rangle_\infty] da \\ &\leq \int_0^\infty \mathbb{P}(\sqrt{\langle M, M \rangle_\infty} \geq a) da + \mathbb{E} \left[\int_0^\infty 1 \wedge \frac{\langle M, M \rangle_\infty}{a^2} da \right] \\ &\leq \mathbb{E} \left[\sqrt{\langle M, M \rangle_\infty} \right] + \mathbb{E} \left[\int_0^{\sqrt{\langle M, M \rangle_\infty}} da \right] + \mathbb{E} \left[\langle M, M \rangle_\infty \int_{\sqrt{\langle M, M \rangle_\infty}}^\infty a^{-2} da \right] \\ &\leq 3 \mathbb{E} \left[\sqrt{\langle M, M \rangle_\infty} \right]. \end{aligned}$$

4. Using the previous question we have that $\sup_{t \geq 0} |M_t| = C < \infty$ which implies that the local martingale M is dominated in L^1 and thus a uniformly integrable martingale.
5. By the same reasoning we have that for every $T > 0$, the process $(M_t)_{t \in [0, T]}$ is dominated in L^1 , that is a martingale on $[0, T]$ for every $T > 0$ and thus a martingale on \mathbb{R}_+ .

Chapter 3

Stochastic Integration

Exercise 3.1

Let M be a local martingale. Show that the family $\{M_\tau, \tau < \infty\}$ is uniformly integrable if, and only if, M is a uniformly integrable martingale.

Solution

Assume first that M is a uniformly integrable martingale, therefore for all stopping time τ we have that $M_\tau = \mathbb{E}[M_\infty | \mathcal{F}_\tau]$ hence $\{M_\tau, \tau < \infty\}$ is uniformly integrable.

Assume now that the family $\{M_\tau, \tau < \infty\}$ is uniformly integrable, in particular the family $\{M_t\}_{t \geq 0}$ is uniformly integrable, we have to prove the martingale property. Let τ_n be a localizing sequence, for every $0 \leq s \leq t$ and every $A \in \mathcal{F}_{s \wedge \tau_n} \subseteq \mathcal{F}_s$, we know that

$$\mathbb{E}[M_{s \wedge \tau_n} \mathbf{1}_A] = \mathbb{E}[M_{t \wedge \tau_n} \mathbf{1}_A].$$

By assumption, we have that $(M_{x \wedge \tau_n})_{x \geq 0}$ is uniformly integrable, so we have L^1 converge in both terms, we have

$$\mathbb{E}[M_s \mathbf{1}_A] = \mathbb{E}[M_t \mathbf{1}_A]$$

for all $A \in \mathcal{F}_s$ and M is a uniformly integrable martingale.

Exercise 3.2

Let M be a bounded local martingale. Show that $\langle M \rangle_\infty < \infty$ a.s.

Solution

Clearly, M is a bounded martingale. Hence $\lim_{t \rightarrow \infty} M_t^2 = M_\infty^2$ in L^1 and $\lim_{t \rightarrow \infty} \mathbb{E}[\langle M \rangle_t] = \mathbb{E}[\langle M \rangle_\infty]$ by monotone convergence. We obtain

$$0 = \mathbb{E}[M_t^2 - \langle M \rangle_t] \xrightarrow[t \rightarrow \infty]{} \mathbb{E}[M_\infty^2 - \langle M \rangle_\infty].$$

This implies that $\mathbb{E}[\langle M \rangle_\infty] = \mathbb{E}[M_\infty^2] < \infty$.

Exercise 3.3

Let B be a standard Brownian motion with $B_0 = x > 0$. Set $T = \inf\{t \geq 0 : B_t = 0\}$. Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ compactly supported. Compute $\mathbb{E} \left[\int_0^T f(B_s) ds \right]$.

Exercise 3.4

Let $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a continuous function. With the help of the stochastic integration by parts formula, check that the process

$$X_t := \int_0^t \exp \left(\int_s^t \alpha(u) du \right) dB_s,$$

satisfies the stochastic differential equation $dX_t = \alpha(t)X_t dt + dB_t$.

Solution

First, we have for every $t \geq 0$

$$\begin{aligned} X_t &= \int_0^t \exp \left(\int_0^t \alpha(u) du - \int_0^s \alpha(u) du \right) dB_s \\ &= \exp \left(\int_0^t \alpha(s) ds \right) \int_0^t \exp \left(- \int_0^s \alpha(u) du \right) dB_s \\ &= e^{Y_t} \int_0^t e^{-Y_s} dB_s \end{aligned}$$

Where $Y_t = \int_0^t \alpha(s) ds$. Now using integration by part we obtain

$$\begin{aligned} dX_t &= d \left(e^{Y_t} \right) \int_0^t e^{-Y_s} dB_s + e^{Y_t} d \left(\int_0^t e^{-Y_s} dB_s \right) \\ &= \alpha(t) e^{Y_t} dt \int_0^t e^{-Y_s} dB_s + e^{Y_t} e^{-Y_t} dB_t. \end{aligned}$$

Hence, $dX_t = \alpha(t)X_t dt + dB_t$.

Exercise 3.5

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuously twice-differentiable. Show that the process

$$X_t := f(B_t) - \frac{1}{2} \int_0^t f''(B_s) ds,$$

is a continuous local martingale. Give a sufficient condition for X to be a martingale.

Exercise 3.6

Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a (deterministic) function in $L_{\text{loc}}^2(\mathbb{R}_+)$ (i.e. $\int_0^t \varphi^2(s) ds < \infty$ for every $t \geq 0$) and $Z^\varphi = (Z_t^\varphi)_{t \geq 0}$ the associated Doléans-Dade exponential process. Check that Z^φ is a martingale.

Solution

Since, $\varphi \in L^2_{\text{loc}}$, $\int_0^t \varphi(s)dB_s$ is well-defined and is a centered gaussian random variable with variance $\int_0^t \varphi(s)^2 ds < \infty$, this implies that $Z_t^\varphi \in L^1$ for every $t \geq 0$. Let $0 \leq s \leq t$ we have

$$\mathbb{E} \left[e^{\int_0^t \varphi(u)dB_u - \frac{1}{2} \int_0^t \varphi(u)^2 du} \middle| \mathcal{F}_s \right] = \mathbb{E} \left[e^{\int_0^t \varphi(u)dB_u - \int_0^s \varphi(u)dB_u} \middle| \mathcal{F}_s \right] e^{\int_0^s \varphi(u)dB_u - \frac{1}{2} \int_0^t \varphi(u)^2 du}.$$

Recall that, $\int_0^t \varphi(u)dB_u - \int_0^s \varphi(u)dB_u = \int_s^t \varphi(u)dB_u$ is independent of \mathcal{F}_s and is a centered gaussian variable with variance $\int_s^t \varphi(s)^2 ds$, we thus obtain

$$\begin{aligned} \mathbb{E} \left[e^{\int_0^t \varphi(u)dB_u - \frac{1}{2} \int_0^t \varphi(u)^2 du} \middle| \mathcal{F}_s \right] &= \mathbb{E} \left[e^{\int_s^t \varphi(u)dB_u} \right] e^{\int_0^s \varphi(u)dB_u - \frac{1}{2} \int_0^t \varphi(u)^2 du} \\ &= e^{\frac{1}{2} \int_s^t \varphi(s)^2 ds} \cdot e^{\int_0^s \varphi(u)dB_u - \frac{1}{2} \int_0^t \varphi(u)^2 du} \\ &= e^{\int_0^s \varphi(u)dB_u - \frac{1}{2} \int_0^s \varphi(u)^2 du}. \end{aligned}$$

Hence the result.

Exercise 3.7

Find a progressive process $X = (X_t)_{t \geq 0}$ such that the process $Z = (Z_t)_{t \geq 0}$ defined by $Z_t = \exp(X_t - B_t^2)$ is a martingale.

Solution

Assume that the process X has the form

$$X_t = \int_0^t b_s ds + \int_0^t \sigma_s dB_s,$$

and $X_0 = 0$, then by Itô's formula we have

$$\begin{aligned} Z_t &= 1 + \int_0^t Z_s dX_s - 2 \int_0^t Z_s B_s dB_s + \frac{1}{2} \int_0^t Z_s d\langle X, X \rangle_s - 2 \int_0^t Z_s B_s d\langle X, B \rangle_s + \int_0^t Z_s (2B_s^2 - 1) ds \\ &= 1 + \int_0^t Z_s (\sigma_s - 2B_s) dB_s + \int_0^t Z_s \left(\frac{1}{2} \sigma_s^2 - 2B_s \sigma_s + 2B_s^2 - 1 + b_s \right) ds. \end{aligned}$$

Then, by taking $X_t = \int_0^t (1 - B_s^2/2) ds + \int_0^t B_s dB_s$, $Z_t = 1 - \int_0^t Z_s B_s dB_s$ is a local martingale. Using the fact that $B_t^2 - t = 2 \int_0^t B_s dB_s$, we have

$$Z_t = e^{-\int_0^t B_s dB_s - \frac{1}{2} \int_0^t B_s^2 ds}.$$

Exercise 3.8

Let X and Y be two (\mathcal{F}_t) independent Brownian motions and let H be a progressive process. We set

$$\begin{aligned} \beta_t &= \int_0^t \cos(H_s) dX_s - \int_0^t \sin(H_s) dY_s, \\ \gamma_t &= \int_0^t \sin(H_s) dX_s + \int_0^t \cos(H_s) dY_s. \end{aligned}$$

Show that β and γ are independent (\mathcal{F}_t) Brownian motions.

Exercise 3.9

Let B be a Brownian motion. Show that $\int_0^t \mathbf{1}_{\{B_s=0\}} dB_s = 0$.

Exercise 3.10

1. Let $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a continuous function. Let $f : \mathbb{R}_+ \rightarrow (0, \infty)$ be a \mathcal{C}^2 function such that $f'' = 2gf$ on \mathbb{R}_+ and $f(0) = 1$, $f'(1) = 0$. We set

$$u(t) := \frac{f'(t)}{2f(t)}, \quad t \geq 0.$$

Show that $u' + 2u^2 = g$ on \mathbb{R}_+ .

2. Let β a (\mathcal{F}_t) standard Brownian motion. Let $x_0 \geq 0$, $a \geq 0$ and X an adapted, continuous and non negative process such that

$$X_t = x_0 + 2 \int_0^t \sqrt{X_s} d\beta_s + at.$$

Show that $u(t)X_t - \int_0^t g(s)X_s ds = u(0)X_0 + \int_0^t u(s)dX_s - 2 \int_0^t u(s)^2 X_s ds$, $t \geq 0$.

3. Set $M_t := u(0)x_0 + 2 \int_0^t u(s)\sqrt{X_s}d\beta_s$, $t \geq 0$. Show that

$$f(t)^{-a/2} \exp \left(u(t)X_t - \int_0^t g(s)X_s ds \right) = \mathcal{E}(M)_t.$$

4. Show that f is non increasing on $[0, 1]$ and show that

$$\mathbb{E} \left[\exp \left(- \int_0^1 g(s)X_s ds \right) \right] = f(1)^{a/2} e^{x_0 f'(0)/2}.$$

5. Show that

$$\mathbb{E} \left[\exp \left(- \frac{\theta^2}{2} \int_0^1 X_s ds \right) \right] = \frac{1}{\cosh(\theta)^{a/2}} \exp \left(- \frac{x_0}{2} \theta \tanh(\theta) \right), \quad \forall \theta \in \mathbb{R}.$$

6. Let B be a standard Brownian motion. For every $x \in \mathbb{R}$, show that,

$$\mathbb{E} \left[\exp \left(- \frac{\theta^2}{2} \int_0^1 (B_s + x)^2 ds \right) \right] = \frac{1}{\cosh(\theta)^{1/2}} \exp \left(- \frac{x^2}{2} \theta \tanh(\theta) \right), \quad \forall \theta \in \mathbb{R}.$$

7. Let B and \tilde{B} be two independent standard Brownian motions. For every $t > 0$ show that $\inf \{s \geq 0 : |B_s| = t\} = \int_0^t B_s^2 ds + \int_0^t \tilde{B}_s^2 ds$ in law.

Solution

4. Use definition of f , that is, we have domination in L^1 for every $t \in [0, 1]$ ($X \geq 0, u \leq 0$).

5. Take $f(t) = \frac{\cosh(\theta(t-1))}{\cosh(\theta)}$.

Exercise 3.11

Let B be a Brownian motion and let $S_t := \sup_{s \in [0, t]} B_s$. We set $X_t := S_t - B_t$.

1. Show that $\int_0^t \mathbb{1}_{\{X_u \neq 0\}} dS_u = 0$.
2. Show that $Y_t := X_t^2 - t$ is a martingale.
3. Let $\tau := \inf\{t \geq 0 : X_t = 1\}$. Compute $\mathbb{E}[\tau]$.

Chapter 4

Stochastic differential equations

Exercise 4.1

Let $M_t := \frac{1}{2}(B_t^2 - t)$, $t \geq 0$.

1. Justify that M is a martingale and express M as a stochastic integral.
2. Show that for all $b \geq 0$, the exponential local martingale $\mathcal{E}(-bM)$ is a martingale. For all $T > 0$, justify that $\mathbb{Q} := \mathcal{E}(-bM)_T \cdot \mathbb{P}$ defines a probability measure.
3. Determine the SDE satisfied by $(B_t)_{t \in [0, T]}$ on \mathbb{Q} . Deduce the distribution of B_t , $t \in [0, T]$ on \mathbb{Q} .
4. Deduce that for all $a, b \geq 0$,

$$\mathbb{E} \left[\exp \left\{ -aB_t^2 - \frac{b^2}{2} \int_0^t B_s^2 ds \right\} \right] = \left(\frac{b}{b \cosh(bt) + 2a \sinh(bt)} \right)^{1/2}.$$

5. Using that for all $\alpha, \beta > 0$ and $s \geq 0$,

$$\frac{1}{\Gamma(\alpha)} \int_0^\infty \beta^\alpha x^{\alpha-1} e^{-(\beta+s)x} dx = \left(\frac{\beta}{s + \beta} \right)^\alpha,$$

compute

$$\mathbb{E} \left[\exp \left\{ -\frac{b^2}{2} \int_0^t B_s^2 ds \right\} \middle| B_t = y \right], \quad b > 0, y \in \mathbb{R}.$$

Solution

1. By Itô's formula we have that $B_t^2 = 2 \int_0^t B_s dB_s + t$, hence $M_t = \int_0^t B_s dB_s$ is a local martingale. Moreover, $\mathbb{E} [\langle M \rangle_t] = \mathbb{E} \left[\int_0^t B_s^2 ds \right] = \int_0^t \mathbb{E} [B_s^2] ds = \frac{t^2}{2} < \infty$, hence M is a martingale.
2. For every $T \geq 0$, $t \in [0, T]$, we have $\mathcal{E}(-bM)_t = e^{-\frac{b}{2}B_t^2 + \frac{bt}{2} - \frac{b^2}{2}\langle M \rangle_t} \leq e^{\frac{bT}{2}}$. Hence, $\mathcal{E}(-bM)$ is a martingale, $\mathbb{E} [\mathcal{E}(-bM)_T] = 1$ and thus \mathbb{Q} define a probability measure.
3. By Girsanov theorem we have that $\beta_t := B_t + b \langle B, M \rangle_t$ is a Brownian motion under \mathbb{Q} , i.e. B solve

$$dB_t = d\beta_t - bB_t dt.$$

We recognize the SDE satisfied by the Ornstein-Uhlenbeck process and thus,

$$B_t = \int_0^t e^{-b(t-s)} d\beta_s.$$

Therefore, B is a centered gaussian process with variance $\frac{1-e^{-2bt}}{2b}$.

4. For every $a, b \geq 0$ we have

$$\begin{aligned} \mathbb{E} \left[\exp \left\{ -aB_t^2 - \frac{b^2}{2} \int_0^t B_s^2 ds \right\} \right] &= \mathbb{E} \left[e^{-aB_t^2 + b(B_t^2 - t)} \mathcal{E}(-bM)_t \right] \\ &= \mathbb{E}_{\mathbb{Q}} \left[e^{(\frac{b}{2} - a)B_t^2 - \frac{b}{2}t} \right] \\ &= \mathbb{E}_{\mathbb{Q}} \left[e^{(\frac{b}{2} - a)(\frac{1-e^{-2bt}}{2b})Z - \frac{b}{2}t} \right] \end{aligned}$$

Where $Z \sim \mathcal{N}(1)$, and the expectation is well defined if $(\frac{b}{2} - a)(\frac{1-e^{-2bt}}{2b}) < \frac{1}{2}$, indeed

$$\left(\frac{b}{2} - a \right) \left(\frac{1 - e^{-2bt}}{2b} \right) \leq \frac{b - 2a}{4b} < \frac{1}{2}.$$

Therefore,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \left[e^{(\frac{b}{2} - a)(\frac{1-e^{-2bt}}{2b})Z - \frac{b}{2}t} \right] &= \frac{e^{-\frac{bt}{2}}}{\sqrt{1 - (\frac{b}{2} - a)(\frac{1-e^{-2bt}}{b})}} \\ &= \frac{1}{\sqrt{e^{bt} (1 - (\frac{b}{2} - a)(\frac{1-e^{-2bt}}{b})}} \\ &= \left(\frac{b}{be^{bt} - (\frac{b}{2} - a)(e^{bt} - e^{-bt})} \right)^{1/2} \\ &= \left(\frac{b}{b \cosh(bt) + 2a \sinh(bt)} \right)^{1/2}. \end{aligned}$$

Exercise 4.2

1. Justify that for all $T < 1$ and $x \in \mathbb{R}$, there exists almost surely a solution to the SDE:

$$X_t^x = x + B_t - \int_0^t \frac{X_s^x}{1-s} ds, \quad t \in [0, T].$$

2. By applying the Itô formula to $(\frac{X_t^0}{1-t})_{t \in [0, T]}$, find an explicit formula for $(X_t^0)_{t \in [0, T]}$.

3. Show that $X_t^x = X_t^0 + x(1-t)$, $t \in [0, T]$ and determine its distribution.

4. Show that X^x can be extended to a continuous process on $[0, 1]$.

5. What does $(X_t^x)_{t \in [0, 1]}$ represent ?

Solution

1. The coefficients $\sigma(t, x) = 1$ and $b(t, x) = -\frac{x}{1-t}$ are Lipschitz in space and bounded in time, thus there exists a unique solution.

2. By Itô formula we have

$$\frac{X_t^0}{1-t} = \int_0^t \frac{1}{1-s} dX_s^0 + \int_0^t \frac{X_s^0}{(1-s)^2} ds = \int_0^t \frac{1}{1-s} dB_s.$$

Hence, $X_t^0 = \int_0^t \frac{1-t}{1-s} dB_s$ and $X_t^0 \sim \mathcal{N}\left(0, \int_0^t \left(\frac{1-t}{1-s}\right)^2 ds\right) = \mathcal{N}(0, t(1-t))$ for every $t \in [0, T]$.

3. Let $t \in [0, T]$,

$$x + B_t - \int_0^t \frac{X_s^0 + x(1-t)}{1-s} ds = B_t - \int_0^t \frac{X_s^0}{1-s} ds + x - \int_0^t x ds = X_t^0 + x(1-t).$$

$X_t^0 + x(1-t)$ solve the SDE and by uniqueness we have that $X_t^x = X_t^0 + x(1-t)$. Moreover $X_t^x \sim \mathcal{N}(x(1-t), t(1-t))$.

4. It suffices to show that X^0 extend to a continuous process on $[0, 1]$. We know that $\frac{X_t^0}{t-1}$ is a centered gaussian process with covariance $\min\left(\frac{t}{1-t}, \frac{s}{1-s}\right)$, thus $X_t^0 = (1-t)B_{\frac{t}{1-t}}$ as processes on $[0, T]$. And by time inversion we have that

$$\lim_{t \rightarrow 1} (1-t)B_{\frac{t}{1-t}} = \lim_{t \rightarrow \infty} \frac{B_t}{t} = 0.$$

5. X^x is a Brownian bridge between x and 0.

Exercise 4.3

Let $b \in \mathbb{R}$, $a > 0$ and set $X_t = B_t - bt$. Let $T = \inf\{t \geq 0 : B_t = a\}$.

1. Find a probability measure \mathbb{Q} on \mathcal{F}_∞ such that $(X_t)_{t \geq 0}$ is a Brownian motion.
2. Deduce the value of $\mathbb{P}(T \leq t)$ and the distribution of $Z = \sup_{t \geq 0} X_t$ under \mathbb{P} .

Exercise 4.4

1. Let $X = (X_t)_{t \geq 0}$ solution to $E(\sigma, b)$ with value in an open set $D \subset \mathbb{R}^d$. Let $\lambda \in \mathbb{R}$. Let $f : D \rightarrow \mathbb{R}$ be twice continuously differentiable and such that $\mathcal{L}f = \lambda f$, where

$$(\mathcal{L}f)(x) := b(x)^\top \nabla f(x) + \frac{1}{2} \text{tr} \left((\sigma \sigma^\top)(x) D^2 f(x) \right).$$

Show that $(f(X_t)e^{-\lambda t})_{t \geq 0}$ is a continuous local martingale.

2. Let $B = (B^1, B^2, B^3)$ be a Brownian motion with value in \mathbb{R}^3 , and $B_0 := a \in \mathbb{R}^3 \setminus \{0\}$. Let $X = \|B\|^2$. Show that X solves some SDE $E(\sigma, b)$ and give the coefficients σ and b .
3. We now assume that $\lambda \geq 0$. Show that $2tf''(t) + 3f'(t) = \lambda f(t)$, $t > 0$, for $f(t) = \frac{\sinh(\sqrt{2\lambda t})}{\sqrt{2\lambda t}}$.
4. Let $x > \|a\|^2$, and let $T_x = \inf\{t \geq 0 : X_t = x\}$. Show that for every $\lambda \geq 0$, we have

$$\mathbb{E} \left[e^{-\lambda T_x} \right] = \frac{\sinh(\sqrt{2\lambda \|a\|^2})}{\sqrt{2\lambda \|a\|^2}} \frac{\sqrt{2\lambda x}}{\sinh(\sqrt{2\lambda x})}.$$

Exercise 4.5

1. Let H, Z and X be continuous semimartingales, such that $X_t = H_t + \int_0^t X_s dZ_s$. Express X as a function of H and Z .

Hint: start with $H \equiv 1$ and use a variation of the constant method as in classical ODE.

2. Solve $X_t = x + B_t - \beta \int_0^t X_s ds$, where $x \in \mathbb{R}$ and $\beta \geq 0$ are constants. The process X is called Ornstein-Uhlenbeck process.

Solution

1. By setting $H \equiv 1$, we have that $X = \mathcal{E}(Z)$, so we need to find a process Y such that $X_t = Y_t \mathcal{E}(Z)_t$. First by Itô formula we have

$$\begin{aligned} \frac{1}{\mathcal{E}(Z)_t} &= 1 - \int_0^t \frac{1}{\mathcal{E}(Z)_s^2} d\mathcal{E}(Z)_s + \int_0^t \frac{1}{\mathcal{E}(Z)_s^3} d\langle \mathcal{E}(Z) \rangle_s \\ &= 1 - \int_0^t \frac{1}{\mathcal{E}(Z)_s} dZ_s + \int_0^t \frac{1}{\mathcal{E}(Z)_s} d\langle Z \rangle_s. \end{aligned}$$

Therefore, using again Itô formula, we have

$$\begin{aligned} dY_t &= d\left(\frac{X_t}{\mathcal{E}(Z)_t}\right) = X_t d\left(\frac{1}{\mathcal{E}(Z)_t}\right) + \frac{dX_t}{\mathcal{E}(Z)_t} + d\left\langle X, \frac{1}{\mathcal{E}(Z)} \right\rangle_t \\ &= X_t \left(-\frac{1}{\mathcal{E}(Z)_t} dZ_t + \frac{1}{\mathcal{E}(Z)_t} d\langle Z \rangle_t\right) + \frac{1}{\mathcal{E}(Z)_t} (dH_t + X_t dZ_t) + d\left\langle X, \frac{1}{\mathcal{E}(Z)} \right\rangle_t \\ &= \frac{1}{\mathcal{E}(Z)_t} (dH_t + X_t d\langle Z \rangle_t) + d\left\langle X, \frac{1}{\mathcal{E}(Z)} \right\rangle_t \\ &= \frac{1}{\mathcal{E}(Z)_t} (dH_t + X_t d\langle Z \rangle_t) - \frac{1}{\mathcal{E}(Z)_t} d\langle H, Z \rangle_t - \frac{X_t}{\mathcal{E}(Z)_t} d\langle Z \rangle_t \\ &= \frac{dH_t - d\langle H, Z \rangle_t}{\mathcal{E}(Z)_t}. \end{aligned}$$

We deduce that

$$X_t = \mathcal{E}(Z)_t \left(H_0 + \int_0^t \frac{1}{\mathcal{E}(Z)_s} dH_s - \int_0^t \frac{1}{\mathcal{E}(Z)_t} d\langle H, Z \rangle_s \right).$$

2. Taking $H_t = x + B_t$ and $Z_t = -\beta t$ in the previous question, we obtain

$$X_t = e^{-\beta t} \left(x + \int_0^t e^{\beta s} dB_s \right).$$

Chapter 5

Exams

5.1 M2MO - 2024

Exercise 1

1. We know that $B_t^2 - t = 2 \int_0^t B_s dB_s$ and that $\mathcal{E}(L)$ is a local martingale where $L_t = -2 \int_0^t B_s dB_s$. Thus we have

$$\mathcal{E}(L)_t = e^{-2 \int_0^t B_s dB_s - \frac{4}{2} \int_0^t B_s^2 ds} = e^{-B_t^2 - t - 2 \int_0^t B_s^2 ds}.$$

We deduce that $a = -2$.

2. The sum of two independent gaussian variables is still gaussian, thus $(\frac{B_t+W_t}{\sqrt{2}})_{t \geq 0}$ is a gaussian process, $\mathbb{E} \left[\frac{B_t+W_t}{\sqrt{2}} \right] = 0$ and for every $t, s \geq 0$

$$\text{Cov} \left(\frac{B_s + W_s}{\sqrt{2}}, \frac{B_t + W_t}{\sqrt{2}} \right) = \frac{1}{2} (\text{Cov}(B_s, B_t) + \text{Cov}(W_s, W_t)) = s \wedge t.$$

We deduce that $(\frac{B_t+W_t}{\sqrt{2}})_{t \geq 0}$ is a Brownian motion and therefore for every $t \geq 0$, $\left\langle \frac{B+W}{\sqrt{2}} \right\rangle_t = t$. On the other hand we have

$$\left\langle \frac{B+W}{\sqrt{2}} \right\rangle_t = \frac{1}{2} (\langle B \rangle_t + 2 \langle B, W \rangle_t + \langle W \rangle_t) = t + \langle B, W \rangle_t.$$

Hence, for every $t \geq 0$, $\langle B, W \rangle_t = 0$.

3. Fix $t \geq 0$, by Girsanov's theorem, it suffice to find a process L such that for every $s \in [0, t]$, $B_s + s^2 = B_s - \langle B, L \rangle_s$ and $(\mathcal{E}(L)_s)_{s \in [0, t]}$ is a martingale. Take $L_s = -2 \int_0^s u dB_u$ and since

$$\mathbb{E} \left[e^{\frac{1}{2} \langle L \rangle_t} \right] = e^{2 \int_0^t s^2 ds} < \infty,$$

by Novikov's criterion, $(\mathcal{E}(L)_s)_{s \in [0, t]}$ is a martingale, let $d\mathbb{Q} = \mathcal{E}(L)_t d\mathbb{P}$ then $B_s + s^2$ is a Brownian motion under \mathbb{Q} .

Exercise 2

1. Set $X_n = B_n - B_{n-1}$, $n \geq 1$ be an i.i.d. sequence of integrable random variables, then by the law of large numbers:

$$\lim_{n \rightarrow \infty} \frac{B_n}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n X_i = \mathbb{E}[X_1] = 0.$$

2. Using basic properties of Brownian motion we have

$$\begin{aligned} \mathbb{P}(L_n > n^{2/3}) &= \mathbb{P}\left(\sup_{t \in [0,1]} |B_t| > n^{2/3}\right) \leq 2\mathbb{P}\left(\sup_{t \in [0,1]} B_t > n^{2/3}\right) \\ &\leq 2\mathbb{P}(|B_1| > n^{2/3}) \leq 2\mathbb{P}(B_1^2 > n^{4/3}) \leq 2\mathbb{P}\left(B_1^2 > \frac{1}{4}n^{4/3}\right). \end{aligned}$$

(I do not see we $\frac{1}{4}$ plays an important role in this exercise...)

3. By Markov's inequality, we have $\mathbb{P}(B_1^2 > \frac{n^{3/4}}{4}) \leq \frac{4}{n^{3/4}} \mathbb{E}[B_1^2] = \frac{4}{n^{3/4}}$. By the previous question we obtain that

$$\mathbb{P}\left(\frac{L_n}{n} > \frac{1}{n^{1/3}}\right) \leq \frac{8}{n^{4/3}}.$$

This prove first that $\frac{L_n}{n}$ converges toward 0 in probability and since $\frac{8}{n^{4/3}}$ is summable, by Borel Cantelli, it converges also almost surely toward 0 as $n \rightarrow \infty$.

4. For every $t > 0$, there exists $n \in [1, t]$ such that $t \in [n, n+1]$ and

$$\frac{|B_t|}{t} \leq \frac{|B_t - B_n|}{t} + \frac{|B_n|}{t} \leq \frac{L_n}{t} + \frac{|B_n|}{t} \leq \frac{L_n}{n} + \frac{|B_n|}{n},$$

taking $t \rightarrow \infty$ then $n \rightarrow \infty$ and by using previous questions we obtain $\lim_{t \rightarrow \infty} \frac{B_t}{t} = 0$ almost surely.

5. The process M is adapted (to the natural filtration of B), for every $t \geq 0$, $\mathbb{E}[|M_t|] \leq \mathbb{E}[e^{B_t}] = e^{\frac{t}{2}} < \infty$ and for every $0 \leq s \leq t$, using that $B_t - B_s$ is independent of \mathcal{F}_s , we have

$$\begin{aligned} \mathbb{E}\left[e^{B_t - \frac{t}{2}} \mid \mathcal{F}_s\right] &= \mathbb{E}\left[e^{B_t - \frac{t}{2}} \mid \mathcal{F}_s\right] = \mathbb{E}\left[e^{B_t - B_s} \mid \mathcal{F}_s\right] e^{B_s - \frac{t}{2}} \\ &= \mathbb{E}\left[e^{B_t - B_s}\right] e^{B_s - \frac{t}{2}} = e^{\frac{t-s}{2}} e^{B_s - \frac{t}{2}} = M_s. \end{aligned}$$

6. Using question 4. we have

$$M_\infty = \lim_{t \rightarrow \infty} e^{B_t - \frac{t}{2}} = \lim_{t \rightarrow \infty} e^{t\left(\frac{B_t}{t} - \frac{1}{2}\right)} = 0.$$

7. If M is a uniformly integrable martingale then $1 = \mathbb{E}[M_0] = \mathbb{E}[M_\infty] = 0$. Contradiction, M is not a uniformly integrable martingale.
8. Again, if $\sup_{t \geq 0} M_t$ is integrable, this will imply that M is uniformly integrable, since M would be dominated in L^1 by $\sup_{t \geq 0} M_t$, hence $\sup_{t \geq 0} M_t$ is not integrable.

Exercise 3

1. Clearly, ψ_n is even. Set $g_n(x) = \int_0^x \rho_n(u)du$, since ρ_n is continuous, g_n is \mathcal{C}^1 and therefore ψ_n is \mathcal{C}^2 . We have that

$$\psi'_n(x) = \frac{x}{|x|} g_n(|x|) \quad \text{and} \quad \psi''_n(x) = \rho_n(|x|).$$

(for \mathcal{C}^2 proof, one has that $\psi''_n(x) = \rho_n(|x|)$ is a continuous function)

And observe that g_n is non decreasing, if $x \leq a_n$, $g_n(x) = 0$ and if $x \geq a_{n-1}$, $g_n(x) = 1$. Thus $|\psi'_n(x)| = g_n(|x|) \leq 1$.

2. For all $y \geq 0$, since $\lim_{n \rightarrow \infty} a_n = 0$

$$\exists N \geq n, \forall n \geq N, a_{n-1} \leq y.$$

We deduce that for every $y \geq 0$

$$\lim_{n \rightarrow \infty} \int_0^y \rho_n(u)du = \lim_{n \rightarrow \infty} \int_{a_n}^{a_{n-1}} \rho_n(u)du = 1.$$

Since g_n is non decreasing, by monotone convergence theorem,

$$\lim_{n \rightarrow \infty} \psi_n(x) = \int_0^{|x|} \lim_{n \rightarrow \infty} \int_0^y \rho_n(u)du = |x|,$$

ψ_n is non decreasing, because g_n it is.

3. For every $n \geq 1$ and $t \geq 0$,

$$\begin{aligned} \mathbb{E} \left[\int_0^t \psi''_n(\Delta_s) (\sigma(X_s) - \sigma(Y_s))^2 ds \right] &\leq \mathbb{E} \left[\int_0^t \rho_n(|\Delta_s|) h(|\Delta_s|)^2 ds \right] \\ &\leq \mathbb{E} \left[\int_0^t \frac{2}{n} ds \right] = \frac{2t}{n}. \end{aligned}$$

4. We apply Itô's formula to the function $(x, y) \mapsto \psi_n(x - y)$:

$$\begin{aligned} d\psi_n(\Delta_t) &= \psi'_n(\Delta_t) dX_t - \psi'_n(\Delta_t) dY_t + \frac{1}{2} \psi''_n(\Delta_t) d\langle X \rangle_t + \frac{1}{2} \psi''_n(\Delta_t) d\langle Y \rangle_t - \psi''_n(\Delta_t) d\langle X, Y \rangle_t \\ &= \psi'_n(\Delta_t) (dX_t - dY_t) + \frac{1}{2} \psi''_n(\Delta_t) (\sigma(X_t) - \sigma(Y_t))^2 dt \\ &= \psi'_n(\Delta_t) (b(X_t) - b(Y_t)) dt + \psi'_n(\Delta_t) (\sigma(X_t) + \sigma(Y_t)) dB_t + \frac{1}{2} \psi''_n(\Delta_t) (\sigma(X_t) - \sigma(Y_t))^2 dt. \end{aligned}$$

The assumptions (1) makes $\psi'_n(\Delta_t) (\sigma(X_t) + \sigma(Y_t)) dB_t$ be a true martingale and thus have a expectation of 0. Using question 1., 3. and the Lipschitz condition on b we obtain

$$\begin{aligned} \mathbb{E} [\psi_n(\Delta_t)] &\leq \mathbb{E} \left[\int_0^t K |\Delta_s| ds \right] + \frac{1}{2} \mathbb{E} \left[\int_0^t \psi''_n(\Delta_s) (\sigma(X_s) - \sigma(Y_s))^2 ds \right] \\ &\leq K \int_0^t \mathbb{E} [|\Delta_s|] ds + \frac{t}{n}. \end{aligned}$$

5. By question 2., we can apply monotone convergence theorem, and passing to the limit

$$\mathbb{E} [|\Delta_t|] \leq K \int_0^t \mathbb{E} [|\Delta_s|] ds,$$

we conclude with Grönwall lemma, that gives us $\mathbb{E} [|\Delta_t|] = 0$ which imply that for every $t \geq 0$, $|X_t - Y_t| = 0$ almost surely. Paths continuity of X and Y ensure that almost surely for every $t \geq 0$, $|X_t - Y_t| = 0$ and we have path uniqueness of the solution of the SDE.

Exercise 4

1. If $x = 0$, we observe that $X_t \equiv 0$ is a solution and by uniqueness, this is the only one.
2. Fix $t \geq 0$, we apply the Itô's formula to the function $(s, x) \mapsto \exp\left(-\frac{\lambda x}{1+\frac{1}{2}(t-s)}\right)$:

$$\begin{aligned} dM_s &= -\frac{\lambda}{1+\frac{\lambda}{2}(t-s)}M_s dX_s + \frac{\lambda^2}{2} \frac{X_s}{(1+\frac{\lambda}{2}(t-s))^2} M_s ds + \frac{1}{2} \frac{\lambda^2}{1+\frac{\lambda}{2}(t-s)} d\langle X \rangle_s \\ &= -\frac{\lambda}{1+\frac{\lambda}{2}(t-s)} M_s \sqrt{X_s} dB_s. \end{aligned}$$

Therefore, $(M_s)_{s \in [0, t]}$ is a local martingale, moreover $M_s \leq 1$ for every $s \in [0, t]$, thus, this is a martingale on $[0, t]$.

3. Obviously,

$$e^{-\frac{\lambda x}{1+\frac{\lambda}{2}t}} = \mathbb{E}[M_0] = \mathbb{E}[M_t] = \mathbb{E}\left[e^{-\lambda X_t}\right].$$

4. Set $T := \inf\{s \geq 0 : X_s = 0\}$, then $\mathbb{P}(X_t = 0) = \mathbb{P}(T \leq t) \dots$
- 5.

$$\begin{aligned} dZ_t &= dX_t + dY_t \\ &= \sqrt{X_t} dB_t + \sqrt{Y_t} dW_t \\ &= \sqrt{X_t + Y_t} \left(\sqrt{\frac{X_t}{X_t + Y_t}} dB_t + \sqrt{\frac{Y_t}{X_t + Y_t}} dW_t \right). \end{aligned}$$

Take $\beta_t = \sqrt{\frac{X_t}{X_t + Y_t}} dB_t + \sqrt{\frac{Y_t}{X_t + Y_t}} dW_t$ which is a local martingale starting at 0 and for every $t \geq 0$ since B and W are independent Brownian motion, we have $\langle B, W \rangle_t = 0$ by **Exercise 1** and

$$\begin{aligned} \langle \beta \rangle_t &= \left\langle \int_0^t \frac{X_s}{X_s + Y_s} dB_s \right\rangle_t + \left\langle \int_0^t \frac{Y_s}{X_s + Y_s} dW_s \right\rangle_t \\ &= \int_0^t \frac{X_s}{X_s + Y_s} ds + \int_0^t \frac{Y_s}{X_s + Y_s} ds = t. \end{aligned}$$

By Lévy's characterization, β is a Brownian motion and $dZ_t = \sqrt{Z_t} d\beta_t$.