October 9th

We move to some discussion of the non-homogeneous case (still 2nd order linear with constant coefficients):

First is the 'Method of Undetermined Coefficients', i.e., guessing. Do some guesses.

An example would be

$$y'' - 2y' + y = e3x$$

Let us first notice the solution to the related homogeneous equation:

$$y'' - 2y' + y = 0$$

has the characteristic equation

$$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0$$

so the general solution to the homogeneous equation is

$$y(x) = c_1 e^x + c_2 x e^x$$

What about the non-homogeneous version? Well, notice that we want to get e^{3x} after taking some derivatives, so we probably need to start with e^{3x}, since that is the antiderivative (well, $\frac{1}{3}e^{3x}$. Let us guess that the solution is of the form $y(x) = A \cdot e^{3x}$ for some constant coefficient A. Putting this into the differential equation, we get

$$\begin{aligned} y'' - 2y' + y &= \left(Ae^{3x}\right)'' - 2\left(Ae^{3x}\right)' + Ae^{3x} \\ &= 9Ae^{3x} - 6Ae^{3x} + Ae^{3x} \\ &= 4Ae^{3x} \end{aligned}$$

So, if we want this to be e^{3x}, we set $A = \frac{1}{4}$

From previous discussion, any solution to this linear differential equation will be this specific solution plus any solution to the associated homogeneous equation, so in the end our general solution is

$$y(x) = \frac{1}{4}e^{3x} + c_1 e^x + c_2 x e^x$$

for arbitrary constants c_1 and c_2.

Another example: Solve the equation

$$y'' - 3y' - 4y = \cos(x)$$

Again, let's get the associated homogeneous equation out of the way:

$$\lambda^2 - 3\lambda - 4 = (\lambda + 1)(\lambda - 4) = 0$$

so $y(x) = c_1 e^{-x} + c_2 e^{4x}$ is the general solution to $y'' - 3y' - 4y = 0$.

Let's do it wrong first: thinking like before, if we want to take a derivative and get $\cos(x)$, we'll need to have $\sin(x)$ in our function. Guessing $y(x) = A\sin(x)$ gives

$$\begin{aligned} y'' - 3y' - 4y &= (A\sin(x))'' - 3\,(A\sin(x))' - 4A\sin(x) \\ &= -A\sin(x) - 3A\cos(x) - 4A\sin(x) \\ &= -5A\sin(x) - 3A\cos(x) \end{aligned}$$

Notice there is no choice of A that would make this equal to $\cos(x)$: if A is not zero, there will be a sin term. The problem is that as we take aren't just taking one derivative, we have the original $y(x)$ term as well as second derivatives on the left side, so we get $\sin(x)$ terms popping up. In order to cancel these terms, we try to add a $\cos(x)$ term as well, since the derivatives there will produce $\sin(x)$ we can use to cancel out.

So, set $y(x) = A\sin(x) + B\cos(x)$ and compute

$$\begin{aligned} y'' - 3y' - 4y &= (A\sin(x) + B\cos(x))'' - 3\,(A\sin(x) + B\cos(x))' \\ &\quad - 4\,(A\sin(x) + B\cos(x)) \\ &= -A\sin(x) - B\cos(x) - 3A\cos(x) + 3B\sin(x) \\ &\quad - 4A\sin(x) - 4B\cos(x) \\ &= (3B - 5A)\sin(x) + (-3A - 5B)\cos(x) \end{aligned}$$

If we want there to be no $\sin(x)$ term, we need $3B - 5A = 0$, or $A = \frac{3}{5}B$ and then to get $\cos(x)$ we need

$$\begin{aligned} -3A - 5B &= 1 \\ -3\frac{3}{5}B - 5B &= 1 \\ \frac{-34}{5}B &= 1 \\ B &= -\frac{5}{34} \\ A &= -\frac{3}{34} \end{aligned}$$

and we get

$$y(x) = -\frac{3}{35}\sin(x) - \frac{5}{34}\cos(x)$$

Of course, for our general solution we have

$$y(x) = -\frac{3}{35}\sin(x) - \frac{5}{34}\cos(x) + c_1 e^{-x} + c_2 e^{4x}$$

There's not too much theory happening: We have some intuition about what functions will lead to the non-homogeneous term when we differentiate, so we take such functions and add them together with some unknown coefficients and put that into the differential equation, hoping to get a solvable system of equations for those coefficients.

Another example, to serve as a warning: Solve the equation

$$y'' - 3y' - 4y = e^{4x}$$

Here we know the homogeneous part from our previous example. Like our first example, we imagine a solution must be something like Ae^{4x}, to get back e^{4x} terms when we differentiate. If we do that:

$$\begin{aligned} y'' - 3y' - 4y &= \left(Ae^{4x}\right)'' - 3\left(Ae^{4x}\right)' - 4\left(Ae^{4x}\right) \\ &= 16Ae^{4x} - 3 \cdot 4Ae^{4x} - 4Ae^{4x} \\ &= e^{4x}\left(16A - 12A - 4A\right) \\ &= e^{4x} \cdot 0 \\ &= 0 \end{aligned}$$

You might have noticed the problem before we started: since e^{4x} is a solution to the associated homogeneous equation, any multiple is still a solution to the homogeneous equation. So Ae^{4x} will make the left-hand side zero, and not give us the correct non-homogeneous solution. What should we do?

Thinking about the product rule, you might notice that xe^{4x} would also give and e^{4x} term when we differentiate, so let's try $y(x) = Axe^{4x}$:

$$\begin{aligned} y'' - 3y' - 4y &= \left(Axe^{4x}\right)'' - 3\left(Axe^{4x}\right)' - 4\left(Axe^{4x}\right) \\ &= 8Ae^{4x} + 16Axe^{4x} - 3Ae^{4x} - 12Axe^{4x} - 4Axe^{4x} \\ &= 5Ae^{4x} \end{aligned}$$

If we want to get e^{4x}, we need $A = \frac{1}{5}$. So a particular solution to our differential equation is

$$y(x) = \frac{1}{5}xe^{4x}$$

There is a less 'guess-and-check' way of thinking about this, which I'll just sketch: suppose we have the differential equation

$$y'' - 3y' - 4y = f(x)$$

for some function $f(x)$. We know the characteristic equation factors

$$\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0$$

If we squint our eyes a bit, think of the λs as representing 'take a derivative', and rewrite in Leibniz notation, we might try to 'factor' like so:

$$\begin{aligned} \frac{d^2}{dx^2}y - 3\frac{d}{dx}y - 4y &= f(x) \\ \left(\frac{d^2}{dx^2} - 3\frac{d}{dx} - 4)\right) y &= f(x) \\ \left(\frac{d}{dx} - 4\right)\left(\frac{d}{dx} + 1\right) y &= f(x) \end{aligned}$$

Now, introduce a new function,

$$u = \left(\frac{d}{dx} + 1\right) y = y' + y$$

Then the above is

$$\left(\frac{d}{dx} - 4\right) u = f(x)$$

or

$$u' - 4u = f(x)$$

which is a first order linear equation we can solve (using an integrating factor).

Once we have solved for the function $u(x)$, the defining equation $y' + y = u(x)$ is then another first order linear equation, which we can solve.

Example:

$$y'' + 3y' - 10y = x$$

has characteristic equation

$$\lambda^2 + 3\lambda - 10 = (\lambda - 2)(\lambda + 5) = 0$$

so we rewrite this equation as or

$$(\frac{d}{dx} - 2)(\frac{d}{dx} + 5)y = x$$

and consider

$$(\frac{d}{dx} - 2)u = x$$

Written more traditionally,

$$u' - 2u = x$$

so

$$\begin{aligned}
e^{-2x}u' - 2e^{-2x}u &= xe^{-2x} \\
\left(e^{-2x}u\right)' &= xe^{-2x} \\
e^{-2x}u &= \int xe^{-2x} \\
e^{-2x}u &= -\frac{1}{4}(1+2x)e^{-2x} + c \\
u(x) &= -\frac{1}{4}(1+2x) + ce^{2x}
\end{aligned}$$

For simplicity, set $c = 0$, so

$$u(x) = -\frac{1}{4}(1 + 2x)$$

Now, we have

$$(\frac{d}{dx}+5)y = u$$

or

$$y' + 5y = -\frac{1}{4}(1+2x)$$

which, using an integrating factor of e^{5x}, leads to

$$(e^{5x}y)' = -\frac{1}{4}(1+2x)e^{5x}$$

Integrating gives

$$e^{5x}y = -\frac{1}{100}e^{5x}(10x+3) + c$$

or

$$y(x) = -\frac{10x+3}{100} + ce^{-5x}$$

Notice the ce^{5x} term can be thought of as solving the associated homogeneous equation. It is not difficult to check that

$$y(x) = -\frac{10x+3}{100}$$

solves the differential equation

$$y'' + 3y' - 10y = x$$

(although, thinking about what sorts of functions would produce an x term from itself and derivatives, you might guess that a linear function will work, and the algebra for checking $Ax+B$ will probably be quicker than solving two first order equations like above: you see I ran out of steam showing details for the second one).