September 11

Last time, we ended on discussion of the exponential function

$$f(x) = 2$$

In particular, we have a definition in words when x is a positive whole number: "multiply x many copies of 2 together". By thinking about exponent laws, we saw that if $f(x)$ means anything for x a fraction, it has to mean a root of some positive integer power of 2. For instance, if $x = \frac{7}{12}$, the exponent law $(2^a)^b = 2^{(a \cdot b)}$ means that

$$\left(2^{\frac{7}{12}}\right)^{12} = 2^{(\frac{7}{12} \cdot 12)} = 2^7 = 128,$$

so

$$2^{\frac{7}{12}} = \sqrt[12]{128},$$

which is a number that we understand.

What does $f(\sqrt{2})$ mean? Can we use the exponent laws to relate it to other numbers we know? It turns out we cannot. Basically, if we wanted to do something to understand $2^{\sqrt{2}}$, we would need some way of relating 2^a to $2^{(a^b)}$ (so that we could put in $\sqrt{2}^2 = 2$, something we already know.) Convince yourself that none of the exponent laws will help here.

So, what do we do? Here's one idea: a finite decimal, like 1.4, is also a fraction, with

$$1.4 = 1 + 4 \cdot \frac{1}{10} = \frac{14}{10},$$

so we know what $f(1.4)$ is. But, $1.4^2 = 1.96$ is very close to 2, so 1.4 is near the square root of 2. Since $f(x)$ is a 'nice function' (given by a formula?), we might hope that since the inputs 1.4 and $\sqrt{2}$ are close to each other, the value $f(1.4)$ is near $f(\sqrt{2})$. Of course we don't expect the exact right answer, because 1.4 isn't *actually* the square root of 2, but if we wanted to get closer we could try $x = 1.41$, or $x = 1.414$, or..... with the hope that as the inputs gets closer and closer to the point we want, the outputs will get closer and closer to the value we want.

x	$f(x) = 2^x$
1	2
1.4	2.63902...
1.41	2.65737...
1.414	2.66475...
1.4142	2.66512...
$\sqrt{2}$	???
1.4143	2.66530...
1.415	2.66660...
1.42	2.67586...
1.5	2.82843...

$$f(\sqrt{2}) = 2^{\sqrt{2}} \approx 2.665144...$$

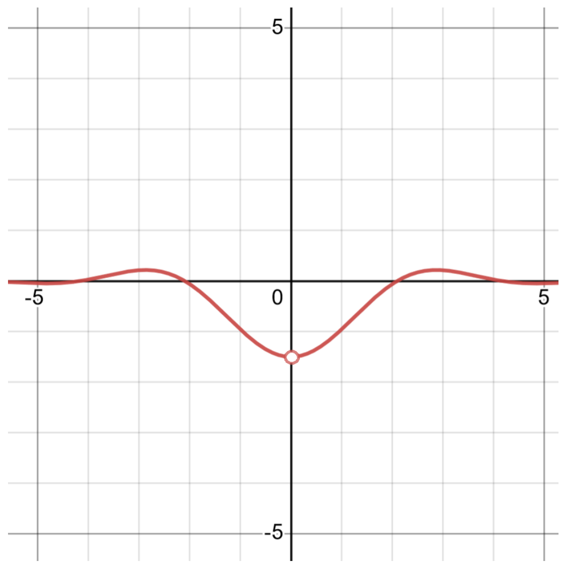
We summarize this idea with the concept of the **limit**.

Definition 1. *The **limit of the function** $f(x)$ **as** x **approaches** a **is equal to** L if we can make $f(x)$ arbitrarily close to L for all x sufficiently close to a.*

This is written as

$$\lim_{x \to a} f(x) = L$$

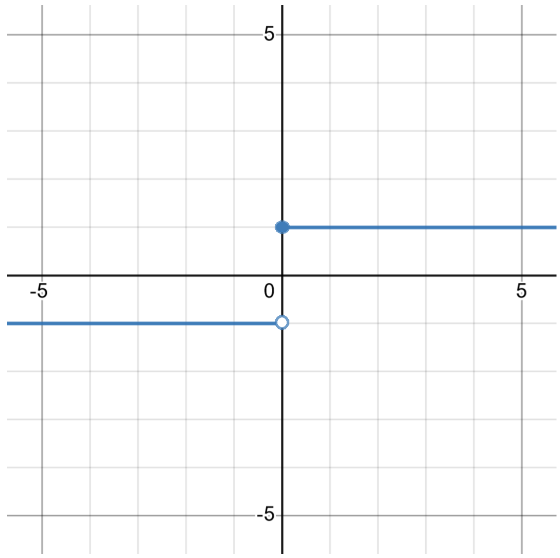
As another example, here is a graph of the function $g(x) = \frac{\cos(2x) - \cos(x)}{x^2}$:



Observe that even though $g(x)$ is not defined for $x = 0$, the graph of the function makes a nice curve, with no jumps or breaks. In particular, there seems to be some 'value' near $x = 0$, where the function gives outputs close to this 'missing value'.

Warning! The limit of a function at a point does not need to exist. For instance, consider the *signum* function

$$sgn(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$



Here, the value of the function at $x = 0$ is $sgn(0) = 1$. If you think about x as a small positive number, getting closer and closer to 0, then since it is positive the value will be $sgn(x) = 1$, so the values are very close to 1! On the other hand, if x is a 'big' negative number (I mean, negative and near 0), then as we move x closer and closer to 0 we get the values $sgn(x) = -1$. So even though the function has an output at $x = 0$, the behavior of the function nearby $x = 0$ is a little undecided. Viewed from inputs to the left, the function wants to go to -1, while inputs from the right want to go to 1.

Definition 2. *The* ***one-sided limit of*** $f(x)$ ***as*** x ***approaches*** a ***from the left is equal to*** L *if we can make*$f(x)$ *arbitrarily close to* L *for all* x *sufficiently close to* a *with* $x < a$.

This is written as

$$\lim_{x \to a^-} f(x) = L$$

We similarly have

Definition 3. *The* ***one-sided limit of*** $f(x)$ ***as*** x ***approaches*** a ***from the left is equal to*** L *if we can make*$f(x)$ *arbitrarily close to* L *for all* x *sufficiently close to* a *with* $x > a$.

This is written as

$$\lim_{x \to a^+} f(x) = L$$

Proposition 1. *The limit of* $f(x)$ *as* x *goes to* a *exists and is equal to* L *if and only if each one-sided limit exists and is equal to* L

Warning! The one-sided limits don't need to exist either.

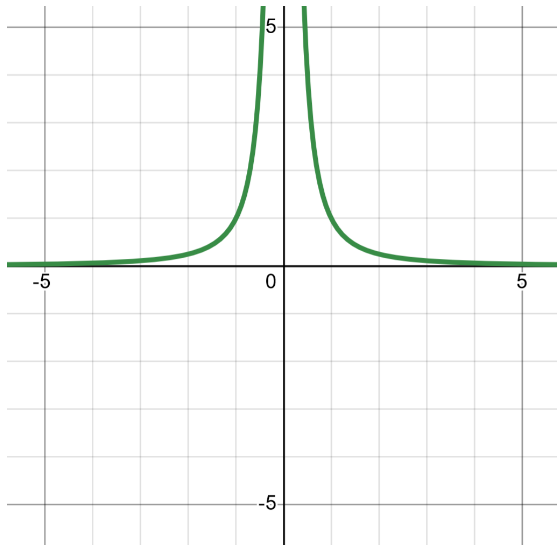
An important point in the above definitions is that L is some real number.

I don't like this terminology, but we have

Definition 4. *The **limit of** $f(x)$ **as** x **approaches** a **is infinite** if the value of f is arbitrarily large for all x sufficiently close to a.*

This is written as $\lim_{x\to a} f(x) = \infty$.

For instance, consider $k(x) = \frac{1}{x^2}$, which has the graph



Similarly, we have

Definition 5. *The **limit of** $f(x)$ **as** x **approaches** a **is negatively infinite** if the value of f is arbitrarily small (in the sense of extremely negative) for all x sufficiently close to a.*

This is written as $\lim_{x\to a} f(x) = \infty$.

In particular, ∞ is not a number. So when we say the limit is infinite, we are saying the limit does not exist. 'The limit is infinite' is really saying that the limit does not exist, but it fails to exist in a certain way.

We can combine the above ideas:

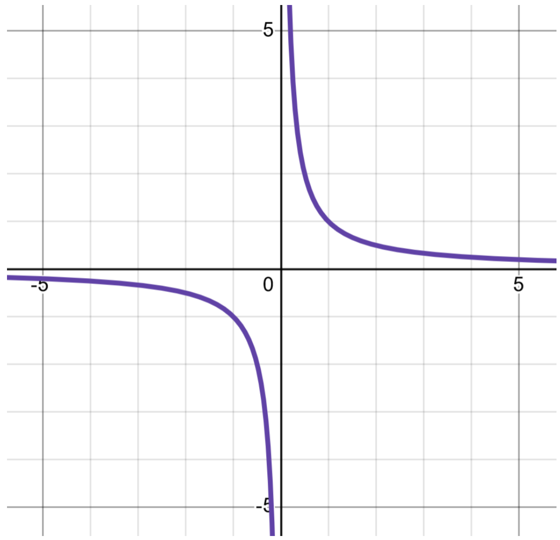
Definition 6. *The **one-sided limit of** $f(x)$ **as** x **approaches** a **from the left is infinite** if the value of f is arbitrarily large for all x sufficiently close to a with $x < a$.*

This is written as

$$\lim_{x\to a^-} f(x) = \infty$$

I'll let you fill in the remaining three combinations yourself.

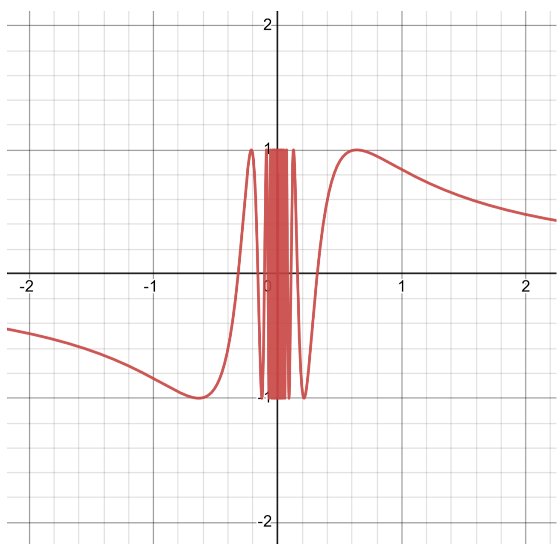
As an example, $r(x) = \frac{1}{x}$:



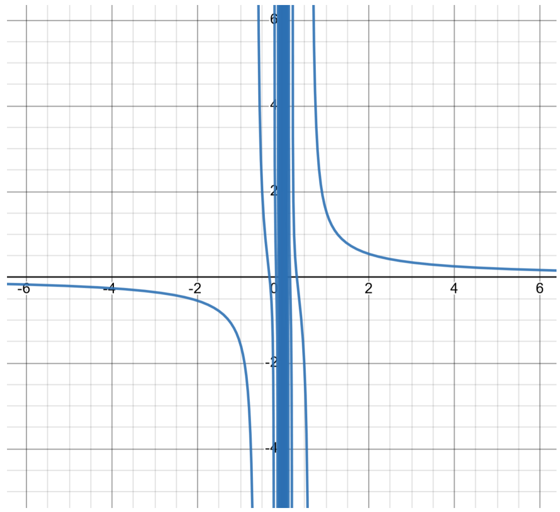
Definition 7. *A function $f(x)$ has a* ***vertical asymptote*** *at $x = a$ if some one-sided limit is infinite or negatively infinite.*

Warning! Disagreement of one-sided limits, and infinite limits, are not the only ways a limit can fail to exist:

Here is the graph of $b(x) = \sin(\frac{1}{x})$:



And here is the graph of $v(x) = \tan(\frac{1}{x})$:



If we wanted, we could give a

Definition 8. *The limit of $f(x)$ as x approaches a is **oscillatory** if there are two numbers L_1 and L_2, not equal, so that there are numbers x as close to a as we want such that the values of $f(x)$ are as close to L_1 or L_2 as we want.*

Notice in the last two graphs, the solid bands of color in the middle are the oscillations getting more and more rapid.