# October 2nd

We open with an example of a second order equation whose characteristic equation has complex roots which are *not* purely imaginary:

Solve the initial value problem

$$\begin{cases} y'' - 2y' + 4y = 0 \\ y(0) = 1 \\ y'(0) = 2 \end{cases}$$

First, we form the characteristic equation

$$\lambda^2 - 2\lambda + 4 = 0 \tag{$\chi$}$$

which has roots

$$\lambda_\pm = 1 \pm \sqrt{3}i$$

So our general solution is

$$\begin{aligned} y(x) &= c_+ \cdot e^{(1+\sqrt{3}i)x} + c_- \cdot e^{(1-\sqrt{3}i)x} \\ &= c_+ \cdot e^x e^{\sqrt{3}ix} + c_- \cdot e^x e^{-\sqrt{3}ix} \end{aligned}$$

If we choose $c_+ = c_- = \frac{1}{2}$ and recall Euler's formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$

we get a solution

$$y_1(x) = e^x \cos(\sqrt{3}x)$$

Similarly, if we choose $c_+ = -c_- = -\frac{i}{2}$, we get a solution

$$y_2(x) = e^x \sin(\sqrt{3}x)$$

So, our general (real-valued) solution can be written as

$$y(x) = a \cdot y_1(x) + b \cdot y_2(x)$$

for some numbers $a, b \in \mathbb{R}$.

To solve the initial value problem, we evaluate $y(0)$ and $y'(0)$ to get a system of equations for $a$ and $b$:

$$\begin{aligned} y(0) &= a \cdot y_1(0) + b \cdot y_2(0) \\ &= ae^x \cos(\sqrt{3}x) + be^x \sin(\sqrt{3}x) \\ &= ae^0 \cos(\sqrt{3} \cdot 0) + be^0 \sin(\sqrt{3} \cdot 0) \\ &= a \cdot 1 \cdot 1 + b \cdot 1 \cdot 0 \\ &= a \end{aligned}$$

If we want $y(0) = 1$, we need $a = 1$. Then

$$\begin{aligned} y'(0) &= a \cdot y_1'(0) + b \cdot y_2'(0) \\ &= a \left(e^0 \cos(\sqrt{3} \cdot 0) - \sqrt{3}e^0 \sin(\sqrt{3} \cdot 0)\right) + b \left(e^0 \sin(\sqrt{3} \cdot 0) + \sqrt{3}e^0 \cos(\sqrt{3} \cdot 0)\right) \\ &= a(1 + 0) + b(0 + \sqrt{3}) \\ &= a + b \cdot \sqrt{3} \\ &= 1 + b\sqrt{3} \end{aligned}$$

If we want $y'(0) = 2$, we need $b = \frac{1}{\sqrt{3}}$

So we arrive at the solution

$$y(x) = e^x \cos(\sqrt{3}x) + \frac{1}{\sqrt{3}} e^x \sin(\sqrt{3}x)$$

which you can check is a solution to the given differential equation.

Now we continue discussing the space of solutions. Here it is fairly clear that $y_1(x)$ and $y_2(x)$ are 'really' different, i.e. one of them is not a multiple of the other. Since we have said (without proof) that the space of solutions to such a second order linear equation is a two-dimensional vector space, this means that $y_1$ and $y_2$ form a **basis** for the space of solutions.

If we are given two solutions, how do we know they form a basis? This is important, because otherwise there may be other solutions which we cannot write using our known solutions.

We introduce a tool which will allow us to analyze linear independence of solutions.

For completeness, we recall some definitions from linear algebra:

**Definition 1.** *A set of functions $f_1, ..., f_n$ is called **linearly dependent** if there is a non-trivial linear combination of them which is equal to zero. A non-trivial linear combination adding to zero means there are numbers $a_1, a_2, ..., a_n$ which are not all equal to zero (non-trivial) so that*

$$a_1 f_1(x) + a_2 f_2(x) + ... + a_n f_n(x) = 0$$

*Otherwise, if the only linear combination of the $f_i$ equal to zero is $a_1 = a_2 = ... = a_n = 0$, then this set of functions is called **linearly independent**.*

As motivation, suppose $f(x)$ and $g(x)$ are two solutions to a differential equation, but they are linearly dependent. That is, there are numbers $a$ and $b$ such that

$$a \cdot f(x) + b \cdot g(x) = 0$$

Now take the derivative. Using the sum and constant multiple rule from calculus (i.e., the fact that differentiation is a **linear operator**, in linear algebra terminology) we have

$$a \cdot f'(x) + b \cdot g'(x) = 0$$

If you have seen linear algebra before, notice these two equations can be written in a matrix form

$$\begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We want to know when $f$ and $g$ are linearly dependent, i.e., when there is a solution to this equation other than $a = b = 0$.

From linear algebra, we know that this depends on the **determinant** of the matrix. In our context, $f$ and $g$ are linearly dependent if and only if the determinant of this matrix is 0.

**Definition 2.** *The **Wronskian** of two functions $f$ and $g$ is*

$$W(f, g)(x) = f(x) \cdot g'(x) - g(x) \cdot f'(x)$$

Notice this Wronskian $W(f, g)$ is a function of $x$.

We translate some facts from linear algebra into our context: given $f, g$ smooth functions defined on an interval $I$, we have

1. If $W(f, g)(x) \neq 0$ for some $x \in I$, then $f$ and $g$ are linearly independent.

2. If $f$ and $g$ are linearly dependent, then $W(f, g)(x) = 0$ for all $x \in I$.

*Proof.* The first follows from the second (it is called the 'contrapositive' statement), since if $f, g$ are not independent they are dependent, and the second fact says the Wronskian must be zero in that case, contradicting the assumption that the Wronskian is not zero somewhere.

To prove the second statement, recall that $f$ and $g$ are linearly dependent if there are non-zero numbers $a$ and $b$ so that

$$a \cdot f(x) + b \cdot g(x) = 0$$

Then

$$f(x) = -\frac{b}{a} g(x),$$

and we compute the Wronskian

$$\begin{aligned} W(f, g)(x) &= f(x) \cdot g'(x) - g(x) \cdot f'(x) \\ &= -\frac{b}{a} g(x) g'(x) - g(x) \left( -\frac{b}{a} g(x) \right)' \\ &= -\frac{b}{a} g(x) g'(x) + \frac{b}{a} g(x) g'(x) \\ &= 0 \end{aligned}$$

□

I think I did not do any examples, but I should have: For $f(x) = \cos(x)$ and $g(x) = \sin(x)$, we have

$$\begin{aligned} W(f, g)(x) &= f(x) g'(x) - g(x) f'(x) \\ &= \cos(x) \cos(x) - \sin(x)(-\sin(x)) \\ &= 1 \end{aligned}$$

so these functions are linearly independent.

For $h(x) = 5^x$ and $k(x) = 5^{x+2}$, we have

$$\begin{aligned} W(h, k)(x) &= h(x) k'(x) - k(x) h'(x) \\ &= 5^x (\ln(5) 5^{x+2}) - 5^{x+2} (\ln(5) 5^x) \\ &= \ln(5) 5^{2x+2} - \ln(5) 5^{2x+2} \qquad = 0 \end{aligned}$$

which makes sense since $k(x) = 5^{x+2} = 25 \cdot 5^x = 25 \cdot h(x)$, so for instance

$$-25h(x) + k(x) = 0$$

is a non-trivial linear combination which is equal to 0.

In the last example I was careful to show that $h, k$ were linearly dependent, instead of saying they are because the Wronskian is 0. That is **not** what the second fact above says, it goes the other way.

You might believe the

**Theorem 1** (**FALSE**). *If $W(f, g)(x) = 0$, then $f, g$ are (**not**) linearly dependent.*

**Counterexample:** Consider $f(x) = x^3$ and $g(x) = |x|^3$. We compute the Wronskian:

$$\begin{aligned} W(f, g)(x) &= f(x) \cdot g'(x) - g(x) \cdot f'(x) \\ &= x^3 \cdot 3x|x| - |x|^3 3x^2 \\ &= 3|x|^5 - 3|x|^5 \\ &= 0 \end{aligned}$$

(you should check that these combinations of absolute values are correct)

So the Wronskian of $f$ and $g$ is zero. If there was a linear combination

$$af(x) + bg(x) = 0$$

then evaluating at $x = 1$ would give

$$af(1) + bg(1) = a + b = 0$$

so $b = -a$, but evaluating at $x = -1$ gives

$$af(-1) + bg(-1) = -a + b = 0$$

so $b = a$.

The only way this can happen is if $a = b = 0$. So there is no non-trivial combination of $f$ and $g$ which is equal to 0.