Math 3002: Problem Set 2

1. Rewrite each equation in the standard form of linear equations. Then solve each differential equation, showing and justifying all work:

   (a)
$$y' = \frac{x - y\cos(x)}{\sin(x)}$$

Solution: By 'standard form', I suppose I meant

$$y' + \frac{\cos(x)}{\sin(x)}y = \frac{x}{\sin(x)}$$

We want to find an integrating factor, i.e. a function $\mu(x)$ so that, multiplying through gives the derivative of a product:

$$\begin{aligned} \mu(x)y' + \mu(x)\frac{\cos(x)}{\sin(x)}y &= \mu(x)\frac{x}{\sin(x)} \\ (\mu(x)\cdot y)' &= \mu(x)\frac{x}{\sin(x)} \end{aligned}$$

To do this, we need $\mu(x)$ to satisfy the differential equation

$$\mu'(x) = \frac{\cos(x)}{\sin(x)}\mu(x),$$

or

$$\begin{aligned} \frac{\mu'}{\mu} &= \frac{\cos(x)}{\sin(x)} \\ (\ln(\mu))' &= \frac{\cos(x)}{\sin(x)} \\ \ln(|\mu|) &= \int \frac{\cos(x)}{\sin(x)}dx = \ln(|\sin(x)|) + c \\ |\mu| &= e^{\ln(|\sin(x)|+c} \\ \mu(x) = C\cdot\sin(x) \end{aligned}$$

Recall we only need a single function μ to make this work, so take $C = 1$. Multiplying through turns our differential equation into

$$\sin(x)y' + \cos(x)y = x$$

or

$$(\sin(x)y)' = x$$

so

$$\sin(x) \cdot y = \frac{1}{2}x^2 + k$$

or

$$y(x) = \frac{\frac{1}{2}x^2 + k}{\sin(x)}$$

which we could write as

$$y(x) = k \cdot \csc(x) + \frac{1}{2}x^2 \csc(x)$$

if we want.

(b)

$$y' = \frac{y + x^2}{2x}$$

Solution: As above, we rewrite to

$$y' - \frac{1}{2x}y = \frac{x}{2}$$

and look for μ such that

$$\mu y' - \mu\frac{1}{2x}y = (\mu y)'$$

so we need $\mu' = -\frac{1}{2x}\mu$.
Dividing and recognizing $\ln(|\mu|)$, we get

$$(\ln(|\mu|))' = -\frac{1}{2x}$$

so

$$\ln(|\mu|) = -\int \frac{1}{2x}dx = -\frac{1}{2}\ln(|x|) + c$$

or

$$\mu(x) = Cx^{-\frac{1}{2}}$$

Set $C = 1$, and using this integrating factor we get

$$\begin{aligned}
x^{-\frac{1}{2}}y' - x^{-\frac{1}{2}}\frac{1}{2x}y &= x^{-\frac{1}{2}}\frac{x}{2} \\
x^{-\frac{1}{2}}y' - \frac{1}{2}x^{-\frac{3}{2}}y &= \frac{1}{2}x^{\frac{1}{2}} \\
\left(x^{-\frac{1}{2}}y\right)' &= \frac{1}{2}x^{\frac{1}{2}} \\
x^{-\frac{1}{2}}y &= \int \frac{1}{2}x^{\frac{1}{2}} \\
x^{-\frac{1}{2}}y &= \frac{1}{3}x^{\frac{3}{2}} + c \\
y &= x^{\frac{1}{2}} \cdot \left(\frac{1}{3}x^{\frac{3}{2}} + c\right) \\
y(x) &= c\sqrt{x} + \frac{1}{3}x^2
\end{aligned}$$

(c)

$$xy' = x\sin(x) + y$$

(hint: integration by parts gives $\int x\sin(x)dx = \sin(x) - x\cos(x) + c$)

Solution: Here, if we rewrite as

$$y' - \frac{1}{x}y = \sin(x)$$

we look for an integrating factor satisfying

$$\mu' = -\frac{1}{x}\mu,$$

so

$$\begin{aligned}
\frac{\mu'}{\mu} &= -\frac{1}{x} \\
(\ln(|\mu|))' &= -\frac{1}{x} \\
\ln(|\mu|) &= -\ln(|x|) + c \\
\mu(x) &= \frac{C}{x}
\end{aligned}$$

Setting $C = 1$, multiplying our ODE gives

$$\frac{1}{x}y' - \frac{1}{x^2}y = \frac{\sin(x)}{x}$$

which is equivalent to

$$\left(\frac{1}{x}y\right)' = \frac{\sin(x)}{x},$$

and eventually we arrive at

$$y(x) = x\int \frac{\sin(x)}{x} + kx$$

This integral does not have a simple expression (see 'sine integral' on wikipedia, or 'sinc function' for the integrand).
I think this was intended to be

$$xy' = x\sin(x) - y,$$

so that the standard form is

$$y' + \frac{1}{x}y = \sin(x)$$

and we get $\mu(x) = x$ as an integrating factor (so actually, we shouldn't have divided by x to begin with, and left it as):

$$\begin{aligned} xy' + y &= x\sin(x) \\ (xy)' &= x\sin(x) \\ xy &= \int x\sin(x) = \sin(x) - x\cos(x) + c \\ y(x) &= \frac{\sin(x)}{x} - \cos(x) + \frac{c}{x} \end{aligned}$$

where we used the hint, or we could do the integration by parts by hand.

(d)

$$x^2y' + xy = 12x^2$$

Solution: Let's think a bit: to put this in standard form, we would divide by x^2. Then the left side would be exactly the same as the previous problem (with the intended sign). Instead, let's divide by x to get

$$xy' + y = 12x$$

or

$$(xy)' = 12x$$

Then integrating gives

$$xy = 6x^2 + k$$

or

$$y(x) = 6x + \frac{k}{x}$$

(e) For each differential equation above, write the corresponding homogeneous version (i.e., replace $b(x)$ with 0, using the notation from class), and find the solution of these homogeneous equations. Recognize anything?

Solution: The homogeneous associates are

(a) $y' + \frac{\cos(x)}{\sin(x)} y = 0$

(b) $y' - \frac{1}{2x} y = 0$

(c) $y' \pm \frac{1}{x} y = 0$

(d) $y' + \frac{1}{x} y = 0$

Without showing my work, the general solutions to these homogeneous equations are

(a) $y(x) = k \csc(x)$

(b) $y(x) = k\sqrt{x}$

(c) $y(x) = kx^{\mp 1}$

(d) $y(x) = \frac{k}{x}$

The thing one is supposed to notice is: every solution to the non-homogeneous versions consists of some specific function of x, plus an arbitrary solution of the associated homogeneous equation. That is, the 'free parameters' part of the discussion is completely given by the homogeneous equation.

It should not be hard to convince yourself of this: as an exercise, consider two solutions of a non-homogeneous linear differential equation, and observe that their difference satisfies the associated homogeneous linear differential equation.

2. Find a solution to the following differential equations:

   (a)
$$y' - 3y = 6$$

Solution: Multiply through by e^{-3x} to obtain

$$e^{-3x}y' - 3e^{-3x}y = 6e^{-3x},$$

or

$$\left(e^{-3x}y\right)' = 6e^{-3x}$$

Integrating, we obtain

$$e^{-3x}y = -2e^{-3x} + c$$

or

$$y(x) = -2 + ce^{3x}$$

for some constant $c \in \mathbb{R}$

We should check this is a solution, but I won't.

   (b)
$$y' - 3y = x$$

Solution: We proceed exactly the same: multiply by e^{-3x} to obtain

$$e^{-3x}y' - 3e^{-3x}y = xe^{-3x},$$

or

$$\left(e^{-3x}y\right)' = xe^{-3x}$$

To integrate the right-hand side, we need to use integration by parts: with $u = x$ and $dv = e^{-3x}dx$, we get

$$\int xe^{-3x}dx = -\frac{1}{3}xe^{-3x} - \int -\frac{1}{3}e^{-3x}dx = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + c,$$

(watch the minus signs...) and we obtain

$$e^{-3x}y = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + c$$

or

$$y(x) = -\frac{x}{3} - \frac{1}{9} + ce^{3x}$$

for some constant $c \in \mathbb{R}$

Here, maybe we should double-check. For simplicity I'll take $c = 1$ (it doesn't change much): if $y(x) = -\frac{x}{3} - \frac{1}{9} + e^{3x}$, then $y'(x) = -\frac{1}{3} + 3e^{3x}$, so

$$\begin{aligned} y' - 3y &= \left(-\frac{1}{3} + 3e^{3x}\right) - 3\left(-\frac{x}{3} - \frac{1}{9} + e^{3x}\right) \\ &= -\frac{1}{3} + 3e^{3x} + x + \frac{1}{3} - 3e^{3x} \\ &= x \end{aligned}$$

as desired.

(c) If $y_a(x)$ is your solution from part (a), and $y_b(x)$ is your solution for part (b), define a new function

$$z(x) = y_a(x) + y_b(x)$$

Find a differential equation which $z(x)$ solves.

Can you relate this question to question 1(e)?

Solution: If y_a solves the first equation, and y_b solves the second equation, and $z(x) = y_a(x) + y_b(x)$, then using the sum rule for differentiation we get

$$\begin{aligned} z'(x) &= y_a'(x) + y_b'(x) \\ &= (3y_a + 6) + (3y_b + x) \\ &= 3y_a + 3y_b + 6 + x \\ &= 3z(x) + 6 + x \\ z'(x) - 3z(x) &= 6 + x \end{aligned}$$

In the second line, we used that y_a and y_b are solutions to the differential equation, solving for their derivatives in terms of the functions themselves.

What you are supposed to notice is z solves a differential equation with the same homogeneous part, and the non-homogeneous part is the sum of the non-homogeneous parts for y_a and y_b.

This is the same as part 1e), where we think about the associated homogeneous equation as having 'non-homogeneous part' equal to 0.

I should also point out, when we add y_a and y_b above with general constants, we get

$$\begin{aligned} y_a(x) + y_b(x) &= -2 + c_a e^{3x} + -\frac{x}{3} - \frac{1}{9} + c_b e^{3x} \\ &= -2 - \frac{x}{3} - \frac{1}{9} + (c_a + c_b)e^{3x} \end{aligned}$$

So there is still really just one parameter.