The electrical power produced by a wind turbine can be estimated by a mathematical function that incorporates several factors. We will explore this function in Exercise 1.2.25 and determine the expected power output of a particular turbine for various wind speeds.
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1 Functions and Models

THE FUNDAMENTAL OBJECTS THAT WE deal with in calculus are functions. This chapter prepares the way for calculus by discussing the basic ideas concerning functions, their graphs, and ways of transforming and combining them. We stress that a function can be represented in different ways: by an equation, in a table, by a graph, or in words. We look at the main types of functions that occur in calculus and describe the process of using these functions as mathematical models of real-world phenomena.

1.1 Four Ways to Represent a Function

Functions

Functions arise whenever one quantity depends on another. Consider the following four situations.

A. The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the equation $A = \pi r^2$. With each positive number r there is associated one value of A, and we say that A is a *function* of r.

B. The human population of the world P depends on the time t. Table 1 gives estimates of the world population P at time t, for certain years. For instance,

$$P \approx 2{,}560{,}000{,}000 \qquad \text{when } t = 1950$$

For each value of the time t there is a corresponding value of P, and we say that P is a function of t.

Table 1 World Population

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080
2010	6870

C. The cost C of mailing an envelope depends on its weight w. Although there is no simple formula that connects w and C, the post office has a rule for determining C when w is known.

D. The vertical acceleration a of the ground as measured by a seismograph during an earthquake is a function of the elapsed time t. Figure 1 shows a graph generated by seismic activity during the Northridge earthquake that shook Los Angeles in 1994. For a given value of t, the graph provides a corresponding value of a.

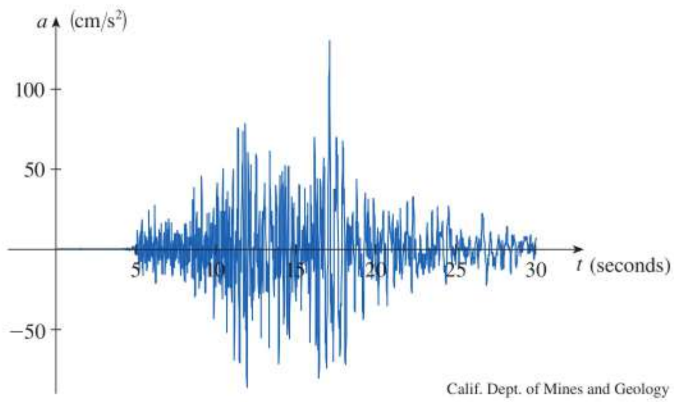


FIGURE 1
Vertical ground acceleration during the Northridge earthquake

Each of these examples describes a rule whereby, given a number (r in Example A), another number (A) is assigned. In each case we say that the second number is a function of the first number. If f represents the rule that connects A to r in Example A, then we express this in **function notation** as $A = f(r)$.

> A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

We usually consider functions for which the sets D and E are sets of real numbers. The set D is called the **domain** of the function. The number $f(x)$ is the **value of f at x** and is read "f of x." The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain. A symbol that represents an arbitrary number in the *domain* of a function f is called an **independent variable**. A symbol that represents a number in the *range* of f is called a **dependent variable**. In Example A, for instance, r is the independent variable and A is the dependent variable.

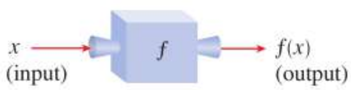


FIGURE 2
Machine diagram for a function f

It's helpful to think of a function as a **machine** (see Figure 2). If x is in the domain of the function f, then when x enters the machine, it's accepted as an **input** and the machine produces an **output** $f(x)$ according to the rule of the function. So we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs. The preprogrammed functions in a calculator are good examples of a function as a machine. For example, if you input a number and press the squaring key, the calculator displays the output, the square of the input.

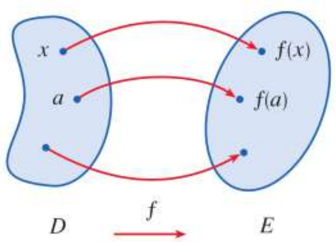


FIGURE 3
Arrow diagram for f

Another way to picture a function is by an **arrow diagram** as in Figure 3. Each arrow connects an element of D to an element of E. The arrow indicates that $f(x)$ is associated with x, $f(a)$ is associated with a, and so on.

Perhaps the most useful method for visualizing a function is its graph. If f is a function with domain D, then its **graph** is the set of ordered pairs

$$\{(x, f(x)) \mid x \in D\}$$

(Notice that these are input-output pairs.) In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.

The graph of a function f gives us a useful picture of the behavior or "life history" of a function. Since the y-coordinate of any point (x, y) on the graph is $y = f(x)$, we can read the value of $f(x)$ from the graph as being the height of the graph above the point x. (See Figure 4.) The graph of f also allows us to picture the domain of f on the x-axis and its range on the y-axis as in Figure 5.

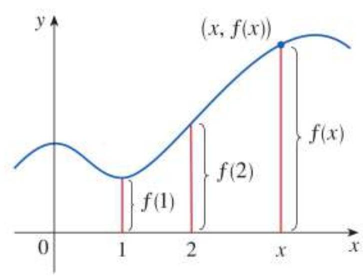


FIGURE 4

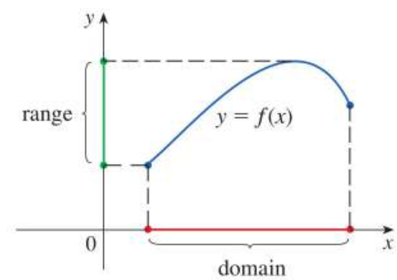


FIGURE 5

EXAMPLE 1 The graph of a function f is shown in Figure 6.
(a) Find the values of $f(1)$ and $f(5)$.
(b) What are the domain and range of f?

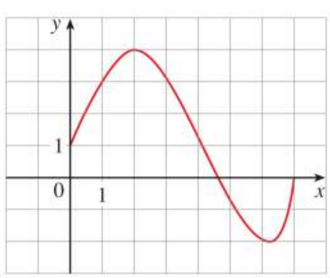


FIGURE 6

SOLUTION

(a) We see from Figure 6 that the point $(1, 3)$ lies on the graph of f, so the value of f at 1 is $f(1) = 3$. (In other words, the point on the graph that lies above $x = 1$ is 3 units above the x-axis.)

When $x = 5$, the graph lies about 0.7 units below the x-axis, so we estimate that $f(5) \approx -0.7$.

The notation for intervals is given in Appendix A.

(b) We see that $f(x)$ is defined when $0 \leqslant x \leqslant 7$, so the domain of f is the closed interval $[0, 7]$. Notice that f takes on all values from -2 to 4, so the range of f is

$$\{y \mid -2 \leqslant y \leqslant 4\} = [-2, 4]$$

■

In calculus, the most common method of defining a function is by an algebraic equation. For example, the equation $y = 2x - 1$ defines y as a function of x. We can express this in function notation as $f(x) = 2x - 1$.

EXAMPLE 2 Sketch the graph and find the domain and range of each function.
(a) $f(x) = 2x - 1$ (b) $g(x) = x^2$

SOLUTION
(a) The equation of the graph is $y = 2x - 1$, and we recognize this as being the equation of a line with slope 2 and y-intercept -1. (Recall the slope-intercept form of the equation of a line: $y = mx + b$. See Appendix B.) This enables us to sketch a portion of the graph of f in Figure 7. The expression $2x - 1$ is defined for all real numbers, so the domain of f is the set of all real numbers, which we denote by $\mathbb{R}$. The graph shows that the range is also $\mathbb{R}$.

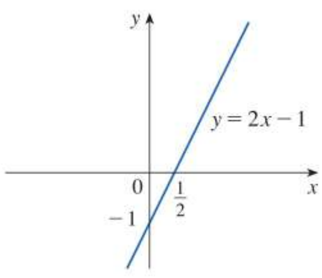


FIGURE 7

(b) Since $g(2) = 2^2 = 4$ and $g(-1) = (-1)^2 = 1$, we could plot the points $(2, 4)$ and $(-1, 1)$, together with a few other points on the graph, and join them to produce the graph (Figure 8). The equation of the graph is $y = x^2$, which represents a parabola (see Appendix C). The domain of g is $\mathbb{R}$. The range of g consists of all values of $g(x)$, that is, all numbers of the form x^2. But $x^2 \geqslant 0$ for all numbers x and any positive number y is a square. So the range of g is $\{y \mid y \geqslant 0\} = [0, \infty)$. This can also be seen from Figure 8. ■

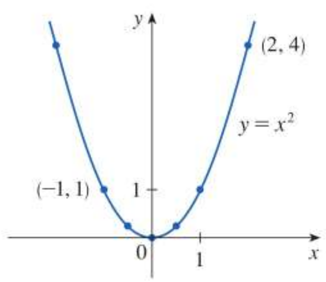


FIGURE 8

EXAMPLE 3 If $f(x) = 2x^2 - 5x + 1$ and $h \neq 0$, evaluate $\dfrac{f(a + h) - f(a)}{h}$.

SOLUTION We first evaluate $f(a + h)$ by replacing x by $a + h$ in the expression for $f(x)$:

$$\begin{aligned} f(a + h) &= 2(a + h)^2 - 5(a + h) + 1 \\ &= 2(a^2 + 2ah + h^2) - 5(a + h) + 1 \\ &= 2a^2 + 4ah + 2h^2 - 5a - 5h + 1 \end{aligned}$$

Then we substitute into the given expression and simplify:

$$\begin{aligned} \frac{f(a + h) - f(a)}{h} &= \frac{(2a^2 + 4ah + 2h^2 - 5a - 5h + 1) - (2a^2 - 5a + 1)}{h} \\ &= \frac{2a^2 + 4ah + 2h^2 - 5a - 5h + 1 - 2a^2 + 5a - 1}{h} \\ &= \frac{4ah + 2h^2 - 5h}{h} = 4a + 2h - 5 \end{aligned}$$ ■

The expression

$$\frac{f(a + h) - f(a)}{h}$$

in Example 3 is called a **difference quotient** and occurs frequently in calculus. As we will see in Chapter 2, it represents the average rate of change of $f(x)$ between $x = a$ and $x = a + h$.

Representations of Functions

We consider four different ways to represent a function:

- verbally (by a description in words)
- numerically (by a table of values)
- visually (by a graph)
- algebraically (by an explicit formula)

If a single function can be represented in all four ways, it's often useful to go from one representation to another to gain additional insight into the function. (In Example 2, for instance, we started with algebraic formulas and then obtained graphs.) But certain functions are described more naturally by one method than by another. With this in mind, let's reexamine the four situations that we considered at the beginning of this section.

A. The most useful representation of the area of a circle as a function of its radius is probably the algebraic formula $A = \pi r^2$ or, in function notation, $A(r) = \pi r^2$. It is also possible to compile a table of values or sketch a graph (half a parabola). Because a circle has to have a positive radius, the domain is $\{r \mid r > 0\} = (0, \infty)$ and the range is also $(0, \infty)$.

Table 2 World Population

t (years since 1900)	Population (millions)
0	1650
10	1750
20	1860
30	2070
40	2300
50	2560
60	3040
70	3710
80	4450
90	5280
100	6080
110	6870

B. We are given a description of the function in words: $P(t)$ is the human population of the world at time t. Let's measure t so that $t = 0$ corresponds to the year 1900. Table 2 provides a convenient representation of this function. If we plot the ordered pairs in the table, we get the graph (called a *scatter plot*) in Figure 9. It too is a useful representation; the graph allows us to absorb all the data at once. What about a formula? Of course, it's impossible to devise an explicit formula that gives the exact human population $P(t)$ at any time t. But it is possible to find an expression for a function that *approximates* $P(t)$. In fact, using methods explained in Section 1.4, we obtain an approximation for the population P:

$$P(t) \approx f(t) = (1.43653 \times 10^9) \cdot (1.01395)^t$$

Figure 10 shows that it is a reasonably good "fit." The function f is called a *mathematical model* for population growth. In other words, it is a function with an explicit formula that approximates the behavior of our given function. We will see, however, that the ideas of calculus can be applied to a table of values; an explicit formula is not necessary.

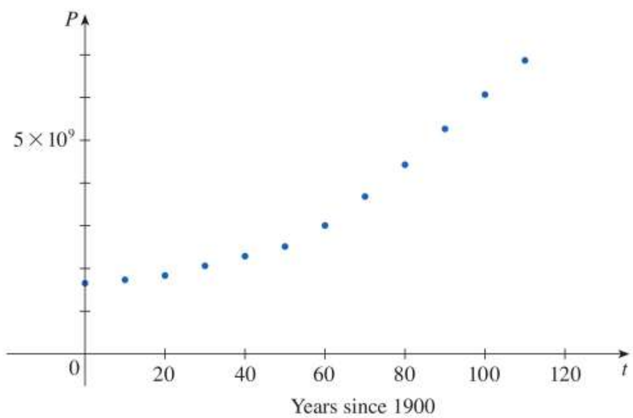


FIGURE 9

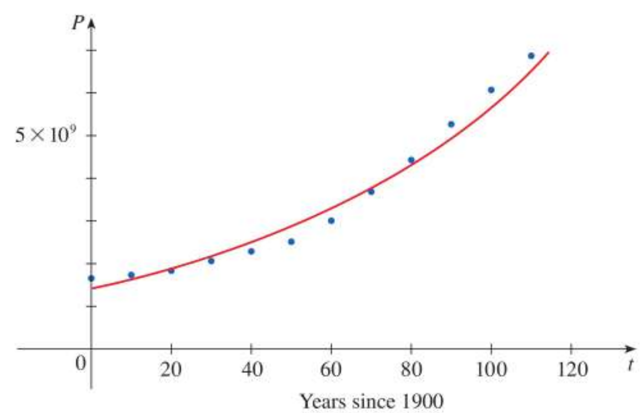


FIGURE 10

The function P is typical of the functions that arise whenever we attempt to apply calculus to the real world. We start with a verbal description of a function. Then we may be able to construct a table of values of the function, perhaps from instrument readings in a scientific experiment. Even though we don't have complete knowledge of the values of the function, we will see throughout the book that it is still possible to perform the operations of calculus on such a function.

A function defined by a table of values is called a *tabular* function.

Table 3

w (ounces)	$C(w)$ (dollars)
$0 < w \leqslant 1$	1.00
$1 < w \leqslant 2$	1.15
$2 < w \leqslant 3$	1.30
$3 < w \leqslant 4$	1.45
$4 < w \leqslant 5$	1.60
⋮	⋮

C. Again, the function is described in words: Let $C(w)$ be the cost of mailing a large envelope with weight w. The rule that the US Postal Service used as of 2019 is as follows: The cost is 1 dollar for up to 1 oz, plus 15 cents for each additional ounce (or less) up to 13 oz. A table of values is the most convenient representation for this function (see Table 3), though it is possible to sketch a graph (see Example 10).

D. The graph shown in Figure 1 is the most natural representation of the vertical acceleration function $a(t)$. It's true that a table of values could be compiled, and it is even possible to devise an approximate formula. But everything a geologist needs to know—amplitudes and patterns—can be seen easily from the graph. (The same is true for the patterns seen in electrocardiograms of heart patients and polygraphs for lie-detection.)

In the next example we sketch the graph of a function that is defined verbally.

EXAMPLE 4 When you turn on a hot-water faucet that is connected to a hot-water tank, the temperature T of the water depends on how long the water has been running. Draw a rough graph of T as a function of the time t that has elapsed since the faucet was turned on.

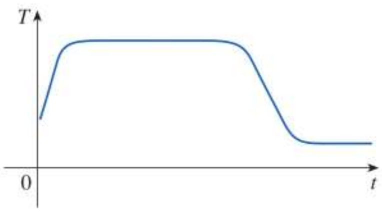


FIGURE 11

SOLUTION The initial temperature of the running water is close to room temperature because the water has been sitting in the pipes. When the water from the hot-water tank starts flowing from the faucet, T increases quickly. In the next phase, T is constant at the temperature of the heated water in the tank. When the tank is drained, T decreases to the temperature of the water supply. This enables us to make the rough sketch of T as a function of t shown in Figure 11. ■

In the following example we start with a verbal description of a function in a physical situation and obtain an explicit algebraic formula. The ability to do this is a useful skill in solving calculus problems that ask for the maximum or minimum values of quantities.

EXAMPLE 5 A rectangular storage container with an open top has a volume of 10 m^3. The length of its base is twice its width. Material for the base costs \$10 per square meter; material for the sides costs \$6 per square meter. Express the cost of materials as a function of the width of the base.

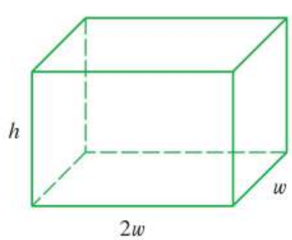


FIGURE 12

SOLUTION We draw a diagram as in Figure 12 and introduce notation by letting w and $2w$ be the width and length of the base, respectively, and h be the height.

The area of the base is $(2w)w = 2w^2$, so the cost, in dollars, of the material for the base is $10(2w^2)$. Two of the sides have area wh and the other two have area $2wh$, so the cost of the material for the sides is $6[2(wh) + 2(2wh)]$. The total cost is therefore

$$C = 10(2w^2) + 6[2(wh) + 2(2wh)] = 20w^2 + 36wh$$

To express C as a function of w alone, we need to eliminate h and we do so by using the fact that the volume is 10 m^3. Thus

$$w(2w)h = 10$$

which gives

$$h = \frac{10}{2w^2} = \frac{5}{w^2}$$

Substituting this into the expression for C, we have

$$C = 20w^2 + 36w\left(\frac{5}{w^2}\right) = 20w^2 + \frac{180}{w}$$

Therefore the equation

$$C(w) = 20w^2 + \frac{180}{w} \qquad w > 0$$

expresses C as a function of w. ■

PS In setting up applied functions as in Example 5, it may be useful to review the principles of problem solving at the end of this chapter, particularly *Step 1: Understand the Problem.*

In the next example we find the domain of a function that is defined algebraically. If a function is given by a formula and the domain is not stated explicitly, we use the following **domain convention**: the domain of the function is the set of all inputs for which the formula makes sense and gives a real-number output.

EXAMPLE 6 Find the domain of each function.

(a) $f(x) = \sqrt{x + 2}$ (b) $g(x) = \dfrac{1}{x^2 - x}$

SOLUTION

(a) Because the square root of a negative number is not defined (as a real number), the domain of f consists of all values of x such that $x + 2 \geqslant 0$. This is equivalent to $x \geqslant -2$, so the domain is the interval $[-2, \infty)$.

(b) Since

$$g(x) = \frac{1}{x^2 - x} = \frac{1}{x(x - 1)}$$

and division by 0 is not allowed, we see that $g(x)$ is not defined when $x = 0$ or $x = 1$. So the domain of g is

$$\{x \mid x \neq 0, x \neq 1\}$$

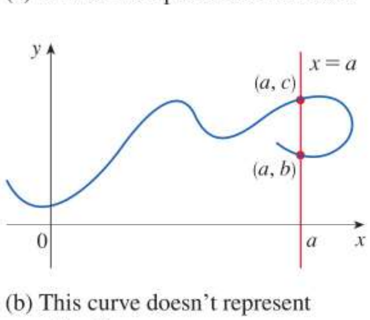
which could also be written in interval notation as

$$(-\infty, 0) \cup (0, 1) \cup (1, \infty)$$

Which Rules Define Functions?

Not every equation defines a function. The equation $y = x^2$ defines y as a function of x because the equation determines exactly one value of y for each value of x. However, the equation $y^2 = x$ does *not* define y as a function of x because some input values x correspond to more than one output y; for instance, for the input $x = 4$ the equation gives the outputs $y = 2$ and $y = -2$.

Similarly, not every table defines a function. Table 3 defined C as a function of w—each package weight w corresponds to exactly one mailing cost. On the other hand, Table 4 does *not* define y as a function of x because some input values x in the table correspond to more than one output y; for instance, the input $x = 5$ gives the outputs $y = 7$ and $y = 8$.



(a) This curve represents a function.

(b) This curve doesn't represent a function.

FIGURE 13

Table 4

x	2	4	5	5	6
y	3	6	7	8	9

What about curves drawn in the xy-plane? Which curves are graphs of functions? The following test gives an answer.

The Vertical Line Test A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

The reason for the truth of the Vertical Line Test can be seen in Figure 13. If each vertical line $x = a$ intersects a curve only once, at (a, b), then exactly one function value is defined by $f(a) = b$. But if a line $x = a$ intersects the curve twice, at (a, b) and (a, c), then the curve can't represent a function because a function can't assign two different values to a.

For example, the parabola $x = y^2 - 2$ shown in Figure 14(a) is not the graph of a function of x because, as you can see, there are vertical lines that intersect the parabola twice. The parabola, however, does contain the graphs of *two* functions of x. Notice that the equation $x = y^2 - 2$ implies $y^2 = x + 2$, so $y = \pm\sqrt{x + 2}$. Thus the upper and lower halves of the parabola are the graphs of the functions $f(x) = \sqrt{x + 2}$ [from Example 6(a)] and $g(x) = -\sqrt{x + 2}$. [See Figures 14(b) and (c).]

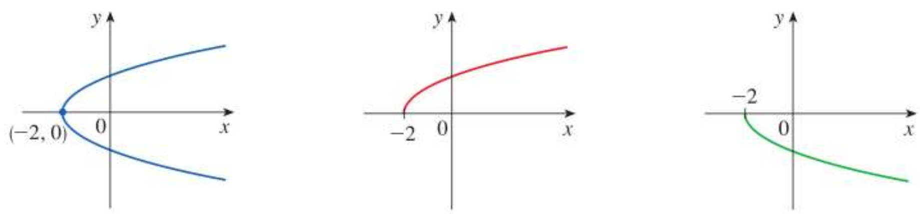


FIGURE 14 (a) $x = y^2 - 2$ (b) $y = \sqrt{x+2}$ (c) $y = -\sqrt{x+2}$

We observe that if we reverse the roles of x and y, then the equation $x = h(y) = y^2 - 2$ *does* define x as a function of y (with y as the independent variable and x as the dependent variable). The graph of the function h is the parabola in Figure 14(a).

Piecewise Defined Functions

The functions in the following four examples are defined by different formulas in different parts of their domains. Such functions are called **piecewise defined functions**.

EXAMPLE 7 A function f is defined by

$$f(x) = \begin{cases} 1 - x & \text{if } x \leqslant -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

Evaluate $f(-2)$, $f(-1)$, and $f(0)$ and sketch the graph.

SOLUTION Remember that a function is a rule. For this particular function the rule is the following: First look at the value of the input x. If it happens that $x \leqslant -1$, then the value of $f(x)$ is $1 - x$. On the other hand, if $x > -1$, then the value of $f(x)$ is x^2. Note that even though two different formulas are used, f is *one* function, not two.

$$\text{Since } -2 \leqslant -1, \text{ we have } f(-2) = 1 - (-2) = 3.$$

$$\text{Since } -1 \leqslant -1, \text{ we have } f(-1) = 1 - (-1) = 2.$$

$$\text{Since } 0 > -1, \text{ we have } f(0) = 0^2 = 0.$$

How do we draw the graph of f? We observe that if $x \leqslant -1$, then $f(x) = 1 - x$, so the part of the graph of f that lies to the left of the vertical line $x = -1$ must coincide with the line $y = 1 - x$, which has slope -1 and y-intercept 1. If $x > -1$, then $f(x) = x^2$, so the part of the graph of f that lies to the right of the line $x = -1$ must coincide with the graph of $y = x^2$, which is a parabola. This enables us to sketch the graph in Figure 15. The solid dot indicates that the point $(-1, 2)$ is included on the graph; the open dot indicates that the point $(-1, 1)$ is excluded from the graph. ■

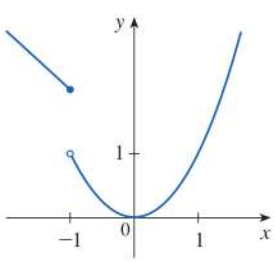


FIGURE 15

The next example of a piecewise defined function is the absolute value function. Recall that the **absolute value** of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line. Distances are always positive or 0, so we have

For a more extensive review of absolute values, see Appendix A.

$$|a| \geqslant 0 \qquad \text{for every number } a$$

For example,

$$|3| = 3 \qquad |-3| = 3 \qquad |0| = 0 \qquad |\sqrt{2} - 1| = \sqrt{2} - 1 \qquad |3 - \pi| = \pi - 3$$

In general, we have

$$\boxed{\begin{aligned} |a| &= a \quad \text{if } a \geqslant 0 \\ |a| &= -a \quad \text{if } a < 0 \end{aligned}}$$

(Remember that if a is negative, then $-a$ is positive.)

EXAMPLE 8 Sketch the graph of the absolute value function $f(x) = |x|$.

SOLUTION From the preceding discussion we know that

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

Using the same method as in Example 7, we see that the graph of f coincides with the line $y = x$ to the right of the y-axis and coincides with the line $y = -x$ to the left of the y-axis (see Figure 16). ■

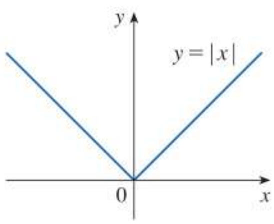


FIGURE 16

EXAMPLE 9 Find a formula for the function f graphed in Figure 17.

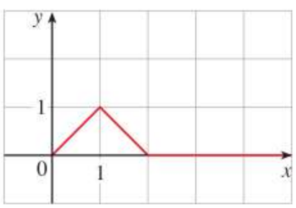


FIGURE 17

The point-slope form of the equation of a line is $y - y_1 = m(x - x_1)$. See Appendix B.

SOLUTION The line through $(0, 0)$ and $(1, 1)$ has slope $m = 1$ and y-intercept $b = 0$, so its equation is $y = x$. Thus, for the part of the graph of f that joins $(0, 0)$ to $(1, 1)$, we have

$$f(x) = x \qquad \text{if } 0 \leqslant x \leqslant 1$$

The line through $(1, 1)$ and $(2, 0)$ has slope $m = -1$, so its point-slope form is

$$y - 0 = (-1)(x - 2) \qquad \text{or} \qquad y = 2 - x$$

So we have

$$f(x) = 2 - x \qquad \text{if } 1 < x \leqslant 2$$

We also see that the graph of f coincides with the x-axis for $x > 2$. Putting this information together, we have the following three-piece formula for f:

$$f(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant 1 \\ 2 - x & \text{if } 1 < x \leqslant 2 \\ 0 & \text{if } x > 2 \end{cases}$$ ■

EXAMPLE 10 In Example C at the beginning of this section we considered the cost $C(w)$ of mailing a large envelope with weight w. In effect, this is a piecewise defined function because, from Table 3, we have

$$C(w) = \begin{cases} 1.00 & \text{if } 0 < w \leqslant 1 \\ 1.15 & \text{if } 1 < w \leqslant 2 \\ 1.30 & \text{if } 2 < w \leqslant 3 \\ 1.45 & \text{if } 3 < w \leqslant 4 \\ \vdots & \end{cases}$$

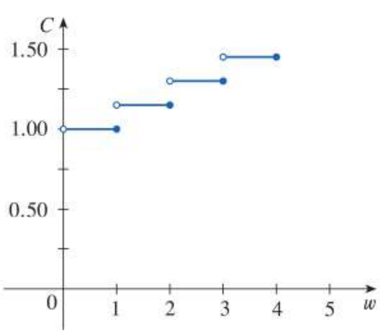


FIGURE 18

The graph is shown in Figure 18. ■

Looking at Figure 18, you can see why a function like the one in Example 10 is called a **step function**.

Even and Odd Functions

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even function**. For instance, the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x)$$

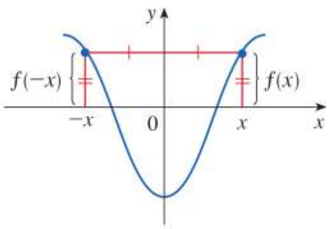


FIGURE 19
An even function

The geometric significance of an even function is that its graph is symmetric with respect to the y-axis (see Figure 19). This means that if we have plotted the graph of f for $x \geqslant 0$, we obtain the entire graph simply by reflecting this portion about the y-axis.

If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**. For example, the function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

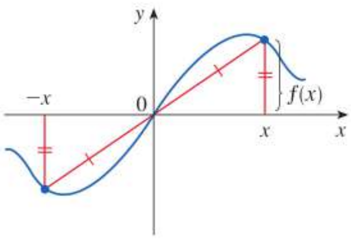


FIGURE 20
An odd function

The graph of an odd function is symmetric about the origin (see Figure 20). If we already have the graph of f for $x \geqslant 0$, we can obtain the entire graph by rotating this portion through 180° about the origin.

EXAMPLE 11 Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ (b) $g(x) = 1 - x^4$ (c) $h(x) = 2x - x^2$

SOLUTION

(a)
$$\begin{aligned} f(-x) &= (-x)^5 + (-x) = (-1)^5x^5 + (-x) \\ &= -x^5 - x = -(x^5 + x) \\ &= -f(x) \end{aligned}$$

Therefore f is an odd function.

(b)
$$g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$$

So g is even.

(c)
$$h(-x) = 2(-x) - (-x)^2 = -2x - x^2$$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd. ■

The graphs of the functions in Example 11 are shown in Figure 21. Notice that the graph of h is symmetric neither about the y-axis nor about the origin.

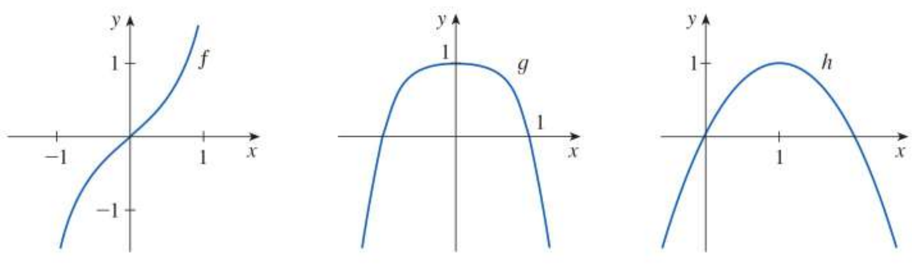


FIGURE 21

Increasing and Decreasing Functions

The graph shown in Figure 22 rises from A to B, falls from B to C, and rises again from C to D. The function f is said to be increasing on the interval $[a, b]$, decreasing on $[b, c]$, and increasing again on $[c, d]$. Notice that if x_1 and x_2 are any two numbers between

a and b with $x_1 < x_2$, then $f(x_1) < f(x_2)$. We use this as the defining property of an increasing function.

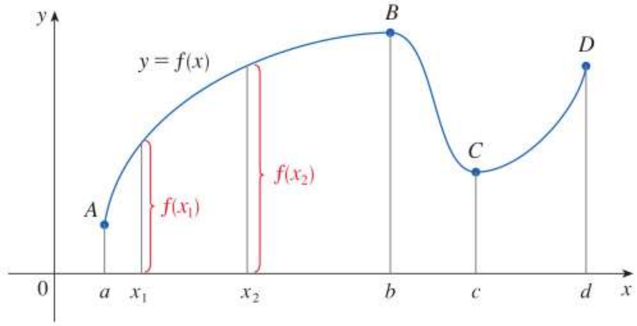


FIGURE 22

> A function f is called **increasing** on an interval I if
>
> $$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$
>
> It is called **decreasing** on I if
>
> $$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

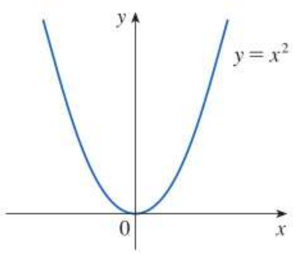


FIGURE 23

In the definition of an increasing function it is important to realize that the inequality $f(x_1) < f(x_2)$ must be satisfied for *every* pair of numbers x_1 and x_2 in I with $x_1 < x_2$.

You can see from Figure 23 that the function $f(x) = x^2$ is decreasing on the interval $(-\infty, 0]$ and increasing on the interval $[0, \infty)$.

1.1 Exercises

1. If $f(x) = x + \sqrt{2 - x}$ and $g(u) = u + \sqrt{2 - u}$, is it true that $f = g$?

2. If

$$f(x) = \frac{x^2 - x}{x - 1} \qquad \text{and} \qquad g(x) = x$$

is it true that $f = g$?

3. The graph of a function g is given.

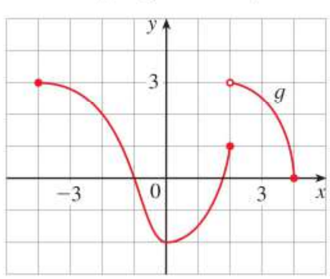
(a) State the values of $g(-2)$, $g(0)$, $g(2)$, and $g(3)$.

(b) For what value(s) of x is $g(x) = 3$?

(c) For what value(s) of x is $g(x) \leqslant 3$?

(d) State the domain and range of g.

(e) On what interval(s) is g increasing?



4. The graphs of f and g are given.

(a) State the values of $f(-4)$ and $g(3)$.

(b) Which is larger, $f(-3)$ or $g(-3)$?

(c) For what values of x is $f(x) = g(x)$?

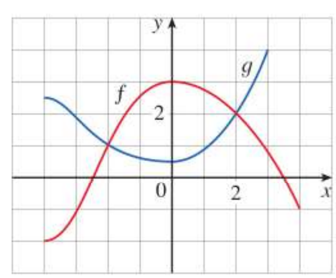
(d) On what interval(s) is $f(x) \leqslant g(x)$?

(e) State the solution of the equation $f(x) = -1$.

(f) On what interval(s) is g decreasing?

(g) State the domain and range of f.

(h) State the domain and range of g.



5. Figure 1 was recorded by an instrument operated by the California Department of Mines and Geology at the

University Hospital of the University of Southern California in Los Angeles. Use it to estimate the range of the vertical ground acceleration function at USC during the Northridge earthquake.

6. In this section we discussed examples of ordinary, everyday functions: population is a function of time, postage cost is a function of package weight, water temperature is a function of time. Give three other examples of functions from everyday life that are described verbally. What can you say about the domain and range of each of your functions? If possible, sketch a rough graph of each function.

7–14 Determine whether the equation or table defines y as a function of x.

7. $3x - 5y = 7$

8. $3x^2 - 2y = 5$

9. $x^2 + (y - 3)^2 = 5$

10. $2xy + 5y^2 = 4$

11. $(y + 3)^3 + 1 = 2x$

12. $2x - |y| = 0$

13.

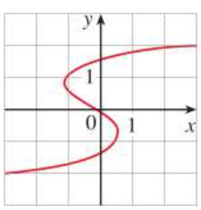
x Height (in)	y Shoe size
72	12
60	8
60	7
63	9
70	10

14.

x Year	y Tuition cost ($)
2016	10,900
2017	11,000
2018	11,200
2019	11,200
2020	11,300

15–18 Determine whether the curve is the graph of a function of x. If it is, state the domain and range of the function.

15.



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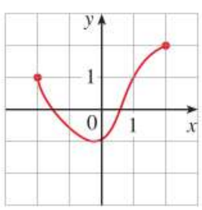

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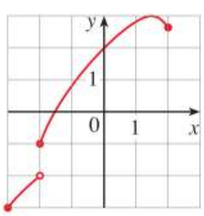
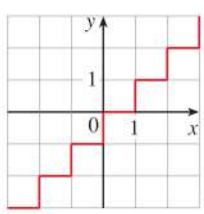


18.



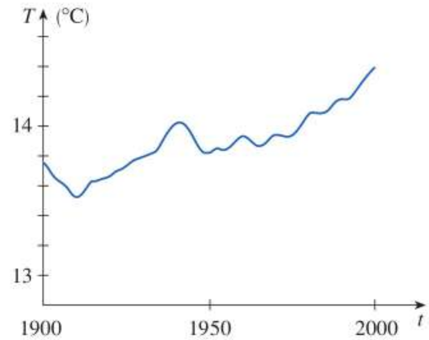
19. Shown is a graph of the global average temperature T during the 20th century. Estimate the following.

(a) The global average temperature in 1950

(b) The year when the average temperature was 14.2°C

(c) The years when the temperature was smallest and largest

(d) The range of T

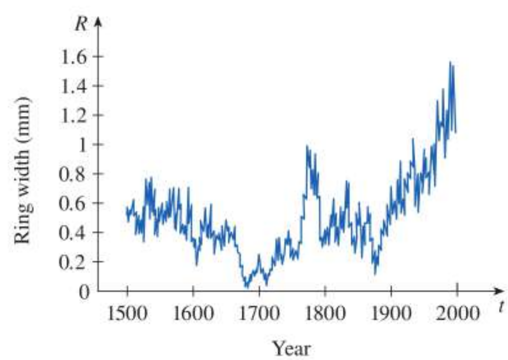


Source: Adapted from *Globe and Mail* [Toronto], 5 Dec. 2009. Print.

20. Trees grow faster and form wider rings in warm years and grow more slowly and form narrower rings in cooler years. The figure shows ring widths of a Siberian pine from 1500 to 2000.

(a) What is the range of the ring width function?

(b) What does the graph tend to say about the temperature of the earth? Does the graph reflect the volcanic eruptions of the mid-19th century?



Source: Adapted from G. Jacoby et al., "Mongolian Tree Rings and 20th-Century Warming," *Science* 273 (1996): 771–73.

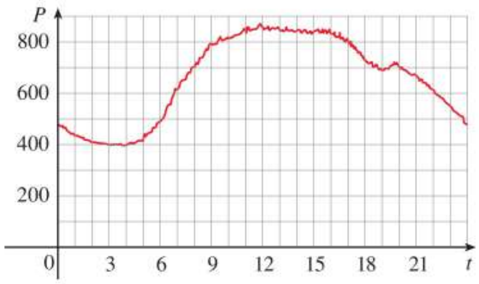
21. You put some ice cubes in a glass, fill the glass with cold water, and then let the glass sit on a table. Describe how the temperature of the water changes as time passes. Then sketch a rough graph of the temperature of the water as a function of the elapsed time.

22. You place a frozen pie in an oven and bake it for an hour. Then you take it out and let it cool. Describe how the temperature of the pie changes as time passes. Then sketch a rough graph of the temperature of the pie as a function of time.

23. The graph shows the power consumption for a day in September in San Francisco. (P is measured in megawatts; t is measured in hours starting at midnight.)

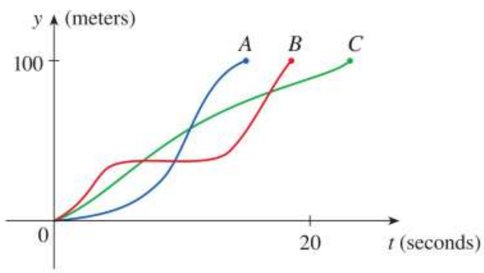
(a) What was the power consumption at 6 AM? At 6 PM?

(b) When was the power consumption the lowest? When was it the highest? Do these times seem reasonable?



Pacific Gas & Electric

24. Three runners compete in a 100-meter race. The graph depicts the distance run as a function of time for each runner. Describe in words what the graph tells you about this race. Who won the race? Did each runner finish the race?



25. Sketch a rough graph of the outdoor temperature as a function of time during a typical spring day.

26. Sketch a rough graph of the number of hours of daylight as a function of the time of year.

27. Sketch a rough graph of the amount of a particular brand of coffee sold by a store as a function of the price of the coffee.

28. Sketch a rough graph of the market value of a new car as a function of time for a period of 20 years. Assume the car is well maintained.

29. A homeowner mows the lawn every Wednesday afternoon. Sketch a rough graph of the height of the grass as a function of time over the course of a four-week period.

30. An airplane takes off from an airport and lands an hour later at another airport, 400 miles away. If t represents the time in minutes since the plane has left the terminal building, let $x(t)$ be the horizontal distance traveled and $y(t)$ be the altitude of the plane.

(a) Sketch a possible graph of $x(t)$.
(b) Sketch a possible graph of $y(t)$.
(c) Sketch a possible graph of the ground speed.
(d) Sketch a possible graph of the vertical velocity.

31. Temperature readings T (in °F) were recorded every two hours from midnight to 2:00 PM in Atlanta on a day in June. The time t was measured in hours from midnight.

t	0	2	4	6	8	10	12	14
T	74	69	68	66	70	78	82	86

(a) Use the readings to sketch a rough graph of T as a function of t.
(b) Use your graph to estimate the temperature at 9:00 AM.

32. Researchers measured the blood alcohol concentration (BAC) of eight adult male subjects after rapid consumption of 30 mL of ethanol (corresponding to two standard alcoholic drinks). The table shows the data they obtained by averaging the BAC (in g/dL) of the eight men.

(a) Use the readings to sketch a graph of the BAC as a function of t.
(b) Use your graph to describe how the effect of alcohol varies with time.

t (hours)	BAC	t (hours)	BAC
0	0	1.75	0.022
0.2	0.025	2.0	0.018
0.5	0.041	2.25	0.015
0.75	0.040	2.5	0.012
1.0	0.033	3.0	0.007
1.25	0.029	3.5	0.003
1.5	0.024	4.0	0.001

Source: Adapted from P. Wilkinson et al., "Pharmacokinetics of Ethanol after Oral Administration in the Fasting State," *Journal of Pharmacokinetics and Biopharmaceutics* 5 (1977): 207–24.

33. If $f(x) = 3x^2 - x + 2$, find $f(2)$, $f(-2)$, $f(a)$, $f(-a)$, $f(a + 1)$, $2f(a)$, $f(2a)$, $f(a^2)$, $[f(a)]^2$, and $f(a + h)$.

34. If $g(x) = \dfrac{x}{\sqrt{x + 1}}$, find $g(0)$, $g(3)$, $5g(a)$, $\frac{1}{2}g(4a)$, $g(a^2)$, $[g(a)]^2$, $g(a + h)$, and $g(x - a)$.

35–38 Evaluate the difference quotient for the given function. Simplify your answer.

35. $f(x) = 4 + 3x - x^2$, $\quad \dfrac{f(3 + h) - f(3)}{h}$

36. $f(x) = x^3$, $\quad \dfrac{f(a + h) - f(a)}{h}$

37. $f(x) = \dfrac{1}{x}$, $\quad \dfrac{f(x) - f(a)}{x - a}$

38. $f(x) = \sqrt{x + 2}$, $\quad \dfrac{f(x) - f(1)}{x - 1}$

39–46 Find the domain of the function.

39. $f(x) = \dfrac{x + 4}{x^2 - 9}$

40. $f(x) = \dfrac{x^2 + 1}{x^2 + 4x - 21}$

41. $f(t) = \sqrt[3]{2t - 1}$

42. $g(t) = \sqrt{3 - t} - \sqrt{2 + t}$

43. $h(x) = \dfrac{1}{\sqrt[4]{x^2 - 5x}}$

44. $f(u) = \dfrac{u + 1}{1 + \dfrac{1}{u + 1}}$

45. $F(p) = \sqrt{2 - \sqrt{p}}$

46. $h(x) = \sqrt{x^2 - 4x - 5}$

47. Find the domain and range and sketch the graph of the function $h(x) = \sqrt{4 - x^2}$.

48. Find the domain and sketch the graph of the function

$$f(x) = \frac{x^2 - 4}{x - 2}$$

49–52 Evaluate $f(-3)$, $f(0)$, and $f(2)$ for the piecewise defined function. Then sketch the graph of the function.

49. $f(x) = \begin{cases} x^2 + 2 & \text{if } x < 0 \\ x & \text{if } x \geqslant 0 \end{cases}$

50. $f(x) = \begin{cases} 5 & \text{if } x < 2 \\ \frac{1}{2}x - 3 & \text{if } x \geqslant 2 \end{cases}$

51. $f(x) = \begin{cases} x + 1 & \text{if } x \leqslant -1 \\ x^2 & \text{if } x > -1 \end{cases}$

52. $f(x) = \begin{cases} -1 & \text{if } x \leqslant 1 \\ 7 - 2x & \text{if } x > 1 \end{cases}$

53–58 Sketch the graph of the function.

53. $f(x) = x + |x|$

54. $f(x) = |x + 2|$

55. $g(t) = |1 - 3t|$

56. $f(x) = \dfrac{|x|}{x}$

57. $f(x) = \begin{cases} |x| & \text{if } |x| \leqslant 1 \\ 1 & \text{if } |x| > 1 \end{cases}$

58. $g(x) = \big||x| - 1\big|$

59–64 Find a formula for the function whose graph is the given curve.

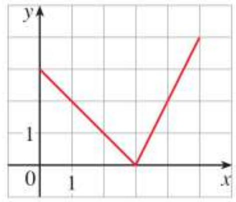
59. The line segment joining the points $(1, -3)$ and $(5, 7)$

60. The line segment joining the points $(-5, 10)$ and $(7, -10)$

61. The bottom half of the parabola $x + (y - 1)^2 = 0$

62. The top half of the circle $x^2 + (y - 2)^2 = 4$

63.



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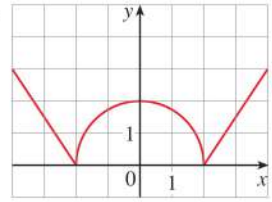


65–70 Find a formula for the described function and state its domain.

65. A rectangle has perimeter 20 m. Express the area of the rectangle as a function of the length of one of its sides.

66. A rectangle has area 16 m^2. Express the perimeter of the rectangle as a function of the length of one of its sides.

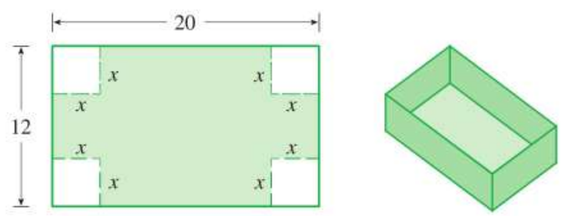
67. Express the area of an equilateral triangle as a function of the length of a side.

68. A closed rectangular box with volume 8 ft^3 has length twice the width. Express the height of the box as a function of the width.

69. An open rectangular box with volume 2 m^3 has a square base. Express the surface area of the box as a function of the length of a side of the base.

70. A right circular cylinder has volume 25 in^3. Express the radius of the cylinder as a function of the height.

71. A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 12 in. by 20 in. by cutting out equal squares of side x at each corner and then folding up the sides as in the figure. Express the volume V of the box as a function of x.



72. A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 30 ft, express the area A of the window as a function of the width x of the window.

73. In a certain state the maximum speed permitted on freeways is 65 mi/h and the minimum speed is 40 mi/h. The fine for violating these limits is \$15 for every mile per hour above the maximum speed or below the minimum speed. Express the amount of the fine F as a function of the driving speed x and graph $F(x)$ for $0 \leqslant x \leqslant 100$.

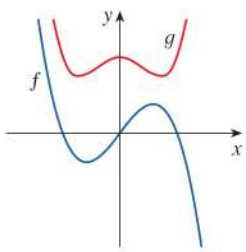
74. An electricity company charges its customers a base rate of \$10 a month, plus 6 cents per kilowatt-hour (kWh) for the first 1200 kWh and 7 cents per kWh for all usage over 1200 kWh. Express the monthly cost E as a function of the amount x of electricity used. Then graph the function E for $0 \leqslant x \leqslant 2000$.

75. In a certain country, income tax is assessed as follows. There is no tax on income up to \$10,000. Any income over \$10,000 is taxed at a rate of 10%, up to an income of \$20,000. Any income over \$20,000 is taxed at 15%.
- (a) Sketch the graph of the tax rate R as a function of the income I.
- (b) How much tax is assessed on an income of \$14,000? On \$26,000?
- (c) Sketch the graph of the total assessed tax T as a function of the income I.

76. (a) If the point (5, 3) is on the graph of an even function, what other point must also be on the graph?
(b) If the point (5, 3) is on the graph of an odd function, what other point must also be on the graph?

77–78 Graphs of f and g are shown. Decide whether each function is even, odd, or neither. Explain your reasoning.

77.



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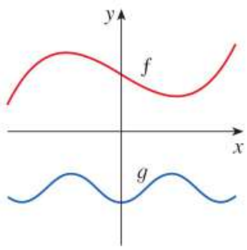
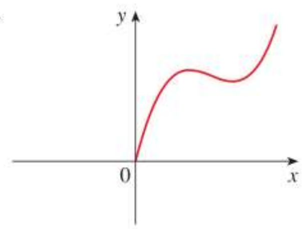


79–80 The graph of a function defined for $x \geqslant 0$ is given. Complete the graph for $x < 0$ to make (a) an even function and (b) an odd function.

79.



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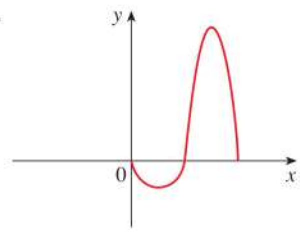


81–86 Determine whether f is even, odd, or neither. You may wish to use a graphing calculator or computer to check your answer visually.

81. $f(x) = \dfrac{x}{x^2 + 1}$

82. $f(x) = \dfrac{x^2}{x^4 + 1}$

83. $f(x) = \dfrac{x}{x + 1}$

84. $f(x) = x|x|$

85. $f(x) = 1 + 3x^2 - x^4$

86. $f(x) = 1 + 3x^3 - x^5$

87. If f and g are both even functions, is $f + g$ even? If f and g are both odd functions, is $f + g$ odd? What if f is even and g is odd? Justify your answers.

88. If f and g are both even functions, is the product fg even? If f and g are both odd functions, is fg odd? What if f is even and g is odd? Justify your answers.

1.2 Mathematical Models: A Catalog of Essential Functions

A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon such as the size of a population, the demand for a product, the speed of a falling object, the concentration of a product in a chemical reaction, the life expectancy of a person at birth, or the cost of emissions reductions. The purpose of the model is to understand the phenomenon and perhaps to make predictions about future behavior.

Given a real-world problem, our first task in the mathematical modeling process is to formulate a mathematical model by identifying and naming the independent and dependent variables and making assumptions that simplify the phenomenon enough to make it mathematically tractable. We use our knowledge of the physical situation and our mathematical skills to obtain equations that relate the variables. In situations where there is no physical law to guide us, we may need to collect data (either from the Internet or a library or by conducting our own experiments) and examine the data in the form of a table in order to discern patterns. From this numerical representation of a function we may wish to obtain a graphical representation by plotting the data. The graph might even suggest a suitable algebraic formula in some cases.

The second stage is to apply the mathematics that we know (such as the calculus that will be developed throughout this book) to the mathematical model that we have formulated in order to derive mathematical conclusions. Then, in the third stage, we take those mathematical conclusions and interpret them as information about the original real-world phenomenon by way of offering explanations or making predictions. The final step is to test our predictions by checking against new real data. If the predictions don't compare well with reality, we need to refine our model or formulate a new model and start the cycle again. Figure 1 illustrates the process of mathematical modeling.

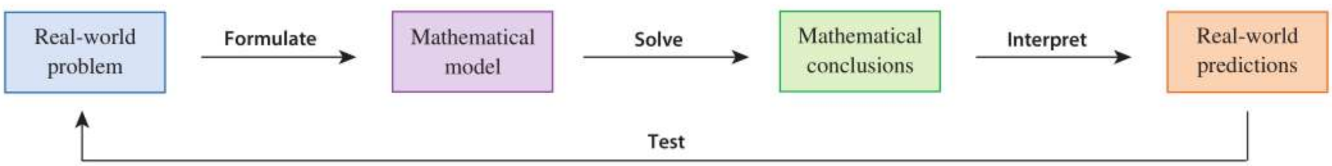


FIGURE 1
The modeling process

A mathematical model is never a completely accurate representation of a physical situation—it is an *idealization*. A good model simplifies reality enough to permit mathematical calculations but is accurate enough to provide valuable conclusions. It is important to realize the limitations of a model.

There are many different types of functions that can be used to model relationships observed in the real world. In what follows, we discuss the behavior and graphs of some of these functions and give examples of situations appropriately modeled by such functions.

Linear Models

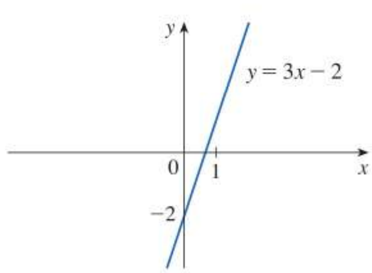
The coordinate geometry of lines is reviewed in Appendix B.

When we say that y is a **linear function** of x, we mean that the graph of the function is a line, so we can use the slope-intercept form of the equation of a line to write a formula for the function as

$$y = f(x) = mx + b$$

where m is the slope of the line and b is the y-intercept.

A characteristic feature of linear functions is that they change at a constant rate. For instance, Figure 2 shows a graph of the linear function $f(x) = 3x - 2$ and a table of sample values. Notice that whenever x increases by 0.1, the value of $f(x)$ increases by 0.3. So $f(x)$ increases three times as fast as x. This means that the slope of the graph of $y = 3x - 2$, namely 3, can be interpreted as the rate of change of y with respect to x.



x	$f(x) = 3x - 2$
1.0	1.0
1.1	1.3
1.2	1.6
1.3	1.9
1.4	2.2
1.5	2.5

FIGURE 2

EXAMPLE 1

(a) As dry air moves upward, it expands and cools. If the ground temperature is 20°C and the temperature at a height of 1 km is 10°C, express the temperature T (in °C) as a function of the height h (in kilometers), assuming that a linear model is appropriate.
(b) Draw the graph of the function in part (a). What does the slope represent?
(c) What is the temperature at a height of 2.5 km?

SOLUTION

(a) Because we are assuming that T is a linear function of h, we can write

$$T = mh + b$$

We are given that $T = 20$ when $h = 0$, so

$$20 = m \cdot 0 + b = b$$

In other words, the y-intercept is $b = 20$.

We are also given that $T = 10$ when $h = 1$, so

$$10 = m \cdot 1 + 20$$

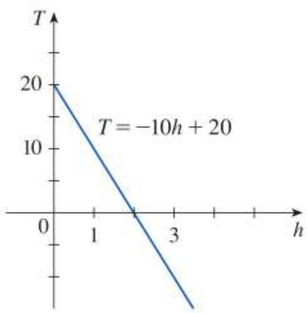


FIGURE 3

The slope of the line is therefore $m = 10 - 20 = -10$ and the required linear function is

$$T = -10h + 20$$

(b) The graph is sketched in Figure 3. The slope is $m = -10$°C/km, and this represents the rate of change of temperature with respect to height.

(c) At a height of $h = 2.5$ km, the temperature is

$$T = -10(2.5) + 20 = -5°\text{C}$$ ■

If there is no physical law or principle to help us formulate a model, we construct an **empirical model**, which is based entirely on collected data. We seek a curve that "fits" the data in the sense that it captures the basic trend of the data points.

Table 1

Year	CO_2 level (in ppm)	Year	CO_2 level (in ppm)
1980	338.7	2000	369.4
1984	344.4	2004	377.5
1988	351.5	2008	385.6
1992	356.3	2012	393.8
1996	362.4	2016	404.2

EXAMPLE 2 Table 1 lists the average carbon dioxide level in the atmosphere, measured in parts per million at Mauna Loa Observatory from 1980 to 2016. Use the data in Table 1 to find a model for the carbon dioxide level.

SOLUTION We use the data in Table 1 to make the scatter plot in Figure 4, where t represents time (in years) and C represents the CO_2 level (in parts per million, ppm).

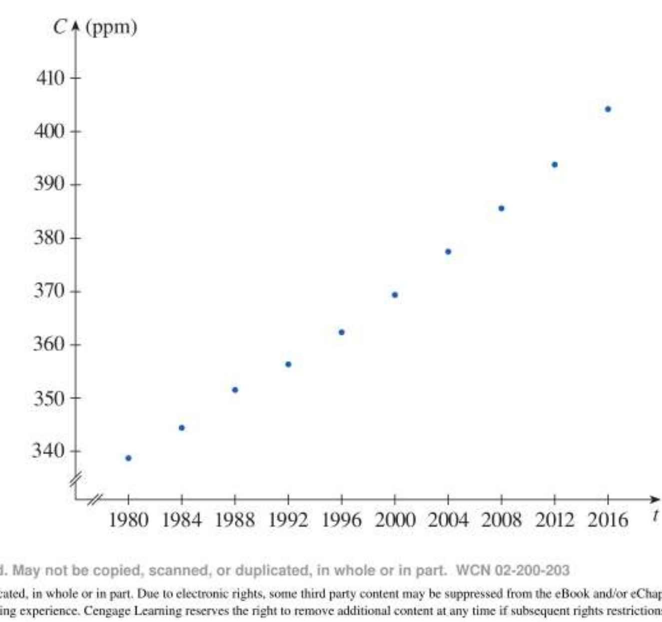


FIGURE 4
Scatter plot for the average CO_2 level

Notice that the data points appear to lie close to a straight line, so it's natural to choose a linear model in this case. But there are many possible lines that approximate these data points, so which one should we use? One possibility is the line that passes through the first and last data points. The slope of this line is

$$\frac{404.2 - 338.7}{2016 - 1980} = \frac{65.5}{36} \approx 1.819$$

We write its equation as

$$C - 338.7 = 1.819(t - 1980)$$

or

$$\boxed{1} \qquad C = 1.819t - 3262.92$$

A computer or graphing calculator finds the regression line by the **method of least squares**, which is to minimize the sum of the squares of the vertical distances between the data points and the line. The details are explained in Exercise 14.7.61.

Equation 1 gives one possible linear model for the carbon dioxide level; it is graphed in Figure 5. Notice that our model gives values higher than most of the actual CO_2 levels. A better linear model is obtained by a procedure from statistics called *linear regression*. Many graphing calculators and computer software applications can determine the regression line for a set of data. One such calculator gives the slope and y-intercept of the regression line for the data from Table 1 as

$$m = 1.78242 \qquad b = -3192.90$$

So our least squares model for the CO_2 level is

$$\boxed{2} \qquad C = 1.78242t - 3192.90$$

In Figure 6 we graph the regression line as well as the data points. Comparing with Figure 5, we see that the regression line gives a better fit.

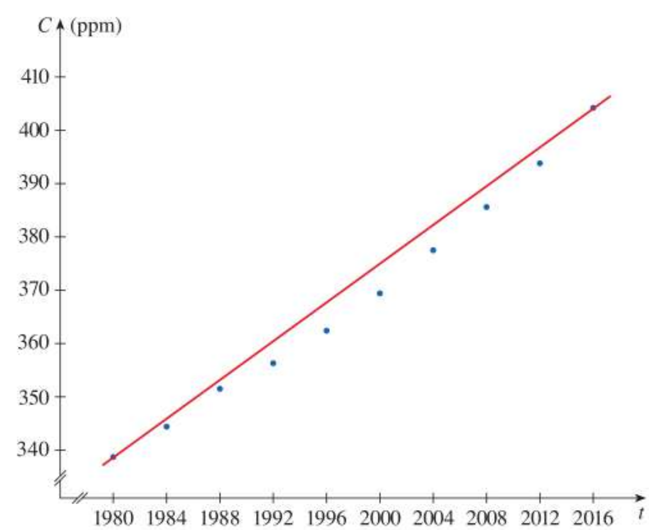


FIGURE 5
Linear model through first and last data points

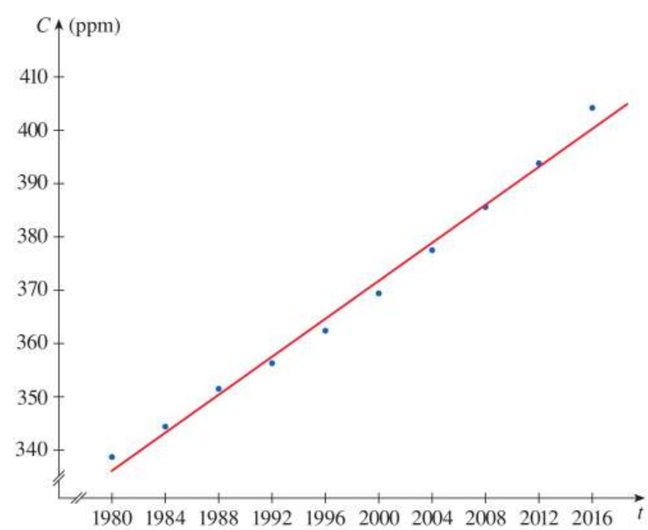


FIGURE 6
The regression line

EXAMPLE 3 Use the linear model given by Equation 2 to estimate the average CO_2 level for 1987 and to predict the level for the year 2025. According to this model, when will the CO_2 level exceed 440 parts per million?

SOLUTION Using Equation 2 with $t = 1987$, we estimate that the average CO_2 level in 1987 was

$$C(1987) = 1.78242(1987) - 3192.90 \approx 348.77$$

This is an example of *interpolation* because we have estimated a value *between* observed values. (In fact, the Mauna Loa Observatory reported that the average CO_2 level in 1987 was 348.93 ppm, so our estimate is quite accurate.)

With $t = 2025$, we get

$$C(2025) = 1.78242(2025) - 3192.90 \approx 416.50$$

So we predict that the average CO_2 level in the year 2025 will be 416.5 ppm. This is an example of *extrapolation* because we have predicted a value *outside* the time frame of observations. Consequently, we are far less certain about the accuracy of our prediction.

Using Equation 2, we see that the CO_2 level exceeds 440 ppm when

$$1.78242t - 3192.90 > 440$$

Solving this inequality, we get

$$t > \frac{3632.9}{1.78242} \approx 2038.18$$

We therefore predict that the CO_2 level will exceed 440 ppm by the year 2038. This prediction is risky because it involves a time quite remote from our observations. In fact, we see from Figure 6 that the trend has been for CO_2 levels to increase rather more rapidly in recent years, so the level might exceed 440 ppm well before 2038. ■

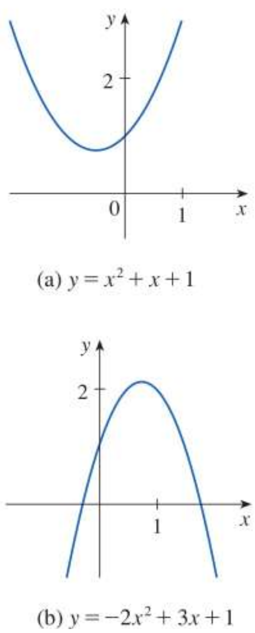


FIGURE 7
The graphs of quadratic functions are parabolas.

Polynomials

A function P is called a **polynomial** if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are constants called the **coefficients** of the polynomial. The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$. If the **leading coefficient** $a_n \neq 0$, then the **degree** of the polynomial is n. For example, the function

$$P(x) = 2x^6 - x^4 + \tfrac{2}{5}x^3 + \sqrt{2}$$

is a polynomial of degree 6.

A polynomial of degree 1 is of the form $P(x) = mx + b$ and so it is a linear function. A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and is called a **quadratic function**. Its graph is always a parabola obtained by shifting the parabola $y = ax^2$, as we will see in Section 1.3. The parabola opens upward if $a > 0$ and downward if $a < 0$. (See Figure 7.)

A polynomial of degree 3 is of the form

$$P(x) = ax^3 + bx^2 + cx + d \qquad a \neq 0$$

and is called a **cubic function**. Figure 8 shows the graph of a cubic function in part (a) and graphs of polynomials of degrees 4 and 5 in parts (b) and (c). We will see later why the graphs have these shapes.

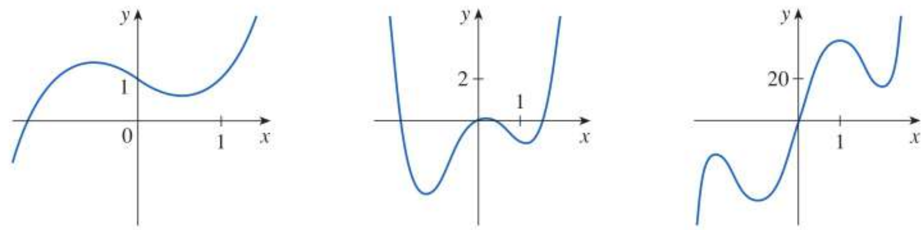


FIGURE 8

(a) $y = x^3 - x + 1$ (b) $y = x^4 - 3x^2 + x$ (c) $y = 3x^5 - 25x^3 + 60x$

Polynomials are commonly used to model various quantities that occur in the natural and social sciences. For instance, in Section 3.7 we will explain why economists often use a polynomial $P(x)$ to represent the cost of producing x units of a commodity. In the following example we use a quadratic function to model the fall of a ball.

Table 2

Time (seconds)	Height (meters)
0	450
1	445
2	431
3	408
4	375
5	332
6	279
7	216
8	143
9	61

EXAMPLE 4 A ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground, and its height h above the ground is recorded at 1-second intervals in Table 2. Find a model to fit the data and use the model to predict the time at which the ball hits the ground.

SOLUTION We draw a scatter plot of the data in Figure 9 and observe that a linear model is inappropriate. But it looks as if the data points might lie on a parabola, so we try a quadratic model instead. Using a graphing calculator or computer algebra system (which uses the least squares method), we obtain the following quadratic model:

$$\boxed{3} \qquad h = 449.36 + 0.96t - 4.90t^2$$

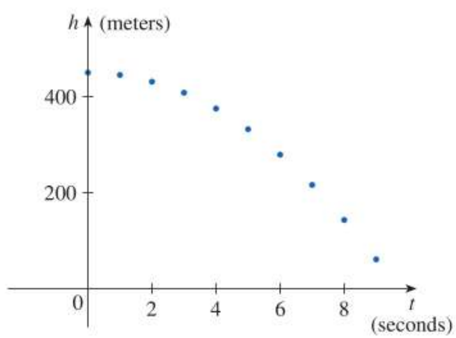


FIGURE 9
Scatter plot for a falling ball

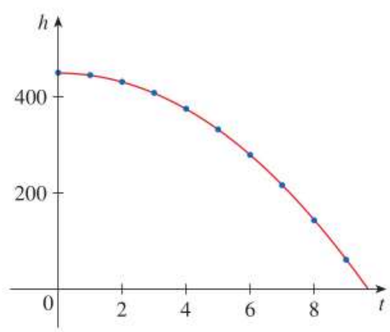


FIGURE 10
Quadratic model for a falling ball

In Figure 10 we plot the graph of Equation 3 together with the data points and see that the quadratic model gives a very good fit.

The ball hits the ground when $h = 0$, so we solve the quadratic equation

$$-4.90t^2 + 0.96t + 449.36 = 0$$

The quadratic formula gives

$$t = \frac{-0.96 \pm \sqrt{(0.96)^2 - 4(-4.90)(449.36)}}{2(-4.90)}$$

The positive root is $t \approx 9.67$, so we predict that the ball will hit the ground after falling about 9.7 seconds. ■

Power Functions

A function of the form $f(x) = x^a$, where a is a constant, is called a **power function**. We consider several cases.

(i) $a = n$, where n is a positive integer

The graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4$, and 5 are shown in Figure 11. (These are polynomials with only one term.) We already know the shape of the graphs of $y = x$ (a line through the origin with slope 1) and $y = x^2$ [a parabola, see Example 1.1.2(b)].

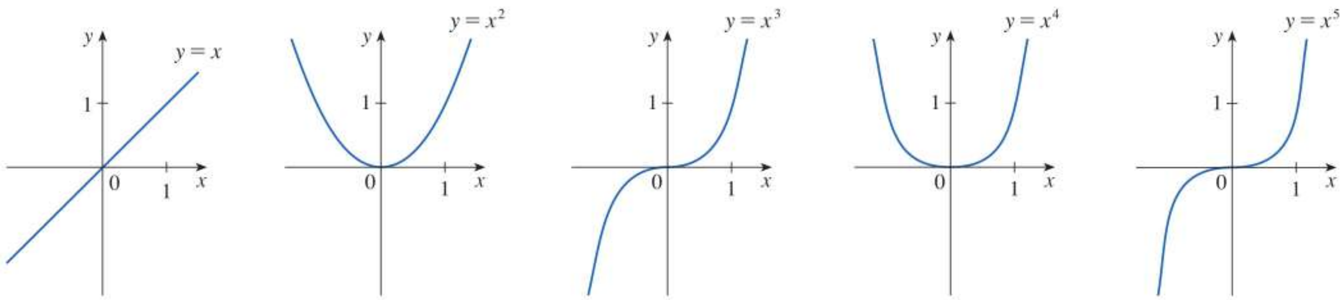


FIGURE 11 Graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4, 5$

The general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd. If n is even, then $f(x) = x^n$ is an even function and its graph is similar to the parabola $y = x^2$. If n is odd, then $f(x) = x^n$ is an odd function and its graph is similar to that of $y = x^3$. Notice from Figure 12, however, that as n increases, the graph of $y = x^n$ becomes flatter near 0 and steeper when $|x| \geqslant 1$. (If x is small, then x^2 is smaller, x^3 is even smaller, x^4 is smaller still, and so on.)

A **family of functions** is a collection of functions whose equations are related. Figure 12 shows two families of power functions, one with even powers and one with odd powers.

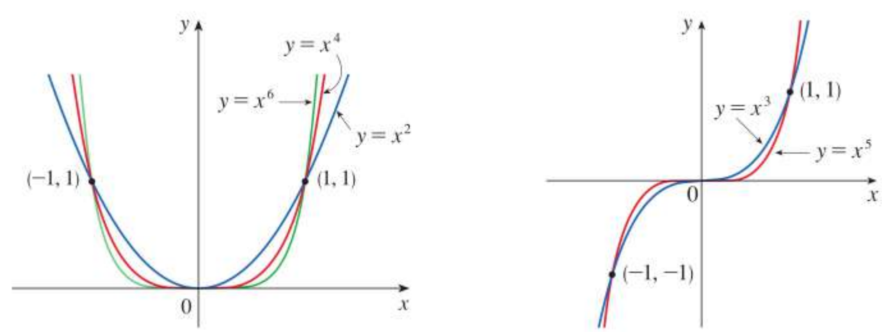
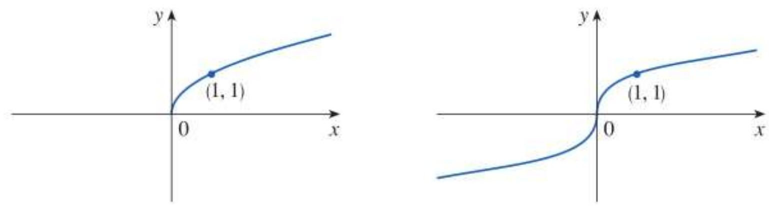


FIGURE 12

(ii) $a = 1/n$, where n is a positive integer

The function $f(x) = x^{1/n} = \sqrt[n]{x}$ is a **root function**. For $n = 2$ it is the square root function $f(x) = \sqrt{x}$, whose domain is $[0, \infty)$ and whose graph is the upper half of the parabola $x = y^2$. [See Figure 13(a).] For other even values of n, the graph of $y = \sqrt[n]{x}$ is similar to that of $y = \sqrt{x}$. For $n = 3$ we have the cube root function $f(x) = \sqrt[3]{x}$ whose domain is $\mathbb{R}$ (recall that every real number has a cube root) and whose graph is shown in Figure 13(b). The graph of $y = \sqrt[n]{x}$ for n odd ($n > 3$) is similar to that of $y = \sqrt[3]{x}$.



(a) $f(x) = \sqrt{x}$

(b) $f(x) = \sqrt[3]{x}$

FIGURE 13
Graphs of root functions

(iii) $a = -1$

The graph of the **reciprocal function** $f(x) = x^{-1} = 1/x$ is shown in Figure 14. Its graph has the equation $y = 1/x$, or $xy = 1$, and is a hyperbola with the coordinate axes as its asymptotes. This function arises in physics and chemistry in connection with Boyle's Law, which says that when the temperature is constant, the volume V of a gas is inversely proportional to the pressure P:

$$V = \frac{C}{P}$$

where C is a constant. Thus the graph of V as a function of P (see Figure 15) has the same general shape as the right half of Figure 14.

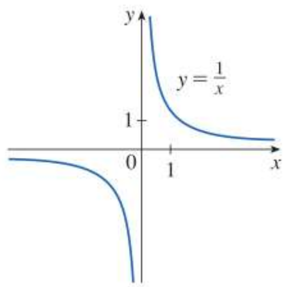


FIGURE 14
The reciprocal function

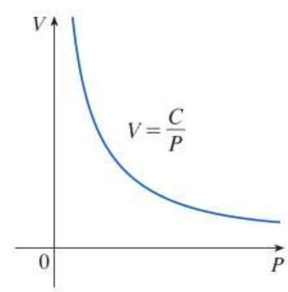


FIGURE 15
Volume as a function of pressure at constant temperature

(iv) $a = -2$

Among the remaining negative powers for the power function $f(x) = x^a$, by far the most important is that of $a = -2$. Many natural laws state that one quantity is inversely proportional to the square of another quantity. In other words, the first quantity is modeled by a function of the form $f(x) = C/x^2$ and we refer to this as an **inverse square law**. For instance, the illumination I of an object by a light source is inversely proportional to the square of the distance x from the source:

$$I = \frac{C}{x^2}$$

where C is a constant. Thus the graph of I as a function of x (see Figure 17) has the same general shape as the right half of Figure 16.

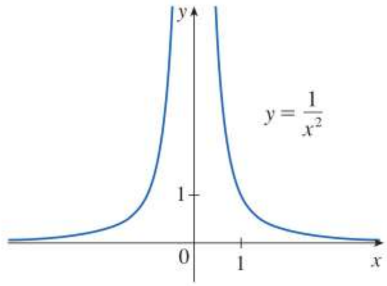


FIGURE 16
The reciprocal of the squaring function

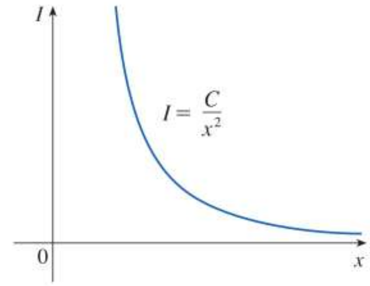


FIGURE 17
Illumination from a light source as a function of distance from the source

Inverse square laws model gravitational force, loudness of sound, and electrostatic force between two charged particles. See Exercise 37 for a geometric reason why inverse square laws often occur in nature.

Power functions are also used to model species-area relationships (Exercises 35–36) and the period of revolution of a planet as a function of its distance from the sun (see Exercise 34).

Rational Functions

A **rational function** f is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. The domain consists of all values of x such that $Q(x) \neq 0$. A simple example of a rational function is the function $f(x) = 1/x$, whose domain is $\{x \mid x \neq 0\}$; this is the reciprocal function graphed in Figure 14. The function

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

is a rational function with domain $\{x \mid x \neq \pm 2\}$. Its graph is shown in Figure 18.

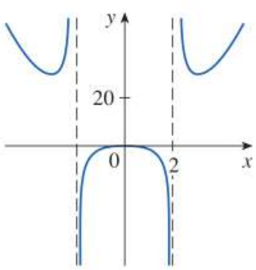


FIGURE 18
$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$

Algebraic Functions

A function f is called an **algebraic function** if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials. Any rational function is automatically an algebraic function. Here are two more examples:

$$f(x) = \sqrt{x^2 + 1} \qquad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x + 1}$$

In Chapter 4 we will sketch a variety of algebraic functions, and we will see that their graphs can assume many different shapes.

An example of an algebraic function occurs in the theory of relativity. The mass of a particle with velocity v is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

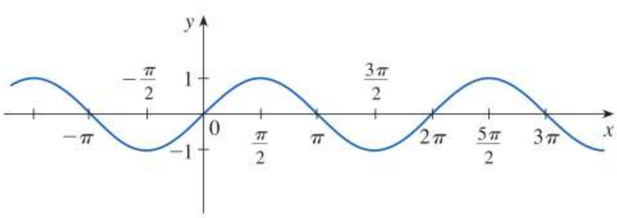
where m_0 is the rest mass of the particle and $c = 3.0 \times 10^5$ km/s is the speed of light in a vacuum.

Functions that are not algebraic are called **transcendental**; these include the trigonometric, exponential, and logarithmic functions.

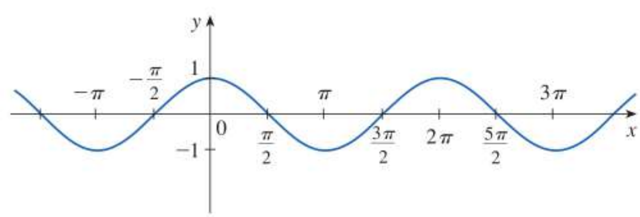
Trigonometric Functions

The Reference Pages are located at the front and back of the book.

Trigonometry and the trigonometric functions are reviewed on Reference Page 2 and also in Appendix D. In calculus the convention is that *radian measure* is always used (except when otherwise indicated). For example, when we use the function $f(x) = \sin x$, it is understood that $\sin x$ means the sine of the angle whose radian measure is x. Thus the graphs of the sine and cosine functions are as shown in Figure 19.



(a) $f(x) = \sin x$



(b) $g(x) = \cos x$

FIGURE 19

Notice that for both the sine and cosine functions the domain is $(-\infty, \infty)$ and the range is the closed interval $[-1, 1]$. Thus, for all values of x, we have

$$\boxed{-1 \leqslant \sin x \leqslant 1 \qquad -1 \leqslant \cos x \leqslant 1}$$

or, in terms of absolute values,

$$|\sin x| \leqslant 1 \qquad |\cos x| \leqslant 1$$

An important property of the sine and cosine functions is that they are periodic functions and have period 2π. This means that, for all values of x,

$$\boxed{\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x}$$

The periodic nature of these functions makes them suitable for modeling repetitive phenomena such as tides, vibrating springs, and sound waves. For instance, in Example 1.3.4 we will see that a reasonable model for the number of hours of daylight in Philadelphia t days after January 1 is given by the function

$$L(t) = 12 + 2.8 \sin\left[\frac{2\pi}{365}(t - 80)\right]$$

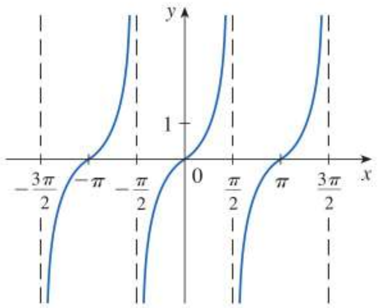


FIGURE 20
$y = \tan x$

EXAMPLE 5 Find the domain of the function $f(x) = \dfrac{1}{1 - 2\cos x}$.

SOLUTION This function is defined for all values of x except for those that make the denominator 0. But

$$1 - 2\cos x = 0 \iff \cos x = \frac{1}{2} \iff x = \frac{\pi}{3} + 2n\pi \quad \text{or} \quad x = \frac{5\pi}{3} + 2n\pi$$

where n is any integer (because the cosine function has period 2π). So the domain of f is the set of all real numbers except for the ones noted above. ■

The tangent function is related to the sine and cosine functions by the equation

$$\tan x = \frac{\sin x}{\cos x}$$

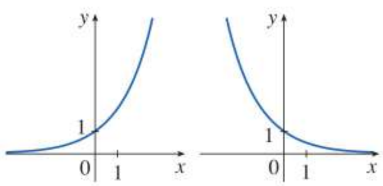
and its graph is shown in Figure 20. It is undefined whenever $\cos x = 0$, that is, when $x = \pm\pi/2, \pm 3\pi/2, \ldots$. Its range is $(-\infty, \infty)$. Notice that the tangent function has period π:

$$\tan(x + \pi) = \tan x \qquad \text{for all } x$$

The remaining three trigonometric functions (cosecant, secant, and cotangent) are the reciprocals of the sine, cosine, and tangent functions. Their graphs are shown in Appendix D.

Exponential Functions

The **exponential functions** are the functions of the form $f(x) = b^x$, where the base b is a positive constant. The graphs of $y = 2^x$ and $y = (0.5)^x$ are shown in Figure 21. In both cases the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.



(a) $y = 2^x$ (b) $y = (0.5)^x$

FIGURE 21

Exponential functions will be studied in detail in Section 1.4, and we will see that they are useful for modeling many natural phenomena, such as when populations grow (if $b > 1$) or decline (if $b < 1$).

Logarithmic Functions

The **logarithmic functions** $f(x) = \log_b x$, where the base b is a positive constant, are the inverse functions of the exponential functions. They will be studied in Section 1.5. Figure 22 shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the function increases slowly when $x > 1$.

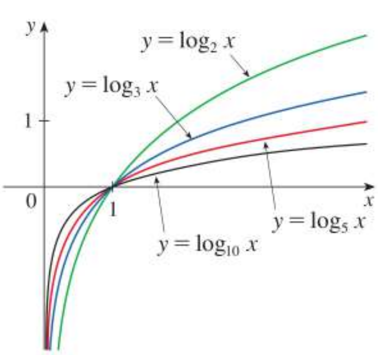


FIGURE 22

EXAMPLE 6 Classify the following functions as one of the types of functions that we have discussed.

(a) $f(x) = 5^x$ (b) $g(x) = x^5$ (c) $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$ (d) $u(t) = 1 - t + 5t^4$

SOLUTION

(a) $f(x) = 5^x$ is an exponential function. (The variable x is the exponent.)

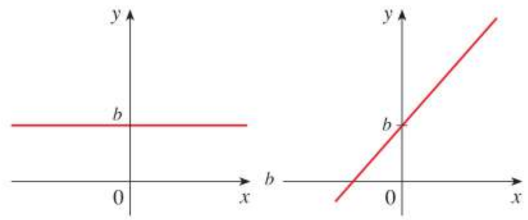
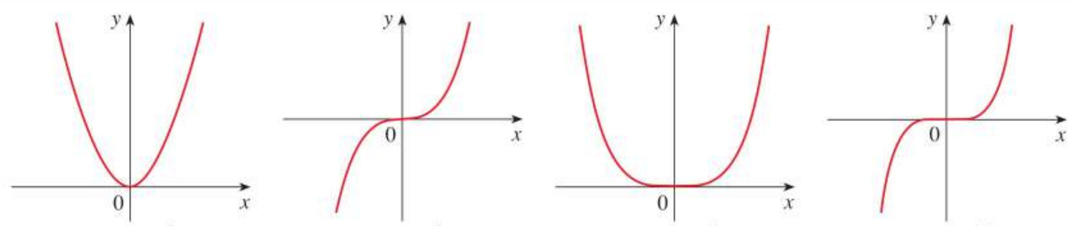
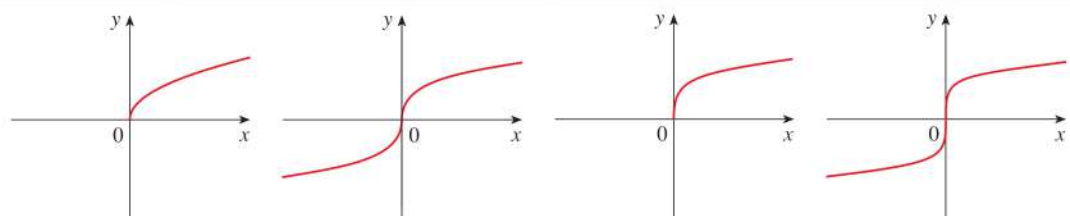
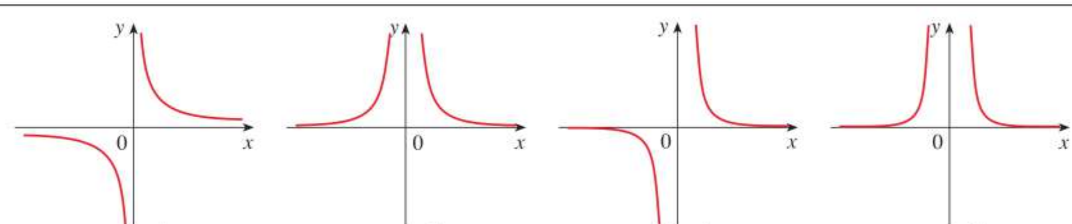
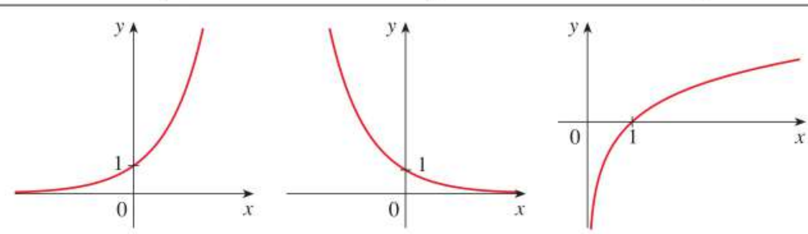
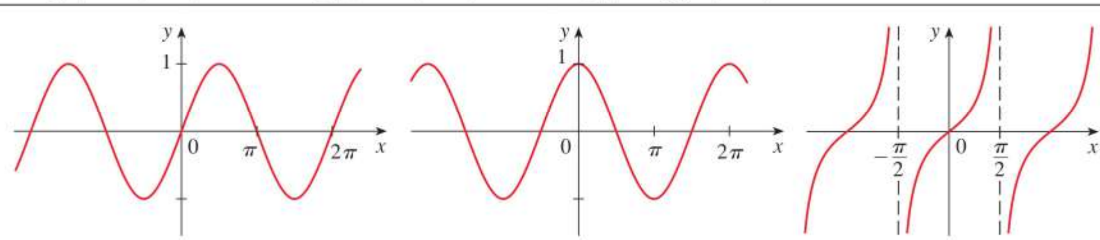
(b) $g(x) = x^5$ is a power function. (The variable x is the base.) We could also consider it to be a polynomial of degree 5.

(c) $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$ is an algebraic function. (It is not a rational function because the denominator is not a polynomial.)

(d) $u(t) = 1 - t + 5t^4$ is a polynomial of degree 4. ■

Table 3 (on the following page) shows a summary of graphs of some families of essential functions that will be used frequently throughout the book.

Table 3 Families of Essential Functions and Their Graphs

Linear Functions $f(x) = mx + b$	 $f(x) = b$ $f(x) = mx + b$
Power Functions $f(x) = x^n$	 $f(x) = x^2$ $f(x) = x^3$ $f(x) = x^4$ $f(x) = x^5$
Root Functions $f(x) = \sqrt[n]{x}$	 $f(x) = \sqrt{x}$ $f(x) = \sqrt[3]{x}$ $f(x) = \sqrt[4]{x}$ $f(x) = \sqrt[5]{x}$
Reciprocal Functions $f(x) = \dfrac{1}{x^n}$	 $f(x) = \dfrac{1}{x}$ $f(x) = \dfrac{1}{x^2}$ $f(x) = \dfrac{1}{x^3}$ $f(x) = \dfrac{1}{x^4}$
Exponential and Logarithmic Functions $f(x) = b^x$ $f(x) = \log_b x$	 $f(x) = b^x \ (b > 1)$ $f(x) = b^x \ (b < 1)$ $f(x) = \log_b x \ (b > 1)$
Trigonometric Functions $f(x) = \sin x$ $f(x) = \cos x$ $f(x) = \tan x$	 $f(x) = \sin x$ $f(x) = \cos x$ $f(x) = \tan x$

1.2 Exercises

1–2 Classify each function as a power function, root function, polynomial (state its degree), rational function, algebraic function, trigonometric function, exponential function, or logarithmic function.

1. (a) $f(x) = x^3 + 3x^2$ (b) $g(t) = \cos^2 t - \sin t$

(c) $r(t) = t^{\sqrt{3}}$ (d) $v(t) = 8^t$

(e) $y = \dfrac{\sqrt{x}}{x^2 + 1}$ (f) $g(u) = \log_{10} u$

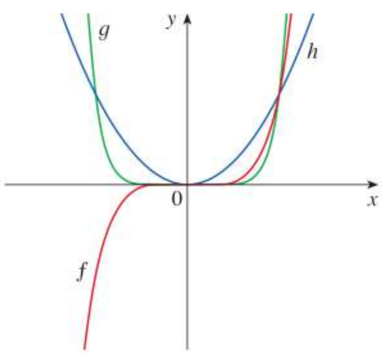
2. (a) $f(t) = \dfrac{3t^2 + 2}{t}$ (b) $h(r) = 2.3^r$

(c) $s(t) = \sqrt{t + 4}$ (d) $y = x^4 + 5$

(e) $g(x) = \sqrt[3]{x}$ (f) $y = \dfrac{1}{x^2}$

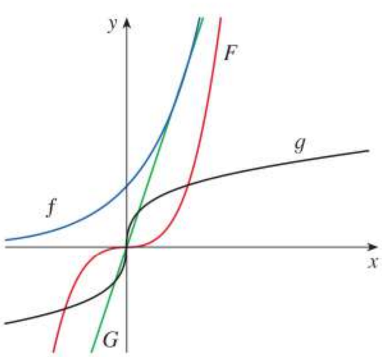
3–4 Match each equation with its graph. Explain your choices. (Don't use a computer or graphing calculator.)

3. (a) $y = x^2$ (b) $y = x^5$ (c) $y = x^8$



4. (a) $y = 3x$ (b) $y = 3^x$

(c) $y = x^3$ (d) $y = \sqrt[3]{x}$



5–6 Find the domain of the function.

5. $f(x) = \dfrac{\cos x}{1 - \sin x}$ **6.** $g(x) = \dfrac{1}{1 - \tan x}$

7. (a) Find an equation for the family of linear functions with slope 2 and sketch several members of the family.
(b) Find an equation for the family of linear functions such that $f(2) = 1$. Sketch several members of the family.
(c) Which function belongs to both families?

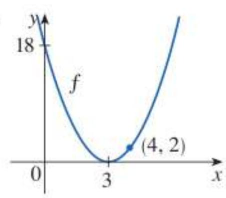
8. What do all members of the family of linear functions $f(x) = 1 + m(x + 3)$ have in common? Sketch several members of the family.

9. What do all members of the family of linear functions $f(x) = c - x$ have in common? Sketch several members of the family.

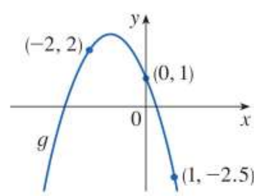
10. Sketch several members of the family of polynomials $P(x) = x^3 - cx^2$. How does the graph change when c changes?

11–12 Find a formula for the quadratic function whose graph is shown.

11.



12.



13. Find a formula for a cubic function f if $f(1) = 6$ and $f(-1) = f(0) = f(2) = 0$.

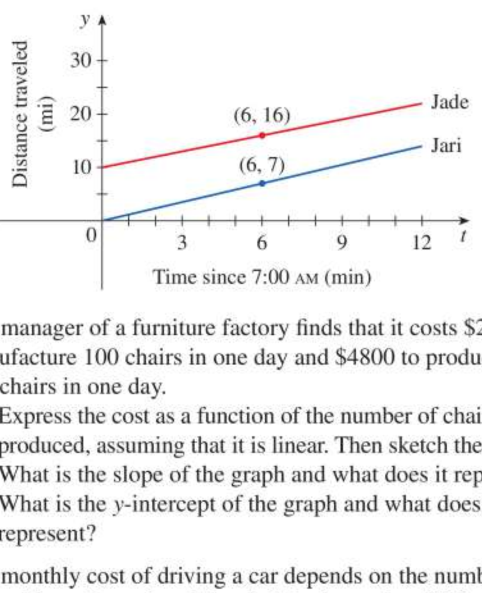
14. Recent studies indicate that the average surface temperature of the earth has been rising steadily. Some scientists have modeled the temperature by the linear function $T = 0.02t + 8.50$, where T is temperature in °C and t represents years since 1900.
(a) What do the slope and T-intercept represent?
(b) Use the equation to predict the earth's average surface temperature in 2100.

15. If the recommended adult dosage for a drug is D (in mg), then to determine the appropriate dosage c for a child of age a, pharmacists use the equation $c = 0.0417D(a + 1)$. Suppose the dosage for an adult is 200 mg.
(a) Find the slope of the graph of c. What does it represent?
(b) What is the dosage for a newborn?

16. The manager of a weekend flea market knows from past experience that if he charges x dollars for a rental space at the market, then the number y of spaces that will be rented is given by the equation $y = 200 - 4x$.
(a) Sketch a graph of this linear function. (Remember that the rental charge per space and the number of spaces rented can't be negative quantities.)
(b) What do the slope, the y-intercept, and the x-intercept of the graph represent?

17. The relationship between the Fahrenheit (F) and Celsius (C) temperature scales is given by the linear function $F = \frac{9}{5}C + 32$.
(a) Sketch a graph of this function.
(b) What is the slope of the graph and what does it represent? What is the F-intercept and what does it represent?

18. Jade and her roommate Jari commute to work each morning, traveling west on I-10. One morning Jade left for work at 6:50 AM, but Jari left 10 minutes later. Both drove at a constant speed. The graphs show the distance (in miles) each of them has traveled on I-10, t minutes after 7:00 AM.
(a) Use the graph to decide which driver is traveling faster.
(b) Find the speed (in mi/h) at which each of them is driving.
(c) Find linear functions f and g that model the distances traveled by Jade and Jari as functions of t (in minutes).



19. The manager of a furniture factory finds that it costs \$2200 to manufacture 100 chairs in one day and \$4800 to produce 300 chairs in one day.
(a) Express the cost as a function of the number of chairs produced, assuming that it is linear. Then sketch the graph.
(b) What is the slope of the graph and what does it represent?
(c) What is the y-intercept of the graph and what does it represent?

20. The monthly cost of driving a car depends on the number of miles driven. Lynn found that in May it cost her \$380 to drive 480 mi and in June it cost her \$460 to drive 800 mi.
(a) Express the monthly cost C as a function of the distance driven d, assuming that a linear relationship gives a suitable model.
(b) Use part (a) to predict the cost of driving 1500 miles per month.
(c) Draw the graph of the linear function. What does the slope represent?
(d) What does the C-intercept represent?
(e) Why does a linear function give a suitable model in this situation?

21. At the surface of the ocean, the water pressure is the same as the air pressure above the water, 15 lb/in^2. Below the surface, the water pressure increases by 4.34 lb/in^2 for every 10 ft of descent.
(a) Express the water pressure as a function of the depth below the ocean surface.
(b) At what depth is the pressure 100 lb/in^2?

22. The resistance R of a wire of fixed length is related to its diameter x by an inverse square law, that is, by a function of the form $R(x) = kx^{-2}$.
(a) A wire of fixed length and 0.005 meters in diameter has a resistance of 140 ohms. Find the value of k.
(b) Find the resistance of a wire made of the same material and of the same length as the wire in part (a) but with a diameter of 0.008 meters.

23. The illumination of an object by a light source is related to the distance from the source by an inverse square law. Suppose that after dark you are sitting in a room with just one lamp, trying to read a book. The light is too dim, so you move your chair halfway to the lamp. How much brighter is the light?

24. The pressure P of a sample of oxygen gas that is compressed at a constant temperature is related to the volume V of gas by a reciprocal function of the form $P = k/V$.
(a) A sample of oxygen gas that occupies 0.671 m^3 exerts a pressure of 39 kPa at a temperature of 293 K (absolute temperature measured on the Kelvin scale). Find the value of k in the given model.
(b) If the sample expands to a volume of 0.916 m^3, find the new pressure.

25. The power output of a wind turbine depends on many factors. It can be shown using physical principles that the power P generated by a wind turbine is modeled by

$$P = kAv^3$$

where v is the wind speed, A is the area swept out by the blades, and k is a constant that depends on air density, efficiency of the turbine, and the design of the wind turbine blades.
(a) If only wind speed is doubled, by what factor is the power output increased?
(b) If only the length of the blades is doubled, by what factor is the power output increased?
(c) For a particular wind turbine, the length of the blades is 30 m and $k = 0.214$ kg/m^3. Find the power output (in watts, $\mathrm{W} = \mathrm{m}^2 \cdot \mathrm{kg/s}^3$) when the wind speed is 10 m/s, 15 m/s, and 25 m/s.

26. Astronomers infer the radiant exitance (radiant flux emitted per unit area) of stars using the Stefan Boltzmann Law:

$$E(T) = (5.67 \times 10^{-8})T^4$$

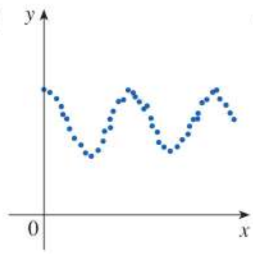
where E is the energy radiated per unit of surface area measured in watts (W) and T is the absolute temperature measured in kelvins (K).

(a) Graph the function E for temperatures T between 100 K and 300 K.

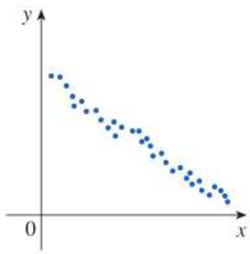
(b) Use the graph to describe the change in energy E as the temperature T increases.

27–28 For each scatter plot, decide what type of function you might choose as a model for the data. Explain your choices.

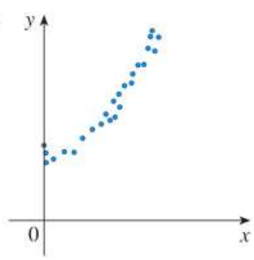
27. (a)



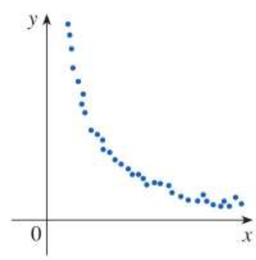
(b)



28. (a)



(b)



29. The table shows (lifetime) peptic ulcer rates (per 100 population) for various family incomes as reported by the National Health Interview Survey.

(a) Make a scatter plot of these data and decide whether a linear model is appropriate.

(b) Find and graph a linear model using the first and last data points.

(c) Find and graph the regression line.

(d) Use the linear model in part (c) to estimate the ulcer rate for people with an income of \$25,000.

(e) According to the model, how likely is someone with an income of \$80,000 to suffer from peptic ulcers?

(f) Do you think it would be reasonable to apply the model to someone with an income of \$200,000?

Income	Ulcer rate (per 100 population)
\$4,000	14.1
\$6,000	13.0
\$8,000	13.4
\$12,000	12.5
\$16,000	12.0
\$20,000	12.4
\$30,000	10.5
\$45,000	9.4
\$60,000	8.2

30. When laboratory rats are exposed to asbestos fibers, some of them develop lung tumors. The table lists the results of several experiments by different scientists.

(a) Find the regression line for the data.

(b) Make a scatter plot and graph the regression line. Does the regression line appear to be a suitable model for the data?

(c) What does the y-intercept of the regression line represent?

Asbestos exposure (fibers/mL)	Percent of mice that develop lung tumors	Asbestos exposure (fibers/mL)	Percent of mice that develop lung tumors
50	2	1600	42
400	6	1800	37
500	5	2000	38
900	10	3000	50
1100	26		

31. Anthropologists use a linear model that relates human femur (thighbone) length to height. The model allows an anthropologist to determine the height of an individual when only a partial skeleton (including the femur) is found. Here we find the model by analyzing the data on femur length and height for the eight males given in the table.

(a) Make a scatter plot of the data.

(b) Find and graph the regression line that models the data.

(c) An anthropologist finds a human femur of length 53 cm. How tall was the person?

Femur length (cm)	Height (cm)	Femur length (cm)	Height (cm)
50.1	178.5	44.5	168.3
48.3	173.6	42.7	165.0
45.2	164.8	39.5	155.4
44.7	163.7	38.0	155.8

32. The table shows average US retail residential prices of electricity from 2000 to 2016, measured in cents per kilowatt hour.

(a) Make a scatter plot. Is a linear model appropriate?

(b) Find and graph the regression line.

(c) Use your linear model from part (b) to estimate the average retail price of electricity in 2005 and 2017.

Years since 2000	Cents/kWh	Years since 2000	Cents/kWh
0	8.24	10	11.54
2	8.44	12	11.88
4	8.95	14	12.52
6	10.40	16	12.90
8	11.26		

Source: US Energy Information Administration

T **33.** The table shows world average daily oil consumption from 1985 to 2015, measured in thousands of barrels per day.

(a) Make a scatter plot and decide whether a linear model is appropriate.

(b) Find and graph the regression line.

(c) Use the linear model to estimate the oil consumption in 2002 and 2017.

Years since 1985	Thousands of barrels of oil per day
0	60,083
5	66,533
10	70,099
15	76,784
20	84,077
25	87,302
30	94,071

Source: US Energy Information Administration

T **34.** The table shows the mean (average) distances d of the planets from the sun (taking the unit of measurement to be the distance from the earth to the sun) and their periods T (time of revolution in years).

(a) Fit a power model to the data.

(b) Kepler's Third Law of Planetary Motion states that "The square of the period of revolution of a planet is proportional to the cube of its mean distance from the sun." Does your model corroborate Kepler's Third Law?

Planet	d	T
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.523	1.881
Jupiter	5.203	11.861
Saturn	9.541	29.457
Uranus	19.190	84.008
Neptune	30.086	164.784

35. It makes sense that the larger the area of a region, the larger the number of species that inhabit the region. Many ecologists have modeled the species-area relation with a power function. In particular, the number of species S of bats living in caves in central Mexico has been related to the surface area A of the caves by the equation $S = 0.7A^{0.3}$.

(a) The cave called *Misión Imposible* near Puebla, Mexico, has a surface area of $A = 60\text{ m}^2$. How many species of bats would you expect to find in that cave?

(b) If you discover that four species of bats live in a cave, estimate the area of the cave.

T **36.** The table shows the number N of species of reptiles and amphibians inhabiting Caribbean islands and the area A of the island in square miles.

(a) Use a power function to model N as a function of A.

(b) The Caribbean island of Dominica has area 291 mi^2. How many species of reptiles and amphibians would you expect to find on Dominica?

Island	A	N
Saba	4	5
Monserrat	40	9
Puerto Rico	3,459	40
Jamaica	4,411	39
Hispaniola	29,418	84
Cuba	44,218	76

37. Suppose that a force or energy originates from a point source and spreads its influence equally in all directions, such as the light from a lightbulb or the gravitational force of a planet. So at a distance r from the source, the intensity I of the force or energy is equal to the source strength S divided by the surface area of a sphere of radius r. Show that I satisfies the inverse square law $I = k/r^2$, where k is a positive constant.

1.3 New Functions from Old Functions

In this section we start with the basic functions we discussed in Section 1.2 and obtain new functions by shifting, stretching, and reflecting their graphs. We also show how to combine pairs of functions by the standard arithmetic operations and by composition.

Transformations of Functions

By applying certain transformations to the graph of a given function we can obtain the graphs of related functions. This will give us the ability to sketch the graphs of many functions quickly by hand. It will also enable us to write equations for given graphs.

Let's first consider **translations** of graphs. If c is a positive number, then the graph of $y = f(x) + c$ is just the graph of $y = f(x)$ shifted upward a distance of c units (because each y-coordinate is increased by the same number c). Likewise, if $g(x) = f(x - c)$, where $c > 0$, then the value of g at x is the same as the value of f at $x - c$ (c units to the left of x). Therefore the graph of $y = f(x - c)$ is just the graph of $y = f(x)$ shifted c units to the right (see Figure 1).

Vertical and Horizontal Shifts Suppose $c > 0$. To obtain the graph of

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward
$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward
$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right
$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

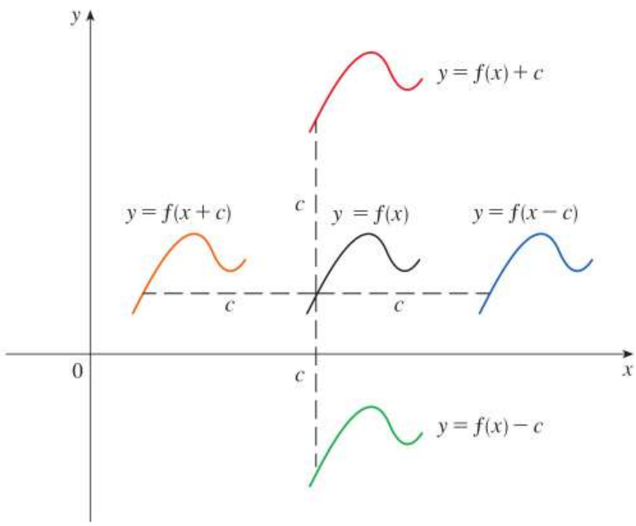


FIGURE 1 Translating the graph of f

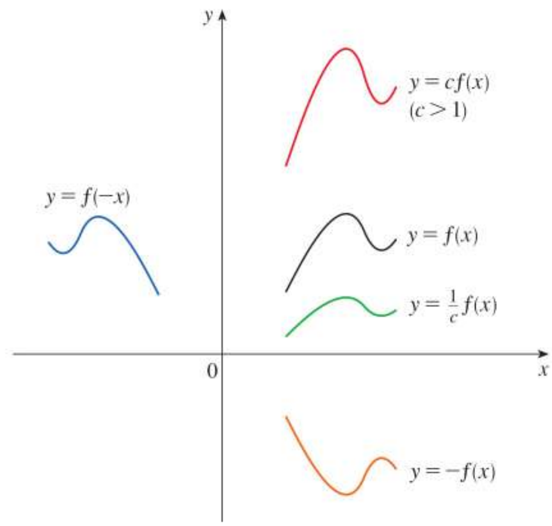


FIGURE 2 Stretching and reflecting the graph of f

Now let's consider the **stretching** and **reflecting** transformations. If $c > 1$, then the graph of $y = cf(x)$ is the graph of $y = f(x)$ stretched by a factor of c in the vertical direction (because each y-coordinate is multiplied by the same number c). The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x-axis because the point (x, y) is replaced by the point $(x, -y)$. (See Figure 2 and the following chart, where the results of other stretching, shrinking, and reflecting transformations are also given.)

Vertical and Horizontal Stretching and Reflecting Suppose $c > 1$. To obtain the graph of

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c
$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c
$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c
$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c
$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis
$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis

Figure 3 illustrates these stretching transformations when applied to the cosine function with $c = 2$. For instance, in order to get the graph of $y = 2\cos x$ we multiply the y-coordinate of each point on the graph of $y = \cos x$ by 2. This means that the graph of $y = \cos x$ gets stretched vertically by a factor of 2.

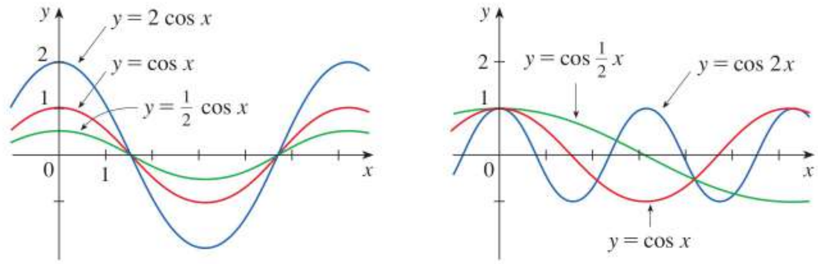
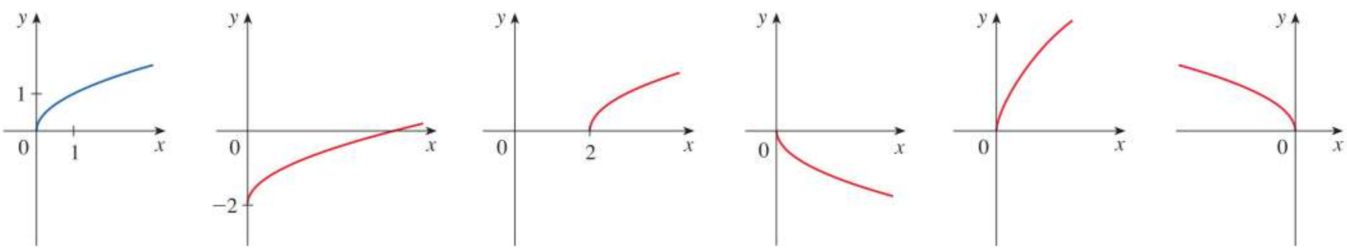


FIGURE 3

EXAMPLE 1 Given the graph of $y = \sqrt{x}$, use transformations to graph $y = \sqrt{x} - 2$, $y = \sqrt{x-2}$, $y = -\sqrt{x}$, $y = 2\sqrt{x}$, and $y = \sqrt{-x}$.

SOLUTION The graph of the square root function $y = \sqrt{x}$, obtained from Figure 1.2.13(a), is shown in Figure 4(a). In the other parts of the figure we sketch $y = \sqrt{x} - 2$ by shifting 2 units downward, $y = \sqrt{x-2}$ by shifting 2 units to the right, $y = -\sqrt{x}$ by reflecting about the x-axis, $y = 2\sqrt{x}$ by stretching vertically by a factor of 2, and $y = \sqrt{-x}$ by reflecting about the y-axis.



(a) $y = \sqrt{x}$ (b) $y = \sqrt{x} - 2$ (c) $y = \sqrt{x-2}$ (d) $y = -\sqrt{x}$ (e) $y = 2\sqrt{x}$ (f) $y = \sqrt{-x}$

FIGURE 4

■

EXAMPLE 2 Sketch the graph of the function $f(x) = x^2 + 6x + 10$.

SOLUTION Completing the square, we write the equation of the graph as

$$y = x^2 + 6x + 10 = (x + 3)^2 + 1$$

This means we obtain the desired graph by starting with the parabola $y = x^2$ and shifting 3 units to the left and then 1 unit upward (see Figure 5).

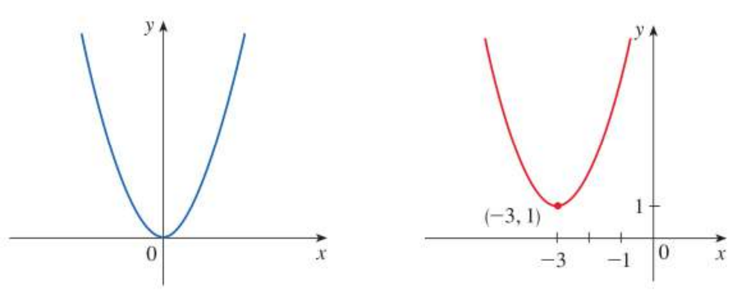


FIGURE 5 (a) $y = x^2$ (b) $y = (x + 3)^2 + 1$

■

EXAMPLE 3 Sketch the graph of each function.

(a) $y = \sin 2x$ (b) $y = 1 - \sin x$

SOLUTION

(a) We obtain the graph of $y = \sin 2x$ from that of $y = \sin x$ by compressing horizontally by a factor of 2. (See Figures 6 and 7.) Because the period of $y = \sin x$ is 2π, the period of $y = \sin 2x$ is $2\pi/2 = \pi$.

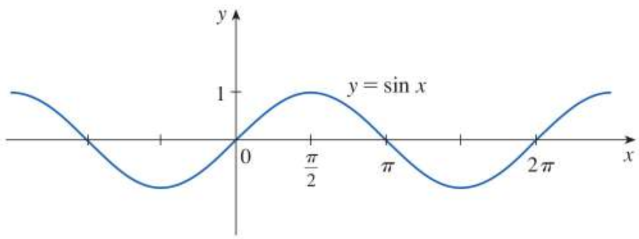


FIGURE 6

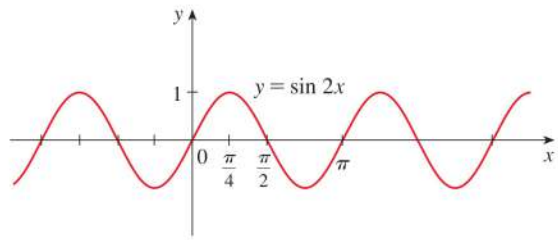


FIGURE 7

(b) To obtain the graph of $y = 1 - \sin x$, we again start with $y = \sin x$. We reflect about the x-axis to get the graph of $y = -\sin x$ and then we shift 1 unit upward to get $y = 1 - \sin x$. (See Figure 8.)

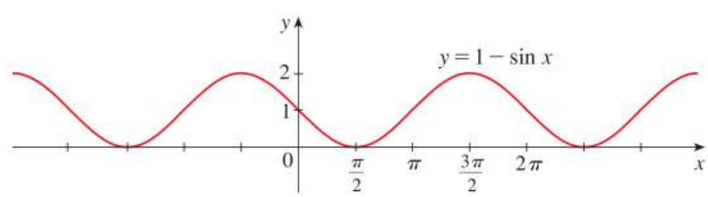


FIGURE 8

■

EXAMPLE 4 Figure 9 shows graphs of the number of hours of daylight as functions of the time of the year at several latitudes. Given that Philadelphia is located at approximately 40°N latitude, find a function that models the length of daylight at Philadelphia.

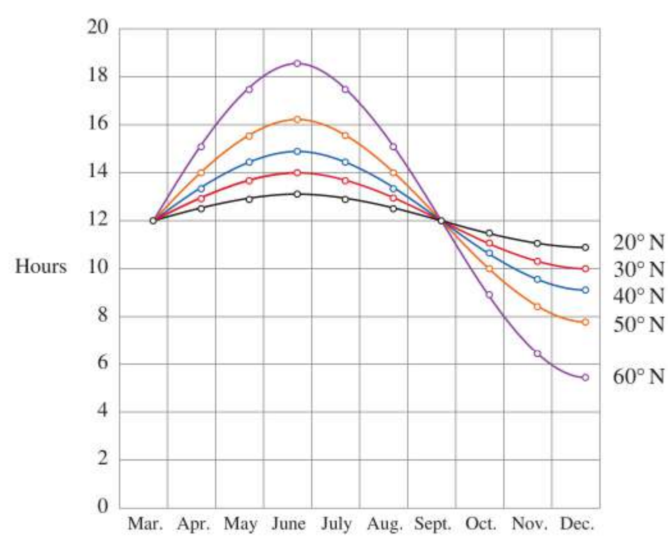


FIGURE 9
Graph of the length of daylight from March 21 through December 21 at various latitudes

Source: Adapted from L. Harrison, *Daylight, Twilight, Darkness and Time* (New York: Silver, Burdett, 1935), 40.

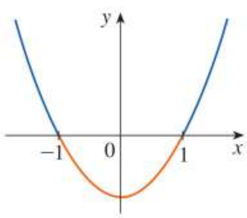
SOLUTION Notice that each curve resembles a shifted and stretched sine function. By looking at the blue curve we see that, at the latitude of Philadelphia, daylight lasts about 14.8 hours on June 21 and 9.2 hours on December 21, so the amplitude of the curve (the factor by which we have to stretch the sine curve vertically) is $\frac{1}{2}(14.8 - 9.2) = 2.8$.

By what factor do we need to stretch the sine curve horizontally if we measure the time t in days? Because there are about 365 days in a year, the period of our model should be 365. But the period of $y = \sin t$ is 2π, so the horizontal stretching factor is $2\pi/365$.

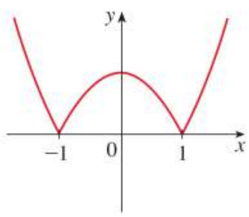
We also notice that the curve begins its cycle on March 21, the 80th day of the year, so we have to shift the curve 80 units to the right. In addition, we shift it 12 units upward. Therefore we model the length of daylight in Philadelphia on the tth day of the year by the function

$$L(t) = 12 + 2.8 \sin\left[\frac{2\pi}{365}(t - 80)\right]$$

Another transformation of some interest is taking the *absolute value* of a function. If $y = |f(x)|$, then according to the definition of absolute value, $y = f(x)$ when $f(x) \geqslant 0$ and $y = -f(x)$ when $f(x) < 0$. This tells us how to get the graph of $y = |f(x)|$ from the graph of $y = f(x)$: the part of the graph that lies above the x-axis remains the same, and the part that lies below the x-axis is reflected about the x-axis.



(a) $y = x^2 - 1$



(b) $y = |x^2 - 1|$

FIGURE 10

EXAMPLE 5 Sketch the graph of the function $y = |x^2 - 1|$.

SOLUTION We first graph the parabola $y = x^2 - 1$ in Figure 10(a) by shifting the parabola $y = x^2$ downward 1 unit. We see that the graph lies below the x-axis when $-1 < x < 1$, so we reflect that part of the graph about the x-axis to obtain the graph of $y = |x^2 - 1|$ in Figure 10(b).

Combinations of Functions

Two functions f and g can be combined to form new functions $f + g$, $f - g$, fg, and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers.

Definition Given two functions f and g, the **sum**, **difference**, **product**, and **quotient** functions are defined by

$$(f + g)(x) = f(x) + g(x) \qquad (f - g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)\,g(x) \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

If the domain of f is A and the domain of g is B, then the domain of $f + g$ (and the domain of $f - g$) is the intersection $A \cap B$ because both $f(x)$ and $g(x)$ have to be defined. For example, the domain of $f(x) = \sqrt{x}$ is $A = [0, \infty)$ and the domain of $g(x) = \sqrt{2 - x}$ is $B = (-\infty, 2]$, so the domain of $(f + g)(x) = \sqrt{x} + \sqrt{2 - x}$ is $A \cap B = [0, 2]$.

The domain of fg is also $A \cap B$. Because we can't divide by 0, the domain of f/g is $\{x \in A \cap B \mid g(x) \neq 0\}$. For instance, if $f(x) = x^2$ and $g(x) = x - 1$, then the domain of the rational function $(f/g)(x) = x^2/(x - 1)$ is $\{x \mid x \neq 1\}$, or $(-\infty, 1) \cup (1, \infty)$.

There is another way of combining two functions to obtain a new function. For example, suppose that $y = f(u) = \sqrt{u}$ and $u = g(x) = x^2 + 1$. Since y is a function of u and u is, in turn, a function of x, it follows that y is ultimately a function of x. We compute this by substitution:

$$y = f(u) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

The procedure is called *composition* because the new function is *composed* of the two given functions f and g.

In general, given any two functions f and g, we start with a number x in the domain of g and calculate $g(x)$. If this number $g(x)$ is in the domain of f, then we can calculate the value of $f(g(x))$. Notice that the output of one function is used as the input to the next function. The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f. It is called the *composition* (or *composite*) of f and g and is denoted by $f \circ g$ ("f circle g").

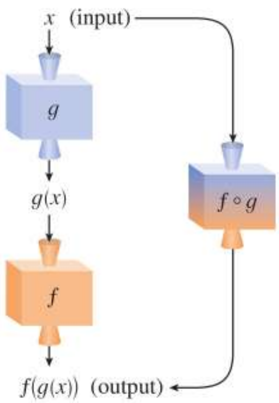


FIGURE 11
The $f \circ g$ machine is composed of the g machine (first) and then the f machine.

Definition Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined. Figure 11 shows how to picture $f \circ g$ in terms of machines.

EXAMPLE 6 If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

SOLUTION We have

$$(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)^2$$
$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 3$$

■

NOTE You can see from Example 6 that, in general, $f \circ g \neq g \circ f$. Remember, the notation $f \circ g$ means that the function g is applied first and then f is applied second. In Example 6, $f \circ g$ is the function that *first* subtracts 3 and *then* squares; $g \circ f$ is the function that *first* squares and *then* subtracts 3.

EXAMPLE 7 If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2 - x}$, find each function and its domain.

(a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$

SOLUTION

(a) $$(f \circ g)(x) = f(g(x)) = f\left(\sqrt{2 - x}\right) = \sqrt{\sqrt{2 - x}} = \sqrt[4]{2 - x}$$

The domain of $f \circ g$ is $\{x \mid 2 - x \geqslant 0\} = \{x \mid x \leqslant 2\} = (-\infty, 2]$.

(b) $$(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x}\right) = \sqrt{2 - \sqrt{x}}$$

If $0 \leqslant a \leqslant b$, then $a^2 \leqslant b^2$.

For $\sqrt{x}$ to be defined we must have $x \geqslant 0$. For $\sqrt{2 - \sqrt{x}}$ to be defined we must have $2 - \sqrt{x} \geqslant 0$, that is, $\sqrt{x} \leqslant 2$, or $x \leqslant 4$. Thus we have $0 \leqslant x \leqslant 4$, so the domain of $g \circ f$ is the closed interval $[0, 4]$.

(c) $$(f \circ f)(x) = f(f(x)) = f\left(\sqrt{x}\right) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$$

The domain of $f \circ f$ is $[0, \infty)$.

(d) $$(g \circ g)(x) = g(g(x)) = g\left(\sqrt{2-x}\right) = \sqrt{2-\sqrt{2-x}}$$

This expression is defined when both $2 - x \geqslant 0$ and $2 - \sqrt{2-x} \geqslant 0$. The first inequality means $x \leqslant 2$, and the second is equivalent to $\sqrt{2-x} \leqslant 2$, or $2 - x \leqslant 4$, or $x \geqslant -2$. Thus $-2 \leqslant x \leqslant 2$, so the domain of $g \circ g$ is the closed interval $[-2, 2]$. ■

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h, then g, and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

EXAMPLE 8 Find $f \circ g \circ h$ if $f(x) = x/(x+1)$, $g(x) = x^{10}$, and $h(x) = x + 3$.

SOLUTION

$$\begin{aligned}(f \circ g \circ h)(x) &= f(g(h(x))) = f(g(x+3)) \\ &= f((x+3)^{10}) = \frac{(x+3)^{10}}{(x+3)^{10}+1}\end{aligned}$$ ■

So far we have used composition to build complicated functions from simpler ones. But in calculus it is often useful to be able to *decompose* a complicated function into simpler ones, as in the following example.

EXAMPLE 9 Given $F(x) = \cos^2(x+9)$, find functions f, g, and h such that $F = f \circ g \circ h$.

SOLUTION Since $F(x) = [\cos(x+9)]^2$, the formula for F says: first add 9, then take the cosine of the result, and finally square. So we let

$$h(x) = x + 9 \qquad g(x) = \cos x \qquad f(x) = x^2$$

Then $$\begin{aligned}(f \circ g \circ h)(x) &= f(g(h(x))) = f(g(x+9)) = f(\cos(x+9)) \\ &= [\cos(x+9)]^2 = F(x)\end{aligned}$$ ■

1.3 Exercises

1. Suppose the graph of f is given. Write equations for the graphs that are obtained from the graph of f as follows.

(a) Shift 3 units upward.

(b) Shift 3 units downward.

(c) Shift 3 units to the right.

(d) Shift 3 units to the left.

(e) Reflect about the x-axis.

(f) Reflect about the y-axis.

(g) Stretch vertically by a factor of 3.

(h) Shrink vertically by a factor of 3.

2. Explain how each graph is obtained from the graph of $y = f(x)$.

(a) $y = f(x) + 8$ (b) $y = f(x + 8)$

(c) $y = 8f(x)$ (d) $y = f(8x)$

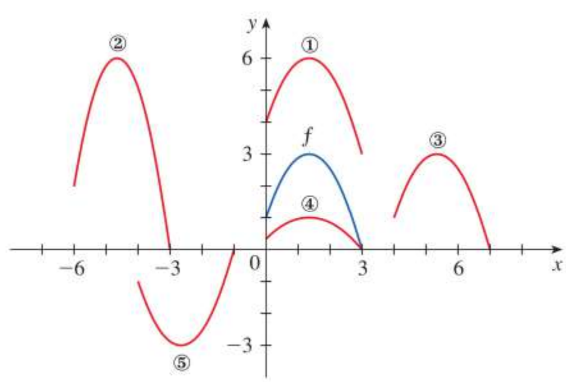
(e) $y = -f(x) - 1$ (f) $y = 8f\left(\frac{1}{8}x\right)$

3. The graph of $y = f(x)$ is given. Match each equation with its graph and give reasons for your choices.

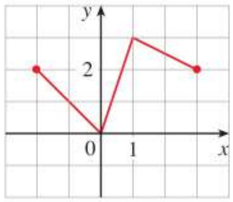
(a) $y = f(x - 4)$ (b) $y = f(x) + 3$

(c) $y = \frac{1}{3}f(x)$ (d) $y = -f(x + 4)$

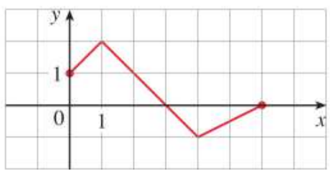
(e) $y = 2f(x + 6)$



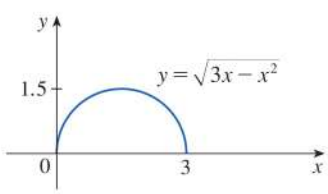
4. The graph of f is given. Draw the graphs of the following functions.
(a) $y = f(x) - 3$ (b) $y = f(x + 1)$
(c) $y = \frac{1}{2}f(x)$ (d) $y = -f(x)$



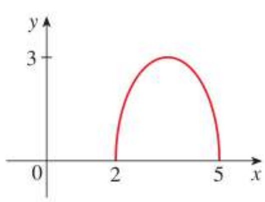
5. The graph of f is given. Use it to graph the following functions.
(a) $y = f(2x)$ (b) $y = f(\frac{1}{2}x)$
(c) $y = f(-x)$ (d) $y = -f(-x)$



6–7 The graph of $y = \sqrt{3x - x^2}$ is given. Use transformations to create a function whose graph is as shown.



6.



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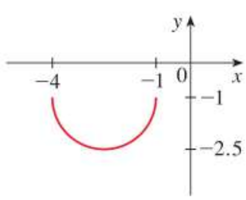


8. (a) How is the graph of $y = 1 + \sqrt{x}$ related to the graph of $y = \sqrt{x}$? Use your answer and Figure 4(a) to sketch the graph of $y = 1 + \sqrt{x}$.
(b) How is the graph of $y = 5 \sin \pi x$ related to the graph of $y = \sin x$? Use your answer and Figure 6 to sketch the graph of $y = 5 \sin \pi x$.

9–26 Graph the function by hand, not by plotting points, but by starting with the graph of one of the standard functions given in Table 1.2.3, and then applying the appropriate transformations.

9. $y = 1 + x^2$ **10.** $y = (x + 1)^2$

11. $y = |x + 2|$ **12.** $y = 1 - x^3$

13. $y = \dfrac{1}{x} + 2$ **14.** $y = -\sqrt{x} - 1$

15. $y = \sin 4x$ **16.** $y = 1 + \dfrac{1}{x^2}$

17. $y = 2 + \sqrt{x + 1}$ **18.** $y = -(x - 1)^2 + 3$

19. $y = x^2 - 2x + 5$ **20.** $y = (x + 1)^3 + 2$

21. $y = 2 - |x|$ **22.** $y = 2 - 2 \cos x$

23. $y = 3 \sin \frac{1}{2}x + 1$ **24.** $y = \dfrac{1}{4}\tan\left(x - \dfrac{\pi}{4}\right)$

25. $y = |\cos \pi x|$ **26.** $y = |\sqrt{x} - 1|$

27. The city of New Orleans is located at latitude 30°N. Use Figure 9 to find a function that models the number of hours of daylight at New Orleans as a function of the time of year. To check the accuracy of your model, use the fact that on March 31 the sun rises at 5:51 AM and sets at 6:18 PM in New Orleans.

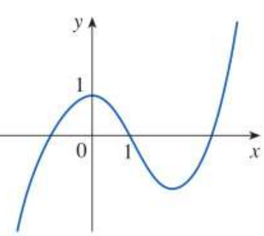
28. A variable star is one whose brightness alternately increases and decreases. For the most visible variable star, Delta Cephei, the time between periods of maximum brightness is 5.4 days, the average brightness (or magnitude) of the star is 4.0, and its brightness varies by ± 0.35 magnitude. Find a function that models the brightness of Delta Cephei as a function of time.

29. Some of the highest tides in the world occur in the Bay of Fundy on the Atlantic Coast of Canada. At Hopewell Cape the water depth at low tide is about 2.0 m and at high tide it is about 12.0 m. The natural period of oscillation is about 12 hours and on a particular day, high tide occurred at 6:45 AM. Find a function involving the cosine function that models the water depth $D(t)$ (in meters) as a function of time t (in hours after midnight) on that day.

30. In a normal respiratory cycle the volume of air that moves into and out of the lungs is about 500 mL. The reserve and residue volumes of air that remain in the lungs occupy about 2000 mL and a single respiratory cycle for an average human takes about 4 seconds. Find a model for the total volume of air $V(t)$ in the lungs as a function of time.

31. (a) How is the graph of $y = f(|x|)$ related to the graph of f?
(b) Sketch the graph of $y = \sin |x|$.
(c) Sketch the graph of $y = \sqrt{|x|}$.

32. Use the given graph of f to sketch the graph of $y = 1/f(x)$. Which features of f are the most important in sketching $y = 1/f(x)$? Explain how they are used.



33–34 Find (a) $f + g$, (b) $f - g$, (c) fg, and (d) f/g and state their domains.

33. $f(x) = \sqrt{25 - x^2}, \quad g(x) = \sqrt{x + 1}$

34. $f(x) = \dfrac{1}{x - 1}, \quad g(x) = \dfrac{1}{x} - 2$

35–40 Find the functions (a) $f \circ g$, (b) $g \circ f$, (c) $f \circ f$, and (d) $g \circ g$ and their domains.

35. $f(x) = x^3 + 5, \quad g(x) = \sqrt[3]{x}$

36. $f(x) = \dfrac{1}{x}, \quad g(x) = 2x + 1$

37. $f(x) = \dfrac{1}{\sqrt{x}}, \quad g(x) = x + 1$

38. $f(x) = \dfrac{x}{x + 1}, \quad g(x) = 2x - 1$

39. $f(x) = \dfrac{2}{x}, \quad g(x) = \sin x$

40. $f(x) = \sqrt{5 - x}, \quad g(x) = \sqrt{x - 1}$

41–44 Find $f \circ g \circ h$.

41. $f(x) = 3x - 2, \quad g(x) = \sin x, \quad h(x) = x^2$

42. $f(x) = |x - 4|, \quad g(x) = 2^x, \quad h(x) = \sqrt{x}$

43. $f(x) = \sqrt{x - 3}, \quad g(x) = x^2, \quad h(x) = x^3 + 2$

44. $f(x) = \tan x, \quad g(x) = \dfrac{x}{x - 1}, \quad h(x) = \sqrt[3]{x}$

45–50 Express the function in the form $f \circ g$.

45. $F(x) = (2x + x^2)^4$

46. $F(x) = \cos^2 x$

47. $F(x) = \dfrac{\sqrt[3]{x}}{1 + \sqrt[3]{x}}$

48. $G(x) = \sqrt[3]{\dfrac{x}{1 + x}}$

49. $v(t) = \sec(t^2)\tan(t^2)$

50. $H(x) = \sqrt{1 + \sqrt{x}}$

51–54 Express the function in the form $f \circ g \circ h$.

51. $R(x) = \sqrt{\sqrt{x} - 1}$

52. $H(x) = \sqrt[8]{2 + |x|}$

53. $S(t) = \sin^2(\cos t)$

54. $H(t) = \cos\left(\sqrt{\tan t} + 1\right)$

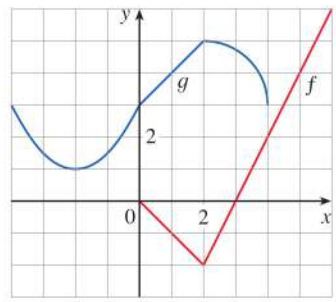
55–56 Use the table to evaluate each expression.

x	1	2	3	4	5	6
$f(x)$	3	1	5	6	2	4
$g(x)$	5	3	4	1	3	2

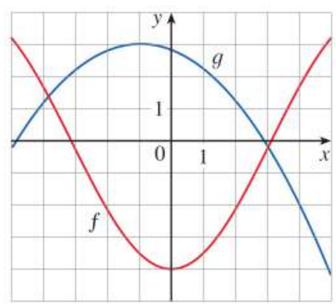
55. (a) $f(g(3))$ (b) $g(f(2))$
(c) $(f \circ g)(5)$ (d) $(g \circ f)(5)$

56. (a) $g(g(g(2)))$ (b) $(f \circ f \circ f)(1)$
(c) $(f \circ f \circ g)(1)$ (d) $(g \circ f \circ g)(3)$

57. Use the given graphs of f and g to evaluate each expression, or explain why it is undefined.
(a) $f(g(2))$ (b) $g(f(0))$ (c) $(f \circ g)(0)$
(d) $(g \circ f)(6)$ (e) $(g \circ g)(-2)$ (f) $(f \circ f)(4)$



58. Use the given graphs of f and g to estimate the value of $f(g(x))$ for $x = -5, -4, -3, \ldots, 5$. Use these estimates to sketch a rough graph of $f \circ g$.



59. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s.
(a) Express the radius r of this circle as a function of the time t (in seconds).
(b) If A is the area of this circle as a function of the radius, find $A \circ r$ and interpret it.

60. A spherical balloon is being inflated and the radius of the balloon is increasing at a rate of 2 cm/s.
(a) Express the radius r of the balloon as a function of the time t (in seconds).
(b) If V is the volume of the balloon as a function of the radius, find $V \circ r$ and interpret it.

61. A ship is moving at a speed of 30 km/h parallel to a straight shoreline. The ship is 6 km from shore and it passes a lighthouse at noon.
(a) Express the distance s between the lighthouse and the ship as a function of d, the distance the ship has traveled since noon; that is, find f so that $s = f(d)$.
(b) Express d as a function of t, the time elapsed since noon; that is, find g so that $d = g(t)$.
(c) Find $f \circ g$. What does this function represent?

62. An airplane is flying at a speed of 350 mi/h at an altitude of one mile and passes directly over a radar station at time $t = 0$.

(a) Express the horizontal distance d (in miles) that the plane has flown as a function of t.

(b) Express the distance s between the plane and the radar station as a function of d.

(c) Use composition to express s as a function of t.

63. The Heaviside Function The *Heaviside function H* is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geqslant 0 \end{cases}$$

It is used in the study of electric circuits to represent the sudden surge of electric current, or voltage, when a switch is instantaneously turned on.

(a) Sketch the graph of the Heaviside function.

(b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and 120 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$.

(c) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 5$ seconds and 240 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$. (Note that starting at $t = 5$ corresponds to a translation.)

64. The Ramp Function The Heaviside function defined in Exercise 63 can also be used to define the *ramp function* $y = ctH(t)$, which represents a gradual increase in voltage or current in a circuit.

(a) Sketch the graph of the ramp function $y = tH(t)$.

(b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and the voltage is gradually increased to 120 volts over a 60-second time interval. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leqslant 60$.

(c) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 7$ seconds and the voltage is gradually increased to 100 volts over a period of 25 seconds. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leqslant 32$.

65. Let f and g be linear functions with equations $f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$. Is $f \circ g$ also a linear function? If so, what is the slope of its graph?

66. If you invest x dollars at 4% interest compounded annually, then the amount $A(x)$ of the investment after one year is $A(x) = 1.04x$. Find $A \circ A$, $A \circ A \circ A$, and $A \circ A \circ A \circ A$. What do these compositions represent? Find a formula for the composition of n copies of A.

67. (a) If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that $f \circ g = h$. (Think about what operations you would have to perform on the formula for g to end up with the formula for h.)

(b) If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, find a function g such that $f \circ g = h$.

68. If $f(x) = x + 4$ and $h(x) = 4x - 1$, find a function g such that $g \circ f = h$.

69. Suppose g is an even function and let $h = f \circ g$. Is h always an even function?

70. Suppose g is an odd function and let $h = f \circ g$. Is h always an odd function? What if f is odd? What if f is even?

71. Let $f(x)$ be a function with domain $\mathbb{R}$.

(a) Show that $E(x) = f(x) + f(-x)$ is an even function.

(b) Show that $O(x) = f(x) - f(-x)$ is an odd function.

(c) Prove that every function $f(x)$ can be written as a sum of an even function and an odd function.

(d) Express the function $f(x) = 2^x + (x - 3)^2$ as a sum of an even function and an odd function.

1.4 Exponential Functions

The function $f(x) = 2^x$ is called an *exponential function* because the variable, x, is the exponent. It should not be confused with the power function $g(x) = x^2$, in which the variable is the base.

Exponential Functions and Their Graphs

In general, an **exponential function** is a function of the form

$$f(x) = b^x$$

where b is a positive constant. Let's recall what this means.

In Appendix G we present an alternative approach to the exponential and logarithmic functions using integral calculus.

If $x = n$, a positive integer, then

$$b^n = \underbrace{b \cdot b \cdot \cdots \cdot b}_{n \text{ factors}}$$

If $x = 0$, then $b^0 = 1$, and if $x = -n$, where n is a positive integer, then

$$b^{-n} = \frac{1}{b^n}$$

If x is a rational number, $x = p/q$, where p and q are integers and $q > 0$, then

$$b^x = b^{p/q} = \sqrt[q]{b^p} = \left(\sqrt[q]{b}\right)^p$$

But what is the meaning of b^x if x is an irrational number? For instance, what is meant by $2^{\sqrt{3}}$ or 5^{π}?

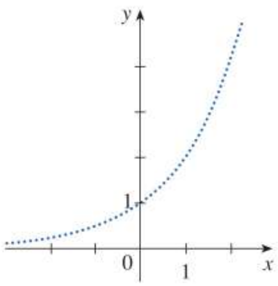


FIGURE 1
Representation of $y = 2^x$, x rational

To help us answer this question we first look at the graph of the function $y = 2^x$, where x is rational. A representation of this graph is shown in Figure 1. We want to enlarge the domain of $y = 2^x$ to include both rational and irrational numbers.

There are holes in the graph in Figure 1 corresponding to irrational values of x. We want to fill in the holes by defining $f(x) = 2^x$, where $x \in \mathbb{R}$, so that f is an increasing function. In particular, since the irrational number $\sqrt{3}$ satisfies

$$1.7 < \sqrt{3} < 1.8$$

we must have
$$2^{1.7} < 2^{\sqrt{3}} < 2^{1.8}$$

and we know what $2^{1.7}$ and $2^{1.8}$ mean because 1.7 and 1.8 are rational numbers. Similarly, if we use better approximations for $\sqrt{3}$, we obtain better approximations for $2^{\sqrt{3}}$:

$$\begin{array}{ccc}
1.73 < \sqrt{3} < 1.74 & \Rightarrow & 2^{1.73} < 2^{\sqrt{3}} < 2^{1.74} \\
1.732 < \sqrt{3} < 1.733 & \Rightarrow & 2^{1.732} < 2^{\sqrt{3}} < 2^{1.733} \\
1.7320 < \sqrt{3} < 1.7321 & \Rightarrow & 2^{1.7320} < 2^{\sqrt{3}} < 2^{1.7321} \\
1.73205 < \sqrt{3} < 1.73206 & \Rightarrow & 2^{1.73205} < 2^{\sqrt{3}} < 2^{1.73206} \\
\vdots \quad \vdots & & \vdots \quad \vdots
\end{array}$$

A proof of this fact is given in J. Marsden and A. Weinstein, *Calculus Unlimited* (Menlo Park, CA: Benjamin/Cummings, 1981).

It can be shown that there is exactly one number that is greater than all of the numbers

$$2^{1.7}, \quad 2^{1.73}, \quad 2^{1.732}, \quad 2^{1.7320}, \quad 2^{1.73205}, \quad \ldots$$

and less than all of the numbers

$$2^{1.8}, \quad 2^{1.74}, \quad 2^{1.733}, \quad 2^{1.7321}, \quad 2^{1.73206}, \quad \ldots$$

We define $2^{\sqrt{3}}$ to be this number. Using the preceding approximation process we can compute it correct to six decimal places:

$$2^{\sqrt{3}} \approx 3.321997$$

Similarly, we can define 2^x (or b^x, if $b > 0$) where x is any irrational number. Figure 2 shows how all the holes in Figure 1 have been filled to complete the graph of the function $f(x) = 2^x$, $x \in \mathbb{R}$.

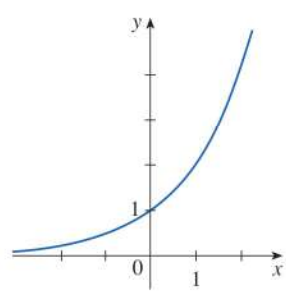


FIGURE 2
$y = 2^x$, x real

The graphs of members of the family of functions $y = b^x$ are shown in Figure 3 for various values of the base b. Notice that all of these graphs pass through the same point (0, 1) because $b^0 = 1$ for $b \neq 0$. Notice also that as the base b gets larger, the exponential function grows more rapidly (for $x > 0$).

If $0 < b < 1$, then b^x approaches 0 as x becomes large. If $b > 1$, then b^x approaches 0 as x decreases through negative values. In both cases the x-axis is a horizontal asymptote. These matters are discussed in Section 2.6.

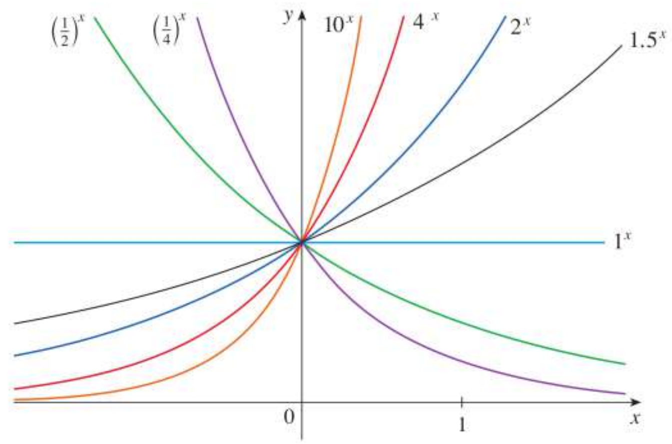


FIGURE 3

You can see from Figure 3 that there are basically three kinds of exponential functions $y = b^x$. If $0 < b < 1$, the exponential function decreases; if $b = 1$, it is a constant; and if $b > 1$, it increases. These three cases are illustrated in Figure 4. Observe that if $b \neq 1$, then the exponential function $y = b^x$ has domain $\mathbb{R}$ and range $(0, \infty)$. Notice also that, since $(1/b)^x = 1/b^x = b^{-x}$, the graph of $y = (1/b)^x$ is just the reflection of the graph of $y = b^x$ about the y-axis.

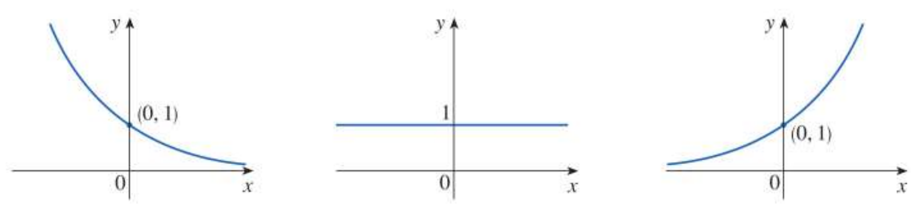


FIGURE 4 (a) $y = b^x$, $0 < b < 1$ (b) $y = 1^x$ (c) $y = b^x$, $b > 1$

One reason for the importance of the exponential function lies in the following properties. If x and y are rational numbers, then these laws are well known from elementary algebra. It can be proved that they remain true for arbitrary real numbers x and y.

www.StewartCalculus.com
For review and practice using the Laws of Exponents, click on *Review of Algebra*.

Laws of Exponents If a and b are positive numbers and x and y are any real numbers, then

1. $b^{x+y} = b^x b^y$ **2.** $b^{x-y} = \dfrac{b^x}{b^y}$ **3.** $(b^x)^y = b^{xy}$ **4.** $(ab)^x = a^x b^x$

EXAMPLE 1 Sketch the graph of the function $y = 3 - 2^x$ and determine its domain and range.

For a review of reflecting and shifting graphs, see Section 1.3.

SOLUTION First we reflect the graph of $y = 2^x$ [shown in Figures 2 and 5(a)] about the x-axis to get the graph of $y = -2^x$ in Figure 5(b). Then we shift the graph of $y = -2^x$

upward 3 units to obtain the graph of $y = 3 - 2^x$ in Figure 5(c). The domain is $\mathbb{R}$ and the range is $(-\infty, 3)$.

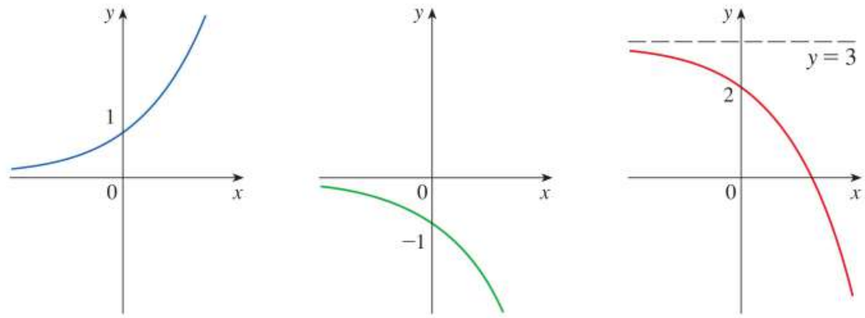


FIGURE 5

(a) $y = 2^x$ (b) $y = -2^x$ (c) $y = 3 - 2^x$

EXAMPLE 2 Use a graphing calculator or computer to compare the exponential function $f(x) = 2^x$ and the power function $g(x) = x^2$. Which function grows more quickly when x is large?

SOLUTION Figure 6 shows both functions graphed in the viewing rectangle $[-2, 6]$ by $[0, 40]$. We see that the graphs intersect three times, but for $x > 4$ the graph of $f(x) = 2^x$ stays above the graph of $g(x) = x^2$. Figure 7 gives a more global view and shows that for large values of x, the exponential function $f(x) = 2^x$ grows far more rapidly than the power function $g(x) = x^2$.

Example 2 shows that $y = 2^x$ increases more quickly than $y = x^2$. To demonstrate just how quickly $f(x) = 2^x$ increases, let's perform the following thought experiment. Suppose we start with a piece of paper a thousandth of an inch thick and we fold it in half 50 times. Each time we fold the paper in half, the thickness of the paper doubles, so the thickness of the resulting paper would be $2^{50}/1000$ inches. How thick do you think that is? It works out to be more than 17 million miles!

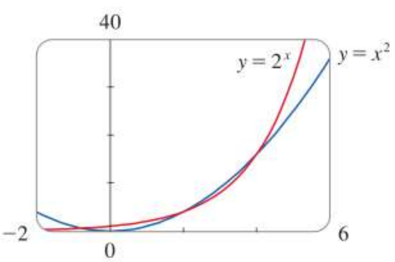


FIGURE 6

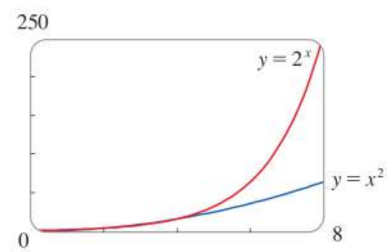


FIGURE 7

Applications of Exponential Functions

The exponential function occurs very frequently in mathematical models of nature and society. Here we indicate briefly how it arises in the description of increasing population or decreasing viral loads. In later chapters we will pursue these and other applications in greater detail.

First we consider a population of bacteria in a homogeneous nutrient medium. Suppose that by sampling the population at certain intervals, it is determined that the population doubles every hour. If the number of bacteria at time t is $p(t)$, where t is measured in hours, and the initial population is $p(0) = 1000$, then we have

$$p(1) = 2p(0) = 2 \times 1000$$
$$p(2) = 2p(1) = 2^2 \times 1000$$
$$p(3) = 2p(2) = 2^3 \times 1000$$

It seems from this pattern that, in general,

$$p(t) = 2^t \times 1000 = (1000)2^t$$

This population function is a constant multiple of the exponential function $y = 2^t$, so it exhibits the rapid growth that we observed in Figure 7. Under ideal conditions (unlimited space and nutrition and absence of disease) this exponential growth is typical of what actually occurs in nature.

Table 1 World Population

t (years since 1900)	Population P (millions)
0	1650
10	1750
20	1860
30	2070
40	2300
50	2560
60	3040
70	3710
80	4450
90	5280
100	6080
110	6870

EXAMPLE 3 Table 1 shows data for the population of the world in the 20th century and Figure 8 shows the corresponding scatter plot.

The pattern of the data points in Figure 8 suggests exponential growth, so we use a graphing calculator (or computer) with exponential regression capability to apply the method of least squares and obtain the exponential model

$$P(t) = (1.43653 \times 10^9) \cdot (1.01395)^t$$

where $t = 0$ corresponds to 1900. Figure 9 shows the graph of this exponential function together with the original data points. We see that the exponential curve fits the data reasonably well. The period of relatively slow population growth is explained by the two world wars and the Great Depression of the 1930s.

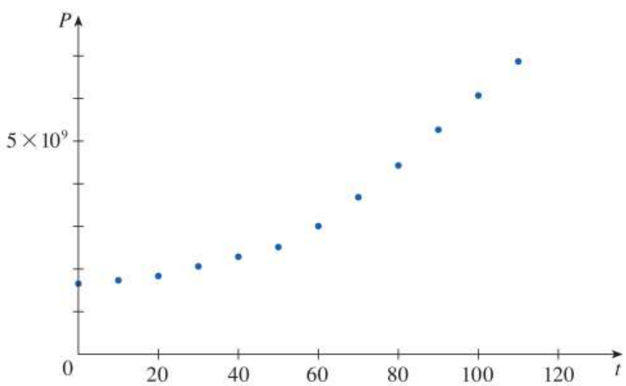


FIGURE 8 Scatter plot for world population growth

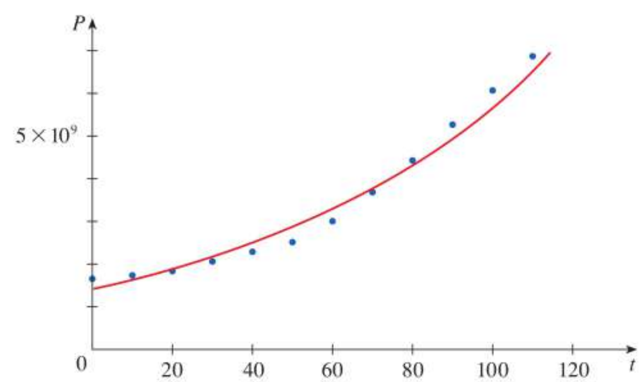


FIGURE 9 Exponential model for world population growth ■

Table 2

t (days)	$V(t)$
1	76.0
4	53.0
8	18.0
11	9.4
15	5.2
22	3.6

EXAMPLE 4 In 1995 a research article was published that detailed the effect of the protease inhibitor ABT-538 on the human immunodeficiency virus HIV-1.[1] Table 2 shows values of the plasma viral load $V(t)$ of patient 303, measured in RNA copies per mL, t days after ABT-538 treatment was begun. The corresponding scatter plot is shown in Figure 10 (on the following page).

The rather dramatic decline of the viral load that we see in Figure 10 reminds us of the graphs of the exponential function $y = b^x$ in Figures 3 and 4(a) for the case where the base b is less than 1. So let's model the function $V(t)$ by an exponential function. Using a graphing calculator or computer to fit the data in Table 2 with an exponential

1. D. Ho et al., "Rapid Turnover of Plasma Virions and CD4 Lymphocytes in HIV-1 Infection," *Nature* 373 (1995): 123–26.

function of the form $y = a \cdot b^t$, we obtain the model

$$V = 96.39785 \cdot (0.818656)^t$$

In Figure 11 we graph this exponential function with the data points and observe that the model represents the viral load reasonably well for the first month of treatment.

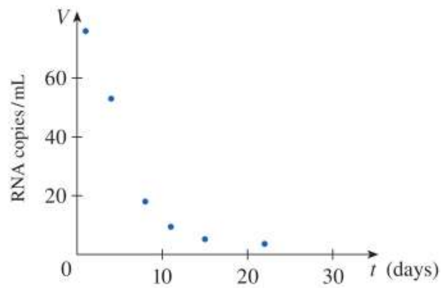


FIGURE 10
Plasma viral load in patient 303

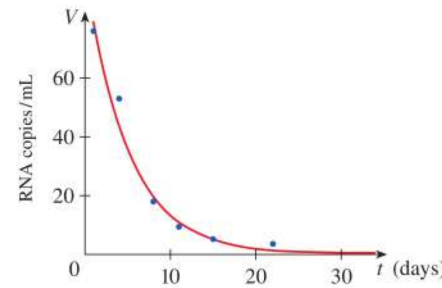


FIGURE 11
Exponential model for viral load

In Example 3 we used an exponential function of the form $y = a \cdot b^t, b > 1$, to model an increasing population and in Example 4 we used $y = a \cdot b^t, b < 1$, to model a decreasing viral load. In Section 3.8 we will explore additional examples of quantities that grow or decline exponentially, including the value of an investment account with compounding interest and the amount of radioactive material that remains as the material decays.

The Number *e*

Of all possible bases for an exponential function, there is one that is most convenient for the purposes of calculus. The choice of a base b is influenced by the way the graph of $y = b^x$ crosses the y-axis. Figures 12 and 13 show the tangent lines to the graphs of $y = 2^x$ and $y = 3^x$ at the point $(0, 1)$. (Tangent lines will be defined precisely in Section 2.7. For present purposes, you can think of the tangent line to an exponential graph at a point as the line that touches the graph only at that point.) If we measure the slopes of these tangent lines at $(0, 1)$, we find that $m \approx 0.7$ for $y = 2^x$ and $m \approx 1.1$ for $y = 3^x$.

It turns out, as we will see in Chapter 3, that some of the formulas of calculus will be greatly simplified if we choose the base b so that the slope of the tangent line to $y = b^x$ at $(0, 1)$ is *exactly* 1. (See Figure 14.) In fact, there *is* such a number and it is denoted by the letter e. (This notation was chosen by the Swiss mathematician Leonhard Euler in 1727, probably because it is the first letter of the word *exponential*.) In view of

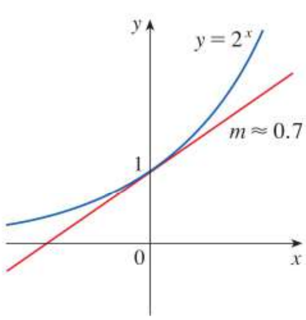


FIGURE 12

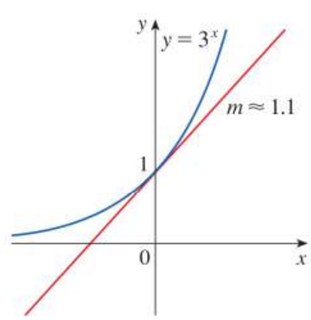


FIGURE 13

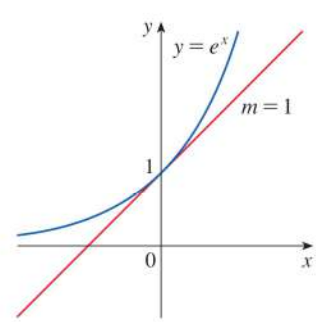


FIGURE 14

Figures 12 and 13, it comes as no surprise that the number e lies between 2 and 3 and the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$. (See Figure 15.) In Chapter 3 we will see that the value of e, correct to five decimal places, is

$$e \approx 2.71828$$

We call the function $f(x) = e^x$ the **natural exponential function**.

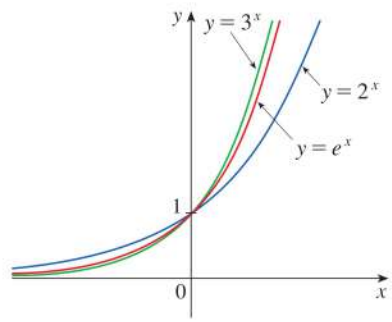
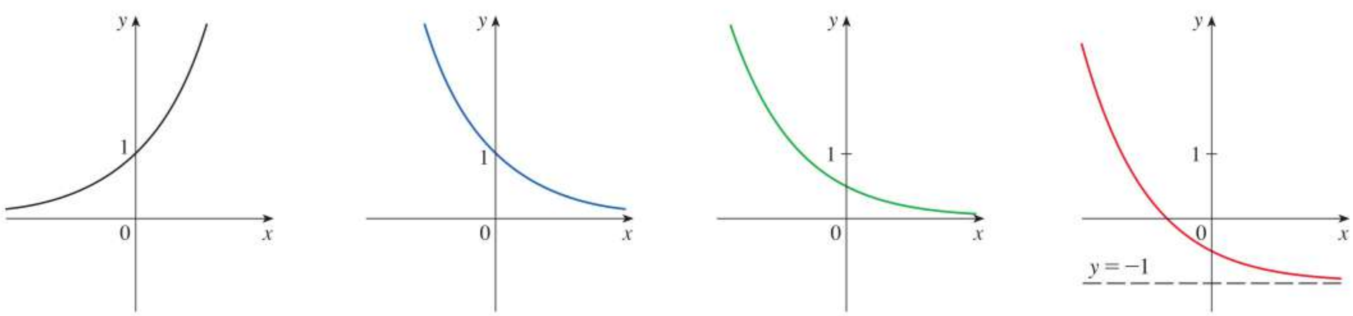


FIGURE 15
The graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$.

EXAMPLE 5 Graph the function $y = \frac{1}{2}e^{-x} - 1$ and state the domain and range.

SOLUTION We start with the graph of $y = e^x$ from Figures 14 and 16(a) and reflect about the y-axis to get the graph of $y = e^{-x}$ in Figure 16(b). (Notice that the tangent line to the graph at the y-intercept has slope -1.) Then we compress the graph vertically by a factor of 2 to obtain the graph of $y = \frac{1}{2}e^{-x}$ in Figure 16(c). Finally, we shift the graph downward one unit to get the desired graph in Figure 16(d). The domain is $\mathbb{R}$ and the range is $(-1, \infty)$.



(a) $y = e^x$ (b) $y = e^{-x}$ (c) $y = \frac{1}{2}e^{-x}$ (d) $y = \frac{1}{2}e^{-x} - 1$

FIGURE 16

■

How far to the right do you think we would have to go for the height of the graph of $y = e^x$ to exceed a million? The next example demonstrates the rapid growth of this function by providing an answer that might surprise you.

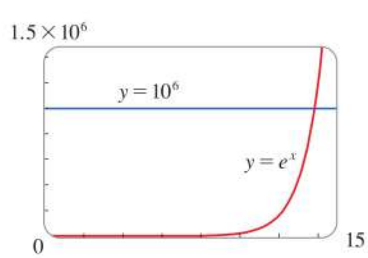


FIGURE 17

EXAMPLE 6 Use a graphing calculator or computer to find the values of x for which $e^x > 1{,}000{,}000$.

SOLUTION In Figure 17 we graph both the function $y = e^x$ and the horizontal line $y = 1{,}000{,}000$. We see that these curves intersect when $x \approx 13.8$. So $e^x > 10^6$ when $x > 13.8$. It is perhaps surprising that the values of the exponential function have already surpassed a million when x is only 14. ■

1.4 Exercises

1–2 Use the Laws of Exponents to rewrite and simplify each expression.

1. (a) $\dfrac{-2^6}{4^3}$ (b) $\dfrac{(-3)^6}{9^6}$ (c) $\dfrac{1}{\sqrt[4]{x^5}}$

(d) $\dfrac{x^3 \cdot x^n}{x^{n+1}}$ (e) $b^3(3b^{-1})^{-2}$ (f) $\dfrac{2x^2y}{(3x^{-2}y)^2}$

2. (a) $\dfrac{\sqrt[3]{4}}{\sqrt[3]{108}}$ (b) $27^{2/3}$ (c) $2x^2(3x^5)^2$

(d) $(2x^{-2})^{-3}x^{-3}$ (e) $\dfrac{3a^{3/2} \cdot a^{1/2}}{a^{-1}}$ (f) $\dfrac{\sqrt{a\sqrt{b}}}{\sqrt[3]{ab}}$

3. (a) Write an equation that defines the exponential function with base $b > 0$.
(b) What is the domain of this function?
(c) If $b \neq 1$, what is the range of this function?
(d) Sketch the general shape of the graph of the exponential function for each of the following cases.
(i) $b > 1$
(ii) $b = 1$
(iii) $0 < b < 1$

4. (a) How is the number e defined?
(b) What is an approximate value for e?
(c) What is the natural exponential function?

5–8 Graph the given functions on a common screen. How are these graphs related?

5. $y = 2^x$, $y = e^x$, $y = 5^x$, $y = 20^x$

6. $y = e^x$, $y = e^{-x}$, $y = 8^x$, $y = 8^{-x}$

7. $y = 3^x$, $y = 10^x$, $y = \left(\frac{1}{3}\right)^x$, $y = \left(\frac{1}{10}\right)^x$

8. $y = 0.9^x$, $y = 0.6^x$, $y = 0.3^x$, $y = 0.1^x$

9–14 Make a rough sketch by hand of the graph of the function. Use the graphs given in Figures 3 and 15 and, if necessary, the transformations of Section 1.3.

9. $g(x) = 3^x + 1$

10. $h(x) = 2\left(\frac{1}{2}\right)^x - 3$

11. $y = -e^{-x}$

12. $y = 4^{x+2}$

13. $y = 1 - \frac{1}{2}e^{-x}$

14. $y = e^{|x|}$

15. Starting with the graph of $y = e^x$, write the equation of the graph that results from
(a) shifting 2 units downward.
(b) shifting 2 units to the right.
(c) reflecting about the x-axis.
(d) reflecting about the y-axis.
(e) reflecting about the x-axis and then about the y-axis.

16. Starting with the graph of $y = e^x$, find the equation of the graph that results from
(a) reflecting about the line $y = 4$.
(b) reflecting about the line $x = 2$.

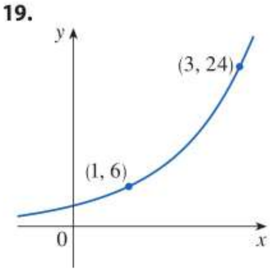
17–18 Find the domain of each function.

17. (a) $f(x) = \dfrac{1 - e^{x^2}}{1 - e^{1-x^2}}$ (b) $f(x) = \dfrac{1 + x}{e^{\cos x}}$

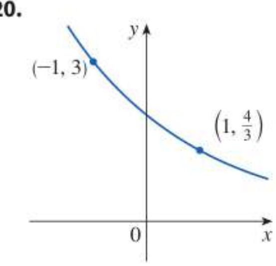
18. (a) $g(t) = \sqrt{10^t - 100}$ (b) $g(t) = \sin(e^t - 1)$

19–20 Find the exponential function $f(x) = Cb^x$ whose graph is given.

19.



20.



21. If $f(x) = 5^x$, show that

$$\frac{f(x + h) - f(x)}{h} = 5^x\left(\frac{5^h - 1}{h}\right)$$

22. Suppose you are offered a job that lasts one month. Which of the following methods of payment do you prefer?
I. One million dollars at the end of the month.
II. One cent on the first day of the month, two cents on the second day, four cents on the third day, and, in general, 2^{n-1} cents on the nth day.

23. Suppose the graphs of $f(x) = x^2$ and $g(x) = 2^x$ are drawn on a coordinate grid where the unit of measurement is 1 inch. Show that at a distance 2 ft to the right of the origin, the height of the graph of f is 48 ft but the height of the graph of g is about 265 mi.

24. Compare the functions $f(x) = x^5$ and $g(x) = 5^x$ by graphing both functions in several viewing rectangles. Find all points of intersection of the graphs correct to one decimal place. Which function grows more rapidly when x is large?

25. Compare the functions $f(x) = x^{10}$ and $g(x) = e^x$ by graphing both functions in several viewing rectangles. When does the graph of g finally surpass the graph of f?

26. Use a graph to estimate the values of x such that $e^x > 1{,}000{,}000{,}000$.

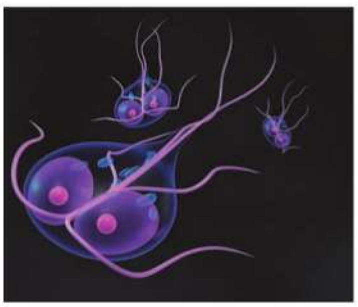
T **27.** A researcher is trying to determine the doubling time for a population of the bacterium *Giardia lamblia*. He starts a culture in a nutrient solution and estimates the bacteria count every four hours. His data are shown in the table.

Time (hours)	0	4	8	12	16	20	24
Bacteria count (CFU/mL)	37	47	63	78	105	130	173

(a) Make a scatter plot of the data.

(b) Use a calculator or computer to find an exponential curve $f(t) = a \cdot b^t$ that models the bacteria population t hours later.

(c) Graph the model from part (b) together with the scatter plot in part (a). Use the graph to estimate how long it takes for the bacteria count to double.


Sebastian Kaulitzki / Shutterstock.com

G. lamblia

T **28.** The table gives the population of the United States, in millions, for the years 1900–2010. Use a graphing calculator (or computer) with exponential regression capability to model the US population since 1900. Use the model to estimate the population in 1925 and to predict the population in the year 2020.

Year	Population
1900	76
1910	92
1920	106
1930	123
1940	131
1950	150
1960	179
1970	203
1980	227
1990	250
2000	281
2010	310

29. A bacteria culture starts with 500 bacteria and doubles in size every half hour.

(a) How many bacteria are there after 3 hours?

(b) How many bacteria are there after t hours?

(c) How many bacteria are there after 40 minutes?

(d) Graph the population function and estimate the time for the population to reach 100,000.

30. A gray squirrel population was introduced in a certain region 18 years ago. Biologists observe that the population doubles every six years, and now the population is 600.

(a) What was the initial squirrel population?

(b) What is the expected squirrel population t years after introduction?

(c) Estimate the expected squirrel population 10 years from now.

31. In Example 4, the patient's viral load V was 76.0 RNA copies per mL after one day of treatment. Use the graph of V in Figure 11 to estimate the additional time required for the viral load to decrease to half that amount.

32. After alcohol is fully absorbed into the body, it is metabolized. Suppose that after consuming several alcoholic drinks earlier in the evening, your blood alcohol concentration (BAC) at midnight is 0.14 g/dL. After 1.5 hours your BAC is half this amount.

(a) Find an exponential model for your BAC t hours after midnight.

(b) Graph your BAC and use the graph to determine when your BAC reaches the legal limit of 0.08 g/dL.

Source: Adapted from P. Wilkinson et al., "Pharmacokinetics of Ethanol after Oral Administration in the Fasting State," *Journal of Pharmacokinetics and Biopharmaceutics* 5 (1977): 207–24.

33. If you graph the function

$$f(x) = \frac{1 - e^{1/x}}{1 + e^{1/x}}$$

you'll see that f appears to be an odd function. Prove it.

34. Graph several members of the family of functions

$$f(x) = \frac{1}{1 + ae^{bx}}$$

where $a > 0$. How does the graph change when b changes? How does it change when a changes?

35. Graph several members of the family of functions

$$f(x) = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$$

where $a > 0$. How does the graph change as a increases?

1.5 Inverse Functions and Logarithms

Inverse Functions

Table 1 gives data from an experiment in which a biologist started a bacteria culture with 100 bacteria in a limited nutrient medium; the size of the bacteria population was recorded at hourly intervals. The number of bacteria N is a function of the time t: $N = f(t)$.

Suppose, however, that the biologist changes her point of view and becomes interested in the time required for the population to reach various levels. In other words, she is thinking of t as a function of N. This function is called the *inverse function* of f, denoted by f^{-1}, and read "f inverse." Here $t = f^{-1}(N)$ is the time required for the population level to reach N. The values of f^{-1} can be found by reading Table 1 from right to left or by consulting Table 2. For instance, $f^{-1}(550) = 6$ because $f(6) = 550$.

Table 1 *N* as a function of *t*

t (hours)	$N = f(t)$ = population at time t
0	100
1	168
2	259
3	358
4	445
5	509
6	550
7	573
8	586

Table 2 *t* as a function of *N*

N	$t = f^{-1}(N)$ = time to reach N bacteria
100	0
168	1
259	2
358	3
445	4
509	5
550	6
573	7
586	8

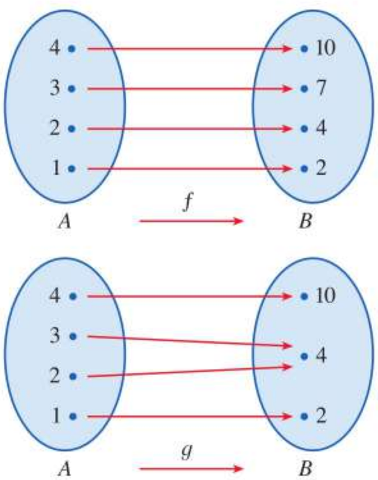


FIGURE 1
f is one-to-one; g is not.

Not all functions possess inverses. Let's compare the functions f and g whose arrow diagrams are shown in Figure 1. Note that f never takes on the same value twice (any two inputs in A have different outputs), whereas g does take on the same value twice (both 2 and 3 have the same output, 4). In symbols,

$$g(2) = g(3)$$

but
$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

Functions that share this property with f are called *one-to-one functions*.

In the language of inputs and outputs, Definition 1 says that f is one-to-one if each output corresponds to only one input.

1 Definition A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

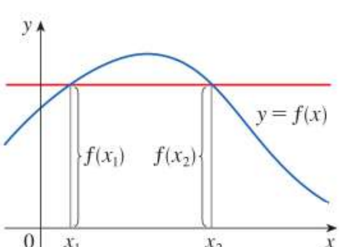


FIGURE 2
This function is not one-to-one because $f(x_1) = f(x_2)$.

If a horizontal line intersects the graph of f in more than one point, then we see from Figure 2 that there are numbers x_1 and x_2 such that $f(x_1) = f(x_2)$. This means that f is not one-to-one.

Therefore we have the following geometric method for determining whether a function is one-to-one.

Horizontal Line Test A function is one-to-one if and only if no horizontal line intersects its graph more than once.

EXAMPLE 1 Is the function $f(x) = x^3$ one-to-one?

SOLUTION 1 If $x_1 \neq x_2$, then $x_1^3 \neq x_2^3$ (two different numbers can't have the same cube). Therefore, by Definition 1, $f(x) = x^3$ is one-to-one.

SOLUTION 2 From Figure 3 we see that no horizontal line intersects the graph of $f(x) = x^3$ more than once. Therefore, by the Horizontal Line Test, f is one-to-one. ■

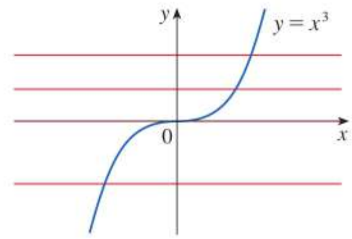


FIGURE 3
$f(x) = x^3$ is one-to-one.

EXAMPLE 2 Is the function $g(x) = x^2$ one-to-one?

SOLUTION 1 This function is not one-to-one because, for instance,

$$g(1) = 1 = g(-1)$$

and so 1 and -1 have the same output.

SOLUTION 2 From Figure 4 we see that there are horizontal lines that intersect the graph of g more than once. Therefore, by the Horizontal Line Test, g is not one-to-one. ■

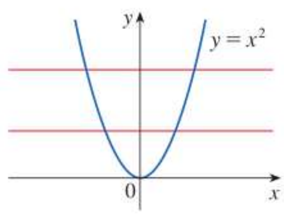


FIGURE 4
$g(x) = x^2$ is not one-to-one.

One-to-one functions are important because they are precisely the functions that possess inverse functions according to the following definition.

> **2 Definition** Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by
>
> $$f^{-1}(y) = x \iff f(x) = y$$
>
> for any y in B.

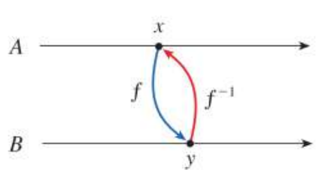


FIGURE 5

This definition says that if f maps x into y, then f^{-1} maps y back into x. (If f were not one-to-one, then f^{-1} would not be uniquely defined.) The arrow diagram in Figure 5 indicates that f^{-1} reverses the effect of f. Note that

$$\boxed{\begin{aligned} \text{domain of } f^{-1} &= \text{range of } f \\ \text{range of } f^{-1} &= \text{domain of } f \end{aligned}}$$

For example, the inverse function of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$ because if $y = x^3$, then

$$f^{-1}(y) = f^{-1}(x^3) = (x^3)^{1/3} = x$$

CAUTION Do not mistake the -1 in f^{-1} for an exponent. Thus

$$f^{-1}(x) \quad \text{does } \textit{not} \text{ mean} \quad \frac{1}{f(x)}$$

The reciprocal $1/f(x)$ could be written as $[f(x)]^{-1}$.

EXAMPLE 3 If f is a one-to-one function and $f(1) = 5$, $f(3) = 7$, and $f(8) = -10$, find $f^{-1}(7)$, $f^{-1}(5)$, and $f^{-1}(-10)$.

SOLUTION From the definition of f^{-1} we have

$$f^{-1}(7) = 3 \quad \text{because} \quad f(3) = 7$$
$$f^{-1}(5) = 1 \quad \text{because} \quad f(1) = 5$$
$$f^{-1}(-10) = 8 \quad \text{because} \quad f(8) = -10$$

The diagram in Figure 6 makes it clear how f^{-1} reverses the effect of f in this case. ■

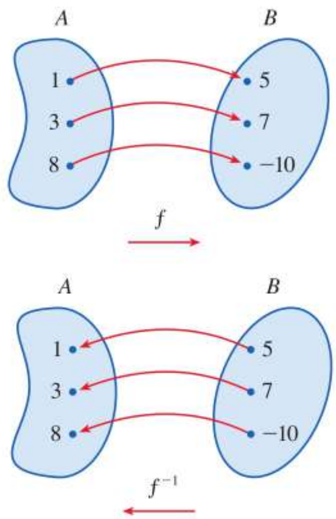


FIGURE 6
The inverse function reverses inputs and outputs.

The letter x is traditionally used as the independent variable, so when we concentrate on f^{-1} rather than on f, we usually reverse the roles of x and y in Definition 2 and write

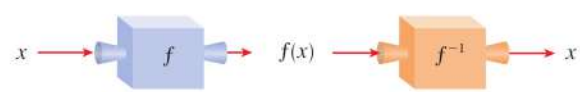
3
$$f^{-1}(x) = y \iff f(y) = x$$

By substituting for y in Definition 2 and substituting for x in (3), we get the following **cancellation equations**:

4
$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$
$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

The first cancellation equation says that if we start with x, apply f, and then apply f^{-1}, we arrive back at x, where we started (see the machine diagram in Figure 7). Thus f^{-1} undoes what f does. The second equation says that f undoes what f^{-1} does.

FIGURE 7



For example, if $f(x) = x^3$, then $f^{-1}(x) = x^{1/3}$ and so the cancellation equations become

$$f^{-1}(f(x)) = (x^3)^{1/3} = x$$
$$f(f^{-1}(x)) = (x^{1/3})^3 = x$$

These equations simply say that the cube function and the cube root function cancel each other when applied in succession.

Now let's see how to compute inverse functions. If we have a function $y = f(x)$ and are able to solve this equation for x in terms of y, then according to Definition 2 we must have $x = f^{-1}(y)$. If we want to call the independent variable x, we then interchange x and y and arrive at the equation $y = f^{-1}(x)$.

5 How to Find the Inverse Function of a One-to-One Function f

STEP 1 Write $y = f(x)$.

STEP 2 Solve this equation for x in terms of y (if possible).

STEP 3 To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$.

EXAMPLE 4 Find the inverse function of $f(x) = x^3 + 2$.

SOLUTION According to (5) we first write

$$y = x^3 + 2$$

Then we solve this equation for x:

$$x^3 = y - 2$$
$$x = \sqrt[3]{y - 2}$$

Finally, we interchange x and y:

$$y = \sqrt[3]{x - 2}$$

Therefore the inverse function is $f^{-1}(x) = \sqrt[3]{x - 2}$. ■

In Example 4, notice how f^{-1} reverses the effect of f. The function f is the rule "Cube, then add 2"; f^{-1} is the rule "Subtract 2, then take the cube root."

The principle of interchanging x and y to find the inverse function also gives us the method for obtaining the graph of f^{-1} from the graph of f. Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}. But we get the point (b, a) from (a, b) by reflecting about the line $y = x$. (See Figure 8.)

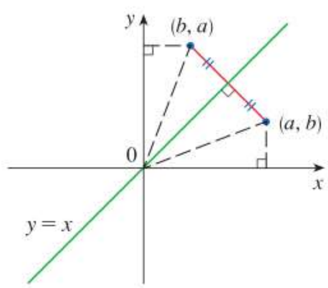


FIGURE 8

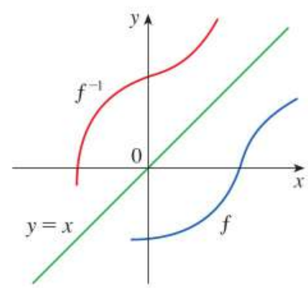


FIGURE 9

Therefore, as illustrated by Figure 9:

> The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

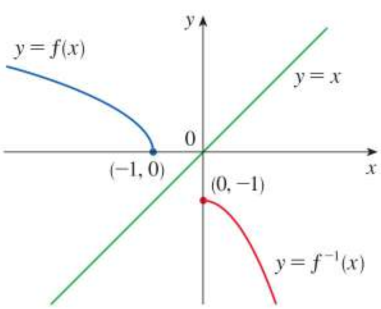


FIGURE 10

EXAMPLE 5 Sketch the graphs of $f(x) = \sqrt{-1 - x}$ and its inverse function using the same coordinate axes.

SOLUTION First we sketch the curve $y = \sqrt{-1 - x}$ (the top half of the parabola $y^2 = -1 - x$, or $x = -y^2 - 1$) and then we reflect about the line $y = x$ to get the graph of f^{-1}. (See Figure 10.) As a check on our graph, notice that the expression for f^{-1} is $f^{-1}(x) = -x^2 - 1$, $x \geqslant 0$. So the graph of f^{-1} is the right half of the parabola $y = -x^2 - 1$ and this seems reasonable from Figure 10. ■

Logarithmic Functions

If $b > 0$ and $b \neq 1$, the exponential function $f(x) = b^x$ is either increasing or decreasing and so it is one-to-one by the Horizontal Line Test. It therefore has an inverse function f^{-1}, which is called the **logarithmic function with base *b*** and is denoted by $\log_b$. If we use the formulation of an inverse function given by (3),

$$f^{-1}(x) = y \iff f(y) = x$$

then we have

$$\boxed{6} \qquad \log_b x = y \iff b^y = x$$

Thus, if $x > 0$, then $\log_b x$ is the exponent to which the base b must be raised to give x. For example, $\log_{10} 0.001 = -3$ because $10^{-3} = 0.001$.

The cancellation equations (4), when applied to the functions $f(x) = b^x$ and $f^{-1}(x) = \log_b x$, become

$$\boxed{7} \qquad \begin{aligned} \log_b(b^x) &= x \quad \text{for every } x \in \mathbb{R} \\ b^{\log_b x} &= x \quad \text{for every } x > 0 \end{aligned}$$

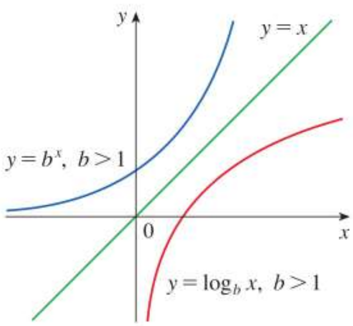


FIGURE 11

The logarithmic function $\log_b$ has domain $(0, \infty)$ and range $\mathbb{R}$. Its graph is the reflection of the graph of $y = b^x$ about the line $y = x$.

Figure 11 shows the case where $b > 1$. (The most important logarithmic functions have base $b > 1$.) The fact that $y = b^x$ is a very rapidly increasing function for $x > 0$ is reflected in the fact that $y = \log_b x$ is a very slowly increasing function for $x > 1$.

Figure 12 shows the graphs of $y = \log_b x$ with various values of the base $b > 1$. Because $\log_b 1 = 0$, the graphs of all logarithmic functions pass through the point $(1, 0)$.

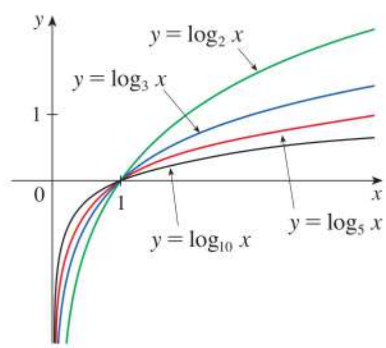


FIGURE 12

The following properties of logarithmic functions follow from the corresponding properties of exponential functions given in Section 1.4.

Laws of Logarithms If x and y are positive numbers, then

1. $\log_b(xy) = \log_b x + \log_b y$
2. $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$
3. $\log_b(x^r) = r \log_b x$ (where r is any real number)

EXAMPLE 6 Use the laws of logarithms to evaluate $\log_2 80 - \log_2 5$.

SOLUTION Using Law 2, we have

$$\log_2 80 - \log_2 5 = \log_2\left(\frac{80}{5}\right) = \log_2 16 = 4$$

because $2^4 = 16$. ■

Natural Logarithms

Notation for Logarithms
Most textbooks in calculus and the sciences, as well as calculators, use the notation $\ln x$ for the natural logarithm and $\log x$ for the "common logarithm," $\log_{10} x$. In the more advanced mathematical and scientific literature and in computer languages, however, the notation $\log x$ usually denotes the natural logarithm.

Of all possible bases b for logarithms, we will see in Chapter 3 that the most convenient choice of a base is the number e, which was defined in Section 1.4. The logarithm with base e is called the **natural logarithm** and has a special notation:

$$\boxed{\log_e x = \ln x}$$

If we set $b = e$ and replace $\log_e$ with "ln" in (6) and (7), then the defining properties of the natural logarithm function become

8 $$\boxed{\ln x = y \iff e^y = x}$$

9 $$\boxed{\begin{aligned} \ln(e^x) &= x \qquad x \in \mathbb{R} \\ e^{\ln x} &= x \qquad x > 0 \end{aligned}}$$

In particular, if we set $x = 1$, we get

$$\boxed{\ln e = 1}$$

Combining Property 9 with Law 3 allows us to write

$$x^r = e^{\ln(x^r)} = e^{r \ln x} \qquad x > 0$$

Thus a power of x can be expressed in an equivalent exponential form; we will find this useful in the chapters to come.

10 $$\boxed{x^r = e^{r \ln x}}$$

EXAMPLE 7 Find x if $\ln x = 5$.

SOLUTION 1 From (8) we see that

$$\ln x = 5 \qquad \text{means} \qquad e^5 = x$$

Therefore $x = e^5$.

(If you have trouble working with the "ln" notation, just replace it by $\log_e$. Then the equation becomes $\log_e x = 5$; so, by the definition of logarithm, $e^5 = x$.)

SOLUTION 2 Start with the equation

$$\ln x = 5$$

and apply the exponential function to both sides of the equation:

$$e^{\ln x} = e^5$$

But the second cancellation equation in (9) says that $e^{\ln x} = x$. Therefore $x = e^5$. ■

EXAMPLE 8 Solve the equation $e^{5-3x} = 10$.

SOLUTION We take natural logarithms of both sides of the equation and use (9):

$$\begin{aligned} \ln(e^{5-3x}) &= \ln 10 \\ 5 - 3x &= \ln 10 \\ 3x &= 5 - \ln 10 \\ x &= \tfrac{1}{3}(5 - \ln 10) \end{aligned}$$

Using a calculator, we can approximate the solution: to four decimal places, $x \approx 0.8991$. ■

The laws of logarithms allow us to expand logarithms of products and quotients as sums and differences of logarithms. These same laws also allow us to combine sums and differences of logarithms into a single logarithmic expression. These processes are illustrated in Examples 9 and 10.

EXAMPLE 9 Use the laws of logarithms to expand $\ln \dfrac{x^2\sqrt{x^2+2}}{3x+1}$.

SOLUTION Using Laws 1, 2, and 3 of logarithms, we have

$$\begin{aligned} \ln \frac{x^2\sqrt{x^2+2}}{3x+1} &= \ln x^2 + \ln\sqrt{x^2+2} - \ln(3x+1) \\ &= 2\ln x + \tfrac{1}{2}\ln(x^2+2) - \ln(3x+1) \end{aligned}$$ ■

EXAMPLE 10 Express $\ln a + \frac{1}{2}\ln b$ as a single logarithm.

SOLUTION Using Laws 3 and 1 of logarithms, we have

$$\begin{aligned} \ln a + \tfrac{1}{2}\ln b &= \ln a + \ln b^{1/2} \\ &= \ln a + \ln\sqrt{b} \\ &= \ln\left(a\sqrt{b}\right) \end{aligned}$$ ■

The following formula shows that logarithms with any base can be expressed in terms of the natural logarithm.

11 Change of Base Formula For any positive number b ($b \neq 1$), we have

$$\log_b x = \frac{\ln x}{\ln b}$$

PROOF Let $y = \log_b x$. Then, from (6), we have $b^y = x$. Taking natural logarithms of both sides of this equation, we get $y \ln b = \ln x$. Therefore

$$y = \frac{\ln x}{\ln b}$$ ■

Formula 11 enables us to use a calculator to compute a logarithm with any base (as shown in the following example). Similarly, Formula 11 allows us to graph any logarithmic function on a graphing calculator or computer (see Exercises 49 and 50).

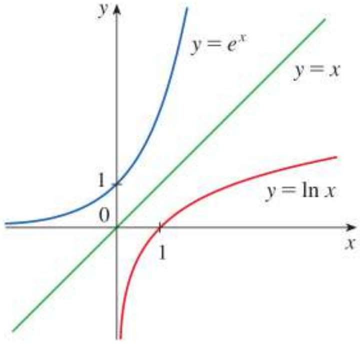


FIGURE 13
The graph of $y = \ln x$ is the reflection of the graph of $y = e^x$ about the line $y = x$.

EXAMPLE 11 Evaluate $\log_8 5$ correct to six decimal places.

SOLUTION Formula 11 gives

$$\log_8 5 = \frac{\ln 5}{\ln 8} \approx 0.773976$$ ■

■ Graph and Growth of the Natural Logarithm

The graphs of the exponential function $y = e^x$ and its inverse function, the natural logarithm function, are shown in Figure 13. In common with all other logarithmic functions with base greater than 1, the natural logarithm is an increasing function defined on $(0, \infty)$ and the y-axis is a vertical asymptote. (This means that the values of $\ln x$ become very large negative as x approaches 0.)

EXAMPLE 12 Sketch the graph of the function $y = \ln(x - 2) - 1$.

SOLUTION We start with the graph of $y = \ln x$ as given in Figure 13. Using the transformations of Section 1.3, we shift it 2 units to the right to get the graph of $y = \ln(x - 2)$ and then we shift it 1 unit downward to get the graph of $y = \ln(x - 2) - 1$. (See Figure 14.)

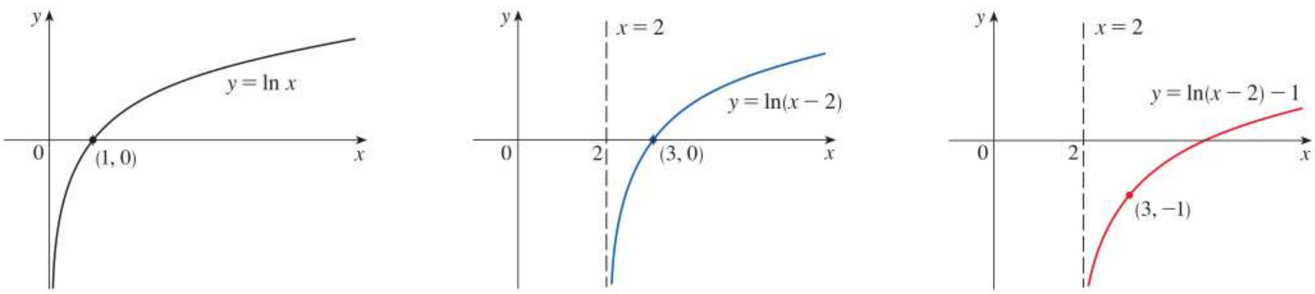


FIGURE 14 ■

Although $\ln x$ is an increasing function, it grows *very* slowly when $x > 1$. In fact, $\ln x$ grows more slowly than any positive power of x. To illustrate this fact, we graph $y = \ln x$ and $y = x^{1/2} = \sqrt{x}$ in Figures 15 and 16. You can see that the graphs initially grow at comparable rates, but eventually the root function far surpasses the logarithm.

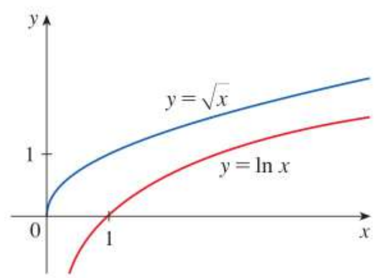


FIGURE 15

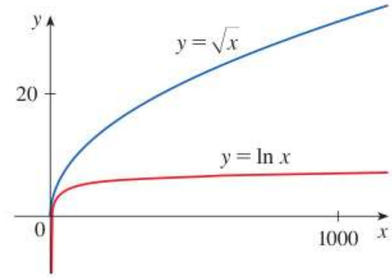


FIGURE 16

■ Inverse Trigonometric Functions

When we try to find the inverse trigonometric functions, we have a slight difficulty: because the trigonometric functions are not one-to-one, they don't have inverse functions. The difficulty is overcome by restricting the domains of these functions so that they become one-to-one.

You can see from Figure 17 that the sine function $y = \sin x$ is not one-to-one (use the Horizontal Line Test). However, if we restrict the domain to the interval $[-\pi/2, \pi/2]$, then the function *is* one-to-one and all values in the range of $y = \sin x$ are attained (see Figure 18). The inverse function of this restricted sine function f exists and is denoted by $\sin^{-1}$ or arcsin. It is called the **inverse sine function** or the **arcsine function**.

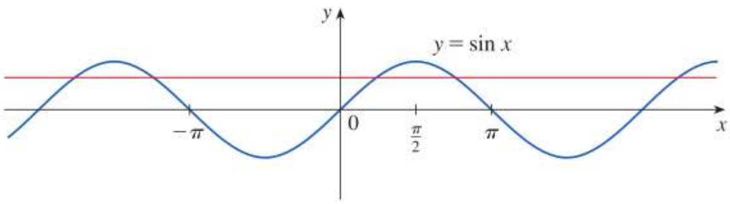


FIGURE 17

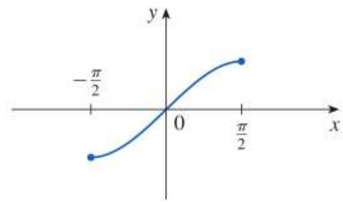


FIGURE 18
$y = \sin x,\ -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

Since the definition of an inverse function says that

$$f^{-1}(x) = y \iff f(y) = x$$

we have

$$\boxed{\sin^{-1}x = y \iff \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}}$$

$\sin^{-1}x \neq \dfrac{1}{\sin x}$

So, if $-1 \leqslant x \leqslant 1$, $\sin^{-1}x$ is the number between $-\pi/2$ and $\pi/2$ whose sine is x.

EXAMPLE 13 Evaluate (a) $\sin^{-1}\left(\frac{1}{2}\right)$ and (b) $\tan\left(\arcsin \frac{1}{3}\right)$.

SOLUTION

(a) We have

$$\sin^{-1}\left(\tfrac{1}{2}\right) = \frac{\pi}{6}$$

because $\sin(\pi/6) = \frac{1}{2}$ and $\pi/6$ lies between $-\pi/2$ and $\pi/2$.

(b) Let $\theta = \arcsin \frac{1}{3}$, so $\sin \theta = \frac{1}{3}$. Then we can draw a right triangle with angle θ as in Figure 19 and deduce from the Pythagorean Theorem that the third side has length $\sqrt{9 - 1} = 2\sqrt{2}$. This enables us to read from the triangle that

$$\tan\left(\arcsin \tfrac{1}{3}\right) = \tan u = \frac{1}{2\sqrt{2}}$$

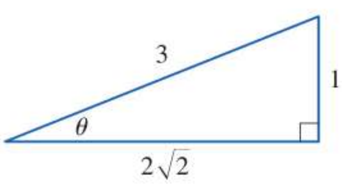


FIGURE 19

The cancellation equations for inverse functions become, in this case,

$$\boxed{\begin{aligned} \sin^{-1}(\sin x) &= x \quad \text{for } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2} \\ \sin(\sin^{-1}x) &= x \quad \text{for } -1 \leqslant x \leqslant 1 \end{aligned}}$$

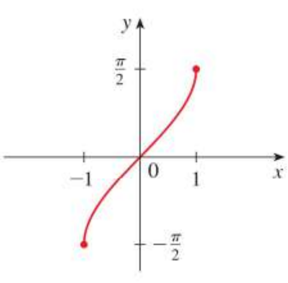


FIGURE 20
$y = \sin^{-1}x = \arcsin x$

The inverse sine function, $\sin^{-1}$, has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$, and its graph, shown in Figure 20, is obtained from that of the restricted sine function (Figure 18) by reflection about the line $y = x$.

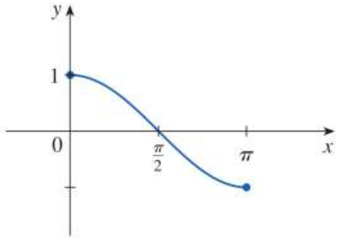


FIGURE 21
$y = \cos x,\ 0 \leqslant x \leqslant \pi$

The **inverse cosine function** is handled similarly. The restricted cosine function $f(x) = \cos x$, $0 \leqslant x \leqslant \pi$, is one-to-one (see Figure 21) and so it has an inverse function denoted by $\cos^{-1}$ or arccos.

$$\cos^{-1}x = y \iff \cos y = x \quad \text{and} \quad 0 \leqslant y \leqslant \pi$$

The cancellation equations are

$$\begin{aligned} \cos^{-1}(\cos x) &= x \quad \text{for } 0 \leqslant x \leqslant \pi \\ \cos(\cos^{-1}x) &= x \quad \text{for } -1 \leqslant x \leqslant 1 \end{aligned}$$

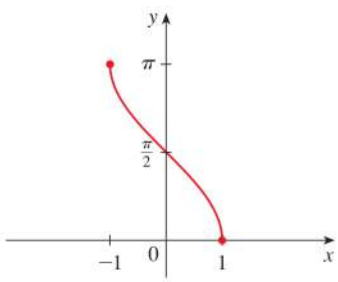


FIGURE 22
$y = \cos^{-1}x = \arccos x$

The inverse cosine function, $\cos^{-1}$, has domain $[-1, 1]$ and range $[0, \pi]$. Its graph is shown in Figure 22.

The tangent function can be made one-to-one by restricting it to the interval $(-\pi/2, \pi/2)$. Thus the **inverse tangent function** is defined as the inverse of the function $f(x) = \tan x$, $-\pi/2 < x < \pi/2$. (See Figure 23.) It is denoted by $\tan^{-1}$ or arctan.

$$\tan^{-1}x = y \iff \tan y = x \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

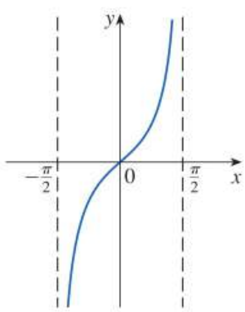


FIGURE 23
$y = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

EXAMPLE 14 Simplify the expression $\cos(\tan^{-1}x)$.

SOLUTION 1 Let $y = \tan^{-1}x$. Then $\tan y = x$ and $-\pi/2 < y < \pi/2$. We want to find $\cos y$ but, since $\tan y$ is known, it is easier to find $\sec y$ first:

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$\sec y = \sqrt{1 + x^2} \qquad \text{(since } \sec y > 0 \text{ for } -\pi/2 < y < \pi/2\text{)}$$

Thus $$\cos(\tan^{-1}x) = \cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1 + x^2}}$$

SOLUTION 2 Instead of using trigonometric identities as in Solution 1, it is perhaps easier to use a diagram. If $y = \tan^{-1}x$, then $\tan y = x$, and we can read from Figure 24 (which illustrates the case $y > 0$) that

$$\cos(\tan^{-1}x) = \cos y = \frac{1}{\sqrt{1 + x^2}}$$ ■

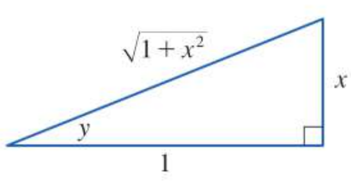


FIGURE 24

The inverse tangent function, $\tan^{-1} = \arctan$, has domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$. Its graph is shown in Figure 25.

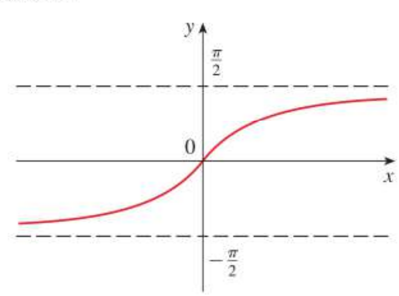


FIGURE 25
$y = \tan^{-1}x = \arctan x$

We know that the lines $x = \pm\pi/2$ are vertical asymptotes of the graph of tan. Since the graph of $\tan^{-1}$ is obtained by reflecting the graph of the restricted tangent function about the line $y = x$, it follows that the lines $y = \pi/2$ and $y = -\pi/2$ are horizontal asymptotes of the graph of $\tan^{-1}$.

The remaining inverse trigonometric functions are not used as frequently and are summarized here.

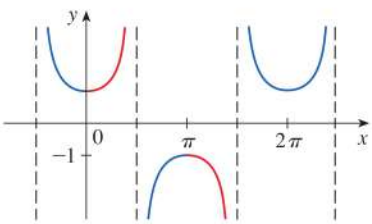


FIGURE 26
$y = \sec x$

12
$$\begin{aligned} &y = \csc^{-1}x\ (|x| \geqslant 1) \iff \csc y = x \text{ and } y \in (0, \pi/2] \cup (\pi, 3\pi/2] \\ &y = \sec^{-1}x\ (|x| \geqslant 1) \iff \sec y = x \text{ and } y \in [0, \pi/2) \cup [\pi, 3\pi/2) \\ &y = \cot^{-1}x\ (x \in \mathbb{R}) \iff \cot y = x \text{ and } y \in (0, \pi) \end{aligned}$$

The choice of intervals for y in the definitions of $\csc^{-1}$ and $\sec^{-1}$ is not universally agreed upon. For instance, some authors use $y \in [0, \pi/2) \cup (\pi/2, \pi]$ in the definition of $\sec^{-1}$. [You can see from the graph of the secant function in Figure 26 that both this choice and the one in (12) will work.]

1.5 Exercises

1. (a) What is a one-to-one function?
(b) How can you tell from the graph of a function whether it is one-to-one?

2. (a) Suppose f is a one-to-one function with domain A and range B. How is the inverse function f^{-1} defined? What is the domain of f^{-1}? What is the range of f^{-1}?
(b) If you are given a formula for f, how do you find a formula for f^{-1}?
(c) If you are given the graph of f, how do you find the graph of f^{-1}?

3–16 A function is given by a table of values, a graph, a formula, or a verbal description. Determine whether it is one-to-one.

3.

x	1	2	3	4	5	6
$f(x)$	1.5	2.0	3.6	5.3	2.8	2.0

4.

x	1	2	3	4	5	6
$f(x)$	1.0	1.9	2.8	3.5	3.1	2.9

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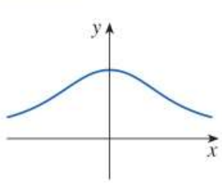

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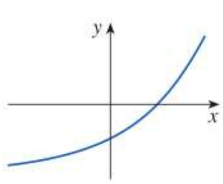

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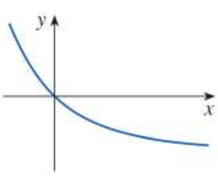

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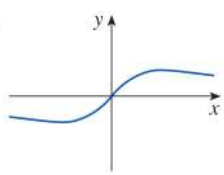


9. $f(x) = 2x - 3$

10. $f(x) = x^4 - 16$

11. $r(t) = t^3 + 4$

12. $g(x) = \sqrt[3]{x}$

13. $g(x) = 1 - \sin x$

14. $f(x) = x^4 - 1, \quad 0 \leqslant x \leqslant 10$

15. $f(t)$ is the height of a football t seconds after kickoff.

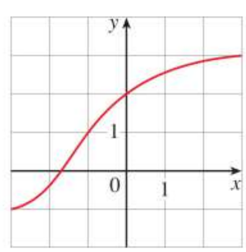
16. $f(t)$ is your height at age t.

17. Assume that f is a one-to-one function.
(a) If $f(6) = 17$, what is $f^{-1}(17)$?
(b) If $f^{-1}(3) = 2$, what is $f(2)$?

18. If $f(x) = x^5 + x^3 + x$, find $f^{-1}(3)$ and $f(f^{-1}(2))$.

19. If $g(x) = 3 + x + e^x$, find $g^{-1}(4)$.

20. The graph of f is given.
(a) Why is f one-to-one?
(b) What are the domain and range of f^{-1}?
(c) What is the value of $f^{-1}(2)$?
(d) Estimate the value of $f^{-1}(0)$.



21. The formula $C = \frac{5}{9}(F - 32)$, where $F \geqslant -459.67$, expresses the Celsius temperature C as a function of the Fahrenheit temperature F. Find a formula for the inverse function and interpret it. What is the domain of the inverse function?

22. In the theory of relativity, the mass of a particle with speed v is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle and c is the speed of light in a vacuum. Find the inverse function of f and explain its meaning.

23–30 Find a formula for the inverse of the function.

23. $f(x) = 1 - x^2, \quad x \geqslant 0$ **24.** $g(x) = x^2 - 2x, \quad x \geqslant 1$

25. $g(x) = 2 + \sqrt{x + 1}$ **26.** $h(x) = \dfrac{6 - 3x}{5x + 7}$

27. $y = e^{1-x}$ **28.** $y = 3\ln(x - 2)$

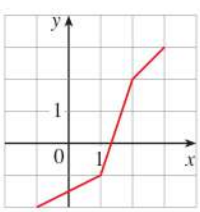
29. $y = \left(2 + \sqrt[3]{x}\right)^5$ **30.** $y = \dfrac{1 - e^{-x}}{1 + e^{-x}}$

31–32 Find an explicit formula for f^{-1} and use it to graph f^{-1}, f, and the line $y = x$ on the same screen. To check your work, see whether the graphs of f and f^{-1} are reflections about the line.

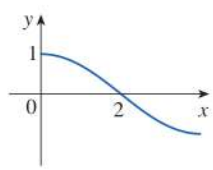
31. $f(x) = \sqrt{4x + 3}$ **32.** $f(x) = 1 + e^{-x}$

33–34 Use the given graph of f to sketch the graph of f^{-1}.

33.



34.



35. Let $f(x) = \sqrt{1 - x^2}$, $0 \leqslant x \leqslant 1$.
(a) Find f^{-1}. How is it related to f?
(b) Identify the graph of f and explain your answer to part (a).

36. Let $g(x) = \sqrt[3]{1 - x^3}$.
(a) Find g^{-1}. How is it related to g?
(b) Graph g. How do you explain your answer to part (a)?

37. (a) How is the logarithmic function $y = \log_b x$ defined?
(b) What is the domain of this function?
(c) What is the range of this function?
(d) Sketch the general shape of the graph of the function $y = \log_b x$ if $b > 1$.

38. (a) What is the natural logarithm?
(b) What is the common logarithm?
(c) Sketch the graphs of the natural logarithm function and the natural exponential function with a common set of axes.

39–42 Find the exact value of each expression.

39. (a) $\log_3 81$ (b) $\log_3\left(\frac{1}{81}\right)$ (c) $\log_9 3$

40. (a) $\ln \dfrac{1}{e^2}$ (b) $\ln \sqrt{e}$ (c) $\ln\left(\ln e^{e^{50}}\right)$

41. (a) $\log_2 30 - \log_2 15$
(b) $\log_3 10 - \log_3 5 - \log_3 18$
(c) $2\log_5 100 - 4\log_5 50$

42. (a) $e^{3\ln 2}$ (b) $e^{-2\ln 5}$ (c) $e^{\ln(\ln e^3)}$

43–44 Use the laws of logarithms to expand each expression.

43. (a) $\log_{10}(x^2 y^3 z)$ (b) $\ln\left(\dfrac{x^4}{\sqrt{x^2 - 4}}\right)$

44. (a) $\ln\sqrt{\dfrac{3x}{x - 3}}$ (b) $\log_2\left[(x^3 + 1)\sqrt[3]{(x - 3)^2}\right]$

45–46 Express as a single logarithm.

45. (a) $\log_{10} 20 - \frac{1}{3}\log_{10} 1000$ (b) $\ln a - 2\ln b + 3\ln c$

46. (a) $3\ln(x - 2) - \ln(x^2 - 5x + 6) + 2\ln(x - 3)$
(b) $c\log_a x - d\log_a y + \log_a z$

47–48 Use Formula 11 to evaluate each logarithm correct to six decimal places.

47. (a) $\log_5 10$ (b) $\log_{15} 12$

48. (a) $\log_3 12$ (b) $\log_{12} 6$

49–50 Use Formula 11 to graph the given functions on a common screen. How are these graphs related?

49. $y = \log_{1.5} x, \quad y = \ln x, \quad y = \log_{10} x, \quad y = \log_{50} x$

50. $y = \ln x, \quad y = \log_8 x, \quad y = e^x, \quad y = 8^x$

51. Suppose that the graph of $y = \log_2 x$ is drawn on a coordinate grid where the unit of measurement is an inch. How many miles to the right of the origin do we have to move before the height of the curve reaches 3 ft?

52. Compare the functions $f(x) = x^{0.1}$ and $g(x) = \ln x$ by graphing both functions in several viewing rectangles. When does the graph of f finally surpass the graph of g?

53–54 Make a rough sketch by hand of the graph of each function. Use the graphs given in Figures 12 and 13 and, if necessary, the transformations of Section 1.3.

53. (a) $y = \log_{10}(x + 5)$ (b) $y = -\ln x$

54. (a) $y = \ln(-x)$ (b) $y = \ln|x|$

55–56

(a) What are the domain and range of f?
(b) What is the x-intercept of the graph of f?
(c) Sketch the graph of f.

55. $f(x) = \ln x + 2$ **56.** $f(x) = \ln(x - 1) - 1$

57–60 Solve each equation for x. Give both an exact value and a decimal approximation, correct to three decimal places.

57. (a) $\ln(4x + 2) = 3$ (b) $e^{2x-3} = 12$

58. (a) $\log_2(x^2 - x - 1) = 2$ (b) $1 + e^{4x+1} = 20$

59. (a) $\ln x + \ln(x - 1) = 0$ (b) $5^{1-2x} = 9$

60. (a) $\ln(\ln x) = 0$ (b) $\dfrac{60}{1 + e^{-x}} = 4$

61–62 Solve each inequality for x.

61. (a) $\ln x < 0$ (b) $e^x > 5$

62. (a) $1 < e^{3x-1} < 2$ (b) $1 - 2\ln x < 3$

63. (a) Find the domain of $f(x) = \ln(e^x - 3)$.
(b) Find f^{-1} and its domain.

64. (a) What are the values of $e^{\ln 300}$ and $\ln(e^{300})$?
(b) Use your calculator to evaluate $e^{\ln 300}$ and $\ln(e^{300})$. What do you notice? Can you explain why the calculator has trouble?

T **65.** Graph the function $f(x) = \sqrt{x^3 + x^2 + x + 1}$ and explain why it is one-to-one. Then use a computer algebra system to find an explicit expression for $f^{-1}(x)$. (Your CAS will produce three possible expressions. Explain why two of them are irrelevant in this context.)

T **66.** (a) If $g(x) = x^6 + x^4$, $x \geqslant 0$, use a computer algebra system to find an expression for $g^{-1}(x)$.
(b) Use the expression in part (a) to graph $y = g(x)$, $y = x$, and $y = g^{-1}(x)$ on the same screen.

67. If a bacteria population starts with 100 bacteria and doubles every three hours, then the number of bacteria after t hours is $n = f(t) = 100 \cdot 2^{t/3}$.
(a) Find the inverse of this function and explain its meaning.
(b) When will the population reach 50,000?

68. The National Ignition Facility at the Lawrence Livermore National Laboratory maintains the world's largest laser facility. The lasers, which are used to start a nuclear fusion reaction, are powered by a capacitor bank that stores a total of about 400 megajoules of energy. When the lasers are fired the capacitors discharge completely and then immediately begin recharging. The charge Q of the capacitors t seconds after the discharge is given by

$$Q(t) = Q_0(1 - e^{-t/a})$$

(The maximum charge capacity is Q_0 and t is measured in seconds.)
(a) Find a formula for the inverse of this function and explain its meaning.
(b) How long does it take to recharge the capacitors to 90% of capacity if $a = 50$?

69–74 Find the exact value of each expression.

69. (a) $\cos^{-1}(-1)$ (b) $\sin^{-1}(0.5)$

70. (a) $\tan^{-1}\sqrt{3}$ (b) $\arctan(-1)$

71. (a) $\csc^{-1}\sqrt{2}$ (b) $\arcsin 1$

72. (a) $\sin^{-1}(-1/\sqrt{2})$ (b) $\cos^{-1}(\sqrt{3}/2)$

73. (a) $\cot^{-1}(-\sqrt{3})$ (b) $\sec^{-1} 2$

74. (a) $\arcsin(\sin(5\pi/4))$ (b) $\cos(2\sin^{-1}(\frac{5}{13}))$

75. Prove that $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$.

76–78 Simplify the expression.

76. $\tan(\sin^{-1}x)$ **77.** $\sin(\tan^{-1}x)$ **78.** $\sin(2\arccos x)$

79-80 Graph the given functions on the same screen. How are these graphs related?

79. $y = \sin x,\ -\pi/2 \leqslant x \leqslant \pi/2;\quad y = \sin^{-1}x;\quad y = x$

80. $y = \tan x,\ -\pi/2 < x < \pi/2;\quad y = \tan^{-1}x;\quad y = x$

81. Find the domain and range of the function

$$g(x) = \sin^{-1}(3x + 1)$$

82. (a) Graph the function $f(x) = \sin(\sin^{-1}x)$ and explain the appearance of the graph.
(b) Graph the function $g(x) = \sin^{-1}(\sin x)$. How do you explain the appearance of this graph?

83. (a) If we shift a curve to the left, what happens to its reflection about the line $y = x$? In view of this geometric principle, find an expression for the inverse of $g(x) = f(x + c)$, where f is a one-to-one function.
(b) Find an expression for the inverse of $h(x) = f(cx)$, where $c \neq 0$.

1 REVIEW

CONCEPT CHECK

Answers to the Concept Check are available at StewartCalculus.com.

1. (a) What is a function? What are its domain and range?
(b) What is the graph of a function?
(c) How can you tell whether a given curve is the graph of a function?

2. Discuss four ways of representing a function. Illustrate your discussion with examples.

3. (a) What is an even function? How can you tell if a function is even by looking at its graph? Give three examples of an even function.
(b) What is an odd function? How can you tell if a function is odd by looking at its graph? Give three examples of an odd function.

4. What is an increasing function?

5. What is a mathematical model?

6. Give an example of each type of function.
(a) Linear function (b) Power function
(c) Exponential function (d) Quadratic function
(e) Polynomial of degree 5 (f) Rational function

7. Sketch by hand, on the same axes, the graphs of the following functions.
(a) $f(x) = x$ (b) $g(x) = x^2$
(c) $h(x) = x^3$ (d) $j(x) = x^4$

8. Draw, by hand, a rough sketch of the graph of each function.
(a) $y = \sin x$ (b) $y = \tan x$ (c) $y = e^x$
(d) $y = \ln x$ (e) $y = 1/x$ (f) $y = |x|$
(g) $y = \sqrt{x}$ (h) $y = \tan^{-1}x$

9. Suppose that f has domain A and g has domain B.
(a) What is the domain of $f + g$?
(b) What is the domain of fg?
(c) What is the domain of f/g?

10. How is the composite function $f \circ g$ defined? What is its domain?

11. Suppose the graph of f is given. Write an equation for each of the graphs that are obtained from the graph of f as follows.
(a) Shift 2 units upward. (b) Shift 2 units downward.
(c) Shift 2 units to the right. (d) Shift 2 units to the left.
(e) Reflect about the x-axis.
(f) Reflect about the y-axis.
(g) Stretch vertically by a factor of 2.
(h) Shrink vertically by a factor of 2.
(i) Stretch horizontally by a factor of 2.
(j) Shrink horizontally by a factor of 2.

12. (a) What is a one-to-one function? How can you tell if a function is one-to-one by looking at its graph?
(b) If f is a one-to-one function, how is its inverse function f^{-1} defined? How do you obtain the graph of f^{-1} from the graph of f?

13. (a) How is the inverse sine function $f(x) = \sin^{-1}x$ defined? What are its domain and range?
(b) How is the inverse cosine function $f(x) = \cos^{-1}x$ defined? What are its domain and range?
(c) How is the inverse tangent function $f(x) = \tan^{-1}x$ defined? What are its domain and range?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If f is a function, then $f(s + t) = f(s) + f(t)$.

2. If $f(s) = f(t)$, then $s = t$.

3. If f is a function, then $f(3x) = 3f(x)$.

4. If the function f has an inverse and $f(2) = 3$, then $f^{-1}(3) = 2$.

5. A vertical line intersects the graph of a function at most once.

6. If f and g are functions, then $f \circ g = g \circ f$.

7. If f is one-to-one, then $f^{-1}(x) = \dfrac{1}{f(x)}$.

8. You can always divide by e^x.

9. If $0 < a < b$, then $\ln a < \ln b$.

10. If $x > 0$, then $(\ln x)^6 = 6 \ln x$.

11. If $x > 0$ and $a > 1$, then $\dfrac{\ln x}{\ln a} = \ln \dfrac{x}{a}$.

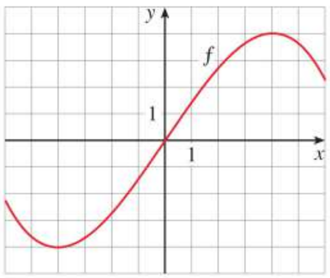
12. $\tan^{-1}(-1) = 3\pi/4$

13. $\tan^{-1}x = \dfrac{\sin^{-1}x}{\cos^{-1}x}$

14. If x is any real number, then $\sqrt{x^2} = x$.

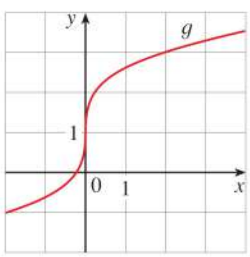
EXERCISES

1. Let f be the function whose graph is given.



(a) Estimate the value of $f(2)$.
(b) Estimate the values of x such that $f(x) = 3$.
(c) State the domain of f.
(d) State the range of f.
(e) On what interval is f increasing?
(f) Is f one-to-one? Explain.
(g) Is f even, odd, or neither even nor odd? Explain.

2. The graph of g is given.



(a) State the value of $g(2)$.
(b) Why is g one-to-one?
(c) Estimate the value of $g^{-1}(2)$.
(d) Estimate the domain of g^{-1}.
(e) Sketch the graph of g^{-1}.

3. If $f(x) = x^2 - 2x + 3$, evaluate the difference quotient

$$\frac{f(a+h) - f(a)}{h}$$

4. Sketch a rough graph of the yield of a crop as a function of the amount of fertilizer used.

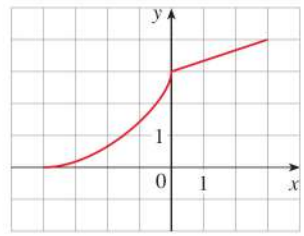
5–8 Find the domain and range of the function. Write your answer in interval notation.

5. $f(x) = 2/(3x - 1)$ **6.** $g(x) = \sqrt{16 - x^4}$

7. $h(x) = \ln(x + 6)$ **8.** $F(t) = 3 + \cos 2t$

9. Suppose that the graph of f is given. Describe how the graphs of the following functions can be obtained from the graph of f.
(a) $y = f(x) + 5$ (b) $y = f(x + 5)$
(c) $y = 1 + 2f(x)$ (d) $y = f(x - 2) - 2$
(e) $y = -f(x)$ (f) $y = f^{-1}(x)$

10. The graph of f is given. Draw the graphs of the following functions.
(a) $y = f(x - 8)$ (b) $y = -f(x)$
(c) $y = 2 - f(x)$ (d) $y = \frac{1}{2}f(x) - 1$
(e) $y = f^{-1}(x)$ (f) $y = f^{-1}(x + 3)$



11–18 Use transformations to sketch the graph of the function.

11. $f(x) = x^3 + 2$ **12.** $f(x) = (x - 3)^2$

13. $y = \sqrt{x + 2}$ **14.** $y = \ln(x + 5)$

15. $g(x) = 1 + \cos 2x$ **16.** $h(x) = -e^x + 2$

17. $s(x) = 1 + 0.5^x$ **18.** $f(x) = \begin{cases} -x & \text{if } x < 0 \\ e^x - 1 & \text{if } x \geq 0 \end{cases}$

19. Determine whether f is even, odd, or neither even nor odd.
(a) $f(x) = 2x^5 - 3x^2 + 2$ (b) $f(x) = x^3 - x^7$
(c) $f(x) = e^{-x^2}$ (d) $f(x) = 1 + \sin x$
(e) $f(x) = 1 - \cos 2x$ (f) $f(x) = (x + 1)^2$

20. Find an expression for the function whose graph consists of the line segment from the point $(-2, 2)$ to the point $(-1, 0)$ together with the top half of the circle with center the origin and radius 1.

21. If $f(x) = \ln x$ and $g(x) = x^2 - 9$, find the functions (a) $f \circ g$, (b) $g \circ f$, (c) $f \circ f$, (d) $g \circ g$, and their domains.

22. Express the function $F(x) = 1/\sqrt{x + \sqrt{x}}$ as a composition of three functions.

23. Life expectancy has improved dramatically in recent decades. The table gives the life expectancy at birth (in years) of males born in the United States. Use a scatter plot to choose an appropriate type of model. Use your model to predict the life span of a male born in the year 2030.

Birth year	Life expectancy	Birth year	Life expectancy
1900	48.3	1960	66.6
1910	51.1	1970	67.1
1920	55.2	1980	70.0
1930	57.4	1990	71.8
1940	62.5	2000	73.0
1950	65.6	2010	76.2

24. A small-appliance manufacturer finds that it costs \$9000 to produce 1000 toaster ovens a week and \$12,000 to produce 1500 toaster ovens a week.

(a) Express the cost as a function of the number of toaster ovens produced, assuming that it is linear. Then sketch the graph.

(b) What is the slope of the graph and what does it represent?

(c) What is the y-intercept of the graph and what does it represent?

25. If $f(x) = 2x + 4^x$, find $f^{-1}(6)$.

26. Find the inverse function of $f(x) = \dfrac{2x+3}{1-5x}$.

27. Use the laws of logarithms to expand each expression.

(a) $\ln x\sqrt{x+1}$ (b) $\log_2 \sqrt{\dfrac{x^2+1}{x-1}}$

28. Express as a single logarithm.

(a) $\frac{1}{2}\ln x - 2\ln(x^2+1)$

(b) $\ln(x-3) + \ln(x+3) - 2\ln(x^2-9)$

29–30 Find the exact value of each expression.

29. (a) $e^{2\ln 5}$ (b) $\log_6 4 + \log_6 54$ (c) $\tan(\arcsin \frac{4}{5})$

30. (a) $\ln \dfrac{1}{e^3}$ (b) $\sin(\tan^{-1} 1)$ (c) $10^{-3\log 4}$

31–36 Solve the equation for x. Give both an exact value and a decimal approximation, correct to three decimal places.

31. $e^{2x} = 3$

32. $\ln x^2 = 5$

33. $e^{e^x} = 10$

34. $\cos^{-1}x = 2$

35. $\tan^{-1}(3x^2) = \dfrac{\pi}{4}$

36. $\ln x - 1 = \ln(5+x) - 4$

37. The viral load for an HIV patient is 52.0 RNA copies/mL before treatment begins. Eight days later the viral load is half of the initial amount.

(a) Find the viral load after 24 days.

(b) Find the viral load $V(t)$ that remains after t days.

(c) Find a formula for the inverse of the function V and explain its meaning.

(d) After how many days will the viral load be reduced to 2.0 RNA copies/mL?

38. The population of a certain species in a limited environment with initial population 100 and carrying capacity 1000 is

$$P(t) = \frac{100{,}000}{100 + 900e^{-t}}$$

where t is measured in years.

(a) Graph this function and estimate how long it takes for the population to reach 900.

(b) Find the inverse of this function and explain its meaning.

(c) Use the inverse function to find the time required for the population to reach 900. Compare with the result of part (a).

Principles of Problem Solving

There are no hard and fast rules that will ensure success in solving problems. However, it is possible to outline some general steps in the problem-solving process and to give some principles that may be useful in the solution of certain problems. These steps and principles are just common sense made explicit. They have been adapted from George Polya's book *How To Solve It*.

1 UNDERSTAND THE PROBLEM

The first step is to read the problem and make sure that you understand it clearly. Ask yourself the following questions:

What is the unknown?

What are the given quantities?

What are the given conditions?

For many problems it is useful to

draw a diagram

and identify the given and required quantities on the diagram.

Usually it is necessary to

introduce suitable notation

In choosing symbols for the unknown quantities we often use letters such as a, b, c, m, n, x, and y, but in some cases it helps to use initials as suggestive symbols; for instance, V for volume or t for time.

2 THINK OF A PLAN

Find a connection between the given information and the unknown that will enable you to calculate the unknown. It often helps to ask yourself explicitly: "How can I relate the given to the unknown?" If you don't see a connection immediately, the following ideas may be helpful in devising a plan.

Try to Recognize Something Familiar Relate the given situation to previous knowledge. Look at the unknown and try to recall a more familiar problem that has a similar unknown.

Try to Recognize Patterns Some problems are solved by recognizing that some kind of pattern is occurring. The pattern could be geometric, or numerical, or algebraic. If you can see regularity or repetition in a problem, you might be able to guess what the continuing pattern is and then prove it.

Use Analogy Try to think of an analogous problem—that is, a similar problem, a related problem—but one that is easier than the original problem. If you can solve the similar, simpler problem, then it might give you the clues you need to solve the original, more difficult problem. For instance, if a problem involves very large numbers, you could first try a similar problem with smaller numbers. Or if the problem involves three-dimensional geometry, you could look for a similar problem in two-dimensional geometry. Or if the problem you start with is a general one, you could first try a special case.

Introduce Something Extra It may sometimes be necessary to introduce something new—an auxiliary aid—to help make the connection between the given and the unknown. For instance, in a problem where a diagram is useful the auxiliary aid could be a new line drawn in a diagram. In a more algebraic problem it could be a new unknown that is related to the original unknown.

Take Cases We may sometimes have to split a problem into several cases and give a different argument for each of the cases. For instance, we often have to use this strategy in dealing with absolute value.

Work Backward Sometimes it is useful to imagine that your problem is solved and then to work backward, step by step, until you arrive at the given data. At this point you may be able to reverse your steps and thereby construct a solution to the original problem. This procedure is commonly used in solving equations. For instance, in solving the equation $3x - 5 = 7$, we suppose that x is a number that satisfies $3x - 5 = 7$ and work backward. We add 5 to each side of the equation and then divide each side by 3 to get $x = 4$. Since each of these steps can be reversed, we have solved the problem.

Establish Subgoals In a complex problem it is often useful to set subgoals (in which the desired situation is only partially fulfilled). If we can first reach these subgoals, then we may be able to build on them to reach our final goal.

Indirect Reasoning Sometimes it is appropriate to attack a problem indirectly. In using proof by contradiction to prove that P implies Q, we assume that P is true and Q is false and try to see why this can't happen. Somehow we have to use this information and arrive at a contradiction to what we absolutely know is true.

Mathematical Induction In proving statements that involve a positive integer n, it is frequently helpful to use the following principle.

Principle of Mathematical Induction Let S_n be a statement about the positive integer n. Suppose that

1. S_1 is true.
2. S_{k+1} is true whenever S_k is true.

Then S_n is true for all positive integers n.

This is reasonable because, since S_1 is true, it follows from condition 2 (with $k = 1$) that S_2 is true. Then, using condition 2 with $k = 2$, we see that S_3 is true. Again using condition 2, this time with $k = 3$, we have that S_4 is true. This procedure can be followed indefinitely.

3 CARRY OUT THE PLAN

In Step 2 a plan was devised. In carrying out that plan we have to check each stage of the plan and write the details that prove that each stage is correct.

4 LOOK BACK

Having completed our solution, it is wise to look back over it, partly to see if we have made errors in the solution and partly to see if we can think of an easier way to solve the problem. Another reason for looking back is that it will familiarize us with the method of solution and this may be useful for solving a future problem. Descartes said, "Every problem that I solved became a rule which served afterwards to solve other problems."

These principles of problem solving are illustrated in the following examples. Before you look at the solutions, try to solve these problems yourself, referring to these principles of problem solving if you get stuck. You may find it useful to refer to this section from time to time as you solve the exercises in the remaining chapters of this book.

EXAMPLE 1 Express the hypotenuse h of a right triangle with area 25 m^2 as a function of its perimeter P.

PS Understand the problem.

SOLUTION Let's first sort out the information by identifying the unknown quantity and the data:

Unknown: hypotenuse h

Given quantities: perimeter P, area 25 m^2

PS Draw a diagram.

It helps to draw a diagram and we do so in Figure 1.

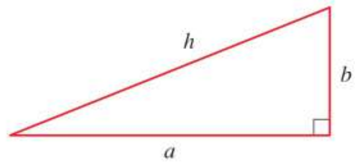


FIGURE 1

PS Connect the given with the unknown.

PS Introduce something extra.

In order to connect the given quantities to the unknown, we introduce two extra variables a and b, which are the lengths of the other two sides of the triangle. This enables us to express the given condition, which is that the triangle is right-angled, by the Pythagorean Theorem:

$$h^2 = a^2 + b^2$$

The other connections among the variables come by writing expressions for the area and perimeter:

$$25 = \tfrac{1}{2}ab \qquad P = a + b + h$$

Since P is given, notice that we now have three equations in the three unknowns a, b, and h:

$$\boxed{1} \qquad h^2 = a^2 + b^2$$

$$\boxed{2} \qquad 25 = \tfrac{1}{2}ab$$

$$\boxed{3} \qquad P = a + b + h$$

PS Relate to the familiar.

Although we have the correct number of equations, they are not easy to solve in a straightforward fashion. But if we use the problem-solving strategy of trying to recognize something familiar, then we can solve these equations by an easier method. Look at the right sides of Equations 1, 2, and 3. Do these expressions remind you of anything familiar? Notice that they contain the ingredients of a familiar formula:

$$(a + b)^2 = a^2 + 2ab + b^2$$

Using this idea, we express $(a + b)^2$ in two ways. From Equations 1 and 2 we have

$$(a + b)^2 = (a^2 + b^2) + 2ab = h^2 + 4(25) = h^2 + 100$$

From Equation 3 we have

$$(a + b)^2 = (P - h)^2 = P^2 - 2Ph + h^2$$

Thus

$$h^2 + 100 = P^2 - 2Ph + h^2$$

$$2Ph = P^2 - 100$$

$$h = \frac{P^2 - 100}{2P}$$

This is the required expression for h as a function of P. ■

As the next example illustrates, it is often necessary to use the problem-solving principle of *taking cases* when dealing with absolute values.

EXAMPLE 2 Solve the inequality $|x - 3| + |x + 2| < 11$.

SOLUTION Recall the definition of absolute value:

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

It follows that

$$|x - 3| = \begin{cases} x - 3 & \text{if } x - 3 \geqslant 0 \\ -(x - 3) & \text{if } x - 3 < 0 \end{cases}$$

$$= \begin{cases} x - 3 & \text{if } x \geqslant 3 \\ -x + 3 & \text{if } x < 3 \end{cases}$$

Similarly

$$|x + 2| = \begin{cases} x + 2 & \text{if } x + 2 \geqslant 0 \\ -(x + 2) & \text{if } x + 2 < 0 \end{cases}$$

$$= \begin{cases} x + 2 & \text{if } x \geqslant -2 \\ -x - 2 & \text{if } x < -2 \end{cases}$$

PS Take cases.

These expressions show that we must consider three cases:

$$x < -2 \qquad -2 \leqslant x < 3 \qquad x \geqslant 3$$

CASE I If $x < -2$, we have

$$\begin{aligned} |x - 3| + |x + 2| &< 11 \\ -x + 3 - x - 2 &< 11 \\ -2x &< 10 \\ x &> -5 \end{aligned}$$

CASE II If $-2 \leqslant x < 3$, the given inequality becomes

$$\begin{aligned} -x + 3 + x + 2 &< 11 \\ 5 &< 11 \qquad \text{(always true)} \end{aligned}$$

CASE III If $x \geqslant 3$, the inequality becomes

$$\begin{aligned} x - 3 + x + 2 &< 11 \\ 2x &< 12 \\ x &< 6 \end{aligned}$$

Combining cases I, II, and III, we see that the inequality is satisfied when $-5 < x < 6$. So the solution is the interval $(-5, 6)$. ■

In the following example we first guess the answer by looking at special cases and recognizing a pattern. Then we prove our conjecture by mathematical induction.

In using the Principle of Mathematical Induction, we follow three steps:

Step 1 Prove that S_n is true when $n = 1$.

Step 2 Assume that S_n is true when $n = k$ and deduce that S_n is true when $n = k + 1$.

Step 3 Conclude that S_n is true for all n by the Principle of Mathematical Induction.

EXAMPLE 3 If $f_0(x) = x/(x+1)$ and $f_{n+1} = f_0 \circ f_n$ for $n = 0, 1, 2, \ldots$, find a formula for $f_n(x)$.

PS Analogy: Try a similar, simpler problem.

SOLUTION We start by finding formulas for $f_n(x)$ for the special cases $n = 1$, 2, and 3.

$$f_1(x) = (f_0 \circ f_0)(x) = f_0(f_0(x)) = f_0\left(\frac{x}{x+1}\right)$$

$$= \frac{\dfrac{x}{x+1}}{\dfrac{x}{x+1}+1} = \frac{\dfrac{x}{x+1}}{\dfrac{2x+1}{x+1}} = \frac{x}{2x+1}$$

$$f_2(x) = (f_0 \circ f_1)(x) = f_0(f_1(x)) = f_0\left(\frac{x}{2x+1}\right)$$

$$= \frac{\dfrac{x}{2x+1}}{\dfrac{x}{2x+1}+1} = \frac{\dfrac{x}{2x+1}}{\dfrac{3x+1}{2x+1}} = \frac{x}{3x+1}$$

$$f_3(x) = (f_0 \circ f_2)(x) = f_0(f_2(x)) = f_0\left(\frac{x}{3x+1}\right)$$

PS Look for a pattern.

$$= \frac{\dfrac{x}{3x+1}}{\dfrac{x}{3x+1}+1} = \frac{\dfrac{x}{3x+1}}{\dfrac{4x+1}{3x+1}} = \frac{x}{4x+1}$$

We notice a pattern: the coefficient of x in the denominator of $f_n(x)$ is $n+1$ in the three cases we have computed. So we make the guess that, in general,

$$\boxed{4} \qquad f_n(x) = \frac{x}{(n+1)x+1}$$

To prove this, we use the Principle of Mathematical Induction. We have already verified that (4) is true for $n = 1$. Assume that it is true for $n = k$, that is,

$$f_k(x) = \frac{x}{(k+1)x+1}$$

Then $$f_{k+1}(x) = (f_0 \circ f_k)(x) = f_0(f_k(x)) = f_0\left(\frac{x}{(k+1)x+1}\right)$$

$$= \frac{\dfrac{x}{(k+1)x+1}}{\dfrac{x}{(k+1)x+1}+1} = \frac{\dfrac{x}{(k+1)x+1}}{\dfrac{(k+2)x+1}{(k+1)x+1}} = \frac{x}{(k+2)x+1}$$

This expression shows that (4) is true for $n = k+1$. Therefore, by mathematical induction, it is true for all positive integers n. ■

Problems

1. One of the legs of a right triangle has length 4 cm. Express the length of the altitude perpendicular to the hypotenuse as a function of the length of the hypotenuse.

2. The altitude perpendicular to the hypotenuse of a right triangle is 12 cm. Express the length of the hypotenuse as a function of the perimeter.

3. Solve the equation $\left|4x - |x+1|\right| = 3$.

4. Solve the inequality $|x-1| - |x-3| \geqslant 5$.

5. Sketch the graph of the function $f(x) = \left|x^2 - 4|x| + 3\right|$.

6. Sketch the graph of the function $g(x) = |x^2 - 1| - |x^2 - 4|$.

7. Draw the graph of the equation $x + |x| = y + |y|$.

8. Sketch the region in the plane consisting of all points (x, y) such that

$$|x - y| + |x| - |y| \leqslant 2$$

9. The notation $\max\{a, b, \ldots\}$ means the largest of the numbers $a, b, \ldots$. Sketch the graph of each function.

(a) $f(x) = \max\{x, 1/x\}$

(b) $f(x) = \max\{\sin x, \cos x\}$

(c) $f(x) = \max\{x^2, 2 + x, 2 - x\}$

10. Sketch the region in the plane defined by each of the following equations or inequalities.

(a) $\max\{x, 2y\} = 1$

(b) $-1 \leqslant \max\{x, 2y\} \leqslant 1$

(c) $\max\{x, y^2\} = 1$

11. Show that if $x > 0$ and $x \neq 1$, then

$$\frac{1}{\log_2 x} + \frac{1}{\log_3 x} + \frac{1}{\log_5 x} = \frac{1}{\log_{30} x}$$

12. Find the number of solutions of the equation $\sin x = \dfrac{x}{100}$.

13. Find the exact value of

$$\sin\frac{\pi}{100} + \sin\frac{2\pi}{100} + \sin\frac{3\pi}{100} + \cdots + \sin\frac{200\pi}{100}$$

14. (a) Show that the function $f(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$ is an odd function.

(b) Find the inverse function of f.

15. Solve the inequality $\ln(x^2 - 2x - 2) \leqslant 0$.

16. Use indirect reasoning to prove that $\log_2 5$ is an irrational number.

17. A driver sets out on a journey. For the first half of the distance she drives at the leisurely pace of 30 mi/h; she drives the second half at 60 mi/h. What is her average speed on this trip?

18. Is it true that $f \circ (g + h) = f \circ g + f \circ h$?

19. Prove that if n is a positive integer, then $7^n - 1$ is divisible by 6.

20. Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

21. If $f_0(x) = x^2$ and $f_{n+1}(x) = f_0(f_n(x))$ for $n = 0, 1, 2, \ldots$, find a formula for $f_n(x)$.

22. (a) If $f_0(x) = \dfrac{1}{2 - x}$ and $f_{n+1} = f_0 \circ f_n$ for $n = 0, 1, 2, \ldots$, find an expression for $f_n(x)$ and use mathematical induction to prove it.

(b) Graph f_0, f_1, f_2, f_3 on the same screen and describe the effects of repeated composition.