Fifth day

We've been discussing solutions, specifically their uniqueness and their domains of definitions.

Let's begin with another example: consider the Initial Value Problem

$$\begin{cases} y' - x\sqrt{y} = 0 \\ y(0) = 0 \end{cases}$$

This equation is separable, namely

$$\begin{aligned} y' - x\sqrt{y} &= 0 \\ y' &= x\sqrt{y} \\ \frac{y'}{\sqrt{y}} &= x \\ \int \frac{dy}{\sqrt{y}} &= \int x dx \\ \frac{y^{\frac{1}{2}}}{\frac{1}{2}} &= \frac{1}{2}x^2 + c \\ 2\sqrt{y} &= \frac{1}{2}x^2 + c \\ y(x) &= \left(\frac{1}{2} \cdot (\frac{1}{2}x^2 + c\right)^2 \\ y(x) &= \frac{1}{16}x^2 + \frac{1}{4}x^2c + \frac{1}{4}c^2 \end{aligned}$$

With our initial condition, we see $c = 0$, so our solution is

$$y(x) = \frac{1}{16}x^4,$$

which is easily checked to solve the differential equation.

Are there any other solutions?

If $y(x) = 0$, then $\sqrt{y} = 0$, and $y' = 0$, so

$$y' - x\sqrt{y} = 0 - x \cdot 0 = 0$$

and we see that the constant 0 function is also a solution.

Notice there is a difference between the constant 0 appearing here and the constant 0 appearing as a solution to $y' = y$, with solutions Ce^x for arbitrary C. In the exponential case, we have a family of solutions to the differential equation, and any initial condition will pick out a single solution, e.g. $y(0) = 0$ would pick out the constant 0 solution. In the previous case, we found a family of solutions to the general differential equation, and the constant 0 was not among them. The initial condition picked out a single solution in that family, but the constant 0 gives another solution, which does not come from our method.

Paul's notes state a theorem about uniqueness of solutions to linear differential equations defined on an interval (where the coefficient functions of the ODE must be continuous), without proof. The proof is essentially the same as our proof the first day of class, about the uniqueness the solution $y(x) = e^x$ to $y' = y$ subject to $y(0) = 1$ (I've been a little loose with the initial conditions):

Suppose $y(x)$ is a solution to the equation

$$\begin{cases} y' + p(x)y = q(x) y(x_0) = y_0, \end{cases}$$

where p, q are continuous on some interval $(x_0 - a, x_0 + b)$.

Differentiate

$$\begin{aligned}
\frac{d}{dx}\left(e^{\int p(x)dx)}y\right) &= e^{\int p(x)dx}y' + p(x)e^{\int p(x)dx}y \\
&= e^{\int p(x)dx}\left(q(x) - p(x)y\right) + p(x)e^{\int p(x)dx}y \\
&= q(x)e^{\int p(x)dx} - p(x)e^{\int p(x)dx}y + p(x)e^{\int p(x)dx}y \\
\frac{d}{dx}\left(e^{\int p(x)dx)}y\right) &= q(x)e^{\int p(x)dx}
\end{aligned}$$

This says $e^{\int p(x)dx)}y$ is an antiderivative of $q(x)e^{\int p(x)dx)}$, so by the fundamental theorem of calculus

$$e^{\int p(x)dx)}y = \int_{x_0}^{x} q(t)e^{\int p(u)du)}dt + c$$

or

$$y(x) = \frac{\int_{x_0}^{x} q(t)e^{\int p(u)du)}dt + c}{e^{\int p(u)du}}$$

(where we have written a specific antiderivative to emphasize the constant of integration.

Then evaluating at x_0 gives

$$y_0 = \frac{c}{e^{\int p(u)du}},$$

so that c is uniquely determined (is this true? What about the denominator, shouldn't there be another constant of integration?).

This is far enough that I'm starting to feel guilty for not talking about the geometry of differential equations. We introduced the word 'semilinear' last time, for equations

$$y'(x) = F(x, y)$$

What does this mean? The derivative of y is given explicitly as a function of x and y. So, given such a semilinear differential equation, we can visualize the function $F(x, y)$ by placing at each point (x, y) a line segment with slope $F(x, y)$. This gives a representation of the differential equation by what is called a **slope field** (or **direction field**).

Here are some slope fields:

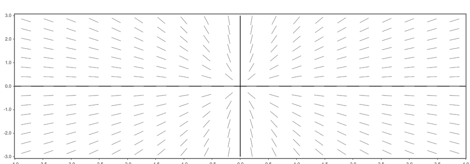

Figure 1: $y' = \frac{y}{x}$

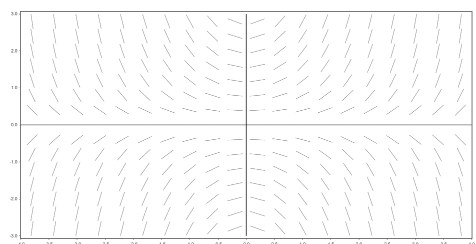

Figure 2: $y' = xy$

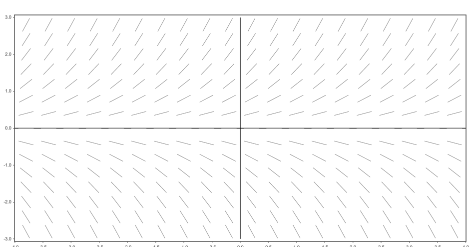

Figure 3: $y' = y$

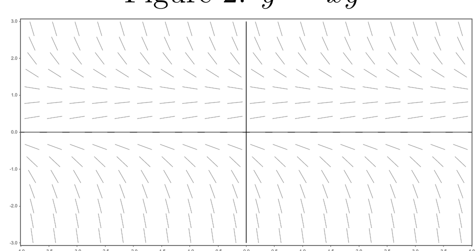

Figure 4: $y' = y(1 - y)$

Then a solution to the differential equation is precisely a function whose graph is given by a curve tangent to the slope field at every point of the curve.

Here are some solutions plotted:

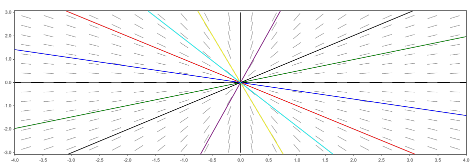

Figure 5: $y' = \frac{y}{x}$

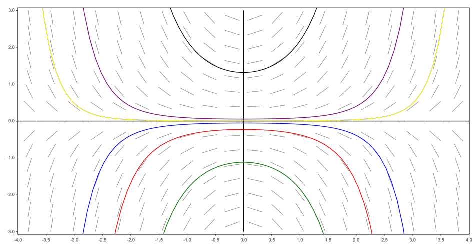

Figure 6: $y' = xy$

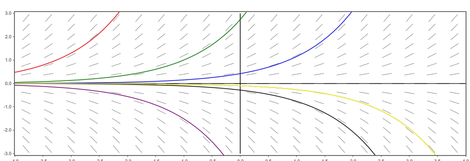

Figure 7: $y' = y$

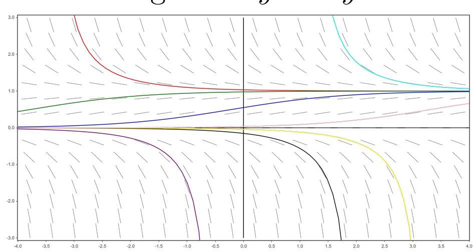

Figure 8: $y' = y(1-y)$

Towards the end, I muttered a bit about how this might lead you think of a practical method for approximating solutions to a differential equation: starting at some point (x_0, y_0), there is a line segment. Follow that line segment for a little bit, until you get to $(x_0 + h, y_0 + F(x_0, y_0) \cdot h$. Repeat this process, and you get a sequence of line segments. The slope of each line segment is approximately equal to what the derivative of a true solution would be. In fact, the slope is exactly equal at the left endpoint, so if h is small enough that the slopes do not change much, our broken line is approximately the graph of a solution. This technique is known as 'Euler's method'.