Third Day

Consider the first order linear **non**-homogeneous differential equation

$$x^2 y' + 2xy = \cos(x)$$

Were it not for that pesky $\cos(x)$ term, we could solve this by the method of last class.

Recall the idea: by rearranging the equation $x^2 y' + 2xy = 0$, we could write the ratio $\frac{y'}{y}$ as a function of x. Then we recognized that $\frac{y'}{y}$ is the 'logarithmic derivative' $(\ln(y))'$, by the chain rule. Then we could integrate and solve for y. Speaking more loosely, we wrote the derivative of some expression involving y as a function of x, which we could then integrate to get a function involving only x and y (not y').

After staring at the non-homogeneous equation for a bit, you might notice that $\frac{d}{dx}[x^2] = 2x$, so if we wrote $u(x) = x^2$ we would have

$$\begin{aligned} x^2 y' + 2xy &= u(x) \cdot y'(x) + u'(x) \cdot y(x) \\ &= (u(x) \cdot y(x))' \end{aligned}$$

recognizing the product rule. Then the original equation becomes

$$\left(x^2 y\right)' = \cos(x)$$

Integrating, we obtain

$$x^2 y = \int \cos(x) dx = \sin(x) + c$$

or

$$y(x) = \frac{\sin(x) + c}{x^2}$$

for some constant $c \in \mathbb{R}$.

We check that this is in fact a solution to the original differential equation:

$$\begin{aligned}
x^2y' + 2xy &= x^2 \frac{d}{dx}\left(\frac{\sin(x)+c}{x^2}\right) + 2x\left(\frac{\sin(x)+c}{x^2}\right)\\
&= x^2 \cdot \frac{x^2 \frac{d}{dx}\left(\sin(x)+c\right) - \left(\sin(x)+c\right)\frac{d}{dx}\left(x^2\right)}{x^4} + \frac{2x\left(\sin(x)+c\right)}{x^2}\\
&= \frac{x^2\cos(x) - 2x\sin(x) - 2cx}{x^2} + \frac{2x\left(\sin(x)+c\right)}{x^2}\\
&= \frac{x^2\cos(x) - 2x\sin(x) - 2cx + 2x\sin(x) + 2cx}{x^2}\\
&= \frac{x^2\cos(x)}{x^2}\\
&= \cos(x)
\end{aligned}$$

Great!

Let's try another: consider the differential equation

$$y' + \sin(x)y = 3x^2$$

Oh, there isn't a product rule going on. If the first term was $-\cos(x)y'$, that would be great, but it isn't. So we can't solve this one in the same way.

Or can we? Recall the form of the product rule, $(uv)' = uv' + u'v$. So if the left-hand side was coming from a product rule, the coefficient of y' would be the factor. Here, the coefficient is 1, so things don't work out, but we could multiply the equation by anything (μ is traditional):

$$\mu(x)y' + \mu(x)\sin(x)y = \mu(x)3x^2$$

In order for this left-hand side to be a product rule, we would need the coefficient of y to be the derivative of the factor, which we just said was the coefficient of y'.

This is a long-winded way of saying

$$(\mu(x)y)' = \mu(x)y' + \mu'(x)y$$

so we need $\mu'(x) = \mu(x)\sin(x)$ in order to write this as a product rule. But this is now a homogeneous linear equation, which we can solve! Namely:

$$\begin{aligned}
\mu' &= \sin(x)\mu \\
\frac{\mu'}{\mu} &= \sin(x) \\
(\ln(|\mu|))' &= \sin(x) \\
\ln(|\mu|) &= \int \sin(x) = -\cos(x) + c \\
\mu(x) &= Ce^{-\cos(x)}
\end{aligned}$$

We leave it to you to check this satisfies $\mu' = \sin(x)\mu$.

Remember our goal: we wanted to find a μ in order to rewrite our differential equation as a product rule. To do this, we only need one such μ, so might as well take our constant as $C = 1$.

Then we rewrite our equation as

$$\begin{aligned}
y' + \sin(x)y &= 3x^2 \\
e^{-\cos(x)}y' + \sin(x)e^{-\cos(x)}y &= 3x^2e^{-\cos(x)} \\
\left(e^{-\cos(x)}y\right)' &= 3x^2e^{-\cos(x)} \\
e^{-\cos(x)}y &= \int 3x^2e^{-\cos(x)} + c \\
y(x) &= e^{\cos(x)}\int 3x^2e^{-\cos(x)} + ce^{\cos(x)}
\end{aligned}$$

I won't attempt the integral, but we can still check that this is a solution by using the fundamental theorem of calculus. Here I wrote out the constant of integration to emphasize that this is still a family of solutions.

Let's formulate this for a general first order linear (non-homogeneous) differential equation. This is known as the **method of integrating factors**, which is what $\mu(x)$ is called (it's a 'factor' (in the product rule) that let's you 'integrate').

Consider an equation

$$y'(x) + p(x)y(x) = q(x)$$

and multiply by a function $\mu(x)$, chosen so that $\mu'(x) = p(x)\mu(x)$. As we have seen, the general solution to this is

$$\mu(x) = e^{\int p(x)dx}$$

The equation then becomes

$$\begin{aligned}
\mu(x)y'(x) + p(x)\mu(x)y(x) &= \mu(x)q(x) \\
(\mu(x)y(x))' &= \mu(x)q(x) \\
\mu(x)y(x) &= \int \mu(x)q(x)dx \\
y(x) &= \frac{\int \mu(x)q(x)dx}{\mu(x)}
\end{aligned}$$

Putting in our formula for $\mu(x)$, this becomes

$$y(x) = \frac{\int q(x)e^{\int p(x)dx}dx}{e^{\int p(x)dx}}$$

Or, in my usual laziness,

$$y = \frac{\int qe^{\int p}}{e^{\int p}}$$

In practice, one remembers the product rule idea and figures it out, rather than memorizing the formula. Remember to watch for different sign conventions, and remember we only need one integrating factor to rewrite the equation, so there is only one free parameter in this general solution, coming from the integral $\int \mu q$