October 11th

I wanted to mention the **Variation of Parameters** formula: given a second order linear non-homogeneous differential equation

$$y'' + p(x)y' + q(x)y = r(t),$$

and linearly independent functions f, g which solve the associated homogeneous equation

$$y'' + p(x)y' + q(x)y = 0$$

a solution to the non-homogeneous equation is given by

$$y(x) = -f(x)\int \frac{g(x)r(x)}{W(f,g)(x)}dx + g(x)\int \frac{f(x)r(x)}{W(f,g)(x)s}dx$$

So, if you can solve a homogeneous equation, you can 'explicitly' solve the non-homogeneous version (where explicitly means involving integration). We will not dwell too much on this.

Then we talked about Harmonic Oscillators: This is just $F = ma$, which is a second order differential equation for position $x(t)$, and $a = \frac{d^2}{dt^2}x(t)$. In general, the force might involve time, position, and potentially the velocity, $F = F(t, x, x')$.

In the case of a spring, relative to some 'equilibrium position' which we take to be $x = 0$, we have the force of the spring which tries to bring us towards equilibrium (magnitude $-kx$, with $k > 0$) and a damping force, e.g. friction working against any motion (which we assume is proportional to the velocity, so magnitude $-cv$). Then

$$F = ma$$

is

$$-kx - cv = ma$$

or

$$ma + cv + kx = 0$$

or

$$m\frac{d^2}{dt^2}x(t) + c\frac{d}{dt}x(t) + kx(t) = 0$$

The associated characteristic equation has roots

$$\lambda_{\pm} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

Remark: since all quantities are positive, $\sqrt{c^2 - 4mk}$ is either less than c or imaginary, so $Re(\lambda_{\pm}) < 0$ no matter what. This means that $\lim_{t\to\infty} x(t) = 0$ (except the case $c = 0$, which gives $Re(\lambda) = 0$ and there is truly periodic motion).

We call the different cases: Underdamped: $c^2 - 4mk < 0$, Critically damped: $c^2 - 4mk = 0$, and Overdamped: $c^2 - 4mk > 0$

Physically, these cases are give rather different behavior: see xmdemo 068 on YouTube. Apparently practical people rewrite this as

$$\frac{d^2}{dt^2}x + 2\zeta\omega_0 \frac{d}{dt}x + \omega_0^2 x = 0$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the 'undamped frequency' (i.e., the frequency of oscillation if $c = 0$) and $\zeta = \frac{c}{2\sqrt{mk}}$ is the 'damping ratio'. One can find various formulas about rate of energy loss and whatnot in these terms.

If a harmonic oscillator is subject to some external force, we call it a forced or driven harmonic oscillator, and it has the form

$$m\frac{d^2}{dt^2}x(t) + c\frac{d}{dt}x(t) + kx(t) = F(t)$$

or

$$\frac{d^2}{dt^2}x + 2\zeta\omega_0 \frac{d}{dt}x + \omega_0^2 x = \frac{F(t)}{m}$$

If $F(t) = F_0 \sin(\omega t)$, then this has an explicit solution

$$x(t) = \frac{F_0}{\sqrt{\frac{k^4}{m^2} + c^2\omega^2 - 2k^2\omega^2 + m^2\omega^4}} \sin(\omega t + \phi$$

for some ϕ, which is maximized when

$$\omega \approx \sqrt{\frac{k - c}{m}}$$

This is known as the **resonant frequency**, and leads to the famous Tacoma bridge collapse. A slightly more substantive explanation of the specific mechanism is given here by the channel Practical Engineering.

Mathematically, this is all stuff we have seen in previous weeks, just given in a physics-y language.

Let's give a more abstract result with the

Theorem 1 (Sturm separation theorem)**.** *If f, g are two linearly independent solutions to a homogeneous second order linear differential equation, with $f(x_0) = f(x_1) = 0$ being two consecutive zeros of f, then $g(x)$ has exactly one zero in the interval (x_0, x_1).*

Proof. Since f, g are linearly independent, their Wronskian is nonzero:

$$W(x) := W(f, g)(x) \neq 0$$

Suppose W is negative (otherwise switch the order of f and g).

In particular since $f(x_0) = f(x_1) = 0$, we have

$$W(x_0) = -f'(x_0)g(x_0)$$

and

$$W(x_1) = -f'(x_1)g(x_1)$$

with $f'(x_0)$ and $g(x_0)$ both positive or both negative. Say both positive for convenience.

Since x_1 is the next consecutive zero after x_0, we must have $f'(x_1) < 0$, hence we must have $g(x_1) < 0$. This says g switches signs on the interval (x_0, x_1), so has a zero by the intermediate value theorem.

There can only be one such zero, otherwise the above argument could be applied with the roles reversed and f would have a zero between these two zeros, contradicting our assumption that x_0, x_1 were consecutive zeros.

Something should be said about accumulation points of zeros: if $(x_i) \to x$ are all zeros of f, then $f'(x) = 0$ (why?), which would lead to $W(x) = 0$, a contradiction. $\square$