Let X be the set

$$\{\infty\} \coprod \mathbb{N} \coprod (\mathbb{N} \times \mathbb{N})$$

and consider the topology τ_{Arens} generated by

$$\begin{aligned}
&\{(a,b) : (a,b) \in \mathbb{N} \times \mathbb{N}\} \\
B_{n,i} &= \{n\} \cup \{(n,k) : k \geq i\} \\
W_m &= \{\infty\} \cup (\bigcup_{m \geq j} B_{m,1} \setminus \{(m,b_1), ...(m,b_{s_m})\})
\end{aligned}$$

This says the pairs are all isolated points, and open sets around n contain all but finitely many pairs with first coordinate n. Finally, the s_m are some finite numbers, depending on m, so an open set containing ∞ all but finitely many of all but finitely many columns.

Definition 1. *The* ***Arens space*** *is* (X, τ_{Arens}).

Claim 1. *In the Arens space, the sequential closure operator is not idempotent. That is, the equation* $scl(scl(A)) = scl(A)$ *is* ***not*** *true.*

Proof. Let $A = \mathbb{N} \times \mathbb{N}$. For any fixed n, we have the sequence $x_i = (n,i)$, which converges to $n \in \mathbb{N}$ (by definition of $B_{n,i}$).

On the other hand, no sequence in $\mathbb{N} \times \mathbb{N}$ can converge to ∞. Suppose $x_i = (a_i, b_i)$ did so. There must be a subsequence $x_{i,j}$ with a_{i_j} going to infinity, otherwise we could remove the finitely many columns appearing in the sequence and get an open set around ∞ which does not contain the sequence. Then, form an open set W_m by setting $s_m = 1$ for all m, and removing (m, b_{i_j}) whenever $a_{i_j} = m$. Since the a_{i_j} go to infinity, the sequence cannot eventually be contained in the neighborhood.

It is clear that $x_i = i$ converges to ∞.

Therefore,

$$scl(\mathbb{N} \times \mathbb{N}) = \mathbb{N} \cup \mathbb{N} \times \mathbb{N}$$

but

$$scl(scl(\mathbb{N} \times \mathbb{N})) = \{\infty\} \cup \mathbb{N} \cup \mathbb{N} \times \mathbb{N}$$

□

Definition 2. *A space* (X, τ) *is called* ***Frechét-Urysohn*** *if the sequential closure operator is idempotent.*

Definition 3. *A set A is called **sequentially closed** if the limit of any sequence in A is also in A.*

Definition 4. *A topological space (X, τ) is called **sequential** if every sequentially closed subset is closed.*

Claim 2. *The Arens space is sequential.*

Proof. Say A is a sequentially closed set, so we must show A is closed. Consider the n for which $\{(n,i)\} \cap A$ is infinite. For any such n, we have $n \in A$ since this gives a convergent sequence. If there are only finitely many such n, then the complement of A is open: if N is the maximum such n, then W_{N+1} (minus finitely many pairs in each column) is most of the complement, and for any n that doesn't appear, some $B_{n,k}$ will cover all but finitely many of the column. Everything in the complement not in one of these sets is a pair (a, b), so any union of them will be open. Thus the complement of A is open, and A is closed. $\square$

Let us give an alternate proof:

Claim 3. *Any first countable space is sequential.*

Proof. Suppose A is a sequentially closed set, and pick x in the closure of A. This means every open set containing x intersects A (otherwise x would be in the interior of the complement of A). Since X is first countable, there is a (descending) sequence U_i of open sets containing x such that any neighborhood V of x contains some U_i. Pick a sequence $a_i \in U_i \cap A$, and we claim $\lim_{i\to\infty} a_i = x$, thus $x \in A$, and A is closed. $\square$

Claim 4. *The quotient of a sequential space is sequential.*

Proof. This is maybe easier to do with the idea of 'sequentially open': Say a set A is **sequentially open** if, whenever (x_i) converges to $x \in A$, there is some N such that $x_n \in A$ for all $n \geq N$. One checks that a set is sequentially open if and only if its complement is sequentially closed, and a space is sequential if and only if every sequentially open set is open.

Suppose $A \subseteq Y$ is a sequentially open set, where $f : X \to Y$ is the projection mapping of a quotient. Then $f^{-1}(A)$ is sequentially open in X: if (x_i) is a sequence converging to $x \in f^{-1}(A)$, then $(y_i) = (f(x_i))$ is a sequence in Y with $\lim(y_i) = f(x) \in A$. Since A is sequentially closed, for some N we have $y_n \in A$ whenever $n \geq N$. This means $x_n \in f^{-1}(A)$ for $n \geq N$, so $f^{-1}(A)$ is sequentially open. Since X is sequential, this means $f^{-1}(A)$ is open, hence A is open in the quotient topology on Y.

$\square$

Now, we present the Arens space as a quotient of a first countable space. Let

$$Y = \left\{(a,b) \in \mathbb{R}^2 : a = \frac{1}{i}, b = \frac{1}{j}, i, j \in \mathbb{N}_+\right\} \cup \left\{(\frac{1}{i}, 0) : i \in \mathbb{N}_+\right\}$$
$$\cup \left\{(\frac{1}{i}, -1)\right\} \cup \{(0,-1)\}$$

with the equivalence relation

$$(\frac{1}{i}, 0) \sim (\frac{1}{i}, -1)$$

Claim 5. *The Arens space is homeomorphic to $Y/\sim$.*

Remark 1. *It is **not** true that the quotient of a first countable space is first countable. We have seen this with the 'infinite wedge of circles', namely $\mathbb{R}/\mathbb{Z}$ (this is the topological quotient, under the equivalence relation $x \sim y$ if $x = y$ or if $x, y \in \mathbb{Z}$, not the group-theoretic quotient).*