# Quotient Topologies

Recall an **equivalence relation** on a set $X$ is a set of pairs, $\sim \subset X \times X$, such that

1. $x \sim x$ (reflexivity)

2. $x \sim y$ iff $y \sim x$ (symmetry)

3. If $x \sim y$ and $y \sim z$, then $x \sim z$ (transitivity)

(where we use the notation $x \sim y$ for $(x, y) \in \sim$)

**Definition 1.** *If $X$ is a set, and $\sim$ is an equivalence relation on $X$, then the* ***quotient set*** *$X/\sim$ is the set of equivalence classes of $\sim$.*
   *That is,*
$$X/\sim = \{\{y \in X : y \sim x\} \in \mathcal{P}(X) : x \in X\}$$

   Notice there is a canonical function $\pi : X \to X/\sim$, called the **projection** associated to $\sim$, given by

$$\pi(x) = \{y \in X : y \sim x\}.$$

**Definition 2.** *If $(X, \tau)$ is a topological space, and $\sim$ is an equivalence relation on $X$, then the* ***quotient topology*** *on $X/\sim$ is giveny by*

$$\tau_\sim = \{U \in \mathcal{P}(X/\sim) : \pi^{-1}(U) \in \tau\}$$

   *That is, a set in the quotient is open if its preimage under the canonical projection is open.*

**Remark 1.** *In particular, the projection map is continuous. It follows that the quotient of a connected or compact space is connected or compact.*

**Example 1.** *If $\mathbb{Z}$ is given the discrete topology, and we take $\mod 3$ as the equivalence relation, then $\mathbb{Z}/3\mathbb{Z}$ has the discrete topology.*
   *In general, any quotient of the discrete topology will be discrete.*

**Example 2.** *If $\mathbb{R}$ is the real numbers with the usual topology, define an equivalence relation $x \sim y$ iff there is some $k \in \mathbb{Z}$ such that $x + k = y$. Then every number is equivalent to a unique number in $[0, 1)$, so take this to model $\mathbb{R}/\sim$.*

For any positive number $t$, consider $\delta < \frac{t}{2}$. This is just so $(t - \delta, t + \delta)$ is completely contained in $(0, 1)$. It's preimage in $\mathbb{R}$ is

$$\bigcup_{k \in \mathbb{Z}} \{x \in \mathbb{R} : t - \delta + k < x < t + \delta + k\},$$

which is open. One should argue that any open set in $\mathbb{R}/\sim$ containing a positive $t$ must contain such an interval, again by considering the preimage in $\mathbb{R}$. The goal here is to say the topology is generated by sufficiently small intervals around the points. But what happens at $0$?

Since any open set in $\mathbb{R}$ containing $0$ must contain some interval $(-\varepsilon_1, \varepsilon_2)$, an open set in $\mathbb{R}/\sim$ must contain $[0, \varepsilon_2) \cup (1 - \varepsilon_1, 1)$, and of course any such set is open.

Then, one should argue, say by thinking about the map $[0, 1) \to \mathbb{C} :: t \mapsto e^{2\pi i t}$, that $\mathbb{R}/\sim$ is homeomorphic to a circle, usually denoted $\mathbb{S}^1$.

**Example 3.** *A fairly simple example that shows quotient topologies are not always nice is called 'the line with two origins'. This is given by a relation defined on two copies of the real line: $X = \mathbb{R} \times \{0\} \cup \mathbb{R} \times \{1\}$, with $(x, a) \sim (y, b)$ if $x = y$ and $x \neq 0$. So, $(t, 0) \sim (t, 1)$ for any $t \neq 0$.*

*In particular, $(0, 0)$ and $(0, 1)$ are in distinct equivalence classes. Any open set $U_0$ containing $(0, 0)$ (in the quotient) must contain an open interval $\{(t, 0) : -\varepsilon_1 < t < \varepsilon_2\}$ in its preimage, and any open set $U_1$ containing $(0, 1)$ (in the quotient) must contain some interval $\{(t, 1) : -\varepsilon_3 < t < \varepsilon_4\}$ in its preimage. Then, for $s = \frac{\min(\varepsilon_2, \varepsilon_4)}{2}$, we have $\{(s, 0), (s, 1)\} \in U_0 \cap U_1$.*

*In particular, $X/\sim$ is not Hausdorff, even though $X$ is.*

**Remark 2.** *Suppose we have a subset $A \subseteq X$. Then we can define an equivalence relation on $X$ by saying every element of $A$ is related, and no elements not in $A$ are related to anything other then themselves (required by reflexivity). In symbols, $x \sim y$ if $x, y \in A$, and if $x \notin A$ then $x \sim y$ only if $y = x$.*

*For instance, $[0, 1]/\{0, 1\}$ is homeomorphic to the circle $\mathbb{S}^1$, similar to example 2.*

*Another good example here is $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}/\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. This is the unit disk quotiented by its boundary, the unit circle. This turns out to be homeomorphic to the 2-sphere $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$*

**Example 4.** *Consider $X = \mathbb{R}$ with the usual topology, and the subset $A = \mathbb{Z}$. Then $X/A$ is an 'infinite wedge of circles'. That is, each interval $[k, k+1]$ in $\mathbb{R}$ becomes a copy of the circle with $k$ and $k+1$ glued together. Then, the gluing points on all these different circles are glued together to a single point.*

*Call the $k$-th circle $C_k = \pi([k, k+1])$, so $X/A = \bigcup_{k \in \mathbb{Z}} C_k$ and $\bigcap_{k \in \mathbb{Z}} = \{p\}$ is a single point, called the 'wedge point' (this point is the equivalence class of $\mathbb{Z}$, but writing $\{\mathbb{Z}\}$ to mean the single point set is potentially confusing).*

*Suppose that $\{U_n\}_{n \in \mathbb{N}}$ is a countable family of open sets which contain the wedge point. This means the preimages $\tilde{U}_n$ are open in $\mathbb{R}$. For each choice of $n$ and $k$, we must have positive numbers $\varepsilon_{n,k}$ such that $(k - \varepsilon_{n,k}, k + \varepsilon_{n,k}) \subseteq \tilde{U}_n$. Now, consider $\delta_k = \frac{\varepsilon_{k,k}}{2}$ and form the open set $\tilde{V} = \cup(k - \delta_k, k + \delta_k)$. We claim that $\tilde{U}_n \not\subseteq \tilde{V}$ for any $n$, so $U_n \not\subseteq V$ for any $n$.*

*What we have just shown is that $X/A$ is not first countable, in particular the wedge point does not have a countable neighborhood basis. This also shows $X/A$ is not metrizable.*

**Remark 3.** *Why did I write $X$ and $A$ instead of $\mathbb{R}$ and $\mathbb{Z}$? Notice that $\mathbb{Z} \subseteq \mathbb{R}$ can also define another equivalence relation, namely the the relation in Example 2. This takes advantage of the group structure $+$ on $\mathbb{R}$, and $\mathbb{Z}$ as a subgroup of $\mathbb{R}$. So, in group theory, the quotient '$\mathbb{R}/\mathbb{Z}$' means something different (Example 2) than what '$\mathbb{R}/\mathbb{Z}$' means in topology (Example 4), viewing $\mathbb{Z}$ as (only) a subspace of $\mathbb{R}$ instead of a subgroup.*

**Proposition 1.** *Suppose $X$ is a topological space, $Y = X/\sim$ some quotient, and $\pi : X \to Y$ the canonical projection. Consider any topological space $Z$. A function $f : Y \to Z$ is continuous if and only if $f \circ \pi : X \to Z$ is continuous.*

*Proof.* This follows from the definition of the quotient topology. $\square$

**Proposition 2.** *If $\sim$ is an equivalence relation on $X$, and $f : X \to Z$ is a continuous function such that $x \sim y$ implies $f(x) = f(y)$, then there is a unique continuous map $\hat{f} : X/\sim \to Z$.*

*Proof.* Since $f$ has the same image on equivalent elements, we get a well-defined map $\hat{f}$ on equivalence classes. It is not hard to check this map is continuous. $\square$

**Remark 4.** *This proposition says that the quotient space satisfies a 'universal property'. Universal here means 'for any function on $X$ which sends equivalent elements to equal images, (something)'*

Some words: the quotient topology is the **finest** topology on $X/\sim$ such that $\pi_\sim$ is continuous. This means that if $\tau'$ is a topology on $X/\sim$ such that $\pi_\sim$ is continuous, then any $\tau'$-open set is open in the quotient topology.