Math 3002: Problem Set 6

1. A couple more second order linear constant coefficient Solve the following initial value problems (explain any trick questions):

   (a)
   $$\begin{cases} y'' + 3y' + 4y = 0 \\ y(0) = 1 \\ y'(0) = -1 \end{cases}$$

   (b)
   $$\begin{cases} y'' + 6y' + 9y = 0 \\ y(0) = 2 \\ y'(0) = 2 \end{cases}$$

   (c)
   $$\begin{cases} y'' - 6y' - 27y = 0 \\ y(0) = 0 \\ y'(0) = 3 \end{cases}$$

2. Consider the differential equation

$$y'' + p(x) \cdot y' + q(x) \cdot y = 0$$

Suppose that $f(x)$ and $g(x)$ are fundamental solutions, i.e., linearly independent. Show that:

(a)

$$p(x) = -\frac{f(x) \cdot g''(x) - g(x) \cdot f''(x)}{W(f,g)(x)}$$

(b)

$$q(x) = \frac{f'(x) \cdot g''(x) - g'(x) \cdot f''(x)}{W(f,g)(x)}$$

(c) Use the above to write a differential equation which has solutions $f(x) = x$ and $g(x) = \sin(x)$.

(d) Use the above to write a differential equation which has solutions $f(x) = e^x$ and $g(x) = \sin(x)$.

3. Show that if $u(x)$ is a solution to the differential equation

$$u'' + q(x)u = 0$$

then $v(x) = \frac{u'(x)}{u(x)}$ is a solution to

$$v' + v^2 + q(x) = 0$$

Use this to solve the equation

$$y' + y^2 + 1 = 0$$

4. Check that $f(x) = x$ solves the differential equation

$$x^2 y'' + 2xy' - 2y = 0$$

Use Abel's theorem to write a differential equation for another (independent) solution, and solve it. Write the general form of a solution to

$$x^2 y'' + 2xy' - 2y = 0$$