Explain, in a (complete) sentence or two, your reasoning.

1. Let f be the function defined by $f(x) = \sin(x) + \cos(x)$, and let g be the function defined by $g(u) = \sin(u) + \cos(u)$, for all real numbers x and u. Then

   A. f and g are exactly the same function

   B. if x and u are different numbers, f and g are different functions

   C. not enough information is given to determine if f and g are the same

Solution: A function is a set of inputs and an assignment of one output to each input. Each of these functions has the same set of inputs, and assigns the same output to the same input. That is what a function is. The names of variables x or u or t, or the names of functions f or g or h, are choices we can make.

2. If $f(x) = \frac{x^2-4}{x-2}$ and $g(x) = x + 2$, then the functions $f(x)$ and $g(x)$ are equal.

   A. True

   B. False

   Graph these functions. Explain.

Solution: False: even though the graphs are nearly identical, the function $f(x)$ is not defined at $x = 2$, so its graph will have a hole there while $g(x)$ will continue as a normal line.

3. You were once exactly 3 feet tall.

   A. True

   B. False

Solution: True, assuming you are at least 3 feet tall. Although we haven't talked about it yet, the function of 'your height' as it depends on 'age' is **continuous**, meaning here that it has no jumps. Since you were shorter than 3 feet tall at some point in your life, there must have been an age between then and now at which you became 3 feet tall, in order to get as tall as you are now.

The **average rate of change between** a **and** b of a function $f(x)$ is defined as $\frac{f(b)-f(a)}{b-a}$. This is the change in the values divided by the change in the inputs, as the input changes from a to b.

Consider the functions

$$g(x) = \frac{1}{10} \cdot 2^x$$

and

$$h(x) = x^2$$

(use just a calculator to answer, only graph at question 8)

4. What is the average rate of change of $g(x)$, and of $h(x)$, between 0 and 10?

Solution: The rate of change of $g(x)$ between 0 and 10 is

$$\frac{g(10)-g(0)}{10-0} = \frac{102.4-0.1}{10} = 10.23,$$

while the 'r.o.c.' between 10 and 20 is

$$\frac{g(20)-g(10)}{20-10} = \frac{104857.6-102.4}{10} = 10475.52$$

(This is about 2^{10} multiplied by the previous number.)

On the other hand, for $h(x)$ the rates of change are

$$\frac{h(10)-h(0)}{10-0} = \frac{100-0}{10} = 10$$

and

$$\frac{h(20)-h(10)}{20-10} = 30,$$

respectively.

(The problem was designed so that the first numbers are close.)

5. What is the average rate of change of g between 0 and 5, and what is the average rate of change between 5 and 10? The same question for $h(x)$.

Solution: As above, we have the rate of change of $g(x)$ between $x=0$ and $x=5$:

$$\frac{g(5)-g(0)}{5-0} = \frac{3.2-0.1}{5} = 1.52$$

and between $x = 5$ and $x = 10$:

$$\frac{g(10) - g(5)}{10 - 5} = \frac{102.4 - 3.2}{5} = 19.84$$

Similarly, for $h(x)$ between $x = 0$ and $x = 5$:

$$\frac{h(5) - h(0)}{5 - 0} = \frac{25 - 0}{5} = 5$$

and between $x = 5$ and $x = 10$:

$$\frac{h(10) - h(5)}{5 - 0} = \frac{100 - 25}{5} = 15$$

6. Make a table of the average rates of change of $g(x)$ and $h(x)$ between 0 and 1, 1 and 2, 2 and 3, and so on, until 9 to 10. Do you see the patterns?

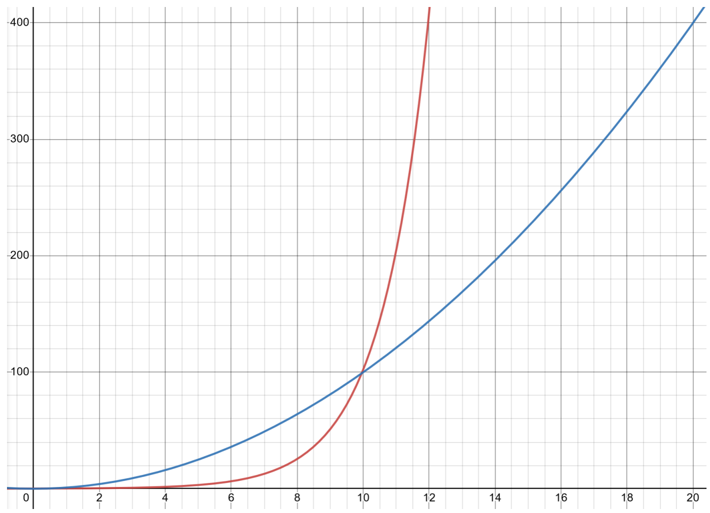
Solution:

Interval	Avg R.o.C. of $g(x)$	Avg R.o.C. of $h(x)$
$(0, 1)$	0.05	1
$(1, 2)$	0.2	3
$(2, 3)$	0.4	5
$(3, 4)$	0.8	7
$(4, 5)$	1.6	9
$(5, 6)$	3.2	11
$(6, 7)$	6.4	13
$(7, 8)$	12.8	15
$(8, 9)$	25.6	17
$(9, 10)$	51.2	19

As for patterns, the rate of change of $g(x)$ seems to double at each step, while that of $h(x)$ seems to increase by 2 each step. (Maybe try comparing with the idea that multiplication is repeated addition, and exponentiation is repeated multiplication...).

7. Graph the functions $g(x)$ and $h(x)$ in the domain between $x = 0$ and $x = 20$. Are they similar? Different? In what ways?

Solution:

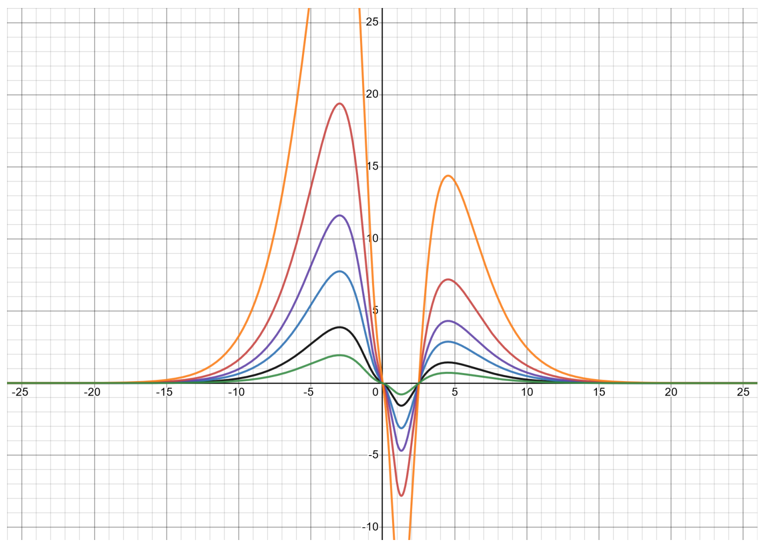


Notice the scale of the axes. Even with this, $g(x)$ (in red) grows so quickly it goes off the chart. Between 0 and 10, even though the functions end up at the same value, the way they get there is very different. While $h(x)$ (in blue) increases at a steady pace, $g(x)$ seems to be much slower, until it starts to increase extremely rapidly.

8. Consider the function
$$f\left(x\right)=\frac{\left(3x^{4}-7x^{3}-3x\right)}{\left(x^{2}\ -\ x+2\right)\left(2^{x}+2^{-x}\right)}.$$

(a) Graph $2\cdot f(x)$. Graph $3\cdot f(x)$. Graph $5\cdot f(x)$. Graph $10\cdot f(x)$. Graph $\frac{1}{2}\cdot f(x)$. Meditate.

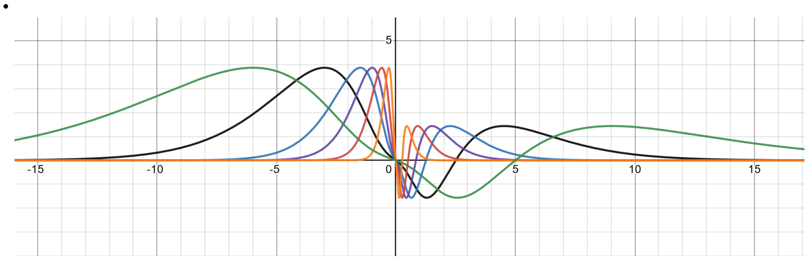
Solution:



Here black is the original, green is the $\frac{1}{2}$, and blue, purple, red, orange are 2, 3, 5, 10.

(b) Graph $f(2x)$. Graph $f(3x)$. Graph $f(5x)$. Graph $f(10x)$. Graph $f(\frac{1}{2}x)$. Meditate.

Solution:



Solution: Multiplying the function by some number, which means multiplying the values by some number, results in stretching or shrinking the graph vertically, while multiplying the input first stretches or shrinks the graph horizontally. Notice how multiplying the input by a bigger number shrinks the horizontal distances much more than multiplying by a smaller number.

For instance, the first green graph is *shorter* because $\frac{1}{2}$ is smaller than 1, but the second green graph is *wider* for the same reason.