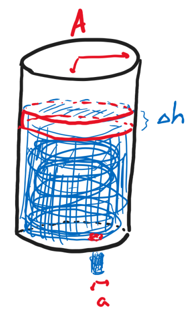
Math 3002: Problem Set 3

1. The point of this question is to give a 'physical' example of non-uniqueness.

   There is a water tank with a hole at the bottom. Derive a differential equation for the height of the water over time. To do this, say the kinetic energy of the water leaving the tank is equal to the (gravitational) potential energy lost from the top of the water moving down.

   The kinetic energy is $\frac{1}{2}mv^2$, where m is the total mass of the water and v is the speed of the water leaving the tank, and the potential energy is mgh, where m is again the total mass of the water moved, h is the height of the water, and g is some constant. So we have

   $$mgh = \frac{1}{2}mv^2$$

   or

   $$2gh = v^2$$

   We want a differential equation for h, so we want to write v in terms of h, h', h'', etc..

   (a) The mass of a fluid is given by the density times the volume. Suppose the density is constant throughout the fluid. Say the hole has area a, and the entire water tank has cross-sectional area A. How much water leaves the tank in some amount of time Δt? How does this affect the height of the water left? Use this to write an equation relating v and h'.

   (b) Use the previous part and the energy discussion above to write a differential equation for the height of the water. Solve this differential equation by separating variables.

   (c) If I tell you the height of the water tank at time $t = 10$ is zero, can you tell me how the height changed over time? Why or why not?

2. The point of this question is to flex your muscles. Solve the initial value problems.

   (a)
   $$\begin{cases} y' = x(y^2+1) \\ y(1) = 1 \end{cases}$$

   (b)
   $$\begin{cases} y' = (x+1)y \\ y(2) = -1 \end{cases}$$

   (c)
   $$\begin{cases} y' = \frac{x^2y-y}{y+1} \\ y(3) = -1 \end{cases}$$

   (d)
   $$\begin{cases} y' = \frac{e^x}{y} \\ y(0) = -1 \end{cases}$$

3. The point of this question is to show a nice substitution, since we won't talk about it much in the course.

   We wish to solve the equation $y' = \frac{y+x}{x}$. Notice this equation is not separable, but we can turn it into a separable equation:

   (a) create a new function,
   $$v(x) = \frac{y(x)}{x}$$

   and evaluate the derivative. Write your answer in terms of x, v, v' and y' (but not y!).

   (b) Write the right-hand side of the original differential equation in terms of v and x. Replace the y' term in your answer to the previous part to get a differential equation in terms of v and x,

   (c) Use the differential equation for v to find a solution y to the original differential equation.

4. The point of this question is to explore how solutions change as we vary through a family of differential equations. Also, partial fractions practice.

   In class, we discussed the logistic equation

   $$\frac{dP}{dt} = P(1-P)$$

   as a simple model for population growth (where we think of 1 here as some 'carrying capacity', if P is close to 1 from below, the population increase slows down, while if P is greater than 1, population decreases until it reaches this carrying capacity.

   We could imagine that, in addition to this carrying capacity, we also have a steady decrease of the population due to some outside factor, such as hunting. Model this by

   $$\frac{dP}{dt} = P(1-P) - h$$

   where h is some constant 'hunting rate'.

   Solve this differential equation, assuming h is smaller than $\frac{1}{4}$. (Hint: to do partial fraction decomposition, you need to factor the roots of the denominator polynomial. The algebra may be easier if you name the roots something like r_+ and r_-, instead of writing down the algebraic expression every time). What happens (mathematically and in terms of the model) if $h < \frac{1}{4}$, $h = \frac{1}{4}$, or $h > \frac{1}{4}$?