Sept 13

We began by considering the function

$$f(x) = \frac{x^2 + 2x - 3}{x - 1}$$

Notice that $x = 1$ is **not** in the domain. Every other number is in the domain. What does the function do 'near' $x = 1$?

Let's compute

$$\begin{aligned} f(1.1) &= \frac{1.1^2 + 2 \cdot 1.1 - 3}{1.1 - 1} \\ &= \frac{1.21 + 2.2 - 3}{0.1} \\ &= \frac{0.41}{0.1} \\ &= 4.1, \end{aligned}$$

and then compute

$$\begin{aligned} f(1.01) &= \frac{1.01^2 + 2 \cdot 1.01 - 3}{1.01 - 1} \\ &= \frac{1.0201 + 2.02 - 3}{1.01 - 1} \\ &= \frac{3.0401}{0.01} \\ &= 4.01, \end{aligned}$$

and then compute

$$\begin{aligned} f(1.001) &= \frac{1.001^2 + 2 \cdot 1.001 - 3}{1.001 - 1} \\ &= \frac{1.002001 + 2.002 - 3}{1.001 - 1} \\ &= \frac{3.004001}{0.001} \\ &= 4.001, \end{aligned}$$

and then...

So, it seems like the values of $f(x)$ are getting closer and closer to 4 as our input x gets closer and closer to 1. Remember that you can talk about limits coming from both the left side and the right side, and they have to agree if we want to talk about 'the' limit. So I should also compute

$$\begin{aligned} f(0.9) &= \frac{0.9^2 + 2 \cdot 0.9 - 3}{0.9 - 1} \\ &= \frac{0.81 + 1.8 - 3}{-0.1} \\ &= \frac{-0.39}{-0.1} \\ &= 3.9 \end{aligned}$$

and then compute..... to convince you the limit as x approaches 1 from either side is 4. Notice that, even if I do a million decimal places, it doesn' really 'prove' that the limit is actually 4. You could imagine the function seems like its going to 4 but actually once you get to a billion decimal places it starts to jump up to 5. It's a little hard to imagine how to *prove* you can control the values of the function, since there are infinite possibilities for the input.

In this case, we can do some algebra:

$$\begin{aligned} \frac{x^2 + 2x - 3}{x - 1} &= \frac{(x-1)(x+3)}{x-1} \\ &\overset{!}{=} x + 3 \end{aligned}$$

Here, it's pretty clear that if x gets close to 1, $x + 3$ will get close to 4. If we want the correct answer to three decimal places, 4.000, then we need for x to be correct to three decimal places, so x between 0.999 and 1.001. In symbols, we want $0.999 < x < 1.001$, or $x \in (0.999, 1.001)$. (Here, I'm counting 3.999 as 'correct up to three decimal places': really I should be saying 'within 0.001 of the correct value').

Why the exclamation point in the above equation? Because that equality is **not** true always. There is one place where it fails, namely when $x = 1$. Then the right side is $3 + 1 = 4$, but the left side involves a division by 0, so is undefined. Since the limit is asking about what the function is doing when inputs *get close* to 1, and the two functions agree everywhere close to 1, the limits are the same. So we are allowed to say the limit of $f(x)$ is the limit of $x + 3$.

Claim I.f we have two functions $f(x)$ and $g(x)$ which are defined near $x = a$, and $f(x) = g(x)$ for every input x except $x = a$, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$$

It is interesting to notice that we don't really need $f(x)$ or $g(x)$ to even be defined at $x = a$. When I say 'near $x = a$', I really mean x near but not equal to a.

A third example: consider

$$k(t) = \frac{t^2 - t}{t^2 + 3t - 10}$$

What are the domain and range of $k(t)$? (This has an interesting answer).

Find the limit

$$\lim_{t\to 2} k(t)$$

Notice in particular that 2 is not in the domain of k.

Factor the polynomials:

$$\begin{aligned}
\lim_{t\to 2} k(t) &= \frac{t^2-2t}{t^2+3t-10} \\
&= \lim_{t\to 2} \frac{t(t-2)}{(t-2)(t+5)} \\
&= \lim_{t\to 2} \frac{t}{t+5} \\
&= \frac{2}{2+5} \\
&= \frac{2}{7}
\end{aligned}$$

In these computations, we have been using 'obvious' properties of limits (which we have not proved, since we haven't really 'proven' any of these limits).

Let's state these properties. The main idea, in words, is: 'If a function is built out of simpler functions in some algebraic way, then the limit of the total function is built out of the limits of the simpler function in the same algebraic way.'

This is phrased as

Theorem 1 (Limit Laws). *Suppose $f(x)$ and $g(x)$ are functions, a is some number such that*

$$\lim_{x\to a} f(x)$$

and

$$\lim_{x\to a} g(x)$$

both exist. We also have some number c. Then:

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

3. $\lim_{x\to a}[c\cdot f(x)] = c\cdot \lim_{x\to a} f(x)$

4. $\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

5. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ ***as long as*** $\lim_{x\to a} g(x) \neq 0$

By using the 4th law over and over, we get that for n a positive integer

6. $\lim_{x\to a}[f(x)^n] = [\lim_{x\to a} f(x)]^n$

and, through a bit more complicated reasoning, we get that for n a positive integer

7. $\lim_{x\to a}[\sqrt[n]{f(x)}] = \sqrt[n]{\lim_{x\to a} f(x)}$

Warning! Law 7 doesn't make sense for n even unless the limit of $f(x)$ we are talking about is positive.

We also need two laws which are simple to prove

8. $\lim_{x\to a} c = c$

9. $\lim_{x\to a} x = a$

This last law is saying 'the limit of x as x approaches a is a'. Since a polynomial is built up from powers of x multiplied by a number and added together, we can use laws 1, 2, 3, and 9 to say that the limit of any polynomial is the same as directly substituting a in for x. By limit law 5, this is also true for rational functions, as long as the denominator is not equal to 0 when we substitute $x = a$.

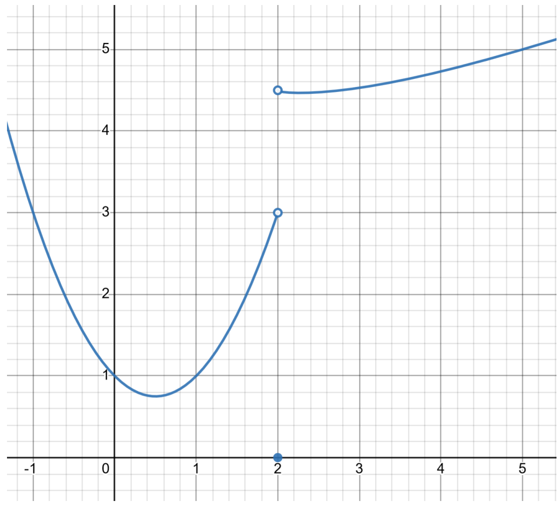
We should label this a

Proposition 1 (Direct susbtitution)**.** *If $f(x)$ is a polynomial and a is any number, or if $f(x)$ is a rational function and a is in its domain, then*

$$\lim_{x\to a} f(x) = f(a)$$

This is also true for algebraic functions(always being careful about domains), exponential functions, and trigonometric functions.

While it is good to be able to manipulate algebraic formulas, one should also spend time thinking about the geometric content Here was the 'challenge' problem I gave at the end of class: The following is the graph of a function $h(x)$.



Before asking the trick question, notice that the limit of $h(x)$ as x approaches 2 does not exist. In this case, 2 is in the domain of h, and we have $h(2) = 0$. Still, the limit does not exist.

The trick question was supposed to be 'Find the limit $\lim_{x\to 0} h(2-x^2)$. The reason it is a trick is that you should notice when x goes to 0, $2-x^2$ goes to 2. We just said the limit of h as x approaches 2 does not exist, so you might think this limit does not exist.

This is false. We are not just taking the limit of $h(x)$ as x goes to 2, we are taking the limit of $h(2-x^2)$ as x goes to 0. In particular, x^2 is always positive, so as x gets closer to 0, $2-x^2$ is closer to 2, but always less than 2. Then, examining the graph, we see that the one-sided (left-sided) limit of $h(x)$ does exist at $x = 2$. Since we are approaching from the left, the whole composite limit exists, and we have

$$\lim_{x\to 0} h(2-x^2) = 3$$

The point of this trick question is to show you that even though the limit laws say limits work perfectly with algebraic operations, they do not always work with composition. Actually, it's not completely clear what we should ask for, but all of these fail:

$$\lim_{x\to a} f \circ g(x) \neq f\left(\lim_{x\to a} g(x)\right)$$

$$\lim_{x\to a} f \circ g(x) \neq \lim_{x\to g(a)} f(x)$$

$$\lim_{x\to a} f \circ g(x) \neq \lim_{y\to[\lim_{x\to a} g(x)]} f(y)$$

(the last one is the 'correct' idea, but it fails in general.)