November 8

We recall

Theorem 1 (Green's Theorem)**.** *Suppose D is a planar region with no holes (i.e.* ***simply connected****) bounded by the simple (i.e. non-self-intersecting) closed curve C. If $\begin{pmatrix} M(x,y) \\ N(x,y) \end{pmatrix}$ is a vector field defined everywhere in D, with continuous partial derivatives, then*

$$\oint_C M dx + N dy = \iint_D \left(\frac{\partial}{\partial x} N - \frac{\partial}{\partial y} M \right) dx dy$$

In the previous class, we saw some examples of nonlinear systems. One of these had closed trajectories, and one did not, which we analyzed on a case-by-case basis. Here we give an abstract criterion to guarantee there are no closed trajectories.

Theorem 2 (Bendixson-Dulac's criterion)**.** *Suppose*

$$\begin{aligned} x' &= f(x,y) \\ x' &= g(x,y) \end{aligned}$$

is a system of ODEs defined on a simply connected region R. If the divergence

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$$

has a constant sign, then this system has no closed trajectory inside of R.

Proof. Suppose $C = C(t)$ is a closed trajectory of this system. For convenience, say the divergence is positive: if

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} > 0$$

then

$$\iint_D \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dx dy > 0$$

On the other hand, by Green's theorem we have

$$\begin{aligned}
\iint_D \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dxdy &= \iint_D \left(\frac{\partial f}{\partial x} - \frac{\partial - g}{\partial y}\right) dxdy \\
&= \oint_C (-gdx + fdy) \\
&= \oint_C (-y'dx + x'dy) \\
&= \int_0^T \left(-y'\frac{dx}{dt} + x'\frac{dy}{dt}\right) dt \\
&= \int_0^T \left(-\frac{dy}{dt} \cdot \frac{dx}{dt} + \frac{dx}{dt} \cdot \frac{dy}{dt}\right) dt \\
&= \int_0^T 0dt \\
&= 0
\end{aligned}$$

This is a contradiction, so our starting assumption that there is a closed trajectory for this system must be false. □

Notice the theorem is phrased in terms of 'constant sign', not 'never zero'. If R is connected, then never being zero is the same as having constant sign, by continuity. However, since our proof uses integration, we can actually get a bit more out of it: we could ask that the divergence have constant sign, except possibly allowing zero at some points in R, or along some curve in R. The point is, the double integral will still be positive even if the integrand is zero on some $<$ 2-dimensional pieces. The proper terminology here is that the divergence is allowed to be 0 on a 'set of measure zero', or 'null set'. Then, the first integral in the proof is still positive, so we still get a contradiction.

For example, the system

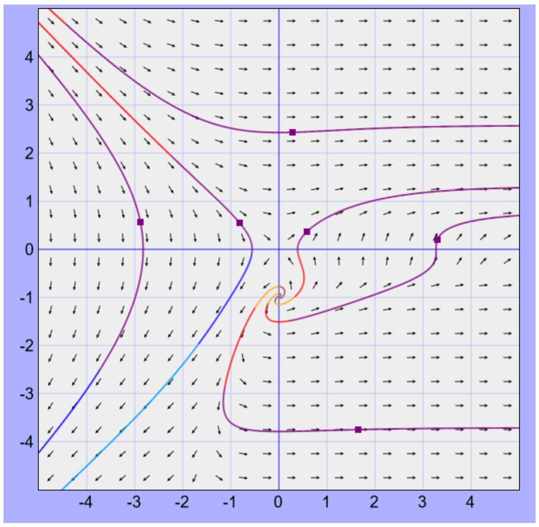
$$\begin{aligned}
x' &= y + y^2 e^x \\
y' &= x
\end{aligned}$$

has divergence

$$\frac{\partial}{\partial x}\left(y + y^2 e^x\right) + \frac{\partial}{\partial y}(x) = y^2 e^x + 0 = y^2 e^x \geq 0$$

which is 0 only on the line $y = 0$. If $C(t)$ was a closed trajectory, it cannot stay in the line $y = 0$ (unless it is the constant solution at $(0, 0)$), so integrating the interior would give a positive number, while Green's theorem would say the integral must be 0.

Thus, this system has no closed trajectories.



$$
\begin{aligned}
x' &= y + y^2 e^x \\
y' &= x
\end{aligned}
$$

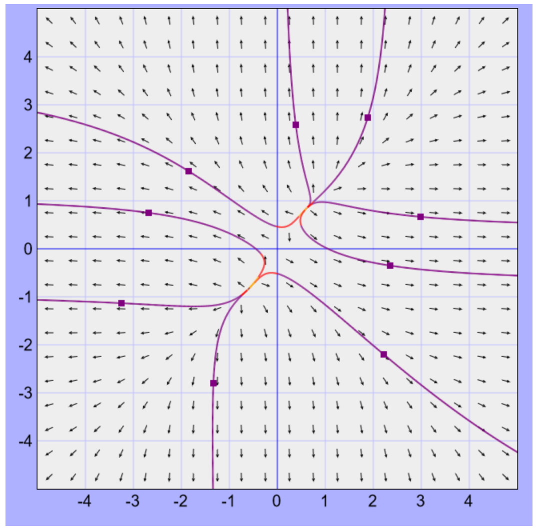
This is the example I gave in class, but I forgot about this subtlety with allowing divergence to be 0 at a point. A more traditional example would have been

$$
\begin{aligned}
x' &= -y + x^3 + x \\
y' &= x + y^3
\end{aligned}
$$

Then the divergence is

$$\frac{\partial}{\partial x}\left(-y + x^3 + x\right) + \frac{\partial}{\partial y}\left(x + y^3\right) = 3x^2 + 1 + 3y^2 = 3(x^2 + y^2) + 1 > 0$$

so this system has no closed trajectories.



$$x' = -y + x^3 + x$$
$$y' = x + y^3$$

Another very interesting example is given by the system

$$x' = -y - x^2$$
$$y' = -x + y^2$$

Here, the divergence is

$$\frac{\partial}{\partial x}\left(-y - x^2\right) + \frac{\partial}{\partial y}\left(-x + y^2\right) = -2x + 2y = 2(y - x)$$

Notice this is positive to the left of the line $y = x$ and negative to the right of the line $y = x$. Thus, Bendixson-Dulac tells us there are no closed trajectories in either of these regions. This is not enough to rule out the existence of *any* closed trajectory though: there might be a closed trajectory which goes back and forth between these regions. Can we rule this out?

Consider the system at the dividing line $y = x$: we have

$$x' = -y - x^2 = -x - x^2$$

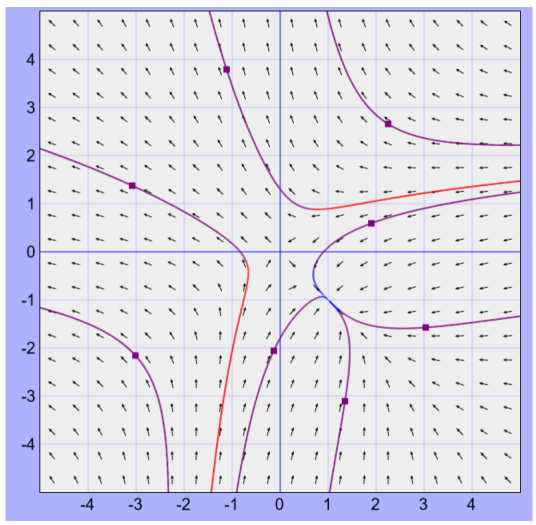
and

$$y' = -x + y^2 = -x + x^2$$

Since $x^2 \geq 0$, we have that $x' < y'$ for any (x, y) on this line. This means that (except for the origin), the vector $\begin{pmatrix} x' \\ y' \end{pmatrix}$ along the line $y = x$ is pointing *into* the region $y > x$.

Thus, whenever a trajectory intersects this line, it must be heading into the region $y > x$. But, if the trajectory is closed, it must eventually go back to the other region, $y < x$, so cross the line $y = x$ heading in the 'wrong' direction, a contradiction.

So this system has no closed trajectories.



$$x' = -y - x^2$$
$$y' = -x + y^2$$

While I didn't get to talk about it much, for completeness's sake I include the

Theorem 3 (Poincaré-Bendixson Theorem)**.** *If R is a bounded region in the plane with no equilibrium point, then any trajectory which stays in R approaches some closed trajectory.*

A closed trajectory which is approached by some nearby solution is called a **limit cycle**.

In fact, this is not the optimal statement, the full Poincaré-Bendixson theorem should say that all trajectories eventually head towards either equilibrium points or closed trajectories.

For culture, we make two comments: First, the Poincaré-Bendixson theorem fails dramatically in higher dimensions. Already in dimension three, the Lorenz attractor is a famous example of a trajectory which stays bounded, but is not periodic. This was discovered in a model for weather, and is the start of 'chaos theory' and the so-called 'butterfly effect'.

Second, even in dimension two, the structure of limit cycles is unknown in general. Hilbert's Sixteenth Problem asks for a bound on the number of limit cycles for a polynomial system, and this question is apparently out of reach today (even showing there are only a finite number is rather difficult, compare to the last example from last class).