MATH 3002 - Exam 1

Name: ____

Instructions: Write cleanly, show all work. Explain any trick questions.

1. Find a function $y(x)$ which solves the following differential equation and satisfies the given condition.

   (a)

$$\begin{cases} y' - \cos(x)y = 0 \\ y(0) = 1 \end{cases}$$

> **Solution:** This is a **first-order**, **linear**, and **homogeneous**, so we can rearrange this equation to get
>
> $$\begin{aligned} y' - \cos(x)y &= 0 \\ y' &= \cos(x)y \\ \frac{y'}{y} &= \cos(x) \end{aligned}$$
>
> (you might notice that all the adjectives above also imply the equation is **separable**)
> Recognise the 'logarithmic derivative' on the left side:
>
> $$(\ln(|y|))' = \cos(x)$$
>
> Then integrating with respect to x gives
>
> $$\ln(|y|) = \int \cos(x)dx = \sin(x) + c$$
>
> where c is some constant of integration. If we exponentiate both sides, we get
>
> $$\begin{aligned} e^{\ln(|y|)} &= e^{\sin(x)+c} \\ |y| &= e^c e^{\sin(x)} \\ y(x) &= Ce^{\sin(x)} \end{aligned}$$
>
> with $C = e^c$. We removed the absolute value signs because we can choose $C = -e^c$ instead if we wanted negative solutions.
> The initial condition given is $y(0) = 1$, so plugging this into our solution gives
>
> $$1 = y(0) = Ce^{\sin(0)} = C \cdot 1 = C$$
>
> Thus, $C = 1$ and our specific solution is
>
> $$y(x) = e^{\sin(x)}$$

(b)

$$\begin{cases} y' - 4xy = 7x \\ y(0) = 2 \end{cases}$$

Solution: This equation is **first-order** and **linear**, but **non-homogeneous**, so we cannot rearrange like we did above.

Instead, we search for an **integrating factor**: a function $\mu(x)$ so that when we multiply through

$$\mu(x)y' - \mu(x)4xy = \mu(x)7x$$

the left-hand side becomes a product rule

$$(\mu(x)y)' = 7x\mu(x)$$

In order for this to happen, doing the product rule means we need

$$\mu'(x) = -4x\mu(x)$$

Which we can solve like the previous example. I'll just tell you a solution is

$$\mu(x) = e^{-2x^2}$$

(there is a C to choose but it doesn't matter, so I chose $C = 1$)

Now when we multiply through the original equation, we get

$$\begin{aligned} e^{-2x^2}y' - 4xe^{-2x^2}y &= 7xe^{-2x^2} \\ \left(e^{-2x^2}y\right)' &= 7xe^{-2x^2} \end{aligned}$$

Integrating both sides with respect to x:

$$e^{-2x^2}y = \int 7xe^{-2x^2}\,dx$$

To integrate the right-hand side, we do a u-substitution with $u = -2x^2$, so that

$du = -4xdx$. Then

$$\begin{aligned} e^{-2x^2}y &= \int 7xe^{-2x^2}dx \\ &= \int -\frac{7}{4}e^u du \\ &= -\frac{7}{4}\int e^u du \\ &= -\frac{7}{4}e^u - \frac{7}{4}c \\ e^{-2x^2}y &= -\frac{7}{4}e^{-2x^2} + C \end{aligned}$$

Here, c is the constant of integration, which ends up getting multiplied. But this is just a different constant number, so I call it C.
Multiplying through by e^{2x^2} to get a formula for y:

$$y(x) = -\frac{7}{4} + Ce^{2x^2}$$

Our initial condition says $y(0) = 2$, so evaluating

$$2 = y(0) = -\frac{7}{4} + Ce^{2\cdot 0^2} = -\frac{7}{4} + C\cdot 1 = -\frac{7}{4} + C$$

leads to $C = \frac{15}{4}$, so our specific solution is

$$y(x) = -\frac{7}{4} + \frac{15}{4}e^{2x^2}$$

(c)

$$\begin{cases} y' - \frac{1}{x}y = 2x^2 + 1 \\ y(1) = 1 \end{cases}$$

Solution: This equation is **first-order** and **linear**, but **non-homogeneous**, so again we look for an integrating factor to turn the left side into a product rule.
We want

$$\mu(x)y' - \frac{1}{x}\mu(x)y = (2x^2+1)\mu(x)$$

to be equivalent to

$$(\mu(x)y)' = (2x^2+1)\mu(x)$$

Applying the product rule shows this is

$$\mu(x)y' + \mu'(x)y = (2x^2+1)\mu(x)$$

so we need

$$\mu'(x) = -\frac{1}{x}\mu(x)$$

Dividing both sides by $\mu(x)$, we obtain

$$\frac{\mu'}{\mu} = -\frac{1}{x}$$

or

$$(\ln|\mu(x)|)' = -\frac{1}{x}$$

Integrating gives

$$\ln(|\mu|) = -\int \frac{1}{x}dx = -\ln(|x|) + c$$

and exponentiating gives

$$|\mu(x)| = e^{-\ln(|x|)+c} = e^c e^{\ln(\frac{1}{|x|})} = C\frac{1}{|x|}$$

Choose $C = 1$, since we only need one solution to this equation to get a μ that works.

This leads to either

$$\mu(x) = \frac{1}{x}$$

or

$$\mu(x) = -\frac{1}{x}$$

Either will work; let's choose $\mu(x) = \frac{1}{x}$.

Then our modified equation becomes

$$\begin{aligned}
y' - \frac{1}{x}y &= 2x^2 + 1 \\
\mu(x)y' - \mu(x)\frac{1}{x}y &= (2x^2+1)\mu(x) \\
\frac{1}{x}y' - \frac{1}{x^2}y &= 2x + \frac{1}{x} \\
\left(\frac{1}{x}y\right)' &= 2x + \frac{1}{x} \\
\frac{1}{x}y &= \int \left(2x + \frac{1}{x}\right) dx \\
\frac{1}{x}y &= x^2 + \ln(|x|) + c \\
y(x) &= x^3 + x\ln(|x|) + cx
\end{aligned}$$

We have the initial condition $y(1) = 1$, so

$$1 = y(1) = 1^3 + 1 \cdot \ln(1) + c = 1 + c$$

thus $c = 0$ and our specific solution is

$$y(x) = x^3 + x \ln(|x|)$$

2. Solve the following initial value problem:

$$\begin{cases} y' = \frac{xy^2}{\sqrt{1-x^2}} \\ y(0) = 1 \end{cases}$$

What is the domain of your solution?

Solution: This equation is **first-order**, but not **linear** since there is a y^2 term. Fortunately, it does turn out to be **separable**. Usually we write $y'(x)$ as $\frac{dy}{dx}$, pretend this is a fraction, and separate all the x and y terms:

$$\begin{aligned} \frac{dy}{dx} &= \frac{xy^2}{\sqrt{1-x^2}} \\ \frac{dy}{y^2} &= \frac{x}{\sqrt{1-x^2}} dx \end{aligned}$$

Integrating both sides (here the 'with respect to' is already built into equation, with our 'pretend its a fraction' idea):

$$\begin{aligned} \int y^{-2} dy &= \int \frac{x}{\sqrt{1-x^2}} dx \\ \frac{y^{-1}}{-1} &= \int -\frac{1}{2} \frac{1}{\sqrt{u}} du \\ -\frac{1}{y} &= -\frac{1}{2} \left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c \right) \\ -\frac{1}{y} &= -u^{\frac{1}{2}} + C \\ -\frac{1}{y} &= -(1-x^2)^{\frac{1}{2}} + C \end{aligned}$$

where we used the substitution $u = 1 - x^2$, so $du = -2xdx$ or $xdx = -\frac{1}{2}du$. Rearranging this to solve for $y(x)$, we get

$$y(x) = \frac{1}{\sqrt{1 - x^2 - C}}$$

Our initial condition says $y(0) = 1$, so that

$$1 = y(0) = \frac{1}{\sqrt{1 - 0^2 - C}} = \frac{1}{1 - C}$$

and we get $C = 0$.

So our specific solution to this initial value problem is

$$y(x) = \frac{1}{\sqrt{1 - x^2}}$$

3. Solve the following initial value problem:

$$\begin{cases} xy^2 + 2 + (x^2 - 3)\, y' = 0 \\ y(-1) = 8 \end{cases}$$

What is the maximal domain on which your solution is continuous?

Warning: typo, coefficient of y' should be $x^2y - 3$

Solution: The equation I meant to write is

$$xy^2 + 2 + \left(x^2y - 3\right) y' = 0$$

This equation is **first-order**, but not **linear**, both because there is a y^2 term and because there is a yy' term (if you expand it out). It is also not easily seen to be separable. Let us check that this equation is **exact**.

First, set $M(x, y) = xy^2 + 2$ and $N(x, y) = x^2y - 3$, thinking of the equation as $y' = -\frac{M(x,y)}{N(x,y)}$. Next, we check that the y-derivative of M matches the x-derivative of N:

$$\frac{\partial}{\partial y} M(x, y) = \frac{\partial}{\partial y} \left(xy^2 + 2\right) = 2xy$$

and

$$\frac{\partial}{\partial x} N(x, y) = \frac{\partial}{\partial x} \left(x^2y - 3\right) = 2xy$$

These are the same, so the equation is exact.

Now, we look for a function $\Psi(x,y)$ so that $\frac{\partial}{\partial x}\Psi(x,y) = M(x,y)$ and $\frac{\partial}{\partial y}\Psi(x,y) = N(x,y)$. Here we are thinking of Ψ as a function of two independent variables x and y.

Integrating the first equation with respect to x gives

$$\begin{aligned}
\frac{\partial}{\partial x}\Psi(x,y) &= M(x,y) \\
&= xy^2 + 2 \\
\Psi(x,y) &= \int \left(xy^2 + 2\right) dx \\
&= \frac{1}{2}x^2y^2 + 2x + f(y)
\end{aligned}$$

Here the 'constant' of integration is a function $f(y)$, which is constant with respect to x.

Similarly,

$$\begin{aligned}
\frac{\partial}{\partial y}\Psi(x,y) &= N(x,y) \\
&= x^2y - 3 \\
\Psi(x,y) &= \int \left(x^2y - 3\right) dx \\
&= \frac{1}{2}x^2y^2 - 3y + g(x)
\end{aligned}$$

where again, the 'constant' $g(x)$ is constant with respect to y.

To get a formula for Ψ, we want to choose $f(y)$ and $g(x)$ so that these expressions agree. Setting $g(x) = 2x$ and $f(y) = -3y$ works, so we get

$$\Psi(x,y) = \frac{1}{2}x^2y^2 + 2x - 3y$$

Just to remind you why we care: if we think of $y = y(x)$, y as a function of x, then the multivariable chain rule says

$$\begin{aligned}
\frac{d}{dx}\Psi(x,y(x)) &= \frac{\partial}{\partial x}\Psi + \frac{\partial}{\partial y}\Psi \cdot \frac{dy}{dx} \\
&= (xy^2 + 2) + (x^2y - 3)y' \\
\frac{d}{dx}\Psi(x,y(x)) &= 0
\end{aligned}$$

so if $y(x)$ is a solution to our differential equation, then $\Psi(x,y)$ is a constant.

So our solution is
$$\frac{1}{2}x^2y^2 + 2x - 3y = c$$
for some constant c. Plugging in our initial condition $y(-1) = 8$, we get
$$\frac{1}{2}(-1)^2 8^2 + 2(-1) - 3 \cdot 8 = 32 - 2 - 24 = 6$$
so $c = 6$, and our particular solution is
$$\frac{1}{2}x^2y^2 + 2x - 3y = 6$$

This implicit equation is sensible for all values of x. If I had asked the question more carefully to get y as a function of x, we could use the quadratic formula on
$$\frac{1}{2}x^2y^2 - 3y + 2x - 6 = 0$$
to obtain
$$y(x) = \frac{3 \pm \sqrt{9 + 12x^2 - 4x^3}}{x^2}$$
Observe the discontinuity as 0, so the maximal (connected) domain for this expression which includes $x = -1$ would be $(-\infty, 0)$.

4. Solve the following initial value problems.

(a)
$$\begin{cases} y'' - 21y' + 90y = 0 \\ y(0) = 1 \\ y'(0) = -12 \end{cases}$$

Solution: This is a **second-order linear homogeneous** equation, with **constant coefficients**. Such an equation has an associated **characteristic equation** whose solutions give us solutions to the differential equation:
$$\lambda^2 - 21\lambda + 90 = 0$$

One can use the quadratic formula:

$$\begin{aligned}
\lambda_{\pm} &= \frac{21 \pm \sqrt{21^2 - 4 \cdot 1 \cdot 90}}{2} \\
&= \frac{21 \pm \sqrt{441 - 360}}{2} \\
&= \frac{21 \pm \sqrt{81}}{2} \\
&= \frac{21 \pm 9}{2} \\
&= 6, 15
\end{aligned}$$

or directly notice $(-6) + (-15) = -21$ and $(-6)(-15) = 90$, so this quadratic factors as

$$(\lambda - 6)(\lambda - 15) = 0$$

This means the general solution to our differential equation is

$$y(x) = c_1 e^{6x} + c_2 e^{15x}$$

for some numbers c_1 and c_2.
Our initial conditions say $y(0) = 1$ and $y'(0) = -12$. Let's first compute the derivative:

$$\begin{aligned}
y'(x) &= \frac{d}{dx}\left(c_1 e^{6x} + c_2 e^{15x}\right) \\
&= \frac{d}{dx}\left(c_1 e^{6x}\right) + \frac{d}{dx}\left(c_2 e^{5x}\right) \\
&= 6c_1 e^{6x} + 15 c_2 e^{15x}
\end{aligned}$$

Now, evaluating $y(x)$ at $x = 0$ gives

$$1 = y(0) = c_1 e^{6 \cdot 0} + c_2 e^{15 \cdot 0} = c_1 \cdot 1 + c_2 \cdot 1 = c_1 + c_2$$

Let's rearrange this as $c_2 = 1 - c_1$.
Evaluating $y'(x)$ at $x = 0$ gives

$$-12 = y'(0) = 6c_1 e^{6 \cdot 0} + 15 c_2 e^{15 \cdot 0} = 6c_1 + 15c_2$$

If we substitute $c_2 = 1 - c_1$ into this, we get

$$-12 = 6c_1 + 15(1 - c_1) = 6c_1 + 15 - 15c_1 = 15 - 9c_1$$

This can be rearranged into
$$9c_1 = 27$$
or
$$c_1 = 3$$
Then we have
$$c_2 = 1 - c_2 = 1 - 3 = -2$$
so our specific solution is
$$y(x) = 3e^{6x} - 2e^{15x}$$

(b)
$$\begin{cases} y'' - 2y' + 10y = 0 \\ y(0) = -1 \\ y'(0) = 3 \end{cases}$$

Solution: Again, this is a **second-order linear homogeneous** equation, with **constant coefficients**, so form its **characteristic equation**:

$$\lambda^2 - 2\lambda + 10 = 0$$

Here, use the quadratic formula:
$$\begin{aligned}
\lambda_{\pm} &= \frac{2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 10}}{2} \\
&= \frac{2 \pm \sqrt{4 - 40}}{2} \\
&= \frac{2 \pm \sqrt{-36}}{2} \\
&= \frac{2 \pm 6i}{2} \\
&= 1 + 3i, 1 - 3i
\end{aligned}$$
So our general solution is
$$y(x) = c_1 e^{(1+3i)x} + c_2 e^{(1-3i)x}$$
for some numbers c_1 and c_2. It's a little annoying that we started only with real numbers, but here have complex numbers in the exponents. Using Euler's formula
$$e^{\alpha + \beta i} = e^{\alpha} \left(\cos(\beta) + i \cdot \sin(\beta)\right)$$

we can rewrite our general solution as

$$y(x) = ae^x \cos(3x) + be^x \sin(3x)$$

for some numbers a and b.
Let us also take the derivative of the function y in this form:

$$\begin{aligned} y'(x) &= \frac{d}{dx}\left(ae^x \cos(3x) + be^x \sin(3x)\right) \\ &= \frac{d}{dx}\left(ae^x \cos(3x)\right) + \frac{d}{dx}\left(be^x \sin(3x)\right) \\ &= ae^x \cos(3x) - 3ae^x \sin(3x) + be^x \sin(3x) + 3be^x \cos(3x) \end{aligned}$$

Evaluating $y(x)$ at $x = 0$ gives

$$-1 = y(0) = ae^0 \cos(3 \cdot 0) + be^0 \sin(3 \cdot 0) = a$$

so that $a = -1$, and evaluating $y'(x)$ at $x = 0$ gives

$$\begin{aligned} 3 = y'(0) &= ae^0 \cos(3 \cdot 0) - 3ae^0 \sin(3 \cdot 0) + be^0 \sin(3 \cdot 0) + 3be^0 \cos(3 \cdot 0) \\ 3 &= a + 3b \\ 3 &= -1 + 3b \\ 4 &= 3b \\ \frac{4}{3} &= b \end{aligned}$$

Thus our final specific solution to this initial value problem is

$$y(x) = -e^x \cos(3x) + \frac{4}{3}e^x \sin(3x)$$