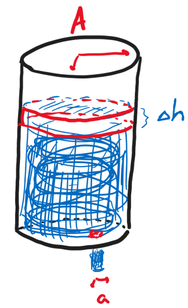
Math 3002: Problem Set 3

1. The point of this question is to give a 'physical' example of non-uniqueness.

   There is a water tank with a hole at the bottom. Derive a differential equation for the height of the water over time. To do this, say the kinetic energy of the water leaving the tank is equal to the (gravitational) potential energy lost from the top of the water moving down.

   The kinetic energy is $\frac{1}{2}mv^2$, where m is the total mass of the water and v is the speed of the water leaving the tank, and the potential energy is mgh, where m is again the total mass of the water moved, h is the height of the water, and g is some constant. So we have

   $$mgh = \frac{1}{2}mv^2$$

   or

   $$2gh = v^2$$

   We want a differential equation for h, so we want to write v in terms of h, h', h'', etc..

   (a) The mass of a fluid is given by the density times the volume. Suppose the density is constant throughout the fluid. Say the hole has area a, and the entire water tank has cross-sectional area A. How much water leaves the tank in some amount of time Δt? How does this affect the height of the water left? Use this to write an equation relating v and h'.

   Solution: The intended solution was: in a small amount of time Δt, the amount of water leaving the tank is shaped like a cylinder with cross-sectional area a, and height given by $v\Delta t$ (since v is the speed of the water leaving the hole). This water must come from somewhere, in particular, this volume should be the same as the volume of water displaced from the top.

   Say that in the same small amount of time Δt, the height changes by Δh (we don't know how much, that is what we are trying to find), so the volume of water lost from the top is $A\Delta h$ and equating these two volumes gives

   $$A\Delta h = av\Delta t$$

or

$$\frac{\Delta h}{\Delta t} = \frac{a}{A} v$$

Taking the limit as the time interval goes to zero, we get

$$\frac{dh}{dt} = \frac{a}{A} v$$

(b) Use the previous part and the energy discussion above to write a differential equation for the height of the water. Solve this differential equation by separating variables.

Solution: The equation given above already writes v in terms of h:

$$v = \sqrt{2gh} = \sqrt{2g}\sqrt{h}$$

Here we separated out the $\sqrt{2g}$ term to emphasize that it is a constant. Putting our previous answer together with this equation for v, we have

$$\begin{aligned} \frac{dh}{dt} &= \frac{a}{A} v \\ \frac{dh}{dt} &= \frac{a\sqrt{2g}}{A}\sqrt{h} \end{aligned}$$

which is a differential equation we know how to solve:

$$\begin{aligned} \frac{dh}{dt} &= \frac{a\sqrt{2g}}{A}\sqrt{h} \\ \frac{\frac{dh}{dt}}{\sqrt{h}} &= \frac{a\sqrt{2g}}{A} \\ \int h^{-\frac{1}{2}} dh &= \int \frac{a\sqrt{2g}}{A} dt \\ \frac{1}{2} h^{\frac{1}{2}} &= \frac{a\sqrt{2g}}{A} t + c \\ h(t) &= \left(\frac{2a\sqrt{2g}}{A} t + c\right)^2 \end{aligned}$$

where $c \in \mathbb{R}$ is some constant.

(c) If I tell you the height of the water tank at time $t = 10$ is zero, can you tell me how the height changed over time? Why or why not?

Solution: This was a bad question all around, my apologies. The answer is 'no' in an obvious way, because you don't know what a and A are (you can look up $g \approx 9.8$ in appropriate units).

The real point of the question was: notice that $h(t) = 0$ is also a solution, which was not picked up by our methods (why not? because our first step was to divide by 0 in that case).

More interestingly, our solutions are quadratic equations, which hit the x-axis at some time t_*, depending on c. This represents the height of the water being zero at that time, i.e., the tank is empty. If t gets bigger than this t_* then our formula says $h(t)$ should start increasing. Physically, this means that after the tank is empty, the water level starts rising again, as the 'no water' leaks out the hole. What? This does not make sense.

The 'actual' solutions should be something like

$$h(t) = \begin{cases} \left(\frac{2a\sqrt{2g}}{A}t + c\right)^2 & t < \frac{-cA}{2a\sqrt{2g}} \\ 0 & \text{else} \end{cases}$$

Physically, this says that once all the water has leaked out, the water level does not change any more.

Let's get to the question that was actually asked: if we know the water tank at time $t = 10$ is zero, can we say how the height changed over time? Not really: maybe $t = 10$ was the moment the water level reached zero, or maybe the leak started twenty three days ago and the water level was zero for a long time before $t = 10$. I should have given units, but the point is obvious if you think about the 'real world': if we find a water tank with a hole in it and no water, how could we possible know when the last drop left the tank? All we know is that the water ran out before we saw it.

2. The point of this question is to flex your muscles. Solve the initial value problems.

   (a)

$$\begin{cases} y' = x(y^2 + 1) \\ y(1) = 1 \end{cases}$$

Solution: This equation is separable, so writing in Leibniz notation we rearrange

$$\begin{aligned}\frac{dy}{dx} &= x(y^2+1)\\ \frac{dy}{y^2+1} &= xdx\\ \int \frac{dy}{y^2+1} &= \int xdx\\ \arctan(y) &= \frac{1}{2}x^2 + c\end{aligned}$$

$$y(x) = \tan(\frac{1}{2}x^2 + c)$$

If we want $y(1) = 1$, then we need

$$\tan(\frac{1}{2} + c) = 1$$

or

$$c = \frac{\pi}{4} - \frac{1}{2}$$

So our final solution is

$$y(x) = \tan(\frac{1}{2}x^2 + \frac{\pi}{4} - \frac{1}{2}$$

(b)

$$\begin{cases} y' = (x+1)y \\ y(2) = -1 \end{cases}$$

Solution: Again, we adopt Leibniz notation:

$$\begin{aligned}\frac{dy}{dx} &= (x+1)y\\ \frac{dy}{y} &= (x+1)dx\\ \int \frac{dy}{y} &= \int (x+1)dx\\ \ln(|y|) &= \frac{1}{2}x^2 + x + c\\ y(x) &= Ce^x e^{\frac{1}{2}x^2}\end{aligned}$$

If we want $y(2) = -1$, then

$$Ce^2 e^{\frac{1}{2}2^2} = Ce^4 = -1$$

so $C = -e^{-4}$ and

$$y(x) = -e^{-4}e^{x}e^{\frac{1}{2}x^2}$$

or alternatively $y(x) = -e^{\frac{1}{2}x^2+x-4}$

(c)

$$\begin{cases} y' = \frac{x^2y-y}{y+1} \\ y(3) = -1 \end{cases}$$

Solution: Notice we can factor y in the numerator, so this is again separable. As above:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x^2y-y}{y+1} \\
\frac{dy}{dx} &= (x^2-1)\frac{y}{y+1} \\
\frac{(y+1)dy}{y} &= (x^2-1)dx \\
(1+\frac{1}{y})dy &= (x^2-1)dx \\
\int(1+\frac{1}{y})dy &= \int(x^2-1)dx \\
y + \ln|y| &= \frac{1}{3}x^3 - x + c \\
ye^y &= Ce^{\frac{1}{3}x^3-x}
\end{aligned}$$

Wait, I recognize the left-hand side of this... it is not possible to express y as a function of x in simple terms. This is another example where I did not double-check that the question I asked is really 'solvable'... my apologies.

(d)

$$\begin{cases} y' = \frac{e^x}{y} \\ y(0) = -1 \end{cases}$$

Solution: As above:

$$\begin{aligned}\frac{dy}{dx} &= \frac{e^x}{y}\\ ydy &= e^x dx\\ \int ydy &= \int e^x dx\\ \frac{1}{2}y^2 &= e^x + c\\ y(x) &= \pm\sqrt{2e^x + k}\end{aligned}$$

If we want $y(0) = -1$, we must choose the negative branch of the square root, and $k = -1$, so

$$y(x) = -\sqrt{2e^x - 1}$$

3. The point of this question is to show a nice substitution, since we won't talk about it much in the course.

   We wish to solve the equation $y' = \frac{y+x}{x}$. Notice this equation is not separable, but we can turn it into a separable equation:

   (a) Create a new function,

   $$v(x) = \frac{y(x)}{x}$$

   and evaluate the derivative. Write your answer in terms of x, v, v' and y' (but not y!).

   Solution: Using the quotient rule, we get

   $$v'(x) = \frac{xy'(x) - y(x)}{x^2} = \frac{y'(x)}{x} - \frac{v(x)}{x}$$

   Rearranging to solve for $y'(x)$, we have

   $$y'(x) = v(x) + xv'(x)$$

   (b) Write the right-hand side of the original differential equation in terms of v and x. Replace the y' term in your answer to the previous part to get a differential equation in terms of v and x,

   Solution: Placing v into our original differential equation gives

   $$y'(x) = \frac{y(x) + x}{x} = \frac{y(x)}{x} + 1 = v(x) + 1$$

and setting this equal to the previous part gives

$$v + 1 = v + xv'$$

or

$$v' = \frac{1}{x}$$

(c) Use the differential equation for v to find a solution y to the original differential equation.

Solution: You can probably recognize the solution to

$$v'(x) = \frac{1}{x}$$

is

$$v(x) = \ln|x| + c,$$

so that

$$y(x) = xv(x) = x\ln|x| + cx$$

4. The point of this question is to explore how solutions change as we vary through a family of differential equations. Also, partial fractions practice.

   In class, we discussed the logistic equation

   $$\frac{dP}{dt} = P(1-P)$$

   as a simple model for population growth (where we think of 1 here as some 'carrying capacity', if P is close to 1 from below, the population increase slows down, while if P is greater than 1, population decreases until it reaches this carrying capacity.

   We could imagine that, in addition to this carrying capacity, we also have a steady decrease of the population due to some outside factor, such as hunting. Model this by

   $$\frac{dP}{dt} = P(1-P) - h$$

   where h is some constant 'hunting rate'.

   Solve this differential equation, assuming h is smaller than $\frac{1}{4}$. (Hint: to do partial fraction decomposition, you need to factor the roots of the denominator polynomial. The algebra may be easier if you name the roots something like r_+ and r_-, instead of writing down the algebraic expression every time). What happens (mathematically and in terms of the model) if $h < \frac{1}{4}$, $h = \frac{1}{4}$, or $h > \frac{1}{4}$?

Solution: This differential equation is still separable:

$$\begin{aligned} \frac{dP}{dt} &= P(1-P) - h \\ \frac{dP}{-P^2+P-h} &= dt \\ \int \frac{dP}{-P^2+P-h} &= \int dt \\ \int \frac{dP}{-P^2+P-h} &= t + c \end{aligned}$$

To integrate the left-hand side, we need to use partial fraction decomposition. To do this, we need to separate the denominator into a product of linear factors.

According to the quadratic formula, the roots of the bottom equation are

$$r_{\pm} = \frac{1 \pm \sqrt{1-4h}}{2}$$

so we need to find numbers a, b so that

$$\frac{1}{-P^2+P-h} = \frac{a}{P-r_+} + \frac{b}{P-r_-}$$

or

$$\begin{aligned} 1 &= a(P-r_-) + b(P-r_+) \\ 1 &= aP - ar_- + bP - br_+ \end{aligned}$$

To cancel the $P(t)$ terms, we need $a = -b$, so

$$1 = ar_+ - ar_- = a(r_+ - r_-) = a\sqrt{1-4h},$$

and

$$a = \frac{1}{\sqrt{1-4h}}$$

Then

$$
\begin{aligned}
\int \frac{dP}{-P^2+P-h} &= t+c \\
\int \frac{1}{(P-r_+)\sqrt{1-4h}} - \frac{1}{(P-r_-)\sqrt{1-4h}} dt &= t+c \\
\frac{1}{\sqrt{1-4h}} \ln|(P-r_+)\sqrt{1-4h}| - \frac{1}{\sqrt{1-4h}} \ln|(P-r_-)\sqrt{1-4h}| &= t+c \\
\ln|(P-r_+)\sqrt{1-4h}| - \ln|(P-r_-)\sqrt{1-4h}| &= t\sqrt{1-4h}+k \\
\ln|\frac{(P-r_+)}{(P-r_-)}| &= t\sqrt{1-4h}+k \\
\frac{(P-r_+)}{(P-r_-)} &= K \cdot e^{t\sqrt{1-4h}} \\
(P-r_+) &= (P-r_-) \cdot K \cdot e^{t\sqrt{1-4h}} \\
P\left(1 - K \cdot e^{t\sqrt{1-4h}}\right) &= -r_- K \cdot e^{t\sqrt{1-4h}} + r_+ \\
P(t) &= \frac{r_+ - r_- K \cdot e^{t\sqrt{1-4h}}}{\left(1 - K \cdot e^{t\sqrt{1-4h}}\right)}
\end{aligned}
$$

I remember working this out before I assigned it, but looking now this is completely unreasonable.

What one is supposed to notice is that for $h < \frac{1}{4}$, we get something like the logistic equation we discussed in class. In this case, notice the right-hand side of the differential equation has two roots, so we have two 'stable' solutions $P(t) = r_+$ and $P(t) = r_-$ (for which K does this occur? It's a little tricky for r_-). On the other hand, for $h = \frac{1}{4}$ there is only one root and only one constant solution $P(t) = r_+ (= r_-)$.

If $h > \frac{1}{4}$, then there are no roots to the right-hand side, and the derivative is always negative. This means that, if the hunting rate is too large, every solution eventually goes to $-\infty$.

Realistically, this means that for $h > \frac{1}{4}$ the population is eventually wiped out.

I was supposed to suggest that you read the Hirsch-Devaney-Smale chapter on the website. This example is in section 1.3, and introduces the idea of 'bifurcation': the solutions to this differential equation change drastically as we change the parameter h.