First Definitions

An important part of modern mathematics is the concept of a 'function'. We think of a knot as an 'abstract' circle which we *put* into space. That is, a kind of function from a circle into three-dimensional space. If we wanted to be formal, we could say

Definition 1. *A* ***parametrized circle in*** $\mathbb{R}^3$ *is a continuous function* $f : [0, 2\pi] \to \mathbb{R}^3$ *such that* $f(0) = f(2\pi)$.

Here, $\mathbb{R}$ means the set of real numbers and $\mathbb{R}^3 == \{(x, y, z) : x, y, z \in \mathbb{R}\}$ means the set of all (ordered) triples of real numbers.

As an example of a parametrized circle, we have the usual unit circle in xy-plane, given by

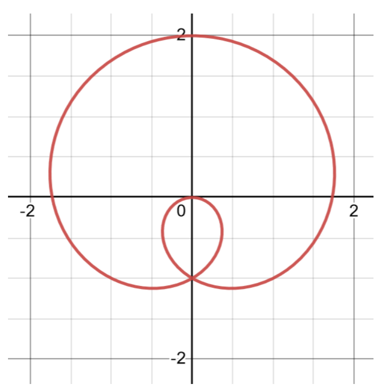
$$f(\theta) = (\cos(\theta), \sin(\theta), 0)$$

We should check that $f(2\pi) = f(0)$, so that this really fits our definition. It is easy to see that, e.g., $(\cos(\theta), 0, \sin(\theta))$ or $(\sin(\theta), 6, \cos(\theta))$ also give parametrized circles in $\mathbb{R}^3$.

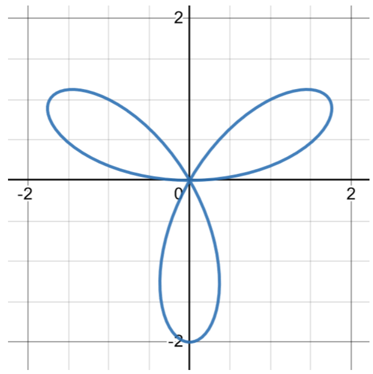
It is traditional to use θ as the circle variable. Prasolov gives a parametrization of a trefoil knot in 3-dimensional space.

Here are some parametrizations which **are not** knots:

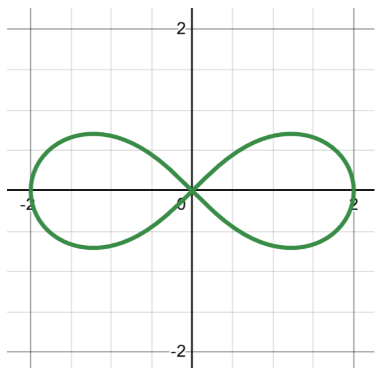
$$f(\theta) = (\cos(\theta) + \sin(2\theta), \sin(\theta) - \cos(2\theta), 0)$$



$$g(\theta) = (\cos(\theta) + \sin(2\theta), \sin(\theta) + \cos(2\theta), 0)$$



$$h(\theta) = (2\sqrt{\cos(2\theta)}\cos(\theta), 2\sqrt{\cos(2\theta)}\sin(\theta), 0)$$



Notice these are all in the xy-plane, so you might say they aren't 'really' 3-dimensional, but that's not the issue I want to focus on. Rather, all of these curves have **self-intersection** (the last example is interesting: it looks like it 'passes through' itself, but that isn't what really happens. Try graphing it for different values of θ).

One way of saying what this 'self-intersection' means is

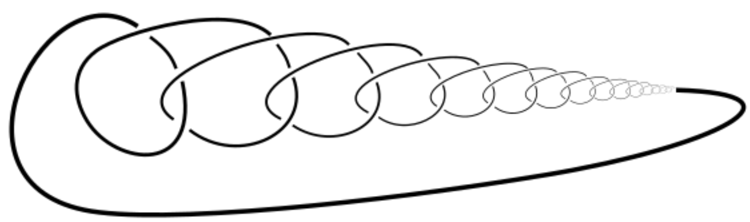
Definition 2. *A function $f : X \to Y$ is **injective** if, for any pair a, b of different elements of X, the values $f(a)$ and $f(b)$ are different.*

More symbolically:

$$\forall a, b \in X : a \neq b \implies f(a) \neq f(b)$$

The symbol $\forall$ means 'for all', so the symbols are read 'for all a, b in X, if $a \neq b$ then $f(a) \neq b$.'

Since knots are supposed to model real physical knots (ignoring things like thickness), we should not allow two points of the knot to be in literally the same place. So, we could define knots as injective functions of the circle. This almost works... except it allows things like this:



(image courtesy of Wikipedia)

It might be fun to study such things, but let's pretend we want a somewhat realistic model of physical knots, so we should exclude this. The kind of knots we want, we call 'tame', and knots like the above are called 'wild'. We want a good definition of 'tame knot'. Try to come up with one.

Here's one attempt, with the idea that 'real' knots have thickness:

Definition 3. *An injective continuous function $f : \mathbb{S}^1 \to \mathbb{R}^3$ from the circle into 3-space is called **tame** if it can be thickened. More precisely, if $\mathbb{D}^2$ denotes the 2-dimensional disk, then $\mathbb{S}^1 \times \mathbb{D}^2$ denotes the solid torus, and we ask that f is able to be extended to an injective function $\hat{f} : \mathbb{S}^1 \times \mathbb{D}^2 \to \mathbb{R}^3$. Here, 'extended' means that $\hat{f}(\theta, c) = f(\theta)$, where c is the center of the disk $\mathbb{D}^2$.*

(In particular, tame functions must be injective and continuous.)

Definition 4. *A **tame knot** is a tame function from the circle into 3-dimensional space.*

So, a (mathematical) knot is tame if it can be represented by some actual knot in physical space.

Smooth and Piecewise-Linear Models

Part of the reason we worried about wild knots above is that the idea of a 'function' is itself very wild, in some sense. Absolutely any way of assigning unique outputs to given inputs will define a function. For instance, the indicator function

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & x = \frac{a}{b} \\ 0 & x \text{ is not a fraction} \end{cases}$$

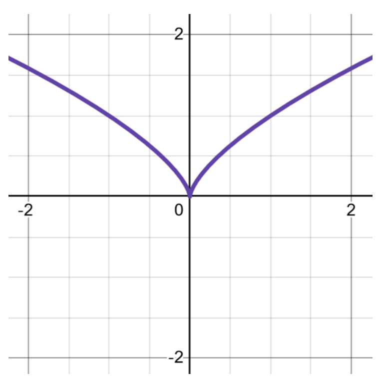
or the 'popcorn'/'Stars under Babylon' function

$$P(x) = \begin{cases} \frac{1}{b} & x = \frac{a}{b} \\ 0 & x \text{ is not a fraction} \end{cases}$$

These functions are not continuous (At least not everywhere. Exercise: Is there any point for which the Stars under Babylon function is continuous at that point?)

You can imagine writing down all sorts of silly rules, but most 'real world' things are not so arbitrary. Temperature doesn't jump wildly as you move around the room, it varies continuously, even smoothly. So we might get the idea that we should not just want our knots to be continuous functions, but differentiable, or smooth (meaning infinitely-differentiable, we can take derivatives as many times as we want). Here is an example of a curve with a smooth parametrization:

$$\gamma(t) = (x(t), y(t)) = (t^3, t^2)$$



Notice that this curve *cannot* be thickened near the cusp point. The width of any thickening would have to get smaller and smaller as it approaches the cusp, to avoid overlapping with the other side of the curve. So the thickening at the cusp point would have width zero, so not be thickened.

Even though the function $\gamma(t)$ has all derivatives, the derivative at $t = 0$ is

$$g'(0) = (2 \cdot 0, 3 \cdot 0^2) = (0, 0)$$

Imagine a car traveling along this path, trying to keep their speed constant. This is impossible, since they have to pull into the cusp and then immediately be moving out, in the opposite direction. The car would have to stop (i.e. its position has derivative 0).

Definition 5. *A function $f : \mathbb{S}^1 \to \mathbb{R}^3$ is an **immersion** if it is smooth and its derivative is always non-zero.*

(Here the derivative f' also has values in $\mathbb{R}^3$, so 'non-zero' means 'not the triple $(0, 0, 0)$').

It turns out that this vanishing derivative is the only issue:

Proposition 1. *An injective immersion of $\mathbb{S}^1$ into $\mathbb{R}^3$ is tame.*

(since an immersion must be smooth, it is continuous.)

We could then try to make a

Definition 6. *A **smooth knot** is an injective immersion from $\mathbb{S}^1$ into $\mathbb{R}^3$.*

This has many advantages, first of all that 'tame'ness is kind of confusing, and it's not clear how you would prove something is tame. Taking derivatives is not that difficult though, so if we are okay only looking at smooth things (which, physically, is pretty reasonable), we have this nice way to check: just make sure the derivative is not 0.

There is another approach to making knots easier to deal with, by looking only at certain kinds of knots. Even smooth functions can be pretty bizarre, and curve around in all sorts of ways. This seems okay since we can also move a knot around in all sorts of ways, but since we can do that why not move it into a particularly nice configuration, with as little curving as possible. Of course, you need to allow some way of curving, to make the knot, but we can allow 'almost no curving' by taking straight lines as our basic pieces, and gluing them together.

Definition 7. *A **piecewise-linear function** from $\mathbb{S}^1$ into $\mathbb{R}^3$ is function such that you can break $\mathbb{S}^1$ up into intervals A_i, $i = 1, 2, 3, ..., n$, and for each i the image $f(A_i)$ is a straight line segment in $\mathbb{R}^3$.*

We might also call this a 'polygonal closed curve', where 'closed' means the start and end points glue together like a circle.

Exercise 1 (hard?). *Any polygonal closed curve with* ≤ 5 *pieces is unknotted.*

Compare with: