Quotient Topologies

Recall an **equivalence relation** on a set X is a set of pairs, $\sim \subset X \times X$, such that

1. $x \sim x$ (reflexivity)
2. $x \sim y$ iff $y \sim x$ (symmetry)
3. If $x \sim y$ and $y \sim z$, then $x \sim z$ (transitivity)

(where we use the notation $x \sim y$ for $(x, y) \in \sim$)

Definition 1. *If X is a set, and $\sim$ is an equivalence relation on X, then the **quotient set** $X/\sim$ is the set of equivalence classes of $\sim$.*

That is,

$$X/\sim = \{\{y \in X : y \sim x\} \in \mathcal{P}(X) : x \in X\}$$

Notice there is a canonical function $\pi : X \to X/\sim$, called the **projection** associated to $\sim$, given by

$$\pi(x) = \{y \in X : y \sim x\}.$$

Definition 2. *If (X, τ) is a topological space, and $\sim$ is an equivalence relation on X, then the **quotient topology** on $X/\sim$ is giveny by*

$$\tau_\sim = \{U \in \mathcal{P}(X/\sim) : \pi^{-1}(U) \in \tau\}$$

That is, a set in the quotient is open if its preimage under the canonical projection is open.

Remark 1. *In particular, the projection map is continuous. It follows that the quotient of a connected or compact space is connected or compact.*

Example 1. *If $\mathbb{Z}$ is given the discrete topology, and we take $\mod 3$ as the equivalence relation, then $\mathbb{Z}/3\mathbb{Z}$ has the discrete topology.*

In general, any quotient of the discrete topology will be discrete.

Example 2. *If $\mathbb{R}$ is the real numbers with the usual topology, define an equivalence relation $x \sim y$ iff there is some $k \in \mathbb{Z}$ such that $x + k = y$. Then every number is equivalent to a unique number in $[0, 1)$, so take this to model $\mathbb{R}/\sim$.*

For any positive number t, consider $\delta < \frac{t}{2}$. This is just so $(t-\delta, t+\delta)$ is completely contained in $(0,1)$. It's preimage in $\mathbb{R}$ is

$$\bigcup_{k\in\mathbb{Z}} \{x \in \mathbb{R} : t-\delta+k < x < t+\delta+k\},$$

which is open. One should argue that any open set in $\mathbb{R}/\sim$ containing a positive t must contain such an interval, again by considering the preimage in $\mathbb{R}$. The goal here is to say the topology is generated by sufficiently small intervals around the points. But what happens at 0?

Since any open set in $\mathbb{R}$ containing 0 must contain some interval $(-\varepsilon_1, \varepsilon_2)$, an open set in $\mathbb{R}/\sim$ must contain $[0,\varepsilon_2) \cup (1-\varepsilon_1, 1)$, and of course any such set is open.

Then, one should argue, say by thinking about the map $[0,1) \to \mathbb{C} :: t \mapsto e^{2\pi i t}$, that $\mathbb{R}/\sim$ is homeomorphic to a circle, usually denoted $\mathbb{S}^1$.

Example 3. *A fairly simple example that shows quotient topologies are not always nice is called 'the line with two origins'. This is given by a relation defined on two copies of the real line: $X = \mathbb{R}\times\{0\} \cup \mathbb{R}\times\{1\}$, with $(x,a) \sim (y,b)$ if $x = y$ and $x \neq 0$. So, $(t,0) \sim (t,1)$ for any $t \neq 0$.*

In particular, $(0,0)$ and $(0,1)$ are in distinct equivalence classes. Any open set U_0 containing $(0,0)$ (in the quotient) must contain an open interval $\{(t,0) : -\varepsilon_1 < t < \varepsilon_2\}$ in its preimage, and any open set U_1 containing $(0,1)$ (in the quotient) must contain some interval $\{(t,1) : -\varepsilon_3 < t < \varepsilon_4\}$ in its preimage. Then, for $s = \frac{\min(\varepsilon_2,\varepsilon_4)}{2}$, we have $\{(s,0),(s,1)\} \in U_0 \cap U_1$.

In particular, $X/\sim$ is not Hausdorff, even though X is.

Remark 2. *Suppose we have a subset $A \subseteq X$. Then we can define an equivalence relation on X by saying every element of A is related, and no elements not in A are related to anything other then themselves (required by reflexivity). In symbols, $x \sim y$ if $x, y \in A$, and if $x \notin A$ then $x \sim y$ only if $y = x$.*

For instance, $[0,1]/\{0,1\}$ is homeomorphic to the circle $\mathbb{S}^1$, similar to example 2.

Another good example here is $\{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}/\{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$. This is the unit disk quotiented by its boundary, the unit circle. This turns out to be homeomorphic to the 2-sphere $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$

Example 4. *Consider* $X = \mathbb{R}$ *with the usual topology, and the subset* $A = \mathbb{Z}$*. Then* X/A *is an 'infinite wedge of circles'. That is, each interval* $[k, k+1]$ *in* $\mathbb{R}$ *becomes a copy of the circle with* k *and* $k+1$ *glued together. Then, the gluing points on all these different circles are glued together to a single point.*

Call the k*-th circle* $C_k = \pi([k, k+1])$*, so* $X/A = \bigcup_{k\in\mathbb{Z}} C_k$ *and* $\bigcap_{k\in\mathbb{Z}} = \{p\}$ *is a single point, called the 'wedge point' (this point is the equivalence class of* $\mathbb{Z}$*, but writing* $\{\mathbb{Z}\}$ *to mean the single point set is potentially confusing).*

Suppose that $\{U_n\}_{n\in\mathbb{N}}$ *is a countable family of open sets which contain the wedge point. This means the preimages* $\tilde{U}_n$ *are open in* $\mathbb{R}$*. For each choice of* n *and* k*, we must have positive numbers* $\varepsilon_{n,k}$ *such that* $(k-\varepsilon_{n,k}, k+\varepsilon_{n,k}) \subseteq \tilde{U}_n$*. Now, consider* $\delta_k = \frac{\varepsilon_{k,k}}{2}$ *and form the open set* $\tilde{V} = \cup(k - \delta_k, k + \delta_k)$*. We claim that* $\tilde{U}_n \not\subseteq \tilde{V}$ *for any* n*, so* $U_n \not\subseteq V$ *for any* n*.*

What we have just shown is that X/A *is not first countable, in particular the wedge point does not have a countable neighborhood basis. This also shows* X/A *is not metrizable.*

Remark 3. *Why did I write* X *and* A *instead of* $\mathbb{R}$ *and* $\mathbb{Z}$*? Notice that* $\mathbb{Z} \subseteq \mathbb{R}$ *can also define another equivalence relation, namely the the relation in Example 2. This takes advantage of the group structure* $+$ *on* $\mathbb{R}$*, and* $\mathbb{Z}$ *as a subgroup of* $\mathbb{R}$*. So, in group theory, the quotient '*$\mathbb{R}/\mathbb{Z}$*' means something different (Example 2) than what '*$\mathbb{R}/\mathbb{Z}$*' means in topology (Example 4), viewing* $\mathbb{Z}$ *as (only) a subspace of* $\mathbb{R}$ *instead of a subgroup.*

Proposition 1. *Suppose* X *is a topological space,* $Y = X/\!\sim$ *some quotient, and* $\pi : X \to Y$ *the canonical projection. Consider any topological space* Z*. A function* $f : Y \to Z$ *is continuous if and only if* $f \circ \pi : X \to Z$ *is continuous.*

Proof. This follows from the definition of the quotient topology. □

Proposition 2. *If* $\sim$ *is an equivalence relation on* X*, and* $f : X \to Z$ *is a continuous function such that* $x \sim y$ *implies* $f(x) = f(y)$*, then there is a unique continuous map* $\hat{f} : X/\!\sim\, \to Z$*.*

Proof. Since f has the same image on equivalent elements, we get a well-defined map $\hat{f}$ on equivalence classes. It is not hard to check this map is continuous. □

Remark 4. *This proposition says that the quotient space satisfies a 'universal property'. Universal here means 'for any function on* X *which sends equivalent elements to equal images, (something)'*

Some words: the quotient topology is the **finest** topology on $X/\sim$ such that $\pi_\sim$ is continuous. This means that if τ' is a topology on $X/\sim$ such that $\pi_\sim$ is continuous, then any τ'-open set is open in the quotient topology.

An important case, alread hinted at in Example 2, is the notion of "quotient by a group action": in Example 2, the group action was $\mathbb{Z} \curvearrowright \mathbb{R} :: k \mapsto$ '$r \mapsto r + k'$

In general, we have

Definition 3. *An **action** of a group G on a 'structure' X (read: on a topological space (X, τ)) is a group homomorphism $\alpha : G \to Aut(X)$ (read: $\alpha : G \to Homeo(X; \tau) = \{f : X \to X : f$ is a homeomorphism (continuous bijection with continuous inverse) w.r.t. $\tau\}$)*

The more common definition is, an action of a group G on a set(/space) X is a map

$$\alpha : G \times X \to X$$

usually written $\alpha(g, x) = g \cdot x$ such that

$$(g \cdot h) \cdot x = g \cdot (h \cdot x)$$

Here I used $g \cdot h$ to mean the group product of g and h. So the definition is that the function α 'behaves nicely' with the group multiplication. Of course, if I'm thinking of X as a topological space, I should ask for α to be continuous. This raises the issue of the topology on G. If G is a finite group, the discrete topology is the most obvious choice. In fact, for any group G, the discrete topology is *a* choice, so why not use it. This gives the notion of a group action on a topological space.

Given an action (of sets) $\alpha : G \times X \to X$, there is an equivalence relation on X given by $x \sim y$ if $y = \alpha(g, x)$ for some $g \in G$. If X is a topological space, the quotient topology of X by this equivalence relation is denoted by X/G (with the α typically left implicit).

Example 5. *Consider $X = \mathbb{R}$. The non-zero real numbers form a group under multiplication, $G = \mathbb{R}^* = \mathbb{R} \setminus \{0\}$. There is an action (on sets) $G \cdot X \to X$ given by $(r, x) \mapsto r \cdot x$, where $r \neq 0$. Then $X/G = \mathbb{R}/\mathbb{R}^*$ is an interesting space.*