Sixth Day

We begin with another separable example:

$$\frac{dy}{dx} = \frac{2}{x\tan(y)}$$

Then rewrite to

$$\tan(y)dy = \frac{2}{x}dx$$

and integrate:

$$\begin{aligned}\int \tan(y)dy &= \int \frac{2}{x}dx \\ -\ln(|\cos(y)|) &= 2\ln(|x|) + c \\ \ln(\frac{1}{|\cos(y)|}) &= \ln(x^2) + c \\ \frac{1}{|\cos(y)|} &= Cx^2 \\ y(x) = \cos^{-1}(\frac{C}{x^2})\end{aligned}$$

Actually, the discussion was a little deficient: $y(x) = \pm\cos^{-1}(\frac{C}{x^2})$. I made some comment about the sign of C, but forgot that it's 'inside' the arccos.

We waved our hands a bit when solving separable equations, treating $\frac{dy}{dx}$ as a fraction which we could split up and manipulate separately, and then integrate.

It turns out separable equations fit into a larger framework:

Definition 1. *A (first order) differential equation is called **exact** if it is of the form*

$$y'(x) = -\frac{M(x,y)}{N(x,y)}$$

for functions M, N satisfying

$$\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y)$$

Notice that separable equations are exact, since if

$$y' = f(x)\cdot g(y) = -\frac{-f(x)}{\frac{1}{g(y)}}$$

we have $\frac{\partial}{\partial y}(-f(x)) = 0 = \frac{\partial}{\partial x}\frac{1}{g(y)}$.

To motivate this definition, we make a slight digression into vector calculus.

Line Integrals and Conservative Vector Fields

Throughout the discussion, we work on some domain $R \subseteq \mathbb{R}^2$ which is simply connected (intuitively, 'has no holes'). We could take the entire plane $R = \mathbb{R}^2$ if we want.

A **vector field** on R is a function $V : R \to \mathbb{R}^2$, which we write

$$V(x, y) = (V_x(x, y), V_y(x, y))$$

The reason we call this function a vector field is because we have in mind that every point $(x, y) \in R$ gets assigned a vector, (V_x, V_y), based at that point.

A **(smooth) path in** R is a (smooth) function $\gamma : [0, 1] \to R$, which we write

$$\gamma(t) = (x(t), y(t))$$

Then the derivative is

$$\dot{\gamma}(t) = \left(\frac{dx}{dt}(t), \frac{dy}{dt}(t)\right)$$

In multivariable calculus, we learn the concept of a **line integral** of a vector field along a path:

$$\int_\gamma V = \int_0^1 V(\gamma(t)) \cdot \dot{\gamma(t)} dt$$

(here $\cdot$ is the direct product of vectors). In other words,

$$\int_\gamma V = \int_0^1 \left(V_x(\gamma(t)) \cdot \frac{dx}{dt} + V_y(\gamma(t)) \cdot \frac{dy}{dt}\right) dt$$

(which people sometimes write as

$$\int_\gamma V = \int_\gamma V_x dx + V_y dy$$

)

Definition 2. *A vector field* V *is* ***conservative*** *if it is the gradient of some function* $\Psi(x, y)$*:*

$$V = \nabla\Psi = \left(\frac{\partial}{\partial x}\Psi, \frac{\partial}{\partial y}\Psi\right)$$

This function Ψ *is called a 'potential' for the vector field* V*.*

Theorem 1. *If* $V = \nabla\Psi$ *is a conservative vector field, then*

$$\int_\gamma V = \Psi(\gamma(1)) - \Psi(\gamma(0))$$

Proof. This is a version of the fundamental theorem of calculus:

$$\begin{aligned}
\int_\gamma v &= \int_0^1 V_x(\gamma(t)) \cdot \frac{dx}{dt} + V_y(\gamma(t)) \cdot \frac{dy}{dt} dt \\
&= \int_0^1 \left(\frac{\partial\Psi}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial\Psi}{\partial y}\frac{dy}{dt}\right) dt \\
&= \int_0^1 \frac{d}{dt}\left(\Psi(\gamma(t))\right) dt \\
&= \Psi(\gamma(t))\big|_{t=0}^{t=1} \\
&= \Psi(\gamma(1)) - \Psi(\gamma(0))
\end{aligned}$$

□

This result is known as 'path-independence': regardless of what path γ you take, the integral only depends on the starting and ending points of the path. I claim many naturally occuring vector fields of interest are conservative, so their line integrals are path-independent.

How can we tell if a vector field is conservative? Suppose $V = \nabla\Psi$. Then $V_x = \frac{\partial\Psi}{\partial x}$ and $V_y = \frac{\partial\Psi}{\partial y}$. We differentiate and appeal to 'Clairaut's theorem', i.e. the equality of mixed partial derivatives:

$$
\begin{aligned}
\frac{\partial}{\partial y} V_x &= \frac{\partial}{\partial y} \frac{\partial \Psi}{\partial x} \\
&= \frac{\partial^2 \Psi}{\partial y \partial x} \\
&= \frac{\partial^2 \Psi}{\partial x \partial y} \\
&= \frac{\partial}{\partial x} \frac{\partial \Psi}{\partial y} \\
\frac{\partial}{\partial y} V_x &= \frac{\partial}{\partial x} V_y
\end{aligned}
$$

So a vector field can only be conservative if the y-derivative of the x-component is equal to the x-derivative of the y-component. It turns out that in simple cases, the reverse is true:

Theorem 2. *If V is a vector field defined on a region $R \subseteq \mathbb{R}^2$, and R is simply connected, then V is conservative if and only if*

$$
\frac{\partial}{\partial y} V_x = \frac{\partial}{\partial x} V_y
$$

We will not define simply-connectedness.

We remark that this is a special case of Green's theorem:

Theorem 3. *If C is a closed curve bounding a nice region D, then*

$$
\int_C V = \int_D \left(\frac{\partial}{\partial x} V_y - \frac{\partial}{\partial y} V_x \right) dxdy
$$

If V is conservative, both sides are zero ('closed' curve means the starting point and ending point are equal).

We are now in a position to describe the meaning of an 'exact differential equation': the exactness condition

$$
\frac{\partial}{\partial y} M = \frac{\partial}{\partial x} N
$$

means that the vector field $V = (M, N)$ is a conservative vector field, so there is a potential function $\Psi(x, y)$ with $V = \nabla \Psi$.

Recall we want to talk about single variable functions, so suppose we have a little piece of such a function $y = y(x)$ which solves our original differential equation, and differentiate Ψ as follows:

$$\begin{aligned}
\frac{d}{dx}\Psi(x, y(x)) &= \frac{\partial \Psi}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial \Psi}{\partial y} \cdot \frac{dy}{dx} \\
&= \frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y} \cdot \frac{dy}{dx} \\
&= M(x, y) + N(x, y) \cdot \frac{dy}{dx} \\
&= 0
\end{aligned}$$

Thus

$$\Psi(x, y(x)) = c$$

for some constant c. This gives an implicit equation relating $y(x)$ to x.

All in all, this discussion gives a sort of recipe for how to solve exact differential equations.

1. Check that the equation is exact
2. Exactness means the vector field $V = (M, N)$ is conservative, so find a potential function.
3. Solutions to the differential equation are (parts of) level curves of the potential.

This is 'almost' a recipe in that we did not talk about how to do step 2. As always, the solution is to integrate.

We finished with an example: consider the equation

$$\frac{dy}{dx} = -\frac{3x^2 + 4y}{4x + 3y^2}$$

so in our notation above, $M(x, y) = 3x^2 + 4y$ and $N(x, y) = 4x + 3y^2$. Then we have

$$\frac{\partial}{\partial y} M = \frac{\partial}{\partial y}\left(3x^2 + 4y\right) = 4$$

and

$$\frac{\partial}{\partial x} N = \frac{\partial}{\partial x}\left(4x + 3y^2\right) = 4$$

so the partial derivatives agree, and this equation is exact.

Now we want to find a function $\Psi(x, y)$ so that

$$\frac{\partial}{\partial x}\Psi = M(x, y) = 3x^2 + 4y$$

Integrating with respect to x leads to

$$\Psi(x, y) = x^3 + 4xy + f(y),$$

where $f(y)$ is a function of y and **not** x (this is the 'constant of integration', since any function of y will vanish if you differentiate with respect to x.

Similarly, we integrate

$$\frac{\partial}{\partial y}\Psi = N(x, y) = 4x + 3y^2$$

to get

$$\Psi(x, y) = 4xy + y^3 + g(x)$$

for some function of x, $g(x)$. Comparing this to the previous statement, we see that

$$\Psi(x, y) = x^3 + 4xy + y^3$$

gives the desired potential function.

So, solutions to the differential equation

$$\frac{dy}{dx} = -\frac{3x^2 + 4y}{4x + 3y^2}$$

are defined implicitly by the equation

$$x^3 + 4xy + y^3 = c$$

for some constant c.

Did not quite get to interpreting this solution, but I claim it's very pretty geometrically.