Fourth Day

We begin with some definitions:

Definition 1. *A (n-th order) differential equation is called* ***semilinear*** *if it is of the form*

$$y^{(n)}(x) = F\left(x, y, y', ..., y^{(n-1)}\right)$$

for some multivariate function F

In particular, for first order equations we have

$$y' = F(x, y)$$

Examples:

$$y' = \cos(x)y - e^{-x}$$

$$y' = y^2 - 3$$

$$y' = xy - xy^2 + x^2y$$

$$y' = x^2e^y$$

Definition 2. *A first order differential equation is called* ***separable*** *if it is of the form*

$$y' = f(x) \cdot g(y)$$

Exercise: which of the above examples are separable?

Definition 3. *A first order differential equation is called* ***autonomous*** *if it is of the form*

$$y' = g(y)$$

Exercise: which of the above examples are autonomous?

Notice that autonomous equations are separable (with $f(x) = 1$), and separable equations are semilinear (with $F(x, y) = f(x) \cdot g(y)$).

With no further ado, we make the

Claim 1. *We can solve any separable first order equation.*

First, let's do some algebraic manipulation and actually 'separate' the y and x variables:

$$\begin{aligned} y' &= f(x) \cdot g(y) \\ \frac{y'}{g(y)} &= f(x) \end{aligned}$$

Now, let's break the rules and pretend $y'(x) = \frac{dy}{dx}$ is a fraction:

$$\begin{aligned} \frac{y'}{g(y)} &= f(x) \\ \frac{\frac{dy}{dx}}{g(y)} &= f(x) \\ \frac{dy}{g(y)} &= f(x)dx \end{aligned}$$

then integrate both sides to obtain

$$\begin{aligned} \frac{dy}{g(y)} &= f(x)dx \\ \int \frac{1}{g(y)}dy &= \int f(x)dx \end{aligned}$$

For convenience, let's write $\tilde{G}(y)$ to mean some antiderivative of $\frac{1}{g(y)}$, and $F(x)$ to mean some antiderivative of $f(x)$ (in class I used $[\int \frac{1}{g}]$ and $[\int f]$, the point is these indefinite integrals are themselves functions, almost, other than the constant of integration (really families of functions)). Then this equation becomes

$$\tilde{G}(y) = F(x) + C$$

This gives an *implicit* equation involving y and x, which can then be solved for the unkown y in terms of x. Warning: there may be choices involved: the equation $y^2 = 25 - x^2$, there are two solutions for $x = 3$, either $y = 4$ or $y = -4$. The same considerations apply here, we may need to restrict ourselves to get a well-defined function.

Let's jump into examples: Find a solution to the differential equation

$$y' = e^y x^2$$

Separating variables leads to

$$e^{-y}y' = x^2$$

or

$$e^{-y}\frac{dy}{dx} = x^2$$

which we pretend means

$$e^{-y}dy = x^2dx$$

Then, integrating both sides gives

$$\begin{aligned}
\int e^{-y}dy &= \int x^2dx \\
-e^{-y} &= \frac{x^3}{3} + C \\
e^{-y} &= C - \frac{x^3}{3} \\
-y &= \ln\left(C - \frac{x^3}{3}\right) \\
y(x) &= \ln\left(\frac{1}{C - \frac{x^3}{3}}\right)
\end{aligned}$$

Since the algebra is a bit more involved, let's check:

$$\begin{aligned}
y'(x) &= \frac{d}{dx}\left(\ln\left(\frac{1}{C - \frac{x^3}{3}}\right)\right) \\
&= \frac{1}{\frac{1}{C-\frac{x^3}{3}}} \cdot \frac{d}{dx}\left(\frac{1}{C - \frac{x^3}{3}}\right) \\
&= \left(C - \frac{x^3}{3}\right) \cdot \left(\frac{0-(-x^2)}{(C - \frac{x^3}{3})^2}\right) \\
&= \frac{x^2}{C - \frac{x^3}{3}} \qquad\qquad = x^2 \cdot e^{-y}
\end{aligned}$$

So this does seem to solve our differential equation. A word of warning, we should notice that this function is only defined for $x < \sqrt[3]{3C}$. There's this domain issue I keep putting off.

Another example: I rambled a bit about bunny rabbits, but the point was that the equation

$$\frac{P'}{P} = k,$$

or

$$P' = kP,$$

is a reasonable model for population growth, in the sense that the change in population should be proportional to the total population (this is saying something like '20% of rabbits have a baby each year', so if there are 100 rabbits there are 20 babies, if there are 1000 rabbits there are 200 babies). This is assuming there is enough space and enough food and so on for any number of rabbits. A more realistic model might include the idea that with a larger number of rabbits, there is also less food to go around, so the proportion of rabbits having children may drop as the malnourished rabbits forgo breeding.

The simplest possible way to account for this would be to have a negative linear term:

$$\frac{P'}{P} = k - P$$

or

$$P' = P \cdot (k - P) = kP - P^2$$

This equation is autonomous, so separable, and we collect all the P terms on one side:

$$\frac{P'}{P(k-P)} = 1$$

and integrate to get

$$\int \frac{P'}{P(k-P)} = \int 1dt$$

To integrate the left-hand side, use the technique of 'partial fraction decomposition': try to find numbers a and b such that

$$\frac{1}{P(k-P)} = \frac{a}{P} + \frac{b}{k-P}$$

Add the fractions to get

$$\frac{1}{P(k-P)} = \frac{ak - aP + bP}{P(k-P)}$$

So we want

$$1 = ak - aP + bP,$$

where P is a function and k is a number.

Since the left side is just a number, we need to cancel the Ps somehow, so we need $a = b$. Then we get $a = \frac{1}{k} = b$, so

$$\frac{1}{P(k-P)} = \frac{\frac{1}{k}}{P} + \frac{\frac{1}{k}}{k-P}$$

Then we integrate

$$\begin{aligned} \int \frac{\frac{1}{k}}{P} + \frac{\frac{1}{k}}{k-P} dP &= \frac{1}{k}\int \frac{1}{P} dP + \frac{1}{k}\int \frac{1}{k-P} dP \\ &= \frac{1}{k}\ln(|P|) - \frac{1}{k}\ln(|k-P|) + C \end{aligned}$$

Plugging this into the above gives

$$\begin{aligned} \int \frac{P'}{P(k-P)} &= \int 1 dt \\ \frac{1}{k}\ln(|P|) - \frac{1}{k}\ln(|k-P|) + C &= t + c_2 \\ \ln(|\frac{P}{k-P}|) &= kt + C \frac{P}{k-P} \qquad = Ce^{kt} P = Ce^{kt}(k-P) \\ P &= Ce^{kt}k - Ce^{kt}P \\ P\left(1 + Ce^{kt}\right) &= Cke^{kt} \\ P(t) &= \frac{Cke^{kt}}{1 + Ce^{kt}} \end{aligned}$$

Again, one should verify that this does indeed solve the original differential equation (exercise).

One final example: Consider

$$y' = y^2$$

Let me go back to Leibnitz notation and hand-waving:

$$\frac{dy}{dx} = y^2$$
$$\frac{dy}{y^2} = dx$$
$$y^{-2}dy = dx$$
$$\int y^{-2}dy = \int dx$$
$$\frac{y^{-1}}{-1} = x + C$$
$$\frac{1}{y} = C - x$$
$$y(x) = \frac{1}{C - x}$$

Notice that $y(C)$ is undefined: any solution of $y' = y^2$ must be undefined for some input. This is sometimes called 'finite time blowup': finite time as in at some finite real number input, blowup as in the solution 'blows up' ('goes to infinity', etc.).

One should compare this to the homework 1.4b, where the differential equation $y' = \frac{1}{x}y = 0$ only has solutions which are undefined away from $x = 0$. This is maybe not so surprising, since the differential equation does not make sense for $x = 0$ (exercise: we got to this differential equation from another one, which seems to make sense for $x = 0$. Is there anything weird going on in the original differential equation?).

On the other hand, $y' = y^2$ seems to make sense for any value of x or y, but the solution still blows up. One might compare this with Paul's theorem in the 'Intervals of Validity' section of the notes.