September 20

Last time, we ended by talking about how a function behaves as the inputs get bigger and bigger ('go off to infinity').

An example is the function

$$r(x) = \frac{3x^2 + 2}{5x^2 + 7}$$

By evaluating $r(x)$ at different "x"s, we can get a table

x	$r(x)$
0	$\frac{2}{7}$
1	$\frac{5}{12}$
2	$\frac{14}{27}$
10	$\frac{302}{507}$
100	$\frac{30002}{50007}$
1000	$\frac{3000002}{5000007}$
1000000...	$\frac{30000...}{50000...}$

In the last entry, the number of digits of the numerator and denominator are the same.

So, I claim that the limit as x goes to infinity of $r(x)$ is $\frac{3}{5}$: for really large x, $3x^2 + 2$ is almost equal to $3x^2$, relative to how big these numbers are. I'll write

$$\lim_{x\to\infty} r(x) = \lim_{x\to\infty} \frac{3x^2+2}{5x^2+7} \approx \lim_{x\to\infty} \frac{3x^2}{5x^2} = \lim_{x\to\infty} \frac{3}{5} = \frac{3}{5}$$

Remember, we're asking how the values behave as x gets bigger and bigger. I like thinking about 10, 100, ... 10^n, because when you put these in a power function, the number of digits behaves in a very understandable way. That is, 10^3 has 4 digits, while $(10^3)^5 = 10^{15}$ has 16 digits. Then it's believable that for x very big, the only x^n that matters is the biggest n. In this case, x^2 just gets much bigger than 2 or 7. Even if there was an x term, it will only have about half as many digits the x^2 term.

I should also point out, $r(-1) = \frac{5}{12}$, and $r(-2) = \frac{14}{27}$, and so on, $r(-x) = r(x)$. This is because all of the terms in $r(x)$ are the input raised to an even power (and constant numbers, which you can think of like $2 \cdot x^0$ if you like). So the behavior as the inputs go to negative infinity is the same.

We also talked through some other examples:

$$q(x) = \frac{3x^5 - 50x^2 + 173}{7x^5 + 70x - 13} \qquad c(x) = \frac{5x^4 + 7x^2 - 2}{3x^4 - 2x^2 + 104}$$

x	$q(x)$
1	1.96875
10	0.4212622754....
100	0.4285642454
1000	0.4285714214...
10000	0.42857142856...
100000	0.428571428571...

x	$c(x)$
1	0.095238095...
10	1.69535848....
100	1.667010549...
1000	1.6666701110...
10000	1.6666667011...
100000	1.6666666670...

$$u(x) = \frac{1}{4 + 2^{-\frac{x}{2}}} \qquad k(t) = \frac{1}{2}\sin\left(\frac{\pi t}{100}\right)$$

x	$u(x)$
1	0.21244472379...
10	0.248062
100	0.24999999999...
1000	≈ 0.25
10000	≈ 0.25
100000	≈ 0.25

t	$k(t)$
1	0.0157053795...
10	0.154508497....
100	0
1000	0
10000	0
100000	0

$$h(x) = \frac{x}{3x^3 - 4x + 2} \qquad l(x) = \frac{2x^3 - 6x + 2}{3x^2 + 7}$$

x	$h(x)$
1	1
10	0.003376097...
100	0.0000333...
1000	$3.33... \times 10^{-7}$
10000	$3.33... \times 10^{-9}$
100000	$3.33... \times 10^{-11}$

x	$l(x)$
1	-0.2
10	6.3257...
100	66.631186...
1000	666.66311...
10000	6666.66631...
100000	66666.66663...

This 'approximately' is important: at no point does $r(x)$ actually become equal to $\frac{3}{5}$. How should we argue that the limit is actually equal to $\frac{3}{5}$? To do this, let me give a precise definition of this 'limit at infinity' idea:

Definition 1. *We say the limit of $f(x)$ as x goes to ∞ is equal to L, written*

$$\lim_{x\to\infty} f(x) = L,$$

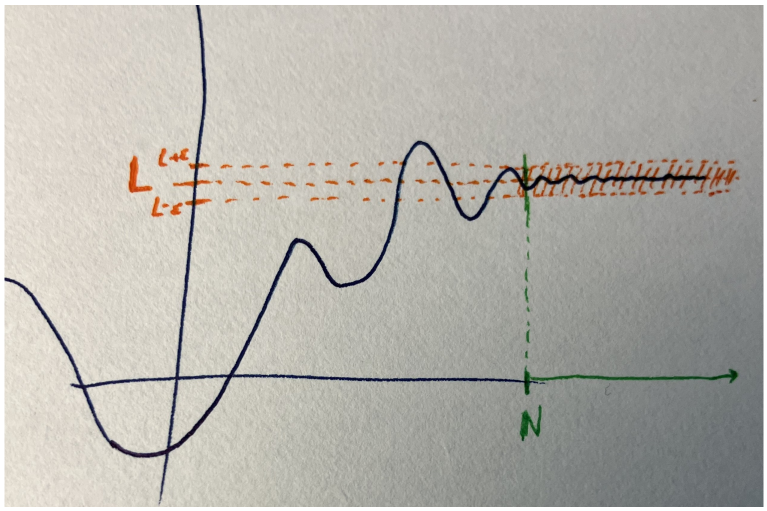
if, for any number $\varepsilon > 0$ there is some number N so that

$$|f(x) - L| < \varepsilon$$

whenever

$$x > N$$

The ε symbol is the Greek letter 'epsilon', which you might think of as 'e' short for 'error'. This definition is saying that for any error (think 'any error, as small as we want') we can find some threshold so that every input above that threshold has value L, up to that error. Pictorially:



If you can do this for any ε, that means however close you want the function to be to L, you can arrange it so that happens.

Since I've given the official definition for this limit, let me give the official definition for the usual limit we have been discussing:

Definition 2. *We say the limit of $f(x)$ ad x goes to a is equal to L, written*

$$\lim_{x \to a} f(x) = L,$$

if, for any number $\varepsilon > 0$ there is some number $\delta > 0$ so that

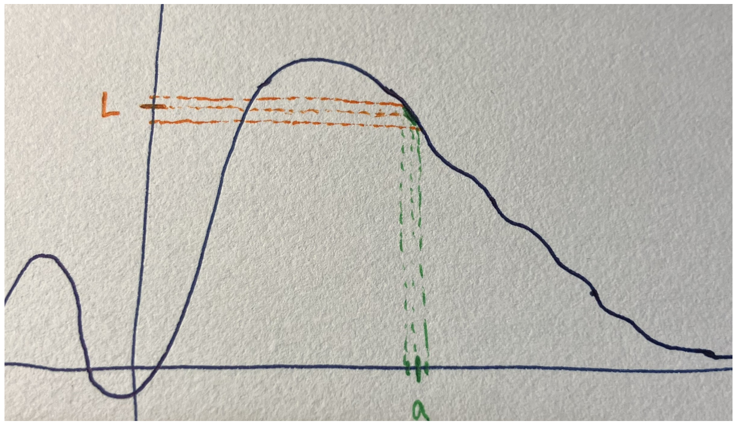
$$|f(x) - L| < \varepsilon$$

whenever

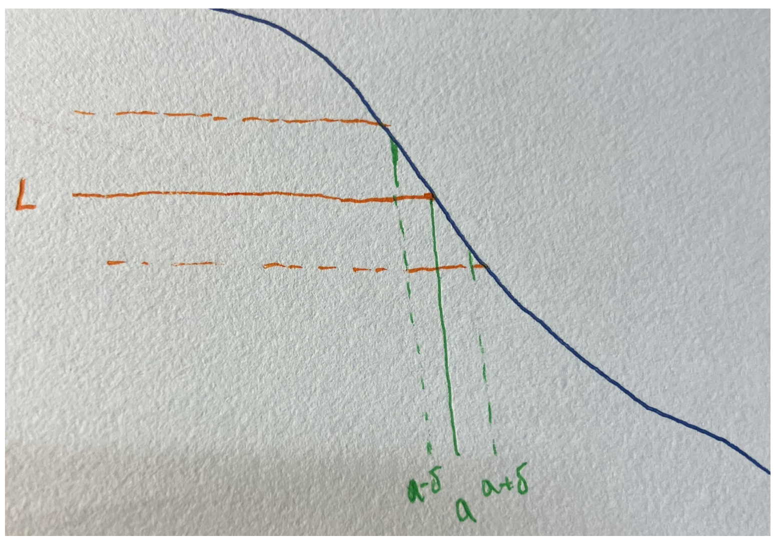
$$0 < |x - a| < \delta$$

The δ symbol is the Greek letter 'delta', which you might think of as 'd' short for 'distance' (or 'differential'...).

Again, this says that for L to be the limit of a function $f(x)$ at $x = a$ means however accurate we want to be, however close to L we want $f(x)$ to be, we can make this happen by choosing x close enough to a. Pictorially:



Zooming in:



Notice that for any x between $x-\delta$ and $x+\delta$, the value of the function is between $L-\varepsilon$ and $L+\varepsilon$.

As an example, let's walk through a "δ-ε proof".

Exercise 1. Define $A(t)=t^2-3t+1$. The value of the function $A(t)$ at $t=2$ is $2^2-3\cdot 2+1=-1$. Let us prove that

$$\lim_{t\to 2} A(t) = -1$$

Let's begin with some scratch work. To use the δ-ε definition of limit, we have some positive number ε given to us and we want to make $|A(t)-(-1)|<\varepsilon$.

Expand this into

$$\begin{aligned}|t^2 - 3t + 1 - (-1)| &< \varepsilon \\ |t^2 - 3t + 2| &< \varepsilon \\ |(t-2)(t-1)| &< \varepsilon\end{aligned}$$

It is a fact that $|ab| = |a||b|$ (Warning! This is only true for multiplication, not addition!). So, we see that the statement we want is the same as

$$|t-2||t-1| < \varepsilon$$

We want to argue that we can make this true if t is close enought to 2. This seems plausible, since $|t-2|$ would be close to 0 then, so if its close enough it will be smaller than whatever positive number ε is.

Since we will consider t close to 2, let's suppose that no matter what, we are looking at t such that $|t-2| < \frac{1}{2}$. This number is not important, I'm just picking one half to be specific. We can rewrite this as $\frac{3}{2} < t < \frac{5}{2}$. Then, subtracting 1 from everything, we get $\frac{1}{2} < t-1 < \frac{3}{2}$. The important thing we want here is that $|t-1| < \frac{3}{2}$, so that $\frac{|t-1|}{\frac{3}{2}} < 1$.

Now we are ready to do the exercise.

Proof. Let $\varepsilon > 0$ be any positive number. We choose $\delta = \min(\frac{1}{2}, \frac{2\varepsilon}{3})$.

Suppose $|t-2| < \delta$. Since $\delta \leq \frac{1}{2}$, this means $|t-1| < \frac{3}{2}$. Then

$$\frac{|t-1|}{\frac{3}{2}}|t-2| < |t-2| < \delta$$

Since we chose δ so that $\delta \leq \frac{2\varepsilon}{3}$, this means

$$\frac{|t-1||t-2|}{\frac{3}{2}} < \frac{2\varepsilon}{3}$$

Multiply both sides by $\frac{3}{2}$ to get

$$|t-1||t-2| < \varepsilon$$

which is what we wanted to show. $\square$

So, if you give me any ε, I can find a δ so that all the inputs which are δ-close to a are ε-close to the limit value. I can control the values of the function to whatever degree of precision you want.