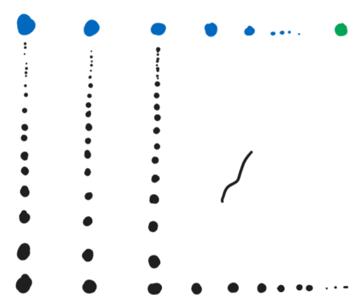
Let $X$ be the set

$$\{\infty\} \coprod \mathbb{N} \coprod (\mathbb{N} \times \mathbb{N})$$

and consider the topology $\tau_{Arens}$ generated by

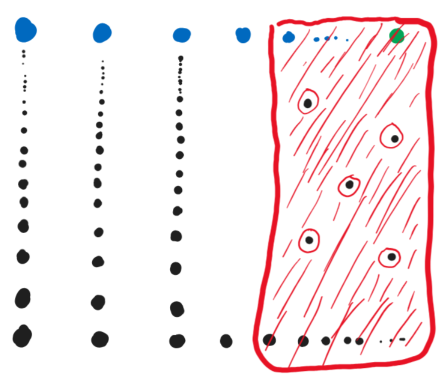
$$\{(a,b) : (a,b) \in \mathbb{N} \times \mathbb{N}\}$$
$$B_{n,i} = \{n\} \cup \{(n,k) : k \geq i\}$$
$$W_m = \{\infty\} \cup (\bigcup_{m \geq j} B_{m,1} \setminus \{(m,b_1), ...(m,b_{s_m})\})$$

**Definition 1.** *The* ***Arens space*** *is* $(X, \tau_{Arens})$.



Here, the green is $\infty$ and the blue are $\mathbb{N}$.

This says the pairs are all isolated points, and open sets around $n$ contain all but finitely many pairs with first coordinate $n$. Finally, the $s_m$ are some finite numbers, depending on $m$, so an open set containing $\infty$ contains all but finitely many of all but finitely many columns.



A neighborhood of $\infty$ must contain almost everything in some red box like so.

**Claim 1.** *In the Arens space, the sequential closure operator is not idempotent. That is, the equation $scl(scl(A)) = scl(A)$ is **not** true.*

*Proof.* Let $A = \mathbb{N} \times \mathbb{N}$. For any fixed $n$, we have the sequence $x_i = (n, i)$, which converges to $n \in \mathbb{N}$ (by definition of $B_{n,i}$).

On the other hand, no sequence in $\mathbb{N} \times \mathbb{N}$ can converge to $\infty$. Suppose $x_i = (a_i, b_i)$ did so. There must be a subsequence $x_{i,j}$ with $a_{i_j}$ going to infinity, otherwise we could remove the finitely many columns appearing in the sequence and get an open set around $\infty$ which does not contain the sequence. Then, form an open set $W_m$ by setting $s_m = 1$ for all $m$, and removing $(m, b_{i_j})$ whenever $a_{i_j} = m$. Since the $a_{i_j}$ go to infinity, the sequence cannot eventually be contained in the neighborhood.

It is clear that $x_i = i$ converges to $\infty$.

Therefore,

$$scl(\mathbb{N} \times \mathbb{N}) = \mathbb{N} \cup \mathbb{N} \times \mathbb{N}$$

but

$$scl(scl(\mathbb{N} \times \mathbb{N})) = \{\infty\} \cup \mathbb{N} \cup \mathbb{N} \times \mathbb{N}$$

□

**Definition 2.** *A space $(X, \tau)$ is called **Frechét-Urysohn** if the sequential closure operator is idempotent.*

**Definition 3.** *A set $A$ is called **sequentially closed** if the limit of any sequence in $A$ is also in $A$.*

**Definition 4.** *A topological space $(X, \tau)$ is called **sequential** if every sequentially closed subset is closed.*

**Claim 2.** *The Arens space is sequential.*

*Proof.* Say $A$ is a sequentially closed set, so we must show $A$ is closed. Consider the $n$ for which $\{(n, i)\} \cap A$ is infinite. For any such $n$, we have $n \in A$ since this gives a convergent sequence. If there are only finitely many such $n$, then the complement of $A$ is open: if $N$ is the maximum such $n$, then $W_{N+1}$ (minus finitely many pairs in each column) is most of the complement, and for any $n$ that doesn't appear, some $B_{n,k}$ will cover all but finitely many of the column. Everything in the complement not in one of these sets is a pair $(a, b)$, so any union of them will be open. Thus the complement of $A$ is open, and $A$ is closed. □

Let us give an alternate proof:

**Claim 3.** *Any first countable space is sequential.*

*Proof.* Suppose $A$ is a sequentially closed set, and pick $x$ in the closure of $A$. This means every open set containing $x$ intersects $A$ (otherwise $x$ would be in the interior of the complement of $A$). Since $X$ is first countable, there is a (descending) sequence $U_i$ of open sets containing $x$ such that any neighborhood $V$ of $x$ contains some $U_i$. Pick a sequence $a_i \in U_i \cap A$, and we claim $\lim_{i\to\infty} a_i = x$, thus $x \in A$, and $A$ is closed. □

**Claim 4.** *The quotient of a sequential space is sequential.*

*Proof.* This is maybe easier to do with the idea of 'sequentially open': Say a set $A$ is **sequentially open** if, whenever $(x_i)$ converges to $x \in A$, there is some $N$ such that $x_n \in A$ for all $n \geq N$. One checks that a set is sequentially open if and only if its complement is sequentially closed, and a space is sequential if and only if every sequentially open set is open.

Suppose $A \subseteq Y$ is a sequentially open set, where $f : X \to Y$ is the projection mapping of a quotient. Then $f^{-1}(A)$ is sequentially open in $X$: if $(x_i)$ is a sequence converging to $x \in f^{-1}(A)$, then $(y_i) = (f(x_i))$ is a sequence in $Y$ with $\lim(y_i) = f(x) \in A$. Since $A$ is sequentially closed, for some $N$ we have $y_n \in A$ whenever $n \geq N$. This means $x_n \in f^{-1}(A)$ for $n \geq N$, so $f^{-1}(A)$ is sequentially open. Since $X$ is sequential, this means $f^{-1}(A)$ is open, hence $A$ is open in the quotient topology on $Y$.

□

Now, we present the Arens space as a quotient of a first countable space. Let

$$Y = \left\{(a,b) \in \mathbb{R}^2 : a = \frac{1}{i}, b = \frac{1}{j}, i, j \in \mathbb{N}_+\right\} \cup \left\{\left(\frac{1}{i}, 0\right) : i \in \mathbb{N}_+\right\}$$
$$\cup \left\{\left(\frac{1}{i}, -1\right)\right\} \cup \{(0, -1)\}$$

with the equivalence relation

$$\left(\frac{1}{i}, 0\right) \sim \left(\frac{1}{i}, -1\right)$$

**Claim 5.** *The Arens space is homeomorphic to $Y/\sim$.*

**Remark 1.** *It is **not** true that the quotient of a first countable space is first countable. We have seen this with the 'infinite wedge of circles', namely $\mathbb{R}/\mathbb{Z}$ (this is the topological quotient, under the equivalence relation $x \sim y$ if $x = y$ or if $x, y \in \mathbb{Z}$, not the group-theoretic quotient).*