Math 3002: Problem Set 4

1. Check that the following differential equations are exact, and solve them.

   (a)

$$\frac{dy}{dx} = \frac{2xe^y + \sin(x)y^2}{2\cos(x)y - x^2e^y}$$

Solution: Clearing the fractions and subtracting, we see

$$\begin{aligned} \frac{dy}{dx} &= \frac{2xe^y + \sin(x)y^2}{2\cos(x)y - x^2e^y} \\ \left(2\cos(x)y - x^2e^y\right) dy &= \left(2xe^y + \sin(x)y^2\right) dx \\ \left(2xe^y + \sin(x)y^2\right) dx + \left(x^2e^y - 2\cos(x)y\right) dy &= 0 \end{aligned}$$

We remark that it would be completely rigorous if we wrote

$$\left(2xe^y + \sin(x)y^2\right) + \left(x^2e^y - 2\cos(x)y\right)\frac{dy}{dx} = 0$$

(or even if we just subtracted the right-hand side and left the $\frac{dy}{dx}$ coefficient as 1, but this is more convenient)

To check exactness, take the y-derivative of the dx coefficient, and the x derivative of the dy coefficient:

$$\frac{\partial}{\partial y}\left(2xe^y + \sin(x)y^2\right) = 2xe^y + 2\sin(x)y$$

and

$$\frac{\partial}{\partial x}\left(x^2e^y - 2\cos(x)y\right) = 2xe^y + 2\sin(x)y$$

Since these are equal, the equation is exact, and there is some $\Psi(x, y)$ with gradient equal to the above expression. We just have to find such a Ψ

We know

$$\frac{\partial}{\partial x}\Psi(x, y) = 2xe^y + \sin(x)y^2$$

so integrating with respect to x gives

$$\Psi(x, y) = x^2e^y - \cos(x)y^2 + f(y)$$

for some function f

Similarly, integrate

$$\frac{\partial}{\partial y}\Psi(x,y) = x^2e^y - 2\cos(x)y$$

with respect to y to obtain

$$\Psi(x,y) = x^2e^y - \cos(x)y^2 + g(x)$$

for some function g.

Choosing $f(y) = g(x) = 0$, we get

$$\Psi(x,y) = x^2e^y - \cos(x)y^2$$

in both expressions, so the differential equation is solved by

$$\Psi(x,y) = c$$

or

$$x^2e^y - \cos(x)y^2 = c$$

for some constant c.

We remark that one cannot easily solve this to get an expression for y in terms of x.

(b)

$$y' = \frac{y^2 - 3x^2}{4y^3 - 2xy}$$

Solution: We begin as before, by separating the differential in Leibniz notation:

$$\begin{aligned}\frac{dy}{dx} &= \frac{y^2 - 3x^2}{4y^3 - 2xy}\\ \left(4y^3 - 2xy\right)dy &= \left(y^2 - 3x^2\right)dx\\ 0 &= \left(y^2 - 3x^2\right)dx + \left(2xy - 4y^3\right)dy\end{aligned}$$

Taking derivatives:

$$\frac{\partial}{\partial y}\left(y^2 - 3x^2\right) = 2y$$

and

$$\frac{\partial}{\partial x}\left(2xy - 4y^3\right) = 2y$$

so the equation is exact.

Now we look for a function $\Psi(x)$ such that

$$\frac{\partial}{\partial x}\Psi(x,y) = y^2 - 3x^2,$$

so integrate to get

$$\Psi(x,y) = xy^2 - x^3 + f(y)$$

We also need

$$\frac{\partial}{\partial y}\Psi(x,y) = 2xy - 4y^3,$$

so integrate to get

$$\Psi(x,y) = xy^2 - y^4 + g(x),$$

Setting these equal gives

$$xy^2 - x^3 + f(y) = xy^2 - y^4 + g(x)$$

so setting $g(x) = -x^3$ and $f(y) = -y^4$, we get

$$\Psi(x,y) = xy^2 - x^3 - y^4$$

So the solutions of the differential equation are

$$xy^2 - x^3 - y^4 = c$$

for some constant $c \in \mathbb{R}$, which we could write as

$$y(x) = \pm\sqrt{\frac{x \pm \sqrt{x^2 - 4x^3 - 4c}}{2}}$$

(using the quadratic formula for y^2), if we wanted to get a *function* y in terms of x.

(c)

$$\frac{dy}{dx} = -\frac{2x\sin(y) - \sin(x)}{x^2\cos(y)}$$

Solution: Let's do slightly different notation this time.
This differential equation is of the form

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)},$$

with
$$\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial y}\left(2x\sin(y) - \sin(x)\right) = 2x\cos(y)$$
and
$$\frac{\partial}{\partial x}N(x,y) = \frac{\partial}{\partial y}\left(x^2\cos(y)\right) = 2x\cos(y)$$
Since
$$\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y),$$
this equation is exact.
We want to find a function $\Psi(x,y)$ with gradient
$$\nabla\Psi = \begin{pmatrix} M \\ N \end{pmatrix}$$
In particular, we need
$$\frac{\partial}{\partial x}\Psi(x,y) = 2x\sin(y) - \sin(x),$$
so integrating with respect to x gives
$$\Psi(x,y) = x^2\sin(y) + \cos(x) + f(y)$$
for some function f of y.
Similarly, to get
$$\frac{\partial}{\partial y}\Psi(x,y) = x^2\cos(y),$$
we integrate and see
$$\Psi(x,y) = x^2\sin(y) + g(x)$$
for some function g of x. Equating these, we get
$$x^2\sin(y) + \cos(x) + f(y) = x^2\sin(y) + g(x),$$
so set $f(y) = 0$ and $g(x) = \cos(x)$, and we get
$$\Psi(x,y) = x^2\sin(y) + \cos(x)$$
Thus, the differential equation is solved by
$$x^2\sin(y) + \cos(x) = c,$$
or, again if we want y as a function of x,
$$y(x) = \arcsin\left(\frac{c - \cos(x)}{x^2}\right)$$

(d)

$$y' = -\frac{x^2 + y^2}{2y(x+y)}$$

Solution: Rewriting in Leibniz notation:

$$\begin{aligned} \frac{dy}{dx} &= -\frac{x^2 + y^2}{2y(x+y)} \\ \left(2xy + 2y^2\right)\frac{dy}{dx} &= -\left(x^2 + y^2\right) \\ \left(x^2 + y^2\right) + \left(2xy + 2y^2\right)\frac{dy}{dx} &= 0 \end{aligned}$$

Taking appropriate derivatives, we have

$$\frac{\partial}{\partial y}\left(x^2 + y^2\right) = 2y$$

and

$$\frac{\partial}{\partial x}\left(2xy + y^2\right) = 2y$$

so this equation is exact.

To find a potential function $\Psi(x, y)$, integrate

$$\frac{\partial}{\partial x}\Psi(x, y) = x^2 + y^2$$

to obtain

$$\Psi(x, y) = \frac{1}{3}x^3 + xy^2 + f(y)$$

and

$$\frac{\partial}{\partial y}\Psi(x, y) = 2xy + y^2$$

to get

$$\Psi(x, y) = xy^2 + \frac{1}{3}y^3 + g(x)$$

Equating these

$$\frac{1}{3}x^3 + xy^2 + f(y) = xy^2 + \frac{1}{3}y^3 + g(x)$$

and setting $f(y) = \frac{1}{3}y^3$, $g(x) = \frac{1}{3}x^3$ gives

$$\Psi(x, y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 + xy^2$$

Thus, the solution to the differential equation is given by

$$\frac{1}{3}x^3 + \frac{1}{3}y^3 + xy^2 = c$$

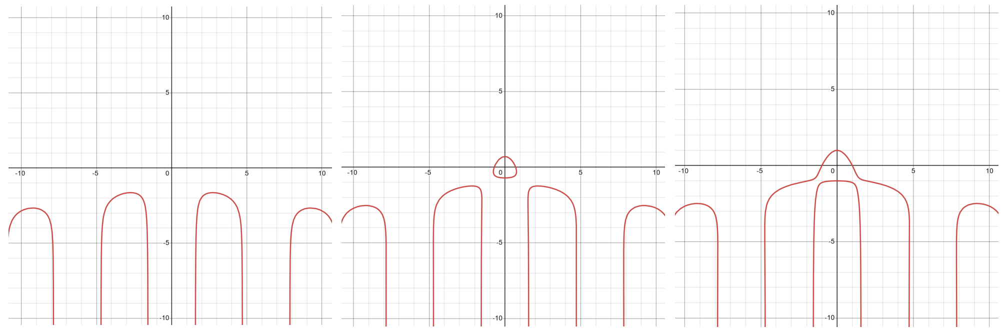
for some real number c.

This is solvable for y as a function of x, but let's not.

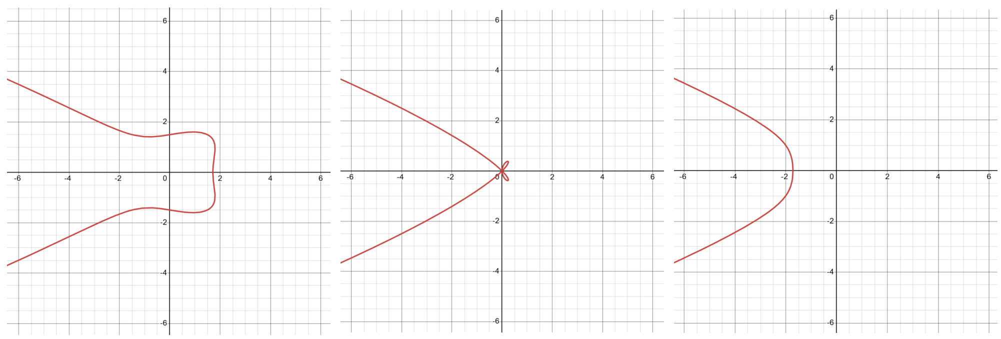
(e) Graph some of the solutions. Are any of them interesting?

Solution:

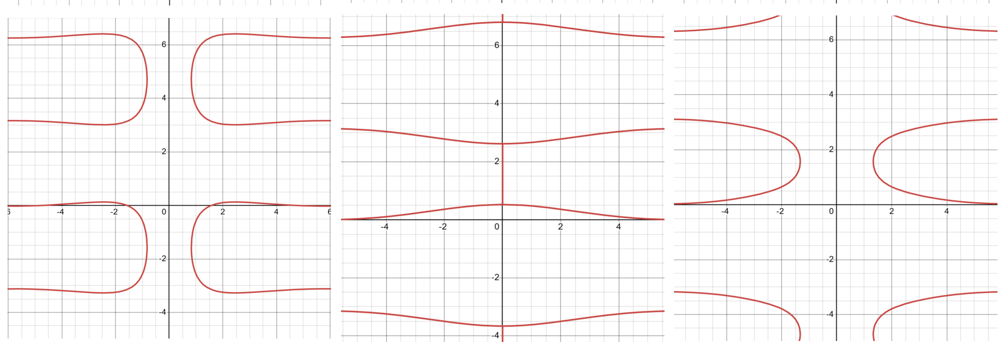
(a):



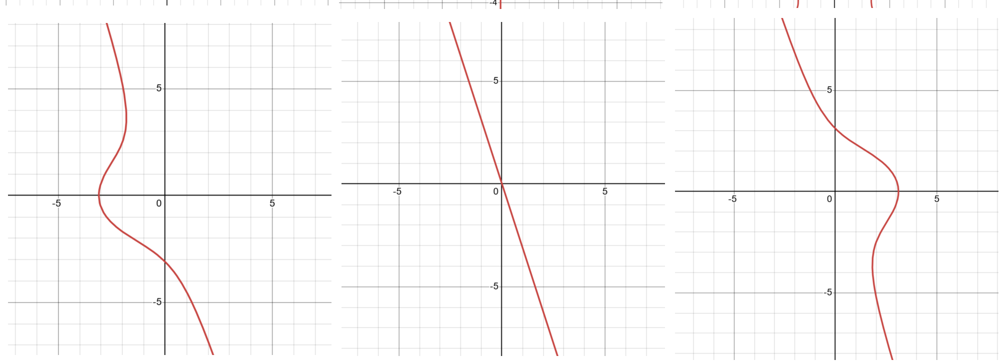
(b):



(c):



(d):



There are various interesting things going on, I would say. In particular, as we vary the constant c the different connected components of the graphs coalesce into different kinds of singularities (that is, points of non-differentiability, in these cases points with multiple tangent lines).

I also particularly like $1(a)$ as we increase the constant c to ∞, where the tops of the different curves combine into a single solution with a singularity at $x = 0$.

2. Check that these differential equations are not exact, but can be made exact by multiplying by the given integrating factor.

   (a)

$$3xy - y^2 + (x^2 - xy)\frac{dy}{dx} = 0$$

   Integrating factor $\mu(x, y) = x$

Solution: With $M(x, y) = 3xy - y^2$ and $N(x, y) = x^2 - xy$, we check

$$\frac{\partial}{\partial y}M = \frac{\partial}{\partial y}\left(3xy - y^2\right) = 3x - 2y$$

while

$$\frac{\partial}{\partial x}N = \frac{\partial}{\partial x}\left(x^2 - xy\right) = 2x - y$$

These are not equal, so the equation is not exact. If we multiply through by x, we get

$$3x^2y - xy^2 + (x^3 - x^2y)\frac{dy}{dx} = 0$$

and check

$$\begin{aligned}\frac{\partial}{\partial y}\left(3x^2y - xy^2\right) &= 3x^2 - 2xy \\ &= \frac{\partial}{\partial y}\left(x^3 - x^2y\right)\end{aligned}$$

so the equation is now exact.

   (b)

$$6xy + 5(x^2 + y)\frac{dy}{dx} = 0$$

   Integrating factor $\mu(x, y) = y^{\frac{2}{3}}$

Solution: We check (failure of) exactness:

$$\frac{\partial}{\partial y}(6xy) = 6x$$

and

$$\frac{\partial}{\partial x}\left(5(x^2 + y)\right) = 10x$$

so this equation is not exact.
Multiplying through by $y^{\frac{2}{3}}$ yields

$$6xy^{\frac{5}{3}} + 5(x^2y^{\frac{2}{3}} + y^{\frac{5}{3}})\frac{dy}{dx} = 0$$

Checking exactness for this:

$$\frac{\partial}{\partial y}\left(6xy^{\frac{5}{3}}\right) = 5xy^{\frac{2}{3}}$$

and

$$\frac{\partial}{\partial x}\left(5(x^2y^{\frac{2}{3}} + y^{\frac{5}{3}})\right) = 10xy^{\frac{2}{3}}$$

so this equation is exact.

3. Consider the differential equation

$$\frac{dy}{dx} = \frac{y}{x}$$

(a) Solve this differential equation.

Solution: I should remark, this equation is not exact. However, the equation *is* separable, so let's separate variables:

$$\frac{dy}{y} = \frac{dx}{x}$$
$$\int \frac{dy}{y} = \int \frac{dx}{x}$$
$$\ln|y| = \ln|x| + c$$
$$y(x) = C \cdot x$$

for some real number C.

(b) Rewrite this equation as $-y + x\frac{dy}{dx} = 0$, and multiply everything by $\frac{1}{x^2+y^2}$. Is the resulting equation exact?

Solution: The new equation is

$$\frac{-y}{x^2+y^2} + \frac{x}{x^2+y^2}\frac{dy}{dx} = 0$$

so let's check

$$\frac{\partial}{\partial y}\frac{-y}{x^2+y^2} = \frac{-(x^2+y^2) - (-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

and

$$\frac{\partial}{\partial x}\frac{x}{x^2+y^2} = \frac{(x^2+y^2)-(x)(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

and we see this equation is exact.

(c) Consider the path $\gamma : [0, 2\pi] \to \mathbb{R}^2$ given by

$$\gamma(t) = (\cos(t), \sin(t)),$$

then integrate the vector field $(M(x, y), N(x, y))$, where M and N are the coefficients from the previous part.

Solution: By 'the previous parts', I mean $M(x, y) = \frac{-y}{x^2+y^2}$ and $N(x, y) = \frac{x}{x^2+y^2}$.

We are given the path $\gamma(t) = (\cos(t), \sin(t))$, which has derivative $\gamma'(t) = (-\sin(t), \cos(t))$, and integrating a vector field along this curve is given by the formula

$$\begin{aligned}
\int_0^{2\pi} \begin{pmatrix} M(\gamma(t)) \\ N(\gamma(t)) \end{pmatrix} \cdot \gamma'(t)dt &= \int_0^{2\pi} \begin{pmatrix} \frac{-\sin(t)}{\sin^2(t)+\cos^2(t)} \\ \frac{\cos(t)}{\sin^2(t)+\cos^2(t)} \end{pmatrix} \cdot \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} dt \\
&= \int_0^{2\pi} \frac{\sin(t)\sin(t)+\cos(t)\cos(t)}{\sin^2(t)+\cos^2(t)} dt \\
&= \int_0^{2\pi} 1dt \\
&= 2\pi
\end{aligned}$$

(d) Is there any contradiction between the previous parts and what we discussed in class?

Solution: In class, we claimed that the integral of a **conservative** vector field (i.e., the coefficients to an exact differential equation) around a closed curve (i.e., a curve with the same starting and ending point) is 0, so there seems to be some issue.

In fact, there is no contradiction: our discussion in class required the vector field to be defined on a region in the plane with no holes, and the curve to be in that region. Here, however, the vector field $\begin{pmatrix} M \\ N \end{pmatrix}$ is not defined for $(x, y) = (0, 0)$, which the curve does go around.

(I believe this is the most important problem in the course, from the perspective of higher mathematics, leading to the idea of (de Rham) **cohomology**.)