September 20

I finished discussing 'Completing the Square', but I don't think I was very clear, so I'll just tell you the punchline:

If you have the quadratic polynomial expression

$$ax^2 + bx + c$$

you can add and subtract $\left(\frac{b}{2\sqrt{a}}\right)^2$ to get

$$\begin{aligned} ax^2 + bx + c &= ax^2 + bx + c + \left(\frac{b}{2\sqrt{a}}\right)^2 - \left(\frac{b}{2\sqrt{a}}\right)^2 \\ &= ax^2 + bx + \left(\frac{b}{2\sqrt{a}}\right)^2 + c - \left(\frac{b}{2\sqrt{a}}\right)^2 \\ &= \left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 + c - \left(\frac{b}{2\sqrt{a}}\right)^2 \end{aligned}$$

This let's us solve for x to make the quadratic polynomial equal to 0 (this looks scary, but ignore it and see the next page):

$$\begin{aligned} ax^2 + bx + c &= 0 \\ \left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 + c - \left(\frac{b}{2\sqrt{a}}\right)^2 &= 0 \\ \left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 &= \left(\frac{b}{2\sqrt{a}}\right)^2 - c \\ \sqrt{a}x + \frac{b}{2\sqrt{a}} &= \sqrt{\left(\frac{b}{2\sqrt{a}}\right)^2 - c} \\ x &= \frac{1}{\sqrt{a}}\left(-\frac{b}{2\sqrt{a}} + \sqrt{\left(\frac{b}{2\sqrt{a}}\right)^2 - c}\right) \\ x &= \frac{1}{\sqrt{a}}\left(-\frac{b}{2\sqrt{a}} + \frac{2\sqrt{a}}{2\sqrt{a}}\sqrt{\left(\frac{b}{2\sqrt{a}}\right)^2 - c}\right) \\ x &= \frac{1}{\sqrt{a}}\left(-\frac{b}{2\sqrt{a}} + \frac{\sqrt{b^2 - 4ac}}{2\sqrt{a}}\right) = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

This is the usual quadratic formula. In the third line, we took a square root. In our class, the symbol $\sqrt{\ }$ always means 'the **positive** square root', so if we don't remember there is also a negative square root, we will miss part of the answer. There can be two solutions to a quadratic equation (sometimes there is only one, like in $(x+1)^2 = x^2+2x+1 = 0$, and sometimes there are no solutions, like $x^2+1=0$).

NOW, I know that algebra was nasty. That's because we jumped in without thinking: a little algebra at the beginning can avoid lots of algebra later. What I mean here is, if we want to solve the equation

$$ax^2 + bx + c = 0,$$

we can divide both sides by a and get the equation

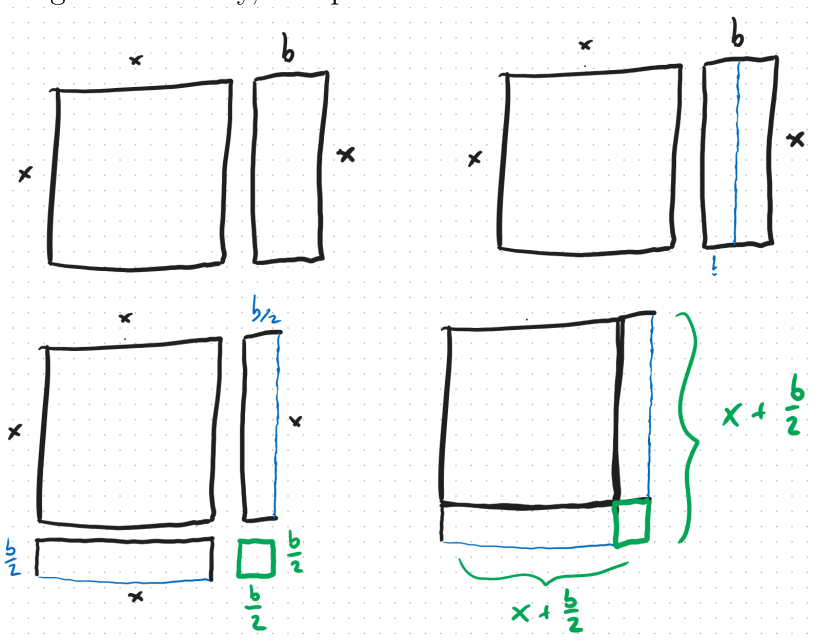
$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Then add and subtract $\left(\frac{b}{2a}\right)^2$:

$$\begin{aligned}
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0\\
x^2 + \frac{b}{a}x + \frac{c}{a} + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 &= 0\\
x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2 &= 0\\
\left(x + \frac{b}{2a}\right)^2 &= \left(\frac{b}{2a}\right)^2 - \frac{c}{a}\\
x &= -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}\\
x &= -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{4a}{4a}\frac{c}{a}}\\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

So again we get the usual quadratic formula. I think it's a little easier to divide by a first. Of course, you can just memorize the quadratic formula, but it's good to know that it comes from somewhere, and you could figure it out.

You might be complaining, how would we know to add and subtract that exact thing? Just guessing, or me telling you what to do? Isn't that the same as memorizing a formula? Well, I also claim there's a reason to think of doing this. Namely, this picture:



Some examples with actual numbers:

Solve $x^2 + 3x + 2 = 0$: Here, $a = 1$ so we don't need to divide, and $b = 3$ so $\frac{b}{2} = \frac{3}{2}$ and $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$. Following the above:

$$\begin{aligned}
x^2 + 3x + 2 &= 0 \\
x^2 + 3x + \frac{9}{4} - \frac{9}{4} + 2 &= 0 \\
&= \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} + 2 = 0 \\
\left(x + \frac{3}{2}\right)^2 &= \frac{1}{4} \\
x &= -\frac{3}{2} \pm \sqrt{\frac{1}{4}} \\
x &= -\frac{3}{2} \pm \frac{1}{2} \\
x &= -2 \text{ or } -1
\end{aligned}$$

(notice you could also factor this equation)

Another example: Solve $x^2+9x+5=0$. Again, $a=1$ so we don't need to worry about it. Here $b=9$ so $\frac{b}{2}=\frac{9}{2}$, and $(\frac{b}{2})^2=\frac{81}{4}$. Adding and subtracting:

$$\begin{aligned}
x^2+9x+5&=0\\
x^2+9x+\frac{81}{4}-\frac{81}{4}+5&=0\\
(x+\frac{9}{2})^2&=\frac{81}{4}-5\\
(x+\frac{9}{2})^2&=\frac{61}{4}\\
x+\frac{9}{2}&=\pm\sqrt{\frac{61}{4}}\\
x&=-\frac{9}{2}\pm\frac{\sqrt{61}}{2}\\
x&=\frac{-9\pm\sqrt{61}}{2}
\end{aligned}$$

You can (and should) check that this is the same as what you get with the quadratic formula.

Another example: Solve $4x^2+2x-8=0$. Here $a=4$, so I find it easier to divide everything by 4:

$$\begin{aligned}
4x^2+2x-8&=0\\
\frac{1}{4}\left(4x^2+2x-8\right)&=\frac{1}{4}\cdot 0=0\\
x^2+\frac{1}{2}x-2&=0
\end{aligned}$$

Now we have a quadratic with leading coefficient 1, and linear coefficient

$b = \frac{1}{2}$, so $(\frac{b}{2})^2 = (\frac{1}{4})^2 = \frac{1}{16}$. Adding and subtracting:

$$\begin{aligned}
x^2 + \frac{1}{2}x - 2 &= 0 \\
x^2 + \frac{1}{2}x + \frac{1}{16} - \frac{1}{16} - 2 &= 0 \\
(x + \frac{1}{4})^2 - \frac{1}{16} - 2 &= 0 \\
(x + \frac{1}{4})^2 &= \frac{33}{16} \\
x + \frac{1}{4} &= \pm\sqrt{\frac{33}{16}} = \pm\frac{\sqrt{33}}{4} \\
x &= \frac{-1 \pm \sqrt{33}}{4}
\end{aligned}$$

Let's check that this works: set $x = \frac{-1+\sqrt{33}}{4}$, and plug in the original equation:

$$\begin{aligned}
4x^2 + 2x - 8 &= 4\left(\frac{-1+\sqrt{33}}{4}\right)^2 + 2\left(\frac{-1+\sqrt{33}}{4}\right) - 8 \\
&= 4\left(\frac{(-1)^2 + 2(-1)\sqrt{33} + \sqrt{33}^2}{4^2}\right) + \frac{-1+\sqrt{33}}{2} - 8 \\
&= \frac{(-1)^2 + 2(-1)\sqrt{33} + \sqrt{33}^2}{4} + \frac{-2+2\sqrt{33}}{4} - \frac{32}{4} \\
&= \frac{1 - 2\sqrt{33} + 33 - 2 + 2\sqrt{33} - 32}{4} \\
&= \frac{0}{4} \\
&= 0
\end{aligned}$$

So $x = \frac{-1+\sqrt{33}}{4}$ is a solution to this polynomial equation. I'll let you check that the other answer we got, with $-\sqrt{33}$ in the numerator, is also a true solution.

Of course, you can also get this answer from the quadratic formula.

Then, I did some examples of turning equations which may not seem quadratic into quadratic equations:

Solve $x = \sqrt{x+6}$.

To do this, square both sides to get

$$x^2 = \sqrt{x+6}^2 = x + 6$$

or

$$x^2 - x - 6 = 0$$

We can factor this by noticing $2 \cdot (-3) = -6$ and $2 + (-3) = -1$, so

$$x^2 - x - 6 = (x-3)(x+2) = 0,$$

So we have either $x = 3$ or $x = -2$. If we think about the original equation, $x = \sqrt{x+6}$, we see $x = 3$ is a solution: $3 = \sqrt{3+6} = \sqrt{9} = 3$.

Is $x = -2$ a solution? Remember we said the symbol $\sqrt{}$ always means the positive square root, so if we check $x = -2$ we get

$$\begin{aligned} -2 &\stackrel{?}{=} \sqrt{(-2)+6} \\ &\stackrel{?}{=} \sqrt{4} \\ -2 &\stackrel{?}{=} 2 \end{aligned}$$

which is **wrong!** When we squared both sides, we forgot about the difference between negative and positive (since the square of a negative number is positive, just like the square of a positive number). This gave us an extra solution, often called an **extraneous solution** to sound fancy.

Another example: Find x so that

$$\sqrt{5x+11} - 1 = x$$

Here, add 1 to both sides to get the $\sqrt{}$ symbol by itself:

$$\sqrt{5x+11} = x + 1$$

then square both sides

$$\begin{aligned} \sqrt{5x+11} &= x + 1 \\ \sqrt{5x+11}^2 &= (x+1)^2 \\ 5x + 11 &= x^2 + 2x + 1 \\ 0 &= x^2 - 3x - 10 \end{aligned}$$

Here, you might again notice that the quadratic factors

$$0 = x^2 - 3x - 10 = (x+2)(x-5)$$

So we get either $x = -2$ or $x = 5$.

Checking $x = 5$: we have

$$\begin{aligned}
\sqrt{5x+11} - 1 &= \sqrt{5 \cdot 5 + 11} - 1 \\
&= \sqrt{36} - 1 \\
&= 6 - 1 \\
&= 5 \\
&= x
\end{aligned}$$

so $x = 5$ really is a solution.

On the other hand, checking $x = -2$:

$$\begin{aligned}
\sqrt{5x+11} - 1 &= \sqrt{5 \cdot (-2) + 11} - 1 \\
&= \sqrt{1} - 1 \\
&= 1 - 1 \\
&= 0 \\
&\neq x
\end{aligned}$$

so $x = -2$ is **not** a solution to our original equation, even though it is a solution to the squared equation.