7

LIMIT CYCLES

7.0 Introduction

A ***limit cycle*** is an isolated closed trajectory. *Isolated* means that neighboring trajectories are not closed; they spiral either toward or away from the limit cycle (Figure 7.0.1).

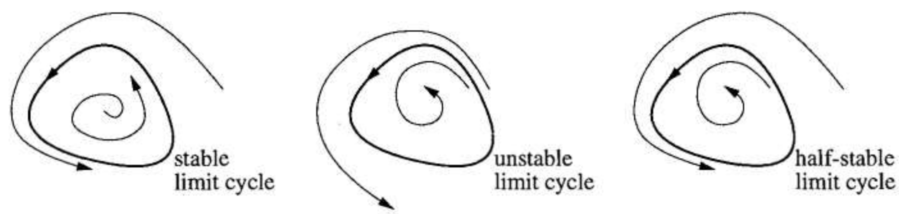

Figure 7.0.1

If all neighboring trajectories approach the limit cycle, we say the limit cycle is *stable* or *attracting*. Otherwise the limit cycle is *unstable*, or in exceptional cases, *half-stable*.

Stable limit cycles are very important scientifically—they model systems that exhibit self-sustained oscillations. In other words, these systems oscillate even in the absence of external periodic forcing. Of the countless examples that could be given, we mention only a few: the beating of a heart; the periodic firing of a pacemaker neuron; daily rhythms in human body temperature and hormone secretion; chemical reactions that oscillate spontaneously; and dangerous self-excited vibrations in bridges and airplane wings. In each case, there is a standard oscillation of some preferred period, waveform, and amplitude. If the system is perturbed slightly, it always returns to the standard cycle.

Limit cycles are inherently nonlinear phenomena; they can't occur in linear systems. Of course, a linear system $\dot{\mathbf{x}} = A\mathbf{x}$ can have closed orbits, but they won't be *isolated*; if $\mathbf{x}(t)$ is a periodic solution, then so is $c\mathbf{x}(t)$ for any constant $c \neq 0$. Hence $\mathbf{x}(t)$ is surrounded by a one-parameter family of closed orbits (Figure 7.0.2).

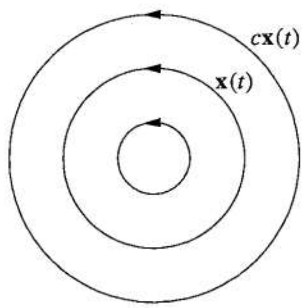

Figure 7.0.2

Consequently, the amplitude of a linear oscillation is set entirely by its initial conditions; any slight disturbance to the amplitude will persist forever. In contrast, limit cycle oscillations are determined by the structure of the system itself.

The next section presents two examples of systems with limit cycles. In the first case, the limit cycle is obvious by inspection, but normally it's difficult to tell whether a given system has a limit cycle, or indeed any closed orbits, from the governing equations alone. Sections 7.2–7.4 present some techniques for ruling out closed orbits or for proving their existence. The remainder of the chapter discusses analytical methods for approximating the shape and period of a closed orbit and for studying its stability.

7.1 Examples

It's straightforward to construct examples of limit cycles if we use polar coordinates.

EXAMPLE 7.1.1: A SIMPLE LIMIT CYCLE

Consider the system

$$\dot{r} = r(1 - r^2), \qquad \dot{\theta} = 1 \tag{1}$$

where $r \geq 0$. The radial and angular dynamics are uncoupled and so can be analyzed separately. Treating $\dot{r} = r(1 - r^2)$ as a vector field on the line, we see that $r^* = 0$ is an unstable fixed point and $r^* = 1$ is stable (Figure 7.1.1).

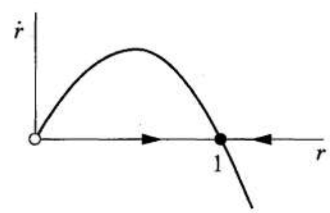

Figure 7.1.1

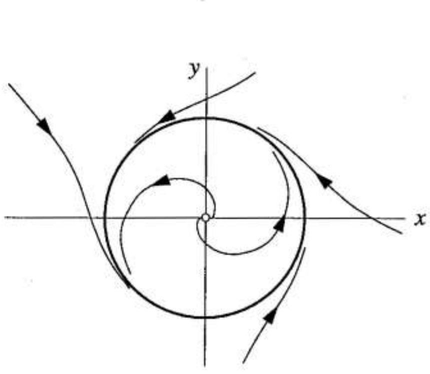

Figure 7.1.2

Hence, back in the phase plane, all trajectories (except $r^* = 0$) approach the unit circle $r^* = 1$ monotonically. Since the motion in the θ-direction is simply rotation at constant angular velocity, we see that all trajectories spiral asymptotically toward a limit cycle at $r = 1$ (Figure 7.1.2).

It is also instructive to plot solutions as functions of t. For instance, in Figure 7.1.3 we plot $x(t) = r(t)\cos\theta(t)$ for a trajectory starting outside the limit cycle. As expected, the solution settles down to a sinusoidal oscillation of constant amplitude, corresponding to the limit cycle solution $x(t) = \cos(t + \theta_0)$ of (1). ■

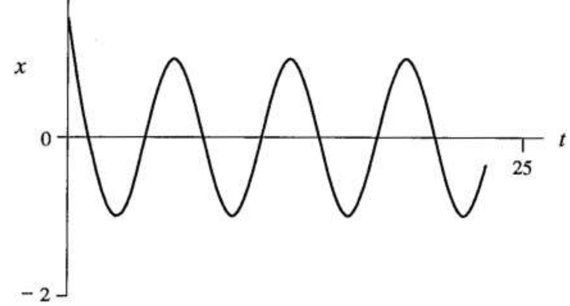

Figure 7.1.3

EXAMPLE 7.1.2: VAN DER POL OSCILLATOR

A less transparent example, but one that played a central role in the development of nonlinear dynamics, is given by the ***van der Pol equation***

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \tag{2}$$

where $\mu \geq 0$ is a parameter. Historically, this equation arose in connection with the nonlinear electrical circuits used in the first radios (see Exercise 7.1.6 for the circuit). Equation (2) looks like a simple harmonic oscillator, but with a ***nonlinear damping*** term $\mu(x^2 - 1)\dot{x}$. This term acts like ordinary positive damping for $|x| > 1$, but like *negative* damping for $|x| < 1$. In other words, it causes large-amplitude oscillations to decay, but it pumps them back up if they become too small.

As you might guess, the system eventually settles into a self-sustained oscillation where the energy dissipated over one cycle balances the energy pumped in. This idea can be made rigorous, and with quite a bit of work, one can prove that *the van der Pol equation has a unique, stable limit cycle for each* $\mu > 0$. This result follows from a more general theorem discussed in Section 7.4.

To give a concrete illustration, suppose we numerically integrate (2) for $\mu = 1.5$, starting from $(x, \dot{x}) = (0.5, 0)$ at $t = 0$. Figure 7.1.4 plots the solution in the phase plane and Figure 7.1.5 shows the graph of $x(t)$. Now, in contrast to Example 7.1.1, the limit cycle is not a circle and the stable waveform is not a sine wave. ■

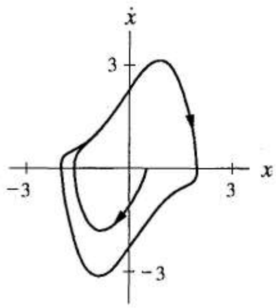

Figure 7.1.4

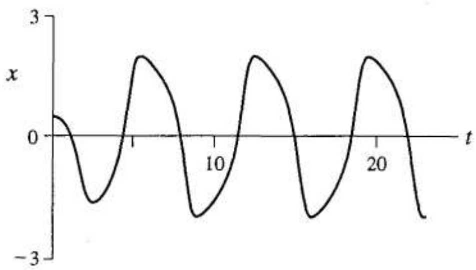

Figure 7.1.5

7.2 Ruling Out Closed Orbits

Suppose we have a strong suspicion, based on numerical evidence or otherwise, that a particular system has no periodic solutions. How could we prove this? In the last chapter we mentioned one method, based on index theory (see Examples 6.8.5 and 6.8.6). Now we present three other ways of ruling out closed orbits. They are of limited applicability, but they're worth knowing about, in case you get lucky.

Gradient Systems

Suppose the system can be written in the form $\dot{\mathbf{x}} = -\nabla V$, for some continuously differentiable, single-valued scalar function $V(\mathbf{x})$. Such a system is called a ***gradient system*** with ***potential function*** V.

Theorem 7.2.1: Closed orbits are impossible in gradient systems.

Proof: Suppose there were a closed orbit. We obtain a contradiction by considering the change in V after one circuit. On the one hand, $\Delta V = 0$ since V is single-valued. But on the other hand,

$$
\begin{aligned}
\Delta V &= \int_0^T \frac{dV}{dt}\,dt \\
&= \int_0^T (\nabla V \cdot \dot{\mathbf{x}})dt \\
&= -\int_0^T \|\dot{\mathbf{x}}\|^2\,dt \\
&< 0
\end{aligned}
$$

(unless $\dot{\mathbf{x}} \equiv \mathbf{0}$, in which case the trajectory is a fixed point, not a closed orbit). This contradiction shows that closed orbits can't exist in gradient systems. ■

The trouble with Theorem 7.2.1 is that most two-dimensional systems are *not* gradient systems. (Although, curiously, all vector fields *on the line* are gradient systems; this gives another explanation for the absence of oscillations noted in Sections 2.6 and 2.7.)

EXAMPLE 7.2.1:

Show that there are no closed orbits for the system $\dot{x} = \sin y$, $\dot{y} = x\cos y$.

Solution: The system is a gradient system with potential function $V(x, y) = -x\sin y$, since $\dot{x} = -\partial V/\partial x$ and $\dot{y} = -\partial V/\partial y$. By Theorem 7.2.1, there are no closed orbits. ■

How can you tell whether a system is a gradient system? And if it is, how do you find its potential function V? See Exercises 7.2.5 and 7.2.6.

Even if the system is not a gradient system, similar techniques may still work, as in the following example. We examine the change in an energy-like function after one circuit around the putative closed orbit, and derive a contradiction.

EXAMPLE 7.2.2:

Show that the nonlinearly damped oscillator $\ddot{x} + (\dot{x})^3 + x = 0$ has no periodic solutions.

Solution: Suppose that there were a periodic solution $x(t)$ of period T. Consider the energy function $E(x, \dot{x}) = \frac{1}{2}(x^2 + \dot{x}^2)$. After one cycle, x and $\dot{x}$ return to their starting values, and therefore $\Delta E = 0$ around any closed orbit.

On the other hand, $\Delta E = \int_0^T \dot{E}\,dt$. If we can show this integral is nonzero, we've reached a contradiction. Note that $\dot{E} = \dot{x}(x + \ddot{x}) = \dot{x}(-\dot{x}^3) = -\dot{x}^4 \le 0$. Therefore $\Delta E = -\int_0^T (\dot{x})^4\,dt \le 0$, with equality only if $\dot{x} \equiv 0$. But $\dot{x} \equiv 0$ would mean the trajectory is a fixed point, contrary to the original assumption that it's a closed orbit. Thus ΔE is *strictly* negative, which contradicts $\Delta E = 0$. Hence there are no periodic solutions. ■

Liapunov Functions

Even for systems that have nothing to do with mechanics, it is occasionally possible to construct an energy-like function that decreases along trajectories. Such a function is called a Liapunov function. If a Liapunov function exists, then closed orbits are forbidden, by the same reasoning as in Example 7.2.2.

To be more precise, consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with a fixed point at $\mathbf{x}^*$. Suppose that we can find a ***Liapunov function***, i.e., a continuously differentiable, real-valued function $V(\mathbf{x})$ with the following properties:

1. $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{x}^*$, and $V(\mathbf{x}^*) = 0$. (We say that V is *positive definite*.)
2. $\dot{V} < 0$ for all $\mathbf{x} \neq \mathbf{x}^*$. (All trajectories flow "downhill" toward $\mathbf{x}^*$.)

Then $\mathbf{x}^*$ is globally asymptotically stable: for all initial conditions, $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$. In particular the system has no closed orbits. (For a proof, see Jordan and Smith 1987.)

The intuition is that all trajectories move monotonically down the graph of $V(\mathbf{x})$ toward $\mathbf{x}^*$ (Figure 7.2.1).

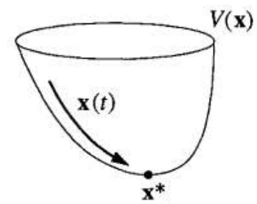

Figure 7.2.1

The solutions can't get stuck anywhere else because if they did, V would stop changing, but by assumption, $\dot{V} < 0$ everywhere except at $\mathbf{x}^*$.

Unfortunately, there is no systematic way to construct Liapunov functions. Divine inspiration is usually required, although sometimes one can work backwards. Sums of squares occasionally work, as in the following example.

EXAMPLE 7.2.3:

By constructing a Liapunov function, show that the system $\dot{x} = -x + 4y$, $\dot{y} = -x - y^3$ has no closed orbits.

Solution: Consider $V(x, y) = x^2 + ay^2$, where a is a parameter to be chosen later. Then $\dot{V} = 2x\dot{x} + 2ay\dot{y} = 2x(-x + 4y) + 2ay(-x - y^3) = -2x^2 + (8 - 2a)xy - 2ay^4$. If we choose $a = 4$, the xy term disappears and $\dot{V} = -2x^2 - 8y^4$. By inspection, $V > 0$ and $\dot{V} < 0$ for all $(x, y) \neq (0, 0)$. Hence $V = x^2 + 4y^2$ is a Liapunov function and so there are no closed orbits. In fact, all trajectories approach the origin as $t \to \infty$. ■

Dulac's Criterion

The third method for ruling out closed orbits is based on Green's theorem, and is known as Dulac's criterion.

Dulac's Criterion: Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ be a continuously differentiable vector field defined on a simply connected subset R of the plane. If there exists a continuously differentiable, real-valued function $g(\mathbf{x})$ such that $\nabla \cdot (g\dot{\mathbf{x}})$ has one sign throughout R, then there are no closed orbits lying entirely in R.

Proof: Suppose there were a closed orbit C lying entirely in the region R. Let A denote the region inside C (Figure 7.2.2). Then Green's theorem yields

$$\iint_A \nabla \cdot (g\dot{\mathbf{x}})\, dA = \oint_C g\dot{\mathbf{x}} \cdot \mathbf{n}\; d\ell$$

where $\mathbf{n}$ is the outward normal and $d\ell$ is the element of arc length along C. Look first at the double integral on the left: it must be *nonzero,* since $\nabla \cdot (g\dot{\mathbf{x}})$ has one sign in R. On the other hand, the line integral on the right equals *zero* since $\dot{\mathbf{x}} \cdot \mathbf{n} = 0$ everywhere, by the assumption that C is a trajectory (the tangent vector $\dot{\mathbf{x}}$ is orthogonal to $\mathbf{n}$). This contradiction implies that no such C can exist. ■

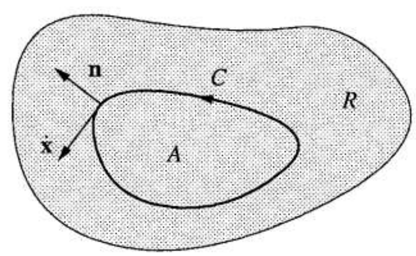

Figure 7.2.2

Dulac's criterion suffers from the same drawback as Liapunov's method: there is no algorithm for finding $g(\mathbf{x})$. Candidates that occasionally work are $g = 1$, $1/x^a y^b$, e^{ax}, and e^{ay}.

EXAMPLE 7.2.4:

Show that the system $\dot{x} = x(2 - x - y)$, $\dot{y} = y(4x - x^2 - 3)$ has no closed orbits in the positive quadrant $x, y > 0$.

Solution: A hunch tells us to pick $g = 1/xy$. Then

$$\begin{aligned}\nabla \cdot (g\dot{\mathbf{x}}) &= \frac{\partial}{\partial x}(g\dot{x}) + \frac{\partial}{\partial y}(g\dot{y}) \\ &= \frac{\partial}{\partial x}\left(\frac{2-x-y}{y}\right) + \frac{\partial}{\partial y}\left(\frac{4x-x^2-3}{x}\right) \\ &= -1/y \\ &< 0.\end{aligned}$$

Since the region $x, y > 0$ is simply connected and g and $\mathbf{f}$ satisfy the required smoothness conditions, Dulac's criterion implies there are no closed orbits in the positive quadrant. ■

EXAMPLE 7.2.5:

Show that the system $\dot{x} = y$, $\dot{y} = -x - y + x^2 + y^2$ has no closed orbits.

Solution: Let $g = e^{-2x}$. Then $\nabla \cdot (g\dot{\mathbf{x}}) = -2e^{-2x}y + e^{-2x}(-1+2y) = -e^{-2x} < 0$. By Dulac's criterion, there are no closed orbits. ■

7.3 Poincaré–Bendixson Theorem

Now that we know how to rule out closed orbits, we turn to the opposite task: finding methods to *establish that closed orbits exist* in particular systems. The following theorem is one of the few results in this direction. It is also one of the key theoretical results in nonlinear dynamics, because it implies that chaos can't occur in the phase plane, as discussed briefly at the end of this section.

Poincaré–Bendixson Theorem: Suppose that:

(1) R is a closed, bounded subset of the plane;

(2) $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a continuously differentiable vector field on an open set containing R;

(3) R does not contain any fixed points; and

(4) There exists a trajectory C that is "confined" in R, in the sense that it starts in R and stays in R for all future time (Figure 7.3.1).

Then either C is a closed orbit, or it spirals toward a closed orbit as $t \to \infty$. In either case, R *contains a closed orbit* (shown as a heavy curve in Figure 7.3.1).

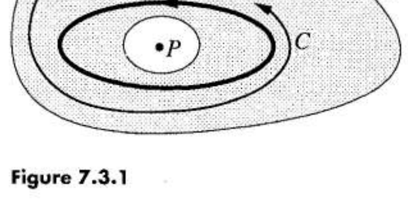

Figure 7.3.1

The proof of this theorem is subtle, and requires some advanced ideas from topology. For details, see Perko (1991), Coddington and Levinson (1955), Hurewicz (1958), or Cesari (1963).

In Figure 7.3.1, we have drawn R as a ring-shaped region because any closed orbit must encircle a fixed point (P in Figure 7.3.1) and no fixed points are allowed in R.

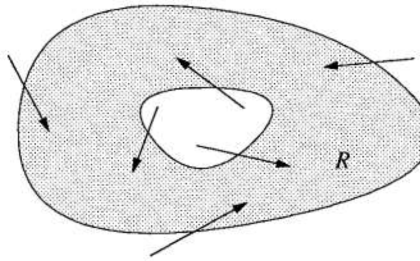

Figure 7.3.2

When applying the Poincaré–Bendixson theorem, it's easy to satisfy conditions (1)–(3); condition (4) is the tough one. How can we be sure that a confined trajectory C exists? The standard trick is to construct a ***trapping region*** R, i.e., a closed connected set such that the vector field points "inward" everywhere on the boundary of R (Figure 7.3.2). Then *all* trajectories in R are confined. If we can also arrange that there are no fixed points in R, then the Poincaré–Bendixson theorem ensures that R contains a closed orbit.

The Poincaré–Bendixson theorem can be difficult to apply in practice. One convenient case occurs when the system has a simple representation in polar coordinates, as in the following example.

EXAMPLE 7.3.1:

Consider the system

$$\begin{aligned}\dot{r} &= r(1-r^2) + \mu r\cos\theta \\ \dot{\theta} &= 1.\end{aligned} \tag{1}$$

When $\mu = 0$, there's a stable limit cycle at $r = 1$, as discussed in Example 7.1.1. Show that a closed orbit still exists for $\mu > 0$, as long as μ is sufficiently small.

Solution: We seek two concentric circles with radii $r_{\min}$ and $r_{\max}$, such that $\dot{r} < 0$ on the outer circle and $\dot{r} > 0$ on the inner circle. Then the annulus $0 < r_{\min} \le r \le r_{\max}$ will be our desired trapping region. Note that there are no fixed points in the annulus since $\dot{\theta} > 0$; hence if $r_{\min}$ and $r_{\max}$ can be found, the Poincaré–Bendixson theorem will imply the existence of a closed orbit.

To find $r_{\min}$, we require $\dot{r} = r(1-r^2) + \mu r\cos\theta > 0$ for all θ. Since $\cos\theta \ge -1$, a sufficient condition for $r_{\min}$ is $1 - r^2 - \mu > 0$. Hence any $r_{\min} < \sqrt{1-\mu}$ will work, as long as $\mu < 1$ so that the square root makes sense. We should choose $r_{\min}$ as large as possible, to hem in the limit cycle as tightly as we can. For instance, we could pick $r_{\min} = 0.999\sqrt{1-\mu}$. (Even $r_{\min} = \sqrt{1-\mu}$ works, but more careful reasoning is required.) By a similar argument, the flow is inward on the outer circle if $r_{\max} = 1.001\sqrt{1+\mu}$.

Therefore a closed orbit exists for all $\mu < 1$, and it lies somewhere in the annulus $0.999\sqrt{1-\mu} < r < 1.001\sqrt{1+\mu}$. ■

The estimates used in Example 7.3.1 are conservative. In fact, the closed orbit can exist even if $\mu \geq 1$. Figure 7.3,3 shows a computer-generated phase portrait of (1) for $\mu = 1$. In Exercise 7.3.8, you're asked to explore what happens for larger μ, and in particular, whether there's a critical μ beyond which the closed orbit disappears. It's also possible to obtain some analytical insight about the closed orbit for small μ (Exercise 7.3.9).

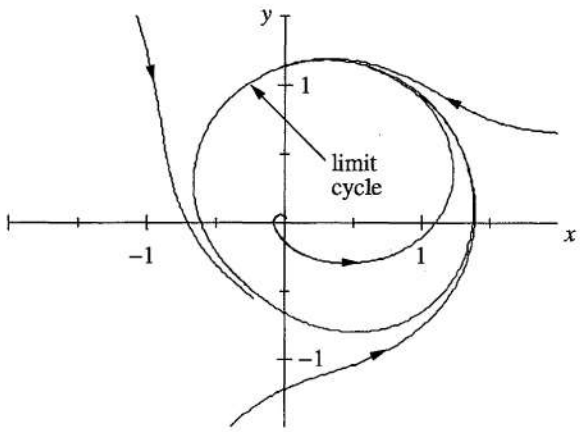

Figure 7.3.3

When polar coordinates are inconvenient, we may still be able to find an appropriate trapping region by examining the system's nullclines, as in the next example.

EXAMPLE 7.3.2:

In the fundamental biochemical process called ***glycolysis***, living cells obtain energy by breaking down sugar. In intact yeast cells as well as in yeast or muscle extracts, glycolysis can proceed in an *oscillatory* fashion, with the concentrations of various intermediates waxing and waning with a period of several minutes. For reviews, see Chance et al. (1973) or Goldbeter (1980).

A simple model of these oscillations has been proposed by Sel'kov (1968). In dimensionless form, the equations are

$$\dot{x} = -x + ay + x^2 y$$
$$\dot{y} = b - ay - x^2 y$$

where x and y are the concentrations of ADP (adenosine diphosphate) and F6P (fructose-6-phosphate), and $a, b > 0$ are kinetic parameters. Construct a trapping region for this system.

Solution: First we find the nullclines. The first equation shows that $\dot{x} = 0$ on the curve $y = x/(a + x^2)$ and the second equation shows that $\dot{y} = 0$ on the curve $y = b/(a + x^2)$. These nullclines are sketched in Figure 7.3.4, along with some representative vectors.

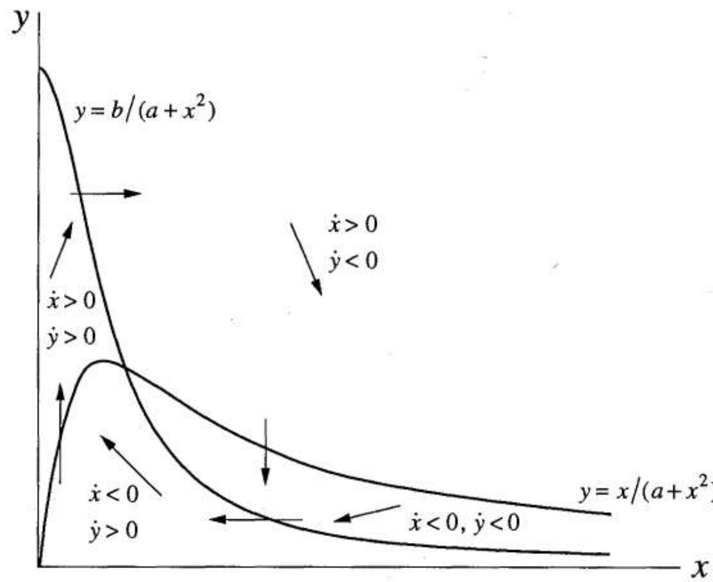

Figure 7.3.4

How did we know how to sketch these vectors? By definition, the arrows are vertical on the $\dot{x} = 0$ nullcline, and horizontal on the $\dot{y} = 0$ nullcline. The direction of flow is determined by the signs of $\dot{x}$ and $\dot{y}$. For instance, in the region above both nullclines, the governing equations imply $\dot{x} > 0$ and $\dot{y} < 0$, so the arrows point down and to the right, as shown in Figure 7.3.4.

Now consider the region bounded by the dashed line shown in Figure 7.3.5. *We claim that it's a trapping region.* To verify this, we have to show that all the vectors on the boundary point into the box. On the horizontal and vertical sides, there's no problem: the claim follows from Figure 7.3.4. The tricky part of the construction is the diagonal line of slope -1 extending from the point $(b, b/a)$ to the nullcline $y = x/(a + x^2)$. Where did this come from?

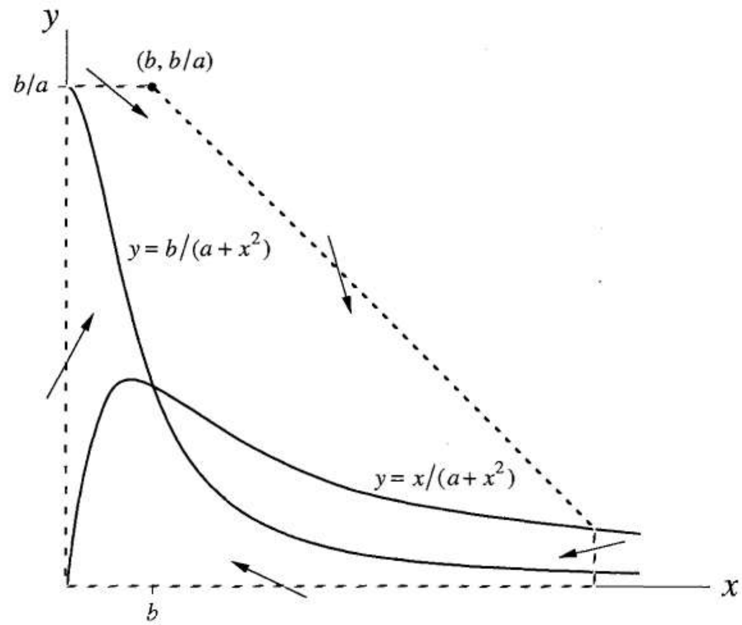

Figure 7.3.5

To get the right intuition, consider $\dot{x}$ and $\dot{y}$ in the limit of very large x. Then $\dot{x} \approx x^2 y$ and $\dot{y} \approx -x^2 y$, so $\dot{y}/\dot{x} = dy/dx \approx -1$ along trajectories. Hence the vector field at large x is roughly parallel to the diagonal line. This suggests that in a more precise calculation, we should compare the sizes of $\dot{x}$ and $-\dot{y}$, for some sufficiently large x.

In particular, consider $\dot{x} - (-\dot{y})$. We find

$$\begin{aligned} \dot{x} - (-\dot{y}) &= -x + ay + x^2 y + (b - ay - x^2 y) \\ &= b - x. \end{aligned}$$

Hence

$$-\dot{y} > \dot{x} \text{ if } x > b.$$

This inequality implies that the vector field points inward on the diagonal line in Figure 7.3.5, because dy/dx is more negative than -1, and therefore the vectors are steeper than the diagonal line. Thus the region is a trapping region, as claimed. ■

Can we conclude that there is a closed orbit inside the trapping region? No! There is a fixed point in the region (at the intersection of the nullclines), and so the conditions of the Poincaré–Bendixson theorem are not satisfied. But if this fixed point is a *repeller*, then we *can* prove the existence of a closed orbit by considering the modified "punctured" region shown in Figure 7.3.6. (The hole is infinitesimal, but drawn larger for clarity.)

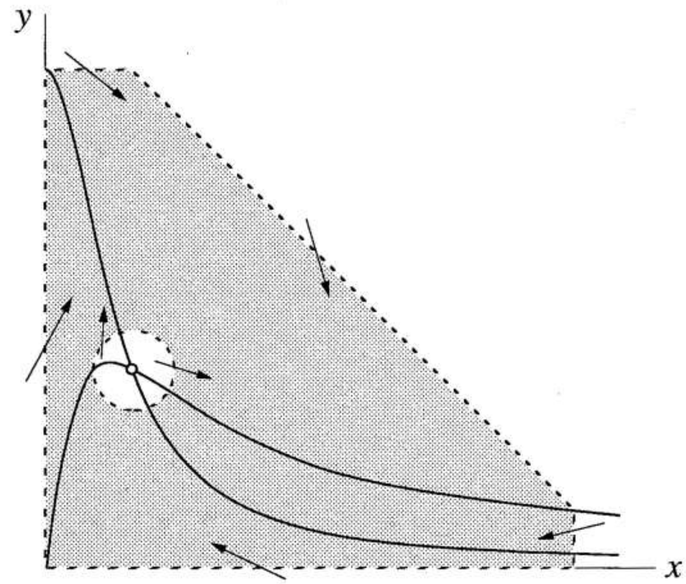

Figure 7.3.6

The repeller drives all neighboring trajectories into the shaded region, and since this region *is* free of fixed points, the Poincaré–Bendixson theorem applies.

Now we find conditions under which the fixed point is a repeller.

EXAMPLE 7.3.3:

Once again, consider the glycolytic oscillator $\dot{x} = -x + ay + x^2 y$, $\dot{y} = b - ay - x^2 y$ of Example 7.3.2. Prove that a closed orbit exists if a and b satisfy an appropriate condition, to be determined. (As before, $a, b > 0$.)

Solution: By the argument above, it suffices to find conditions under which the fixed point is a repeller, i.e., an unstable node or spiral. In general, the Jacobian is

$$A = \begin{pmatrix} -1+2xy & a+x^2 \\ -2xy & -(a+x^2) \end{pmatrix}.$$

After some algebra, we find that at the fixed point

$$x^* = b, \qquad y^* = \frac{b}{a+b^2},$$

the Jacobian has determinant $\Delta = a + b^2 > 0$ and trace

$$\tau = -\frac{b^4 + (2a-1)b^2 + (a+a^2)}{a+b^2}.$$

Hence the fixed point is unstable for $\tau > 0$, and stable for $\tau < 0$. The dividing line $\tau = 0$ occurs when

$$b^2 = \tfrac{1}{2}\left(1 - 2a \pm \sqrt{1-8a}\right).$$

This defines a curve in (a, b) space, as shown in Figure 7.3.7.

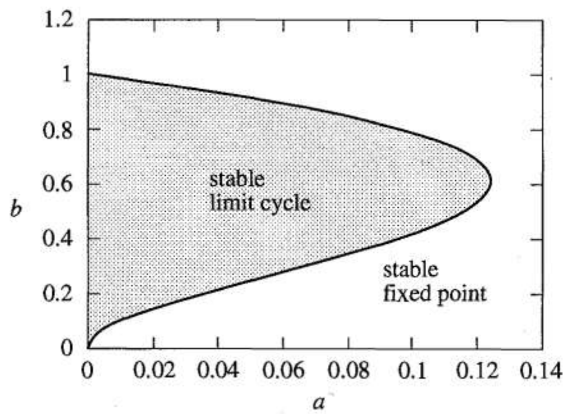

Figure 7.3.7

For parameters in the region corresponding to $\tau > 0$, we are guaranteed that the system has a closed orbit—numerical integration shows that it is actually a stable limit cycle. Figure 7.3.8 shows a computer-generated phase portrait for the typical case $a = 0.08$, $b = 0.6$. ■

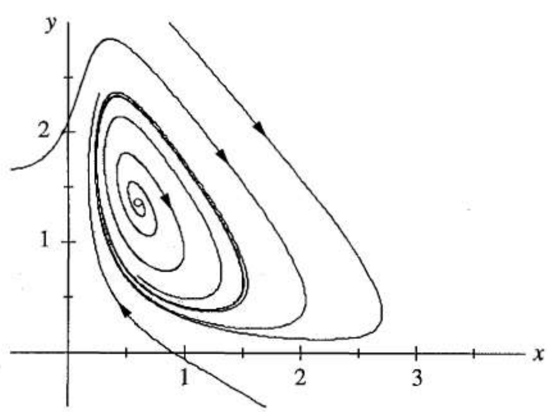

Figure 7.3.8

No Chaos in the Phase Plane

The Poincaré–Bendixson theorem is one of the central results of nonlinear dynamics. It says that the dynamical possibilities in the phase plane are very limited: if a trajectory is confined to a closed, bounded region that contains no fixed points, then the trajectory must eventually approach a closed orbit. Nothing more complicated is possible.

This result depends crucially on the two-dimensionality of the plane. In higher-dimensional systems ($n \geq 3$), the Poincaré–Bendixson theorem no longer applies, and something radically new can happen: trajectories may wander around forever in a bounded region without settling down to a fixed point or a closed orbit. In some cases, the trajectories are attracted to a complex geometric object called a *strange attractor*, a fractal set on which the motion is aperiodic and sensitive to tiny changes in the initial conditions. This sensitivity makes the motion unpredictable in the long run. We are now face to face with *chaos*. We'll discuss this fascinating topic soon enough, but for now you should appreciate that the Poincaré–Bendixson theorem implies that chaos can never occur in the phase plane.

7.4 Liénard Systems

In the early days of nonlinear dynamics, say from about 1920 to 1950, there was a great deal of research on nonlinear oscillations. The work was initially motivated by the development of radio and vacuum tube technology, and later it took on a mathematical life of its own. It was found that many oscillating circuits could be modeled by second-order differential equations of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \qquad (1)$$

now known as ***Liénard's equation***. This equation is a generalization of the van der Pol oscillator $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$ mentioned in Section 7.1. It can also be interpreted mechanically as the equation of motion for a unit mass subject to a nonlinear damping force $-f(x)\dot{x}$ and a nonlinear restoring force $-g(x)$.

Liénard's equation is equivalent to the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -g(x) - f(x)y. \end{aligned} \qquad (2)$$

The following theorem states that this system has a unique, stable limit cycle under appropriate hypotheses on f and g. For a proof, see Jordan and Smith (1987), Grimshaw (1990), or Perko (1991).

Liénard's Theorem: Suppose that $f(x)$ and $g(x)$ satisfy the following conditions:

(1) $f(x)$ and $g(x)$ are continuously differentiable for all x;
(2) $g(-x) = -g(x)$ for all x (i.e., $g(x)$ is an *odd* function);
(3) $g(x) > 0$ for $x > 0$;
(4) $f(-x) = f(x)$ for all x (i.e., $f(x)$ is an *even* function);
(5) The odd function $F(x) = \int_0^x f(u)\,du$ has exactly one positive zero at $x = a$, is negative for $0 < x < a$, is positive and nondecreasing for $x > a$, and $F(x) \to \infty$ as $x \to \infty$.

Then the system (2) has a unique, stable limit cycle surrounding the origin in the phase plane.

This result should seem plausible. The assumptions on $g(x)$ mean that the restoring force acts like an ordinary spring, and tends to reduce any displacement, whereas the assumptions on $f(x)$ imply that the damping is negative at small $|x|$ and positive at large $|x|$. Since small oscillations are pumped up and large oscillations are damped down, it is not surprising that the system tends to settle into a self-sustained oscillation of some intermediate amplitude.

EXAMPLE 7.4.1:

Show that the van der Pol equation has a unique, stable limit cycle.

Solution: The van der Pol equation $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$ has $f(x) = \mu(x^2 - 1)$ and $g(x) = x$, so conditions (1)–(4) of Liénard's theorem are clearly satisfied. To check condition (5), notice that

$$F(x) = \mu\left(\tfrac{1}{3}x^3 - x\right) = \tfrac{1}{3}\mu x(x^2 - 3).$$

Hence condition (5) is satisfied for $a = \sqrt{3}$. Thus the van der Pol equation has a unique, stable limit cycle. ■

There are several other classical results about the existence of periodic solutions for Liénard's equation and its relatives. See Stoker (1950), Minorsky (1962), Andronov et al. (1973), and Jordan and Smith (1987).

7.5 Relaxation Oscillations

It's time to change gears. So far in this chapter, we have focused on a qualitative question: Given a particular two-dimensional system, does it have any periodic solutions? Now we ask a quantitative question: Given that a closed orbit exists, what can we say about its shape and period? In general, such problems can't be solved exactly, but we can still obtain useful approximations if some parameter is large or small.

We begin by considering the van der Pol equation

$$\ddot{x}+\mu(x^2-1)\dot{x}+x=0$$

for $\mu \gg 1$. In this *strongly nonlinear* limit, we'll see that the limit cycle consists of an extremely slow buildup followed by a sudden discharge, followed by another slow buildup, and so on. Oscillations of this type are often called ***relaxation oscillations***, because the "stress" accumulated during the slow buildup is "relaxed" during the sudden discharge. Relaxation oscillations occur in many other scientific contexts, from the stick-slip oscillations of a bowed violin string to the periodic firing of nerve cells driven by a constant current (Edelstein–Keshet 1988, Murray 1989, Rinzel and Ermentrout 1989).

EXAMPLE 7.5.1:

Give a phase plane analysis of the van der Pol equation for $\mu \gg 1$.

Solution: It proves convenient to introduce different phase plane variables from the usual "$\dot{x}=y$, $\dot{y}=\ldots$". To motivate the new variables, notice that

$$\ddot{x}+\mu\dot{x}(x^2-1)=\frac{d}{dt}\left(\dot{x}+\mu\left[\tfrac{1}{3}x^3-x\right]\right).$$

So if we let

$$F(x)=\tfrac{1}{3}x^3-x, \qquad w=\dot{x}+\mu F(x), \tag{1}$$

the van der Pol equation implies that

$$\dot{w}=\ddot{x}+\mu\dot{x}(x^2-1)=-x. \tag{2}$$

Hence the van der Pol equation is equivalent to (1), (2), which may be rewritten as

$$\begin{aligned}\dot{x}&=w-\mu F(x)\\ \dot{w}&=-x.\end{aligned} \tag{3}$$

One further change of variables is helpful. If we let

$$y=\frac{w}{\mu}$$

then (3) becomes

$$\begin{aligned}\dot{x}&=\mu\left[\,y-F(x)\,\right]\\ \dot{y}&=-\tfrac{1}{\mu}x.\end{aligned} \tag{4}$$

Now consider a typical trajectory in the (x, y) phase plane. The nullclines are the key to understanding the motion. We claim that all trajectories behave like that shown in Figure 7.5.1; starting from any point except the origin, the trajectory zaps horizontally onto the ***cubic nullcline*** $y = F(x)$. Then it crawls down the nullcline until it comes to the knee (point B in Figure 7.5.1), after which it zaps over to the other branch of the cubic at C. This is followed by another crawl along the cubic

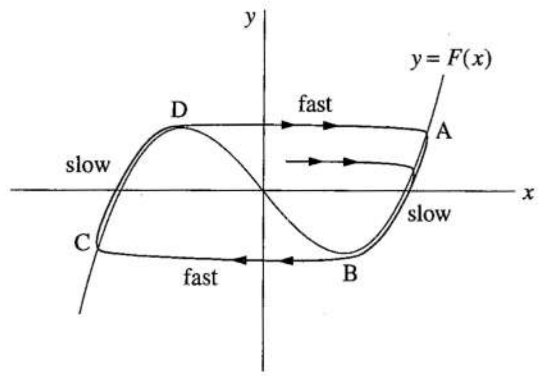

Figure 7.5.1

until the trajectory reaches the next jumping-off point at D, and the motion continues periodically after that.

To justify this picture, suppose that the initial condition is not too close to the cubic nullcline, i.e., suppose $y - F(x) \sim O(1)$. Then (4) implies $|\dot{x}| \sim O(\mu) \gg 1$ whereas $|\dot{y}| \sim O(\mu^{-1}) \ll 1$; hence the velocity is enormous in the horizontal direction and tiny in the vertical direction, so trajectories move practically horizontally. If the initial condition is *above* the nullcline, then $y - F(x) > 0$ and therefore $\dot{x} > 0$; thus the trajectory moves sideways *toward* the nullcline. However, once the trajectory gets so close that $y - F(x) \sim O(\mu^{-2})$, then $\dot{x}$ and $\dot{y}$ become comparable, both being $O(\mu^{-1})$. What happens then? The trajectory crosses the nullcline vertically, as shown in Figure 7.5.1, and then moves slowly along the backside of the branch, with a velocity of size $O(\mu^{-1})$, until it reaches the knee and can jump sideways again. ■

This analysis shows that the limit cycle has two ***widely separated time scales***: the crawls require $\Delta t \sim O(\mu)$ and the jumps require $\Delta t \sim O(\mu^{-1})$. Both time scales are apparent in the waveform of $x(t)$ shown in Figure 7.5.2, obtained by numerical integration of the van der Pol equation for $\mu = 10$ and initial condition $(x_0, y_0) = (2, 0)$.

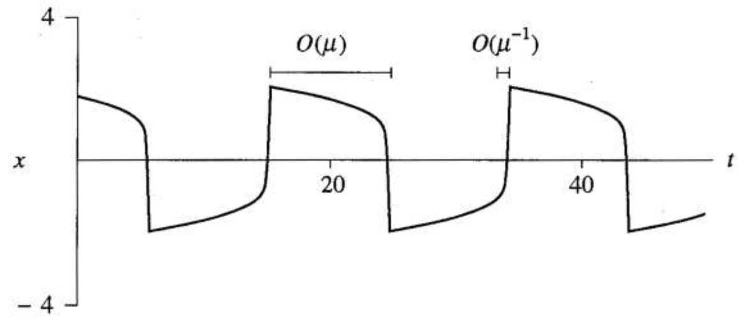

Figure 7.5.2

EXAMPLE 7.5.2:

Estimate the period of the limit cycle for the van der Pol equation for $\mu >> 1$.

Solution: The period T is essentially the time required to travel along the two ***slow branches***, since the time spent in the jumps is negligible for large μ. By symmetry, the time spent on each branch is the same. Hence $T \approx 2\int_{t_A}^{t_B} dt$. To derive an expression for *dt*, note that on the slow branches, $y \approx F(x)$ and thus

$$\frac{dy}{dt} \approx F'(x)\frac{dx}{dt} = (x^2-1)\frac{dx}{dt}.$$

But since $dy/dt = -x/\mu$ from (4), we find $dx/dt = -x/\mu(x^2-1)$. Therefore

$$dt \approx -\frac{\mu(x^2-1)}{x}\,dx \tag{5}$$

on a slow branch. As you can check (Exercise 7.5.1), the positive branch begins at $x_A = 2$ and ends at $x_B = 1$. Hence

$$T \approx 2\int_2^1 \frac{-\mu}{x}(x^2-1)\,dx = 2\mu\left[\frac{x^2}{2} - \ln x\right]_1^2 = \mu\left[3 - 2\ln 2\right], \tag{6}$$

which is $O(\mu)$ as expected. ■

The formula (6) can be refined. With much more work, one can show that $T \approx \mu[3-2\ln 2] + 2\alpha\mu^{-1/3} + \ldots$, where $\alpha \approx 2.338$ is the smallest root of $\mathrm{Ai}(-\alpha) = 0$. Here $\mathrm{Ai}(x)$ is a special function called the Airy function. This correction term comes from an estimate of the time required to turn the corner between the jumps and the crawls. See Grimshaw (1990, pp. 161–163) for a readable derivation of this wonderful formula, discovered by Mary Cartwright (1952). See also Stoker (1950) for more about relaxation oscillations.

One last remark: We have seen that a relaxation oscillation has two time scales that operate *sequentially*—a slow buildup is followed by a fast discharge. In the next section we will encounter problems where two time scales operate *concurrently*, and that makes the problems a bit more subtle.

7.6 Weakly Nonlinear Oscillators

This section deals with equations of the form

$$\ddot{x} + x + \varepsilon h(x, \dot{x}) = 0 \tag{1}$$

where $0 \le \varepsilon << 1$ and $h(x, \dot{x})$ is an arbitrary smooth function. Such equations represent small perturbations of the linear oscillator $\ddot{x} + x = 0$ and are therefore called ***weakly nonlinear oscillators***. Two fundamental examples are the van der Pol equation

$$\ddot{x} + x + \varepsilon(x^2 - 1)\dot{x} = 0, \tag{2}$$

(now in the limit of small nonlinearity), and the ***Duffing equation***

$$\ddot{x} + x + \varepsilon x^3 = 0. \tag{3}$$

To illustrate the kinds of phenomena that can arise, Figure 7.6.1 shows a computer-generated solution of the van der Pol equation in the $(x, \dot{x})$ phase plane, for $\varepsilon = 0.1$ and an initial condition close to the origin. The trajectory is a slowly winding spiral; it takes many cycles for the amplitude to grow substantially. Eventually

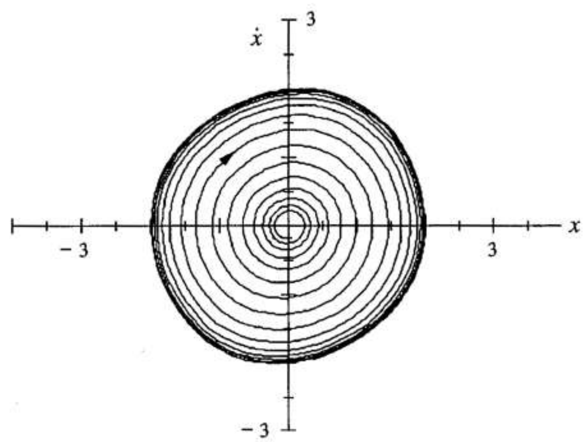

Figure 7.6.1

the trajectory asymptotes to an approximately circular limit cycle whose radius is close to 2.

We'd like to be able to predict the shape, period, and radius of this limit cycle. Our analysis will exploit the fact that the oscillator is "close to" a simple harmonic oscillator, which we understand completely.

Regular Perturbation Theory and Its Failure

As a first approach, we seek solutions of (1) in the form of a power series in ε. Thus if $x(t,\varepsilon)$ is a solution, we expand it as

$$x(t,\varepsilon) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \ldots, \tag{4}$$

where the unknown functions $x_k(t)$ are to be determined from the governing equation and the initial conditions. The hope is that all the important information is captured by the first few terms—ideally, the first *two*—and that the higher-order terms represent only tiny corrections. This technique is called ***regular perturbation theory***. It works well on certain classes of problems (for instance, Exercise 7.3.9), but as we'll see, it runs into trouble here.

To expose the source of the difficulties, we begin with a practice problem that can be solved exactly. Consider the weakly damped linear oscillator

$$\ddot{x} + 2\varepsilon\dot{x} + x = 0, \tag{5}$$

with initial conditions

$$x(0) = 0, \qquad \dot{x}(0) = 1. \tag{6}$$

Using the techniques of Chapter 5, we find the exact solution

$$x(t,\varepsilon) = \left(1-\varepsilon^2\right)^{-1/2} e^{-\varepsilon t} \sin\left[\left(1-\varepsilon^2\right)^{1/2} t\right]. \tag{7}$$

Now let's solve the same problem using perturbation theory. Substitution of (4) into (5) yields

$$\frac{d^2}{dt^2}\left(x_0 + \varepsilon x_1 + \ldots\right) + 2\varepsilon\frac{d}{dt}\left(x_0 + \varepsilon x_1 + \ldots\right) + \left(x_0 + \varepsilon x_1 + \ldots\right) = 0. \tag{8}$$

If we group the terms according to powers of ε, we get

$$\left[\ddot{x}_0 + x_0\right] + \varepsilon\left[\ddot{x}_1 + 2\dot{x}_0 + x_1\right] + O(\varepsilon^2) = 0. \tag{9}$$

Since (9) is supposed to hold for *all* sufficiently small ε, the coefficients of each power of ε must vanish separately. Thus we find

$$O(1): \ddot{x}_0 + x_0 = 0 \tag{10}$$

$$O(\varepsilon): \ddot{x}_1 + 2\dot{x}_0 + x_1 = 0. \tag{11}$$

(We're ignoring the $O(\varepsilon^2)$ and higher equations, in the optimistic spirit mentioned earlier.)

The appropriate initial conditions for these equations come from (6). At $t = 0$, (4) implies that $0 = x_0(0) + \varepsilon x_1(0) + \ldots$; this holds for all ε, so

$$x_0(0) = 0, \qquad x_1(0) = 0. \tag{12}$$

By applying a similar argument to $\dot{x}(0)$ we obtain

$$\dot{x}_0(0) = 1, \qquad \dot{x}_1(0) = 0. \tag{13}$$

Now we solve the initial-value problems one by one; they fall like dominoes. The solution of (10), subject to the initial conditions $x_0(0) = 0$, $\dot{x}_0(0) = 1$, is

$$x_0(t) = \sin t. \tag{14}$$

Plugging this solution into (11) gives

$$\ddot{x}_1 + x_1 = -2\cos t. \tag{15}$$

Here's the first sign of trouble: the right-hand side of (15) is a ***resonant*** forcing. The solution of (15) subject to $x_1(0) = 0$, $\dot{x}_1(0) = 0$ is

$$x_1(t) = -t\sin t, \tag{16}$$

which is a ***secular*** term, i.e., a term that *grows* without bound as $t \to \infty$.

In summary, the solution of (5), (6) according to perturbation theory is

$$x(t,\varepsilon) = \sin t - \varepsilon t \sin t + O(\varepsilon^2). \tag{17}$$

How does this compare with the exact solution (7)? In Exercise 7.6.1, you are asked to show that the two formulas agree in the following sense: If (7) is expanded as power series in ε, the first two terms are given by (17). In fact, (17) is the beginning of a *convergent* series expansion for the true solution. For any fixed t, (17) provides a good approximation as long as ε is small enough—specifically, we need $\varepsilon t \ll 1$ so that the correction term (which is actually $O(\varepsilon^2 t^2)$) is negligible.

But normally we are interested in the behavior for *fixed* ε, not fixed t. In that case we can only expect the perturbation approximation to work for times $t \ll O(1/\varepsilon)$. To illustrate this limitation, Figure 7.6.2 plots the exact solution (7) and the perturbation series (17) for $\varepsilon = 0.1$. As expected, the perturbation series works reasonably well if $t \ll \frac{1}{\varepsilon} = 10$, but it breaks down after that.

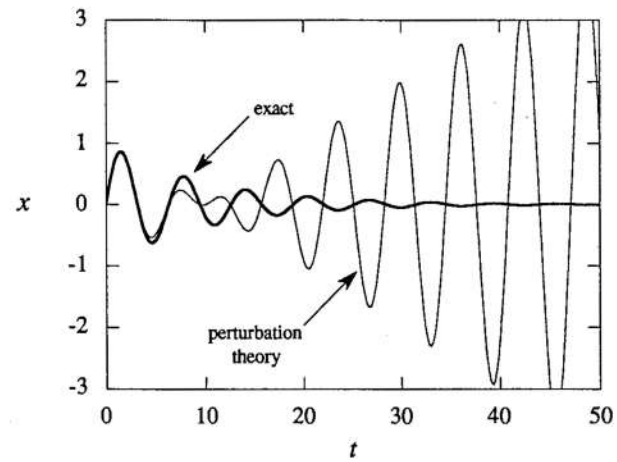

Figure 7.6.2

In many situations we'd like our approximation to capture the true solution's qualitative behavior for all t, or at least for large t. By this criterion, (17) is a failure, as Figure 7.6.2 makes obvious. There are two major problems:

1. The true solution (7) exhibits ***two time scales***: a *fast time* $t \sim O(1)$ for the sinusoidal oscillations and a *slow time* $t \sim 1/\varepsilon$ over which the amplitude decays. Equation (17) completely misrepresents the slow time scale behavior. In particular, because of the secular term $t\sin t$, (17) falsely suggests that the solution grows with time whereas we know from (7) that the amplitude $A = \left(1-\varepsilon^2\right)^{-1/2} e^{-\varepsilon t}$ decays exponentially.

   The discrepancy occurs because $e^{-\varepsilon t} = 1 - \varepsilon t + O(\varepsilon^2 t^2)$, so to this order in ε, it appears (incorrectly) that the amplitude increases with t. To get the correct result, we'd need to calculate an *infinite* number of terms in the series. That's worthless; we want series approximations that work well with just one or two terms.
2. The frequency of the oscillations in (7) is $\omega = \left(1-\varepsilon^2\right)^{1/2} \approx 1 - \frac{1}{2}\varepsilon^2$, which is shifted slightly from the frequency $\omega = 1$ of (17). After a *very* long time $t \sim O(1/\varepsilon^2)$, this frequency error will have a significant cumulative effect. Note that this is a third, *super-slow* time scale!

Two-Timing

The elementary example above reveals a more general truth: There are going to be (at least) two time scales in weakly nonlinear oscillators. We've already met this phenomenon in Figure 7.6.1, where the amplitude of the spiral grew very slowly compared to the cycle time. An analytical method called ***two-timing*** builds in the fact of two time scales from the start, and produces better approximations than regular perturbation theory. In fact, more than two times can be used, but we'll stick to the simplest case.

To apply two-timing to (1), let $\tau = t$ denote the fast $O(1)$ time, and let $T = \varepsilon t$ denote the slow time. We'll treat these two times as if they were *independent* variables. In particular, functions of the slow time T will be regarded as *constants* on the fast time scale τ. It's hard to justify this idea rigorously, but it works! (Here's an analogy: it's like saying that your height is constant on the time scale of a day. Of course, over many months or years your height can change dramatically, especially if you're an infant or a pubescent teenager, but over one day your height stays constant, to a good approximation.)

Now we turn to the mechanics of the method. We expand the solution of (1) as a series

$$x(t,\varepsilon) = x_0(\tau, T) + \varepsilon x_1(\tau, T) + O(\varepsilon^2). \tag{18}$$

The time derivatives in (1) are transformed using the chain rule:

$$\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial \tau} + \frac{\partial x}{\partial T}\frac{\partial T}{\partial t} = \frac{\partial x}{\partial \tau} + \varepsilon \frac{\partial x}{\partial T}. \tag{19}$$

A subscript notation for differentiation is more compact; thus we write (19) as

$$\dot{x} = \partial_\tau x + \varepsilon \partial_T x. \tag{20}$$

After substituting (18) into (20) and collecting powers of ε, we find

$$\dot{x} = \partial_\tau x_0 + \varepsilon\left(\partial_T x_0 + \partial_\tau x_1\right) + O(\varepsilon^2). \tag{21}$$

Similarly,

$$\ddot{x} = \partial_{\tau\tau} x_0 + \varepsilon\left(\partial_{\tau\tau} x_1 + 2\partial_{T\tau} x_0\right) + O(\varepsilon^2). \tag{22}$$

To illustrate the method, let's apply it to our earlier test problem.

EXAMPLE 7.6.1:

Use two-timing to approximate the solution to the damped linear oscillator $\ddot{x} + 2\varepsilon\dot{x} + x = 0$, with initial conditions $x(0) = 0$, $\dot{x}(0) = 1$.

Solution: After substituting (21) and (22) for $\dot{x}$ and $\ddot{x}$, we get

$$\partial_{\tau\tau} x_0 + \varepsilon\left(\partial_{\tau\tau} x_1 + 2\partial_{T\tau} x_0\right) + 2\varepsilon\partial_\tau x_0 + x_0 + \varepsilon x_1 + O(\varepsilon^2) = 0. \tag{23}$$

Collecting powers of ε yields a pair of differential equations:

$$O(1): \partial_{\tau\tau} x_0 + x_0 = 0 \tag{24}$$

$$O(\varepsilon): \partial_{\tau\tau}x_1 + 2\,\partial_{T\tau}x_0 + 2\partial_\tau x_0 + x_1 = 0. \tag{25}$$

Equation (24) is just a simple harmonic oscillator. Its general solution is

$$x_0 = A\sin\tau + B\cos\tau, \tag{26}$$

but now comes the interesting part: *The "constants" A and B are actually functions of the slow time T*. Here we are invoking the above-mentioned ideas that τ and T should be regarded as independent variables, with functions of T behaving like constants on the fast time scale τ.

To determine $A(T)$ and $B(T)$, we need to go to the next order of ε. Substituting (26) into (25) gives

$$\begin{aligned}\partial_{\tau\tau}x_1 + x_1 &= -2\left(\partial_{T\tau}x_0 + \partial_\tau x_0\right)\\ &= -2\left(A' + A\right)\cos\tau \;+\; 2\left(B' + B\right)\sin\tau\end{aligned} \tag{27}$$

where the prime denotes differentiation with respect to T.

Now we face the same predicament that ruined us after (15). As in that case, the right-hand side of (27) is a resonant forcing that will produce *secular terms* like $\tau\sin\tau$ and $\tau\cos\tau$ in the solution for x_1. These terms would lead to a convergent but useless series expansion for x. Since we want an approximation free from secular terms, *we set the coefficients of the resonant terms to zero*—this manuever is characteristic of all two-timing calculations. Here it yields

$$A' + A = 0 \tag{28}$$

$$B' + B = 0. \tag{29}$$

The solutions of (28) and (29) are

$$\begin{aligned}A(T) &= A(0)\,e^{-T}\\ B(T) &= B(0)\,e^{-T}.\end{aligned}$$

The last step is to find the initial values $A(0)$ and $B(0)$. They are determined by (18), (26), and the given initial conditions $x(0) = 0$, $\dot{x}(0) = 1$, as follows. Equation (18) gives $0 = x(0) = x_0(0,0) + \varepsilon x_1(0,0) + O(\varepsilon^2)$. To satisfy this equation for *all* sufficiently small ε, we must have

$$x_0(0,0) = 0 \tag{30}$$

and $x_1(0,0) = 0$. Similarly,

$$1 = \dot{x}(0) = \partial_\tau x_0(0,0) + \varepsilon\left(\partial_T x_0(0,0) + \partial_\tau x_1(0,0)\right) + O(\varepsilon^2)$$

so

$$\partial_\tau x_0(0,0) = 1 \tag{31}$$

and $\partial_T x_0(0,0) + \partial_\tau x_1(0,0) = 0$. Combining (26) and (30) we find $B(0) = 0$; hence $B(T) \equiv 0$. Similarly, (26) and (31) imply $A(0) = 1$, so $A(T) = e^{-T}$. Thus (26) becomes

$$x_0(\tau, T) = e^{-T} \sin \tau . \tag{32}$$

Hence

$$\begin{aligned} x &= e^{-T} \sin \tau + O(\varepsilon) \\ &= e^{-\varepsilon t} \sin t + O(\varepsilon) \end{aligned} \tag{33}$$

is the approximate solution predicted by two-timing. ■

Figure 7.6.3 compares the two-timing solution (33) to the exact solution (7) for $\varepsilon = 0.1$. The two curves are almost indistinguishable, even though ε is not terribly small. This is a characteristic feature of the method—it often works better than it has any right to.

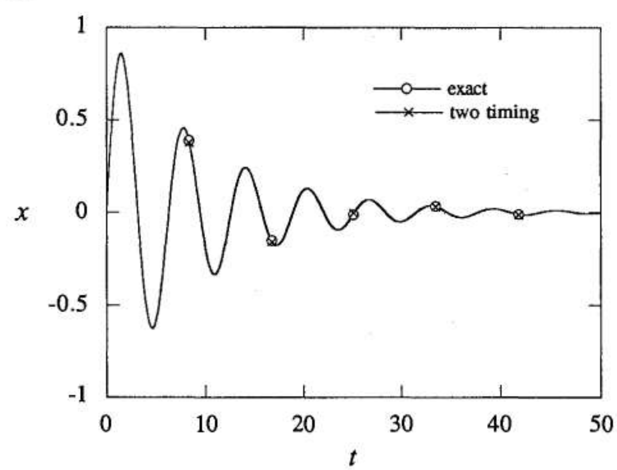

Figure 7.6.3

If we wanted to go further with Example 7.6.1, we could either solve for x_1 and higher-order corrections, or introduce a super-slow time $\Im = \varepsilon^2 t$ to investigate the long-term phase shift caused by the $O(\varepsilon^2)$ error in frequency. But Figure 7.6.3 shows that we already have a good approximation.

OK, enough practice problems! Now that we have calibrated the method, let's unleash it on a genuine nonlinear problem.

EXAMPLE 7.6.2:

Use two-timing to show that the van der Pol oscillator (2) has a stable limit cycle that is nearly circular, with a radius $= 2 + O(\varepsilon)$ and a frequency $\omega = 1 + O(\varepsilon^2)$.

Solution: The equation is $\ddot{x} + x + \varepsilon(x^2 - 1)\dot{x} = 0$. Using (21) and (22) and collecting powers of ε, we find the following equations:

$$O(1): \partial_{\tau\tau} x_0 + x_0 = 0 \tag{34}$$

$$O(\varepsilon): \partial_{\tau\tau} x_1 + x_1 = -2\partial_{\tau T} x_0 - (x_0{}^2 - 1)\partial_\tau x_0 . \tag{35}$$

As always, the $O(1)$ equation is a simple harmonic oscillator. Its general solution can be written as (26), or alternatively, as

$$x_0 = r(T)\cos(\tau + \phi(T)) \tag{36}$$

where $r(T)$ and $\phi(T)$ are the ***slowly-varying amplitude and phase*** of x_0.

To find equations governing r and ϕ, we insert (36) into (35). This yields

$$\begin{aligned} \partial_{\tau\tau} x_1 + x_1 = &-2\big(r' \sin(\tau + \phi) + r\phi' \cos(\tau + \phi)\big) \\ &- r\sin(\tau + \phi)\big[r^2 \cos^2(\tau + \phi) - 1\big]. \end{aligned} \tag{37}$$

As before, we need to avoid resonant terms on the right-hand side. These are terms proportional to $\cos(\tau + \phi)$ and $\sin(\tau + \phi)$. Some terms of this form already appear explicitly in (37). But—and this is the important point—there is also a resonant term lurking in $\sin(\tau + \phi)\cos^2(\tau + \phi)$, because of the trigonometric identity

$$\sin(\tau + \phi)\cos^2(\tau + \phi) = \tfrac{1}{4}\big[\sin(\tau + \phi) + \sin 3(\tau + \phi)\big]. \tag{38}$$

(Exercise 7.6.10 reminds you how to derive such identities, but usually we won't need them—shortcuts are available, as we'll see.) After substituting (38) into (37), we get

$$\begin{aligned} \partial_{\tau\tau} x_1 + x_1 = &\big[-2r' + r - \tfrac{1}{4} r^3\big] \sin(\tau + \phi) \\ &+ \big[-2r\phi'\big] \cos(\tau + \phi) - \tfrac{1}{4} r^3 \sin 3(\tau + \phi) . \end{aligned} \tag{39}$$

To avoid secular terms, we require

$$-2r' + r - \tfrac{1}{4} r^3 = 0 \tag{40}$$

$$-2r\phi' = 0 . \tag{41}$$

First consider (40). It may be rewritten as a vector field

$$r' = \tfrac{1}{8} r(4 - r^2) \tag{42}$$

on the half-line $r \geq 0$. Following the methods of Chapter 2 or Example 7.1.1, we see that $r^* = 0$ is an unstable fixed point and $r^* = 2$ is a stable fixed point. Hence $r(T) \to 2$ as $T \to \infty$. Secondly, (41) implies $\phi' = 0$, so $\phi(T) = \phi_0$ for some constant ϕ_0. Hence $x_0(\tau, T) \to 2\cos(\tau + \phi_0)$ and therefore

$$x(t) \to 2\cos(t+\phi_0) + O(\varepsilon) \tag{43}$$

as $t \to \infty$. Thus $x(t)$ approaches a stable limit cycle of radius $= 2 + O(\varepsilon)$.

To find the frequency implied by (43), let $\theta = t + \phi(T)$ denote the argument of the cosine. Then the angular frequency ω is given by

$$\omega = \frac{d\theta}{dt} = 1 + \frac{d\phi}{dT}\frac{dT}{dt} = 1 + \varepsilon\phi' = 1, \tag{44}$$

through first order in ε. Hence $\omega = 1 + O(\varepsilon^2)$; if we want an explicit formula for this $O(\varepsilon^2)$ correction term, we'd need to introduce a super-slow time $\Im = \varepsilon^2 t$, or we could use the Poincaré–Lindstedt method, as discussed in the exercises. ■

Averaged Equations

The same steps occur again and again in problems about weakly nonlinear oscillators. We can save time by deriving some general formulas.

Consider the equation for a general weakly nonlinear oscillator:

$$\ddot{x} + x + \varepsilon h(x, \dot{x}) = 0. \tag{45}$$

The usual two-timing substitutions give

$$O(1): \partial_{\tau\tau} x_0 + x_0 = 0 \tag{46}$$

$$O(\varepsilon): \partial_{\tau\tau} x_1 + x_1 = -2\partial_{\tau T} x_0 - h \tag{47}$$

where now $h = h(x_0, \partial_\tau x_0)$. As in Example 7.6.2, the solution of the $O(1)$ equation is

$$x_0 = r(T)\cos(\tau + \phi(T)). \tag{48}$$

Our goal is to derive differential equations for r' and ϕ', analogous to (40) and (41). We'll find these equations by insisting, as usual, that there be no terms proportional to $\cos(\tau+\phi)$ and $\sin(\tau+\phi)$ on the right-hand side of (47). Substituting (48) into (47), we see that this right-hand side is

$$2[r'\sin(\tau+\phi) + r\phi'\cos(\tau+\phi)] - h \tag{49}$$

where now $h = h(r\cos(\tau+\phi),\ -r\sin(\tau+\phi))$.

To extract the terms in h proportional to $\cos(\tau+\phi)$ and $\sin(\tau+\phi)$, we borrow some ideas from Fourier analysis. (If you're unfamiliar with Fourier analysis, don't worry—we'll derive all that we need in Exercise 7.6.12.) Notice that h is a 2π-periodic function of $\tau + \phi$. Let

$$\theta = \tau + \phi.$$

Fourier analysis tells us that $h(\theta)$ can be written as a ***Fourier series***

$$h(\theta)=\sum_{k=0}^{\infty} a_k \cos k\theta+\sum_{k=1}^{\infty} b_k \sin k\theta \tag{50}$$

where the ***Fourier coefficients*** are given by

$$\begin{aligned}
a_0 &= \tfrac{1}{2\pi}\int_0^{2\pi} h(\theta)\,d\theta \\
a_k &= \tfrac{1}{\pi}\int_0^{2\pi} h(\theta)\cos k\theta\,d\theta\,,\quad k\ge 1 \\
b_k &= \tfrac{1}{\pi}\int_0^{2\pi} h(\theta)\sin k\theta\,d\theta\,,\quad k\ge 1\,.
\end{aligned} \tag{51}$$

Hence (49) becomes

$$2\left[r'\sin\theta+r\phi'\cos\theta\,\right]-\sum_{k=0}^{\infty} a_k \cos k\theta-\sum_{k=1}^{\infty} b_k \sin k\theta\ . \tag{52}$$

The only resonant terms in (52) are $\left[2r'-b_1\right]\sin\theta$ and $\left[2r\phi'-a_1\right]\cos\theta$. Therefore, to avoid secular terms we need $r'=b_1/2$ and $r\phi'=a_1/2$. Using the expressions in (51) for a_1 and b_1, we obtain

$$\begin{aligned}
r' &= \tfrac{1}{2\pi}\int_0^{2\pi} h(\theta)\sin\theta\,d\theta \equiv \langle h\sin\theta\rangle \\
r\phi' &= \tfrac{1}{2\pi}\int_0^{2\pi} h(\theta)\cos\theta\,d\theta \equiv \langle h\cos\theta\rangle
\end{aligned} \tag{53}$$

where the angled brackets $\langle\cdot\rangle$ denote an average over one cycle of θ.

The equations in (53) are called the ***averaged*** or ***slow-time equations***. To use them, we write out $h=h(r\cos(\tau+\phi),\ -r\sin(\tau+\phi))=h(r\cos\theta\,,\,-r\sin\theta)$ explicitly, and then compute the relevant averages over the fast variable θ, treating the slow variable r as constant. Here are some averages that appear often:

$$\begin{gathered}
\langle\cos\rangle=\langle\sin\rangle=0\,,\quad \langle\sin\ \cos\rangle=0\,,\quad \left\langle\cos^3\right\rangle=\left\langle\sin^3\right\rangle=0\,,\quad \left\langle\cos^{2n+1}\right\rangle=\left\langle\sin^{2n+1}\right\rangle=0\,, \\
\left\langle\cos^2\right\rangle=\left\langle\sin^2\right\rangle=\tfrac{1}{2}\,,\quad \left\langle\cos^4\right\rangle=\left\langle\sin^4\right\rangle=\tfrac{3}{8}\,,\quad \left\langle\cos^2\sin^2\right\rangle=\tfrac{1}{8}\,, \\
\left\langle\cos^{2n}\right\rangle=\left\langle\sin^{2n}\right\rangle=\tfrac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)}\,,\quad n\ge 1\,.
\end{gathered} \tag{54}$$

Other averages can either be derived from these, or found by direct integration. For instance,

$$\left\langle\cos^2\sin^4\right\rangle=\left\langle(1-\sin^2)\sin^4\right\rangle=\left\langle\sin^4\right\rangle-\left\langle\sin^6\right\rangle=\tfrac{3}{8}-\tfrac{15}{48}=\tfrac{1}{16}$$

and

$$\left\langle \cos^3 \sin \right\rangle = \tfrac{1}{2\pi}\int_0^{2\pi} \cos^3\theta \,\sin\theta \,d\theta = -\tfrac{1}{2\pi}\left[\cos^4\theta\right]_0^{2\pi} = 0.$$

EXAMPLE 7.6.3:

Consider the van der Pol equation $\ddot{x} + x + \varepsilon(x^2 - 1)\dot{x} = 0$, subject to the initial conditions $x(0) = 1$, $\dot{x}(0) = 0$. Find the averaged equations, and then solve them to obtain an approximate formula for $x(t,\varepsilon)$. Compare your result to a numerical solution of the full equation, for $\varepsilon = 0.1$.

Solution: The van der Pol equation has $h = (x^2 - 1)\dot{x} = (r^2\cos^2\theta - 1)(-r\sin\theta)$. Hence (53) becomes

$$\begin{aligned} r' &= \langle h\sin\theta\rangle = \left\langle (r^2\cos^2\theta - 1)(-r\sin\theta)\,\sin\theta \right\rangle \\ &= r\left\langle \sin^2\theta \right\rangle - r^3\left\langle \cos^2\theta\,\sin^2\theta \right\rangle \\ &= \tfrac{1}{2}r - \tfrac{1}{8}r^3 \end{aligned}$$

and

$$\begin{aligned} r\phi' &= \langle h\cos\theta\rangle = \left\langle (r^2\cos^2\theta - 1)(-r\sin\theta)\,\cos\theta \right\rangle \\ &= r\left\langle \sin\theta\,\cos\theta \right\rangle - r^3\left\langle \cos^3\theta\,\sin\theta \right\rangle \\ &= 0 - 0 = 0. \end{aligned}$$

These equations match those found in Example 7.6.2, as they should.

The initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$ imply $r(0) \approx \sqrt{x(0)^2 + \dot{x}(0)^2} = 1$ and $\phi(0) \approx \tan^{-1}\left(\dot{x}(0)/x(0)\right) - \tau = 0 - 0 = 0$. Since $\phi' = 0$, we find $\phi(T) \equiv 0$. To find $r(T)$, we solve $r' = \frac{1}{2}r - \frac{1}{8}r^3$ subject to $r(0) = 1$. The differential equation separates to

$$\int \frac{8\,dr}{r(4 - r^2)} = \int dT.$$

After integrating by partial fractions and using $r(0) = 1$, we find

$$r(T) = 2\left(1 + 3e^{-T}\right)^{-1/2}. \qquad (55)$$

Hence

$$\begin{aligned} x(t,\varepsilon) &\sim x_0(\tau, T) + O(\varepsilon) \\ &= \frac{2}{\sqrt{1 + 3e^{-\varepsilon t}}}\cos t + O(\varepsilon). \end{aligned} \qquad (56)$$

Equation (56) describes the transient dynamics of the oscillator as it spirals out to its limit cycle. Notice that $r(T) \to 2$ as $T \to \infty$, as in Example 7.6.2.

In Figure 7.6.4 we plot the "exact" solution of the van der Pol equation, obtained by numerical integration for $\varepsilon = 0.1$ and initial conditions $x(0) = 1$, $\dot{x}(0) = 0$. For comparison, the slowly-varying amplitude $r(T)$ predicted by (55) is also shown. The agreement is striking. Alternatively, we could have plotted the whole solution (56) instead of just its envelope; then the two curves would be virtually indistinguishable, like those in Figure 7.6.3. ■

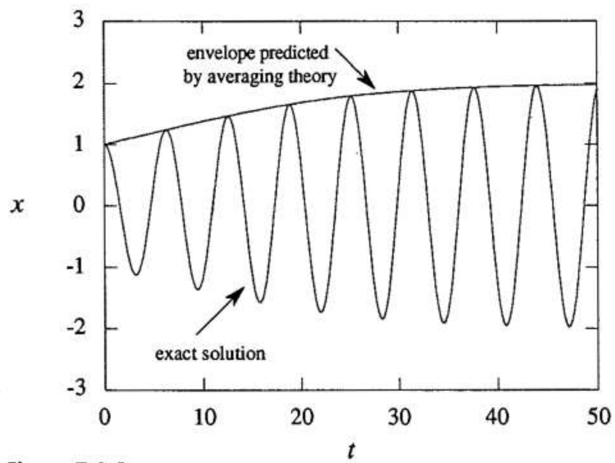

Figure 7.6.4

Now we consider an example in which the frequency of an oscillator depends on its amplitude. This is a common phenomenon, and one that is intrinsically *nonlinear*—it cannot occur for linear oscillators.

EXAMPLE 7.6.4:

Find an approximate relation between the amplitude and frequency of the Duffing oscillator $\ddot{x} + x + \varepsilon x^3 = 0$, where ε can have either sign. Interpret the results physically.

Solution: Here $h = x^3 = r^3 \cos^3 \theta$. Equation (53) becomes

$$r' = \langle h \sin \theta \rangle = r^3 \left\langle \cos^3 \theta \, \sin \theta \right\rangle = 0$$

and

$$r\phi' = \langle h \cos \theta \rangle = r^3 \left\langle \cos^4 \theta \right\rangle = \tfrac{3}{8} r^3 .$$

Hence $r(T) \equiv a$, for some constant a, and $\phi' = \frac{3}{8} a^2$. As in Example 7.6.2, the frequency ω is given by

$$\omega = 1 + \varepsilon\phi' = 1 + \tfrac{3}{8}\varepsilon a^2 + O(\varepsilon^2). \qquad (57)$$

Now for the physical interpretation. The Duffing equation describes the undamped motion of a unit mass attached to a nonlinear spring with restoring force $F(x) = -x - \varepsilon x^3$. We can use our intuition about ordinary linear springs if we write $F(x) = -kx$, where the spring stiffness is now dependent on x:

$$k = k(x) = 1 + \varepsilon x^2.$$

Suppose $\varepsilon > 0$. Then the spring gets *stiffer* as the displacement x increases—this is called a ***hardening spring.*** On physical grounds we'd expect it to *increase* the frequency of the oscillations, consistent with (57). For $\varepsilon < 0$ we have a ***softening spring***, exemplified by the pendulum (Exercise 7.6.15).

It also makes sense that $r' = 0$. The Duffing equation is a conservative system and for all ε sufficiently small, it has a *nonlinear center* at the origin (Exercise 6.5.13). Since all orbits close to the origin are periodic, there can be no long-term change in amplitude, consistent with $r' = 0$. ■

Validity of Two-Timing

We conclude with a few comments about the validity of the two-timing method. The rule of thumb is that the one-term approximation x_0 will be within $O(\varepsilon)$ of the true solution x for all times up to and including $t \sim O(1/\varepsilon)$, assuming that both x and x_0 start from the same initial condition. If x is a periodic solution, the situation is even better: x_0 remains within $O(\varepsilon)$ of x for *all* t.

But for precise statements and rigorous results about these matters, and for discussions of the subtleties that can occur, you should consult more advanced treatments, such as Guckenheimer and Holmes (1983) or Grimshaw (1990). Those authors use the *method of averaging*, an alternative approach that yields the same results as two-timing. See Exercise 7.6.25 for an introduction to this powerful technique.

Also, we have been very loose about the sense in which our formulas approximate the true solutions. The relevant notion is that of *asymptotic* approximation. For introductions to asymptotics, see Lin and Segel (1988) or Bender and Orszag (1978).

EXERCISES FOR CHAPTER 7

7.1 Examples

Sketch the phase portrait for each of the following systems. (As usual, r, θ denote polar coordinates.)

7.1.1 $\dot{r} = r^3 - 4r,\ \dot{\theta} = 1$ **7.1.2** $\dot{r} = r(1-r^2)(9-r^2),\ \dot{\theta} = 1$

7.1.3 $\dot{r} = r(1-r^2)(4-r^2),\ \dot{\theta} = 2-r^2$ **7.1.4** $\dot{r} = r\sin r,\ \dot{\theta} = 1$

7.1.5 (From polar to Cartesian coordinates) Show that the system $\dot{r} = r(1-r^2)$, $\dot{\theta} = 1$ is equivalent to

$$\dot{x} = x - y - x(x^2+y^2), \qquad \dot{y} = x + y - y(x^2+y^2),$$

where $x = r\cos\theta$, $y = r\sin\theta$. (Hint: $\dot{x} = \frac{d}{dt}(r\cos\theta) = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$.)

7.1.6 (Circuit for van der Pol oscillator) Figure 1 shows the "tetrode multivibrator" circuit used in the earliest commercial radios and analyzed by van der Pol. In van der Pol's day, the active element was a vacuum tube; today it would be a semiconductor device. It acts like an ordinary resistor when I is high, but like a negative resistor (energy source) when I is low. Its current-voltage characteristic $V = f(I)$ resembles a cubic function, as discussed below.

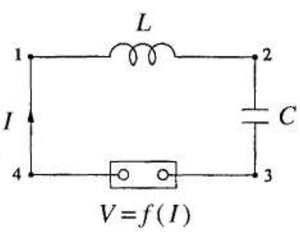

Figure 1

Suppose a source of current is attached to the circuit and then withdrawn. What equations govern the subsequent evolution of the current and the various voltages?

a) Let $V = V_{32} = -V_{23}$ denote the voltage drop from point 3 to point 2 in the circuit. Show that $\dot{V} = -I/C$ and $V = L\dot{I} + f(I)$.

b) Show that the equations in (a) are equivalent to

$$\frac{dw}{d\tau} = -x, \qquad \frac{dx}{d\tau} = w - \mu F(x)$$

where $x = L^{1/2}I$, $w = C^{1/2}V$, $\tau = (LC)^{-1/2}t$, and $F(x) = f(L^{-1/2}x)$.

In Section 7.5, we'll see that this system for (w, x) is equivalent to the van der Pol equation, if $F(x) = \frac{1}{3}x^3 - x$. Thus the circuit produces self-sustained oscillations.

7.1.7 (Waveform) Consider the system $\dot{r} = r(4-r^2)$, $\dot{\theta} = 1$, and let $x(t) = r(t)\cos\theta(t)$. Given the initial condition $x(0) = 0.1$, $y(0) = 0$, sketch the approximate waveform of $x(t)$, *without* obtaining an explicit expression for it.

7.1.8 (A circular limit cycle) Consider $\ddot{x} + a\dot{x}(x^2 + \dot{x}^2 - 1) + x = 0$, where $a > 0$.

a) Find and classify all the fixed points.

b) Show that the system has a circular limit cycle, and find its amplitude and period.

c) Determine the stability of the limit cycle.

d) Give an argument which shows that the limit cycle is unique, i.e., there are no other periodic trajectories.

7.1.9 (Circular pursuit problem) A dog at the center of circular pond sees a duck swimming along the edge. The dog chases the duck by always swimming straight toward it. In other words, the dog's velocity vector always lies along the line connecting it to the duck. Meanwhile, the duck takes evasive action by swimming around the circumference as fast as it can, always moving counterclockwise.

a) Assuming the pond has unit radius and both animals swim at the same constant speed, derive a pair of differential equations for the path of the dog. (Hint: Use the coordinate system shown in Figure 2 and find equations for $dR/d\theta$ and $d\phi/d\theta$.) Analyze the system. Can you solve it explicitly? Does the dog ever catch the duck?

b) Now suppose the dog swims k times faster than the duck. Derive the differential equations for the dog's path.

c) If $k = \frac{1}{2}$, what does the dog end up doing in the long run?

Note: This problem has a long and intriguing history, dating back to the mid-1800s at least. It is much more difficult than similar ***pursuit problems***—there is no known solution for the path of the dog in part (a), in terms of elementary functions. See Davis (1962, pp. 113–125) for a nice analysis and a guide to the literature.

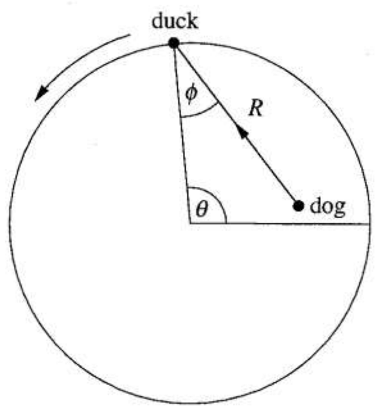

Figure 2

7.2 Ruling Out Closed Orbits

Plot the phase portraits of the following gradient systems $\dot{\mathbf{x}} = -\nabla V$.

7.2.1 $V = x^2 + y^2$ **7.2.2** $V = x^2 - y^2$ **7.2.3** $V = e^x \sin y$

7.2.4 Show that all vector fields on the line are gradient systems. Is the same true of vector fields on the circle?

7.2.5 Let $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$ be a smooth vector field defined on the phase plane.

a) Show that if this is a gradient system, then $\partial f/\partial y = \partial g/\partial x$.

b) Is the condition in (a) also sufficient?

7.2.6 Given that a system is a gradient system, here's how to find its potential function V. Suppose that $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$. Then $\dot{\mathbf{x}} = -\nabla V$ implies

$f(x,y) = -\partial V/\partial x$ and $g(x,y) = -\partial V/\partial y$. These two equations may be "partially integrated" to find V. Use this procedure to find V for the following gradient systems.

a) $\dot{x} = y^2 + y\cos x, \quad \dot{y} = 2xy + \sin x$

b) $\dot{x} = 3x^2 - 1 - e^{2y}, \quad \dot{y} = -2xe^{2y}$

7.2.7 Consider the system $\dot{x} = y + 2xy$, $\dot{y} = x + x^2 - y^2$.

a) Show that $\partial f/\partial y = \partial g/\partial x$. (Then Exercise 7.2.5(a) implies this is a gradient system.)

b) Find V.

c) Sketch the phase portrait.

7.2.8 Show that the trajectories of a gradient system always cross the equipotentials at right angles (except at fixed points).

7.2.9 For each of the following systems, decide whether it is a gradient system. If so, find V and sketch the phase portrait. On a separate graph, sketch the equipotentials $V = \text{constant}$. (If the system is not a gradient system, go on to the next question.)

a) $\dot{x} = y + x^2 y, \quad \dot{y} = -x + 2xy$

b) $\dot{x} = 2x, \quad \dot{y} = 8y$

c) $\dot{x} = -2xe^{x^2+y^2}, \quad \dot{y} = -2ye^{x^2+y^2}$

7.2.10 Show that the system $\dot{x} = y - x^3$, $\dot{y} = -x - y^3$ has no closed orbits, by constructing a Liapunov function $V = ax^2 + by^2$ with suitable a, b.

7.2.11 Show that $V = ax^2 + 2bxy + cy^2$ is positive definite if and only if $a > 0$ and $ac - b^2 > 0$. (This is a useful criterion that allows us to test for positive definiteness when the quadratic form V includes a "cross term" $2bxy$.)

7.2.12 Show that $\dot{x} = -x + 2y^3 - 2y^4$, $\dot{y} = -x - y + xy$ has no periodic solutions. (Hint: Choose a, m, and n such that $V = x^m + ay^n$ is a Liapunov function.)

7.2.13 Recall the competition model

$$\dot{N}_1 = r_1 N_1(1 - N_1/K_1) - b_1 N_1 N_2, \qquad \dot{N}_2 = r_2 N_2(1 - N_2/K_2) - b_2 N_1 N_2,$$

of Exercise 6.4.6. Using Dulac's criterion with the weighting function $g = (N_1 N_2)^{-1}$, show that the system has no periodic orbits in the first quadrant $N_1, N_2 > 0$.

7.2.14 Consider $\dot{x} = x^2 - y - 1$, $\dot{y} = y(x-2)$.

a) Show that there are three fixed points and classify them.

b) By considering the three straight lines through pairs of fixed points, show that there are no closed orbits.
c) Sketch the phase portrait.

7.2.15 Consider the system $\dot{x} = x(2 - x - y)$, $\dot{y} = y(4x - x^2 - 3)$. We know from Example 7.2.4 that this system has no closed orbits.
a) Find the three fixed points and classify them.
b) Sketch the phase portrait.

7.2.16 If R is not simply connected, then the conclusion of Dulac's criterion is no longer valid. Find a counterexample.

7.2.17 Assume the hypotheses of Dulac's criterion, except now suppose that R is topologically equivalent to an annulus, i.e., it has exactly one hole in it. Using Green's theorem, show that there exists *at most* one closed orbit in R. (This result can be useful sometimes as a way of proving that a closed orbit is unique.)

7.3 Poincaré–Bendixson Theorem

7.3.1 Consider $\dot{x} = x - y - x(x^2 + 5y^2)$, $\dot{y} = x + y - y(x^2 + y^2)$.
a) Classify the fixed point at the origin.
b) Rewrite the system in polar coordinates, using $r\dot{r} = x\dot{x} + y\dot{y}$ and $\dot{\theta} = (x\dot{y} - y\dot{x})/r^2$.
c) Determine the circle of maximum radius, r_1, centered on the origin such that all trajectories have a radially *outward* component on it.
d) Determine the circle of minimum radius, r_2, centered on the origin such that all trajectories have a radially *inward* component on it.
e) Prove that the system has a limit cycle somewhere in the trapping region $r_1 \le r \le r_2$.

7.3.2 Using numerical integration, compute the limit cycle of Exercise 7.3.1 and verify that it lies in the trapping region you constructed.

7.3.3 Show that the system $\dot{x} = x - y - x^3$, $\dot{y} = x + y - y^3$ has a periodic solution.

7.3.4 Consider the system

$$\dot{x} = x(1 - 4x^2 - y^2) - \tfrac{1}{2}y(1 + x), \qquad \dot{y} = y(1 - 4x^2 - y^2) + 2x(1 + x).$$

a) Show that the origin is an unstable fixed point.
b) By considering $\dot{V}$, where $V = (1 - 4x^2 - y^2)^2$, show that all trajectories approach the ellipse $4x^2 + y^2 = 1$ as $t \to \infty$.

7.3.5 Show that the system $\dot{x} = -x - y + x(x^2 + 2y^2)$, $\dot{y} = x - y + y(x^2 + 2y^2)$ has at least one periodic solution.

7.3.6 Consider the oscillator equation $\ddot{x} + F(x, \dot{x})\dot{x} + x = 0$, where $F(x, \dot{x}) < 0$ if $r \le a$ and $F(x, \dot{x}) > 0$ if $r \ge b$, where $r^2 = x^2 + \dot{x}^2$.

a) Give a physical interpretation of the assumptions on F.
b) Show that there is at least one closed orbit in the region $a < r < b$.

7.3.7 Consider $\dot{x} = y + ax(1 - 2b - r^2)$, $\dot{y} = -x + ay(1 - r^2)$, where a and b are parameters ($0 < a \le 1$, $0 \le b < \frac{1}{2}$) and $r^2 = x^2 + y^2$.

a) Rewrite the system in polar coordinates.
b) Prove that there is at least one limit cycle, and that if there are several, they all have the same period $T(a, b)$.
c) Prove that for $b = 0$ there is only one limit cycle.

7.3.8 Recall the system $\dot{r} = r(1 - r^2) + \mu r \cos\theta$, $\dot{\theta} = 1$ of Example 7.3.1. Using the computer, plot the phase portrait for various values of $\mu > 0$. Is there a critical value μ_c at which the closed orbit ceases to exist? If so, estimate it. If not, prove that a closed orbit exists for *all* $\mu > 0$.

7.3.9 (Series approximation for a closed orbit) In Example 7.3.1, we used the Poincaré–Bendixson Theorem to prove that the system $\dot{r} = r(1 - r^2) + \mu r \cos\theta$, $\dot{\theta} = 1$ has a closed orbit in the annulus $\sqrt{1-\mu} < r < \sqrt{1+\mu}$ for all $\mu < 1$.

a) To approximate the shape $r(\theta)$ of the orbit for $\mu \ll 1$, assume a power series solution of the form $r(\theta) = 1 + \mu r_1(\theta) + O(\mu^2)$. Substitute the series into a differential equation for $dr/d\theta$. Neglect all $O(\mu^2)$ terms, and thereby derive a simple differential equation for $r_1(\theta)$. Solve this equation explicitly for $r_1(\theta)$. (The approximation technique used here is called regular perturbation theory; see Section 7.6.)
b) Find the maximum and minimum r on your approximate orbit, and hence show that it lies in the annulus $\sqrt{1-\mu} < r < \sqrt{1+\mu}$, as expected.
c) Use a computer to calculate $r(\theta)$ numerically for various small μ, and plot the results on the same graph as your analytical approximation for $r(\theta)$. How does the maximum error depend on μ?

7.3.10 Consider the two-dimensional system $\dot{\mathbf{x}} = A\mathbf{x} - r^2\mathbf{x}$, where $r = \|\mathbf{x}\|$ and A is a 2×2 constant real matrix with complex eigenvalues $\alpha \pm i\omega$. Prove that there exists at least one limit cycle for $\alpha > 0$ and that there are none for $\alpha < 0$.

7.3.11 (Cycle graphs) Suppose $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a smooth vector field on $\mathbf{R}^2$. An improved version of the Poincaré–Bendixson theorem states that if a trajectory is trapped in a compact region, then it must approach a fixed point, a limit cycle, or something exotic called a ***cycle graph*** (an invariant set containing a finite number of fixed points connected by a finite number of trajectories, all oriented either clockwise or counterclockwise). Cycle graphs are rare in practice; here's a contrived but simple example.

a) Plot the phase portrait for the system

$$\dot{r} = r(1-r^2)\left[r^2\sin^2\theta + (r^2\cos^2\theta - 1)^2\right]$$
$$\dot{\theta} = r^2\sin^2\theta + (r^2\cos^2\theta - 1)^2$$

where r, θ are polar coordinates. (Hint: Note the common factor in the two equations; examine where it vanishes.)

b) Sketch x vs. t for a trajectory starting away from the unit circle. What happens as $t \to \infty$?

7.4 Liénard Systems

† **7.4.1** Show that the equation $\ddot{x} + \mu(x^2-1)\dot{x} + \tanh x = 0$, for $\mu > 0$, has exactly one periodic solution, and classify its stability.

† **7.4.2** Consider the equation $\ddot{x} + \mu(x^4-1)\dot{x} + x = 0$.

a) Prove that the system has a unique stable limit cycle if $\mu > 0$.

b) Using a computer, plot the phase portrait for the case $\mu = 1$.

c) If $\mu < 0$, does the system still have a limit cycle? If so, is it stable or unstable?

7.5 Relaxation Oscillations

7.5.1 For the van der Pol oscillator with $\mu >> 1$, show that the positive branch of the cubic nullcline begins at $x_A = 2$ and ends at $x_B = 1$.

7.5.2 In Example 7.5.1, we used a tricky phase plane (often called the ***Liénard plane***) to analyze the van der Pol oscillator for $\mu >> 1$. Try to redo the analysis in the standard phase plane where $\dot{x} = y$, $\dot{y} = -x - \mu(x^2-1)$. What is the advantage of the Liénard plane?

7.5.3 Estimate the period of the limit cycle of $\ddot{x} + k(x^2-4)\dot{x} + x = 1$ for $k >> 1$.

7.5.4 (Piecewise-linear nullclines) Consider the equation $\ddot{x} + \mu f(x)\dot{x} + x = 0$, where $f(x) = -1$ for $|x| < 1$ and $f(x) = 1$ for $|x| \geq 1$.

a) Show that the system is equivalent to $\dot{x} = \mu(y - F(x))$, $\dot{y} = -x/\mu$, where $F(x)$ is the piecewise-linear function

$$F(x) = \begin{cases} x+2, & x \leq -1 \\ -x, & |x| \leq 1 \\ x-2, & x \geq 1. \end{cases}$$

b) Graph the nullclines.
c) Show that the system exhibits relaxation oscillations for $\mu >> 1$, and plot the limit cycle in the (x, y) plane.
d) Estimate the period of the limit cycle for $\mu >> 1$.

7.5.5 Consider the equation $\ddot{x} + \mu\left(|x| - 1\right)\dot{x} + x = 0$. Find the approximate period of the limit cycle for $\mu >> 1$.

7.5.6 (Biased van der Pol) Suppose the van der Pol oscillator is biased by a constant force: $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = a$, where a can be positive, negative, or zero. (Assume $\mu > 0$ as usual.)
a) Find and classify all the fixed points.
b) Plot the nullclines in the Liénard plane. Show that if they intersect on the *middle* branch of the cubic nullcline, the corresponding fixed point is unstable.
c) For $\mu >> 1$, show that the system has a stable limit cycle if and only if $|a| < a_c$, where a_c is to be determined. (Hint: Use the Liénard plane.)
d) Sketch the phase portrait for a slightly greater than a_c. Show that the system is *excitable* (it has a globally attracting fixed point, but certain disturbances can send the system on a long excursion through phase space before returning to the fixed point; compare Exercise 4.5.3.)

This system is closely related to the Fitzhugh–Nagumo model of neural activity; see Murray (1989) or Edelstein–Keshet (1988) for an introduction.

7.5.7 (Cell cycle) Tyson (1991) proposed an elegant model of the cell division cycle, based on interactions between the proteins cdc2 and cyclin. He showed that the model's mathematical essence is contained in the following set of dimensionless equations:

$$\dot{u} = b(v-u)(\alpha + u^2) - u, \qquad \dot{v} = c - u,$$

where u is proportional to the concentration of the active form of a cdc2-cyclin complex, and v is proportional to the total cyclin concentration (monomers and dimers). The parameters $b >> 1$ and $\alpha << 1$ are fixed and satisfy $8\alpha b < 1$, and c is adjustable.
a) Sketch the nullclines.
b) Show that the system exhibits relaxation oscillations for $c_1 < c < c_2$, where c_1 and c_2 are to be determined approximately. (It is too hard to find c_1 and c_2 exactly, but a good approximation can be achieved if you assume $8\alpha b << 1$.)
c) Show that the system is excitable if c is slightly less than c_1.

7.6 Weakly Nonlinear Oscillators

7.6.1 Show that if (7.6.7) is expanded as a power series in ε, we recover (7.6.17).

7.6.2 (Calibrating regular perturbation theory) Consider the initial value problem $\ddot{x} + x + \varepsilon x = 0$, with $x(0) = 1$, $\dot{x}(0) = 0$.

a) Obtain the exact solution to the problem.

b) Using regular perturbation theory, find x_0, x_1, and x_2 in the series expansion $x(t,\varepsilon) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + O(\varepsilon^3)$.

c) Does the perturbation solution contain secular terms? Did you expect to see any? Why?

7.6.3 (More calibration) Consider the initial value problem $\ddot{x} + x = \varepsilon$, with $x(0) = 1$, $\dot{x}(0) = 0$.

a) Solve the problem exactly.

b) Using regular perturbation theory, find x_0, x_1, and x_2 in the series expansion $x(t,\varepsilon) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + O(\varepsilon^3)$.

c) Explain why the perturbation solution does or doesn't contain secular terms.

For each of the following systems $\ddot{x} + x + \varepsilon h(x,\dot{x}) = 0$, with $0 < \varepsilon << 1$, calculate the averaged equations (7.6.53) and analyze the long-term behavior of the system. Find the amplitude and frequency of any limit cycles for the original system. If possible, solve the averaged equations explicitly for $x(t,\varepsilon)$, given the initial conditions $x(0) = a$, $\dot{x}(0) = 0$.

7.6.4 $h(x,\dot{x}) = x$

7.6.5 $h(x,\dot{x}) = x\dot{x}^2$

7.6.6 $h(x,\dot{x}) = x\dot{x}$

7.6.7 $h(x,\dot{x}) = (x^4 - 1)\dot{x}$

7.6.8 $h(x,\dot{x}) = (|x| - 1)\dot{x}$

7.6.9 $h(x,\dot{x}) = (x^2 - 1)\dot{x}^3$

7.6.10 Derive the identity $\sin\theta \cos^2\theta = \frac{1}{4}[\sin\theta + \sin 3\theta]$ as follows: Use the complex representations

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i},$$

multiply everything out, and then collect terms. This is always the most straightforward method of deriving such identities, and you don't have to remember any others.

7.6.11 (Higher harmonics) Notice the third harmonic $\sin 3(\tau + \phi)$ in Equation (7.6.39). The generation of *higher harmonics* is a characteristic feature of nonlinear systems. To find the effect of such terms, return to Example 7.6.2 and solve for x_1, assuming that the original system had initial conditions $x(0) = 2$, $\dot{x}(0) = 0$.

7.6.12 (Deriving the Fourier coefficients) This exercise leads you through the derivation of the formulas (7.6.51) for the Fourier coefficients. For convenience, let brackets denote the average of a function: $\langle f(\theta)\rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta$ for any 2π-periodic function f. Let k and m be arbitrary integers.

a) Using integration by parts, complex exponentials, trig identities, or otherwise, derive the ***orthogonality relations***

$$\langle \cos k\theta \, \sin m\theta \rangle = 0 \text{, for all } k, m \, ;$$

$$\langle \cos k\theta \, \cos m\theta \rangle = \langle \sin k\theta \, \sin m\theta \rangle = 0 \text{, for all } k \neq m \, ;$$

$$\langle \cos^2 k\theta \rangle = \langle \sin^2 k\theta \rangle = \tfrac{1}{2} \text{, for } k \neq 0 \, .$$

b) To find a_k for $k \neq 0$, multiply both sides of (7.6.50) by $\cos m\theta$ and average both sides term by term over the interval $[0, 2\pi]$. Now using the orthogonality relations from part (a), show that *all the terms on the right-hand side cancel out, except the $k = m$ term*! Deduce that $\langle h(\theta)\cos k\theta \rangle = \frac{1}{2}a_k$, which is equivalent to the formula for a_k in (7.6.51).

c) Similarly, derive the formulas for b_k and a_0.

7.6.13 (Exact period of a conservative oscillator) Consider the Duffing oscillator $\ddot{x} + x + \varepsilon x^3 = 0$, where $0 < \varepsilon << 1$, $x(0) = a$, and $\dot{x}(0) = 0$.

a) Using conservation of energy, express the oscillation period $T(\varepsilon)$ as a certain integral.

b) Expand the integrand as a power series in ε, and integrate term by term to obtain an approximate formula $T(\varepsilon) = c_0 + c_1\varepsilon + c_2\varepsilon^2 + O(\varepsilon^3)$. Find c_0, c_1, c_2 and check that c_0, c_1 are consistent with (7.6.57).

7.6.14 (Computer test of two-timing) Consider the equation $\ddot{x} + \varepsilon \dot{x}^3 + x = 0$.

a) Derive the averaged equations.

b) Given the initial conditions $x(0) = a$, $\dot{x}(0) = 0$, solve the averaged equations and thereby find an approximate formula for $x(t, \varepsilon)$.

c) Solve $\ddot{x} + \varepsilon \dot{x}^3 + x = 0$ numerically for $a = 1$, $\varepsilon = 2$, $0 \le t \le 50$, and plot the result on the same graph as your answer to part (b). Notice the impressive agreement, even though ε is not small!

7.6.15 (Pendulum) Consider the pendulum equation $\ddot{x} + \sin x = 0$.

a) Using the method of Example 7.6.4, show that the frequency of small oscillations of amplitude $a << 1$ is given by $\omega \approx 1 - \frac{1}{16}a^2$. (Hint: $\sin x \approx x - \frac{1}{6}x^3$, where $\frac{1}{6}x^3$ is a "small" perturbation.)

b) Is this formula for ω consistent with the exact results obtained in Exercise 6.7.4?

7.6.16 (Amplitude of the van der Pol oscillator via Green's theorem) Here's another way to determine the radius of the nearly circular limit cycle of the van der

Pol oscillator $\ddot{x}+\varepsilon\dot{x}(x^2-1)+x=0$, in the limit $\varepsilon \ll 1$. Assume that the limit cycle is a circle of unknown radius a about the origin, and invoke the normal form of Green's theorem (i.e., the 2-D divergence theorem):

$$\oint_C \mathbf{v}\cdot\mathbf{n}\,d\ell = \iint_A \nabla\cdot\mathbf{v}\,dA$$

where C is the cycle and A is the region enclosed. By substituting $\mathbf{v}=\dot{\mathbf{x}}=(\dot{x},\dot{y})$ and evaluating the integrals, show that $a\approx 2$.

7.6.17 (Playing on a swing) A simple model for a child playing on a swing is

$$\ddot{x}+(1+\varepsilon\gamma+\varepsilon\cos 2t)\sin x=0$$

where ε and γ are parameters, and $0<\varepsilon\ll 1$. The variable x measures the angle between the swing and the downward vertical. The term $1+\varepsilon\gamma+\varepsilon\cos 2t$ models the effects of gravity and the periodic pumping of the child's legs at approximately twice the natural frequency of the swing. The question is: Starting near the fixed point $x=0$, $\dot{x}=0$, can the child get the swing going by pumping her legs this way, or does she need a push?

a) For small x, the equation may be replaced by $\ddot{x}+(1+\varepsilon\gamma+\varepsilon\cos 2t)x=0$. Show that the averaged equations (7.6.53) become

$$r'=\tfrac{1}{4}r\sin 2\phi, \qquad \phi'=\tfrac{1}{2}(\gamma+\tfrac{1}{2}\cos 2\phi),$$

where $x=r\cos\theta=r(T)\cos(t+\phi(T))$, $\dot{x}=-r\sin\theta=-r(T)\sin(t+\phi(T))$, and prime denotes differentiation with respect to slow time $T=\varepsilon t$. Hint: To average terms like $\cos 2t\,\cos\theta\,\sin\theta$ over one cycle of θ, recall that $t=\theta-\phi$ and use trig identities:

$$\begin{aligned}\langle\cos 2t\,\cos\theta\,\sin\theta\rangle &= \tfrac{1}{2}\langle\cos(2\theta-2\phi)\,\sin 2\theta\rangle\\ &= \tfrac{1}{2}\langle(\cos 2\theta\,\cos 2\phi+\sin 2\theta\,\sin 2\phi)\,\sin 2\theta\rangle\\ &= \tfrac{1}{4}\sin 2\phi.\end{aligned}$$

b) Show that the fixed point $r=0$ is unstable to exponentially growing oscillations, i.e., $r(T)=r_0e^{kT}$ with $k>0$, if $|\gamma|<\gamma_c$ where γ_c is to be determined. (Hint: For r near 0, $\phi'\gg r'$ so ϕ equilibrates relatively rapidly.)

c) For $|\gamma|<\gamma_c$, write a formula for the growth rate k in terms of γ.

d) How do the solutions to the averaged equations behave if $|\gamma|>\gamma_c$?

e) Interpret the results physically.

7.6.18 (Mathieu equation and a super-slow time scale) Consider the ***Mathieu equation*** $\ddot{x}+(a+\varepsilon\cos t)x=0$ with $a\approx 1$. Using two-timing with a slow time

$T=\varepsilon^2 t$, show that the solution becomes unbounded as $t\to\infty$ if $1-\frac{1}{12}\varepsilon^2+O(\varepsilon^4)\le a\le 1+\frac{5}{12}\varepsilon^2+O(\varepsilon^4)$.

7.6.19 (Poincaré–Lindstedt method) This exercise guides you through an improved version of perturbation theory known as the ***Poincaré–Lindstedt method***. Consider the Duffing equation $\ddot{x}+x+\varepsilon x^3=0$, where $0<\varepsilon<<1$, $x(0)=a$, and $\dot{x}(0)=0$. We know from phase plane analysis that the true solution $x(t,\varepsilon)$ is periodic; our goal is to find an approximate formula for $x(t,\varepsilon)$ that is valid for all t. The key idea is to regard the frequency ω as *unknown* in advance, and to solve for it by demanding that $x(t,\varepsilon)$ contains no secular terms.

a) Define a new time $\tau=\omega t$ such that the solution has period 2π with respect to τ. Show that the equation transforms to $\omega^2 x''+x+\varepsilon x^3=0$.

b) Let $x(\tau,\varepsilon)=x_0(\tau)+\varepsilon x_1(\tau)+\varepsilon^2 x_2(\tau)+O(\varepsilon^3)$ and $\omega=1+\varepsilon\omega_1+\varepsilon^2\omega_2+O(\varepsilon^3)$. (We know already that $\omega_0=1$ since the solution has frequency $\omega=1$ when $\varepsilon=0$.) Substitute these series into the differential equation and collect powers of ε. Show that

$$O(1):\ x_0''+x_0=0$$
$$O(\varepsilon):\ x_1''+x_1=-2\omega_1 x_0''-x_0^3.$$

c) Show that the initial conditions become $x_0(0)=a$, $\dot{x}_0(0)=0$; $x_k(0)=\dot{x}_k(0)=0$ for all $k>0$.

d) Solve the $O(1)$ equation for x_0.

e) Show that after substitution of x_0 and the use of a trigonometric identity, the $O(\varepsilon)$ equation becomes $x_1''+x_1=(2\omega_1 a-\frac{3}{4}a^3)\cos\tau-\frac{1}{4}a^3\cos 3\tau$. Hence, *to avoid secular terms*, we need $\omega_1=\frac{3}{8}a^2$.

f) Solve for x_1.

Two comments: (1) This exercise shows that the Duffing oscillator has a frequency that depends on amplitude: $\omega=1+\frac{3}{8}\varepsilon a^2+O(\varepsilon^2)$, in agreement with (7.6.57). (2) The Poincaré–Lindstedt method is good for approximating periodic solutions, but that's *all* it can do; if you want to explore transients or nonperiodic solutions, you can't use this method. Use two-timing or averaging theory instead.

7.6.20 Show that if we had used regular perturbation to solve Exercise 7.6.19, we would have obtained $x(t,\varepsilon)=a\cos t+\varepsilon a^3\left[-\frac{3}{8}t\sin t+\frac{1}{32}(\cos 3t-\cos t)\right]+O(\varepsilon^2)$. Why is this solution inferior?

7.6.21 Using the Poincaré–Lindstedt method, show that the frequency of the limit cycle for the van der Pol oscillator $\ddot{x}+\varepsilon(x^2-1)\dot{x}+x=0$ is given by

$\omega = 1 - \frac{1}{16}\varepsilon^2 + O(\varepsilon^3)$.

7.6.22 (Asymmetric spring) Use the Poincaré–Lindstedt method to find the first few terms in the expansion for the solution of $\ddot{x} + x + \varepsilon x^2 = 0$, with $x(0) = a$, $\dot{x}(0) = 0$. Show that the center of oscillation is at $x \approx \frac{1}{2}\varepsilon a^2$, approximately.

7.6.23 Find the approximate relation between amplitude and frequency for the periodic solutions of $\ddot{x} - \varepsilon x\dot{x} + x = 0$.

7.6.24 (Computer algebra) Using Mathematica, Maple, or some other computer algebra package, apply the Poincaré–Lindstedt method to the problem $\ddot{x} + x - \varepsilon x^3 = 0$, with $x(0) = a$, and $\dot{x}(0) = 0$. Find the frequency ω of periodic solutions, up to and including the $O(\varepsilon^3)$ term.

7.6.25 (The method of averaging) Consider the weakly nonlinear oscillator $\ddot{x} + x + \varepsilon h(x, \dot{x}, t) = 0$. Let $x(t) = r(t)\cos(t + \phi(t))$, $\dot{x} = -r(t)\sin(t + \phi(t))$. This change of variables should be regarded as a definition of $r(t)$ and $\phi(t)$.

a) Show that $\dot{r} = \varepsilon h \sin(t + \phi)$, $r\dot{\phi} = \varepsilon h \cos(t + \phi)$. (Hence r and ϕ are slowly varying for $0 < \varepsilon \ll 1$, and thus $x(t)$ is a sinusoidal oscillation modulated by a slowly drifting amplitude and phase.)

b) Let $\langle r\rangle(t) = \bar{r}(t) = \frac{1}{2\pi}\int_{t-\pi}^{t+\pi} r(\tau)\, d\tau$ denote the running average of r over one cycle of the sinusoidal oscillation. Show that $d\langle r\rangle/dt = \langle dr/dt\rangle$, i.e., it doesn't matter whether we differentiate or time-average first.

c) Show that $d\langle r\rangle/dt = \varepsilon\left\langle h[r\cos(t + \phi), -r\sin(t + \phi), t]\sin(t + \phi)\right\rangle$.

d) The result of part (c) is exact, but not helpful because the left-hand side involves $\langle r\rangle$ whereas the right-hand side involves r. Now comes the key approximation: replace r and ϕ by their averages over one cycle. Show that $r(t) = \bar{r}(t) + O(\varepsilon)$ and $\phi(t) = \bar{\phi}(t) + O(\varepsilon)$, and therefore

$$d\bar{r}/dt = \varepsilon\left\langle h[\bar{r}\cos(t + \bar{\phi}), -\bar{r}\sin(t + \bar{\phi}), t]\sin(t + \bar{\phi})\right\rangle + O(\varepsilon^2)$$
$$\bar{r}\, d\bar{\phi}/dt = \varepsilon\left\langle h[\bar{r}\cos(t + \bar{\phi}), -\bar{r}\sin(t + \bar{\phi}), t]\cos(t + \bar{\phi})\right\rangle + O(\varepsilon^2)$$

where the barred quantities are to be treated as constants inside the averages. These equations are just the *averaged equations* (7.6.53), derived by a different approach in the text. It is customary to drop the overbars; one usually doesn't distinguish between slowly varying quantities and their averages.

7.6.26 (Calibrating the method of averaging) Consider the equation $\dot{x} = -\varepsilon x \sin^2 t$, with $0 \le \varepsilon \ll 1$ and $x = x_0$ at $t = 0$.

a) Find the *exact* solution to the equation.

b) Let $\bar{x}(t) = \frac{1}{2\pi}\int_{t-\pi}^{t+\pi} x(\tau)\,d\tau$. Show that $x(t) = \bar{x}(t) + O(\varepsilon)$. Use the method of averaging to find an approximate differential equation satisfied by $\bar{x}$, and solve it.

c) Compare the results of parts (a) and (b); how large is the error incurred by averaging?