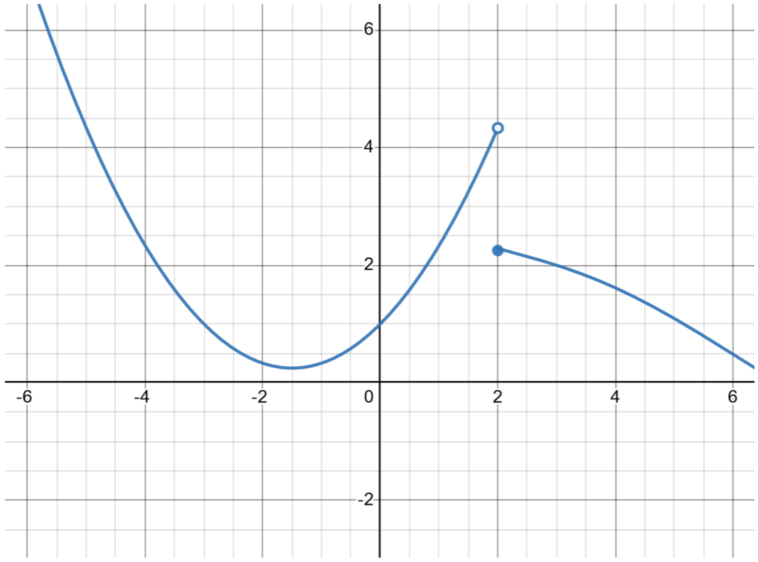
1. I have two functions, $f(x)$ and $g(x)$. Here is the graph of $f(x)$:



Unfortunately, I lost the graph of $g(x)$. Which of the following is true?

A. The limit $\lim_{x\to 2}[f(x)+g(x)]$ exists.

B. The limit $\lim_{x\to 2}[f(x)+g(x)]$ does not exist.

C. There is not enough information to tell if $\lim_{x\to 2}[f(x)+g(x)]$ exists.

Solution: The graph shows that the limit of $f(x)$ as x goes to 2 does not exist. But this does not mean that the limit of $f(x)+g(x)$ does not exist: If $g(x)$ also does not have a limit, but the values jump up by 2 instead of down by 2, then when you add $f+g$ the jumps will cancel out and the limit will exist. On the other hand, if $g(x)$ is a nice continuous function (no jumps), then the jump in $f(x)$ will make a jump in $f(x)+g(x)$.

2. Given two infinite decimals $a = 0.3939393939...$ and $b = 0.67766777666...$, their sum $a+b$:

A. is not defined because the sum of a rational and irrational number is not defined.

B. is not a number because not all infinite decimals are real numbers.

C. can be defined precisely by using successively better approximations

D. is not a real number because the pattern may not be predictable indefinitely.

Solution: I'm not sure I like this question anymore.

The relevance for this course is; the decimal numbers are like a kind of limit. More and more decimals give you more and more accuracy towards the actual number. The point of this question is, if you want to know 1000 decimals of $a+b$, you can get that by knowing sufficiently many decimals of of a and b (you would need 1001, because of possible carrying).

3. "Whether or not $\lim_{x\to a} f(x)$ exists depends on how $f(a)$ is defined" is true
   A. Sometimes
   B. Always
   C. Never

Solution: In general, the limit of $f(x)$ as x goes to a does not depend on $f(a)$. It does not even matter if $f(a)$ is defined. The limit is asking 'how do the function values behave as the input gets close to a', regardless of the actual value at $x = a$.

4. What is the maximum number of horizontal asymptotes that a function can have?
   A. One
   B. Two
   C. Three
   D. There is no maximum number.

Solution: Recall that a horizontal asymptote is a horizontal line, which is given by the value which is the limit of the function as x goes to infinity or negative infinity. So, there can be at most two horizontal asymptotes: the one corresponding to $\lim_{x\to\infty} f(x)$ and the one corresponding to $\lim_{x\to-\infty} f(x)$.

Notice there don't have to be two asymptotes: if one of these limits does not exist, there would only be one horizontal asymptote, while if neither limit existed there would be no horizontal asymptotes.

5. Find $\displaystyle\lim_{x\to 3}\frac{x^3-3x^2-x+3}{x^2-9}$, or explain why the limit does not exist.

Solution: This requires some algebraic manipulation: remember the difference of squares, $x^2-a^2=(x-a)(x+a)$. Applying this to the bottom, and factoring the top, we get

$$\lim_{x\to 3}\frac{x^3-3x^2-x+3}{(x-3)(x+3)}=\lim_{x\to 3}\frac{x^2\cdot(x-3)+(-1)\cdot(x-3)}{(x-3)(x+3)}$$

Observe that $\frac{x^2\cdot(x-3)+(-1)\cdot(x-3)}{(x-3)(x+3)}=\frac{x^2-1}{(x+3)}$, as long as $x\neq 3$. Remember that for any f the limit $\lim_{x\to 3}f(x)$ does not depend on the value at $x=3$. So these functions have the same limit. Remember that the limit of a quotient is equal to the quotient of the individual limits, as long as the limit of the denominator is nonzero. Thus

$$\begin{aligned}\lim_{x\to 3}\frac{x^3-3x^2-x+3}{x^2-9}&=\lim_{x\to 3}\frac{x^2-1}{(x+3)}\\&=\frac{\lim\limits_{x\to 3}x^2-1}{\lim\limits_{x\to 3}x+3}\\&=\frac{9-1}{3+3}=\frac{8}{6}=\frac{4}{3}\end{aligned}$$

In the last line, we used the limit laws for sums and powers. Alternatively, we could say that polynomials are continuous, which means the limit as x approaches some c is exactly the value at $x=c$. So direct substitution works for the numerator and denominator.

6. Find $\lim\limits_{x\to\infty} \dfrac{\sqrt{16x^4+52x-307}}{2x^2+2x+2}$, or explain why it does not exist.

Solution: Notice that when we say $\lim_{x\to\infty} 2x^2+2x+2=\infty$, we mean the limit does not exist. That means the limit laws do not apply. We cannot say that the limit of this quotient is the quotient of the limits (what would $\frac{\infty}{\infty}$ even mean?).

Instead, we try to factor things:

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{16x^4+52x-307}}{2x^2+2x+2} &= \lim_{x\to\infty}\left(\frac{\sqrt{16x^4\left(1+\frac{52x}{16x^4}-\frac{307}{16x^4}\right)}}{2x^2\left(1+\frac{2x}{2x^2}+\frac{2}{2x^2}\right)}\right)\\
&= \lim_{x\to\infty}\left(\frac{\sqrt{16x^4}\cdot\sqrt{\left(1+\frac{52x}{16x^4}-\frac{307}{16x^4}\right)}}{2x^2\left(1+\frac{2x}{2x^2}+\frac{2}{2x^2}\right)}\right)\\
&= \lim_{x\to\infty}\left(\left(\frac{4x^2}{2x^2}\right)\cdot\frac{\sqrt{\left(1+\frac{52x}{16x^4}-\frac{307}{16x^4}\right)}}{\left(1+\frac{2x}{2x^2}+\frac{2}{2x^2}\right)}\right)\\
&= \lim_{x\to\infty}\left(2\cdot\frac{\sqrt{\left(1+\frac{13}{4x^3}-\frac{307}{16x^4}\right)}}{\left(1+\frac{1}{x}+\frac{1}{x^2}\right)}\right)
\end{aligned}$$

I want to point out two things: first, everything that I have done is *inside* the limit. It is all algebra. Only when we want to split up the limit do we need to worry about which limits exists so we can use the limit laws. The second thing is, we still have to follow the rules of algebra. A big rule is you are not allowed to divide by zero, but $2x^2$ is 0 when $x=0$. Since we are interested in the limit as x goes to infinity, we only need to know our function values when x is big. To be careful we should say something about this, so this is us saying something.

Now we use that the limit of a constant times a function is the constant times the limit of the function (limit law 3 in Stewart, you could also use law 4) and law 5, that the limit of a quotient is the quotient of the limits, to get

$$\lim_{x\to\infty}\left(2\cdot\frac{\sqrt{\left(1+\frac{13}{4x^3}-\frac{307}{16x^4}\right)}}{\left(1+\frac{1}{x}+\frac{1}{x^2}\right)}\right)=2\cdot\lim_{x\to\infty}\frac{\sqrt{\left(1+\frac{13}{4x^3}-\frac{307}{16x^4}\right)}}{\left(1+\frac{1}{x}+\frac{1}{x^2}\right)}$$

$$=2\cdot\frac{\lim_{x\to\infty}\sqrt{\left(1+\frac{13}{4x^3}-\frac{307}{16x^4}\right)}}{\lim_{x\to\infty}\left(1+\frac{1}{x}+\frac{1}{x^2}\right)}$$

Stewart describes the 12th limit law, the 'root law' (but also think about the sentence after example 5 on page 119), which we can apply, together with the sum law, to get:

$$\lim_{x\to\infty}\frac{\sqrt{16x^4+52x-307}}{2x^2+2x+2}=2\cdot\frac{\lim_{x\to\infty}\sqrt{\left(1+\frac{13}{4x^3}-\frac{307}{16x^4}\right)}}{\lim_{x\to\infty}\left(1+\frac{1}{x}+\frac{1}{x^2}\right)}$$

$$=2\cdot\frac{\sqrt{\lim_{x\to\infty}\left(1+\frac{13}{4x^3}-.\frac{307}{16x^4}\right)}}{\lim_{x\to\infty}\left(1+\frac{1}{x}+\frac{1}{x^2}\right)}$$

$$=2\cdot\frac{\sqrt{\lim_{x\to\infty}1+\lim_{x\to\infty}\frac{13}{4x^3}-\lim_{x\to\infty}\frac{307}{16x^4}}}{\lim_{x\to\infty}1+\lim_{x\to\infty}\frac{1}{x}+\lim_{x\to\infty}\frac{1}{x^2}}$$

$$=2\cdot\frac{\sqrt{1+0-0}}{1+0+0}$$

$$=2\cdot\frac{1}{1}=2$$

Uff. At the end we also needed to know that $\lim_{x\to\infty}\frac{1}{x^n}=0$ for any positive integer n (in fact, this is true for any positive number, it doesn't have to be an integer).

7. $\lim_{x \to -3} \dfrac{x^3 - 3x^2 - x + 3}{x^2 - 9}$

Solution: We would like to use the quotient limit law, but the limit of the denominator is 0 while the limit of the numerator is

$$(-3)^3 - 3(-3)^2 - (-3) + 3 = -27 - 27 + 3 + 3 = -54 + 6 = -48$$

Intuitively, you should think that since the numerator is always near -48, while the number we are dividing by gets smaller and smaller, the absolute value of this function gets bigger and bigger as x gets closer to -3. So this limit will probably not exist, but go off to infinity. A little more formally:

First think about the numerator: since it is a polynomial, the limit as x goes to a is the same as the value at a. So, when x is close to -3, the value of the numerator is near -48. I want to only think about x close enough to -3 that the value of the numerator is between -40 and -50. I can do this by the definition of the limit.

Next, I think about what the denominator is doing. Since $(-3)^2 - 9 = 0$, and $x^2 - 9$ is a polynomial, we know we can make the values of $x^2 - 3$ close to 0 by making x close to -3.

Putting these together, I can choose some 'close enough' so that, for x close enough to -3, two things happen. The value of $x^2 - 9$ is between, say, $\frac{-40}{1000}$ and $\frac{40}{1000}$, and the value of the numerator is between -40 and -50. Then we can bound the fraction

$$\begin{aligned} \left|\frac{x^3 - 3x^2 - x + 3}{x^2 - 9}\right| &\geq \frac{|x^3 - 3x^2 - x + 3|}{|x^2 - 9|} \\ &\geq \frac{40}{\frac{40}{1000}} = 1000 \end{aligned}$$

In other words, there is some neighborhood of -3 so that every x in the neighborhood has a value bigger than 1000 or less than -1000. If we wanted, we could do the same analysis to make the values bigger than 1000000, or 10^9, or any number. So the limit of this expression cannot exist.

We could even go a bit further: if x is slightly greater than -3, say -2.9, then $x^2 - 9$ will be negative, so the function value will be positive. On the other hand, if x is slightly less than -3, say -3.1, then $x^2 - 9$ will be positive, so the function value will be negative. So it is not even true that the limit is infinity. Rather, we have two different one-sided limits:

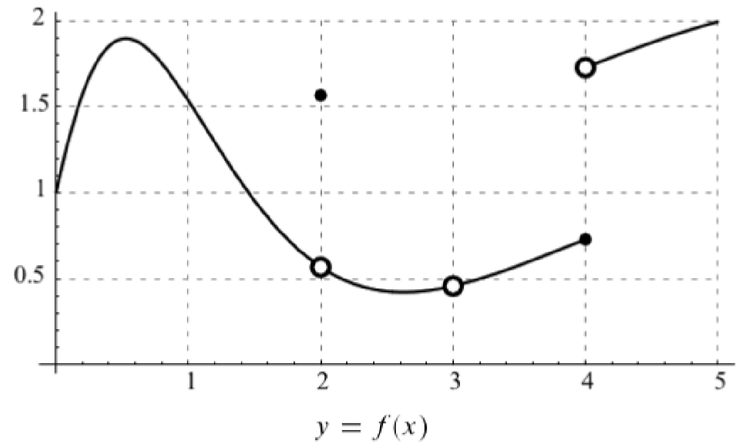
$$\lim_{x \to -3^-} f(x) = -\infty$$

and

$$\lim_{x \to -3^+} f(x) = \infty$$

(Remember, saying the limit is infinite is a special way of saying the limit does not exist. Infinity is not a number, and the limit is supposed to be a number.)

8. Use the following graph to find the values:



If the value does not exist, write DNE.

(a) $\lim_{x\to 1^-} f(x)$ __1.5__

(b) $\lim_{x\to 1^+} f(x)$ __1.5__

(c) $\lim_{x\to 1} f(x)$ __1.5__

(d) $f(1)$ __1.5__

(e) $\lim_{x\to 2^-} f(x)$ __0.55__

(f) $\lim_{x\to 2^+} f(x)$ __0.55__

(g) $\lim_{x\to 2} f(x)$ __0.55__

(h) $f(2)$ __1.55__

(i) $\lim_{x\to 3^-} f(x)$ __0.45__

(j) $\lim_{x\to 3^+} f(x)$ __0.45__

(k) $\lim_{x\to 3} f(x)$ __0.45__

(l) $f(3)$ __DNE__

(m) $\lim_{x\to 4^-} f(x)$ __0.75__

(n) $\lim_{x\to 4^+} f(x)$ __1.75__

(o) $\lim_{x\to 4} f(x)$ __DNE__

(p) $f(4)$ __0.75__

Solution: Whether you thought it was 0.55 or 0.6 or really just 0.5 doesn't matter.