



In the project following Section 3.4 you will calculate the distance from an airport runway at which a pilot should start descent for a smooth landing.

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3

Differentiation Rules

WE HAVE SEEN HOW TO INTERPRET derivatives as slopes and rates of change. We have used the definition of a derivative to calculate the derivatives of functions defined by formulas. But it would be tedious if we always had to use the definition, so in this chapter we develop rules for finding derivatives without having to use the definition directly. These differentiation rules enable us to calculate with relative ease the derivatives of polynomials, rational functions, algebraic functions, exponential and logarithmic functions, and trigonometric and inverse trigonometric functions. We then use these rules to solve problems involving rates of change and the approximation of functions.

3.1 Derivatives of Polynomials and Exponential Functions

In this section we learn how to differentiate constant functions, power functions, polynomials, and exponential functions.

Constant Functions

Let's start with the simplest of all functions, the constant function $f(x) = c$. The graph of this function is the horizontal line $y = c$, which has slope 0, so we must have $f'(x) = 0$. (See Figure 1.) A formal proof, from the definition of a derivative, is also easy:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} = \lim_{h \rightarrow 0} 0 = 0$$

In Leibniz notation, we write this rule as follows.

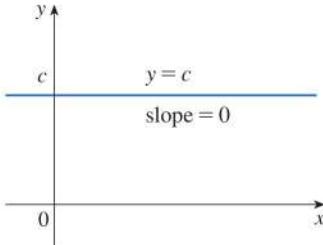


FIGURE 1

The graph of $f(x) = c$ is the line $y = c$, so $f'(x) = 0$.

Derivative of a Constant Function

$$\frac{d}{dx}(c) = 0$$

Power Functions

We next look at the functions $f(x) = x^n$, where n is a positive integer. If $n = 1$, the graph of $f(x) = x$ is the line $y = x$, which has slope 1. (See Figure 2.) So

1

$$\frac{d}{dx}(x) = 1$$

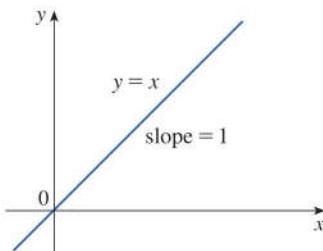


FIGURE 2

The graph of $f(x) = x$ is the line $y = x$, so $f'(x) = 1$.

(You can also verify Equation 1 from the definition of a derivative.) We have already investigated the cases $n = 2$ and $n = 3$. In fact, in Section 2.8 (Exercises 19 and 20) we found that

2

$$\frac{d}{dx}(x^2) = 2x \quad \frac{d}{dx}(x^3) = 3x^2$$

For $n = 4$ we find the derivative of $f(x) = x^4$ as follows:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x + h)^4 - x^4}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h} \\ &= \lim_{h \rightarrow 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} \\ &= \lim_{h \rightarrow 0} (4x^3 + 6x^2h + 4xh^2 + h^3) = 4x^3 \end{aligned}$$

Thus

3

$$\frac{d}{dx}(x^4) = 4x^3$$

Comparing the equations in (1), (2), and (3), we see a pattern emerging. It seems to be a reasonable guess that, when n is a positive integer, $(d/dx)(x^n) = nx^{n-1}$. This turns out to be true. We prove it in two ways; the second proof uses the Binomial Theorem.

The Power Rule If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

FIRST PROOF

$$x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + \dots + xa^{n-2} + a^{n-1})$$

can be verified simply by multiplying out the right-hand side (or by summing the second factor as a geometric series). If $f(x) = x^n$, we can use Equation 2.7.5 for $f'(a)$ and the equation above to write

$$\begin{aligned} f'(a) &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} \\ &= \lim_{x \rightarrow a} (x^{n-1} + x^{n-2}a + \dots + xa^{n-2} + a^{n-1}) \\ &= a^{n-1} + a^{n-2}a + \dots + aa^{n-2} + a^{n-1} \\ &= na^{n-1} \end{aligned}$$

SECOND PROOF

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x + h)^n - x^n}{h}$$

The Binomial Theorem is given on Reference Page 1.

In finding the derivative of x^4 we had to expand $(x + h)^4$. Here we need to expand $(x + h)^n$ and we use the Binomial Theorem to do so:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \dots + nh^{n-1} + h^n \right] - x^n}{h} \\ &= \lim_{h \rightarrow 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \dots + nh^{n-1} + h^n}{h} \\ &= \lim_{h \rightarrow 0} \left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \dots + nh^{n-2} + h^{n-1} \right] \\ &= nx^{n-1} \end{aligned}$$

because every term except the first has h as a factor and therefore approaches 0. ■

We illustrate the Power Rule using various notations in Example 1.

EXAMPLE 1

- (a) If $f(x) = x^6$, then $f'(x) = 6x^5$. (b) If $y = x^{1000}$, then $y' = 1000x^{999}$.

(c) If $y = t^4$, then $\frac{dy}{dt} = 4t^3$. (d) $\frac{d}{dr}(r^3) = 3r^2$

What about power functions with negative integer exponents? In Exercise 69 we ask you to verify from the definition of a derivative that

$$\frac{d}{dx} \left(\frac{1}{x} \right) = -\frac{1}{x^2}$$

We can rewrite this equation as

$$\frac{d}{dx}(x^{-1}) = (-1)x^{-2}$$

and so the Power Rule is true when $n = -1$. In fact, we will show in the next section [Exercise 3.2.66(c)] that it holds for all negative integers.

What if the exponent is a fraction? In Example 2.8.3 we found that

$$\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$$

which can be written as

$$\frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2}$$

This shows that the Power Rule is true even when the exponent is $\frac{1}{2}$. In fact, we will show in Section 3.6 that it is true for all real exponents n .

The Power Rule (General Version) If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

EXAMPLE 2 Differentiate:

(a) $f(x) = \frac{1}{x^2}$ (b) $y = \sqrt[3]{x^2}$

SOLUTION In each case we rewrite the function as a power of x .

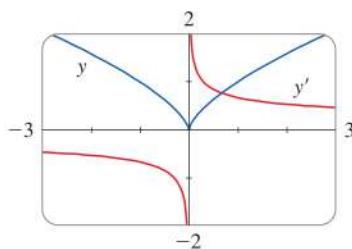


FIGURE 3

$$y = \sqrt[3]{x^2}$$

$$f'(x) = \frac{d}{dx}(x^{-2}) = -2x^{-2-1} = -2x^{-3} = -\frac{2}{x^3}$$

$$(b) \quad \frac{dy}{dx} = \frac{d}{dx} (\sqrt[3]{x^2}) = \frac{d}{dx} (x^{2/3}) = \frac{2}{3} x^{(2/3)-1} = \frac{2}{3} x^{-1/3}$$

Figure 3 shows the function y in Example 2(b) and its derivative y' . Notice that y is not differentiable at 0 (y' is not defined there). Also observe that the function y is increasing when y' is positive and is decreasing when y' is negative. In Chapter 4 we will prove that, in general, *a function increases when its derivative is positive and decreases when its derivative is negative*.

The Power Rule enables us to find tangent lines without having to resort to the definition of a derivative. It also enables us to find *normal lines*. The **normal line** to a curve C at a point P is the line through P that is perpendicular to the tangent line at P . (In the study of optics, the Law of Reflection involves the angle between a light ray and the normal line to a lens.)

EXAMPLE 3 Find equations of the tangent line and normal line to the curve $y = x\sqrt{x}$ at the point $(1, 1)$. Illustrate by graphing the curve and these lines.

SOLUTION The derivative of $f(x) = x\sqrt{x} = x^{3/2}$ is

$$f'(x) = \frac{3}{2}x^{(3/2)-1} = \frac{3}{2}x^{1/2} = \frac{3}{2}\sqrt{x}$$

So the slope of the tangent line at $(1, 1)$ is $f'(1) = \frac{3}{2}$. Therefore an equation of the tangent line is

$$y - 1 = \frac{3}{2}(x - 1) \quad \text{or} \quad y = \frac{3}{2}x - \frac{1}{2}$$

The normal line is perpendicular to the tangent line, so its slope is the negative reciprocal of $\frac{3}{2}$, that is, $-\frac{2}{3}$. Thus an equation of the normal line is

$$y - 1 = -\frac{2}{3}(x - 1) \quad \text{or} \quad y = -\frac{2}{3}x + \frac{5}{3}$$

We graph the curve and its tangent line and normal line in Figure 4. ■

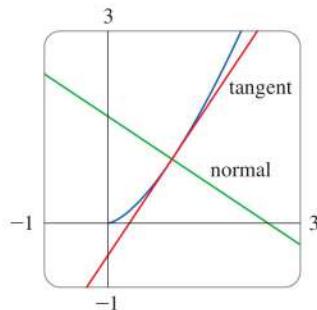
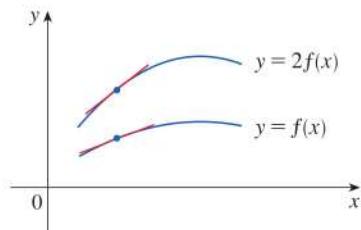


FIGURE 4
 $y = x\sqrt{x}$

■ New Derivatives from Old

When new functions are formed from old functions by addition, subtraction, or multiplication by a constant, their derivatives can be calculated in terms of derivatives of the old functions. In particular, the following formula says that *the derivative of a constant times a function is the constant times the derivative of the function*.

Geometric Interpretation of the Constant Multiple Rule



Multiplying by $c = 2$ stretches the graph vertically by a factor of 2. All the rises have been doubled but the runs stay the same. So the slopes are also doubled.

The Constant Multiple Rule If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c \frac{d}{dx}f(x)$$

PROOF Let $g(x) = cf(x)$. Then

$$\begin{aligned} g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \rightarrow 0} \frac{cf(x+h) - cf(x)}{h} \\ &= \lim_{h \rightarrow 0} c \left[\frac{f(x+h) - f(x)}{h} \right] \\ &= c \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (\text{by Limit Law 3}) \\ &= cf'(x) \end{aligned}$$

EXAMPLE 4

$$(a) \frac{d}{dx}(3x^4) = 3 \frac{d}{dx}(x^4) = 3(4x^3) = 12x^3$$

$$(b) \frac{d}{dx}(-x) = \frac{d}{dx}[(-1)x] = (-1) \frac{d}{dx}(x) = -1(1) = -1$$

The next rule tells us that *the derivative of a sum (or difference) of functions is the sum (or difference) of the derivatives.*

Using prime notation, we can write the Sum and Difference Rules as

$$(f + g)' = f' + g'$$

$$(f - g)' = f' - g'$$

The Sum and Difference Rules If f and g are both differentiable, then

$$\frac{d}{dx} [f(x) + g(x)] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

$$\frac{d}{dx} [f(x) - g(x)] = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$$

PROOF To prove the Sum Rule, we let $F(x) = f(x) + g(x)$. Then

$$\begin{aligned} F'(x) &= \lim_{h \rightarrow 0} \frac{F(x + h) - F(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[f(x + h) + g(x + h)] - [f(x) + g(x)]}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{f(x + h) - f(x)}{h} + \frac{g(x + h) - g(x)}{h} \right] \\ &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x + h) - g(x)}{h} \quad (\text{by Limit Law 1}) \\ &= f'(x) + g'(x) \end{aligned}$$

To prove the Difference Rule, we write $f - g$ as $f + (-1)g$ and apply the Sum Rule and the Constant Multiple Rule. ■

The Sum Rule can be extended to the sum of any number of functions. For instance, using this theorem twice, we get

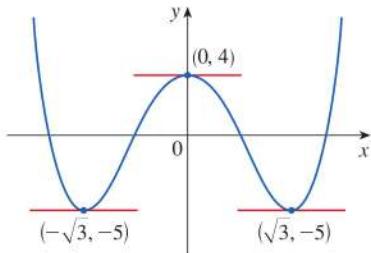
$$(f + g + h)' = [(f + g) + h]' = (f + g)' + h' = f' + g' + h'$$

The Constant Multiple Rule, the Sum Rule, and the Difference Rule can be combined with the Power Rule to differentiate any polynomial, as the following three examples demonstrate.

EXAMPLE 5

$$\begin{aligned} \frac{d}{dx} (x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5) \\ &= \frac{d}{dx} (x^8) + 12 \frac{d}{dx} (x^5) - 4 \frac{d}{dx} (x^4) + 10 \frac{d}{dx} (x^3) - 6 \frac{d}{dx} (x) + \frac{d}{dx} (5) \\ &= 8x^7 + 12(5x^4) - 4(4x^3) + 10(3x^2) - 6(1) + 0 \\ &= 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6 \end{aligned}$$



**FIGURE 5**

The curve $y = x^4 - 6x^2 + 4$ and its horizontal tangents

EXAMPLE 6 Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

SOLUTION Horizontal tangents occur where the derivative is zero. We have

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^4) - 6 \frac{d}{dx}(x^2) + \frac{d}{dx}(4) \\ &= 4x^3 - 12x + 0 = 4x(x^2 - 3)\end{aligned}$$

Thus $dy/dx = 0$ if $x = 0$ or $x^2 - 3 = 0$, that is, $x = \pm\sqrt{3}$. So the given curve has horizontal tangents when $x = 0, \sqrt{3}$, and $-\sqrt{3}$. The corresponding points are $(0, 4)$, $(\sqrt{3}, -5)$, and $(-\sqrt{3}, -5)$. (See Figure 5.) ■

EXAMPLE 7 The equation of motion of a particle is $s = 2t^3 - 5t^2 + 3t + 4$, where s is measured in centimeters and t in seconds. Find the acceleration as a function of time. What is the acceleration after 2 seconds?

SOLUTION The velocity and acceleration are

$$\begin{aligned}v(t) &= \frac{ds}{dt} = 6t^2 - 10t + 3 \\ a(t) &= \frac{dv}{dt} = 12t - 10\end{aligned}$$

The acceleration after 2 seconds is $a(2) = 14$ cm/s². ■

Exponential Functions

Let's try to compute the derivative of the exponential function $f(x) = b^x$ using the definition of a derivative:

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{b^{x+h} - b^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{b^x b^h - b^x}{h} = \lim_{h \rightarrow 0} \frac{b^x(b^h - 1)}{h}\end{aligned}$$

The factor b^x doesn't depend on h , so we can take it in front of the limit:

$$f'(x) = b^x \lim_{h \rightarrow 0} \frac{b^h - 1}{h}$$

Notice that the limit is the value of the derivative of f at 0, that is,

$$\lim_{h \rightarrow 0} \frac{b^h - 1}{h} = f'(0)$$

Therefore we have shown that if the exponential function $f(x) = b^x$ is differentiable at 0, then it is differentiable everywhere and

■

$$f'(x) = f'(0)b^x$$

This equation says that *the rate of change of any exponential function is proportional to the function itself.* (The slope is proportional to the height.)

h	$\frac{2^h - 1}{h}$	$\frac{3^h - 1}{h}$
0.1	0.71773	1.16123
0.01	0.69556	1.10467
0.001	0.69339	1.09922
0.0001	0.69317	1.09867
0.00001	0.69315	1.09862

Numerical evidence for the existence of $f'(0)$ is given in the table at the left for the cases $b = 2$ and $b = 3$. (Values are stated correct to four decimal places.) It can be proved that the limits exist and

$$\text{for } b = 2, \quad f'(0) = \lim_{h \rightarrow 0} \frac{2^h - 1}{h} \approx 0.693$$

$$\text{for } b = 3, \quad f'(0) = \lim_{h \rightarrow 0} \frac{3^h - 1}{h} \approx 1.099$$

Thus, from Equation 4 we have

$$[5] \quad \frac{d}{dx}(2^x) \approx (0.693)2^x \quad \frac{d}{dx}(3^x) \approx (1.099)3^x$$

Of all possible choices for the base b in Equation 4, the simplest differentiation formula occurs when $f'(0) = 1$. In view of the estimates of $f'(0)$ for $b = 2$ and $b = 3$, it seems reasonable that there is a number b between 2 and 3 for which $f'(0) = 1$. It is traditional to denote this value by the letter e . (In fact, that is how we introduced e in Section 1.4.) Thus we have the following definition.

In Exercise 1 we will see that e lies between 2.7 and 2.8. Later we will be able to show that, correct to five decimal places,

$$e \approx 2.71828$$

Definition of the Number e

$$e \text{ is the number such that } \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$$

Geometrically, this means that of all possible exponential functions $y = b^x$, the function $f(x) = e^x$ is the one whose tangent line at $(0, 1)$ has a slope $f'(0)$ that is exactly 1. (See Figures 6 and 7.)

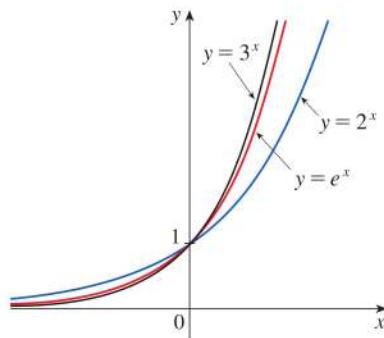


FIGURE 6

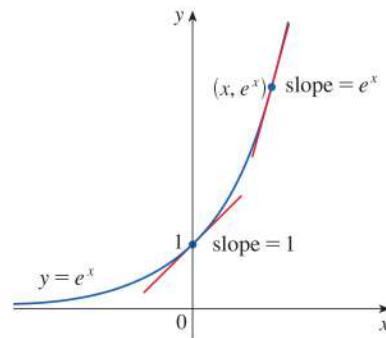


FIGURE 7

If we put $b = e$ and, therefore, $f'(0) = 1$ in Equation 4, it becomes the following important differentiation formula.

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

Thus the exponential function $f(x) = e^x$ has the property that it is its own derivative. The geometrical significance of this fact is that the slope of a tangent line to the curve $y = e^x$ at a point (x, e^x) is equal to the y-coordinate of the point (see Figure 7).

EXAMPLE 8 If $f(x) = e^x - x$, find f' and f'' . Compare the graphs of f and f' .

SOLUTION Using the Difference Rule, we have

$$f'(x) = \frac{d}{dx}(e^x - x) = \frac{d}{dx}(e^x) - \frac{d}{dx}(x) = e^x - 1$$

In Section 2.8 we defined the second derivative as the derivative of f' , so

$$f''(x) = \frac{d}{dx}(e^x - 1) = \frac{d}{dx}(e^x) - \frac{d}{dx}(1) = e^x$$

The function f and its derivative f' are graphed in Figure 8. Notice that f has a horizontal tangent when $x = 0$; this corresponds to the fact that $f'(0) = 0$. Notice also that, for $x > 0$, $f'(x)$ is positive and f is increasing. When $x < 0$, $f'(x)$ is negative and f is decreasing.

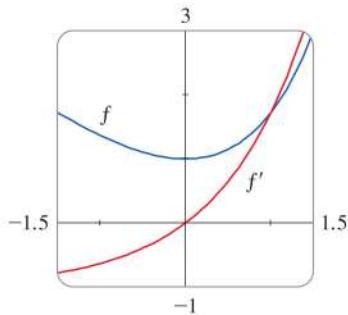


FIGURE 8

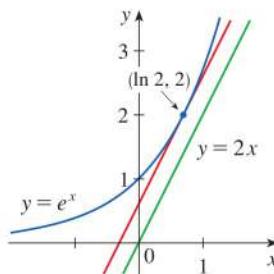


FIGURE 9

EXAMPLE 9 At what point on the curve $y = e^x$ is the tangent line parallel to the line $y = 2x$?

SOLUTION Since $y = e^x$, we have $y' = e^x$. Let the x -coordinate of the point in question be a . Then the slope of the tangent line at that point is e^a . This tangent line will be parallel to the line $y = 2x$ if it has the same slope, that is, 2. Equating slopes, we get

$$e^a = 2 \quad a = \ln 2$$

Therefore the required point is $(a, e^a) = (\ln 2, 2)$. (See Figure 9.)

3.1 Exercises

1. (a) How is the number e defined?
 (b) Use a calculator to estimate the values of the limits

$$\lim_{h \rightarrow 0} \frac{2.7^h - 1}{h} \quad \text{and} \quad \lim_{h \rightarrow 0} \frac{2.8^h - 1}{h}$$

correct to two decimal places. What can you conclude about the value of e ?

2. (a) Sketch, by hand, the graph of the function $f(x) = e^x$, paying particular attention to how the graph crosses the y -axis. What is the slope of the tangent line at that point?
 (b) What types of functions are $f(x) = e^x$ and $g(x) = x^e$? Compare the differentiation formulas for f and g .
 (c) Which of the two functions in part (b) grows more rapidly when x is large?

- 3–34 Differentiate the function.

- | | |
|--|--|
| 3. $g(x) = 4x + 7$ | 4. $g(t) = 5t + 4t^2$ |
| 5. $f(x) = x^{75} - x + 3$ | 6. $g(x) = \frac{7}{4}x^2 - 3x + 12$ |
| 7. $f(t) = -2e^t$ | 8. $F(t) = t^3 + e^3$ |
| 9. $W(v) = 1.8v^{-3}$ | 10. $r(z) = z^{-5} - z^{1/2}$ |
| 11. $f(x) = x^{3/2} + x^{-3}$ | 12. $V(t) = t^{-3/5} + t^4$ |
| 13. $s(t) = \frac{1}{t} + \frac{1}{t^2}$ | 14. $r(t) = \frac{a}{t^2} + \frac{b}{t^4}$ |
| 15. $y = 2x + \sqrt{x}$ | 16. $h(w) = \sqrt{2w} - \sqrt{2}$ |

17. $g(x) = \frac{1}{\sqrt{x}} + \sqrt[4]{x}$

18. $W(t) = \sqrt{t} - 2e^t$

19. $f(x) = x^3(x + 3)$

20. $F(t) = (2t - 3)^2$

21. $y = 3e^x + \frac{4}{\sqrt[3]{x}}$

22. $S(R) = 4\pi R^2$

23. $f(x) = \frac{3x^2 + x^3}{x}$

24. $y = \frac{\sqrt{x} + x}{x^2}$

25. $G(r) = \frac{3r^{3/2} + r^{5/2}}{r}$

26. $G(t) = \sqrt{5t} + \frac{\sqrt{7}}{t}$

27. $j(x) = x^{2.4} + e^{2.4}$

28. $k(r) = e^r + r^e$

29. $F(z) = \frac{A + Bz + Cz^2}{z^2}$

30. $G(q) = (1 + q^{-1})^2$

31. $D(t) = \frac{1 + 16t^2}{(4t)^3}$

32. $f(v) = \frac{\sqrt[3]{v} - 2ve^v}{v}$

33. $P(w) = \frac{2w^2 - w + 4}{\sqrt{w}}$

34. $y = e^{x+1} + 1$

35–36 Find dy/dx and dy/dt .

35. $y = tx^2 + t^3x$

36. $y = \frac{t}{x^2} + \frac{x}{t}$

37–40 Find an equation of the tangent line to the curve at the given point.

37. $y = 2x^3 - x^2 + 2, (1, 3)$

38. $y = 2e^x + x, (0, 2)$

39. $y = x + \frac{2}{x}, (2, 3)$

40. $y = \sqrt[4]{x} - x, (1, 0)$

41–42 Find equations of the tangent line and normal line to the curve at the given point.

41. $y = x^4 + 2e^x, (0, 2)$

42. $y = x^{3/2}, (1, 1)$

43–44 Find an equation of the tangent line to the curve at the given point. Illustrate by graphing the curve and the tangent line on the same screen.

43. $y = 3x^2 - x^3, (1, 2)$

44. $y = x - \sqrt{x}, (1, 0)$

45–46 Find $f'(x)$. Compare the graphs of f and f' and use them to explain why your answer is reasonable.

45. $f(x) = x^4 - 2x^3 + x^2$

46. $f(x) = x^5 - 2x^3 + x - 1$

47. (a) Graph the function

$$f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30$$

in the viewing rectangle $[-3, 5]$ by $[-10, 50]$.

- (b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of f' . (See Example 2.8.1.)
 (c) Calculate $f'(x)$ and use this expression to graph f' . Compare with your sketch in part (b).

48. (a) Graph the function $g(x) = e^x - 3x^2$ in the viewing rectangle $[-1, 4]$ by $[-8, 8]$.

- (b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of g' . (See Example 2.8.1.)
 (c) Calculate $g'(x)$ and use this expression to graph g' . Compare with your sketch in part (b).

49–50 Find the first and second derivatives of the function.

49. $f(x) = 0.001x^5 - 0.02x^3$ 50. $G(r) = \sqrt{r} + \sqrt[3]{r}$

51–52 Find the first and second derivatives of the function. Check to see that your answers are reasonable by comparing the graphs of f , f' , and f'' .

51. $f(x) = 2x - 5x^{3/4}$ 52. $f(x) = e^x - x^3$

53. The equation of motion of a particle is $s = t^3 - 3t$, where s is in meters and t is in seconds. Find

- (a) the velocity and acceleration as functions of t ,
 (b) the acceleration after 2 s, and
 (c) the acceleration when the velocity is 0.

54. The equation of motion of a particle is $s = t^4 - 2t^3 + t^2 - t$, where s is in meters and t is in seconds.

- (a) Find the velocity and acceleration as functions of t .
 (b) Find the acceleration after 1 s.
 (c) Graph the position, velocity, and acceleration functions on the same screen.

55. Biologists have proposed a cubic polynomial to model the length L of Alaskan rockfish at age A :

$$L = 0.0155A^3 - 0.372A^2 + 3.95A + 1.21$$

where L is measured in inches and A in years. Calculate

$$\left. \frac{dL}{dA} \right|_{A=12}$$

and interpret your answer.

56. The number of tree species S in a given area A in a forest reserve has been modeled by the power function

$$S(A) = 0.882A^{0.842}$$

where A is measured in square meters. Find $S'(100)$ and interpret your answer.

85. Let

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ mx + b & \text{if } x > 2 \end{cases}$$

Find the values of m and b that make f differentiable everywhere.

86. Find numbers a and b such that the given function g is differentiable at 1.

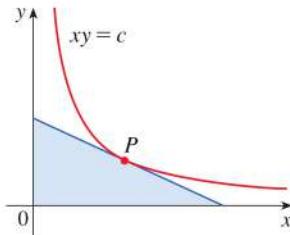
$$g(x) = \begin{cases} ax^3 - 3x & \text{if } x \leq 1 \\ bx^2 + 2 & \text{if } x > 1 \end{cases}$$

87. Evaluate $\lim_{x \rightarrow 1} \frac{x^{1000} - 1}{x - 1}$.

88. A tangent line is drawn to the hyperbola $xy = c$ at a point P as shown in the figure.

(a) Show that the midpoint of the line segment cut from this tangent line by the coordinate axes is P .

(b) Show that the triangle formed by the tangent line and the coordinate axes always has the same area, no matter where P is located on the hyperbola.



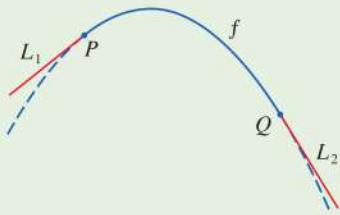
89. Draw a diagram showing two perpendicular lines that intersect on the y -axis and are both tangent to the parabola $y = x^2$. Where do these lines intersect?

90. Sketch the parabolas $y = x^2$ and $y = x^2 - 2x + 2$. Do you think there is a line that is tangent to both curves? If so, find its equation. If not, why not?

91. If $c > \frac{1}{2}$, how many lines through the point $(0, c)$ are normal lines to the parabola $y = x^2$? What if $c \leq \frac{1}{2}$?

APPLIED PROJECT

BUILDING A BETTER ROLLER COASTER



Suppose you are asked to design the first ascent and drop for a new roller coaster. By studying photographs of your favorite coasters, you decide to make the slope of the ascent 0.8 and the slope of the drop -1.6 . You decide to connect these two straight stretches $y = L_1(x)$ and $y = L_2(x)$ with part of a parabola $y = f(x) = ax^2 + bx + c$, where x and $f(x)$ are measured in feet. For the track to be smooth there can't be abrupt changes in direction, so you want the linear segments L_1 and L_2 to be tangent to the parabola at the transition points P and Q . (See the figure.) To simplify the equations, you decide to place the origin at P .

- Suppose the horizontal distance between P and Q is 100 ft. Write equations in a , b , and c that will ensure that the track is smooth at the transition points.
 - Solve the equations in part (a) for a , b , and c to find a formula for $f(x)$.
 - Plot L_1 , f , and L_2 to verify graphically that the transitions are smooth.
 - Find the difference in elevation between P and Q .
- The solution in Problem 1 might *look* smooth, but it might not *feel* smooth because the piecewise defined function [consisting of $L_1(x)$ for $x < 0$, $f(x)$ for $0 \leq x \leq 100$, and $L_2(x)$ for $x > 100$] doesn't have a continuous second derivative. So you decide to improve the design by using a quadratic function $g(x) = ax^2 + bx + c$ only on the interval $10 \leq x \leq 90$ and connecting it to the linear functions by means of two cubic functions:

$$g(x) = kx^3 + lx^2 + mx + n \quad 0 \leq x < 10$$

$$h(x) = px^3 + qx^2 + rx + s \quad 90 < x \leq 100$$

- Write a system of equations in 11 unknowns that ensures that the functions and their first two derivatives agree at the transition points.
- Solve the system of equations in part (a) to find formulas for $g(x)$, $h(x)$, and $f(x)$.
- Plot L_1 , g , h , and L_2 , and compare with the plot in Problem 1(c).



3.2 | The Product and Quotient Rules

The formulas of this section enable us to differentiate new functions that are formed from old functions by using multiplication or division.

■ The Product Rule

By analogy with the Sum and Difference Rules, we might be tempted to guess, as Leibniz did three centuries ago, that the derivative of a product is the product of the derivatives. We can see, however, that this guess is wrong by looking at a particular example.

Let $f(x) = x$ and $g(x) = x^2$. Then the Power Rule gives $f'(x) = 1$ and $g'(x) = 2x$. But $\square (fg)(x) = x^3$, so $(fg)'(x) = 3x^2$. Thus $(fg)' \neq f'g'$. The correct formula was discovered by Leibniz (soon after his false start) and is called the Product Rule.

Before stating the Product Rule, let's see how we might discover it. We start by assuming that $u = f(x)$ and $v = g(x)$ are both positive differentiable functions. Then we can interpret the product uv as an area of a rectangle (see Figure 1). If x changes by an amount Δx , then the corresponding changes in u and v are

Δv	$u \Delta v$	$\Delta u \Delta v$
v	uv	$v \Delta u$
	u	Δu

FIGURE 1

The geometry of the Product Rule

Recall that in Leibniz notation the definition of a derivative can be written as

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

1 $\Delta(uv) = (u + \Delta u)(v + \Delta v) - uv = u \Delta v + v \Delta u + \Delta u \Delta v$
= the sum of the three shaded areas

If we divide by Δx , we get

$$\frac{\Delta(uv)}{\Delta x} = u \frac{\Delta v}{\Delta x} + v \frac{\Delta u}{\Delta x} + \Delta u \frac{\Delta v}{\Delta x}$$

If we now let $\Delta x \rightarrow 0$, we get the derivative of uv :

$$\begin{aligned} \frac{d}{dx}(uv) &= \lim_{\Delta x \rightarrow 0} \frac{\Delta(uv)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \left(u \frac{\Delta v}{\Delta x} + v \frac{\Delta u}{\Delta x} + \Delta u \frac{\Delta v}{\Delta x} \right) \\ &= u \lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x} + v \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} + \left(\lim_{\Delta x \rightarrow 0} \Delta u \right) \left(\lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x} \right) \\ &= u \frac{dv}{dx} + v \frac{du}{dx} + 0 \cdot \frac{dv}{dx} \end{aligned}$$

2 $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

(Notice that $\Delta u \rightarrow 0$ as $\Delta x \rightarrow 0$ since f is differentiable and therefore continuous.)

Although we started by assuming (for the geometric interpretation) that all the quantities are positive, we notice that Equation 1 is always true. (The algebra is valid whether u , v , Δu , and Δv are positive or negative.) So we have proved Equation 2, known as the Product Rule, for all differentiable functions u and v .

In prime notation the Product Rule is written as

$$(fg)' = fg' + gf'$$

The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x) \frac{d}{dx}[g(x)] + g(x) \frac{d}{dx}[f(x)]$$

In words, the Product Rule says that *the derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.*

EXAMPLE 1

- If $f(x) = xe^x$, find $f'(x)$.
- Find the n th derivative, $f^{(n)}(x)$.

SOLUTION

- By the Product Rule, we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}(xe^x) \\ &= x \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x) \\ &= xe^x + e^x \cdot 1 = (x + 1)e^x \end{aligned}$$

- Using the Product Rule a second time, we get

$$\begin{aligned} f''(x) &= \frac{d}{dx}[(x + 1)e^x] \\ &= (x + 1) \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x + 1) \\ &= (x + 1)e^x + e^x \cdot 1 = (x + 2)e^x \end{aligned}$$

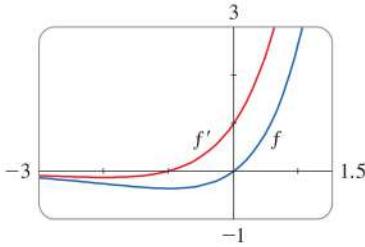
Further applications of the Product Rule give

$$f'''(x) = (x + 3)e^x \quad f^{(4)}(x) = (x + 4)e^x$$

In fact, each successive differentiation adds another term e^x , so

$$f^{(n)}(x) = (x + n)e^x$$

FIGURE 2



In Example 2, a and b are constants. It is customary in mathematics to use letters near the beginning of the alphabet to represent constants and letters near the end of the alphabet to represent variables.

EXAMPLE 2

Differentiate the function $f(t) = \sqrt{t}(a + bt)$.

SOLUTION 1

Using the Product Rule, we have

$$\begin{aligned} f'(t) &= \sqrt{t} \frac{d}{dt}(a + bt) + (a + bt) \frac{d}{dt}(\sqrt{t}) \\ &= \sqrt{t} \cdot b + (a + bt) \cdot \frac{1}{2}t^{-1/2} \\ &= b\sqrt{t} + \frac{a + bt}{2\sqrt{t}} = \frac{a + 3bt}{2\sqrt{t}} \end{aligned}$$

SOLUTION 2

If we first use the laws of exponents to rewrite $f(t)$, then we can proceed directly without using the Product Rule.

$$f(t) = a\sqrt{t} + bt\sqrt{t} = at^{1/2} + bt^{3/2}$$

$$f'(t) = \frac{1}{2}at^{-1/2} + \frac{3}{2}bt^{1/2}$$

which is equivalent to the answer given in Solution 1.

Example 2 shows that it is sometimes easier to simplify a product of functions before differentiating than to use the Product Rule. In Example 1, however, the Product Rule is the only possible method.

EXAMPLE 3 If $f(x) = \sqrt{x} g(x)$, where $g(4) = 2$ and $g'(4) = 3$, find $f'(4)$.

SOLUTION Applying the Product Rule, we get

$$\begin{aligned} f'(x) &= \frac{d}{dx} [\sqrt{x} g(x)] = \sqrt{x} \frac{d}{dx}[g(x)] + g(x) \frac{d}{dx}[\sqrt{x}] \\ &= \sqrt{x} g'(x) + g(x) \cdot \frac{1}{2} x^{-1/2} \\ &= \sqrt{x} g'(x) + \frac{g(x)}{2\sqrt{x}} \end{aligned}$$

$$\text{So } f'(4) = \sqrt{4} g'(4) + \frac{g(4)}{2\sqrt{4}} = 2 \cdot 3 + \frac{2}{2 \cdot 2} = 6.5 \quad \blacksquare$$

The Quotient Rule

We find a rule for differentiating the quotient of two differentiable functions $u = f(x)$ and $v = g(x)$ in much the same way that we found the Product Rule. If x , u , and v change by amounts Δx , Δu , and Δv , then the corresponding change in the quotient u/v is

$$\begin{aligned} \Delta \left(\frac{u}{v} \right) &= \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{(u + \Delta u)v - u(v + \Delta v)}{v(v + \Delta v)} \\ &= \frac{v\Delta u - u\Delta v}{v(v + \Delta v)} \end{aligned}$$

$$\text{so } \frac{d}{dx} \left(\frac{u}{v} \right) = \lim_{\Delta x \rightarrow 0} \frac{\Delta(u/v)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{v \frac{\Delta u}{\Delta x} - u \frac{\Delta v}{\Delta x}}{v(v + \Delta v)}$$

As $\Delta x \rightarrow 0$, $\Delta v \rightarrow 0$ also, because $v = g(x)$ is differentiable and therefore continuous. Thus, using the Limit Laws, we get

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} - u \lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x}}{v \lim_{\Delta x \rightarrow 0} (v + \Delta v)} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

In prime notation the Quotient Rule is written as

$$\left(\frac{f}{g} \right)' = \frac{gf' - fg'}{g^2}$$

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx}[f(x)] - f(x) \frac{d}{dx}[g(x)]}{[g(x)]^2}$$

In words, the Quotient Rule says that the *derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator*.

The Quotient Rule and the other differentiation formulas enable us to compute the derivative of any rational function, as the next example illustrates.

Figure 3 shows the graphs of the function of Example 4 and its derivative. Notice that when y grows rapidly (near $-\sqrt[3]{6} \approx -1.8$), y' is large. And when y grows slowly, y' is near 0.

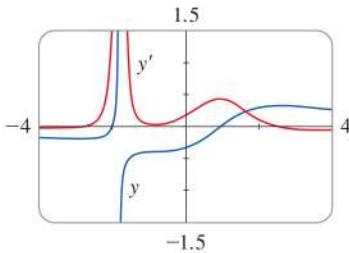


FIGURE 3

EXAMPLE 4 Let $y = \frac{x^2 + x - 2}{x^3 + 6}$. Then

$$\begin{aligned} y' &= \frac{(x^3 + 6) \frac{d}{dx}(x^2 + x - 2) - (x^2 + x - 2) \frac{d}{dx}(x^3 + 6)}{(x^3 + 6)^2} \\ &= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2)(3x^2)}{(x^3 + 6)^2} \\ &= \frac{(2x^4 + x^3 + 12x + 6) - (3x^4 + 3x^3 - 6x^2)}{(x^3 + 6)^2} \\ &= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2} \end{aligned}$$

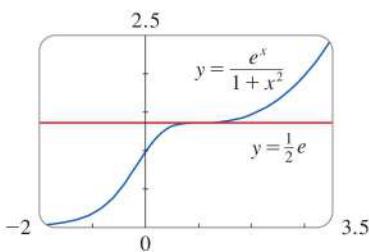


FIGURE 4

EXAMPLE 5 Find an equation of the tangent line to the curve $y = e^x/(1 + x^2)$ at the point $(1, \frac{1}{2}e)$.

SOLUTION According to the Quotient Rule, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{(1 + x^2) \frac{d}{dx}(e^x) - e^x \frac{d}{dx}(1 + x^2)}{(1 + x^2)^2} \\ &= \frac{(1 + x^2)e^x - e^x(2x)}{(1 + x^2)^2} = \frac{e^x(1 - 2x + x^2)}{(1 + x^2)^2} \\ &= \frac{e^x(1 - x)^2}{(1 + x^2)^2} \end{aligned}$$

So the slope of the tangent line at $(1, \frac{1}{2}e)$ is

$$\left. \frac{dy}{dx} \right|_{x=1} = 0$$

This means that the tangent line at $(1, \frac{1}{2}e)$ is horizontal and its equation is $y = \frac{1}{2}e$. (See Figure 4.)

NOTE Don't use the Quotient Rule *every* time you see a quotient. Sometimes it's easier to rewrite a quotient first to put it in a form that is simpler for the purpose of differentiation. For instance, although it is possible to differentiate the function

$$F(x) = \frac{3x^2 + 2\sqrt{x}}{x}$$

using the Quotient Rule, it is much easier to perform the division first and write the function as

$$F(x) = 3x + 2x^{-1/2}$$

before differentiating.

We summarize the differentiation formulas we have learned so far as follows.

Table of Differentiation Formulas

$$\begin{array}{lll} \frac{d}{dx}(c) = 0 & \frac{d}{dx}(x^n) = nx^{n-1} & \frac{d}{dx}(e^x) = e^x \\ (cf)' = cf' & (f+g)' = f' + g' & (f-g)' = f' - g' \\ (fg)' = fg' + gf' & \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2} & \end{array}$$

3.2 Exercises

1. Find the derivative of $f(x) = (1 + 2x^2)(x - x^2)$ in two ways: by using the Product Rule and by performing the multiplication first. Do your answers agree?

2. Find the derivative of the function

$$F(x) = \frac{x^4 - 5x^3 + \sqrt{x}}{x^2}$$

in two ways: by using the Quotient Rule and by simplifying first. Show that your answers are equivalent. Which method do you prefer?

3–30 Differentiate.

3. $y = (4x^2 + 3)(2x + 5)$

4. $y = (10x^2 + 7x - 2)(2 - x^2)$

5. $y = x^3 e^x$

6. $y = (e^x + 2)(2e^x - 1)$

7. $f(x) = (3x^2 - 5x)e^x$

8. $g(x) = (x + 2\sqrt{x})e^x$

9. $y = \frac{x}{e^x}$

10. $y = \frac{e^x}{1 - e^x}$

11. $g(t) = \frac{3 - 2t}{5t + 1}$

12. $G(u) = \frac{6u^4 - 5u}{u + 1}$

13. $f(t) = \frac{5t}{t^3 - t - 1}$

14. $F(x) = \frac{1}{2x^3 - 6x^2 + 5}$

15. $y = \frac{s - \sqrt{s}}{s^2}$

16. $y = \frac{\sqrt{x}}{\sqrt{x} + 1}$

17. $J(u) = \left(\frac{1}{u} + \frac{1}{u^2}\right)\left(u + \frac{1}{u}\right)$

18. $h(w) = (w^2 + 3w)(w^{-1} - w^{-4})$

19. $H(u) = (u - \sqrt{u})(u + \sqrt{u})$

20. $f(z) = (1 - e^z)(z + e^z)$

21. $V(t) = (t + 2e^t)\sqrt{t}$

22. $W(t) = e^t(1 + te^t)$

23. $y = e^p(p + p\sqrt{p})$

24. $h(r) = \frac{ae^r}{b + e^r}$

25. $f(t) = \frac{\sqrt[3]{t}}{t - 3}$

26. $y = (z^2 + e^z)\sqrt{z}$

27. $f(x) = \frac{x^2 e^x}{x^2 + e^x}$

28. $F(t) = \frac{At}{Bt^2 + Ct^3}$

29. $f(x) = \frac{x}{x + \frac{c}{x}}$

30. $f(x) = \frac{ax + b}{cx + d}$

31–34 Find $f'(x)$ and $f''(x)$.

31. $f(x) = x^2 e^x$

32. $f(x) = \sqrt{x}e^x$

33. $f(x) = \frac{x}{x^2 - 1}$

34. $f(x) = \frac{x}{1 + \sqrt{x}}$

35–36 Find an equation of the tangent line to the given curve at the specified point.

35. $y = \frac{x^2}{1 + x}, \quad (1, \frac{1}{2})$

36. $y = \frac{1 + x}{1 + e^x}, \quad (0, \frac{1}{2})$

37–38 Find equations of the tangent line and normal line to the given curve at the specified point.

37. $y = \frac{3x}{1 + 5x^2}, \quad (1, \frac{1}{2})$

38. $y = x + xe^x, \quad (0, 0)$

- 39.** (a) The curve $y = 1/(1 + x^2)$ is called a **witch of Maria Agnesi**. Find an equation of the tangent line to this curve at the point $(-1, \frac{1}{2})$.

(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

- 40.** (a) The curve $y = x/(1 + x^2)$ is called a **serpentine**. Find an equation of the tangent line to this curve at the point $(3, 0.3)$.

(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

- 41.** (a) If $f(x) = (x^3 - x)e^x$, find $f'(x)$.

(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f' .

- 42.** (a) If $f(x) = (x^2 - 1)/(x^2 + 1)$, find $f'(x)$ and $f''(x)$.

(b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f , f' , and f'' .

- 43.** If $f(x) = x^2/(1 + x)$, find $f''(1)$.

- 44.** If $g(x) = x/e^x$, find $g^{(n)}(x)$.

- 45.** Suppose that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$. Find the following values.

(a) $(fg)'(5)$ (b) $(f/g)'(5)$ (c) $(g/f)'(5)$

- 46.** Suppose that $f(4) = 2$, $g(4) = 5$, $f'(4) = 6$, and $g'(4) = -3$. Find $h'(4)$.

(a) $h(x) = 3f(x) + 8g(x)$ (b) $h(x) = f(x)g(x)$

(c) $h(x) = \frac{f(x)}{g(x)}$ (d) $h(x) = \frac{g(x)}{f(x) + g(x)}$

- 47.** If $f(x) = e^x g(x)$, where $g(0) = 2$ and $g'(0) = 5$, find $f'(0)$.

- 48.** If $h(2) = 4$ and $h'(2) = -3$, find

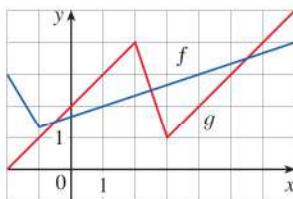
$$\left. \frac{d}{dx} \left(\frac{h(x)}{x} \right) \right|_{x=2}$$

- 49.** If $g(x) = xf(x)$, where $f(3) = 4$ and $f'(3) = -2$, find an equation of the tangent line to the graph of g at the point where $x = 3$.

- 50.** If $f(2) = 10$ and $f'(x) = x^2 f(x)$ for all x , find $f''(2)$.

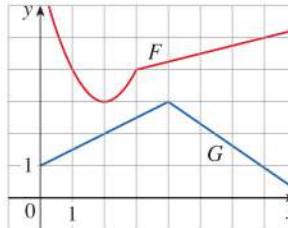
- 51.** If f and g are the functions whose graphs are shown, let $u(x) = f(x)g(x)$ and $v(x) = f(x)/g(x)$.

(a) Find $u'(1)$. (b) Find $v'(4)$.



- 52.** Let $P(x) = F(x)G(x)$ and $Q(x) = F(x)/G(x)$, where F and G are the functions whose graphs are shown.

(a) Find $P'(2)$. (b) Find $Q'(7)$.



- 53.** If g is a differentiable function, find an expression for the derivative of each of the following functions.

(a) $y = xg(x)$ (b) $y = \frac{x}{g(x)}$
 (c) $y = \frac{g(x)}{x}$

- 54.** If f is a differentiable function, find an expression for the derivative of each of the following functions.

(a) $y = x^2 f(x)$ (b) $y = \frac{f(x)}{x^2}$
 (c) $y = \frac{x^2}{f(x)}$ (d) $y = \frac{1 + xf(x)}{\sqrt{x}}$

- 55.** How many tangent lines to the curve $y = x/(x + 1)$ pass through the point $(1, 2)$? At which points do these tangent lines touch the curve?

- 56.** Find equations of the tangent lines to the curve

$$y = \frac{x - 1}{x + 1}$$

that are parallel to the line $x - 2y = 2$.

- 57.** Find $R'(0)$, where

$$R(x) = \frac{x - 3x^3 + 5x^5}{1 + 3x^3 + 6x^6 + 9x^9}$$

Hint: Instead of finding $R'(x)$ first, let $f(x)$ be the numerator and $g(x)$ be the denominator of $R(x)$, and compute $R'(0)$ from $f(0)$, $f'(0)$, $g(0)$, and $g'(0)$.

- 58.** Use the method of Exercise 57 to compute $Q'(0)$, where

$$Q(x) = \frac{1 + x + x^2 + xe^x}{1 - x + x^2 - xe^x}$$

- 59.** In this exercise we estimate the rate at which the total personal income is rising in Boulder, Colorado. In 2015, the population of this city was 107,350 and the population was increasing by roughly 1960 people per year. The average annual income was \$60,220 per capita, and this average

was increasing at about \$2250 per year (a little above the national average of about \$1810 yearly). Use the Product Rule and these figures to estimate the rate at which total personal income was rising in Boulder in 2015. Explain the meaning of each term in the Product Rule.

- 60.** A manufacturer produces bolts of a fabric with a fixed width. The quantity q of this fabric (measured in yards) that is sold is a function of the selling price p (in dollars per yard), so we can write $q = f(p)$. Then the total revenue earned with selling price p is $R(p) = pf(p)$.
- What does it mean to say that $f(20) = 10,000$ and $f'(20) = -350$?
 - Assuming the values in part (a), find $R'(20)$ and interpret your answer.

- 61.** The Michaelis-Menten equation for the enzyme chymotrypsin is

$$v = \frac{0.14[S]}{0.015 + [S]}$$

where v is the rate of an enzymatic reaction and $[S]$ is the concentration of a substrate S . Calculate $dv/d[S]$ and interpret it.

- 62.** The *biomass* $B(t)$ of a fish population is the total mass of the members of the population at time t . It is the product of the number of individuals $N(t)$ in the population and the average mass $M(t)$ of a fish at time t . In the case of guppies, breeding occurs continually. Suppose that at time $t = 4$ weeks the population is 820 guppies and is growing at a rate of 50 guppies per week, while the average mass is 1.2 g and is increasing at a rate of 0.14 g/week. At what rate is the biomass increasing when $t = 4$?

63. Extended Product Rule The Product Rule can be extended to the product of three functions.

- Use the Product Rule twice to prove that if f , g , and h are differentiable, then $(fgh)' = f'gh + fg'h + fgh'$.
- Taking $f = g = h$ in part (a), show that

$$\frac{d}{dx} [f(x)]^3 = 3[f(x)]^2 f'(x)$$

- Use part (b) to differentiate $y = e^{3x}$.

- 64.** (a) If $F(x) = f(x)g(x)$, where f and g have derivatives of all orders, show that $F'' = f''g + 2f'g' + fg''$.

- (b) Find similar formulas for F''' and $F^{(4)}$.

- (c) Guess a formula for $F^{(n)}$.

- 65.** Find expressions for the first five derivatives of $f(x) = x^2e^x$. Do you see a pattern in these expressions? Guess a formula for $f^{(n)}(x)$ and prove it using mathematical induction.

- 66. Reciprocal Rule** If g is differentiable, the *Reciprocal Rule* says that

$$\frac{d}{dx} \left[\frac{1}{g(x)} \right] = -\frac{g'(x)}{[g(x)]^2}$$

- Use the Quotient Rule to prove the Reciprocal Rule.
- Use the Reciprocal Rule to differentiate the function in Exercise 14.
- Use the Reciprocal Rule to verify that the Power Rule is valid for negative integers, that is,

$$\frac{d}{dx} (x^{-n}) = -nx^{-n-1}$$

for all positive integers n .

3.3 | Derivatives of Trigonometric Functions

A review of the trigonometric functions is given in Appendix D.

Before starting this section, you might need to review the trigonometric functions. In particular, it is important to remember that when we talk about the function f defined for all real numbers x by

$$f(x) = \sin x$$

it is understood that $\sin x$ means the sine of the angle whose *radian* measure is x . A similar convention holds for the other trigonometric functions \cos , \tan , \csc , \sec , and \cot . Recall from Section 2.5 that all of the trigonometric functions are continuous at every number in their domains.

■ Derivatives of the Trigonometric Functions

If we sketch the graph of the function $f(x) = \sin x$ and use the interpretation of $f'(x)$ as the slope of the tangent to the sine curve in order to sketch the graph of f' (see

Exercise 2.8.16), then it looks as if the graph of f' may be the same as the cosine curve (see Figure 1).

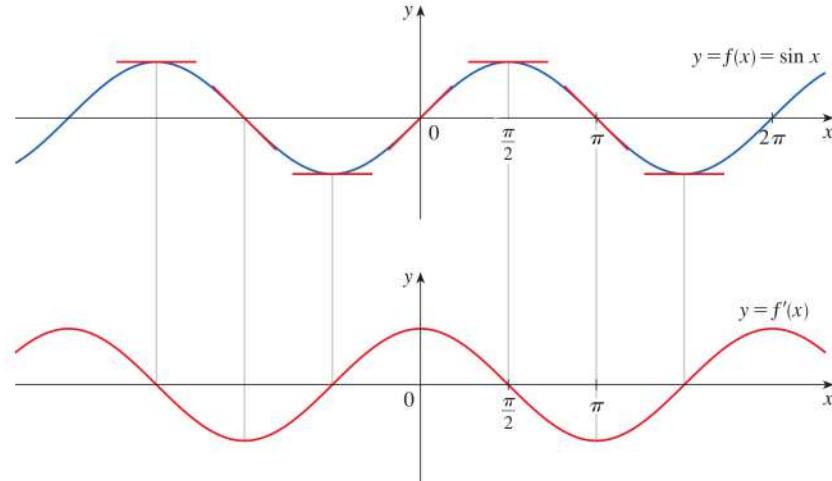


FIGURE 1

Let's try to confirm our guess that if $f(x) = \sin x$, then $f'(x) = \cos x$. From the definition of a derivative, we have

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin(x + h) - \sin x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \quad (\text{using the addition formula for sine; see Appendix D}) \\
 &= \lim_{h \rightarrow 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right] \\
 &= \lim_{h \rightarrow 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right] \\
 \boxed{1} \quad &= \lim_{h \rightarrow 0} \sin x \cdot \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + \lim_{h \rightarrow 0} \cos x \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h}
 \end{aligned}$$

Two of these four limits are easy to evaluate. Because we regard x as a constant when computing a limit as $h \rightarrow 0$, we have

$$\lim_{h \rightarrow 0} \sin x = \sin x \quad \text{and} \quad \lim_{h \rightarrow 0} \cos x = \cos x$$

Later in this section we will prove that

$$\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \quad \text{and} \quad \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = 0$$

Putting these limits into (1), we get

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \sin x \cdot \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + \lim_{h \rightarrow 0} \cos x \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h} \\ &= (\sin x) \cdot 0 + (\cos x) \cdot 1 = \cos x \end{aligned}$$

So we have proved the formula for the derivative of the sine function:

2

$$\frac{d}{dx}(\sin x) = \cos x$$

EXAMPLE 1 Differentiate $y = x^2 \sin x$.

SOLUTION Using the Product Rule and Formula 2, we have

$$\begin{aligned} \frac{dy}{dx} &= x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2) \\ &= x^2 \cos x + 2x \sin x \end{aligned}$$

■

Using the same methods as in the proof of Formula 2, we can prove (see Exercise 26) that

3

$$\frac{d}{dx}(\cos x) = -\sin x$$

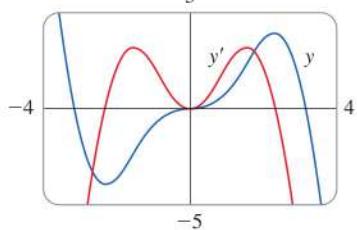


FIGURE 2

The tangent function can also be differentiated by using the definition of a derivative, but it is easier to use the Quotient Rule together with Formulas 2 and 3:

$$\begin{aligned} \frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\ &= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \\ &= \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} = \sec^2 x \quad (\cos^2 x + \sin^2 x = 1) \end{aligned}$$

4

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

The derivatives of the remaining trigonometric functions, csc, sec, and cot, can also be found easily using the Quotient Rule (see Exercises 23–25). We collect all the

differentiation formulas for trigonometric functions in the following table. Remember that they are valid only when x is measured in radians.

When you memorize this table, it is helpful to notice that the minus signs go with the derivatives of the "cofunctions," that is, cosine, cosecant, and cotangent.

Derivatives of Trigonometric Functions

$$\begin{array}{ll} \frac{d}{dx}(\sin x) = \cos x & \frac{d}{dx}(\csc x) = -\csc x \cot x \\ \frac{d}{dx}(\cos x) = -\sin x & \frac{d}{dx}(\sec x) = \sec x \tan x \\ \frac{d}{dx}(\tan x) = \sec^2 x & \frac{d}{dx}(\cot x) = -\csc^2 x \end{array}$$

EXAMPLE 2 Differentiate $f(x) = \frac{\sec x}{1 + \tan x}$. For what values of x does the graph of f have a horizontal tangent?

SOLUTION The Quotient Rule gives

$$\begin{aligned} f'(x) &= \frac{(1 + \tan x) \frac{d}{dx}(\sec x) - \sec x \frac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2} \\ &= \frac{(1 + \tan x) \sec x \tan x - \sec x \cdot \sec^2 x}{(1 + \tan x)^2} \\ &= \frac{\sec x (\tan x + \tan^2 x - \sec^2 x)}{(1 + \tan x)^2} \\ &= \frac{\sec x (\tan x - 1)}{(1 + \tan x)^2} \quad (\sec^2 x = \tan^2 x + 1) \end{aligned}$$

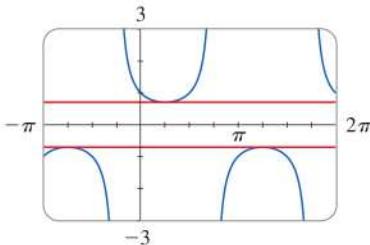


FIGURE 3

The horizontal tangents in Example 2

Because $\sec x$ is never 0, we see that $f'(x) = 0$ when $\tan x = 1$, and this occurs when $x = \pi/4 + n\pi$, where n is an integer (see Figure 3). ■

Trigonometric functions are often used in modeling real-world phenomena. In particular, vibrations, waves, elastic motions, and other quantities that vary in a periodic manner can be described using trigonometric functions. In the following example we discuss an instance of simple harmonic motion.

EXAMPLE 3 An object fastened to the end of a vertical spring is stretched 4 cm beyond its rest position and released at time $t = 0$. (See Figure 4 and note that the downward direction is positive.) Its position at time t is

$$s = f(t) = 4 \cos t$$

Find the velocity and acceleration at time t and use them to analyze the motion of the object.

SOLUTION The velocity and acceleration are

$$\begin{aligned} v &= \frac{ds}{dt} = \frac{d}{dt}(4 \cos t) = 4 \frac{d}{dt}(\cos t) = -4 \sin t \\ a &= \frac{dv}{dt} = \frac{d}{dt}(-4 \sin t) = -4 \frac{d}{dt}(\sin t) = -4 \cos t \end{aligned}$$

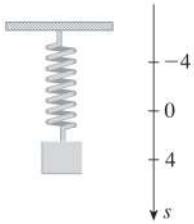


FIGURE 4

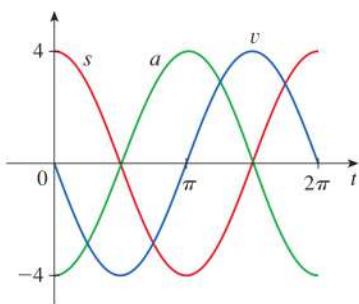


FIGURE 5

The object oscillates from the lowest point ($s = 4 \text{ cm}$) to the highest point ($s = -4 \text{ cm}$). The period of the oscillation is 2π , the period of $\cos t$.

The speed is $|v| = 4|\sin t|$, which is greatest when $|\sin t| = 1$, that is, when $\cos t = 0$. So the object moves fastest as it passes through its equilibrium position ($s = 0$). Its speed is 0 when $\sin t = 0$, that is, at the highest and lowest points.

The acceleration $a = -4 \cos t = 0$ when $s = 0$. It has greatest magnitude at the highest and lowest points. See the graphs in Figure 5.

EXAMPLE 4 Find the 27th derivative of $\cos x$.

SOLUTION The first few derivatives of $f(x) = \cos x$ are as follows:

$$\begin{aligned}f'(x) &= -\sin x \\f''(x) &= -\cos x \\f'''(x) &= \sin x \\f^{(4)}(x) &= \cos x \\f^{(5)}(x) &= -\sin x\end{aligned}$$

PS

We see that the successive derivatives occur in a cycle of length 4 and, in particular, $f^{(n)}(x) = \cos x$ whenever n is a multiple of 4. Therefore

$$f^{(24)}(x) = \cos x$$

and, differentiating three more times, we have

$$f^{(27)}(x) = \sin x$$

■ Two Special Trigonometric Limits

In proving the formula for the derivative of sine we used two special limits, which we now prove.

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

PROOF Assume first that θ lies between 0 and $\pi/2$. Figure 6(a) shows a sector of a circle with center O , central angle θ , and radius 1. BC is drawn perpendicular to OA . By the definition of radian measure, we have $\text{arc } AB = \theta$. Also $|BC| = |OB| \sin \theta = \sin \theta$. From the diagram we see that

$$|BC| < |AB| < \text{arc } AB$$

$$\text{Therefore } \sin \theta < \theta \quad \text{so} \quad \frac{\sin \theta}{\theta} < 1$$

Let the tangent lines at A and B intersect at E . You can see from Figure 6(b) that the circumference of a circle is smaller than the length of a circumscribed polygon, and so $\text{arc } AB < |AE| + |EB|$. Thus

$$\begin{aligned}\theta &= \text{arc } AB < |AE| + |EB| \\&< |AE| + |ED| \\&= |AD| = |OA| \tan \theta \\&= \tan \theta\end{aligned}$$

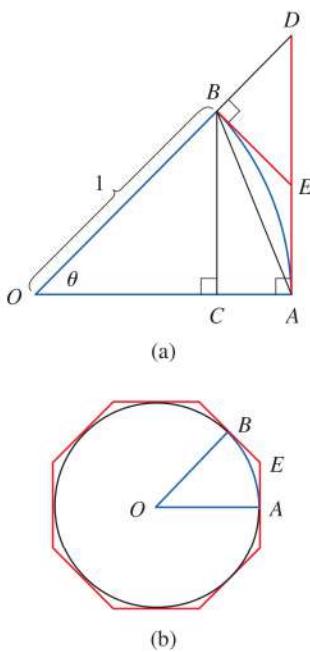


FIGURE 6

(In Appendix F the inequality $\theta \leq \tan \theta$ is proved directly from the definition of the length of an arc without resorting to geometric intuition as we did here.) Therefore we have

$$\theta < \frac{\sin \theta}{\cos \theta}$$

so $\cos \theta < \frac{\sin \theta}{\theta} < 1$

We know that $\lim_{\theta \rightarrow 0} 1 = 1$ and $\lim_{\theta \rightarrow 0} \cos \theta = 1$, so by the Squeeze Theorem, we have

$$\lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta} = 1 \quad (0 < \theta < \pi/2)$$

But the function $(\sin \theta)/\theta$ is an even function, so its right and left limits must be equal. Hence, we have

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

so we have proved Equation 5. ■

The first special limit we considered concerned the sine function. The following special limit involves cosine.

6

$$\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = 0$$

PROOF We multiply numerator and denominator by $\cos \theta + 1$ in order to put the function in a form in which we can use limits that we know.

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} &= \lim_{\theta \rightarrow 0} \left(\frac{\cos \theta - 1}{\theta} \cdot \frac{\cos \theta + 1}{\cos \theta + 1} \right) = \lim_{\theta \rightarrow 0} \frac{\cos^2 \theta - 1}{\theta (\cos \theta + 1)} \\ &= \lim_{\theta \rightarrow 0} \frac{-\sin^2 \theta}{\theta (\cos \theta + 1)} = -\lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{\cos \theta + 1} \right) \\ &= -\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\cos \theta + 1} \\ &= -1 \cdot \left(\frac{0}{1 + 1} \right) = 0 \quad (\text{by Equation 5}) \end{aligned}$$
■

EXAMPLE 5 Find $\lim_{x \rightarrow 0} \frac{\sin 7x}{4x}$.

SOLUTION In order to apply Equation 5, we first rewrite the function by multiplying and dividing by 7:

Note that $\sin 7x \neq 7 \sin x$.

$$\frac{\sin 7x}{4x} = \frac{7}{4} \left(\frac{\sin 7x}{7x} \right)$$

If we let $\theta = 7x$, then $\theta \rightarrow 0$ as $x \rightarrow 0$, so by Equation 5 we have

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin 7x}{4x} &= \frac{7}{4} \lim_{x \rightarrow 0} \left(\frac{\sin 7x}{7x} \right) \\ &= \frac{7}{4} \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = \frac{7}{4} \cdot 1 = \frac{7}{4}\end{aligned}$$

EXAMPLE 6 Calculate $\lim_{x \rightarrow 0} x \cot x$.

SOLUTION Here we divide numerator and denominator by x :

$$\begin{aligned}\lim_{x \rightarrow 0} x \cot x &= \lim_{x \rightarrow 0} \frac{x \cos x}{\sin x} \\ &= \lim_{x \rightarrow 0} \frac{\cos x}{\frac{\sin x}{x}} = \frac{\lim_{x \rightarrow 0} \cos x}{\lim_{x \rightarrow 0} \frac{\sin x}{x}} \\ &= \frac{\cos 0}{1} \quad (\text{by the continuity of cosine and Equation 5}) \\ &= 1\end{aligned}$$

EXAMPLE 7 Find $\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\sin \theta}$.

SOLUTION In order to use Equations 5 and 6 we divide numerator and denominator by θ :

$$\begin{aligned}\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\sin \theta} &= \lim_{\theta \rightarrow 0} \frac{\frac{\cos \theta - 1}{\theta}}{\frac{\sin \theta}{\theta}} \\ &= \frac{\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta}}{\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}} = \frac{0}{1} = 0\end{aligned}$$

3.3 | Exercises

1–22 Differentiate.

- | | | | |
|---|---|---|--|
| 1. $f(x) = 3 \sin x - 2 \cos x$ | 2. $f(x) = \tan x - 4 \sin x$ | 13. $f(\theta) = \frac{\sin \theta}{1 + \cos \theta}$ | 14. $y = \frac{\cos x}{1 - \sin x}$ |
| 3. $y = x^2 + \cot x$ | 4. $y = 2 \sec x - \csc x$ | 15. $y = \frac{x}{2 - \tan x}$ | 16. $f(t) = \frac{\cot t}{e^t}$ |
| 5. $h(\theta) = \theta^2 \sin \theta$ | 6. $g(x) = 3x + x^2 \cos x$ | 17. $f(w) = \frac{1 + \sec w}{1 - \sec w}$ | 18. $y = \frac{\sin t}{1 + \tan t}$ |
| 7. $y = \sec \theta \tan \theta$ | 8. $y = \sin \theta \cos \theta$ | 19. $y = \frac{t \sin t}{1 + t}$ | 20. $g(z) = \frac{z}{\sec z + \tan z}$ |
| 9. $f(\theta) = (\theta - \cos \theta) \sin \theta$ | 10. $g(\theta) = e^\theta (\tan \theta - \theta)$ | 21. $f(\theta) = \theta \cos \theta \sin \theta$ | 22. $f(t) = te^t \cot t$ |
| 11. $H(t) = \cos^2 t$ | 12. $f(x) = e^x \sin x + \cos x$ | | |

23. Show that $\frac{d}{dx}(\csc x) = -\csc x \cot x$.

24. Show that $\frac{d}{dx}(\sec x) = \sec x \tan x$.

25. Show that $\frac{d}{dx}(\cot x) = -\csc^2 x$.

26. Prove, using the definition of a derivative, that if $f(x) = \cos x$, then $f'(x) = -\sin x$.

27–30 Find an equation of the tangent line to the curve at the given point.

27. $y = \sin x + \cos x, (0, 1)$

28. $y = x + \sin x, (\pi, \pi)$

29. $y = e^x \cos x + \sin x, (0, 1)$

30. $y = \frac{1 + \sin x}{\cos x}, (\pi, -1)$

31. (a) Find an equation of the tangent line to the curve $y = 2x \sin x$ at the point $(\pi/2, \pi)$.

(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

32. (a) Find an equation of the tangent line to the curve $y = 3x + 6 \cos x$ at the point $(\pi/3, \pi + 3)$.

(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

33. (a) If $f(x) = \sec x - x$, find $f'(x)$.

(b) Check to see that your answer to part (a) is reasonable by graphing both f and f' for $|x| < \pi/2$.

34. (a) If $f(x) = e^x \cos x$, find $f'(x)$ and $f''(x)$.

(b) Check to see that your answers to part (a) are reasonable by graphing f , f' , and f'' .

35. If $g(\theta) = \frac{\sin \theta}{\theta}$, find $g'(\theta)$ and $g''(\theta)$.

36. If $f(t) = \sec t$, find $f''(\pi/4)$.

37. (a) Use the Quotient Rule to differentiate the function

$$f(x) = \frac{\tan x - 1}{\sec x}$$

(b) Simplify the expression for $f(x)$ by writing it in terms of $\sin x$ and $\cos x$, and then find $f'(x)$.

(c) Show that your answers to parts (a) and (b) are equivalent.

38. Suppose $f(\pi/3) = 4$ and $f'(\pi/3) = -2$, and let

$g(x) = f(x) \sin x$ and $h(x) = (\cos x)/f(x)$. Find

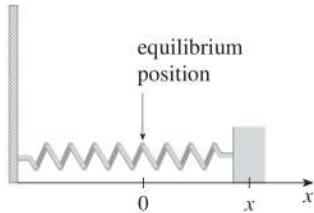
(a) $g'(\pi/3)$ (b) $h'(\pi/3)$

39–40 For what values of x does the graph of f have a horizontal tangent?

39. $f(x) = x + 2 \sin x$

40. $f(x) = e^x \cos x$

- 41.** A mass on a spring vibrates horizontally on a smooth level surface (see the figure). Its equation of motion is $x(t) = 8 \sin t$, where t is in seconds and x in centimeters.
- (a) Find the velocity and acceleration at time t .
- (b) Find the position, velocity, and acceleration of the mass at time $t = 2\pi/3$. In what direction is it moving at that time?



- 42.** An elastic band is hung on a hook and a mass is hung on the lower end of the band. When the mass is pulled downward and then released, it vibrates vertically. The equation of motion is $s = 2 \cos t + 3 \sin t$, $t \geq 0$, where s is measured in centimeters and t in seconds. (Take the positive direction to be downward.)
- (a) Find the velocity and acceleration at time t .
- (b) Graph the velocity and acceleration functions.
- (c) When does the mass pass through the equilibrium position for the first time?
- (d) How far from its equilibrium position does the mass travel?
- (e) When is the speed the greatest?

- 43.** A ladder 10 ft long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall and let x be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does x change with respect to θ when $\theta = \pi/3$?

- 44.** An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin \theta + \cos \theta}$$

where μ is a constant called the *coefficient of friction*.

- (a) Find the rate of change of F with respect to θ .
- (b) When is this rate of change equal to 0?
- (c) If $W = 50$ lb and $\mu = 0.6$, draw the graph of F as a function of θ and use it to locate the value of θ for which $dF/d\theta = 0$. Is the value consistent with your answer to part (b)?

45–60 Find the limit.

45. $\lim_{x \rightarrow 0} \frac{\sin 5x}{3x}$

46. $\lim_{x \rightarrow 0} \frac{\sin x}{\sin \pi x}$

47. $\lim_{t \rightarrow 0} \frac{\sin 3t}{\sin t}$

48. $\lim_{x \rightarrow 0} \frac{\sin^2 3x}{x}$

49. $\lim_{x \rightarrow 0} \frac{\sin x - \sin x \cos x}{x^2}$

50. $\lim_{x \rightarrow 0} \frac{1 - \sec x}{2x}$

51. $\lim_{x \rightarrow 0} \frac{\tan 2x}{x}$

52. $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\tan 7\theta}$

53. $\lim_{x \rightarrow 0} \frac{\sin 3x}{5x^3 - 4x}$

54. $\lim_{x \rightarrow 0} \frac{\sin 3x \sin 5x}{x^2}$

55. $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta + \tan \theta}$

56. $\lim_{x \rightarrow 0} \csc x \sin(\sin x)$

57. $\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{2\theta^2}$

58. $\lim_{x \rightarrow 0} \frac{\sin(x^2)}{x}$

59. $\lim_{x \rightarrow \pi/4} \frac{1 - \tan x}{\sin x - \cos x}$

60. $\lim_{x \rightarrow 1} \frac{\sin(x-1)}{x^2 + x - 2}$

61–62 Find the given derivative by finding the first few derivatives and observing the pattern that occurs.

61. $\frac{d^{99}}{dx^{99}} (\sin x)$

62. $\frac{d^{35}}{dx^{35}} (x \sin x)$

63. Find constants A and B such that the function

$y = A \sin x + B \cos x$ satisfies the differential equation $y'' + y' - 2y = \sin x$.

64. (a) Evaluate $\lim_{x \rightarrow \infty} x \sin \frac{1}{x}$.

(b) Evaluate $\lim_{x \rightarrow 0} x \sin \frac{1}{x}$.

(c) Illustrate parts (a) and (b) by graphing $y = x \sin(1/x)$.

65. Differentiate each trigonometric identity to obtain a new (or familiar) identity.

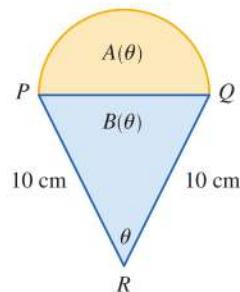
(a) $\tan x = \frac{\sin x}{\cos x}$

(b) $\sec x = \frac{1}{\cos x}$

(c) $\sin x + \cos x = \frac{1 + \cot x}{\csc x}$

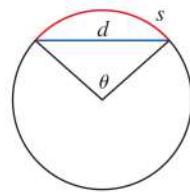
66. A semicircle with diameter PQ sits on an isosceles triangle PQR to form a region shaped like a two-dimensional ice-cream cone, as shown in the figure. If $A(\theta)$ is the area of the semicircle and $B(\theta)$ is the area of the triangle, find

$$\lim_{\theta \rightarrow 0^+} \frac{A(\theta)}{B(\theta)}$$



67. The figure shows a circular arc of length s and a chord of length d , both subtended by a central angle θ . Find

$$\lim_{\theta \rightarrow 0^+} \frac{s}{d}$$



68. Let $f(x) = \frac{x}{\sqrt{1 - \cos 2x}}$.

- (a) Graph f . What type of discontinuity does it appear to have at 0?
(b) Calculate the left and right limits of f at 0. Do these values confirm your answer to part (a)?

3.4 | The Chain Rule

Suppose you are asked to differentiate the function

$$F(x) = \sqrt{x^2 + 1}$$

The differentiation formulas you learned in the previous sections of this chapter do not enable you to calculate $F'(x)$.

Observe that F is a composite function. In fact, if we let $y = f(u) = \sqrt{u}$ and let $u = g(x) = x^2 + 1$, then we can write $y = F(x) = f(g(x))$, that is, $F = f \circ g$. We know how to differentiate both f and g , so it would be useful to have a rule that tells us how to find the derivative of $F = f \circ g$ in terms of the derivatives of f and g .

See Section 1.3 for a review of composite functions.

■ The Chain Rule

It turns out that the derivative of the composite function $f \circ g$ is the product of the derivatives of f and g . This fact is one of the most important of the differentiation rules and is called the *Chain Rule*. It seems plausible if we interpret derivatives as rates of change. Regard du/dx as the rate of change of u with respect to x , dy/du as the rate of change of y with respect to u , and dy/dx as the rate of change of y with respect to x . If u changes twice as fast as x and y changes three times as fast as u , then it seems reasonable that y changes six times as fast as x , and so we expect that dy/dx is the product of dy/du and du/dx .

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

1

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

2

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

Formula 2 is easy to remember because if we think of dy/du and du/dx as quotients, then we could cancel du ; however, du has not been defined and du/dx should not be considered as an actual quotient.

James Gregory

The first person to formulate the Chain Rule was the Scottish mathematician James Gregory (1638–1675), who also designed the first practical reflecting telescope. Gregory discovered the basic ideas of calculus at about the same time as Newton. He became the first Professor of Mathematics at the University of St. Andrews and later held the same position at the University of Edinburgh. But one year after accepting that position, he died at the age of 36.

COMMENTS ON THE PROOF OF THE CHAIN RULE Let Δu be the change in u corresponding to a change of Δx in x , that is,

$$\Delta u = g(x + \Delta x) - g(x)$$

Then the corresponding change in y is

$$\Delta y = f(u + \Delta u) - f(u)$$

It is tempting to write

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} \\ 3 &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} \\ &= \lim_{\Delta u \rightarrow 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} && \text{(Note that } \Delta u \rightarrow 0 \text{ as } \Delta x \rightarrow 0 \text{ because } g \text{ is continuous.)} \\ &= \frac{dy}{du} \frac{du}{dx} \end{aligned}$$

The only flaw in this reasoning is that in (3) it might happen that $\Delta u = 0$ (even when $\Delta x \neq 0$) and, of course, we can't divide by 0. Nonetheless, this reasoning does at least suggest that the Chain Rule is true. A full proof of the Chain Rule is given at the end of this section. ■

EXAMPLE 1 Find $F'(x)$ if $F(x) = \sqrt{x^2 + 1}$.

SOLUTION 1 (using Formula 1): At the beginning of this section we expressed F as $F(x) = (f \circ g)(x) = f(g(x))$ where $f(u) = \sqrt{u}$ and $g(x) = x^2 + 1$. Since

$$f'(u) = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}} \quad \text{and} \quad g'(x) = 2x$$

we have

$$F'(x) = f'(g(x)) \cdot g'(x)$$

$$= \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x = \frac{x}{\sqrt{x^2 + 1}}$$

SOLUTION 2 (using Formula 2): If we let $u = x^2 + 1$ and $y = \sqrt{u}$, then

$$F'(x) = \frac{dy}{du} \frac{du}{dx} = \frac{1}{2\sqrt{u}}(2x) = \frac{1}{2\sqrt{x^2 + 1}}(2x) = \frac{x}{\sqrt{x^2 + 1}}$$

When using Formula 2 we should bear in mind that dy/dx refers to the derivative of y when y is considered as a function of x (the derivative of y with respect to x), whereas dy/du refers to the derivative of y when considered as a function of u (the derivative of y with respect to u). For instance, in Example 1, y can be considered as a function of x ($y = \sqrt{x^2 + 1}$) and also as a function of u ($y = \sqrt{u}$). Note that

$$\frac{dy}{dx} = F'(x) = \frac{x}{\sqrt{x^2 + 1}} \quad \text{whereas} \quad \frac{dy}{du} = f'(u) = \frac{1}{2\sqrt{u}}$$

NOTE In using the Chain Rule we work from the outside to the inside. Formula 1 says that we differentiate the outer function f [at the inner function $g(x)$] and then we multiply by the derivative of the inner function.

$$\frac{d}{dx} \underbrace{f}_{\substack{\text{outer} \\ \text{function}}} \underbrace{(g(x))}_{\substack{\text{evaluated} \\ \text{at inner} \\ \text{function}}} = \underbrace{f'}_{\substack{\text{derivative} \\ \text{of outer} \\ \text{function}}} \underbrace{(g(x))}_{\substack{\text{evaluated} \\ \text{at inner} \\ \text{function}}} \cdot \underbrace{g'(x)}_{\substack{\text{derivative} \\ \text{of inner} \\ \text{function}}}$$

EXAMPLE 2 Differentiate (a) $y = \sin(x^2)$ and (b) $y = \sin^2 x$.

SOLUTION

(a) If $y = \sin(x^2)$, then the outer function is the sine function and the inner function is the squaring function, so the Chain Rule gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \underbrace{\sin}_{\substack{\text{outer} \\ \text{function}}} \underbrace{(x^2)}_{\substack{\text{evaluated} \\ \text{at inner} \\ \text{function}}} = \underbrace{\cos}_{\substack{\text{derivative} \\ \text{of outer} \\ \text{function}}} \underbrace{(x^2)}_{\substack{\text{evaluated} \\ \text{at inner} \\ \text{function}}} \cdot \underbrace{2x}_{\substack{\text{derivative} \\ \text{of inner} \\ \text{function}}} \\ &= 2x \cos(x^2) \end{aligned}$$

(b) Note that $\sin^2 x = (\sin x)^2$. Here the outer function is the squaring function and the inner function is the sine function. So

$$\frac{dy}{dx} = \frac{d}{dx} \underbrace{(\sin x)^2}_{\substack{\text{inner} \\ \text{function}}} = \underbrace{2}_{\substack{\text{derivative} \\ \text{of outer} \\ \text{function}}} \cdot \underbrace{(\sin x)}_{\substack{\text{evaluated} \\ \text{at inner} \\ \text{function}}} \cdot \underbrace{\cos x}_{\substack{\text{derivative} \\ \text{of inner} \\ \text{function}}}$$

The answer can be left as $2 \sin x \cos x$ or written as $\sin 2x$ (by a trigonometric identity known as the double-angle formula).

See Reference Page 2 or Appendix D.

In Example 2(a) we combined the Chain Rule with the rule for differentiating the sine function. In general, if $y = \sin u$, where u is a differentiable function of x , then, by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \cos u \frac{du}{dx}$$

Thus

$$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$$

In a similar fashion, all of the formulas for differentiating trigonometric functions can be combined with the Chain Rule.

Let's make explicit the special case of the Chain Rule where the outer function f is a power function. If $y = [g(x)]^n$, then we can write $y = f(u) = u^n$ where $u = g(x)$. By using the Chain Rule and then the Power Rule, we get

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = nu^{n-1} \frac{du}{dx} = n[g(x)]^{n-1}g'(x)$$

4 The Power Rule Combined with the Chain Rule If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$$

Alternatively,

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

Notice that the derivative we found in Example 1 could be calculated by taking $n = \frac{1}{2}$ in Rule 4.

EXAMPLE 3 Differentiate $y = (x^3 - 1)^{100}$.

SOLUTION Taking $u = g(x) = x^3 - 1$ and $n = 100$ in (4), we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^3 - 1)^{100} = 100(x^3 - 1)^{99} \frac{d}{dx}(x^3 - 1) \\ &= 100(x^3 - 1)^{99} \cdot 3x^2 = 300x^2(x^3 - 1)^{99}\end{aligned}$$

EXAMPLE 4 Find $f'(x)$ if $f(x) = \frac{1}{\sqrt[3]{x^2 + x + 1}}$.

SOLUTION First rewrite f : $f(x) = (x^2 + x + 1)^{-1/3}$

$$\begin{aligned}\text{Thus } f'(x) &= -\frac{1}{3}(x^2 + x + 1)^{-4/3} \frac{d}{dx}(x^2 + x + 1) \\ &= -\frac{1}{3}(x^2 + x + 1)^{-4/3}(2x + 1)\end{aligned}$$

EXAMPLE 5 Find the derivative of the function

$$g(t) = \left(\frac{t-2}{2t+1} \right)^9$$

SOLUTION Combining the Power Rule, Chain Rule, and Quotient Rule, we get

$$\begin{aligned} g'(t) &= 9\left(\frac{t-2}{2t+1}\right)^8 \frac{d}{dt}\left(\frac{t-2}{2t+1}\right) \\ &= 9\left(\frac{t-2}{2t+1}\right)^8 \frac{(2t+1)\cdot 1 - 2(t-2)}{(2t+1)^2} = \frac{45(t-2)^8}{(2t+1)^{10}} \end{aligned}$$

EXAMPLE 6 Differentiate $y = (2x+1)^5(x^3-x+1)^4$.

SOLUTION In this example we must use the Product Rule before using the Chain Rule:

The graphs of the functions y and y' in Example 6 are shown in Figure 1. Notice that y' is large when y increases rapidly and $y' = 0$ when y has a horizontal tangent. So our answer appears to be reasonable.

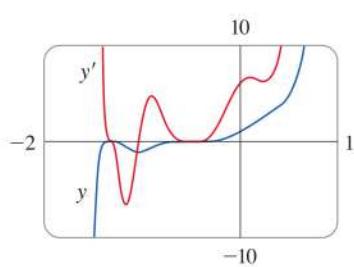


FIGURE 1

More generally, the Chain Rule gives

$$\frac{dy}{dx}(e^u) = e^u \frac{du}{dx}$$

$$\begin{aligned} \frac{dy}{dx} &= (2x+1)^5 \frac{d}{dx}(x^3-x+1)^4 + (x^3-x+1)^4 \frac{d}{dx}(2x+1)^5 \\ &= (2x+1)^5 \cdot 4(x^3-x+1)^3 \frac{d}{dx}(x^3-x+1) \\ &\quad + (x^3-x+1)^4 \cdot 5(2x+1)^4 \frac{d}{dx}(2x+1) \\ &= 4(2x+1)^5(x^3-x+1)^3(3x^2-1) + 5(x^3-x+1)^4(2x+1)^4 \cdot 2 \end{aligned}$$

Noticing that each term has the common factor $2(2x+1)^4(x^3-x+1)^3$, we could factor it out and write the answer as

$$\frac{dy}{dx} = 2(2x+1)^4(x^3-x+1)^3(17x^3+6x^2-9x+3)$$

EXAMPLE 7 Differentiate $y = e^{\sin x}$.

SOLUTION Here the inner function is $g(x) = \sin x$ and the outer function is the exponential function $f(x) = e^x$. So, by the Chain Rule,

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\sin x}) = e^{\sin x} \frac{d}{dx}(\sin x) = e^{\sin x} \cos x$$

The reason for the name “Chain Rule” becomes clear when we make a longer chain by adding another link. Suppose that $y = f(u)$, $u = g(x)$, and $x = h(t)$ where f , g , and h are differentiable functions. Then, to compute the derivative of y with respect to t , we use the Chain Rule twice:

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = \frac{dy}{du} \frac{du}{dx} \frac{dx}{dt}$$

EXAMPLE 8 If $f(x) = \sin(\cos(\tan x))$, then

$$\begin{aligned} f'(x) &= \cos(\cos(\tan x)) \frac{d}{dx} \cos(\tan x) \\ &= \cos(\cos(\tan x)) [-\sin(\tan x)] \frac{d}{dx} (\tan x) \\ &= -\cos(\cos(\tan x)) \sin(\tan x) \sec^2 x \end{aligned}$$

Notice that we used the Chain Rule twice. ■

EXAMPLE 9 Differentiate $y = e^{\sec 3\theta}$.

SOLUTION The outer function is the exponential function, the middle function is the secant function, and the inner function is the tripling function. So we have

$$\begin{aligned} \frac{dy}{d\theta} &= e^{\sec 3\theta} \frac{d}{d\theta} (\sec 3\theta) \\ &= e^{\sec 3\theta} \sec 3\theta \tan 3\theta \frac{d}{d\theta} (3\theta) \\ &= 3e^{\sec 3\theta} \sec 3\theta \tan 3\theta \end{aligned}$$

Derivatives of General Exponential Functions

We can use the Chain Rule to differentiate an exponential function with any base $b > 0$. Recall from Equation 1.5.10 that we can write

$$b^x = e^{(\ln b)x}$$

and then the Chain Rule gives

$$\begin{aligned} \frac{d}{dx} (b^x) &= \frac{d}{dx} (e^{(\ln b)x}) = e^{(\ln b)x} \frac{d}{dx} [(\ln b)x] \\ &= e^{(\ln b)x} (\ln b) = b^x \ln b \end{aligned}$$

because $\ln b$ is a constant. So we have the formula

5

$$\frac{d}{dx} (b^x) = b^x \ln b$$

Don't confuse Formula 5 (where x is the exponent) with the Power Rule (where x is the base):

$$\frac{d}{dx} (x^n) = nx^{n-1}$$

EXAMPLE 10 Find the derivative of each of the functions.

$$(a) g(x) = 2^x \quad (b) h(x) = 5^{x^2}$$

SOLUTION

(a) We use Formula 5 with $b = 2$:

$$g'(x) = \frac{d}{dx} (2^x) = 2^x \ln 2$$

This is consistent with the estimate

$$\frac{d}{dx} (2^x) \approx (0.693)2^x$$

we gave in Section 3.1 because $\ln 2 \approx 0.693147$.

(b) The outer function is an exponential function and the inner function is the squaring function, so we use Formula 5 and the Chain Rule to get

$$h'(x) = \frac{d}{dx}(5^{x^2}) = 5^{x^2} \ln 5 \cdot \frac{d}{dx}(x^2) = 2x \cdot 5^{x^2} \ln 5$$

■ How to Prove the Chain Rule

Recall that if $y = f(x)$ and x changes from a to $a + \Delta x$, we define the increment of y as

$$\Delta y = f(a + \Delta x) - f(a)$$

According to the definition of a derivative, we have

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = f'(a)$$

So if we denote by ε the difference between $\Delta y / \Delta x$ and $f'(a)$, we obtain

$$\lim_{\Delta x \rightarrow 0} \varepsilon = \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta y}{\Delta x} - f'(a) \right) = f'(a) - f'(a) = 0$$

$$\text{But } \varepsilon = \frac{\Delta y}{\Delta x} - f'(a) \Rightarrow \Delta y = f'(a) \Delta x + \varepsilon \Delta x$$

If we define ε to be 0 when $\Delta x = 0$, then ε becomes a continuous function of Δx . Thus, for a differentiable function f , we can write

$$[6] \quad \Delta y = f'(a) \Delta x + \varepsilon \Delta x \quad \text{where } \varepsilon \rightarrow 0 \text{ as } \Delta x \rightarrow 0$$

and ε is a continuous function of Δx . This property of differentiable functions is what enables us to prove the Chain Rule.

PROOF OF THE CHAIN RULE Suppose $u = g(x)$ is differentiable at a and $y = f(u)$ is differentiable at $b = g(a)$. If Δx is an increment in x and Δu and Δy are the corresponding increments in u and y , then we can use Equation 6 to write

$$[7] \quad \Delta u = g'(a) \Delta x + \varepsilon_1 \Delta x = [g'(a) + \varepsilon_1] \Delta x$$

where $\varepsilon_1 \rightarrow 0$ as $\Delta x \rightarrow 0$. Similarly

$$[8] \quad \Delta y = f'(b) \Delta u + \varepsilon_2 \Delta u = [f'(b) + \varepsilon_2] \Delta u$$

where $\varepsilon_2 \rightarrow 0$ as $\Delta u \rightarrow 0$. If we now substitute the expression for Δu from Equation 7 into Equation 8, we get

$$\begin{aligned} \Delta y &= [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1] \Delta x \\ \text{so } \frac{\Delta y}{\Delta x} &= [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1] \end{aligned}$$

As $\Delta x \rightarrow 0$, Equation 7 shows that $\Delta u \rightarrow 0$. Taking the limit as $\Delta x \rightarrow 0$, we get

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1] \\ &= f'(b)g'(a) = f'(g(a))g'(a) \end{aligned}$$

This proves the Chain Rule. ■

3.4 Exercises

1–6 Write the composite function in the form $f(g(x))$. [Identify the inner function $u = g(x)$ and the outer function $y = f(u)$.] Then find the derivative dy/dx .

1. $y = (5 - x^4)^3$

2. $y = \sqrt{x^3 + 2}$

3. $y = \sin(\cos x)$

4. $y = \tan(x^2)$

5. $y = e^{\sqrt{x}}$

6. $y = \sqrt[3]{e^x + 1}$

7–52 Find the derivative of the function.

7. $f(x) = (2x^3 - 5x^2 + 4)^5$ 8. $f(x) = (x^5 + 3x^2 - x)^{50}$

9. $f(x) = \sqrt{5x + 1}$

10. $f(x) = \frac{1}{\sqrt[3]{x^2 - 1}}$

11. $g(t) = \frac{1}{(2t + 1)^2}$

12. $F(t) = \left(\frac{1}{2t + 1}\right)^4$

13. $f(\theta) = \cos(\theta^2)$

14. $g(\theta) = \cos^2 \theta$

15. $g(x) = e^{x^2-x}$

16. $y = 5^{\sqrt{x}}$

17. $y = x^2 e^{-3x}$

18. $f(t) = t \sin \pi t$

19. $f(t) = e^{at} \sin bt$

20. $A(r) = \sqrt{r} \cdot e^{r^2+1}$

21. $F(x) = (4x + 5)^3(x^2 - 2x + 5)^4$

22. $G(z) = (1 - 4z)^2 \sqrt{z^2 + 1}$

23. $y = \sqrt{\frac{x}{x+1}}$

24. $y = \left(x + \frac{1}{x}\right)^5$

25. $y = e^{\tan \theta}$

26. $f(t) = 2^{t^3}$

27. $g(u) = \left(\frac{u^3 - 1}{u^3 + 1}\right)^8$

28. $s(t) = \sqrt{\frac{1 + \sin t}{1 + \cos t}}$

29. $r(t) = 10^{2\sqrt{t}}$

30. $f(z) = e^{z/(z-1)}$

31. $H(r) = \frac{(r^2 - 1)^3}{(2r + 1)^5}$

32. $J(\theta) = \tan^2(n\theta)$

33. $F(t) = e^{t \sin 2t}$

34. $F(t) = \frac{t^2}{\sqrt{t^3 + 1}}$

35. $G(x) = 4^{C/x}$

36. $U(y) = \left(\frac{y^4 + 1}{y^2 + 1}\right)^5$

37. $f(x) = \sin x \cos(1 - x^2)$

38. $g(x) = e^{-x} \cos(x^2)$

39. $F(t) = \tan \sqrt{1 + t^2}$

40. $G(z) = (1 + \cos^2 z)^3$

41. $y = \sin^2(x^2 + 1)$

42. $y = e^{\sin 2x} + \sin(e^{2x})$

43. $g(x) = \sin\left(\frac{e^x}{1 + e^x}\right)$

44. $f(t) = e^{1/t} \sqrt{t^2 - 1}$

45. $f(t) = \tan(\sec(\cos t))$

46. $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

47. $f(x) = e^{\sin^2(x^2)}$

48. $y = 2^{3^{4^x}}$

49. $y = (3^{\cos(x^2)} - 1)^4$

50. $y = \sin(\theta + \tan(\theta + \cos \theta))$

51. $y = \cos \sqrt{\sin(\tan \pi x)}$

52. $y = \sin^3(\cos(x^2))$

53–56 Find y' and y'' .

53. $y = \cos(\sin 3\theta)$

54. $y = (1 + \sqrt{x})^3$

55. $y = \sqrt{\cos x}$

56. $y = e^{e^x}$

57–60 Find an equation of the tangent line to the curve at the given point.

57. $y = 2^x$, $(0, 1)$

58. $y = \sqrt{1 + x^3}$, $(2, 3)$

59. $y = \sin(\sin x)$, $(\pi, 0)$

60. $y = xe^{-x^2}$, $(0, 0)$

61. (a) Find an equation of the tangent line to the curve $y = 2/(1 + e^{-x})$ at the point $(0, 1)$.

 (b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

62. (a) The curve $y = |x|/\sqrt{2 - x^2}$ is called a *bullet-nose curve*. Find an equation of the tangent line to this curve at the point $(1, 1)$.

 (b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

63. (a) If $f(x) = x\sqrt{2 - x^2}$, find $f'(x)$.

 (b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f' .

64. The function $f(x) = \sin(x + \sin 2x)$, $0 \leq x \leq \pi$, arises in applications to frequency modulation (FM) synthesis.

- (a) Use a graph of f produced by a calculator or computer to make a rough sketch of the graph of f' .
 (b) Calculate $f'(x)$ and use this expression, with a calculator or computer, to graph f' . Compare with your sketch in part (a).

65. Find all points on the graph of the function

$f(x) = 2 \sin x + \sin^2 x$ at which the tangent line is horizontal.

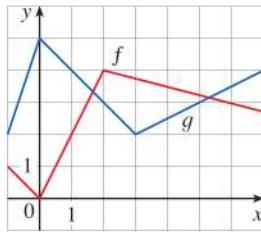
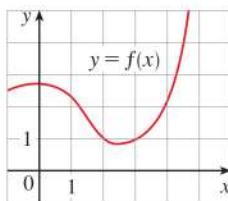
66. At what point on the curve $y = \sqrt{1 + 2x}$ is the tangent line perpendicular to the line $6x + 2y = 1$?

67. If $F(x) = f(g(x))$, where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, and $g'(5) = 6$, find $F'(5)$.

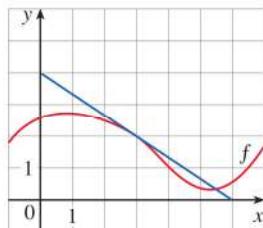
68. If $h(x) = \sqrt{4 + 3f(x)}$, where $f(1) = 7$ and $f'(1) = 4$, find $h'(1)$.

69. A table of values for f , g , f' , and g' is given.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

- (a) If $h(x) = f(g(x))$, find $h'(1)$.
 (b) If $H(x) = g(f(x))$, find $H'(1)$.
70. Let f and g be the functions in Exercise 69.
 (a) If $F(x) = f(f(x))$, find $F'(2)$.
 (b) If $G(x) = g(g(x))$, find $G'(3)$.
71. If f and g are the functions whose graphs are shown, let $u(x) = f(g(x))$, $v(x) = g(f(x))$, and $w(x) = g(g(x))$. Find each derivative, if it exists. If it does not exist, explain why.
 (a) $u'(1)$ (b) $v'(1)$ (c) $w'(1)$
- 
72. If f is the function whose graph is shown, let $h(x) = f(f(x))$ and $g(x) = f(x^2)$. Use the graph of f to estimate the value of each derivative.
 (a) $h'(2)$ (b) $g'(2)$
- 

73. If $g(x) = \sqrt{f(x)}$, where the graph of f is shown, evaluate $g'(3)$.



74. Suppose f is differentiable on \mathbb{R} and α is a real number. Let $F(x) = f(x^\alpha)$ and $G(x) = [f(x)]^\alpha$. Find expressions for (a) $F'(x)$ and (b) $G'(x)$.

75. Suppose f is differentiable on \mathbb{R} . Let $F(x) = f(e^x)$ and $G(x) = e^{f(x)}$. Find expressions for (a) $F'(x)$ and (b) $G'(x)$.

76. Let $g(x) = e^{cx} + f(x)$ and $h(x) = e^{kx}f(x)$, where $f(0) = 3$, $f'(0) = 5$, and $f''(0) = -2$.

- (a) Find $g'(0)$ and $g''(0)$ in terms of c .
 (b) In terms of k , find an equation of the tangent line to the graph of h at the point where $x = 0$.

77. Let $r(x) = f(g(h(x)))$, where $h(1) = 2$, $g(2) = 3$, $h'(1) = 4$, $g'(2) = 5$, and $f'(3) = 6$. Find $r'(1)$.

78. If g is a twice differentiable function and $f(x) = xg(x^2)$, find f'' in terms of g , g' , and g'' .

79. If $F(x) = f(3f(4f(x)))$, where $f(0) = 0$ and $f'(0) = 2$, find $F'(0)$.

80. If $F(x) = f(xf(xf(x)))$, where $f(1) = 2$, $f(2) = 3$, $f'(1) = 4$, $f'(2) = 5$, and $f'(3) = 6$, find $F'(1)$.

81. Show that the function $y = e^{2x}(A \cos 3x + B \sin 3x)$ satisfies the differential equation $y'' - 4y' + 13y = 0$.

82. For what values of r does the function $y = e^{rx}$ satisfy the differential equation $y'' - 4y' + y = 0$?

83. Find the 50th derivative of $y = \cos 2x$.

84. Find the 1000th derivative of $f(x) = xe^{-x}$.

85. The displacement of a particle on a vibrating string is given by the equation

$$s(t) = 10 + \frac{1}{4} \sin(10\pi t)$$

where s is measured in centimeters and t in seconds. Find the velocity of the particle after t seconds.

86. If the equation of motion of a particle is given by $s = A \cos(\omega t + \delta)$, the particle is said to undergo *simple harmonic motion*.

- (a) Find the velocity of the particle at time t .
 (b) When is the velocity 0?

87. A Cepheid variable star is a star whose brightness alternately increases and decreases. The most easily visible such star is Delta Cephei, for which the interval between times of maximum brightness is 5.4 days. The average brightness of this star is 4.0 and its brightness changes by ± 0.35 . In view of these data, the brightness of Delta Cephei at time t , where t is measured in days, has been modeled by the function

$$B(t) = 4.0 + 0.35 \sin\left(\frac{2\pi t}{5.4}\right)$$

- (a) Find the rate of change of the brightness after t days.
 (b) Find, correct to two decimal places, the rate of increase after one day.

- 88.** In Example 1.3.4 we arrived at a model for the length of daylight (in hours) in Philadelphia on the t th day of the year:

$$L(t) = 12 + 2.8 \sin \left[\frac{2\pi}{365}(t - 80) \right]$$

Use this model to compare how the number of hours of daylight is increasing in Philadelphia on March 21 ($t = 80$) and May 21 ($t = 141$).

- 89.** The motion of a spring that is subject to a frictional force or a damping force (such as a shock absorber in a car) is often modeled by the product of an exponential function and a sine or cosine function. Suppose the equation of motion of a point on such a spring is

$$s(t) = 2e^{-1.5t} \sin 2\pi t$$

where s is measured in centimeters and t in seconds. Find the velocity after t seconds and graph both the position and velocity functions for $0 \leq t \leq 2$.

- 90.** Under certain circumstances a rumor spreads according to the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

where $p(t)$ is the proportion of the population that has heard the rumor at time t and a and k are positive constants. [In Section 9.4 we will see that this is a reasonable equation for $p(t)$.]

- (a) Find $\lim_{t \rightarrow \infty} p(t)$ and interpret your answer.
 (b) Find the rate of spread of the rumor.
 (c) Graph p for the case $a = 10$, $k = 0.5$ with t measured in hours. Use the graph to estimate how long it will take for 80% of the population to have heard the rumor.

- 91.** The average blood alcohol concentration (BAC) of eight male subjects was measured after consumption of 15 mL of ethanol (corresponding to one alcoholic drink). The resulting data were modeled by the concentration function

$$C(t) = 0.00225te^{-0.0467t}$$

where t is measured in minutes after consumption and C is measured in g/dL.

- (a) How rapidly was the BAC increasing after 10 minutes?
 (b) How rapidly was it decreasing half an hour later?

Source: Adapted from P. Wilkinson et al., "Pharmacokinetics of Ethanol after Oral Administration in the Fasting State," *Journal of Pharmacokinetics and Biopharmaceutics* 5 (1977): 207–24.

- 92.** Air is being pumped into a spherical weather balloon. At any time t , the volume of the balloon is $V(t)$ and its radius is $r(t)$.
 (a) What do the derivatives dV/dr and dV/dt represent?
 (b) Express dV/dt in terms of dr/dt .

- 93.** A particle moves along a straight line with displacement $s(t)$, velocity $v(t)$, and acceleration $a(t)$. Show that

$$a(t) = v(t) \frac{dv}{ds}$$

Explain the difference between the meanings of the derivatives dv/dt and dv/ds .

- 94.** The table gives the US population from 1790 to 1860.

Year	Population	Year	Population
1790	3,929,000	1830	12,861,000
1800	5,308,000	1840	17,063,000
1810	7,240,000	1850	23,192,000
1820	9,639,000	1860	31,443,000

- T** (a) Fit an exponential function to the data. Graph the data points and the exponential model. How good is the fit?
 (b) Estimate the rates of population growth in 1800 and 1850 by averaging slopes of secant lines.
 (c) Use the exponential model in part (a) to estimate the rates of growth in 1800 and 1850. Compare these estimates with the ones in part (b).
 (d) Use the exponential model to predict the population in 1870. Compare with the actual population of 38,558,000. Can you explain the discrepancy?

- 95.** Use the Chain Rule to prove the following.

- (a) The derivative of an even function is an odd function.
 (b) The derivative of an odd function is an even function.

- 96.** Use the Chain Rule and the Product Rule to give an alternative proof of the Quotient Rule.

[Hint: Write $f(x)/g(x) = f(x)[g(x)]^{-1}$.]

- 97.** Use the Chain Rule to show that if θ is measured in degrees, then

$$\frac{d}{d\theta} (\sin \theta) = \frac{\pi}{180} \cos \theta$$

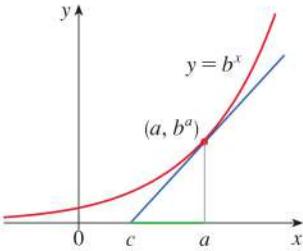
(This gives one reason for the convention that radian measure is always used when dealing with trigonometric functions in calculus: the differentiation formulas would not be as simple if we used degree measure.)

- 98.** (a) Write $|x| = \sqrt{x^2}$ and use the Chain Rule to show that

$$\frac{d}{dx} |x| = \frac{x}{|x|}$$

- (b) If $f(x) = |\sin x|$, find $f'(x)$ and sketch the graphs of f and f' . Where is f not differentiable?
 (c) If $g(x) = \sin |x|$, find $g'(x)$ and sketch the graphs of g and g' . Where is g not differentiable?

99. Let c be the x -intercept of the tangent line to the curve $y = b^x$ ($b > 0, b \neq 1$) at the point (a, b^a) . Show that the distance between the points $(a, 0)$ and $(c, 0)$ is the same for all values of a .



100. On every exponential curve $y = b^x$ ($b > 0, b \neq 1$), there is exactly one point (x_0, y_0) at which the tangent line to the curve passes through the origin. Show that in every case, $y_0 = e$. [Hint: You may wish to use Formula 1.5.10.]

101. If $F = f \circ g \circ h$, where f, g , and h are differentiable functions, use the Chain Rule to show that

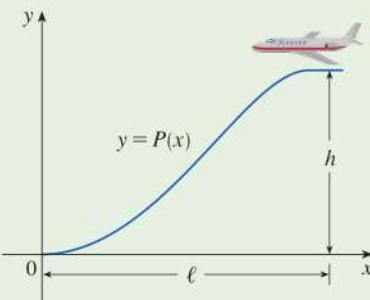
$$F'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$

102. If $F = f \circ g$, where f and g are twice differentiable functions, use the Chain Rule and the Product Rule to show that the second derivative of F is given by

$$F''(x) = f''(g(x)) \cdot [g'(x)]^2 + f'(g(x)) \cdot g''(x)$$

APPLIED PROJECT

WHERE SHOULD A PILOT START DESCENT?



An approach path for an aircraft landing is shown in the figure and satisfies the following conditions:

- The cruising altitude is h when descent starts at a horizontal distance ℓ from touchdown at the origin.
 - The pilot must maintain a constant horizontal speed v throughout descent.
 - The absolute value of the vertical acceleration should not exceed a constant k (which is much less than the acceleration due to gravity).
- Find a cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$ that satisfies condition (i) by imposing suitable conditions on $P(x)$ and $P'(x)$ at the start of descent and at touchdown.
 - Use conditions (ii) and (iii) to show that

$$\frac{6hv^2}{\ell^2} \leq k$$

- Suppose that an airline decides not to allow vertical acceleration of a plane to exceed $k = 860 \text{ mi/h}^2$. If the cruising altitude of a plane is 35,000 ft and the speed is 300 mi/h, how far away from the airport should the pilot start descent?

4. Graph the approach path if the conditions stated in Problem 3 are satisfied.

3.5

Implicit Differentiation

■ Implicitly Defined Functions

The functions that we have met so far can be described by expressing one variable explicitly in terms of another variable—for example,

$$y = \sqrt{x^3 + 1} \quad \text{or} \quad y = x \sin x$$

or, in general, $y = f(x)$. Some functions, however, are defined implicitly by a relation

between x and y such as

1

$$x^2 + y^2 = 25$$

or

2

$$x^3 + y^3 = 6xy$$

In some cases it is possible to solve such an equation for y as an explicit function (or several functions) of x . For instance, if we solve Equation 1 for y , we get $y = \pm\sqrt{25 - x^2}$, so two of the functions determined by the implicit Equation 1 are $f(x) = \sqrt{25 - x^2}$ and $g(x) = -\sqrt{25 - x^2}$. The graphs of f and g are the upper and lower semicircles of the circle $x^2 + y^2 = 25$. (See Figure 1.)

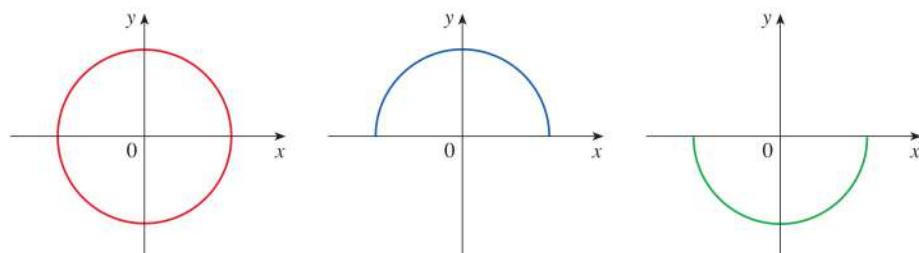


FIGURE 1

(a) $x^2 + y^2 = 25$

(b) $f(x) = \sqrt{25 - x^2}$

(c) $g(x) = -\sqrt{25 - x^2}$

It's not easy to solve Equation 2 for y explicitly as a function of x by hand. (Even if we use technology, the resulting expressions are very complicated.) Nonetheless, (2) is the equation of a curve called the **folium of Descartes** shown in Figure 2 and it implicitly defines y as several functions of x . The graphs of three such functions are shown in Figure 3. When we say that f is a function defined implicitly by Equation 2, we mean that the equation

$$x^3 + [f(x)]^3 = 6xf(x)$$

is true for all values of x in the domain of f .

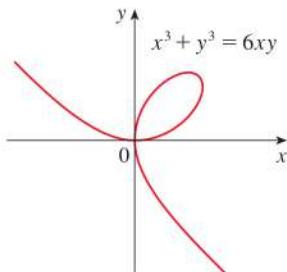


FIGURE 2 The folium of Descartes

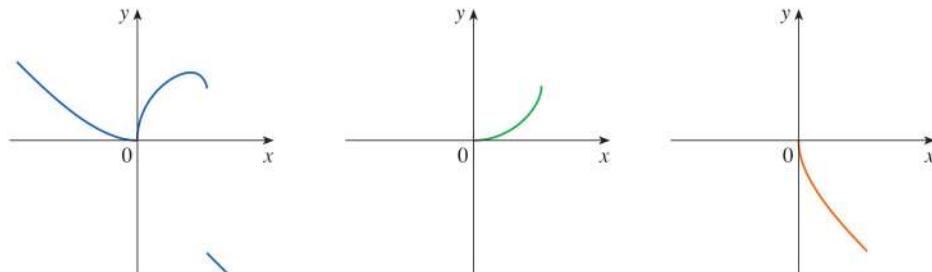


FIGURE 3 Graphs of three functions defined implicitly by the folium of Descartes

■ Implicit Differentiation

Fortunately, we don't need to solve an equation for y in terms of x in order to find the derivative of y . Instead we can use the method of **implicit differentiation**. This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for dy/dx . In the examples and exercises of this section it is always assumed that the given equation determines y implicitly as a differentiable function of x so that the method of implicit differentiation can be applied.

EXAMPLE 1

If $x^2 + y^2 = 25$, find $\frac{dy}{dx}$. Then find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$.

SOLUTION 1

Differentiate both sides of the equation $x^2 + y^2 = 25$:

$$\begin{aligned}\frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(25) \\ \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= 0\end{aligned}$$

Remembering that y is a function of x and using the Chain Rule, we have

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \frac{dy}{dx} = 2y \frac{dy}{dx}$$

$$\text{Thus } 2x + 2y \frac{dy}{dx} = 0$$

Now we solve this equation for dy/dx :

$$\frac{dy}{dx} = -\frac{x}{y}$$

At the point $(3, 4)$ we have $x = 3$ and $y = 4$, so

$$\frac{dy}{dx} = -\frac{3}{4}$$

An equation of the tangent to the circle at $(3, 4)$ is therefore

$$y - 4 = -\frac{3}{4}(x - 3) \quad \text{or} \quad 3x + 4y = 25$$

SOLUTION 2

Solving the equation $x^2 + y^2 = 25$ for y , we get $y = \pm\sqrt{25 - x^2}$. The point $(3, 4)$ lies on the upper semicircle $y = \sqrt{25 - x^2}$ and so we consider the function $f(x) = \sqrt{25 - x^2}$. Differentiating f using the Chain Rule, we have

$$\begin{aligned}f'(x) &= \frac{1}{2}(25 - x^2)^{-1/2} \frac{d}{dx}(25 - x^2) \\ &= \frac{1}{2}(25 - x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{25 - x^2}}\end{aligned}$$

At the point $(3, 4)$ we have

$$f'(3) = -\frac{3}{\sqrt{25 - 3^2}} = -\frac{3}{4}$$

and, as in Solution 1, an equation of the tangent is $3x + 4y = 25$. ■

Example 1 illustrates that even when it is possible to solve an equation explicitly for y in terms of x , it may be easier to use implicit differentiation.

NOTE 1 The expression $dy/dx = -x/y$ in Solution 1 gives the derivative in terms of both x and y . It is correct no matter which function y is determined by the given equation. For instance, for $y = f(x) = \sqrt{25 - x^2}$ we have

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{\sqrt{25 - x^2}}$$

whereas for $y = g(x) = -\sqrt{25 - x^2}$ we have

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{-\sqrt{25 - x^2}} = \frac{x}{\sqrt{25 - x^2}}$$

EXAMPLE 2

- (a) Find y' if $x^3 + y^3 = 6xy$.
- (b) Find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point $(3, 3)$.
- (c) At what point in the first quadrant is the tangent line horizontal?

SOLUTION

- (a) Differentiating both sides of $x^3 + y^3 = 6xy$ with respect to x , regarding y as a function of x , and using the Chain Rule on the term y^3 and the Product Rule on the term $6xy$, we get

$$3x^2 + 3y^2y' = 6xy' + 6y$$

or

$$x^2 + y^2y' = 2xy' + 2y$$

We now solve for y' :

$$y^2y' - 2xy' = 2y - x^2$$

$$(y^2 - 2x)y' = 2y - x^2$$

$$y' = \frac{2y - x^2}{y^2 - 2x}$$

- (b) When $x = y = 3$,

$$y' = \frac{2 \cdot 3 - 3^2}{3^2 - 2 \cdot 3} = -1$$

and a glance at Figure 4 confirms that this is a reasonable value for the slope at $(3, 3)$. So an equation of the tangent to the folium at $(3, 3)$ is

$$y - 3 = -1(x - 3) \quad \text{or} \quad x + y = 6$$

- (c) The tangent line is horizontal if $y' = 0$. Using the expression for y' from part (a), we see that $y' = 0$ when $2y - x^2 = 0$ (provided that $y^2 - 2x \neq 0$). Substituting $y = \frac{1}{2}x^2$ in the equation of the curve, we get

$$x^3 + \left(\frac{1}{2}x^2\right)^3 = 6x\left(\frac{1}{2}x^2\right)$$

which simplifies to $x^6 = 16x^3$. Since $x \neq 0$ in the first quadrant, we have $x^3 = 16$. If $x = 16^{1/3} = 2^{4/3}$, then $y = \frac{1}{2}(2^{8/3}) = 2^{5/3}$. Thus the tangent is horizontal at $(2^{4/3}, 2^{5/3})$, which is approximately $(2.5198, 3.1748)$. Looking at Figure 5, we see that our answer is reasonable.

We can use either of the notations dy/dx or y' for the derivative of y with respect to x .

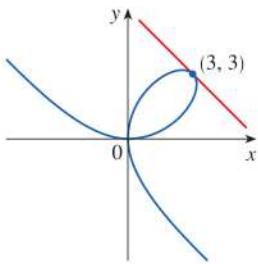


FIGURE 4

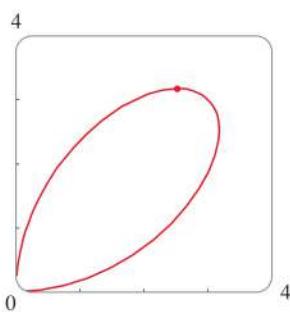


FIGURE 5

NOTE 2 There is a formula for the three solutions of a cubic equation that is like the quadratic formula but much more complicated. If we use this formula (or a computer) to solve the equation $x^3 + y^3 = 6xy$ for y in terms of x , we get three functions determined by the equations:

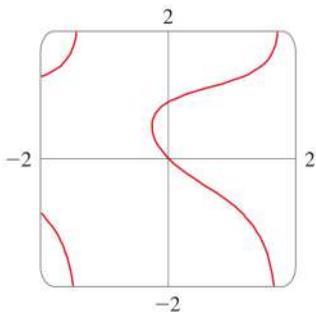
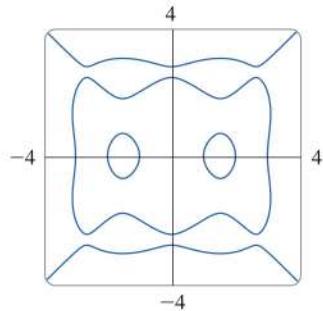
$$y = f(x) = \sqrt[3]{-\frac{1}{2}x^3 + \sqrt{\frac{1}{4}x^6 - 8x^3}} + \sqrt[3]{-\frac{1}{2}x^3 - \sqrt{\frac{1}{4}x^6 - 8x^3}}$$

and

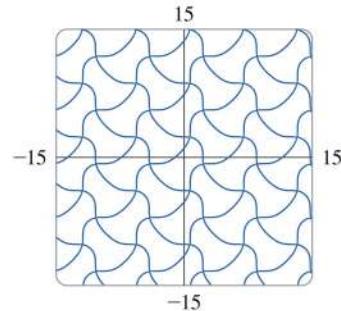
$$y = \frac{1}{2} \left[-f(x) \pm \sqrt{-3} \left(\sqrt[3]{-\frac{1}{2}x^3 + \sqrt{\frac{1}{4}x^6 - 8x^3}} - \sqrt[3]{-\frac{1}{2}x^3 - \sqrt{\frac{1}{4}x^6 - 8x^3}} \right) \right]$$

Abel and Galois

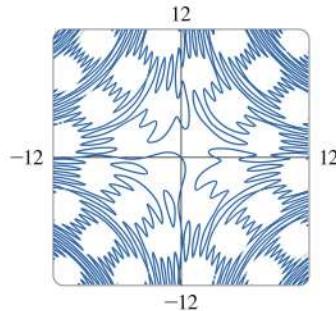
The Norwegian mathematician Niels Abel proved in 1824 that no general formula in terms of radicals can be given for the solutions of the equation $p(x) = 0$, where p is a polynomial of degree 5 with integer coefficients. Later the French mathematician Evariste Galois proved that it is impossible to find a general formula in terms of radicals for the solutions of an equation $p(x) = 0$, where p is a polynomial of degree $n \geq 5$.

**FIGURE 6****FIGURE 7**

$$(x^2 - 1)(x^2 - 4)(x^2 - 9) \\ = y^2(y^2 - 4)(y^2 - 9)$$

**FIGURE 8**

$$\cos(x - \sin y) = \sin(y - \sin x)$$

**FIGURE 9**

$$\sin(xy) = \sin x + \sin y$$

■ Second Derivatives of Implicit Functions

The following example shows how to find the second derivative of a function that is defined implicitly.

EXAMPLE 4 Find y'' if $x^4 + y^4 = 16$.

SOLUTION Differentiating the equation implicitly with respect to x , we get

$$4x^3 + 4y^3y' = 0$$

Solving for y' gives

3

$$y' = -\frac{x^3}{y^3}$$

To find y'' we differentiate this expression for y' using the Quotient Rule and remembering that y is a function of x :

$$\begin{aligned}y'' &= \frac{d}{dx} \left(-\frac{x^3}{y^3} \right) = -\frac{y^3(d/dx)(x^3) - x^3(d/dx)(y^3)}{(y^3)^2} \\&= -\frac{y^3 \cdot 3x^2 - x^3(3y^2y')}{y^6}\end{aligned}$$

If we now substitute Equation 3 into this expression, we get

$$\begin{aligned}y'' &= -\frac{3x^2y^3 - 3x^3y^2\left(-\frac{x^3}{y^3}\right)}{y^6} \\&= -\frac{3(x^2y^4 + x^6)}{y^7} = -\frac{3x^2(y^4 + x^4)}{y^7}\end{aligned}$$

But the values of x and y must satisfy the original equation $x^4 + y^4 = 16$. So the answer simplifies to

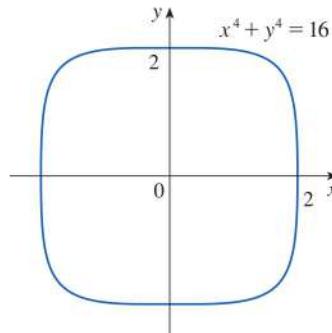
$$y'' = -\frac{3x^2(16)}{y^7} = -48 \frac{x^2}{y^7}$$

■

Figure 10 shows the graph of the curve $x^4 + y^4 = 16$ of Example 4. Notice that it's a stretched and flattened version of the circle $x^2 + y^2 = 4$. For this reason it's sometimes called a *fat circle*. It starts out very steep on the left but quickly becomes very flat. This can be seen from the expression

$$y' = -\frac{x^3}{y^3} = -\left(\frac{x}{y}\right)^3$$

FIGURE 10



3.5 | Exercises

1–4

- (a) Find y' by implicit differentiation.
- (b) Solve the equation explicitly for y and differentiate to get y' in terms of x .
- (c) Check that your solutions to parts (a) and (b) are consistent by substituting the expression for y into your solution for part (a).

1. $5x^2 - y^3 = 7$

2. $6x^4 + y^5 = 2x$

3. $\sqrt{x} + \sqrt{y} = 1$

4. $\frac{2}{x} - \frac{1}{y} = 4$

5–22 Find dy/dx by implicit differentiation.

5. $x^2 - 4xy + y^2 = 4$

6. $2x^2 + xy - y^2 = 2$

7. $x^4 + x^2y^2 + y^3 = 5$

8. $x^3 - xy^2 + y^3 = 1$

9. $\frac{x^2}{x+y} = y^2 + 1$

10. $xe^y = x - y$

11. $\sin x + \cos y = 2x - 3y$

12. $e^x \sin y = x + y$

13. $\sin(x+y) = \cos x + \cos y$

14. $\tan(x-y) = 2xy^3 + 1$

15. $y \cos x = x^2 + y^2$

16. $\sin(xy) = \cos(x+y)$

17. $2xe^y + ye^x = 3$

18. $\sin x \cos y = x^2 - 5y$

19. $\sqrt{x+y} = x^4 + y^4$

20. $xy = \sqrt{x^2 + y^2}$

21. $e^{x/y} = x - y$

22. $\cos(x^2 + y^2) = xe^y$

23. If $f(x) + x^2[f(x)]^3 = 10$ and $f(1) = 2$, find $f'(1)$.

24. If $g(x) + x \sin g(x) = x^2$, find $g'(0)$.

25–26 Regard y as the independent variable and x as the dependent variable and use implicit differentiation to find dx/dy .

25. $x^4y^2 - x^3y + 2xy^3 = 0$ 26. $y \sec x = x \tan y$

27–36 Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

27. $ye^{\sin x} = x \cos y, (0, 0)$

28. $\tan(x + y) + \sec(x - y) = 2, (\pi/8, \pi/8)$

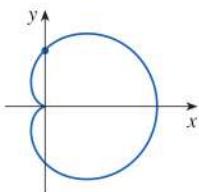
29. $x^{2/3} + y^{2/3} = 4, (-3\sqrt{3}, 1)$ (astroid)

30. $y^2(6 - x) = x^3, (2, \sqrt{2})$ (cissoid of Diocles)

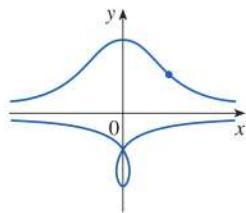
31. $x^2 - xy - y^2 = 1, (2, 1)$ (hyperbola)

32. $x^2 + 2xy + 4y^2 = 12, (2, 1)$ (ellipse)

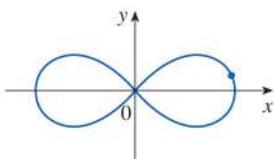
33. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2, (0, \frac{1}{2})$ (cardioid)



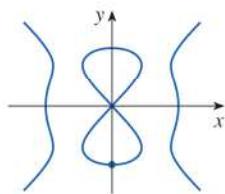
34. $x^2y^2 = (y + 1)^2(4 - y^2), (2\sqrt{3}, 1)$ (conchoid of Nicomedes)



35. $2(x^2 + y^2)^2 = 25(x^2 - y^2), (3, 1)$ (lemniscate)



36. $y^2(y^2 - 4) = x^2(x^2 - 5), (0, -2)$ (devil's curve)



- 37.** (a) The curve with equation $y^2 = 5x^4 - x^2$ is called a **kampyle of Eudoxus**. Find an equation of the tangent line to this curve at the point $(1, 2)$.

- (b) Illustrate part (a) by graphing the curve and the tangent line on a common screen. (Graph the implicitly defined curve if possible, or you can graph the upper and lower halves separately.)

- 38.** (a) The curve with equation $y^2 = x^3 + 3x^2$ is called the **Tschirnhausen cubic**. Find an equation of the tangent line to this curve at the point $(1, -2)$.

- (b) At what points does this curve have horizontal tangents? (c) Illustrate parts (a) and (b) by graphing the curve and the tangent lines on a common screen.

- 39–42** Find y'' by implicit differentiation. Simplify where possible.

39. $x^2 + 4y^2 = 4$

40. $x^2 + xy + y^2 = 3$

41. $\sin y + \cos x = 1$

42. $x^3 - y^3 = 7$

- 43.** If $xy + e^y = e$, find the value of y'' at the point where $x = 0$.

- 44.** If $x^2 + xy + y^3 = 1$, find the value of y''' at the point where $x = 1$.

- 45.** Fanciful shapes can be created by using software that can graph implicitly defined curves.

- (a) Graph the curve with equation

$$y(y^2 - 1)(y - 2) = x(x - 1)(x - 2)$$

At how many points does this curve have horizontal tangents? Estimate the x -coordinates of these points.

- (b) Find equations of the tangent lines at the points $(0, 1)$ and $(0, 2)$.
(c) Find the exact x -coordinates of the points in part (a).
(d) Create even more fanciful curves by modifying the equation in part (a).

- 46.** (a) The curve with equation

$$2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2$$

has been likened to a bouncing wagon. Graph this curve and discover why.

- (b) At how many points does this curve have horizontal tangent lines? Find the x -coordinates of these points.

- 47.** Find the points on the lemniscate in Exercise 35 where the tangent is horizontal.

- 48.** Show by implicit differentiation that the tangent line to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) has equation

$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1$$

49. Find an equation of the tangent line to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) .

50. Show that the sum of the x - and y -intercepts of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is equal to c .

51. Show, using implicit differentiation, that any tangent line at a point P to a circle with center O is perpendicular to the radius OP .

52. The Power Rule can be proved using implicit differentiation for the case where n is a rational number, $n = p/q$, and $y = f(x) = x^n$ is assumed beforehand to be a differentiable function. If $y = x^{p/q}$, then $y^q = x^p$. Use implicit differentiation to show that

$$y' = \frac{p}{q} x^{(p/q)-1}$$

53–56 Orthogonal Trajectories Two curves are *orthogonal* if their tangent lines are perpendicular at each point of intersection. Show that the given families of curves are *orthogonal trajectories* of each other; that is, every curve in one family is orthogonal to every curve in the other family. Sketch both families of curves on the same axes.

53. $x^2 + y^2 = r^2, \quad ax + by = 0$

54. $x^2 + y^2 = ax, \quad x^2 + y^2 = by$

55. $y = cx^2, \quad x^2 + 2y^2 = k$

56. $y = ax^3, \quad x^2 + 3y^2 = b$

57. Show that the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the hyperbola $x^2/A^2 - y^2/B^2 = 1$ are orthogonal trajectories if $A^2 < a^2$ and $a^2 - b^2 = A^2 + B^2$ (so the ellipse and hyperbola have the same foci).

58. Find the value of the number a such that the families of curves $y = (x + c)^{-1}$ and $y = a(x + k)^{1/3}$ are orthogonal trajectories.

59. The *van der Waals equation* for n moles of a gas is

$$\left(P + \frac{n^2a}{V^2}\right)(V - nb) = nRT$$

where P is the pressure, V is the volume, and T is the temperature of the gas. The constant R is the universal gas constant and a and b are positive constants that are characteristic of a particular gas.

- (a) If T remains constant, use implicit differentiation to find dV/dP .
 (b) Find the rate of change of volume with respect to pressure of 1 mole of carbon dioxide at a volume of $V = 10$ L and a pressure of $P = 2.5$ atm. Use $a = 3.592 \text{ L}^2\text{-atm}/\text{mole}^2$ and $b = 0.04267 \text{ L}/\text{mole}$.

60. (a) Use implicit differentiation to find y' if

$$x^2 + xy + y^2 + 1 = 0$$



- (b) Plot the curve in part (a). What do you see? Prove that what you see is correct.
 (c) In view of part (b), what can you say about the expression for y' that you found in part (a)?

61. The equation $x^2 - xy + y^2 = 3$ represents a “rotated ellipse,” that is, an ellipse whose axes are not parallel to the coordinate axes. Find the points at which this ellipse crosses the x -axis and show that the tangent lines at these points are parallel.

62. (a) Where does the normal line to the ellipse $x^2 - xy + y^2 = 3$ at the point $(-1, 1)$ intersect the ellipse a second time?



- (b) Illustrate part (a) by graphing the ellipse and the normal line.

63. Find all points on the curve $x^2y^2 + xy = 2$ where the slope of the tangent line is -1 .

64. Find equations of both the tangent lines to the ellipse $x^2 + 4y^2 = 36$ that pass through the point $(12, 3)$.

65. Use implicit differentiation to find dy/dx for the equation

$$\frac{x}{y} = y^2 + 1 \quad y \neq 0$$

and for the equivalent equation

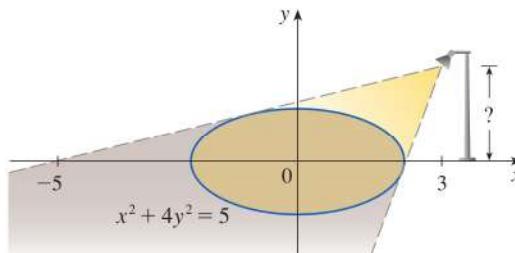
$$x = y^3 + y \quad y \neq 0$$

Show that although the expressions you get for dy/dx look different, they agree for all points that satisfy the given equation.

66. The *Bessel function* of order 0, $y = J(x)$, satisfies the differential equation $xy'' + y' + xy = 0$ for all values of x and its value at 0 is $J(0) = 1$.

- (a) Find $J'(0)$.
 (b) Use implicit differentiation to find $J''(0)$.

67. The figure shows a lamp located three units to the right of the y -axis and a shadow created by the elliptical region $x^2 + 4y^2 \leq 5$. If the point $(-5, 0)$ is on the edge of the shadow, how far above the x -axis is the lamp located?



DISCOVERY PROJECT**FAMILIES OF IMPLICIT CURVES**

In this project you will explore the changing shapes of implicitly defined curves as you vary the constants in a family, and determine which features are common to all members of the family.

1. Consider the family of curves

$$y^2 - 2x^2(x + 8) = c[(y + 1)^2(y + 9) - x^2]$$

- By graphing the curves with $c = 0$ and $c = 2$, determine how many points of intersection there are. (You might have to zoom in to find all of them.)
- Now add the curves with $c = 5$ and $c = 10$ to your graphs in part (a). What do you notice? What about other values of c ?

2. (a) Graph several members of the family of curves

$$x^2 + y^2 + cx^2y^2 = 1$$

Describe how the graph changes as you change the value of c .

- What happens to the curve when $c = -1$? Describe what appears on the screen. Can you prove it algebraically?
- Find y' by implicit differentiation. For the case $c = -1$, is your expression for y' consistent with what you discovered in part (b)?

3.6 | Derivatives of Logarithmic and Inverse Trigonometric Functions

In this section we use implicit differentiation to find derivatives of logarithmic functions and of inverse trigonometric functions.

■ Derivatives of Logarithmic Functions

In Appendix F we prove that if f is a one-to-one differentiable function, then its inverse function f^{-1} is also differentiable, except where its tangents are vertical. This is plausible because, geometrically, we can think of a differentiable function as one whose graph has no corner or cusp. We obtain the graph of f^{-1} by reflecting the graph of f about the line $y = x$, so the graph of f^{-1} has no corner or cusp either. (Note that if f has a horizontal tangent at a point, then f^{-1} has a vertical tangent at the corresponding reflected point and so f^{-1} is not differentiable there.)

Because the logarithmic function $y = \log_b x$ is the inverse of the exponential function $y = b^x$, which we know is differentiable from Section 3.1, it follows that the logarithmic function is also differentiable. We now state and prove the formula for the derivative of a logarithmic function.

1

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$$

PROOF Let $y = \log_b x$. Then

$$b^y = x$$

Formula 3.4.5 says that

$$\frac{d}{dx}(b^x) = b^x \ln b$$

Differentiating this equation implicitly with respect to x , and using Formula 3.4.5, we get

$$(b^y \ln b) \frac{dy}{dx} = 1$$

and so

$$\frac{dy}{dx} = \frac{1}{b^y \ln b} = \frac{1}{x \ln b}$$

If we put $b = e$ in Formula 1, then the factor $\ln b$ on the right side becomes $\ln e = 1$ and we get the formula for the derivative of the natural logarithmic function $\log_e x = \ln x$:

2

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

By comparing Formulas 1 and 2, we see one of the main reasons that natural logarithms (logarithms with base e) are used in calculus: the differentiation formula is simplest when $b = e$ because $\ln e = 1$.

EXAMPLE 1 Differentiate $y = \ln(x^3 + 1)$.

SOLUTION To use the Chain Rule, we let $u = x^3 + 1$. Then $y = \ln u$, so

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{1}{u} \frac{du}{dx} = \frac{1}{x^3 + 1} (3x^2) = \frac{3x^2}{x^3 + 1}$$

In general, if we combine Formula 2 with the Chain Rule as in Example 1, we get

3

$$\frac{d}{dx}(\ln u) = \frac{1}{u} \frac{du}{dx}$$

or

$$\frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$$

EXAMPLE 2 Find $\frac{d}{dx} \ln(\sin x)$.

SOLUTION Using (3), we have

$$\frac{d}{dx} \ln(\sin x) = \frac{1}{\sin x} \frac{d}{dx}(\sin x) = \frac{1}{\sin x} \cos x = \cot x$$

EXAMPLE 3 Differentiate $f(x) = \sqrt{\ln x}$.

SOLUTION This time the logarithm is the inner function, so the Chain Rule gives

$$f'(x) = \frac{1}{2}(\ln x)^{-1/2} \frac{d}{dx}(\ln x) = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$$

EXAMPLE 4 Differentiate $f(x) = \log_{10}(2 + \sin x)$.

SOLUTION Using Formula 1 with $b = 10$, we have

$$\begin{aligned} f'(x) &= \frac{d}{dx} \log_{10}(2 + \sin x) \\ &= \frac{1}{(2 + \sin x) \ln 10} \frac{d}{dx} (2 + \sin x) \\ &= \frac{\cos x}{(2 + \sin x) \ln 10} \end{aligned}$$

Figure 1 shows the graph of the function f of Example 5 together with the graph of its derivative. It gives a visual check on our calculation. Notice that $f'(x)$ is large negative when f is decreasing rapidly.

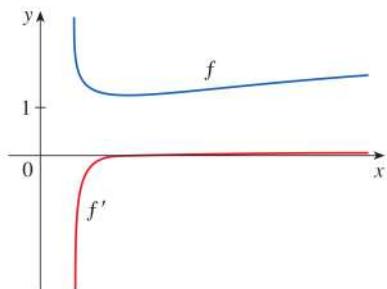


FIGURE 1

Figure 2 shows the graph of the function $f(x) = \ln|x|$ in Example 6 and its derivative $f'(x) = 1/x$. Notice that when x is small, the graph of $y = \ln|x|$ is steep and so $f'(x)$ is large (positive or negative).

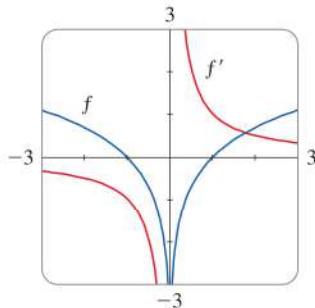


FIGURE 2

EXAMPLE 5 Find $\frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}}$.

SOLUTION 1

$$\begin{aligned} \frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}} &= \frac{1}{x+1} \frac{d}{dx} \frac{x+1}{\sqrt{x-2}} \\ &= \frac{\sqrt{x-2}}{x+1} \frac{\sqrt{x-2} \cdot 1 - (x+1)\left(\frac{1}{2}\right)(x-2)^{-1/2}}{x-2} \\ &= \frac{x-2 - \frac{1}{2}(x+1)}{(x+1)(x-2)} \\ &= \frac{x-5}{2(x+1)(x-2)} \end{aligned}$$

SOLUTION 2 If we first expand the given function using the laws of logarithms, then the differentiation becomes easier:

$$\begin{aligned} \frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}} &= \frac{d}{dx} [\ln(x+1) - \frac{1}{2} \ln(x-2)] \\ &= \frac{1}{x+1} - \frac{1}{2} \left(\frac{1}{x-2} \right) \end{aligned}$$

(This answer can be left as written, but if we used a common denominator we would see that it gives the same answer as in Solution 1.)

EXAMPLE 6 Find $f'(x)$ if $f(x) = \ln|x|$.

SOLUTION Since

$$f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

it follows that

$$f'(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ \frac{1}{-x}(-1) = \frac{1}{x} & \text{if } x < 0 \end{cases}$$

Thus $f'(x) = 1/x$ for all $x \neq 0$.

The result of Example 6 is worth remembering:

4

$$\frac{d}{dx} \ln|x| = \frac{1}{x}$$

■ Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called **logarithmic differentiation**.

EXAMPLE 7 Differentiate $y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

SOLUTION We take logarithms of both sides of the equation and use the Laws of Logarithms to simplify:

$$\ln y = \frac{3}{4} \ln x + \frac{1}{2} \ln(x^2 + 1) - 5 \ln(3x + 2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y} \frac{dy}{dx} = \frac{3}{4} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{2x}{x^2 + 1} - 5 \cdot \frac{3}{3x + 2}$$

Solving for dy/dx , we get

$$\frac{dy}{dx} = y \left(\frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2} \right)$$

If we hadn't used logarithmic differentiation in Example 7, we would have had to use both the Quotient Rule and the Product Rule. The resulting calculation would have been horrendous.

Because we have an explicit expression for y , we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5} \left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2} \right)$$

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to expand the expression.
2. Differentiate implicitly with respect to x .
3. Solve the resulting equation for y' and replace y by $f(x)$.

If $f(x) < 0$ for some values of x , then $\ln f(x)$ is not defined, but we can still use logarithmic differentiation by first writing $|y| = |f(x)|$ and then using Equation 4. We illustrate this procedure by proving the general version of the Power Rule, as promised in Section 3.1. Recall that the general version of the Power Rule states that if n is any real number and $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

PROOF OF THE POWER RULE (GENERAL VERSION) Let $y = x^n$ and use logarithmic differentiation:

$$\ln|y| = \ln|x|^n = n \ln|x| \quad x \neq 0$$

Therefore

$$\frac{y'}{y} = \frac{n}{x}$$

Hence

$$y' = n \frac{y}{x} = n \frac{x^n}{x} = nx^{n-1}$$

If $x = 0$, we can show that $f'(0) = 0$ for $n > 1$ directly from the definition of a derivative.

☐ You should distinguish carefully between the Power Rule $[(x^n)' = nx^{n-1}]$, where the base is variable and the exponent is constant, and the rule for differentiating exponential functions $[(b^x)' = b^x \ln b]$, where the base is constant and the exponent is variable.

In general there are four cases for exponents and bases:

Constant base, constant exponent

$$1. \frac{d}{dx}(b^n) = 0 \quad (b \text{ and } n \text{ are constants})$$

Variable base, constant exponent

$$2. \frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}f'(x)$$

Constant base, variable exponent

$$3. \frac{d}{dx}[b^{g(x)}] = b^{g(x)}(\ln b)g'(x)$$

Variable base, variable exponent

4. To find $(d/dx)[f(x)]^{g(x)}$, logarithmic differentiation can be used, as in the next example.

EXAMPLE 8 Differentiate $y = x^{\sqrt{x}}$.

SOLUTION 1 Since both the base and the exponent are variable, we use logarithmic differentiation:

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x} \ln x$$

$$\frac{y'}{y} = \sqrt{x} \cdot \frac{1}{x} + (\ln x) \frac{1}{2\sqrt{x}}$$

$$y' = y \left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}} \right) = x^{\sqrt{x}} \left(\frac{2 + \ln x}{2\sqrt{x}} \right)$$

SOLUTION 2 Another method is to use Equation 1.5.10 to write $x^{\sqrt{x}} = e^{\sqrt{x} \ln x}$:

$$\frac{d}{dx}(x^{\sqrt{x}}) = \frac{d}{dx}(e^{\sqrt{x} \ln x}) = e^{\sqrt{x} \ln x} \frac{d}{dx}(\sqrt{x} \ln x)$$

$$= x^{\sqrt{x}} \left(\frac{2 + \ln x}{2\sqrt{x}} \right) \quad (\text{as in Solution 1})$$

Figure 3 illustrates Example 8 by showing the graphs of $f(x) = x^{\sqrt{x}}$ and its derivative.

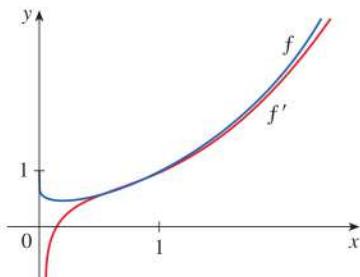


FIGURE 3

The Number e as a Limit

We have shown that if $f(x) = \ln x$, then $f'(x) = 1/x$. Thus $f'(1) = 1$. We now use this fact to express the number e as a limit.

From the definition of a derivative as a limit, we have

$$\begin{aligned} f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \rightarrow 0} \frac{f(1+x) - f(1)}{x} \\ &= \lim_{x \rightarrow 0} \frac{\ln(1+x) - \ln 1}{x} = \lim_{x \rightarrow 0} \frac{1}{x} \ln(1+x) \\ &= \lim_{x \rightarrow 0} \ln(1+x)^{1/x} \end{aligned}$$

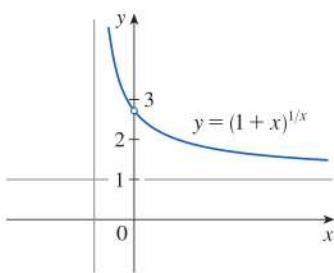


FIGURE 4

x	$(1 + x)^{1/x}$
0.1	2.59374246
0.01	2.70481383
0.001	2.71692393
0.0001	2.71814593
0.00001	2.71826824
0.000001	2.71828047
0.0000001	2.71828169
0.00000001	2.71828181

Because $f'(1) = 1$, we have

$$\lim_{x \rightarrow 0} \ln(1 + x)^{1/x} = 1$$

Then, by Theorem 2.5.8 and the continuity of the exponential function, we have

$$e = e^1 = e^{\lim_{x \rightarrow 0} \ln(1 + x)^{1/x}} = \lim_{x \rightarrow 0} e^{\ln(1 + x)^{1/x}} = \lim_{x \rightarrow 0} (1 + x)^{1/x}$$

5

$$e = \lim_{x \rightarrow 0} (1 + x)^{1/x}$$

Formula 5 is illustrated by the graph of the function $y = (1 + x)^{1/x}$ in Figure 4 and a table of values for small values of x . This illustrates the fact that, correct to seven decimal places,

$$e \approx 2.7182818$$

If we put $n = 1/x$ in Formula 5, then $n \rightarrow \infty$ as $x \rightarrow 0^+$ and so an alternative expression for e is

6

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

Derivatives of Inverse Trigonometric Functions

The inverse trigonometric functions were reviewed in Section 1.5. We discussed their continuity in Section 2.5 and their asymptotes in Section 2.6. Here we use implicit differentiation to find their derivatives. At the beginning of this section we observed that if f is a one-to-one differentiable function, then its inverse function f^{-1} is also differentiable (except where its tangents are vertical). Because the trigonometric functions—with the restricted domains that we used to define their inverses—are one-to-one and differentiable, it follows that the inverse trigonometric functions are also differentiable.

Recall the definition of the arcsine function:

$$y = \sin^{-1} x \quad \text{means} \quad \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

Differentiating $\sin y = x$ implicitly with respect to x , we obtain

$$\cos y \frac{dy}{dx} = 1 \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\cos y}$$

Now $\cos y \geq 0$ because $-\pi/2 \leq y \leq \pi/2$, so

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2} \quad (\cos^2 y + \sin^2 y = 1)$$

Therefore

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$$

Figure 5 shows the graph of $f(x) = \tan^{-1}x$ and its derivative $f'(x) = 1/(1+x^2)$. Notice that f is increasing and $f'(x)$ is always positive. The fact that $\tan^{-1}x \rightarrow \pm\pi/2$ as $x \rightarrow \pm\infty$ is reflected in the fact that $f'(x) \rightarrow 0$ as $x \rightarrow \pm\infty$.

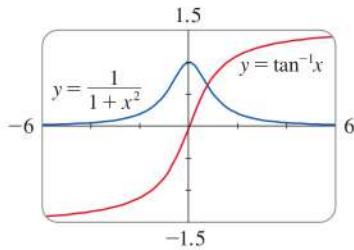


FIGURE 5

The formulas for the derivatives of $\csc^{-1}x$ and $\sec^{-1}x$ depend on the definitions that are used for these functions. See Exercise 82.

The formula for the derivative of the arctangent function is derived in a similar way. If $y = \tan^{-1}x$, then $\tan y = x$. Differentiating $\tan y = x$ implicitly with respect to x , we have

$$\sec^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$

The inverse trigonometric functions $\sin^{-1}x$ and $\tan^{-1}x$ occur most frequently. The derivatives of the remaining four are given in the following table. The proofs of the formulas are left as exercises.

Derivatives of Inverse Trigonometric Functions

$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$
$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$
$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$	$\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$

EXAMPLE 9 Differentiate (a) $y = \frac{1}{\sin^{-1}x}$ and (b) $f(x) = x \arctan \sqrt{x}$.

SOLUTION

$$(a) \quad \frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x)^{-1} = -(\sin^{-1}x)^{-2} \frac{d}{dx}(\sin^{-1}x)$$

$$= -\frac{1}{(\sin^{-1}x)^2 \sqrt{1-x^2}}$$

$$(b) \quad f'(x) = x \frac{1}{1+(\sqrt{x})^2} \left(\frac{1}{2}x^{-1/2} \right) + \arctan \sqrt{x}$$

$$= \frac{\sqrt{x}}{2(1+x)} + \arctan \sqrt{x}$$

Recall that $\arctan x$ is an alternative notation for $\tan^{-1}x$.

EXAMPLE 10 Differentiate $g(x) = \sec^{-1}(x^2)$.

SOLUTION

$$g'(x) = \frac{1}{x^2 \sqrt{(x^2)^2 - 1}} (2x) = \frac{2}{x \sqrt{x^4 - 1}}$$

3.6 Exercises

1. Explain why the natural logarithmic function $y = \ln x$ is used much more frequently in calculus than the other logarithmic functions $y = \log_b x$.

2–26 Differentiate the function.

2. $g(t) = \ln(3 + t^2)$

3. $f(x) = \ln(x^2 + 3x + 5)$

5. $f(x) = \sin(\ln x)$

7. $f(x) = \ln \frac{1}{x}$

9. $g(x) = \ln(xe^{-2x})$

11. $F(t) = (\ln t)^2 \sin t$

13. $y = \log_8(x^2 + 3x)$

15. $F(s) = \ln \ln s$

17. $T(z) = 2^z \log_2 z$

19. $y = \ln |3 - 2x^5|$

21. $y = \ln(e^{-x} + xe^{-x})$

23. $h(x) = e^{x^2 + \ln x}$

25. $y = \ln \frac{x^a}{b^x}$

4. $f(x) = x \ln x - x$

6. $f(x) = \ln(\sin^2 x)$

8. $y = \frac{1}{\ln x}$

10. $g(t) = \sqrt{1 + \ln t}$

12. $p(t) = \ln \sqrt{t^2 + 1}$

14. $y = \log_{10} \sec x$

16. $P(v) = \frac{\ln v}{1 - v}$

18. $g(t) = \ln \frac{t(t^2 + 1)^4}{\sqrt[3]{2t - 1}}$

20. $y = \ln(\csc x - \cot x)$

22. $g(x) = e^{x^2 \ln x}$

24. $y = \ln \sqrt{\frac{1+2x}{1-2x}}$

26. $y = \log_2(x \log_5 x)$

27. Show that $\frac{d}{dx} \ln(x + \sqrt{x^2 + 1}) = \frac{1}{\sqrt{x^2 + 1}}$.

28. Show that $\frac{d}{dx} \ln \sqrt{\frac{1 - \cos x}{1 + \cos x}} = \csc x$.

29–32 Find y' and y'' .

29. $y = \sqrt{x} \ln x$

30. $y = \frac{\ln x}{1 + \ln x}$

31. $y = \ln |\sec x|$

32. $y = \ln(1 + \ln x)$

33–36 Differentiate f and find the domain of f .

33. $f(x) = \frac{x}{1 - \ln(x - 1)}$

34. $f(x) = \sqrt{2 + \ln x}$

35. $f(x) = \ln(x^2 - 2x)$

36. $f(x) = \ln \ln \ln x$

37. If $f(x) = \ln(x + \ln x)$, find $f'(1)$.

38. If $f(x) = \cos(\ln x^2)$, find $f'(1)$.

39–40 Find an equation of the tangent line to the curve at the given point.

39. $y = \ln(x^2 - 3x + 1)$, $(3, 0)$

40. $y = x^2 \ln x$, $(1, 0)$

41 If $f(x) = \sin x + \ln x$, find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f' .

42 Find equations of the tangent lines to the curve $y = (\ln x)/x$ at the points $(1, 0)$ and $(e, 1/e)$. Illustrate by graphing the curve and its tangent lines.

43 Let $f(x) = cx + \ln(\cos x)$. For what value of c is $f'(\pi/4) = 6$?

44 Let $f(x) = \log_b(3x^2 - 2)$. For what value of b is $f'(1) = 3$?

45–56 Use logarithmic differentiation to find the derivative of the function.

45. $y = (x^2 + 2)^2(x^4 + 4)^4$

46. $y = \frac{e^{-x} \cos^2 x}{x^2 + x + 1}$

47. $y = \sqrt{\frac{x-1}{x^4+1}}$

48. $y = \sqrt{x} e^{x^2-x}(x+1)^{2/3}$

49. $y = x^x$

50. $y = x^{1/x}$

51. $y = x^{\sin x}$

52. $y = (\sqrt{x})^x$

53. $y = (\cos x)^x$

54. $y = (\sin x)^{\ln x}$

55. $y = x^{\ln x}$

56. $y = (\ln x)^{\cos x}$

57 Find y' if $y = \ln(x^2 + y^2)$.

58 Find y' if $x^y = y^x$.

59 Find a formula for $f^{(n)}(x)$ if $f(x) = \ln(x - 1)$.

60. Find $\frac{d^9}{dx^9} (x^8 \ln x)$.

61 Use the definition of a derivative to prove that

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$$

62 Show that $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for any $x > 0$.

63–78 Find the derivative of the function. Simplify where possible.

63. $f(x) = \sin^{-1}(5x)$

64. $g(x) = \sec^{-1}(e^x)$

65. $y = \tan^{-1} \sqrt{x-1}$

66. $y = \tan^{-1}(x^2)$

67. $y = (\tan^{-1} x)^2$

68. $g(x) = \arccos \sqrt{x}$

69. $h(x) = (\arcsin x) \ln x$

70. $g(t) = \ln(\arctan(t^4))$

71. $f(z) = e^{\arcsin(z^2)}$

72. $y = \tan^{-1}(x - \sqrt{1+x^2})$

73. $h(t) = \cot^{-1}(t) + \cot^{-1}(1/t)$

74. $R(t) = \arcsin(1/t)$

75. $y = x \sin^{-1}x + \sqrt{1 - x^2}$

76. $y = \cos^{-1}(\sin^{-1}t)$

77. $y = \tan^{-1}\left(\frac{x}{a}\right) + \ln\sqrt{\frac{x-a}{x+a}}$

78. $y = \arctan\sqrt{\frac{1-x}{1+x}}$

79–80 Find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f' .

79. $f(x) = \sqrt{1 - x^2} \arcsin x$ 80. $f(x) = \arctan(x^2 - x)$

81. Prove the formula for $(d/dx)(\cos^{-1}x)$ by the same method as for $(d/dx)(\sin^{-1}x)$.

82. (a) One way of defining $\sec^{-1}x$ is to say that $y = \sec^{-1}x \iff \sec y = x$ and $0 \leq y < \pi/2$ or $\pi \leq y < 3\pi/2$. Show that, with this definition,

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

(b) Another way of defining $\sec^{-1}x$ that is sometimes used is to say that $y = \sec^{-1}x \iff \sec y = x$ and $0 \leq y \leq \pi$, $y \neq \pi/2$. Show that, with this definition,

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$$

83. **Derivatives of Inverse Functions** Suppose that f is a one-to-one differentiable function and its inverse function f^{-1} is also differentiable. Use implicit differentiation to show that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

provided that the denominator is not 0.

84–86 Use the formula in Exercise 83.

84. If $f(4) = 5$ and $f'(4) = \frac{2}{3}$, find $(f^{-1})'(5)$.

85. If $f(x) = x + e^x$, find $(f^{-1})'(1)$.

86. If $f(x) = x^3 + 3 \sin x + 2 \cos x$, find $(f^{-1})'(2)$.

87. Suppose that f and g are differentiable functions and let $h(x) = f(x)^{g(x)}$. Use logarithmic differentiation to derive the formula

$$h' = g \cdot f^{g-1} \cdot f' + (\ln f) \cdot f^g \cdot g'$$

88. Use the formula in Exercise 87 to find the derivative.

- (a) $h(x) = x^3$ (b) $h(x) = 3^x$ (c) $h(x) = (\sin x)^x$

3.7 | Rates of Change in the Natural and Social Sciences

We know that if $y = f(x)$, then the derivative dy/dx can be interpreted as the rate of change of y with respect to x . In this section we examine some of the applications of this idea to physics, chemistry, biology, economics, and other sciences.

Let's recall from Section 2.7 the basic idea behind rates of change. If x changes from x_1 to x_2 , then the change in x is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is the **average rate of change of y with respect to x** over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line PQ in Figure 1. Its limit as $\Delta x \rightarrow 0$ is the derivative $f'(x_1)$, which can therefore be interpreted as the **instantaneous rate of change of y with respect to x** or the slope of the tangent line at $P(x_1, f(x_1))$. Using Leibniz notation, we write the process in the form

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

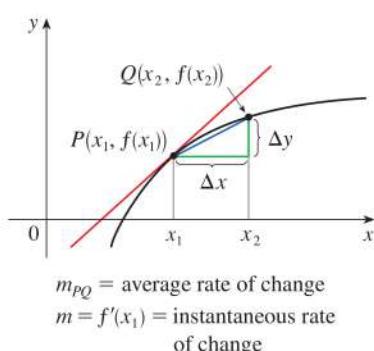


FIGURE 1

Whenever the function $y = f(x)$ has a specific interpretation in one of the sciences, its derivative will have a specific interpretation as a rate of change. (As we discussed in Section 2.7, the units for dy/dx are the units for y divided by the units for x .) We now look at some of these interpretations in the natural and social sciences.

■ Physics

If $s = f(t)$ is the position function of a particle that is moving in a straight line, then $\Delta s/\Delta t$ represents the average velocity over a time period Δt , and $v = ds/dt$ represents the instantaneous **velocity** (the rate of change of displacement with respect to time). The instantaneous rate of change of velocity with respect to time is **acceleration**: $a(t) = v'(t) = s''(t)$. This was discussed in Sections 2.7 and 2.8, but now that we know the differentiation formulas, we are able to more easily solve problems involving the motion of objects.

EXAMPLE 1 The position of a particle is given by the equation

$$s = f(t) = t^3 - 6t^2 + 9t$$

where t is measured in seconds and s in meters.

- (a) Find the velocity at time t .
- (b) What is the velocity after 2 s? After 4 s?
- (c) When is the particle at rest?
- (d) When is the particle moving forward (that is, in the positive direction)?
- (e) Draw a diagram to represent the motion of the particle.
- (f) Find the total distance traveled by the particle during the first five seconds.
- (g) Find the acceleration at time t and after 4 s.
- (h) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 5$.
- (i) When is the particle speeding up? When is it slowing down?

SOLUTION

- (a) The velocity function is the derivative of the position function:

$$s = f(t) = t^3 - 6t^2 + 9t$$

$$v(t) = \frac{ds}{dt} = 3t^2 - 12t + 9$$

- (b) The velocity after 2 s means the instantaneous velocity when $t = 2$, that is,

$$v(2) = \left. \frac{ds}{dt} \right|_{t=2} = 3(2)^2 - 12(2) + 9 = -3 \text{ m/s}$$

The velocity after 4 s is

$$v(4) = 3(4)^2 - 12(4) + 9 = 9 \text{ m/s}$$

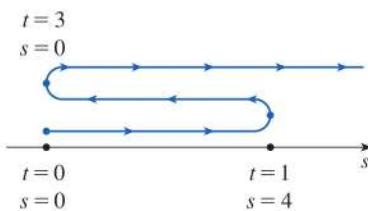
- (c) The particle is at rest when $v(t) = 0$, that is,

$$3t^2 - 12t + 9 = 3(t^2 - 4t + 3) = 3(t - 1)(t - 3) = 0$$

and this is true when $t = 1$ or $t = 3$. Thus the particle is at rest after 1 s and after 3 s.

- (d) The particle moves in the positive direction when $v(t) > 0$, that is,

$$3t^2 - 12t + 9 = 3(t - 1)(t - 3) > 0$$

**FIGURE 2**

This inequality is true when both factors are positive ($t > 3$) or when both factors are negative ($t < 1$). Thus the particle moves in the positive direction in the time intervals $t < 1$ and $t > 3$. It moves backward (in the negative direction) when $1 < t < 3$.

- (e) Using the information from part (d) we make a schematic sketch in Figure 2 of the motion of the particle back and forth along a line (the s -axis).
(f) Because of what we learned in parts (d) and (e), we need to calculate the distances traveled during the time intervals $[0, 1]$, $[1, 3]$, and $[3, 5]$ separately.

The distance traveled in the first second is

$$|f(1) - f(0)| = |4 - 0| = 4 \text{ m}$$

From $t = 1$ to $t = 3$ the distance traveled is

$$|f(3) - f(1)| = |0 - 4| = 4 \text{ m}$$

From $t = 3$ to $t = 5$ the distance traveled is

$$|f(5) - f(3)| = |20 - 0| = 20 \text{ m}$$

The total distance is $4 + 4 + 20 = 28 \text{ m}$.

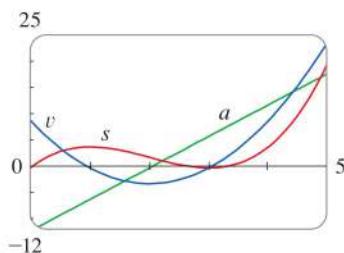
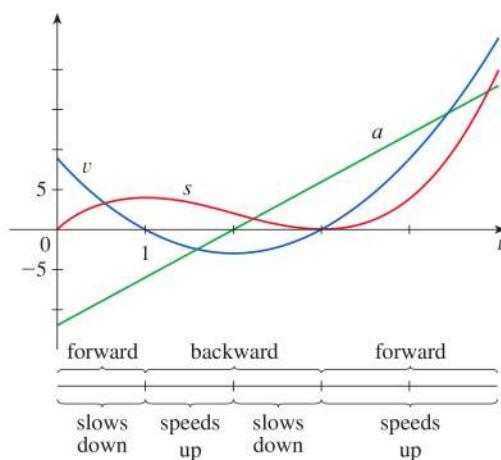
- (g) The acceleration is the derivative of the velocity function:

$$a(t) = \frac{d^2s}{dt^2} = \frac{dv}{dt} = 6t - 12$$

$$a(4) = 6(4) - 12 = 12 \text{ m/s}^2$$

- (h) Figure 3 shows the graphs of s , v , and a .

- (i) The particle speeds up when the velocity is positive and increasing (v and a are both positive) and also when the velocity is negative and decreasing (v and a are both negative). In other words, the particle speeds up when the velocity and acceleration have the same sign. (The particle is pushed in the same direction it is moving.) From Figure 3 we see that this happens when $1 < t < 2$ and when $t > 3$. The particle slows down when v and a have opposite signs, that is, when $0 \leq t < 1$ and when $2 < t < 3$. Figure 4 summarizes the motion of the particle.

**FIGURE 3****FIGURE 4**

EXAMPLE 2 If a rod or piece of wire is homogeneous, then its *linear density* is uniform and is defined as the mass per unit length ($\rho = m/l$) and measured in kilograms per meter. Suppose, however, that the rod is not homogeneous but that its mass measured from its left end to a point x is $m = f(x)$, as shown in Figure 5.



FIGURE 5

The mass of the part of the rod that lies between $x = x_1$ and $x = x_2$ is given by $\Delta m = f(x_2) - f(x_1)$, so the average density of that part of the rod is

$$\text{average density} = \frac{\Delta m}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

If we now let $\Delta x \rightarrow 0$ (that is, $x_2 \rightarrow x_1$), we are computing the average density over smaller and smaller intervals. The **linear density** ρ at x_1 is the limit of these average densities as $\Delta x \rightarrow 0$; that is, the linear density is the rate of change of mass with respect to length. Symbolically,

$$\rho = \lim_{\Delta x \rightarrow 0} \frac{\Delta m}{\Delta x} = \frac{dm}{dx}$$

Thus the linear density of the rod is the derivative of mass with respect to length.

For instance, if $m = f(x) = \sqrt{x}$, where x is measured in meters and m in kilograms, then the average density of the part of the rod given by $1 \leq x \leq 1.2$ is

$$\frac{\Delta m}{\Delta x} = \frac{f(1.2) - f(1)}{1.2 - 1} = \frac{\sqrt{1.2} - 1}{0.2} \approx 0.48 \text{ kg/m}$$

while the density right at $x = 1$ is

$$\rho = \left. \frac{dm}{dx} \right|_{x=1} = \left. \frac{1}{2\sqrt{x}} \right|_{x=1} = 0.50 \text{ kg/m}$$

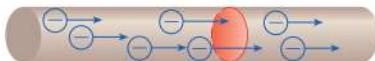


FIGURE 6

EXAMPLE 3 A current exists whenever electric charges move. Figure 6 shows part of a wire and electrons moving through a plane surface, shaded red. If ΔQ is the net charge that passes through this surface during a time period Δt , then the average current during this time interval is defined as

$$\text{average current} = \frac{\Delta Q}{\Delta t} = \frac{Q_2 - Q_1}{t_2 - t_1}$$

If we take the limit of this average current over smaller and smaller time intervals, we get what is called the **current** I at a given time t_1 :

$$I = \lim_{\Delta t \rightarrow 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$$

Thus the current is the rate at which charge flows through a surface. It is measured in units of charge per unit time (often coulombs per second, called amperes).

Velocity, density, and current are not the only rates of change that are important in physics. Others include power (the rate at which work is done), the rate of heat flow, temperature gradient (the rate of change of temperature with respect to position), and the rate of decay of a radioactive substance in nuclear physics.

■ Chemistry

EXAMPLE 4 A chemical reaction results in the formation of one or more substances (called *products*) from one or more starting materials (called *reactants*). For instance, the “equation”



indicates that two molecules of hydrogen and one molecule of oxygen form two molecules of water. Let’s consider the reaction



where A and B are the reactants and C is the product. The **concentration** of a reactant A is the number of moles (1 mole = 6.022×10^{23} molecules) per liter and is denoted by [A]. The concentration varies during a reaction, so [A], [B], and [C] are all functions of time (t). The average rate of reaction of the product C over a time interval $t_1 \leq t \leq t_2$ is

$$\frac{\Delta[\text{C}]}{\Delta t} = \frac{[\text{C}](t_2) - [\text{C}](t_1)}{t_2 - t_1}$$

But chemists are interested in the **instantaneous rate of reaction** because it gives information about the mechanism of the chemical reaction. The instantaneous rate of reaction is obtained by taking the limit of the average rate of reaction as the time interval Δt approaches 0:

$$\text{rate of reaction} = \lim_{\Delta t \rightarrow 0} \frac{\Delta[\text{C}]}{\Delta t} = \frac{d[\text{C}]}{dt}$$

Since the concentration of the product increases as the reaction proceeds, the derivative $d[\text{C}]/dt$ will be positive, and so the rate of reaction of C is positive. The concentrations of the reactants, however, decrease during the reaction, so, to make the rates of reaction of A and B positive numbers, we put minus signs in front of the derivatives $d[\text{A}]/dt$ and $d[\text{B}]/dt$. Since [A] and [B] each decrease at the same rate that [C] increases, we have

$$\text{rate of reaction} = \frac{d[\text{C}]}{dt} = -\frac{d[\text{A}]}{dt} = -\frac{d[\text{B}]}{dt}$$

More generally, it turns out that for a reaction of the form



we have

$$-\frac{1}{a} \frac{d[\text{A}]}{dt} = -\frac{1}{b} \frac{d[\text{B}]}{dt} = \frac{1}{c} \frac{d[\text{C}]}{dt} = \frac{1}{d} \frac{d[\text{D}]}{dt}$$

The rate of reaction can be determined from data and graphical methods. In some cases there are explicit formulas for the concentrations as functions of time that enable us to compute the rate of reaction (see Exercise 26). ■

EXAMPLE 5 One of the quantities of interest in thermodynamics is compressibility. If a given substance is kept at a constant temperature, then its volume V depends on its pressure P . We can consider the rate of change of volume with respect to pressure—namely, the derivative dV/dP . As P increases, V decreases, so $dV/dP < 0$. The **compressibility** is defined by introducing a minus sign and dividing this derivative by the volume V :

$$\text{isothermal compressibility} = \beta = -\frac{1}{V} \frac{dV}{dP}$$

Thus β measures how fast, per unit volume, the volume of a substance decreases as the pressure on it increases at constant temperature.

For instance, the volume V (in cubic meters) of a sample of air at 25°C was found to be related to the pressure P (in kilopascals) by the equation

$$V = \frac{5.3}{P}$$

The rate of change of V with respect to P when $P = 50$ kPa is

$$\begin{aligned} \left. \frac{dV}{dP} \right|_{P=50} &= -\left. \frac{5.3}{P^2} \right|_{P=50} \\ &= -\frac{5.3}{2500} = -0.00212 \text{ m}^3/\text{kPa} \end{aligned}$$

The compressibility at that pressure is

$$\beta = -\left. \frac{1}{V} \frac{dV}{dP} \right|_{P=50} = \frac{0.00212}{\frac{5.3}{50}} = 0.02 \text{ (m}^3/\text{kPa})/\text{m}^3$$

Biology

EXAMPLE 6 Let $n = f(t)$ be the number of individuals in an animal or plant population at time t . The change in the population size between the times $t = t_1$ and $t = t_2$ is $\Delta n = f(t_2) - f(t_1)$, and so the average rate of growth during the time period $t_1 \leq t \leq t_2$ is

$$\text{average rate of growth} = \frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

The **instantaneous rate of growth** is obtained from this average rate of growth by letting the time period Δt approach 0:

$$\text{growth rate} = \lim_{\Delta t \rightarrow 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}$$

Strictly speaking, this is not quite accurate because the actual graph of a population function $n = f(t)$ would be a step function that is discontinuous whenever a birth or death occurs and therefore not differentiable. However, for a large animal

or plant population, we can replace the graph by a smooth approximating curve as in Figure 7.

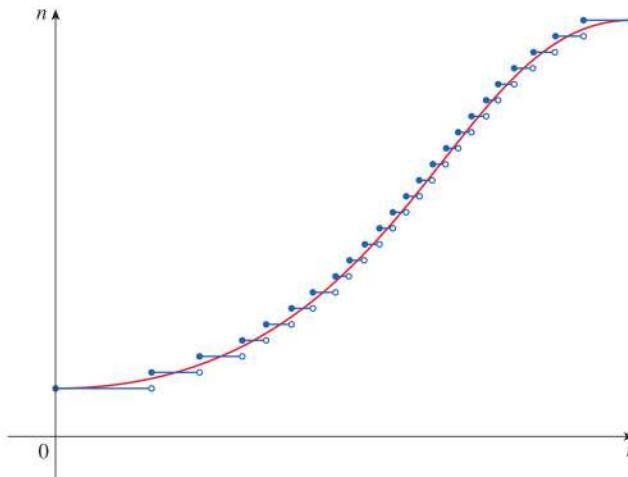


FIGURE 7

A smooth curve approximating a growth function

To be more specific, consider a population of bacteria in a homogeneous nutrient medium. Suppose that by sampling the population at certain intervals it is determined that the population doubles every hour. If the initial population is n_0 and the time t is measured in hours, then

$$f(1) = 2f(0) = 2n_0$$

$$f(2) = 2f(1) = 2^2 n_0$$

$$f(3) = 2f(2) = 2^3 n_0$$

and, in general,

$$f(t) = 2^t n_0$$

The population function is $n = n_0 2^t$.

In Section 3.4 we showed that

$$\frac{d}{dx} (b^x) = b^x \ln b$$

So the rate of growth of the bacteria population at time t is

$$\frac{dn}{dt} = \frac{d}{dt} (n_0 2^t) = n_0 2^t \ln 2$$

For example, suppose that we start with an initial population of $n_0 = 100$ bacteria. Then the rate of growth after 4 hours is

$$\left. \frac{dn}{dt} \right|_{t=4} = 100 \cdot 2^4 \ln 2 = 1600 \ln 2 \approx 1109$$

This means that, after 4 hours, the bacteria population is growing at a rate of about 1109 bacteria per hour. ■

Science Photo Library / Alamy Stock Photo



E. coli bacteria measure about 2 micrometers (μm) long and $0.75 \mu\text{m}$ wide. The image was produced with a scanning electron microscope.

EXAMPLE 7 When we consider the flow of blood through a blood vessel, such as a vein or artery, we can model the shape of the blood vessel by a cylindrical tube with radius R and length l as illustrated in Figure 8.

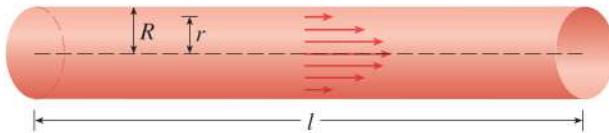


FIGURE 8

Blood flow in an artery

Because of friction at the walls of the tube, the velocity v of the blood is greatest along the central axis of the tube and decreases as the distance r from the axis increases until v becomes 0 at the wall. The relationship between v and r is given by the **law of laminar flow**, which was experimentally derived by the French physicist Jean Léonard Marie Poiseuille in 1838. This law states that

$$\boxed{1} \quad v = \frac{P}{4\eta l} (R^2 - r^2)$$

For more detailed information, see W. Nichols, M. O'Rourke, and C. Vlachopoulos (eds.), *McDonald's Blood Flow in Arteries: Theoretical, Experimental, and Clinical Principles*, 6th ed. (Boca Raton, FL, 2011).

where η is the viscosity of the blood and P is the pressure difference between the ends of the tube. If P and l are constant, then v is a function of r with domain $[0, R]$.

The average rate of change of the velocity as we move from $r = r_1$ outward to $r = r_2$ is given by

$$\frac{\Delta v}{\Delta r} = \frac{v(r_2) - v(r_1)}{r_2 - r_1}$$

and if we let $\Delta r \rightarrow 0$, we obtain the **velocity gradient**, that is, the instantaneous rate of change of velocity with respect to r :

$$\text{velocity gradient} = \lim_{\Delta r \rightarrow 0} \frac{\Delta v}{\Delta r} = \frac{dv}{dr}$$

Using Equation 1, we obtain

$$\frac{dv}{dr} = \frac{P}{4\eta l} (0 - 2r) = -\frac{Pr}{2\eta l}$$

For one of the smaller human arteries we can take $\eta = 0.027$, $R = 0.008$ cm, $l = 2$ cm, and $P = 4000$ dynes/cm², which gives

$$\begin{aligned} v &= \frac{4000}{4(0.027)^2} (0.000064 - r^2) \\ &\approx 1.85 \times 10^4 (6.4 \times 10^{-6} - r^2) \end{aligned}$$

At $r = 0.002$ cm the blood is flowing at a speed of

$$v(0.002) \approx 1.85 \times 10^4 (64 \times 10^{-6} - 4 \times 10^{-6}) = 1.11 \text{ cm/s}$$

and the velocity gradient at that point is

$$\left. \frac{dv}{dr} \right|_{r=0.002} = -\frac{4000(0.002)}{2(0.027)^2} \approx -74 \text{ (cm/s)/cm}$$

To get a feeling for what this statement means, let's change our units from centimeters

to micrometers ($1 \text{ cm} = 10,000 \mu\text{m}$). Then the radius of the artery is $80 \mu\text{m}$. The velocity at the central axis is $11,850 \mu\text{m/s}$, which decreases to $11,110 \mu\text{m/s}$ at a distance of $r = 20 \mu\text{m}$. The fact that $dv/dr = -74 (\mu\text{m/s})/\mu\text{m}$ means that, when $r = 20 \mu\text{m}$, the velocity is decreasing at a rate of about $74 \mu\text{m/s}$ for each micrometer that we proceed away from the center. ■

Economics

EXAMPLE 8 Suppose $C(x)$ is the total cost that a company incurs in producing x units of a certain commodity. The function C is called a **cost function**. If the number of items produced is increased from x_1 to x_2 , then the additional cost is $\Delta C = C(x_2) - C(x_1)$, and the average rate of change of the cost is

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1} = \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}$$

The limit of this quantity as $\Delta x \rightarrow 0$, that is, the instantaneous rate of change of cost with respect to the number of items produced, is called the **marginal cost** by economists:

$$\text{marginal cost} = \lim_{\Delta x \rightarrow 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}$$

[Since x often takes on only integer values, it may not make literal sense to let Δx approach 0, but we can always replace $C(x)$ by a smooth approximating function as in Example 6.]

Taking $\Delta x = 1$ and n large (so that Δx is small compared to n), we have

$$C'(n) \approx C(n + 1) - C(n)$$

Thus the marginal cost of producing n units is approximately equal to the cost of producing one more unit [the $(n + 1)$ st unit].

It is often appropriate to represent a total cost function by a polynomial

$$C(x) = a + bx + cx^2 + dx^3$$

where a represents the overhead cost (rent, heat, maintenance) and the other terms represent the cost of raw materials, labor, and so on. (The cost of raw materials may be proportional to x , but labor costs might depend partly on higher powers of x because of overtime costs and inefficiencies involved in large-scale operations.)

For instance, suppose a company has estimated that the cost (in dollars) of producing x items is

$$C(x) = 10,000 + 5x + 0.01x^2$$

Then the marginal cost function is

$$C'(x) = 5 + 0.02x$$

The marginal cost at the production level of 500 items is

$$C'(500) = 5 + 0.02(500) = \$15/\text{item}$$

This gives the rate at which costs are increasing with respect to the production level when $x = 500$ and predicts the cost of the 501st item.

The actual cost of producing the 501st item is

$$\begin{aligned} C(501) - C(500) &= [10,000 + 5(501) + 0.01(501)^2] \\ &\quad - [10,000 + 5(500) + 0.01(500)^2] \\ &= \$15.01 \end{aligned}$$

Notice that $C'(500) \approx C(501) - C(500)$. ■

Economists also study marginal demand, marginal revenue, and marginal profit, the derivatives of the demand, revenue, and profit functions. These will be considered in Chapter 4 after we have developed techniques for finding the maximum and minimum values of functions.

■ Other Sciences

Rates of change occur in all the sciences. A geologist is interested in knowing the rate at which an intruded body of molten rock cools by conduction of heat into surrounding rocks. An engineer wants to know the rate at which water flows out of a reservoir. An urban geographer is interested in the rate of change of the population density in a city as the distance from the city center increases. A meteorologist is concerned with the rate of change of atmospheric pressure with respect to height (see Exercise 3.8.19).

In psychology, those interested in learning theory study the learning curve, which graphs the performance $P(t)$ of someone learning a skill as a function of the training time t . Of particular interest is the rate at which performance improves as time passes, that is, dP/dt . Psychologists have also studied the phenomenon of memory and have developed models for the rate of memory retention (see Exercise 42). They also study the difficulty involved in performing certain tasks and the rate at which difficulty increases when a given parameter is changed (see Exercise 43).

In sociology, differential calculus is used in analyzing the spread of rumors (or innovations or fads or fashions). If $p(t)$ denotes the proportion of a population that knows a rumor by time t , then the derivative dp/dt represents the rate of spread of the rumor (see Exercise 3.4.90).

■ A Single Idea, Many Interpretations

Velocity, density, current, power, and temperature gradient in physics; rate of reaction and compressibility in chemistry; rate of growth and blood velocity gradient in biology; marginal cost and marginal profit in economics; rate of heat flow in geology; rate of improvement of performance in psychology; rate of spread of a rumor in sociology—these are all special cases of a single mathematical concept, the derivative.

All of these different applications of the derivative illustrate the fact that part of the power of mathematics lies in its abstractness. A single abstract mathematical concept (such as the derivative) can have different interpretations in each of the sciences. When we develop the properties of the mathematical concept once and for all, we can then turn around and apply these results to all of the sciences. This is much more efficient than developing properties of special concepts in each separate science. The French mathematician Joseph Fourier (1768–1830) put it succinctly: “Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them.”

3.7 Exercises

- 1–4** A particle moves according to a law of motion $s = f(t)$, $t \geq 0$, where t is measured in seconds and s in feet.
- Find the velocity at time t .
 - What is the velocity after 1 second?
 - When is the particle at rest?
 - When is the particle moving in the positive direction?
 - Find the total distance traveled during the first 6 seconds.
 - Draw a diagram like Figure 2 to illustrate the motion of the particle.
 - Find the acceleration at time t and after 1 second.
- (h) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 6$.
- (i) When is the particle speeding up? When is it slowing down?

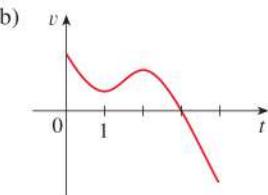
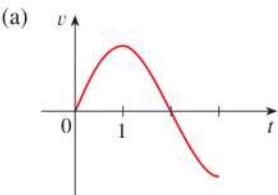
1. $f(t) = t^3 - 9t^2 + 24t$

2. $f(t) = \frac{9t}{t^2 + 9}$

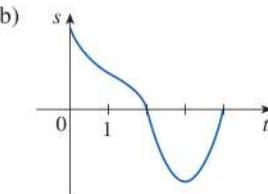
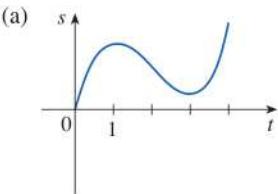
3. $f(t) = \sin(\pi t/2)$

4. $f(t) = t^2 e^{-t}$

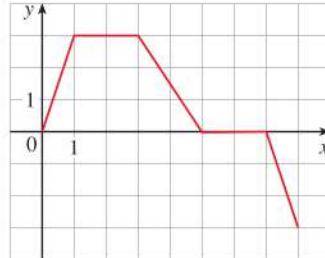
5. Graphs of the *velocity* functions of two particles are shown, where t is measured in seconds. When is each particle speeding up? When is it slowing down? Explain.



6. Graphs of the *position* functions of two particles are shown, where t is measured in seconds. When is the velocity of each particle positive? When is it negative? When is each particle speeding up? When is it slowing down? Explain.



7. Suppose that the graph of the velocity function of a particle is as shown in the figure, where t is measured in seconds. When is the particle traveling forward (in the positive direction)? When is it traveling backward? What is happening when $5 < t < 7$?



8. For the particle described in Exercise 7, sketch a graph of the acceleration function. When is the particle speeding up? When is it slowing down? When is it traveling at a constant speed?
9. The height (in meters) of a projectile shot vertically upward from a point 2 m above ground level with an initial velocity of 24.5 m/s is $h = 2 + 24.5t - 4.9t^2$ after t seconds.
- Find the velocity after 2 s and after 4 s.
 - When does the projectile reach its maximum height?
 - What is the maximum height?
 - When does it hit the ground?
 - With what velocity does it hit the ground?
10. If a ball is thrown vertically upward with a velocity of 80 ft/s, then its height after t seconds is $s = 80t - 16t^2$.
- What is the maximum height reached by the ball?
 - What is the velocity of the ball when it is 96 ft above the ground on its way up? On its way down?
11. If a rock is thrown vertically upward from the surface of Mars with velocity 15 m/s, its height after t seconds is $h = 15t - 1.86t^2$.
- What is the velocity of the rock after 2 s?
 - What is the velocity of the rock when its height is 25 m on its way up? On its way down?
12. A particle moves with position function
- $$s = t^4 - 4t^3 - 20t^2 + 20t \quad t \geq 0$$
- At what time does the particle have a velocity of 20 m/s?
 - At what time is the acceleration 0? What is the significance of this value of t ?
13. (a) A company makes computer chips from square wafers of silicon. A process engineer wants to keep the side length of a wafer very close to 15 mm and needs to know how the area $A(x)$ of a wafer changes when the side length x changes. Find $A'(15)$ and explain its meaning in this situation.
- (b) Show that the rate of change of the area of a square with respect to its side length is half its perimeter. Try to explain geometrically why this is true by drawing a square whose side length x is increased by an amount Δx . How can you approximate the resulting change in area ΔA if Δx is small?

- 14.** (a) Sodium chloride crystals are easy to grow in the shape of cubes by allowing a solution of water and sodium chloride to evaporate slowly. If V is the volume of such a cube with side length x , calculate dV/dx when $x = 3$ mm and explain its meaning.
 (b) Show that the rate of change of the volume of a cube with respect to its edge length is equal to half the surface area of the cube. Explain geometrically why this result is true by arguing by analogy with Exercise 13(b).

15. (a) Find the average rate of change of the area of a circle with respect to its radius r as r changes from
 (i) 2 to 3 (ii) 2 to 2.5 (iii) 2 to 2.1
 (b) Find the instantaneous rate of change when $r = 2$.
 (c) Show that the rate of change of the area of a circle with respect to its radius (at any r) is equal to the circumference of the circle. Try to explain geometrically why this is true by drawing a circle whose radius is increased by an amount Δr . How can you approximate the resulting change in area ΔA if Δr is small?

16. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s. Find the rate at which the area within the circle is increasing after (a) 1 s, (b) 3 s, and (c) 5 s. What can you conclude?

17. A spherical balloon is being inflated. Find the rate of increase of the surface area ($S = 4\pi r^2$) with respect to the radius r when r is (a) 1 ft, (b) 2 ft, and (c) 3 ft. What conclusion can you make?

18. (a) The volume of a growing spherical cell is $V = \frac{4}{3}\pi r^3$, where the radius r is measured in micrometers ($1 \mu\text{m} = 10^{-6}$ m). Find the average rate of change of V with respect to r when r changes from
 (i) 5 to 8 μm (ii) 5 to 6 μm (iii) 5 to 5.1 μm
 (b) Find the instantaneous rate of change of V with respect to r when $r = 5 \mu\text{m}$.
 (c) Show that the rate of change of the volume of a sphere with respect to its radius is equal to its surface area. Explain geometrically why this result is true. Argue by analogy with Exercise 15(c).

19. The mass of the part of a metal rod that lies between its left end and a point x meters to the right is $3x^2$ kg. Find the linear density (see Example 2) when x is (a) 1 m, (b) 2 m, and (c) 3 m. Where is the density the highest? The lowest?

20. If a cylindrical water tank holds 5000 gallons, and the water drains from the bottom of the tank in 40 minutes, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V = 5000\left(1 - \frac{1}{40}t\right)^2 \quad 0 \leq t \leq 40$$

Find the rate at which water is draining from the tank after (a) 5 min, (b) 10 min, (c) 20 min, and (d) 40 min. At what time is the water flowing out the fastest? The slowest? Summarize your findings.

21. The quantity of charge Q in coulombs (C) that has passed through a point in a wire up to time t (measured in seconds) is given by $Q(t) = t^3 - 2t^2 + 6t + 2$. Find the current when (a) $t = 0.5$ s and (b) $t = 1$ s. (See Example 3. The unit of current is an ampere [1 A = 1 C/s].) At what time is the current lowest?

22. Newton's Law of Gravitation says that the magnitude F of the force exerted by a body of mass m on a body of mass M is

$$F = \frac{GmM}{r^2}$$

where G is the gravitational constant and r is the distance between the bodies.

(a) Find dF/dr and explain its meaning. What does the minus sign indicate?
 (b) Suppose it is known that the earth attracts an object with a force that decreases at the rate of 2 N/km when $r = 20,000$ km. How fast does this force change when $r = 10,000$ km?

23. The force F acting on a body with mass m and velocity v is the rate of change of momentum: $F = (d/dt)(mv)$. If m is constant, this becomes $F = ma$, where $a = dv/dt$ is the acceleration. But in the theory of relativity, the mass of a particle varies with v as follows: $m = m_0/\sqrt{1 - v^2/c^2}$ where m_0 is the mass of the particle at rest and c is the speed of light. Show that

$$F = \frac{m_0a}{(1 - v^2/c^2)^{3/2}}$$

24. Some of the highest tides in the world occur in the Bay of Fundy on the Atlantic Coast of Canada. At Hopewell Cape the water depth at low tide is about 2.0 m and at high tide it is about 12.0 m. The natural period of oscillation is a little more than 12 hours and on a day in June, high tide occurred at 6:45 AM. This helps explain the following model for the water depth D (in meters) as a function of the time t (in hours after midnight) on that day:

$$D(t) = 7 + 5 \cos[0.503(t - 6.75)]$$

How fast was the tide rising (or falling) at the following times?
 (a) 3:00 AM (b) 6:00 AM
 (c) 9:00 AM (d) Noon

25. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the product of the pressure and the volume remains constant: $PV = C$.

(a) Find the rate of change of volume with respect to pressure.
 (b) A sample of gas is in a container at low pressure and is steadily compressed at constant temperature for 10 minutes. Is the volume decreasing more rapidly at the beginning or the end of the 10 minutes? Explain.
 (c) Prove that the isothermal compressibility (see Example 5) is given by $\beta = 1/P$.

- 26.** If, in Example 4, one molecule of the product C is formed from one molecule of the reactant A and one molecule of the reactant B, and the initial concentrations of A and B have a common value $[A] = [B] = a$ moles/L, then

$$[C] = a^2kt/(akt + 1)$$

where k is a constant.

- (a) Find the rate of reaction at time t .
 (b) Show that if $x = [C]$, then

$$\frac{dx}{dt} = k(a - x)^2$$

- (c) What happens to the concentration as $t \rightarrow \infty$?
 (d) What happens to the rate of reaction as $t \rightarrow \infty$?
 (e) What do the results of parts (c) and (d) mean in practical terms?

- 27.** In Example 6 we considered a bacteria population that doubles every hour. Suppose that another population of bacteria triples every hour and starts with 400 bacteria. Find an expression for the number n of bacteria after t hours and use it to estimate the rate of growth of the bacteria population after 2.5 hours.

- 28.** The number of yeast cells in a laboratory culture increases rapidly initially but levels off eventually. The population is modeled by the function

$$n = f(t) = \frac{a}{1 + be^{-0.7t}}$$

where t is measured in hours. At time $t = 0$ the population is 20 cells and is increasing at a rate of 12 cells/hour. Find the values of a and b . According to this model, what happens to the yeast population in the long run?

- T 29.** The table gives the world population $P(t)$, in millions, where t is measured in years and $t = 0$ corresponds to the year 1900.

t	Population (millions)	t	Population (millions)
0	1650	60	3040
10	1750	70	3710
20	1860	80	4450
30	2070	90	5280
40	2300	100	6080
50	2560	110	6870

- (a) Estimate the rate of population growth in 1920 and in 1980 by averaging the slopes of two secant lines.
 (b) Use a graphing calculator or computer to find a cubic function (a third-degree polynomial) that models the data.
 (c) Use your model in part (b) to find a model for the rate of population growth.
 (d) Use part (c) to estimate the rates of growth in 1920 and 1980. Compare with your estimates in part (a).

- (e) In Section 1.1 we modeled $P(t)$ with the exponential function

$$f(t) = (1.43653 \times 10^9) \cdot (1.01395)^t$$

Use this model to find a model for the rate of population growth.

- (f) Use your model in part (e) to estimate the rate of growth in 1920 and 1980. Compare with your estimates in parts (a) and (d).
 (g) Estimate the rate of growth in 1985.

- T 30.** The table shows how the average age of first marriage of Japanese women has varied since 1950.

t	$A(t)$	t	$A(t)$
1950	23.0	1985	25.5
1955	23.8	1990	25.9
1960	24.4	1995	26.3
1965	24.5	2000	27.0
1970	24.2	2005	28.0
1975	24.7	2010	28.8
1980	25.2	2015	29.4

- (a) Use a graphing calculator or computer to model these data with a fourth-degree polynomial.
 (b) Use part (a) to find a model for $A'(t)$.
 (c) Estimate the rate of change of marriage age for women in 1990.
 (d) Graph the data points and the models for A and A' .

- 31.** Refer to the law of laminar flow given in Example 7. Consider a blood vessel with radius 0.01 cm, length 3 cm, pressure difference 3000 dynes/cm², and viscosity $\eta = 0.027$.
 (a) Find the velocity of the blood along the centerline
 $r = 0$, at radius $r = 0.005$ cm, and at the wall
 $r = R = 0.01$ cm.
 (b) Find the velocity gradient at $r = 0$, $r = 0.005$, and $r = 0.01$.
 (c) Where is the velocity the greatest? Where is the velocity changing most?

- 32.** The frequency of vibrations of a vibrating violin string is given by

$$f = \frac{1}{2L} \sqrt{\frac{T}{\rho}}$$

where L is the length of the string, T is its tension, and ρ is its linear density. [See Chapter 11 in D. E. Hall, *Musical Acoustics*, 3rd ed. (Pacific Grove, CA, 2002).]

- (a) Find the rate of change of the frequency with respect to
 (i) the length (when T and ρ are constant),
 (ii) the tension (when L and ρ are constant), and
 (iii) the linear density (when L and T are constant).
 (b) The pitch of a note (how high or low the note sounds) is determined by the frequency f . (The higher the frequency, the higher the pitch.) Use the signs of the

derivatives in part (a) to determine what happens to the pitch of a note

- (i) when the effective length of a string is decreased by placing a finger on the string so a shorter portion of the string vibrates,
- (ii) when the tension is increased by turning a tuning peg,
- (iii) when the linear density is increased by switching to another string.

- 33.** Suppose that the cost (in dollars) for a company to produce x pairs of a new line of jeans is

$$C(x) = 2000 + 3x + 0.01x^2 + 0.00002x^3$$

- (a) Find the marginal cost function.
- (b) Find $C'(100)$ and explain its meaning. What does it predict?
- (c) Compare $C'(100)$ with the cost of manufacturing the 101st pair of jeans.

- 34.** The cost function for a certain commodity is

$$C(q) = 84 + 0.16q - 0.0006q^2 + 0.000003q^3$$

- (a) Find and interpret $C'(100)$.
- (b) Compare $C'(100)$ with the cost of producing the 101st item.

- 35.** If $p(x)$ is the total value of the production when there are x workers in a plant, then the *average productivity* of the workforce at the plant is

$$A(x) = \frac{p(x)}{x}$$

- (a) Find $A'(x)$. Why does the company want to hire more workers if $A'(x) > 0$?
- (b) Show that $A'(x) > 0$ if $p'(x)$ is greater than the average productivity.

- 36.** If R denotes the reaction of the body to some stimulus of strength x , the *sensitivity* S is defined to be the rate of change of the reaction with respect to x . A particular example is that when the brightness x of a light source is increased, the eye reacts by decreasing the area R of the pupil. The experimental formula

$$R = \frac{40 + 24x^{0.4}}{1 + 4x^{0.4}}$$

has been used to model the dependence of R on x when R is measured in square millimeters and x is measured in appropriate units of brightness.

- (a) Find the sensitivity.
- (b) Illustrate part (a) by graphing both R and S as functions of x . Comment on the values of R and S at low levels of brightness. Is this what you would expect?



- 37.** Patients undergo dialysis treatment to remove urea from their blood when their kidneys are not functioning properly. Blood is diverted from the patient through a machine that

filters out urea. Under certain conditions, the duration of dialysis required, given that the initial urea concentration is $c > 1$, is given by the equation

$$t = \ln\left(\frac{3c + \sqrt{9c^2 - 8c}}{2}\right)$$

Calculate the derivative of t with respect to c and interpret it.

- 38.** Invasive species often display a wave of advance as they colonize new areas. Mathematical models based on random dispersal and reproduction have demonstrated that the speed with which such waves move is given by the function $f(r) = 2\sqrt{Dr}$, where r is the reproductive rate of individuals and D is a parameter quantifying dispersal. Calculate the derivative of the wave speed with respect to the reproductive rate r and explain its meaning.
- 39.** The gas law for an ideal gas at absolute temperature T (in kelvins), pressure P (in atmospheres), and volume V (in liters) is $PV = nRT$, where n is the number of moles of the gas and $R = 0.0821$ is the gas constant. Suppose that, at a certain instant, $P = 8.0$ atm and is increasing at a rate of 0.10 atm/min and $V = 10$ L and is decreasing at a rate of 0.15 L/min. Find the rate of change of T with respect to time at that instant if $n = 10$ mol.

- 40.** In a fish farm, a population of fish is introduced into a pond and harvested regularly. A model for the rate of change of the fish population is given by the equation

$$\frac{dP}{dt} = r_0\left(1 - \frac{P(t)}{P_c}\right)P(t) - \beta P(t)$$

where r_0 is the birth rate of the fish, P_c is the maximum population that the pond can sustain (called the *carrying capacity*), and β is the percentage of the population that is harvested.

- (a) What value of dP/dt corresponds to a stable population?
- (b) If the pond can sustain 10,000 fish, the birth rate is 5%, and the harvesting rate is 4%, find the stable population level.
- (c) What happens if β is raised to 5%?

- 41.** In the study of ecosystems, *predator-prey models* are often used to study the interaction between species. Consider populations of tundra wolves, given by $W(t)$, and caribou, given by $C(t)$, in northern Canada. The interaction has been modeled by the equations

$$\frac{dC}{dt} = aC - bCW \quad \frac{dW}{dt} = -cW + dCW$$

- (a) What values of dC/dt and dW/dt correspond to stable populations?
- (b) How would the statement “The caribou go extinct” be represented mathematically?
- (c) Suppose that $a = 0.05$, $b = 0.001$, $c = 0.05$, and $d = 0.0001$. Find all population pairs (C, W) that lead to stable populations. According to this model, is it possible for the two species to live in balance or will one or both species become extinct?

- 42.** Hermann Ebbinghaus (1850–1909) pioneered the study of memory. A 2011 article in the *Journal of Mathematical Psychology* presents the mathematical model

$$R(t) = a + b(1 + ct)^{-\beta}$$

for the *Ebbinghaus forgetting curve*, where $R(t)$ is the fraction of memory retained t days after learning a task; a , b , and c are experimentally determined constants between 0 and 1; β is a positive constant; and $R(0) = 1$. The constants depend on the type of task being learned.

- (a) What is the rate of change of retention t days after a task is learned?
 - (b) Do you forget how to perform a task faster soon after learning it or a long time after you have learned it?
 - (c) What fraction of memory is permanent?
- 43.** The difficulty of “acquiring a target” (such as using a mouse to click on an icon on a computer screen) depends on the ratio

between the distance D to the target and width W of the target. According to *Fitts’ law*, the index I of difficulty is modeled by

$$I = \log_2 \left(\frac{2D}{W} \right)$$

This law is used for designing products that involve human-computer interactions.

- (a) If W is held constant, what is the rate of change of I with respect to D ? Does this rate increase or decrease with increasing values of D ?
- (b) If D is held constant, what is the rate of change of I with respect to W ? What does the negative sign in your answer indicate? Does this rate increase or decrease with increasing values of W ?
- (c) Do your answers to parts (a) and (b) agree with your intuition?

3.8 | Exponential Growth and Decay

In many natural phenomena, quantities grow or decay at a rate proportional to their size. For instance, if $y = f(t)$ is the number of individuals in a population of animals or bacteria at time t , then it seems reasonable to expect that the rate of growth $f'(t)$ is proportional to the population $f(t)$; that is, $f'(t) = kf(t)$ for some constant k . Indeed, under ideal conditions (unlimited environment, adequate nutrition, immunity to disease) the mathematical model given by the equation $f'(t) = kf(t)$ predicts what actually happens fairly accurately. Another example occurs in nuclear physics: the mass of a radioactive substance decays at a rate proportional to the mass. In chemistry, the rate of a unimolecular first-order reaction is proportional to the concentration of the substance. In finance, the value of a savings account with continuously compounded interest increases at a rate proportional to that value.

In general, if $y(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to its size $y(t)$ at any time, then

1

$$\frac{dy}{dt} = ky$$

where k is a constant. Equation 1 is sometimes called the **law of natural growth** (if $k > 0$) or the **law of natural decay** (if $k < 0$). It is called a *differential equation* because it involves an unknown function y and its derivative dy/dt .

It’s not hard to think of a solution of Equation 1. This equation asks us to find a function whose derivative is a constant multiple of itself. We have met such functions in this chapter. Any exponential function of the form $y(t) = Ce^{kt}$, where C is a constant, satisfies

$$\frac{dy}{dt} = C(ke^{kt}) = k(Ce^{kt}) = ky$$

We will see in Section 9.4 that *any* function that satisfies $dy/dt = ky$ must be of the form $y = Ce^{kt}$. To see the significance of the constant C , we observe that

$$y(0) = Ce^{k \cdot 0} = C$$

Therefore C is the **initial value** of the function.

2 Theorem The only solutions of the differential equation $dy/dt = ky$ are the exponential functions

$$y(t) = y(0)e^{kt}$$

■ Population Growth

What is the significance of the proportionality constant k ? In the context of population growth, where $P(t)$ is the size of a population at time t , we can write

$$\boxed{3} \quad \frac{dP}{dt} = kP \quad \text{or} \quad \frac{dP/dt}{P} = k$$

The quantity

$$\frac{dP/dt}{P}$$

is the growth rate divided by the population size; it is called the **relative growth rate**. According to (3), instead of saying “the growth rate is proportional to population size” we could say “the relative growth rate is constant.” Then Theorem 2 says that a population with constant relative growth rate must grow exponentially. Notice that the relative growth rate k appears as the coefficient of t in the exponential function Ce^{kt} . For instance, if

$$\frac{dP}{dt} = 0.02P$$

and t is measured in years, then the relative growth rate is $k = 0.02$ and the population grows at a relative rate of 2% per year. If the population at time 0 is P_0 , then the expression for the population is

$$P(t) = P_0 e^{0.02t}$$

EXAMPLE 1 Use the fact that the world population was 2560 million in 1950 and 3040 million in 1960 to model the population of the world in the second half of the 20th century. (Assume that the growth rate is proportional to the population size.) What is the relative growth rate? Use the model to estimate the world population in 1993 and to predict the population in the year 2025.

SOLUTION We measure the time t in years and let $t = 0$ in the year 1950. We measure the population $P(t)$ in millions of people. Then $P(0) = 2560$ and $P(10) = 3040$. Since we are assuming that $dP/dt = kP$, Theorem 2 gives

$$P(t) = P(0)e^{kt} = 2560e^{kt}$$

$$P(10) = 2560e^{10k} = 3040$$

$$k = \frac{1}{10} \ln \frac{3040}{2560} \approx 0.017185$$

The relative growth rate is about 1.7% per year and the model is

$$P(t) = 2560e^{0.017185t}$$

We estimate that the world population in 1993 was

$$P(43) = 2560e^{0.017185(43)} \approx 5360 \text{ million}$$

The model predicts that the population in 2025 will be

$$P(75) = 2560e^{0.017185(75)} \approx 9289 \text{ million}$$

The graph in Figure 1 shows that the model is fairly accurate to the end of the 20th century (the dots represent the actual population), so the estimate for 1993 is quite reliable. But the prediction for 2025 may not be so accurate.

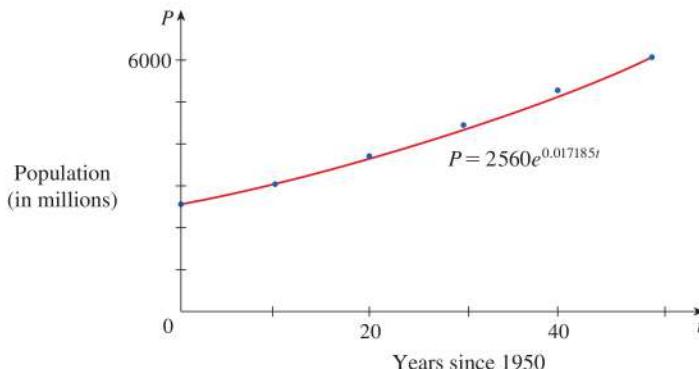


FIGURE 1

A model for world population growth in the second half of the 20th century

■ Radioactive Decay

Radioactive substances decay by spontaneously emitting radiation. If $m(t)$ is the mass remaining from an initial mass m_0 of the substance after time t , then the relative decay rate

$$-\frac{dm/dt}{m}$$

has been found experimentally to be constant. (Since dm/dt is negative, the relative decay rate is positive.) It follows that

$$\frac{dm}{dt} = km$$

where k is a negative constant. In other words, radioactive substances decay at a rate proportional to the remaining mass. This means that we can use Theorem 2 to show that the mass decays exponentially:

$$m(t) = m_0 e^{kt}$$

Physicists express the rate of decay in terms of **half-life**, the time required for half of any given quantity to decay.

EXAMPLE 2 The half-life of radium-226 is 1590 years.

- (a) A sample of radium-226 has mass 100 mg. Find a formula for the mass of the sample that remains after t years.
- (b) Find the mass remaining after 1000 years correct to the nearest milligram.
- (c) When will the mass be reduced to 30 mg?

SOLUTION

- (a) Let $m(t)$ be the mass of radium-226 (in milligrams) that remains after t years. Then $dm/dt = km$ and $m(0) = 100$, so Theorem 2 gives

$$m(t) = m(0)e^{kt} = 100e^{kt}$$

In order to determine the value of k , we use the fact that $m(1590) = \frac{1}{2}(100)$. Thus

$$100e^{1590k} = 50 \quad \text{so} \quad e^{1590k} = \frac{1}{2}$$

and

$$1590k = \ln \frac{1}{2} = -\ln 2$$

$$k = -\frac{\ln 2}{1590}$$

Therefore

$$m(t) = 100e^{-(\ln 2)t/1590}$$

We could use the fact that $e^{\ln 2} = 2$ to write the expression for $m(t)$ in the alternative form

$$m(t) = 100 \times 2^{-t/1590}$$

(b) The mass remaining after 1000 years is

$$m(1000) = 100e^{-(\ln 2)1000/1590} \approx 65 \text{ mg}$$

(c) We want to find the value of t such that $m(t) = 30$, that is,

$$100e^{-(\ln 2)t/1590} = 30 \quad \text{or} \quad e^{-(\ln 2)t/1590} = 0.3$$

We solve this equation for t by taking the natural logarithm of both sides:

$$-\frac{\ln 2}{1590}t = \ln 0.3$$

Thus

$$t = -1590 \frac{\ln 0.3}{\ln 2} \approx 2762 \text{ years}$$

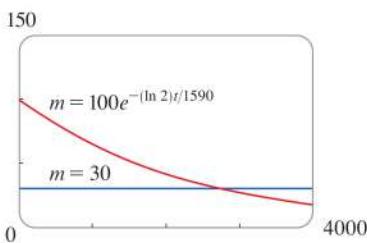


FIGURE 2

As a check on our work in Example 2, we use a calculator or computer to draw the graph of $m(t)$ in Figure 2 together with the horizontal line $m = 30$. These curves intersect when $t \approx 2800$, and this agrees with the answer to part (c). ■

■ Newton's Law of Cooling

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large. (This law also applies to warming.) If we let $T(t)$ be the temperature of the object at time t and T_s be the temperature of the surroundings, then we can formulate Newton's Law of Cooling as a differential equation:

$$\frac{dT}{dt} = k(T - T_s)$$

where k is a constant. This equation is not quite the same as Equation 1, so we make the change of variable $y(t) = T(t) - T_s$. Because T_s is constant, we have $y'(t) = T'(t)$ and so the equation becomes

$$\frac{dy}{dt} = ky$$

We can then use Theorem 2 to find an expression for y , from which we can find T .

EXAMPLE 3 A bottle of iced tea at room temperature (72°F) is placed in a refrigerator where the temperature is 44°F . After half an hour the tea has cooled to 61°F .

- (a) What is the temperature of the tea after another half hour?
- (b) How long does it take for the tea to cool to 50°F ?

SOLUTION

(a) Let $T(t)$ be the temperature of the tea after t minutes. The surrounding temperature is $T_s = 44^\circ\text{F}$, so Newton's Law of Cooling states that

$$\frac{dT}{dt} = k(T - 44)$$

If we let $y = T - 44$, then $y(0) = T(0) - 44 = 72 - 44 = 28$, so y satisfies

$$\frac{dy}{dt} = ky \quad y(0) = 28$$

and by (2) we have $y(t) = y(0)e^{kt} = 28e^{kt}$

We are given that $T(30) = 61$, so $y(30) = 61 - 44 = 17$ and

$$28e^{30k} = 17 \quad e^{30k} = \frac{17}{28}$$

Taking logarithms, we have

$$k = \frac{\ln(\frac{17}{28})}{30} \approx -0.01663$$

Thus

$$y(t) = 28e^{-0.01663t}$$

$$T(t) = 44 + 28e^{-0.01663t}$$

$$T(60) = 44 + 28e^{-0.01663(60)} \approx 54.3$$

So after another half hour the tea has cooled to about 54°F .

(b) We have $T(t) = 50$ when

$$44 + 28e^{-0.01663t} = 50$$

$$e^{-0.01663t} = \frac{6}{28}$$

$$t = \frac{\ln(\frac{6}{28})}{-0.01663} \approx 92.6$$

The tea cools to 50°F after about 1 hour 33 minutes. ■

Notice that in Example 3 we have

$$\lim_{t \rightarrow \infty} T(t) = \lim_{t \rightarrow \infty} (44 + 28e^{-0.01663t}) = 44 + 28 \cdot 0 = 44$$

This means that, as expected, the temperature of the tea approaches the ambient temperature inside the refrigerator. The graph of the temperature function is shown in Figure 3.

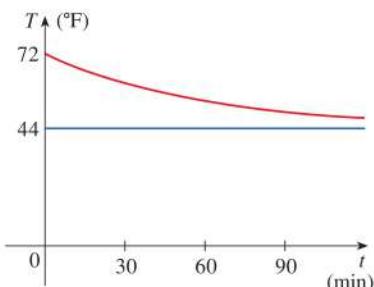


FIGURE 3

■ Continuously Compounded Interest

EXAMPLE 4 If \$5000 is invested at 2% interest, compounded annually, then after 1 year the investment is worth $\$5000(1.02) = \5100.00 , after 2 years it's worth $[\$5000(1.02)](1.02) = \5202.00 , and after t years it's worth $\$5000(1.02)^t$. In general, if an amount A_0 is invested at an interest rate r ($r = 0.02$ in this example), then after t years it's worth $A_0(1 + r)^t$. Usually, interest is compounded more frequently, say, n times per year. Then in each compounding period the rate is r/n , and there are nt compounding periods in t years, so the value of the investment is

$$A = A_0 \left(1 + \frac{r}{n}\right)^{nt}$$

For instance, after 3 years at 2% interest a \$5000 investment will be worth

$$\$5000(1.02)^3 = \$5306.04 \text{ with annual compounding}$$

$$\$5000(1.01)^6 = \$5307.60 \text{ with semiannual compounding}$$

$$\$5000(1.005)^{12} = \$5308.39 \text{ with quarterly compounding}$$

$$\$5000\left(1 + \frac{0.02}{12}\right)^{36} = \$5308.92 \text{ with monthly compounding}$$

$$\$5000\left(1 + \frac{0.02}{365}\right)^{365 \cdot 3} = \$5309.17 \text{ with daily compounding}$$

You can see that the interest paid increases as the number of compounding periods (n) increases. If we let $n \rightarrow \infty$, then we will be compounding the interest **continuously** and the value of the investment will be

$$\begin{aligned} A(t) &= \lim_{n \rightarrow \infty} A_0 \left(1 + \frac{r}{n}\right)^{nt} \\ &= \lim_{n \rightarrow \infty} A_0 \left[\left(1 + \frac{r}{n}\right)^{n/r} \right]^{rt} \\ &= A_0 \left[\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^{n/r} \right]^{rt} \\ &= A_0 \left[\lim_{m \rightarrow \infty} \left(1 + \frac{1}{m}\right)^m \right]^{rt} \quad (\text{where } m = n/r) \end{aligned}$$

Equation 3.6.6:

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

But the limit in this expression is equal to the number e (see Equation 3.6.6). So with continuous compounding of interest at interest rate r , the amount after t years is

$$A(t) = A_0 e^{rt}$$

If we differentiate this equation, we get

$$\frac{dA}{dt} = rA_0 e^{rt} = rA(t)$$

which says that, with continuous compounding of interest, the rate of increase of an investment is proportional to its size.

Returning to the example of \$5000 invested for 3 years at 2% interest, we see that with continuously compounding of interest the value of the investment will be

$$A(3) = \$5000e^{(0.02)3} = \$5309.18$$

Notice how close this is to the amount we calculated for daily compounding, \$5309.17. But the amount is easier to compute if we use continuous compounding. ■

3.8 Exercises

- A population of the yeast cell *Saccharomyces cerevisiae* (a yeast used for fermentation) develops with a constant relative growth rate of 0.4159 per hour. The initial population consists of 3.8 million cells. Find the population size after 2 hours.
- A common inhabitant of human intestines is the bacterium *Escherichia coli*, named after the German pediatrician Theodor Escherich, who identified it in 1885. A cell of this bacterium in a nutrient-broth medium divides into two cells every 20 minutes. The initial population of a culture is 50 cells.
 - Find the relative growth rate.
 - Find an expression for the number of cells after t hours.
 - Find the number of cells after 6 hours.
 - Find the rate of growth after 6 hours.
 - When will the population reach a million cells?
- A culture of the bacterium *Salmonella enteritidis* initially contains 50 cells. When introduced into a nutrient broth, the culture grows at a rate proportional to its size. After 1.5 hours the population has increased to 975.
 - Find an expression for the number of bacteria after t hours.
 - Find the number of bacteria after 3 hours.
 - Find the rate of growth after 3 hours.
 - After how many hours will the population reach 250,000?
- A bacteria culture grows with constant relative growth rate. The bacteria count was 400 after 2 hours and 25,600 after 6 hours.
 - What is the relative growth rate? Express your answer as a percentage.
 - What was the initial size of the culture?
 - Find an expression for the number of bacteria after t hours.
 - Find the number of bacteria after 4.5 hours.
 - Find the rate of growth after 4.5 hours.
 - When will the population reach 50,000?
- The table gives estimates of the world population, in millions, from 1750 to 2000.

Year	Population (millions)	Year	Population (millions)
1750	790	1900	1650
1800	980	1950	2560
1850	1260	2000	6080

- Use the exponential model and the population figures for 1750 and 1800 to predict the world population in 1900 and 1950. Compare with the actual figures.

- Use the exponential model and the population figures for 1850 and 1900 to predict the world population in 1950. Compare with the actual population.
- Use the exponential model and the population figures for 1900 and 1950 to predict the world population in 2000. Compare with the actual population and try to explain the discrepancy.
- The table gives census data for the population of Indonesia, in millions, during the second half of the 20th century.

Year	Population (millions)
1950	83
1960	100
1970	122
1980	150
1990	182
2000	214

- Assuming the population grows at a rate proportional to its size, use the census data for 1950 and 1960 to predict the population in 1980. Compare with the actual figure.
- Use the census data for 1960 and 1980 to predict the population in 2000. Compare with the actual population.
- Use the census data for 1980 and 2000 to predict the population in 2010 and compare with the actual population of 243 million.
- Use the model in part (c) to predict the population in 2025. Do you think the prediction will be too high or too low? Why?

- Experiments show that if the chemical reaction



takes place at 45°C, the rate of reaction of dinitrogen pentoxide is proportional to its concentration as follows:

$$-\frac{d[\text{N}_2\text{O}_5]}{dt} = 0.0005[\text{N}_2\text{O}_5]$$

- Find an expression for the concentration $[\text{N}_2\text{O}_5]$ after t seconds if the initial concentration is C .
- How long will the reaction take to reduce the concentration of N_2O_5 to 90% of its original value?

- 8.** Strontium-90 has a half-life of 28 days.
- A sample has initial mass 50 mg. Find a formula for the mass remaining after t days.
 - Find the mass remaining after 40 days.
 - How long does it take the sample to decay to a mass of 2 mg?
 - Sketch the graph of the mass function.
- 9.** The half-life of cesium-137 is 30 years. Suppose we have a 100-mg sample.
- Find the mass that remains after t years.
 - How much of the sample remains after 100 years?
 - After how long will only 1 mg remain?
- 10.** A sample of einsteinium-252 decayed to 64.3% of its original mass after 300 days.
- What is the half-life of einsteinium-252?
 - How long would it take the sample to decay to one-third of its original mass?
- 11–13 Radiocarbon Dating** Scientists can determine the age of ancient objects by the method of *radiocarbon dating*. The bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope of carbon, ^{14}C , with a half-life of about 5730 years. Vegetation absorbs carbon dioxide through the atmosphere and animal life assimilates ^{14}C through food chains. When a plant or animal dies, it stops replacing its carbon and the amount of ^{14}C present begins to decrease through radioactive decay. Therefore the level of radioactivity must also decay exponentially.
- 11.** A discovery revealed a parchment fragment that had about 74% as much ^{14}C radioactivity as does plant material on the earth today. Estimate the age of the parchment.
- 12.** Dinosaur fossils are too old to be reliably dated using carbon-14. Suppose we had a 68-million-year-old dinosaur fossil. What fraction of the living dinosaur's ^{14}C would be remaining today? Suppose the minimum detectable mass is 0.1%. What is the maximum age of a fossil that could be dated using ^{14}C ?
- 13.** Dinosaur fossils are often dated by using an element other than carbon, such as potassium-40, that has a longer half-life (in this case, approximately 1.25 billion years). Suppose the minimum detectable mass is 0.1% and a dinosaur is dated with ^{40}K to be 68 million years old. Is such a dating possible? In other words, what is the maximum age of a fossil that could be dated using ^{40}K ?
- 14.** A curve passes through the point $(0, 5)$ and has the property that the slope of the curve at every point P is twice the y -coordinate of P . What is the equation of the curve?
- 15.** A roast turkey is removed from an oven when its temperature has reached 185°F and is placed on a table in a room where the ambient temperature is 75°F .
- If the temperature of the turkey is 150°F after half an hour, what is the temperature after 45 minutes?
 - When will the turkey have cooled to 100°F ?
- 16.** In a murder investigation, the temperature of the corpse was 32.5°C at 1:30 PM and 30.3°C an hour later. Normal body temperature is 37.0°C and the ambient temperature was 20.0°C . When did the murder take place?
- 17.** When a cold drink is taken from a refrigerator, its temperature is 5°C . After 25 minutes in a 20°C room its temperature has increased to 10°C .
- What is the temperature of the drink after 50 minutes?
 - When will its temperature reach 15°C ?
- 18.** A freshly brewed cup of coffee has temperature 95°C in a 20°C room. When its temperature is 70°C , it is cooling at a rate of 1°C per minute. When does this occur?
- 19.** The rate of change of atmospheric pressure P with respect to altitude h is proportional to P , provided that the temperature is constant. At 15°C the pressure is 101.3 kPa at sea level and 87.14 kPa at $h = 1000$ m.
- What is the pressure at an altitude of 3000 m?
 - What is the pressure at the top of Mount McKinley, at an altitude of 6187 m?
- 20.** (a) If \$2500 is borrowed at 4.5% interest, find the amounts due at the end of 3 years if the interest is compounded (i) annually, (ii) quarterly, (iii) monthly, (iv) weekly, (v) daily, (vi) hourly, and (vii) continuously.
-
- Suppose \$2500 is borrowed and the interest is compounded continuously. If $A(t)$ is the amount due after t years, where $0 \leq t \leq 3$, graph $A(t)$ for each of the interest rates 5%, 6%, and 7% on a common screen.
- 21.** (a) If \$4000 is invested at 1.75% interest, find the value of the investment at the end of 5 years if the interest is compounded (i) annually, (ii) semiannually, (iii) monthly, (iv) weekly, (v) daily, and (vi) continuously.
- If $A(t)$ is the amount of the investment at time t for the case of continuous compounding, write a differential equation and an initial condition satisfied by $A(t)$.
- 22.** (a) How long will it take an investment to double in value if the interest rate is 3%, compounded continuously?
- What is the equivalent annual interest rate?

APPLIED PROJECT**CONTROLLING RED BLOOD CELL LOSS DURING SURGERY**

Kent Weakley / Shutterstock.com

A typical volume of blood in the human body is about 5 L. A certain percentage of that volume (called the *hematocrit*) consists of red blood cells (RBCs); typically the hematocrit is about 45% in males. Suppose that a surgery takes four hours and a male patient bleeds 2.5 L of blood. During surgery the patient's blood volume is maintained at 5 L by injection of saline solution, which mixes quickly with the blood but dilutes it so that the hematocrit decreases as time passes.

- Assuming that the rate of RBC loss is proportional to the volume of RBCs, determine the patient's volume of RBCs by the end of the operation.
- A procedure called *acute normovolemic hemodilution* (ANH) has been developed to minimize RBC loss during surgery. In this procedure blood is extracted from the patient before the operation and replaced with saline solution. This dilutes the patient's blood, resulting in fewer RBCs being lost by bleeding during surgery. The extracted blood is then returned to the patient after surgery. Only a certain amount of blood can be extracted, however, because the RBC concentration can never be allowed to drop below 25% during surgery. What is the maximum amount of blood that can be extracted in the ANH procedure for the surgery described in this project?
- What is the RBC loss without the ANH procedure? What is the loss if the procedure is carried out with the volume calculated in Problem 2?

3.9**Related Rates**

If we are pumping air into a balloon, both the volume and the radius of the balloon are increasing and their rates of increase are related to each other. But it is much easier to measure directly the rate of increase of the volume than the rate of increase of the radius.

In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

EXAMPLE 1 Air is being pumped into a spherical balloon so that its volume increases at a rate of $100 \text{ cm}^3/\text{s}$. How fast is the radius of the balloon increasing when the diameter is 50 cm?

SOLUTION We start by identifying two things:

the *given information*:

the rate of increase of the volume of air is $100 \text{ cm}^3/\text{s}$

and the *unknown*:

the rate of increase of the radius when the diameter is 50 cm

In order to express these quantities mathematically, we introduce some suggestive notation:

Let V be the volume of the balloon and let r be its radius.

PS According to the Principles of Problem Solving discussed following Chapter 1, the first step is to understand the problem. This includes reading the problem carefully, identifying the given and the unknown, and introducing suitable notation.

The key thing to remember is that rates of change are derivatives. In this problem, the volume and the radius are both functions of the time t . The rate of increase of the volume with respect to time is the derivative dV/dt , and the rate of increase of the radius is dr/dt . We can therefore restate the given and the unknown as follows:

$$\text{Given: } \frac{dV}{dt} = 100 \text{ cm}^3/\text{s}$$

$$\text{Unknown: } \frac{dr}{dt} \text{ when } r = 25 \text{ cm}$$

In order to connect dV/dt and dr/dt , we first relate V and r by a formula—in this case, the formula for the volume of a sphere:

$$V = \frac{4}{3}\pi r^3$$

P5 The second stage of problem solving is to think of a plan for connecting the given and the unknown.

Notice that, although dV/dt is constant, dr/dt is *not* constant.

In order to use the given information, we differentiate each side of this equation with respect to t . To differentiate the right side, we need to use the Chain Rule:

$$\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Now we solve for the unknown quantity:

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \frac{dV}{dt}$$

If we put $r = 25$ and $dV/dt = 100$ in this equation, we obtain

$$\frac{dr}{dt} = \frac{1}{4\pi(25)^2} 100 = \frac{1}{25\pi}$$

The radius of the balloon is increasing at the rate of $1/(25\pi) \approx 0.0127$ cm/s when the diameter is 50 cm. ■

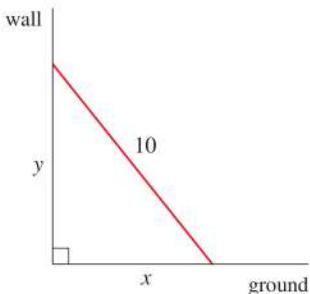


FIGURE 1

EXAMPLE 2 A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 4 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

SOLUTION We first draw a diagram and label it as in Figure 1. Let x feet be the distance from the bottom of the ladder to the wall and y feet the distance from the top of the ladder to the ground. Note that x and y are both functions of t (time, measured in seconds).

We are given that $dx/dt = 4$ ft/s and we are asked to find dy/dt when $x = 6$ ft (see Figure 2). In this problem, the relationship between x and y is given by the Pythagorean Theorem:

$$x^2 + y^2 = 100$$

Differentiating each side with respect to t using the Chain Rule, we have

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

Solving this equation for the desired rate, we obtain

$$\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}$$

When $x = 6$, the Pythagorean Theorem gives $y = 8$ and so, substituting these values and $dx/dt = 4$, we have

$$\frac{dy}{dt} = -\frac{6}{8}(4) = -3 \text{ ft/s}$$

FIGURE 2

The fact that dy/dt is negative means that the distance from the top of the ladder to the ground is *decreasing* at a rate of 3 ft/s. In other words, the top of the ladder is sliding down the wall at a rate of 3 ft/s.

PS Look back: What have we learned from Examples 1 and 2 that will help us solve future problems?

WARNING A common error is to substitute the given numerical information (for quantities that vary with time) too early. This should be done only *after* the differentiation. (Step 7 follows Step 6.) For instance, in Example 1 we dealt with general values of r until we finally substituted $r = 25$ at the last step. (If we had put $r = 25$ earlier, we would have gotten $dV/dt = 0$, which is clearly wrong.)

Problem Solving Strategy

It is useful to recall some of the problem-solving principles and adapt them to related rates in light of our experience in Examples 1 and 2:

1. Read the problem carefully.
2. Draw a diagram if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution (see Example 3 below).
6. Use the Chain Rule to differentiate both sides of the equation with respect to t .
7. Substitute the given information into the resulting equation and solve for the unknown rate.

See also Principles of Problem Solving following Chapter 1.

The following examples further illustrate this strategy.

EXAMPLE 3 A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of $2 \text{ m}^3/\text{min}$, find the rate at which the water level is rising when the water is 3 m deep.

SOLUTION We first sketch the cone and label it as in Figure 3. Let V , r , and h be the volume of the water, the radius of the surface, and the height of the water at time t , where t is measured in minutes.

We are given that $dV/dt = 2 \text{ m}^3/\text{min}$ and we are asked to find dh/dt when h is 3 m. The quantities V and h are related by the equation

$$V = \frac{1}{3}\pi r^2 h$$

but it is very useful to express V as a function of h alone. In order to eliminate r , we use the similar triangles in Figure 3 to write

$$\frac{r}{h} = \frac{2}{4} \quad r = \frac{h}{2}$$

and the expression for V becomes

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

Now we can differentiate each side with respect to t :

$$\frac{dV}{dt} = \frac{\pi}{4}h^2 \frac{dh}{dt}$$

so

$$\frac{dh}{dt} = \frac{4}{\pi h^2} \frac{dV}{dt}$$

Substituting $h = 3$ m and $dV/dt = 2 \text{ m}^3/\text{min}$, we have

$$\frac{dh}{dt} = \frac{4}{\pi(3)^2} \cdot 2 = \frac{8}{9\pi}$$

The water level is rising at a rate of $8/(9\pi) \approx 0.28 \text{ m/min}$. ■

EXAMPLE 4 Car A is traveling west at 50 mi/h and car B is traveling north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?

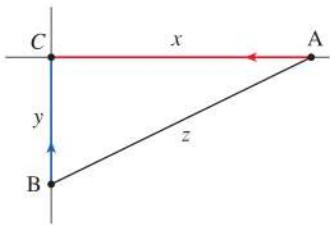


FIGURE 4

SOLUTION We draw Figure 4, where C is the intersection of the roads. At a given time t , let x be the distance from car A to C , let y be the distance from car B to C , and let z be the distance between the cars, where x , y , and z are measured in miles.

We are given that $dx/dt = -50 \text{ mi/h}$ and $dy/dt = -60 \text{ mi/h}$. (The derivatives are negative because x and y are decreasing.) We are asked to find dz/dt . The equation that relates x , y , and z is given by the Pythagorean Theorem:

$$z^2 = x^2 + y^2$$

Differentiating each side with respect to t , we have

$$\begin{aligned} 2z \frac{dz}{dt} &= 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \\ \frac{dz}{dt} &= \frac{1}{z} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right) \end{aligned}$$

When $x = 0.3$ mi and $y = 0.4$ mi, the Pythagorean Theorem gives $z = 0.5$ mi, so

$$\frac{dz}{dt} = \frac{1}{0.5} [0.3(-50) + 0.4(-60)] = -78 \text{ mi/h}$$

The cars are approaching each other at a rate of 78 mi/h. ■

EXAMPLE 5 A man walks along a straight path at a speed of 4 ft/s. A spotlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the spotlight rotating when the man is 15 ft from the point on the path closest to the light?

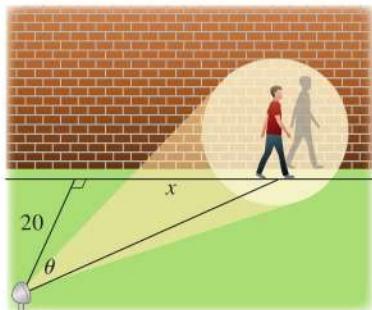


FIGURE 5

SOLUTION We draw Figure 5 and let x be the distance from the man to the point on the path closest to the spotlight. We let θ be the angle between the beam of the light and the perpendicular to the path.

We are given that $dx/dt = 4 \text{ ft/s}$ and are asked to find $d\theta/dt$ when $x = 15$. The equation that relates x and θ can be written from Figure 5:

$$\frac{x}{20} = \tan \theta \quad x = 20 \tan \theta$$

Differentiating each side with respect to t , we get

$$\frac{dx}{dt} = 20 \sec^2 \theta \frac{d\theta}{dt}$$

so

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{1}{20} \cos^2 \theta \frac{dx}{dt} \\ &= \frac{1}{20} \cos^2 \theta (4) = \frac{1}{5} \cos^2 \theta \end{aligned}$$

When $x = 15$, the length of the beam is 25, so $\cos \theta = \frac{20}{25} = \frac{4}{5}$ and

$$0.128 \frac{\text{rad}}{\text{s}} \times \frac{1 \text{ rotation}}{2\pi \text{ rad}} \times \frac{60 \text{ s}}{1 \text{ min}} \\ \approx 1.22 \text{ rotations per min}$$

$$\frac{d\theta}{dt} = \frac{1}{5} \left(\frac{4}{5} \right)^2 = \frac{16}{125} = 0.128$$

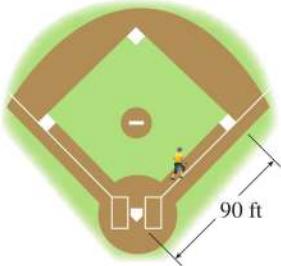
The spotlight is rotating at a rate of 0.128 rad/s. ■

3.9 Exercises

1. (a) If V is the volume of a cube with edge length x and the cube expands as time passes, find dV/dt in terms of dx/dt .
 (b) If the length of the edge of a cube is increasing at a rate of 4 cm/s, how fast is the volume of the cube increasing when the edge length is 15 cm?
 2. (a) If A is the area of a circle with radius r and the circle expands as time passes, find dA/dt in terms of dr/dt .
 (b) Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 2 m/s, how fast is the area of the spill increasing when the radius is 30 m?
 3. Each side of a square is increasing at a rate of 6 cm/s. At what rate is the area of the square increasing when the area of the square is 16 cm^2 ?
 4. The radius of a sphere is increasing at a rate of 4 mm/s. How fast is the volume increasing when the diameter is 80 mm?
 5. The radius of a spherical ball is increasing at a rate of 2 cm/min. At what rate is the surface area of the ball increasing when the radius is 8 cm?
 6. The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?
 7. A cylindrical tank with radius 5 m is being filled with water at a rate of 3 m^3/min . How fast is the height of the water increasing?
 8. The area of a triangle with sides of lengths a and b and contained angle θ is $A = \frac{1}{2}ab \sin \theta$. (See Formula 6 in Appendix D.)
 (a) If $a = 2$ cm, $b = 3$ cm, and θ increases at a rate of 0.2 rad/min, how fast is the area increasing when $\theta = \pi/3$?
 (b) If $a = 2$ cm, b increases at a rate of 1.5 cm/min, and θ increases at a rate of 0.2 rad/min, how fast is the area increasing when $b = 3$ cm and $\theta = \pi/3$?
 (c) If a increases at a rate of 2.5 cm/min, b increases at a rate of 1.5 cm/min, and θ increases at a rate of 0.2 rad/min, how fast is the area increasing when $a = 2$ cm, $b = 3$ cm, and $\theta = \pi/3$?
 9. Suppose $4x^2 + 9y^2 = 25$, where x and y are functions of t .
 (a) If $dy/dt = \frac{1}{3}$, find dx/dt when $x = 2$ and $y = 1$.
 (b) If $dx/dt = 3$, find dy/dt when $x = -2$ and $y = 1$.
 10. If $x^2 + y^2 + z^2 = 9$, $dx/dt = 5$, and $dy/dt = 4$, find dz/dt when $(x, y, z) = (2, 2, 1)$.
 11. The weight w of an astronaut (in pounds) is related to her height h above the surface of the earth (in miles) by

$$w = w_0 \left(\frac{3960}{3960 + h} \right)^2$$
 where w_0 is the weight of the astronaut on the surface of the earth. If the astronaut weighs 130 pounds on earth and is in a rocket, being propelled upward at a speed of 12 mi/s, find the rate at which her weight is changing (in lb/s) when she is 40 miles above the earth's surface.
 12. A particle is moving along a hyperbola $xy = 8$. As it reaches the point $(4, 2)$, the y -coordinate is decreasing at a rate of 3 cm/s. How fast is the x -coordinate of the point changing at that instant?
- 13–16**
- (a) What quantities are given in the problem?
 (b) What is the unknown?
 (c) Draw a picture of the situation for any time t .
 (d) Write an equation that relates the quantities.
 (e) Finish solving the problem.
 13. A plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when the plane is 2 mi away from the station.
 14. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min , find the rate at which the diameter decreases when the diameter is 10 cm.
 15. A street light is mounted at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole?
 16. At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 PM?

17. Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?
18. A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing when he is 4 m from the building?
19. A man starts walking north at 4 ft/s from a point P . Five minutes later a woman starts walking south at 5 ft/s from a point 500 ft due east of P . At what rate are the people moving apart 15 min after the woman starts walking?
20. A baseball diamond is a square with side 90 ft. A batter hits the ball and runs toward first base with a speed of 24 ft/s.
- At what rate is his distance from second base decreasing when he is halfway to first base?
 - At what rate is his distance from third base increasing at the same moment?

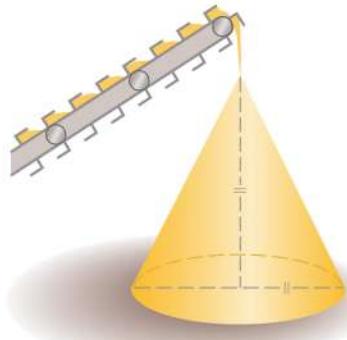


21. The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm^2/min . At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2 ?
22. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?

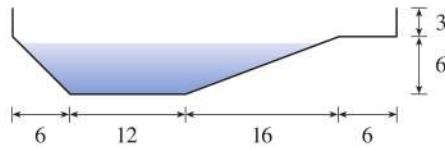


- 23–24** Use the fact that the distance (in meters) a dropped stone falls after t seconds is $d = 4.9t^2$.
23. A woman stands near the edge of a cliff and drops a stone over the edge. Exactly one second later she drops another stone. One second after that, how fast is the distance between the two stones changing?
24. Two men stand 10 m apart on level ground near the edge of a cliff. One man drops a stone and one second later the other man drops a stone. One second after that, how fast is the distance between the two stones changing?

25. Water is leaking out of an inverted conical tank at a rate of 10,000 cm^3/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank.
26. A particle moves along the curve $y = 2 \sin(\pi x/2)$. As the particle passes through the point $(\frac{1}{3}, 1)$, its x -coordinate increases at a rate of $\sqrt{10}$ cm/s. How fast is the distance from the particle to the origin changing at this instant?
27. A water trough is 10 m long and a cross-section has the shape of an isosceles trapezoid that is 30 cm wide at the bottom, 80 cm wide at the top, and has height 50 cm. If the trough is being filled with water at the rate of 0.2 m^3/min , how fast is the water level rising when the water is 30 cm deep?
28. A trough is 10 ft long and its ends have the shape of isosceles triangles that are 3 ft across at the top and have a height of 1 ft. If the trough is being filled with water at a rate of 12 ft^3/min , how fast is the water level rising when the water is 6 inches deep?
29. Gravel is being dumped from a conveyor belt at a rate of 30 ft^3/min , and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?



30. A swimming pool is 20 ft wide, 40 ft long, 3 ft deep at the shallow end, and 9 ft deep at its deepest point. A cross-section is shown in the figure. If the pool is being filled at a rate of 0.8 ft^3/min , how fast is the water level rising when the depth at the deepest point is 5 ft?

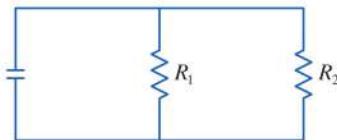


31. The sides of an equilateral triangle are increasing at a rate of 10 cm/min. At what rate is the area of the triangle increasing when the sides are 30 cm long?

32. A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string has been let out?
33. A car is traveling north on a straight road at 20 m/s and a drone is flying east at 6 m/s at an elevation of 25 m. At one instant the drone passes directly over the car. How fast is the distance between the drone and the car changing 5 seconds later?
34. If the minute hand of a clock has length r (in centimeters), find the rate at which it sweeps out area as a function of r .
35. How fast is the angle between the ladder and the ground changing in Example 2 when the bottom of the ladder is 6 ft from the wall?
36. According to the model we used to solve Example 2, what happens as the top of the ladder approaches the ground? Is the model appropriate for small values of y ?
37. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P and volume V satisfy the equation $PV = C$, where C is a constant. Suppose that at a certain instant the volume is 600 cm³, the pressure is 150 kPa, and the pressure is increasing at a rate of 20 kPa/min. At what rate is the volume decreasing at this instant?
38. A faucet is filling a hemispherical basin of diameter 60 cm with water at a rate of 2 L/min. Find the rate at which the water is rising in the basin when it is half full. [Use the following facts: 1 L is 1000 cm³. The volume of the portion of a sphere with radius r from the bottom to a height h is $V = \pi(rh^2 - \frac{1}{3}h^3)$, as we will show in Chapter 6.]
39. If two resistors with resistances R_1 and R_2 are connected in parallel, as shown in the figure, then the total resistance R , measured in ohms (Ω), is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

If R_1 and R_2 are increasing at rates of 0.3 Ω /s and 0.2 Ω /s, respectively, how fast is R changing when $R_1 = 80 \Omega$ and $R_2 = 100 \Omega$?



40. When air expands adiabatically (without gaining or losing heat), its pressure P and volume V are related by the equation $PV^{1.4} = C$, where C is a constant. Suppose that at a certain instant the volume is 400 cm³ and the pressure is 80 kPa and is decreasing at a rate of 10 kPa/min. At what rate is the volume increasing at this instant?

41. Two straight roads diverge from an intersection at an angle of 60° . Two cars leave the intersection at the same time, the first traveling down one road at 40 mi/h and the second traveling down the other road at 60 mi/h. How fast is the distance between the cars changing after half an hour? [Hint: Use the Law of Cosines (Formula 21 in Appendix D).]
42. Brain weight B as a function of body weight W in fish has been modeled by the power function $B = 0.007W^{2/3}$, where B and W are measured in grams. A model for body weight as a function of body length L (measured in centimeters) is $W = 0.12L^{2.53}$. If, over 10 million years, the average length of a certain species of fish evolved from 15 cm to 20 cm at a constant rate, how fast was this species' brain growing when its average length was 18 cm?
43. Two sides of a triangle have lengths 12 m and 15 m. The angle between them is increasing at a rate of $2^\circ/\text{min}$. How fast is the length of the third side increasing when the angle between the sides of fixed length is 60° ? [Hint: Use the Law of Cosines (Formula 21 in Appendix D).]
44. Two carts, A and B, are connected by a rope 39 ft long that passes over a pulley P . (See the figure.) The point Q is on the floor 12 ft directly beneath P and between the carts. Cart A is being pulled away from Q at a speed of 2 ft/s. How fast is cart B moving toward Q at the instant when cart A is 5 ft from Q ?
-
45. A television camera is positioned 4000 ft from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. Also, the mechanism for focusing the camera has to take into account the increasing distance from the camera to the rising rocket. Let's assume the rocket rises vertically and its speed is 600 ft/s when it has risen 3000 ft.
- (a) How fast is the distance from the television camera to the rocket changing at that moment?
- (b) If the television camera is always kept aimed at the rocket, how fast is the camera's angle of elevation changing at that same moment?
46. A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P ?
47. A plane flies horizontally at an altitude of 5 km and passes directly over a tracking telescope on the ground. When the

- angle of elevation is $\pi/3$, this angle is decreasing at a rate of $\pi/6$ rad/min. How fast is the plane traveling at that time?
48. A Ferris wheel with a radius of 10 m is rotating at a rate of one revolution every 2 minutes. How fast is a rider rising when his seat is 16 m above ground level?
49. A plane flying with a constant speed of 300 km/h passes over a ground radar station at an altitude of 1 km and climbs at an angle of 30° . At what rate is the distance from the plane to the radar station increasing a minute later?
50. Two people start from the same point. One walks east at 3 mi/h and the other walks northeast at 2 mi/h. How fast is the distance between the people changing after 15 minutes?

51. A runner sprints around a circular track of radius 100 m at a constant speed of 7 m/s. The runner's friend is standing at a distance 200 m from the center of the track. How fast is the distance between the friends changing when the distance between them is 200 m?
52. The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?
53. Suppose that the volume V of a rolling snowball increases so that dV/dt is proportional to the surface area of the snowball at time t . Show that the radius r increases at a constant rate, that is, dr/dt is constant.

3.10 | Linear Approximations and Differentials

We have seen that a curve lies very close to its tangent line near the point of tangency. In fact, by zooming in toward a point on the graph of a differentiable function, we noticed that the graph looks more and more like its tangent line. (See Figure 2.7.2.) This observation is the basis for a method of finding approximate values of functions.

Linearization and Approximation

It might be easy to calculate a value $f(a)$ of a function, but difficult (or even impossible) to compute nearby values of f . So we settle for the easily computed values of the linear function L whose graph is the tangent line of f at $(a, f(a))$. (See Figure 1.)

In other words, we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a . An equation of this tangent line is

$$y = f(a) + f'(a)(x - a)$$

The linear function whose graph is this tangent line, that is,

1

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a . The approximation $f(x) \approx L(x)$ or

2

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the **linear approximation** or **tangent line approximation** of f at a .

EXAMPLE 1 Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

SOLUTION The derivative of $f(x) = (x+3)^{1/2}$ is

$$f'(x) = \frac{1}{2}(x+3)^{-1/2} = \frac{1}{2\sqrt{x+3}}$$

and so we have $f(1) = 2$ and $f'(1) = \frac{1}{4}$. Putting these values into Equation 1, we see

that the linearization is

$$L(x) = f(1) + f'(1)(x - 1) = 2 + \frac{1}{4}(x - 1) = \frac{7}{4} + \frac{x}{4}$$

The corresponding linear approximation (2) is

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4} \quad (\text{when } x \text{ is near 1})$$

In particular, we have

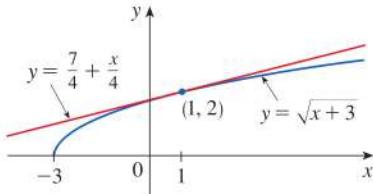


FIGURE 2

The linear approximation is illustrated in Figure 2. We see that, indeed, the tangent line approximation is a good approximation to the given function when x is near 1. We also see that our approximations are overestimates because the tangent line lies above the curve.

Of course, a calculator could give us approximations for $\sqrt{3.98}$ and $\sqrt{4.05}$, but the linear approximation gives an approximation *over an entire interval*. ■

In the following table we compare the estimates from the linear approximation in Example 1 with the true values. Notice from this table, and also from Figure 2, that the tangent line approximation gives good estimates when x is close to 1 but the accuracy of the approximation deteriorates when x is farther away from 1.

	x	From $L(x)$	Actual value
$\sqrt{3.9}$	0.9	1.975	1.97484176...
$\sqrt{3.98}$	0.98	1.995	1.99499373...
$\sqrt{4}$	1	2	2.00000000...
$\sqrt{4.05}$	1.05	2.0125	2.01246117...
$\sqrt{4.1}$	1.1	2.025	2.02484567...
$\sqrt{5}$	2	2.25	2.23606797...
$\sqrt{6}$	3	2.5	2.44948974...

How good is the approximation that we obtained in Example 1? The next example shows that by using a graphing calculator or computer we can determine an interval throughout which a linear approximation provides a specified accuracy.

EXAMPLE 2 For what values of x is the linear approximation

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$$

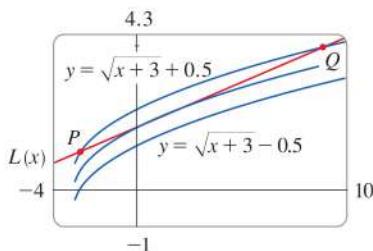
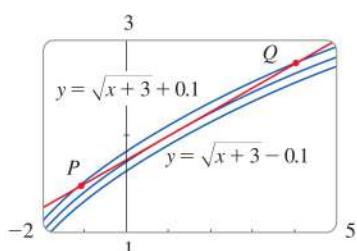
accurate to within 0.5? What about accuracy to within 0.1?

SOLUTION Accuracy to within 0.5 means that the functions should differ by less than 0.5:

$$\left| \sqrt{x+3} - \left(\frac{7}{4} + \frac{x}{4} \right) \right| < 0.5$$

Equivalently, we could write

$$\sqrt{x+3} - 0.5 < \frac{7}{4} + \frac{x}{4} < \sqrt{x+3} + 0.5$$

**FIGURE 3****FIGURE 4**

This says that the linear approximation should lie between the curves obtained by shifting the curve $y = \sqrt{x+3}$ upward and downward by an amount 0.5. Figure 3 shows the tangent line $y = (7+x)/4$ intersecting the upper curve $y = \sqrt{x+3} + 0.5$ at P and Q . We estimate that the x -coordinate of P is about -2.66 and the x -coordinate of Q is about 8.66 . Thus we see from the graph that the approximation

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$$

is accurate to within 0.5 when $-2.6 < x < 8.6$. (We have rounded the smaller value up and the larger value down.)

Similarly, from Figure 4 we see that the approximation is accurate to within 0.1 when $-1.1 < x < 3.9$. ■

■ Applications to Physics

Linear approximations are often used in physics. In analyzing the consequences of an equation, a physicist sometimes needs to simplify a function by replacing it with its linear approximation. For instance, in deriving a formula for the period of a pendulum, physics textbooks obtain an expression involving $\sin \theta$ and then replace $\sin \theta$ by θ with the remark that $\sin \theta$ is very close to θ if θ is not too large. You can verify that the linearization of the function $f(x) = \sin x$ at $a = 0$ is $L(x) = x$ and so the linear approximation at 0 is

$$\sin x \approx x$$

(see Exercise 50). So, in effect, the derivation of the formula for the period of a pendulum uses the tangent line approximation for the sine function.

Another example occurs in the theory of optics, where light rays that arrive at shallow angles relative to the optical axis are called *paraxial rays*. In paraxial (or Gaussian) optics, both $\sin \theta$ and $\cos \theta$ are replaced by their linearizations. In other words, the linear approximations

$$\sin \theta \approx \theta \quad \text{and} \quad \cos \theta \approx 1$$

are used because θ is close to 0. The results of calculations made with these approximations became the basic theoretical tool used to design lenses. (See *Optics*, 5th ed., by Eugene Hecht [Boston, 2017], p. 164.)

In Section 11.11 we will present several other applications of the idea of linear approximations to physics and engineering.

■ Differentials

The ideas behind linear approximations are sometimes formulated in the terminology and notation of *differentials*. If $y = f(x)$, where f is a differentiable function, then the **differential** dx is an independent variable; that is, dx can be given the value of any real number. The **differential** dy is then defined in terms of dx by the equation

3

$$dy = f'(x) dx$$

So dy is a dependent variable; it depends on the values of x and dx . If dx is given a specific value and x is taken to be some specific number in the domain of f , then the numerical value of dy is determined.

If $dx \neq 0$, we can divide both sides of Equation 3 by dx to obtain

$$\frac{dy}{dx} = f'(x)$$

We have seen similar equations before, but now the left side can genuinely be interpreted as a ratio of differentials.

The geometric meaning of differentials is shown in Figure 5. Let $P(x, f(x))$ and $Q(x + \Delta x, f(x + \Delta x))$ be points on the graph of f and let $dx = \Delta x$. The corresponding change in y is

$$\Delta y = f(x + \Delta x) - f(x)$$

The slope of the tangent line PR is the derivative $f'(x)$. Thus the directed distance from S to R is $f'(x) dx = dy$. Therefore dy represents the amount that the tangent line rises or falls (the change in the linearization), whereas Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by an amount dx .

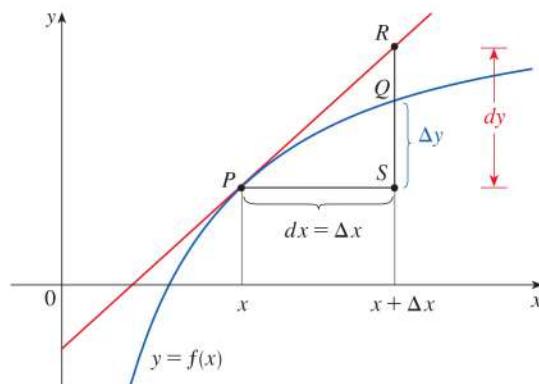


FIGURE 5

EXAMPLE 3 Compare the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes (a) from 2 to 2.05 and (b) from 2 to 2.01.

SOLUTION

(a) We have

$$f(2) = 2^3 + 2^2 - 2(2) + 1 = 9$$

$$f(2.05) = (2.05)^3 + (2.05)^2 - 2(2.05) + 1 = 9.717625$$

$$\Delta y = f(2.05) - f(2) = 0.717625$$

In general,

$$dy = f'(x) dx = (3x^2 + 2x - 2) dx$$

When $x = 2$ and $dx = \Delta x = 0.05$, this becomes

$$dy = [3(2)^2 + 2(2) - 2]0.05 = 0.7$$

$$(b) \quad f(2.01) = (2.01)^3 + (2.01)^2 - 2(2.01) + 1 = 9.140701$$

$$\Delta y = f(2.01) - f(2) = 0.140701$$

When $dx = \Delta x = 0.01$,

$$dy = [3(2)^2 + 2(2) - 2]0.01 = 0.14$$

Figure 6 shows the function in Example 3 and a comparison of dy and Δy when $a = 2$. The viewing rectangle is $[1.8, 2.5]$ by $[6, 18]$.

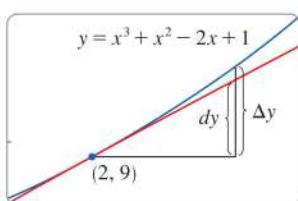


FIGURE 6

Notice that the approximation $\Delta y \approx dy$ becomes better as Δx becomes smaller in Example 3. Notice also that dy was easier to compute than Δy .

In the notation of differentials, the linear approximation $f(x) \approx f(a) + f'(a)(x - a)$ can be written as

$$f(a + dx) \approx f(a) + dy$$

by taking $dx = x - a$, so $x = a + dx$. For instance, for the function $f(x) = \sqrt{x+3}$ in Example 1, we have

$$dy = f'(x) dx = \frac{dx}{2\sqrt{x+3}}$$

If $a = 1$ and $dx = \Delta x = 0.05$, then

$$dy = \frac{0.05}{2\sqrt{1+3}} = 0.0125$$

and $\sqrt{4.05} = f(1.05) = f(1 + 0.05) \approx f(1) + dy = 2.0125$

just as we found in Example 1.

Our final example illustrates the use of differentials in estimating the errors that occur because of approximate measurements.

EXAMPLE 4 The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

SOLUTION If the radius of the sphere is r , then its volume is $V = \frac{4}{3}\pi r^3$. If the error in the measured value of r is denoted by $dr = \Delta r$, then the corresponding error in the calculated value of V is ΔV , which can be approximated by the differential

$$dV = 4\pi r^2 dr$$

When $r = 21$ and $dr = 0.05$, this becomes

$$dV = 4\pi(21)^2 0.05 \approx 277$$

The maximum error in the calculated volume is about 277 cm³. ■

NOTE Although the possible error in Example 4 may appear to be rather large, a better picture of the error is given by the **relative error**, which is computed by dividing the error by the total volume:

$$\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} = 3 \frac{dr}{r}$$

Thus the relative error in the volume is about three times the relative error in the radius. In Example 4 the relative error in the radius is approximately $dr/r = 0.05/21 \approx 0.0024$ and it produces a relative error of about 0.007 in the volume. The errors could also be expressed as percentage errors of 0.24% in the radius and 0.7% in the volume.

3.10 Exercises

- 1–4** Find the linearization $L(x)$ of the function at a .

1. $f(x) = x^3 - x^2 + 3$, $a = -2$
2. $f(x) = e^{3x}$, $a = 0$
3. $f(x) = \sqrt[3]{x}$, $a = 8$
4. $f(x) = \cos 2x$, $a = \pi/6$

5. Find the linear approximation of the function $f(x) = \sqrt{1-x}$ at $a = 0$ and use it to approximate the numbers $\sqrt{0.9}$ and $\sqrt{0.99}$. Illustrate by graphing f and the tangent line.
6. Find the linear approximation of the function $g(x) = \sqrt[3]{1+x}$ at $a = 0$ and use it to approximate the numbers $\sqrt[3]{0.95}$ and $\sqrt[3]{1.1}$. Illustrate by graphing g and the tangent line.

- 7–10** Verify the given linear approximation at $a = 0$. Then determine the values of x for which the linear approximation is accurate to within 0.1.

7. $\tan^{-1}x \approx x$

8. $(1+x)^{-3} \approx 1 - 3x$

9. $\sqrt[4]{1+2x} \approx 1 + \frac{1}{2}x$

10. $\frac{2}{1+e^x} \approx 1 - \frac{1}{2}x$

- 11–18** Find the differential of the function.

11. $y = e^{5x}$

12. $y = \sqrt{1-t^4}$

13. $y = \frac{1+2u}{1+3u}$

14. $y = \theta^2 \sin 2\theta$

15. $y = \frac{1}{x^2 - 3x}$

16. $y = \sqrt{1 + \cos \theta}$

17. $y = \ln(\sin \theta)$

18. $y = \frac{e^x}{1-e^x}$

- 19–22** (a) Find the differential dy and (b) evaluate dy for the given values of x and dx .

19. $y = e^{x/10}, \quad x = 0, \quad dx = 0.1$

20. $y = \cos \pi x, \quad x = \frac{1}{3}, \quad dx = -0.02$

21. $y = \sqrt{3+x^2}, \quad x = 1, \quad dx = -0.1$

22. $y = \frac{x+1}{x-1}, \quad x = 2, \quad dx = 0.05$

- 23–26** Compute Δy and dy for the given values of x and $dx = \Delta x$. Then sketch a diagram like Figure 5 showing the line segments with lengths dx , dy , and Δy .

23. $y = x^2 - 4x, \quad x = 3, \quad \Delta x = 0.5$

24. $y = x - x^3, \quad x = 0, \quad \Delta x = -0.3$

25. $y = \sqrt{x-2}, \quad x = 3, \quad \Delta x = 0.8$

26. $y = e^x, \quad x = 0, \quad \Delta x = 0.5$

- 27–30** Compare the values of Δy and dy if x changes from 1 to 1.05. What if x changes from 1 to 1.01? Does the approximation $\Delta y \approx dy$ become better as Δx gets smaller?

27. $f(x) = x^4 - x + 1$

28. $f(x) = e^{2x-2}$

29. $f(x) = \sqrt{5-x}$

30. $f(x) = \frac{1}{x^2 + 1}$

- 31–36** Use a linear approximation (or differentials) to estimate the given number.

31. $(1.999)^4$

32. $1/4.002$

33. $\sqrt[3]{1001}$

34. $\sqrt{100.5}$

35. $e^{0.1}$

36. $\cos 29^\circ$

- 37–39** Explain, in terms of linear approximations or differentials, why the approximation is reasonable.

37. $\ln 1.04 \approx 0.04$

38. $\sqrt{4.02} \approx 2.005$

39. $\frac{1}{9.98} \approx 0.1002$

- 40.** Let $f(x) = (x-1)^2$ $g(x) = e^{-2x}$

and $h(x) = 1 + \ln(1-2x)$

- (a) Find the linearizations of f , g , and h at $a = 0$. What do you notice? How do you explain what happened?
 (b) Graph f , g , and h and their linear approximations. For which function is the linear approximation best? For which is it worst? Explain.

- 41.** The edge of a cube was found to be 30 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum possible error, relative error, and percentage error in computing (a) the volume of the cube and (b) the surface area of the cube.

- 42.** The radius of a circular disk is given as 24 cm with a maximum error in measurement of 0.2 cm.

- (a) Use differentials to estimate the maximum error in the calculated area of the disk.
(b) What is the relative error and the percentage error?

- 43.** The circumference of a sphere was measured to be 84 cm with a possible error of 0.5 cm.

- (a) Use differentials to estimate the maximum error in the calculated surface area. What is the relative error?
(b) Use differentials to estimate the maximum error in the calculated volume. What is the relative error?

- 44.** Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.

- 45.** (a) Use differentials to find a formula for the approximate volume of a thin cylindrical shell with height h , inner radius r , and thickness Δr .
(b) What is the error involved in using the formula from part (a)?

- 46.** One side of a right triangle is known to be 20 cm long and the opposite angle is measured as 30° , with a possible error of $\pm 1^\circ$.

- (a) Use differentials to estimate the error in computing the length of the hypotenuse.
(b) What is the percentage error?

- 47.** If a current I passes through a resistor with resistance R , Ohm's Law states that the voltage drop is $V = RI$. If V is constant and R is measured with a certain error, use differentials to show that the relative error in calculating I is approximately the same (in magnitude) as the relative error in R .

- 48.** When blood flows along a blood vessel, the flux F (the volume of blood per unit time that flows past a given point) is proportional to the fourth power of the radius R of the blood vessel:

$$F = kR^4$$

(This is known as Poiseuille's Law; we will show why it is true in Section 8.4.) A partially clogged artery can be expanded by an operation called angioplasty, in which a balloon-tipped catheter is inflated inside the artery in order to widen it and restore normal blood flow.

Show that the relative change in F is about four times the relative change in R . How will a 5% increase in the radius affect the flow of blood?

- 49.** Establish the following rules for working with differentials (where c denotes a constant and u and v are functions of x).

$$\begin{array}{ll} \text{(a)} \ dc = 0 & \text{(b)} \ d(cu) = c \ du \\ \text{(c)} \ d(u + v) = du + dv & \text{(d)} \ d(uv) = u \ dv + v \ du \\ \text{(e)} \ d\left(\frac{u}{v}\right) = \frac{v \ du - u \ dv}{v^2} & \text{(f)} \ d(x^n) = nx^{n-1} dx \end{array}$$

- 50.** In physics textbooks, the period T of a pendulum of length L is often given as $T \approx 2\pi\sqrt{L/g}$, provided that the pendulum swings through a relatively small arc. In the course of deriving this formula, the equation $a_T = -g \sin \theta$ for the tangential acceleration of the bob of the pendulum is obtained, and then $\sin \theta$ is replaced by θ with the remark that for small angles, θ (in radians) is very close to $\sin \theta$.

- (a)** Verify the linear approximation at 0 for the sine function:

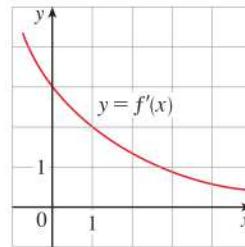
$$\sin \theta \approx \theta$$

- (b)** If $\theta = \pi/18$ (equivalent to 10°) and we approximate $\sin \theta$ by θ , what is the percentage error?

- (c)** Use a graph to determine the values of θ for which $\sin \theta$ and θ differ by less than 2%. What are the values in degrees?

- 51.** Suppose that the only information we have about a function f is that $f(1) = 5$ and the graph of its derivative is as shown.

- (a)** Use a linear approximation to estimate $f(0.9)$ and $f(1.1)$.
(b) Are your estimates in part (a) too large or too small? Explain.



- 52.** Suppose that we don't have a formula for $g(x)$ but we know that $g(2) = -4$ and $g'(x) = \sqrt{x^2 + 5}$ for all x .

- (a)** Use a linear approximation to estimate $g(1.95)$ and $g(2.05)$.
(b) Are your estimates in part (a) too large or too small? Explain.

DISCOVERY PROJECT | POLYNOMIAL APPROXIMATIONS

The tangent line approximation $L(x)$ is the best first-degree (linear) approximation to $f(x)$ near $x = a$ because $f(x)$ and $L(x)$ have the same rate of change (derivative) at a . For a better approximation than a linear one, let's try a second-degree (quadratic) approximation $P(x)$. In other words, we approximate a curve by a parabola instead of by a straight line. To make sure that the approximation is a good one, we stipulate the following:

- (i) $P(a) = f(a)$ (P and f should have the same value at a .)
- (ii) $P'(a) = f'(a)$ (P and f should have the same rate of change at a .)
- (iii) $P''(a) = f''(a)$ (The slopes of P and f should change at the same rate at a .)

1. Find the quadratic approximation $P(x) = A + Bx + Cx^2$ to the function $f(x) = \cos x$ that satisfies conditions (i), (ii), and (iii) with $a = 0$. Graph P , f , and the linear approximation $L(x) = 1$ on a common screen. Comment on how well the functions P and L approximate f .
2. Determine the values of x for which the quadratic approximation $f(x) \approx P(x)$ in Problem 1 is accurate to within 0.1. [Hint: Graph $y = P(x)$, $y = \cos x - 0.1$, and $y = \cos x + 0.1$ on a common screen.]
3. To approximate a function f by a quadratic function P near a number a , it is best to write P in the form

$$P(x) = A + B(x - a) + C(x - a)^2$$

Show that the quadratic function that satisfies conditions (i), (ii), and (iii) is

$$P(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2$$

4. Find the quadratic approximation to $f(x) = \sqrt{x+3}$ near $a = 1$. Graph f , the quadratic approximation, and the linear approximation from Example 3.10.2 on a common screen. What do you conclude?
5. Instead of being satisfied with a linear or quadratic approximation to $f(x)$ near $x = a$, let's try to find better approximations with higher-degree polynomials. We look for an n th-degree polynomial

$$T_n(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \cdots + c_n(x - a)^n$$

such that T_n and its first n derivatives have the same values at $x = a$ as f and its first n derivatives. By differentiating repeatedly and setting $x = a$, show that these conditions are satisfied if $c_0 = f(a)$, $c_1 = f'(a)$, $c_2 = \frac{1}{2}f''(a)$, and in general

$$c_k = \frac{f^{(k)}(a)}{k!}$$

where $k! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots k$. The resulting polynomial

$$T_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

is called the **n th-degree Taylor polynomial of f centered at a** . (We will study Taylor polynomials in more detail in Chapter 11.)

6. Find the 8th-degree Taylor polynomial centered at $a = 0$ for the function $f(x) = \cos x$. Graph f together with the Taylor polynomials T_2 , T_4 , T_6 , T_8 in the viewing rectangle $[-5, 5]$ by $[-1.4, 1.4]$ and comment on how well they approximate f .

3.11

Hyperbolic Functions

■ Hyperbolic Functions and Their Derivatives

Certain combinations of the exponential functions e^x and e^{-x} arise so frequently in mathematics and its applications that they deserve to be given special names. In many ways they are analogous to the trigonometric functions, and they have the same relationship to the hyperbola that the trigonometric functions have to the circle. For this reason they are collectively called **hyperbolic functions** and individually called **hyperbolic sine**, **hyperbolic cosine**, and so on.

Definition of the Hyperbolic Functions

$\sinh x = \frac{e^x - e^{-x}}{2}$	$\csc x = \frac{1}{\sinh x}$
$\cosh x = \frac{e^x + e^{-x}}{2}$	$\sech x = \frac{1}{\cosh x}$
$\tanh x = \frac{\sinh x}{\cosh x}$	$\coth x = \frac{\cosh x}{\sinh x}$

The graphs of hyperbolic sine and hyperbolic cosine can be sketched using graphical addition as in Figures 1 and 2.

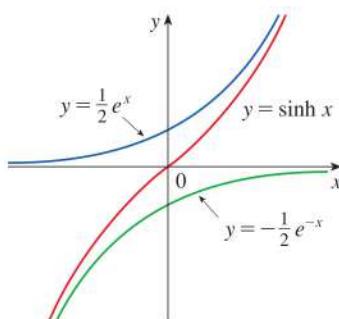


FIGURE 1
 $y = \sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

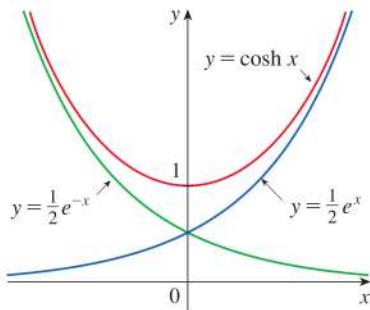


FIGURE 2
 $y = \cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

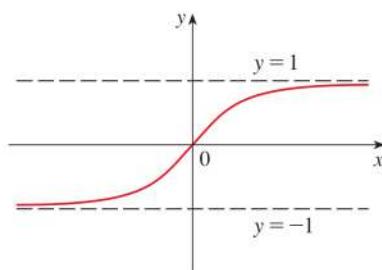


FIGURE 3
 $y = \tanh x$

Note that \sinh has domain \mathbb{R} and range \mathbb{R} , whereas \cosh has domain \mathbb{R} and range $[1, \infty)$. The graph of \tanh is shown in Figure 3. It has horizontal asymptotes $y = \pm 1$. (See Exercise 27.)

Some of the mathematical uses of hyperbolic functions will be seen in Chapter 7. Applications to science and engineering occur whenever an entity such as light, velocity, electricity, or radioactivity is gradually absorbed or extinguished because the decay can be represented by hyperbolic functions. The most famous application is the use of hyperbolic cosine to describe the shape of a hanging wire. It can be proved that if a heavy flexible cable (such as an overhead power line) is suspended between two points at the same height, then it takes the shape of a curve with equation

$$y = c + a \cosh(x/a)$$

called a *catenary* (see Figure 4). (The Latin word *catena* means “chain.”)

Another application of hyperbolic functions occurs in the description of ocean waves: the velocity of a water wave with length L moving across a body of water with depth d is modeled by the function

$$v = \sqrt{\frac{gL}{2\pi} \tanh\left(\frac{2\pi d}{L}\right)}$$

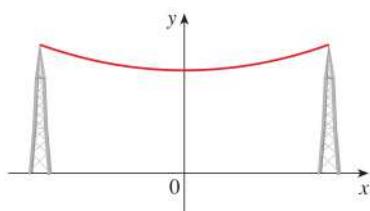


FIGURE 4
A catenary $y = c + a \cosh(x/a)$

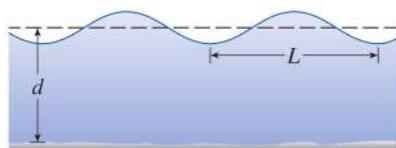


FIGURE 5
Idealized water wave

where g is the acceleration due to gravity (see Figure 5 and Exercise 57).

The hyperbolic functions satisfy a number of identities that are similar to well-known trigonometric identities. We list some of them here and leave most of the proofs to the exercises.

Hyperbolic Identities

$\sinh(-x) = -\sinh x$	$\cosh(-x) = \cosh x$
$\cosh^2 x - \sinh^2 x = 1$	$1 - \tanh^2 x = \operatorname{sech}^2 x$
$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$	
$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$	



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The Gateway Arch in St. Louis was designed using a hyperbolic cosine function (see Exercise 56).

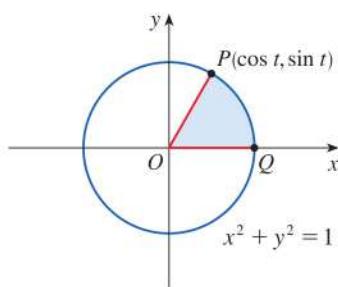


FIGURE 6

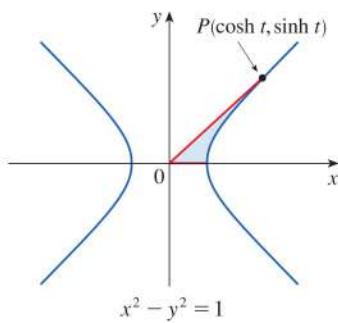


FIGURE 7

EXAMPLE 1 Prove (a) $\cosh^2 x - \sinh^2 x = 1$ and (b) $1 - \tanh^2 x = \operatorname{sech}^2 x$.

SOLUTION

$$\begin{aligned}\text{(a)} \quad \cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \\ &= \frac{4}{4} = 1\end{aligned}$$

(b) We start with the identity proved in part (a):

$$\cosh^2 x - \sinh^2 x = 1$$

If we divide both sides by $\cosh^2 x$, we get

$$1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

or

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

The identity proved in Example 1(a) gives a clue to the reason for the name “hyperbolic” functions:

If t is any real number, then the point $P(\cosh t, \sinh t)$ lies on the right branch of the hyperbola $x^2 - y^2 = 1$ because $\cosh^2 t - \sinh^2 t = 1$ and $\cosh t \geq 1$. In fact, t can be interpreted as the radian measure of $\angle POQ$ in Figure 6. For this reason the trigonometric functions are sometimes called *circular* functions.

Likewise, if t is any real number, then the point $P(\cos t, \sin t)$ lies on the right branch of the hyperbola $x^2 - y^2 = 1$ because $\cosh^2 t - \sinh^2 t = 1$ and $\cosh t \geq 1$. This time, t does not represent the measure of an angle. However, it turns out that t represents twice the area of the shaded hyperbolic sector in Figure 7, just as in the trigonometric case t represents twice the area of the shaded circular sector in Figure 6.

The derivatives of the hyperbolic functions are easily computed. For example,

$$\frac{d}{dx} (\sinh x) = \frac{d}{dx} \left(\frac{e^x - e^{-x}}{2} \right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

We list the differentiation formulas for the hyperbolic functions as Table 1. The remaining proofs are left as exercises. Note the analogy with the differentiation formulas for trigonometric functions, but note that the signs are different in some cases.

1 Derivatives of Hyperbolic Functions

$\frac{d}{dx} (\sinh x) = \cosh x$	$\frac{d}{dx} (\cosh x) = \sinh x$	$\frac{d}{dx} (\operatorname{csch} x) = -\operatorname{csch} x \coth x$	$\frac{d}{dx} (\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
$\frac{d}{dx} (\tanh x) = \operatorname{sech}^2 x$	$\frac{d}{dx} (\coth x) = -\operatorname{csch}^2 x$		

Any of these differentiation rules can be combined with the Chain Rule, as in the next example.

EXAMPLE 2 If $y = \cosh \sqrt{x}$, find dy/dx .

SOLUTION Using (1) and the Chain Rule, we have

$$\frac{dy}{dx} = \frac{d}{dx}(\cosh \sqrt{x}) = \sinh \sqrt{x} \cdot \frac{d}{dx} \sqrt{x} = \frac{\sinh \sqrt{x}}{2\sqrt{x}}$$

■

■ Inverse Hyperbolic Functions and Their Derivatives

You can see from Figures 1 and 3 that sinh and tanh are one-to-one functions and so they have inverse functions denoted by \sinh^{-1} and \tanh^{-1} . Figure 2 shows that cosh is not one-to-one, but if we restrict the domain to the interval $[0, \infty)$, then the function $y = \cosh x$ is one-to-one and attains all the values in its range $[1, \infty)$. The inverse hyperbolic cosine function is defined as the inverse of this restricted function.

2

$$\begin{aligned} y = \sinh^{-1} x &\iff \sinh y = x \\ y = \cosh^{-1} x &\iff \cosh y = x \quad \text{and} \quad y \geq 0 \\ y = \tanh^{-1} x &\iff \tanh y = x \end{aligned}$$

The remaining inverse hyperbolic functions are defined similarly (see Exercise 32).

We sketch the graphs of \sinh^{-1} , \cosh^{-1} , and \tanh^{-1} in Figures 8, 9, and 10 by referring to Figures 1, 2, and 3.

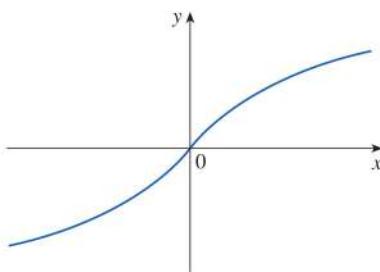


FIGURE 8 $y = \sinh^{-1} x$
domain = \mathbb{R} range = \mathbb{R}

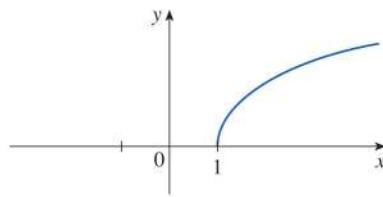


FIGURE 9 $y = \cosh^{-1} x$
domain = $[1, \infty)$ range = $[0, \infty)$

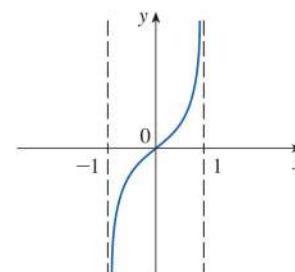


FIGURE 10 $y = \tanh^{-1} x$
domain = $(-1, 1)$ range = \mathbb{R}

Since the hyperbolic functions are defined in terms of exponential functions, it's not surprising to learn that the inverse hyperbolic functions can be expressed in terms of logarithms. In particular, we have:

3

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}) \quad x \in \mathbb{R}$$

4

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \quad x \geq 1$$

5

$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \quad -1 < x < 1$$

Formula 3 is proved in Example 3. The proofs of Formulas 4 and 5 are requested in Exercises 30 and 31.

EXAMPLE 3 Show that $\sinh^{-1}x = \ln(x + \sqrt{x^2 + 1})$.

SOLUTION Let $y = \sinh^{-1}x$. Then

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

so

$$e^y - 2x - e^{-y} = 0$$

or, multiplying by e^y ,

$$e^{2y} - 2xe^y - 1 = 0$$

This is really a quadratic equation in e^y :

$$(e^y)^2 - 2x(e^y) - 1 = 0$$

Solving by the quadratic formula, we get

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$$

Note that $e^y > 0$, but $x - \sqrt{x^2 + 1} < 0$ (because $x < \sqrt{x^2 + 1}$). Thus the minus sign is inadmissible and we have

$$e^y = x + \sqrt{x^2 + 1}$$

Therefore

$$y = \ln(e^y) = \ln(x + \sqrt{x^2 + 1})$$

This shows that

$$\sinh^{-1}x = \ln(x + \sqrt{x^2 + 1})$$

(See Exercise 29 for another method.) ■

Notice that the formulas for the derivatives of $\tanh^{-1}x$ and $\coth^{-1}x$ appear to be identical. But the domains of these functions have no numbers in common: $\tanh^{-1}x$ is defined for $|x| < 1$, whereas $\coth^{-1}x$ is defined for $|x| > 1$.

6 Derivatives of Inverse Hyperbolic Functions

$\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}$	$\frac{d}{dx}(\cosh^{-1}x) = -\frac{1}{ x \sqrt{x^2+1}}$
$\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}}$	$\frac{d}{dx}(\sech^{-1}x) = -\frac{1}{x\sqrt{1-x^2}}$
$\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}$	$\frac{d}{dx}(\coth^{-1}x) = \frac{1}{1-x^2}$

The inverse hyperbolic functions are all differentiable because the hyperbolic functions are differentiable (see Appendix F). The formulas in Table 6 can be proved either by the method for inverse functions or by differentiating Formulas 3, 4, and 5.

EXAMPLE 4 Prove that $\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}$.

SOLUTION 1 Let $y = \sinh^{-1}x$. Then $\sinh y = x$. If we differentiate this equation implicitly with respect to x , we get

$$\cosh y \frac{dy}{dx} = 1$$

Since $\cosh^2 y - \sinh^2 y = 1$ and $\cosh y \geq 0$, we have $\cosh y = \sqrt{1 + \sinh^2 y}$, so

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}$$

SOLUTION 2 From Equation 3 (proved in Example 3), we have

$$\begin{aligned}\frac{d}{dx}(\sinh^{-1}x) &= \frac{d}{dx} \ln(x + \sqrt{x^2 + 1}) \\ &= \frac{1}{x + \sqrt{x^2 + 1}} \frac{d}{dx}(x + \sqrt{x^2 + 1}) \\ &= \frac{1}{x + \sqrt{x^2 + 1}} \left(1 + \frac{x}{\sqrt{x^2 + 1}}\right) \\ &= \frac{\sqrt{x^2 + 1} + x}{(x + \sqrt{x^2 + 1})\sqrt{x^2 + 1}} \\ &= \frac{1}{\sqrt{x^2 + 1}}\end{aligned}$$

EXAMPLE 5 Find $\frac{d}{dx}[\tanh^{-1}(\sin x)]$.

SOLUTION Using Table 6 and the Chain Rule, we have

$$\begin{aligned}\frac{d}{dx}[\tanh^{-1}(\sin x)] &= \frac{1}{1 - (\sin x)^2} \frac{d}{dx}(\sin x) \\ &= \frac{1}{1 - \sin^2 x} \cos x = \frac{\cos x}{\cos^2 x} = \sec x\end{aligned}$$

3.11 Exercises

1–6 Find the numerical value of each expression.

- | | |
|--------------------------------|--------------------|
| 1. (a) $\sinh 0$ | (b) $\cosh 0$ |
| 2. (a) $\tanh 0$ | (b) $\tanh 1$ |
| 3. (a) $\cosh(\ln 5)$ | (b) $\cosh 5$ |
| 4. (a) $\sinh 4$ | (b) $\sinh(\ln 4)$ |
| 5. (a) $\operatorname{sech} 0$ | (b) $\cosh^{-1} 1$ |
| 6. (a) $\sinh 1$ | (b) $\sinh^{-1} 1$ |

7. Write $8 \sinh x + 5 \cosh x$ in terms of e^x and e^{-x} .

8. Write $2e^{2x} + 3e^{-2x}$ in terms of $\sinh 2x$ and $\cosh 2x$.

9. Write $\sinh(\ln x)$ as a rational function of x .

10. Write $\cosh(4 \ln x)$ as a rational function of x .

11–23 Prove the identity.

11. $\sinh(-x) = -\sinh x$
(This shows that \sinh is an odd function.)

12. $\cosh(-x) = \cosh x$
(This shows that \cosh is an even function.)

13. $\cosh x + \sinh x = e^x$

14. $\cosh x - \sinh x = e^{-x}$

15. $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

16. $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

17. $\coth^2 x - 1 = \operatorname{csch}^2 x$

18. $\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

19. $\sinh 2x = 2 \sinh x \cosh x$

20. $\cosh 2x = \cosh^2 x + \sinh^2 x$

21. $\tanh(\ln x) = \frac{x^2 - 1}{x^2 + 1}$

22. $\frac{1 + \tanh x}{1 - \tanh x} = e^{2x}$

23. $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$
(n any real number)

24. If $\tanh x = \frac{12}{13}$, find the values of the other hyperbolic functions at x .

25. If $\cosh x = \frac{5}{3}$ and $x > 0$, find the values of the other hyperbolic functions at x .

26. (a) Use the graphs of \sinh , \cosh , and \tanh in Figures 1–3 to draw the graphs of csch , sech , and \coth .

(b) Check the graphs that you sketched in part (a) by using a graphing calculator or computer to produce them.

27. Use the definitions of the hyperbolic functions to find each of the following limits.

(a) $\lim_{x \rightarrow \infty} \tanh x$

(b) $\lim_{x \rightarrow -\infty} \tanh x$

(c) $\lim_{x \rightarrow \infty} \sinh x$

(d) $\lim_{x \rightarrow -\infty} \sinh x$

(e) $\lim_{x \rightarrow \infty} \operatorname{sech} x$

(f) $\lim_{x \rightarrow \infty} \coth x$

(g) $\lim_{x \rightarrow 0^+} \coth x$

(h) $\lim_{x \rightarrow 0^-} \coth x$

(i) $\lim_{x \rightarrow -\infty} \operatorname{csch} x$

(j) $\lim_{x \rightarrow \infty} \frac{\sinh x}{e^x}$

28. Prove the formulas given in Table 1 for the derivatives of the functions (a) \cosh , (b) \tanh , (c) csch , (d) sech , and (e) \coth .

29. Give an alternative solution to Example 3 by letting $y = \sinh^{-1} x$ and then using Exercise 13 and Example 1(a) with x replaced by y .

30. Prove Equation 4.

31. Prove Equation 5 using (a) the method of Example 3 and (b) Exercise 22 with x replaced by y .

32. For each of the following functions (i) give a definition like those in (2), (ii) sketch the graph, and (iii) find a formula similar to Equation 3.

(a) csch^{-1} (b) sech^{-1} (c) \coth^{-1}

33. Prove the formula given in Table 6 for the derivative of each of the following functions.

(a) \cosh^{-1} (b) \tanh^{-1} (c) \coth^{-1}

34. Prove the formula given in Table 6 for the derivative of each of the following functions.

(a) sech^{-1} (b) csch^{-1}

35–53 Find the derivative. Simplify where possible.

35. $f(x) = \cosh 3x$

36. $f(x) = e^x \cosh x$

37. $h(x) = \sinh(x^2)$

38. $g(x) = \sinh^2 x$

39. $G(t) = \sinh(\ln t)$

40. $F(t) = \ln(\sinh t)$

41. $f(x) = \tanh \sqrt{x}$

42. $H(v) = e^{\tanh 2v}$

43. $y = \operatorname{sech} x \tanh x$

44. $y = \operatorname{sech}(\tanh x)$

45. $g(t) = t \coth \sqrt{t^2 + 1}$

46. $f(t) = \frac{1 + \sinh t}{1 - \sinh t}$

47. $f(x) = \sinh^{-1}(-2x)$

48. $g(x) = \tanh^{-1}(x^3)$

49. $y = \cosh^{-1}(\sec \theta)$, $0 \leq \theta < \pi/2$

50. $y = \operatorname{sech}^{-1}(\sin \theta)$, $0 < \theta < \pi/2$

51. $G(u) = \cosh^{-1} \sqrt{1 + u^2}$, $u > 0$

52. $y = x \tanh^{-1} x + \ln \sqrt{1 - x^2}$

53. $y = x \sinh^{-1}(x/3) - \sqrt{9 + x^2}$

54. Show that $\frac{d}{dx} \sqrt[4]{\frac{1 + \tanh x}{1 - \tanh x}} = \frac{1}{2} e^{x/2}$.

55. Show that $\frac{d}{dx} \arctan(\tanh x) = \operatorname{sech} 2x$.

56. **The Gateway Arch** The Gateway Arch in St. Louis was designed by Eero Saarinen and was constructed using the equation

$$y = 211.49 - 20.96 \cosh 0.03291765x$$

for the central curve of the arch, where x and y are measured in meters and $|x| \leq 91.20$.

(a) Graph the central curve.

(b) What is the height of the arch at its center?

(c) At what points is the height 100 m?

(d) What is the slope of the arch at the points in part (c)?

- 57.** If a water wave with length L moves with velocity v in a body of water with depth d , then

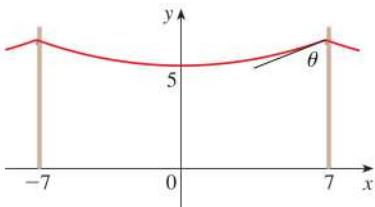
$$v = \sqrt{\frac{gL}{2\pi} \tanh\left(\frac{2\pi d}{L}\right)}$$

where g is the acceleration due to gravity. (See Figure 5.) Explain why the approximation

$$v \approx \sqrt{\frac{gL}{2\pi}}$$

is appropriate in deep water.

- 58.** A flexible cable always hangs in the shape of a catenary $y = c + a \cosh(x/a)$, where c and a are constants and $a > 0$ (see Figure 4 and Exercise 60). Graph several members of the family of functions $y = a \cosh(x/a)$. How does the graph change as a varies?
- 59.** A telephone line hangs between two poles 14 m apart in the shape of the catenary $y = 20 \cosh(x/20) - 15$, where x and y are measured in meters.
- Find the slope of this curve where it meets the right-hand pole.
 - Find the angle θ between the line and the pole.



- 60.** Using principles from physics it can be shown that when a cable is hung between two poles, it takes the shape of a curve $y = f(x)$ that satisfies the differential equation

$$\frac{d^2y}{dx^2} = \frac{\rho g}{T} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

where ρ is the linear density of the cable, g is the acceleration due to gravity, T is the tension in the cable at its lowest point, and the coordinate system is chosen appropriately. Verify that the function

$$y = f(x) = \frac{T}{\rho g} \cosh\left(\frac{\rho g x}{T}\right)$$

is a solution of this differential equation.

- 61.** A cable with linear density $\rho = 2 \text{ kg/m}$ is strung from the tops of two poles that are 200 m apart.

- Use Exercise 60 to find the tension T so that the cable is 60 m above the ground at its lowest point. How tall are the poles?
- If the tension is doubled, what is the new low point of the cable? How tall are the poles now?

- 62.** A model for the velocity of a falling object after time t is

$$v(t) = \sqrt{\frac{mg}{k}} \tanh\left(t \sqrt{\frac{gk}{m}}\right)$$

where m is the mass of the object, $g = 9.8 \text{ m/s}^2$ is the acceleration due to gravity, k is a constant, t is measured in seconds, and v in m/s.

- Calculate the terminal velocity of the object, that is, $\lim_{t \rightarrow \infty} v(t)$.
- If a person skydives from a plane, the value of the constant k depends on his or her position. For a “belly-to-earth” position, $k = 0.515 \text{ kg/s}$, but for a “feet-first” position, $k = 0.067 \text{ kg/s}$. If a 60-kg person descends in belly-to-earth position, what is the terminal velocity? What about feet-first?

Source: L. Long et al., “How Terminal Is Terminal Velocity?” American Mathematical Monthly 113 (2006): 752–55.

- 63.** (a) Show that any function of the form

$$y = A \sinh mx + B \cosh mx$$

satisfies the differential equation $y'' = m^2 y$.

- Find $y = y(x)$ such that $y'' = 9y$, $y(0) = -4$, and $y'(0) = 6$.

- 64.** If $x = \ln(\sec \theta + \tan \theta)$, show that $\sec \theta = \cosh x$.

- 65.** At what point of the curve $y = \cosh x$ does the tangent have slope 1?

- 66.** Investigate the family of functions

$$f_n(x) = \tanh(n \sin x)$$

where n is a positive integer. Describe what happens to the graph of f_n when n becomes large.

- 67.** Show that if $a \neq 0$ and $b \neq 0$, then there exist numbers α and β such that $ae^x + be^{-x}$ equals either

$$\alpha \sinh(x + \beta) \quad \text{or} \quad \alpha \cosh(x + \beta)$$

In other words, almost every function of the form $f(x) = ae^x + be^{-x}$ is a shifted and stretched hyperbolic sine or hyperbolic cosine function.

3 REVIEW

CONCEPT CHECK

- State each differentiation rule both in symbols and in words.
 - The Power Rule
 - The Constant Multiple Rule
 - The Sum Rule
 - The Difference Rule
 - The Product Rule
 - The Quotient Rule
 - The Chain Rule
- State the derivative of each function.

(a) $y = x^n$	(b) $y = e^x$	(c) $y = b^x$
(d) $y = \ln x$	(e) $y = \log_b x$	(f) $y = \sin x$
(g) $y = \cos x$	(h) $y = \tan x$	(i) $y = \csc x$
(j) $y = \sec x$	(k) $y = \cot x$	(l) $y = \sin^{-1} x$
(m) $y = \cos^{-1} x$	(n) $y = \tan^{-1} x$	(o) $y = \sinh x$
(p) $y = \cosh x$	(q) $y = \tanh x$	(r) $y = \sinh^{-1} x$
(s) $y = \cosh^{-1} x$	(t) $y = \tanh^{-1} x$	
- (a) How is the number e defined?
 (b) Express e as a limit.
 (c) Why is the natural exponential function $y = e^x$ used more often in calculus than the other exponential functions $y = b^x$?
- (d) Why is the natural logarithmic function $y = \ln x$ used more often in calculus than the other logarithmic functions $y = \log_b x$?
- (e) Explain how implicit differentiation works.
 (f) Explain how logarithmic differentiation works.
- Give several examples of how the derivative can be interpreted as a rate of change in physics, chemistry, biology, economics, or other sciences.
- (a) Write a differential equation that expresses the law of natural growth.
 (b) Under what circumstances is this an appropriate model for population growth?
 (c) What are the solutions of this equation?
- (a) Write an expression for the linearization of f at a .
 (b) If $y = f(x)$, write an expression for the differential dy .
 (c) If $dx = \Delta x$, draw a picture showing the geometric meanings of Δy and dy .

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If f and g are differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

2. If f and g are differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g'(x)$$

3. If f and g are differentiable, then

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

4. If f is differentiable, then $\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$.

5. If f is differentiable, then $\frac{d}{dx}f(\sqrt{x}) = \frac{f'(x)}{2\sqrt{x}}$.

6. If $y = e^2$, then $y' = 2e$.

7. $\frac{d}{dx}(10^x) = x10^{x-1}$

8. $\frac{d}{dx}(\ln 10) = \frac{1}{10}$

9. $\frac{d}{dx}(\tan^2 x) = \frac{d}{dx}(\sec^2 x)$

10. $\frac{d}{dx}|x^2 + x| = |2x + 1|$

11. The derivative of a polynomial is a polynomial.

12. If $f(x) = (x^6 - x^4)^5$, then $f^{(31)}(x) = 0$.

13. The derivative of a rational function is a rational function.

14. An equation of the tangent line to the parabola $y = x^2$ at $(-2, 4)$ is $y - 4 = 2x(x + 2)$.

15. If $g(x) = x^5$, then $\lim_{x \rightarrow 2} \frac{g(x) - g(2)}{x - 2} = 80$.

EXERCISES

1–54 Calculate y' .

1. $y = (x^2 + x^3)^4$

3. $y = \frac{x^2 - x + 2}{\sqrt{x}}$

5. $y = x^2 \sin \pi x$

7. $y = \frac{t^4 - 1}{t^4 + 1}$

9. $y = \ln(x \ln x)$

11. $y = \sqrt{x} \cos \sqrt{x}$

13. $y = \frac{e^{1/x}}{x^2}$

15. $y + x \cos y = x^2 y$

17. $y = \sqrt{\arctan x}$

19. $y = \tan\left(\frac{t}{1+t^2}\right)$

21. $y = 3^{x \ln x}$

23. $y = (1 - x^{-1})^{-1}$

25. $\sin(xy) = x^2 - y$

27. $y = \log_5(1 + 2x)$

29. $y = \ln \sin x - \frac{1}{2} \sin^2 x$

31. $y = x \tan^{-1}(4x)$

33. $y = \ln |\sec 5x + \tan 5x|$

35. $y = \cot(3x^2 + 5)$

37. $y = \sin(\tan \sqrt{1+x^3})$

39. $y = 5 \arctan(1/x)$

40. $y = \sin^{-1}(\cos \theta), \quad 0 < \theta < \pi$

41. $y = x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2)$

42. $y = \ln(\arcsin x^2)$

44. $y + \ln y = xy^2$

2. $y = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt[3]{x^3}}$

4. $y = \frac{\tan x}{1 + \cos x}$

6. $y = x \cos^{-1} x$

8. $xe^y = y \sin x$

10. $y = e^{mx} \cos nx$

12. $y = (\arcsin 2x)^2$

14. $y = \ln \sec x$

16. $y = \left(\frac{u-1}{u^2+u+1}\right)^4$

18. $y = \cot(\csc x)$

20. $y = e^{x \sec x}$

22. $y = \sec(1 + x^2)$

24. $y = 1/\sqrt[3]{x} + \sqrt{x}$

26. $y = \sqrt{\sin \sqrt{x}}$

28. $y = (\cos x)^x$

30. $y = \frac{(x^2 + 1)^4}{(2x + 1)^3(3x - 1)^5}$

32. $y = e^{\cos x} + \cos(e^x)$

34. $y = 10^{\tan \pi \theta}$

36. $y = \sqrt{t \ln(t^4)}$

38. $y = x \sec^{-1} x$

46. $y = \frac{(x + \lambda)^4}{x^4 + \lambda^4}$

47. $y = x \sinh(x^2)$

48. $y = \frac{\sin mx}{x}$

49. $y = \ln(\cosh 3x)$

50. $y = \ln \left| \frac{x^2 - 4}{2x + 5} \right|$

51. $y = \cosh^{-1}(\sinh x)$

52. $y = x \tanh^{-1} \sqrt{x}$

53. $y = \cos(e^{\sqrt{\tan 3x}})$

54. $y = \sin^2(\cos \sqrt{\sin \pi x})$

55. If $f(t) = \sqrt{4t + 1}$, find $f''(2)$.56. If $g(\theta) = \theta \sin \theta$, find $g''(\pi/6)$.57. Find y'' if $x^6 + y^6 = 1$.58. Find $f^{(n)}(x)$ if $f(x) = 1/(2 - x)$.59. Use mathematical induction to show that if $f(x) = xe^x$, then $f^{(n)}(x) = (x + n)e^x$. (Note: See Principles of Problem Solving following Chapter 1.)

60. Evaluate $\lim_{t \rightarrow 0} \frac{t^3}{\tan^3(2t)}$.

61–63 Find an equation of the tangent line to the curve at the given point.

61. $y = 4 \sin^2 x, \quad (\pi/6, 1)$

62. $y = \frac{x^2 - 1}{x^2 + 1}, \quad (0, -1)$

63. $y = \sqrt{1 + 4 \sin x}, \quad (0, 1)$

64–65 Find equations of the tangent line and normal line to the curve at the given point.

64. $x^2 + 4xy + y^2 = 13, \quad (2, 1)$

65. $y = (2 + x)e^{-x}, \quad (0, 2)$

66. If $f(x) = xe^{\sin x}$, find $f'(x)$. Graph f and f' on the same screen and comment.67. (a) If $f(x) = x\sqrt{5-x}$, find $f'(x)$.(b) Find equations of the tangent lines to the curve $y = x\sqrt{5-x}$ at the points $(1, 2)$ and $(4, 4)$.

67. (c) Illustrate part (b) by graphing the curve and tangent lines on the same screen.

67. (d) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f' .

- 68.** (a) If $f(x) = 4x - \tan x$, $-\pi/2 < x < \pi/2$, find f' and f'' .

(b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f , f' , and f'' .

- 69.** At what points on the curve $y = \sin x + \cos x$, $0 \leq x \leq 2\pi$, is the tangent line horizontal?

- 70.** Find the points on the ellipse $x^2 + 2y^2 = 1$ where the tangent line has slope 1.

- 71.** If $f(x) = (x - a)(x - b)(x - c)$, show that

$$\frac{f'(x)}{f(x)} = \frac{1}{x - a} + \frac{1}{x - b} + \frac{1}{x - c}$$

- 72.** (a) By differentiating the double-angle formula

$$\cos 2x = \cos^2 x - \sin^2 x$$

obtain the double-angle formula for the sine function.

- (b) By differentiating the addition formula

$$\sin(x + a) = \sin x \cos a + \cos x \sin a$$

obtain the addition formula for the cosine function.

- 73.** Suppose that

$$f(1) = 2 \quad f'(1) = 3 \quad f(2) = 1 \quad f'(2) = 2$$

$$g(1) = 3 \quad g'(1) = 1 \quad g(2) = 1 \quad g'(2) = 4$$

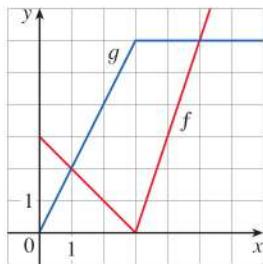
- (a) If $S(x) = f(x) + g(x)$, find $S'(1)$.

- (b) If $P(x) = f(x)g(x)$, find $P'(2)$.

- (c) If $Q(x) = f(x)/g(x)$, find $Q'(1)$.

- (d) If $C(x) = f(g(x))$, find $C'(2)$.

- 74.** If f and g are the functions whose graphs are shown, let $P(x) = f(x)g(x)$, $Q(x) = f(x)/g(x)$, and $C(x) = f(g(x))$. Find (a) $P'(2)$, (b) $Q'(2)$, and (c) $C'(2)$.



- 75–82** Find f' in terms of g' .

75. $f(x) = x^2g(x)$

76. $f(x) = g(x^2)$

77. $f(x) = [g(x)]^2$

78. $f(x) = g(g(x))$

79. $f(x) = g(e^x)$

80. $f(x) = e^{g(x)}$

81. $f(x) = \ln |g(x)|$

82. $f(x) = g(\ln x)$

- 83–85** Find h' in terms of f' and g' .

83. $h(x) = \frac{f(x)g(x)}{f(x) + g(x)}$

84. $h(x) = \sqrt{\frac{f(x)}{g(x)}}$

85. $h(x) = f(g(\sin 4x))$

- 86.** (a) Graph the function $f(x) = x - 2 \sin x$ in the viewing rectangle $[0, 8]$ by $[-2, 8]$.

- (b) On which interval is the average rate of change larger: $[1, 2]$ or $[2, 3]$?

- (c) At which value of x is the instantaneous rate of change larger: $x = 2$ or $x = 5$?

- (d) Check your visual estimates in part (c) by computing $f'(x)$ and comparing the numerical values of $f'(2)$ and $f'(5)$.

- 87.** At what point on the curve

$$y = [\ln(x + 4)]^2$$

is the tangent horizontal?

- 88.** (a) Find an equation of the tangent line to the curve $y = e^x$ that is parallel to the line $x - 4y = 1$.

- (b) Find an equation of the tangent to the curve $y = e^x$ that passes through the origin.

- 89.** Find a parabola $y = ax^2 + bx + c$ that passes through the point $(1, 4)$ and whose tangent lines at $x = -1$ and $x = 5$ have slopes 6 and -2 , respectively.

- 90.** The function $C(t) = K(e^{-at} - e^{-bt})$, where a , b , and K are positive constants and $b > a$, is used to model the concentration at time t of a drug injected into the bloodstream.

- (a) Show that $\lim_{t \rightarrow \infty} C(t) = 0$.

- (b) Find $C'(t)$, the rate of change of drug concentration in the blood.

- (c) When is this rate equal to 0?

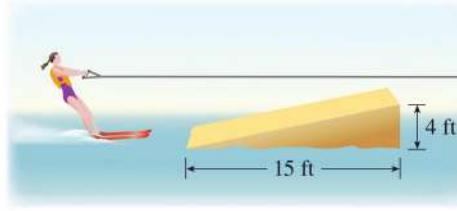
- 91.** An equation of motion of the form $s = Ae^{-ct} \cos(\omega t + \delta)$ represents damped oscillation of an object. Find the velocity and acceleration of the object.

- 92.** A particle moves along a horizontal line so that its coordinate at time t is $x = \sqrt{b^2 + c^2 t^2}$, $t \geq 0$, where b and c are positive constants.

- (a) Find the velocity and acceleration functions.

- (b) Show that the particle always moves in the positive direction.

- 93.** A particle moves on a vertical line so that its coordinate at time t is $y = t^3 - 12t + 3$, $t \geq 0$.
- Find the velocity and acceleration functions.
 - When is the particle moving upward and when is it moving downward?
 - Find the distance that the particle travels in the time interval $0 \leq t \leq 3$.
 - Graph the position, velocity, and acceleration functions for $0 \leq t \leq 3$.
 - When is the particle speeding up? When is it slowing down?
- 94.** The volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height.
- Find the rate of change of the volume with respect to the height if the radius is constant.
 - Find the rate of change of the volume with respect to the radius if the height is constant.
- 95.** The mass of part of a wire is $x(1 + \sqrt{x})$ kilograms, where x is measured in meters from one end of the wire. Find the linear density of the wire when $x = 4$ m.
- 96.** The cost, in dollars, of producing x units of a certain commodity is
- $$C(x) = 920 + 2x - 0.02x^2 + 0.00007x^3$$
- Find the marginal cost function.
 - Find $C'(100)$ and explain its meaning.
 - Compare $C'(100)$ with the cost of producing the 101st item.
- 97.** A bacteria culture contains 200 cells initially and grows at a rate proportional to its size. After half an hour the population has increased to 360 cells.
- Find the number of cells after t hours.
 - Find the number of cells after 4 hours.
 - Find the rate of growth after 4 hours.
 - When will the population reach 10,000?
- 98.** Cobalt-60 has a half-life of 5.24 years.
- Find the mass that remains from a 100-mg sample after 20 years.
 - How long would it take for the mass to decay to 1 mg?
- 99.** Let $C(t)$ be the concentration of a drug in the bloodstream. As the body eliminates the drug, $C(t)$ decreases at a rate that is proportional to the amount of the drug that is present at the time. Thus $C'(t) = -kC(t)$, where k is a positive number called the *elimination constant* of the drug.
- If C_0 is the concentration at time $t = 0$, find the concentration at time t .
 - If the body eliminates half the drug in 30 hours, how long does it take to eliminate 90% of the drug?

- 100.** A cup of hot chocolate has temperature 80°C in a room kept at 20°C . After half an hour the hot chocolate cools to 60°C .
- What is the temperature of the chocolate after another half hour?
 - When will the chocolate have cooled to 40°C ?
- 101.** The volume of a cube is increasing at a rate of $10 \text{ cm}^3/\text{min}$. How fast is the surface area increasing when the length of an edge is 30 cm?
- 102.** A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of $2 \text{ cm}^3/\text{s}$, how fast is the water level rising when the water is 5 cm deep?
- 103.** A balloon is rising at a constant speed of 5 ft/s. A boy is cycling along a straight road at a speed of 15 ft/s. When he passes under the balloon, it is 45 ft above him. How fast is the distance between the boy and the balloon increasing 3 s later?
- 104.** A waterskier skis over the ramp shown in the figure at a speed of 30 ft/s. How fast is she rising as she leaves the ramp?
- 
- 105.** The angle of elevation of the sun is decreasing at a rate of 0.25 rad/h . How fast is the shadow cast by a 400-ft-tall building increasing when the angle of elevation of the sun is $\pi/6$?
- 106.**
- Find the linear approximation to $f(x) = \sqrt{25 - x^2}$ near 3.
 - Illustrate part (a) by graphing f and the linear approximation.
 - For what values of x is the linear approximation accurate to within 0.1?
- 107.**
- Find the linearization of $f(x) = \sqrt[3]{1 + 3x}$ at $a = 0$. State the corresponding linear approximation and use it to give an approximate value for $\sqrt[3]{1.03}$.
 - Determine the values of x for which the linear approximation given in part (a) is accurate to within 0.1.
- 108.** Evaluate dy if $y = x^3 - 2x^2 + 1$, $x = 2$, and $dx = 0.2$.

- 109.** A window has the shape of a square surmounted by a semi-circle. The base of the window is measured as having width 60 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum error possible in computing the area of the window.

110–112 Express the limit as a derivative and evaluate.

110. $\lim_{x \rightarrow 1} \frac{x^{17} - 1}{x - 1}$

111. $\lim_{h \rightarrow 0} \frac{\sqrt[4]{16+h} - 2}{h}$

112. $\lim_{\theta \rightarrow \pi/3} \frac{\cos \theta - 0.5}{\theta - \pi/3}$

113. Evaluate $\lim_{x \rightarrow 0} \frac{\sqrt{1 + \tan x} - \sqrt{1 + \sin x}}{x^3}$.

- 114.** Suppose f is a differentiable function such that $f(g(x)) = x$ and $f'(x) = 1 + [f(x)]^2$. Show that $g'(x) = 1/(1 + x^2)$.

- 115.** Find $f'(x)$ if it is known that

$$\frac{d}{dx} [f(2x)] = x^2$$

- 116.** Show that the length of the portion of any tangent line to the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ cut off by the coordinate axes is constant.

Problems Plus

Try to solve the following examples yourself before reading the solutions.

EXAMPLE 1 How many lines are tangent to both of the parabolas $y = -1 - x^2$ and $y = 1 + x^2$? Find the coordinates of the points at which these tangents touch the parabolas.

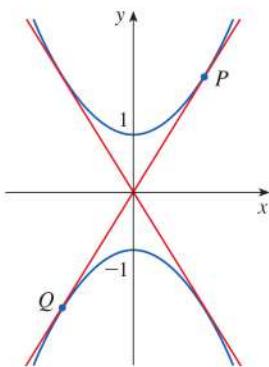


FIGURE 1

SOLUTION To gain insight into this problem, it is essential to draw a diagram. So we sketch the parabolas $y = 1 + x^2$ (which is the standard parabola $y = x^2$ shifted 1 unit upward) and $y = -1 - x^2$ (which is obtained by reflecting the first parabola about the x -axis). If we try to draw a line tangent to both parabolas, we soon discover that there are only two possibilities, as illustrated in Figure 1.

Let P be a point at which one of these tangents touches the upper parabola and let a be its x -coordinate. (The choice of notation for the unknown is important. Of course we could have used b or c or x_0 or x_1 instead of a . However, it's not advisable to use x in place of a because that x could be confused with the variable x in the equation of the parabola.) Then, since P lies on the parabola $y = 1 + x^2$, its y -coordinate must be $1 + a^2$. Because of the symmetry shown in Figure 1, the coordinates of the point Q where the tangent touches the lower parabola must be $(-a, -(1 + a^2))$.

To use the given information that the line is a tangent, we equate the slope of the line PQ to the slope of the tangent line at P . We have

$$m_{PQ} = \frac{1 + a^2 - (-1 - a^2)}{a - (-a)} = \frac{1 + a^2}{a}$$

If $f(x) = 1 + x^2$, then the slope of the tangent line at P is $f'(a) = 2a$. Thus the condition that we need to use is that

$$\frac{1 + a^2}{a} = 2a$$

Solving this equation, we get $1 + a^2 = 2a^2$, so $a^2 = 1$ and $a = \pm 1$. Therefore the points are $(1, 2)$ and $(-1, -2)$. By symmetry, the two remaining points are $(-1, 2)$ and $(1, -2)$. ■

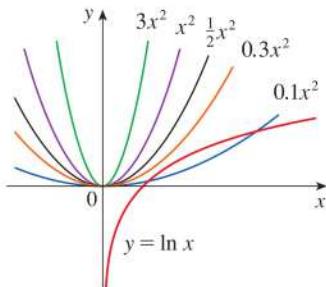


FIGURE 2

EXAMPLE 2 For what values of c does the equation $\ln x = cx^2$ have exactly one solution?

SOLUTION One of the most important principles of problem solving is to draw a diagram, even if the problem as stated doesn't explicitly mention a geometric situation. Our present problem can be reformulated geometrically as follows: for what values of c does the curve $y = \ln x$ intersect the curve $y = cx^2$ in exactly one point?

Let's start by graphing $y = \ln x$ and $y = cx^2$ for various values of c . We know that, for $c \neq 0$, $y = cx^2$ is a parabola that opens upward if $c > 0$ and downward if $c < 0$. Figure 2 shows the parabolas $y = cx^2$ for several positive values of c . Most of them don't intersect $y = \ln x$ at all and one intersects twice. We have the feeling that there must be a value of c (somewhere between 0.1 and 0.3) for which the curves intersect exactly once, as in Figure 3.

To find that particular value of c , we let a be the x -coordinate of the single point of intersection. In other words, $\ln a = ca^2$, so a is the unique solution of the given equation. We see from Figure 3 that the curves just touch, so they have a common tangent line when $x = a$. That means the curves $y = \ln x$ and $y = cx^2$ have the same slope when $x = a$. Therefore

$$\frac{1}{a} = 2ca$$

FIGURE 3

Solving the equations $\ln a = ca^2$ and $1/a = 2ca$, we get

$$\ln a = ca^2 = c \cdot \frac{1}{2c} = \frac{1}{2}$$

Thus $a = e^{1/2}$ and

$$c = \frac{\ln a}{a^2} = \frac{\ln e^{1/2}}{e} = \frac{1}{2e}$$

For negative values of c we have the situation illustrated in Figure 4: all parabolas $y = cx^2$ with negative values of c intersect $y = \ln x$ exactly once. And let's not forget about $c = 0$: the curve $y = 0x^2 = 0$ is just the x -axis, which intersects $y = \ln x$ exactly once.

To summarize, the required values of c are $c = 1/(2e)$ and $c \leq 0$. ■

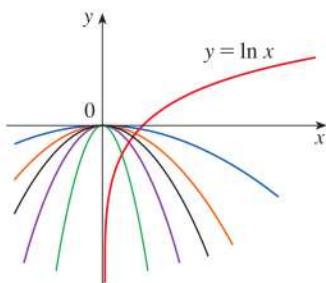
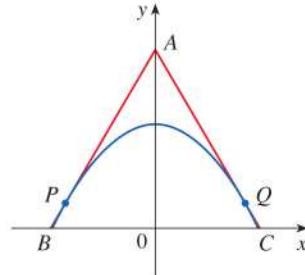


FIGURE 4

Problems

1. Find points P and Q on the parabola $y = 1 - x^2$ so that the triangle ABC formed by the x -axis and the tangent lines at P and Q is an equilateral triangle (see the figure).



2. Find the point where the curves $y = x^3 - 3x + 4$ and $y = 3(x^2 - x)$ are tangent to each other, that is, have a common tangent line. Illustrate by sketching both curves and the common tangent.

3. Show that the tangent lines to the parabola $y = ax^2 + bx + c$ at any two points with x -coordinates p and q must intersect at a point whose x -coordinate is halfway between p and q .

4. Show that $\frac{d}{dx} \left(\frac{\sin^2 x}{1 + \cot x} + \frac{\cos^2 x}{1 + \tan x} \right) = -\cos 2x$.

5. If $f(x) = \lim_{t \rightarrow x} \frac{\sec t - \sec x}{t - x}$, find the value of $f'(\pi/4)$.

6. Find the values of the constants a and b such that

$$\lim_{x \rightarrow 0} \frac{\sqrt[3]{ax + b} - 2}{x} = \frac{5}{12}$$

7. Show that $\sin^{-1}(\tanh x) = \tan^{-1}(\sinh x)$.

8. A car is traveling at night along a highway shaped like a parabola with its vertex at the origin (see the figure). The car starts at a point 100 m west and 100 m north of the origin and travels in an easterly direction. There is a statue located 100 m east and 50 m north of the origin. At what point on the highway will the car's headlights illuminate the statue?

9. Prove that $\frac{d^n}{dx^n} (\sin^4 x + \cos^4 x) = 4^{n-1} \cos(4x + n\pi/2)$.

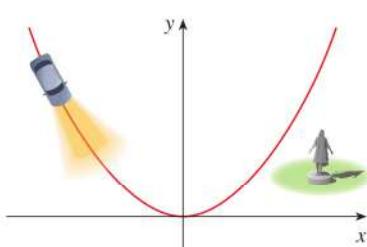
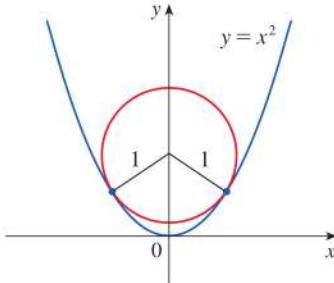


FIGURE FOR PROBLEM 8

- 10.** If f is differentiable at a , where $a > 0$, evaluate the following limit in terms of $f'(a)$:

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}}$$

- 11.** The figure shows a circle with radius 1 inscribed in the parabola $y = x^2$. Find the center of the circle.



- 12.** Find all values of c such that the parabolas $y = 4x^2$ and $x = c + 2y^2$ intersect each other at right angles.

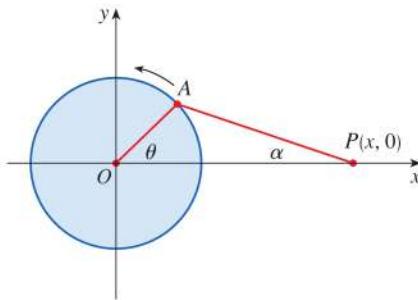
- 13.** How many lines are tangent to both of the circles $x^2 + y^2 = 4$ and $x^2 + (y - 3)^2 = 1$? At what points do these tangent lines touch the circles?

- 14.** If $f(x) = \frac{x^{46} + x^{45} + 2}{1 + x}$, calculate $f^{(46)}(3)$. Express your answer using factorial notation:

$$n! = 1 \cdot 2 \cdot 3 \cdots \cdots (n - 1) \cdot n$$

- 15.** The figure shows a rotating wheel with radius 40 cm and a connecting rod AP with length 1.2 m. The pin P slides back and forth along the x -axis as the wheel rotates counter-clockwise at a rate of 360 revolutions per minute.

- (a) Find the angular velocity of the connecting rod, $d\alpha/dt$, in radians per second, when $\theta = \pi/3$.
(b) Express the distance $x = |OP|$ in terms of θ .
(c) Find an expression for the velocity of the pin P in terms of θ .



- 16.** Tangent lines T_1 and T_2 are drawn at two points P_1 and P_2 on the parabola $y = x^2$ and they intersect at a point P . Another tangent line T is drawn at a point between P_1 and P_2 ; it intersects T_1 at Q_1 and T_2 at Q_2 . Show that

$$\frac{|PQ_1|}{|PP_1|} + \frac{|PQ_2|}{|PP_2|} = 1$$

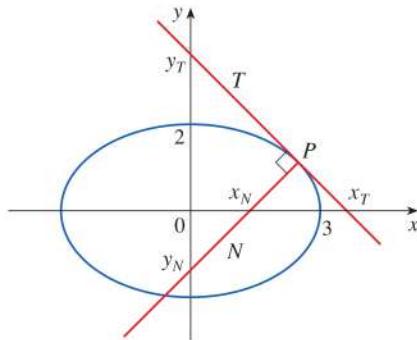
- 17.** Show that

$$\frac{d^n}{dx^n} (e^{ax} \sin bx) = r^n e^{ax} \sin(bx + n\theta)$$

where a and b are positive numbers, $r^2 = a^2 + b^2$, and $\theta = \tan^{-1}(b/a)$.

18. Evaluate $\lim_{x \rightarrow \pi} \frac{e^{\sin x} - 1}{x - \pi}$.

19. Let T and N be the tangent and normal lines to the ellipse $x^2/9 + y^2/4 = 1$ at any point P on the ellipse in the first quadrant. Let x_T and y_T be the x - and y -intercepts of T and x_N and y_N be the intercepts of N . As P moves along the ellipse in the first quadrant (but not on the axes), what values can x_T , y_T , x_N , and y_N take on? First try to guess the answers just by looking at the figure. Then use calculus to solve the problem and see how good your intuition is.



20. Evaluate $\lim_{x \rightarrow 0} \frac{\sin(3+x)^2 - \sin 9}{x}$.

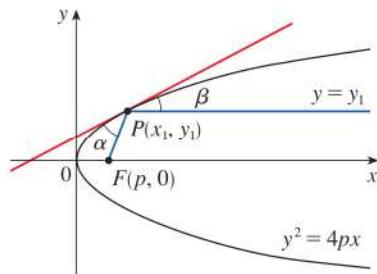
21. (a) Use the identity for $\tan(x - y)$ (see Equation 15b in Appendix D) to show that if two lines L_1 and L_2 intersect at an angle α , then

$$\tan \alpha = \frac{m_2 - m_1}{1 + m_1 m_2}$$

where m_1 and m_2 are the slopes of L_1 and L_2 , respectively.

- (b) The **angle between the curves** C_1 and C_2 at a point of intersection P is defined to be the angle between the tangent lines to C_1 and C_2 at P (if these tangent lines exist). Use part (a) to find, correct to the nearest degree, the angle between each pair of curves at each point of intersection.
- (i) $y = x^2$ and $y = (x - 2)^2$
 - (ii) $x^2 - y^2 = 3$ and $x^2 - 4x + y^2 + 3 = 0$

22. Let $P(x_1, y_1)$ be a point on the parabola $y^2 = 4px$ with focus $F(p, 0)$. Let α be the angle between the parabola and the line segment FP , and let β be the angle between the horizontal line $y = y_1$ and the parabola as in the figure. Prove that $\alpha = \beta$. (Thus, by a principle of geometrical optics, light from a source placed at F will be reflected along a line parallel to the x -axis. This explains why *paraboloids*, the surfaces obtained by rotating parabolas about their axes, are used as the shape of some automobile headlights and mirrors for telescopes.)



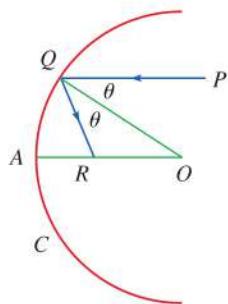


FIGURE FOR PROBLEM 23

- 23.** Suppose that we replace the parabolic mirror of Problem 22 by a spherical mirror. Although the mirror has no focus, we can show the existence of an *approximate* focus. In the figure, C is a semicircle with center O . A ray of light coming in toward the mirror parallel to the axis along the line PQ will be reflected to the point R on the axis so that $\angle PQO = \angle OQR$ (the angle of incidence is equal to the angle of reflection). What happens to the point R as P is taken closer and closer to the axis?

- 24.** If f and g are differentiable functions with $f(0) = g(0) = 0$ and $g'(0) \neq 0$, show that

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)}$$

- 25.** Evaluate $\lim_{x \rightarrow 0} \frac{\sin(a+2x) - 2\sin(a+x) + \sin a}{x^2}$.

- T 26.** (a) The cubic function $f(x) = x(x-2)(x-6)$ has three distinct zeros: 0, 2, and 6. Graph f and its tangent lines at the *average* of each pair of zeros. What do you notice?
 (b) Suppose the cubic function $f(x) = (x-a)(x-b)(x-c)$ has three distinct zeros: a , b , and c . Prove, with the help of a computer algebra system, that a tangent line drawn at the average of the zeros a and b intersects the graph of f at the third zero.

- 27.** For what value of k does the equation $e^{2x} = k\sqrt{x}$ have exactly one solution?

- 28.** For which positive numbers a is it true that $a^x \geq 1 + x$ for all x ?

- 29.** If

$$y = \frac{x}{\sqrt{a^2 - 1}} - \frac{2}{\sqrt{a^2 - 1}} \arctan \frac{\sin x}{a + \sqrt{a^2 - 1} + \cos x}$$

show that $y' = \frac{1}{a + \cos x}$.

- 30.** Given an ellipse $x^2/a^2 + y^2/b^2 = 1$, where $a \neq b$, find the equation of the set of all points from which there are two tangents to the curve whose slopes are (a) reciprocals and (b) negative reciprocals.

- 31.** Find the two points on the curve $y = x^4 - 2x^2 - x$ that have a common tangent line.

- 32.** Suppose that three points on the parabola $y = x^2$ have the property that their normal lines intersect at a common point. Show that the sum of their x -coordinates is 0.

- 33.** A *lattice point* in the plane is a point with integer coordinates. Suppose that circles with radius r are drawn using all lattice points as centers. Find the smallest value of r such that any line with slope $\frac{2}{5}$ intersects some of these circles.

- 34.** A cone of radius r centimeters and height h centimeters is lowered point first at a rate of 1 cm/s into a tall cylinder of radius R centimeters that is partially filled with water. How fast is the water level rising at the instant the cone is completely submerged?

- 35.** A container in the shape of an inverted cone has height 16 cm and radius 5 cm at the top. It is partially filled with a liquid that oozes through the sides at a rate proportional to the area of the container that is in contact with the liquid. (The surface area of a cone is $\pi r l$, where r is the radius and l is the slant height.) If we pour the liquid into the container at a rate of 2 cm³/min, then the height of the liquid decreases at a rate of 0.3 cm/min when the height is 10 cm. If our goal is to keep the liquid at a constant height of 10 cm, at what rate should we pour the liquid into the container?