Sets

In mathematics, we usually phrase things in terms of **sets**, and **structures** on sets.

We will not formally define what we mean by the word 'set'. A set is a collection of things. Sometimes we call the things in a set 'elements' of the set (or 'in' the set). Other words we could use are 'collection', or 'family', or 'class'.[1]

Example 1. *The* ***empty set****, written as* $\varnothing = \{\}$*, is the set of no things.*

Example 2. *The set of (lowercase) letters in the English alphabet is*

$$\{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$$

Since there are only finitely many of them, I can just list them out.

Example 3. *The set of natural numbers, often written using the symbol* $\mathbb{N}$*, is the collection of whole numbers including zero. We could write:*

$$\mathbb{N} = \{0, 1, 2, 3, 4, ...\}$$

Since I can't write an infinite list of things, I write "..." to indicate some pattern continues. If we wanted to be more formal, we would have to say what sorts of things we are allowed to write down.

Example 4. *The set of integers, written using the symbol* $\mathbb{Z}$ *(for the German word for number, 'Zahlen'), is the collection of positive and negative counting numbers (and zero):*

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, ...\}$$

We write $2 \in \mathbb{N}$ to mean '2 is an element of $\mathbb{N}$'. We also write things like $-2 \notin \mathbb{N}$, to mean -2 is not a natural number.

Example 5. *The set of rational numbers, also called fractions, written using the symbol* $\mathbb{Q}$ *(for 'quotient'), is the collection of fractions*

$$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}/\sim$$

[1] A similar word we **will not** use is 'group', which has a specific mathematical definition, even though in ordinary language the word group is synonymous with the word set.

What does the $\sim$ symbol mean? Well, it's referring to the fact that $\frac{3}{2}$ and $\frac{6}{4}$ are 'the same number'. So fractions aren't 'just' writing down symbols with a line in between, we have to say when the symbols mean the same thing. Let's save discussion of $\sim$ for later.

Example 6. *The set of real numbers, written using the symbol $\mathbb{R}$ (for 'real'), is the collection of infinite strings of decimals*

$$\mathbb{R} = \{a_0.a_1a_2a_3a_4... : a_0 \in \mathbb{Z}, a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}\}/\sim$$

Here, the $\sim$ is needed to say that

$$1.00000000.... = 0.9999999999999......$$

Notice we have to use the ... not just because there are infinitely many things in the set, but each thing also is kind of 'infinitary'. If we ask to many precise questions about this, it leads to some bizarre (and interesting!) places.

Functions

Definition 1. *A **function** with **domain** X and **range** Y is a way of assigning an single element of Y to every element of X.*

We typically write $f : X \to Y$ to mean 'f' is a function with domain X and range Y, and write $f(x) = y$ if $y \in Y$ is the element f assigns to $x \in X$.

Remark 1. *Three important things to remember are (1) every element of X gets assigned some element of Y, (2) each element of X is assigned a unique element of Y, and (3) the domain and range are part of the data of the function.*

Example 7. *Thinking about (1), we didn't say that every element of Y gets assigned to something in X. The function $f : X \to Y$ where $X = Y = \mathbb{R}$ and $f(x) = x^2$ is an example where $-1 \in Y$ is not assigned to any $x \in X$. Another way to say this is $f(x) = -1$ has no solution.*

*Thinking about (2), this is why we define $\sqrt{x}$ to be the **positive** solution to $y^2 = x$. We could talk about 'multi-valued functions', but at this level it doesn't really matter (you can always think about it as different functions, $\sqrt{(}x)$ and $-\sqrt{x}$).*

Thinking about (3), the functions $f : \mathbb{N} \to \mathbb{N}$, $f(x) = x + 1$ is different than the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + 1$, even if they are given by the 'same formula'.

With regards to (1), we give the definition

Definition 2. *A function is* ***surjective*** *if every element of y is assigned to some element in X.*

In symbols, we might write

$$\forall y \in Y; \exists x \in X : f(x) = y$$

where $\forall$ is read as 'for all' (the 'universal quantifier' symbol) and $\exists$ is read as 'there exists' (the 'existential quantifier' symbol). So this reads "For all elements $y \in Y$ there exists some element $x \in X$ such that $f(x) = y$.

Definition 3. *A function is* ***injective*** *if no two elements of X are assigned the same element of Y.*

In symbols, we might write

$$\forall x_1, x_2 \in X : \; f(x_1) = f(x_2) \implies x_1 = x_2$$

This reads "For all x_1 and x_2 in the set X, if $f(x_1) = f(x_2)$, then we must have $x_1 = x_2$."

Definition 4. *A function is* ***bijective*** *if it is injective and surjective.*

Example 8. *If X is any set, then the* ***identity function****, which we write $Id_X : X \to X$, is the function that assigns x to x. Identity functions are bijections.*

Definition 5. *If $f : X \to Y$ and $g : Y \to Z$ are two functions, then their* ***composition****, written $g \circ f : X \to Z$, is a function with domain same as f and range same as g, defined by*

$$g \circ f(x) = g(f(x))$$

Claim 1. *A function $f : X \to Y$ is injective if and only if there is some function $g : Y \to X$ such that*

$$f \circ g = Id_Y$$

Claim 2. *A function $f : X \to Y$ is surjective if and only if there is some function $g : Y \to X$ such that*

$$g \circ f = Id_X$$

Claim 3. *A function $f : X \to Y$ is bijective if and only if there is some function $g : Y \to X$ such that*

$$f \circ g = Id_Y$$

and

$$g \circ f = Id_X$$

Proof. If there exists such a g, then claim 1 and the first equation say f is injective, while claim 2 and the second equation say that f is surjective. Thus, f is bijective.

Now, suppose we know that f is bijective. Notice that the two equations do **not** follow immediately from the first two claims. Claim 1 says there is some g making the first equation true, while claim 2 says there is **some** h making the first equation true. We don't know that the left-side inverse is the same function as the right-side inverse. We must prove they are the same.

To do this, we need to notice that a function does not change if you compose it with an identity function. That is,

Lemma 1. *For any function $p : Y \to X$, it is true that*

$$p \circ Id_Y = p$$

and

$$Id_X \circ p = p$$

Proof. This is from the definition of composition. For any x, $p \circ Id_X(x) = p(Id_X(x)) = p(x)$, so the functions are the same. □

Then we show g and h in the previous paragraph are the same:

$$\begin{aligned} h &= h \circ Id_Y \\ &= h \circ (f \circ g) \\ &= (h \circ f) \circ g \\ &= Id_X \circ g \\ &= g \end{aligned}$$

□

What we have really showed is that if the (two-sided) inverse of a function exists, it is unique.

Continuity

Recall from calculus that a function is **continuous** at a number c if

$$\lim_{x \to c} f(x) = f(c)$$

That is, the function 'respects limits'.

For this definition to make sense, we need to know what the meaning of the limit is. In calculus, the definition is

$$\lim_{x \to c} f(x) = L$$

if for any positive number ε, there is a positive number δ so that whenever $|x - c| < \delta$ we have $|f(x) - L| < \varepsilon$.

Since we deal with numbers, can measure how close together things are.

A set just means a collection of things. If we want to make sense of limits and continuity, we need to put more structure on our basic set.

Inspired by the idea of closeness, one option is the idea of

Definition 6. *A **metric space** is a set X together with a metric $d : X \times X \to \mathbb{R}$ (that is, a function from pairs in X to the set of real numbers) which satisfies the following axioms:*

1. $d(x, x) = 0$
2. $d(x, y) \geq 0$, *with* $d(x, y) = 0$ *only if* $x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, y) + d(y, z) \geq d(x, z)$

Example 9. *The real numbers $\mathbb{R}$ form a metric space, with the metric given by absolute value: $d(x, y) = |x - y|$. (Exercise: check axiom 4).*

The Cartesian plane $\mathbb{R}^2$ is also a metric space, with metric given by the Pythagorean theorem,

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Similarly, there is a metric on lists of real numbers of length n, $\mathbb{R}^n$, given by

$$d((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2}$$

Notice the change in meaning of the symbols x and y. In $\mathbb{R}^2$, I was using them to refer to the first or second part of a pair. Then there was a 'point 1' and a 'point 2'. In $\mathbb{R}^n$, I don't want to keep track of n different letters, so I think about 'point x' and 'point y', with the first, second, third, ... thing in the list being labelled by the number. This is just notation, but it reflects a way in thinking.

Now we can say

Definition 7. *Suppose $f : (X, d_X) \to (Y, d_Y)$ is a function between metric spaces. We say the **limit** of $f(x)$ as x approaches c is L (where $x, c \in X$ and $L \in Y$) if for any $\varepsilon > 0$ there exists some $\delta > 0$ such that $d_X(x, c) < \delta$ implies $d_Y(f(x), L) < \varepsilon$.*

This is written as

$$\lim_{x \to c} f(x) = L$$

*We say f is **continuous at** c if $\lim_{x\to c} f(x) = f(c)$, and we say f is **continuous** if it is continuous for every $c \in X$*

We say an example of a continuous function from $[0, 1) \cup (2, 3]$ to $[0, 2]$

Claim 4. *The composition of continuous functions is continuous.*

Definition 8. *A function f is a **homeomorphism** if it is a continuous bijection with continuous inverse.*

Example 10. *There is a homeomorphism from the open interval $(-1, 1)$ to $\mathbb{R}$, given by $x \mapsto \tan(\frac{\varphi}{2}x)$*

Exercises

Exercise 1. *Give arguments for Claims 1 and 2.*

Exercise 2. *Find a homeomorphism between any two open intervals (a, b) and (c, d) in the real numbers (including $\pm\infty$ endpoints).*

Exercise 3. *Is there a homeomorphism between an open interval (a, b) and a closed interval $[c, d]$? Try to find one, or give an argument that none can exist.*

Exercise 4. *Is there a homeomorphism between the closed disk $X = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and the closed square $Y = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1$ and $|y| \leq 1\}$? Try to describe one.*

Exercise 5. *Give an argument for the claim 'The composition of two continuous functions is continuous.', or find a counterexample.*

Exercise 6. *Is there a homeomorphism between the real numbers* $\mathbb{R}$ *and the Cartesian plane* $\mathbb{R}^2$*? Try to find one, or give an argument that none can exist.*

Exercise 7. *Is there a homeomorphism between the Cartesian plane* $\mathbb{R}^2$ *and Cartesian space* $\mathbb{R}^3 = \{(x, y, z)\}$*? Try to find one, or give an argument that none can exist.*

(This is a trick question, think about it, don't try to 'solve' it)