November 13

For the next few lectures, let's use t as our independent variable, so functions $f(t)$, $g(t)$, $y(t)$, etc.

The function e^t, or e^{st} for any positive number s, goes to 0 'very quickly', e.g. faster than any polynomial This can be expressed as the fact that

$$\lim_{t\to\infty} \frac{t^n}{e^{st}} = 0,$$

or

$$\lim_{t\to\infty} t^n e^{-st} = 0,$$

(for any $n \in \mathbb{Z}$).

In this way, e^{-st} 'controls' the value of any polynomial, keeping it close to zero. In fact, the values are close enough to zero that

$$\int_0^\infty t^n e^{-st} dt$$

exists. We might imagine that 'most' functions grow slower than e^{st} in this way.

Definition 1. *The **Laplace transform** of a function $f(t)$ is the function of s defined by*

$$\mathscr{L}(f)(s) = \int_0^\infty f(t)e^{-st}dt,$$

for any s such that this integral converges.

Common notations are $F(s)$ or $\bar{f}(s)$.

If $f(t) = e^{at}$, for some number a, then the Laplace transform is

$$\begin{aligned}
\mathscr{L}(f)(s) &= \int_0^\infty f(t)e^{-st}dt \\
&= \int_0^\infty e^{at}e^{-st}dt \\
&= \int_0^\infty e^{(a-s)t}dt \\
&= \frac{1}{a-s} \cdot e^{(a-s)t}|_{t=0}^{\infty} \\
\mathscr{L}(f)(s) &= \begin{cases} \frac{1}{s-a} & s > a \\ \text{undefined} & s \le a \end{cases}
\end{aligned}$$

Definition 2. *We say $f(t)$ has **exponential order (of order** a**)** if there exist constants C, a such that*

$$|f(t)| \leq Ce^{at}$$

Examples are e^{at}, $\sin(at)$, and at^n.

Proposition 1. *If $f(t)$ has exponential order (of order a), then $f(t)$ has a Laplace transform (defined for $s > a$).*

Proposition 2. *If $f(t)$ has exponential order, then*

$$\lim_{s\to\infty} \mathscr{L}(f)(s) = 0$$

So, for instance, $\frac{s}{s+1}$ is not the Laplace transform of any function with exponential order.

Remark 1. *The function e^{t^2} is **not** of exponential order.*

The first fundamental property of the Laplace transform is

Proposition 3. *The transform $\mathscr{L}$ is a **linear operator**:*

1. $\mathscr{L}(f+g) = \mathscr{L}(f) + \mathscr{L}(g)$

2. $\mathscr{L}(c \cdot f) = c \cdot \mathscr{L}(f)$ *for any constant c*

There are many other linear operators of functions: differentiation, multiplication by a fixed function, and integration are examples.

The second fundamental property of the Laplace transform is

Proposition 4. *The Laplace transform takes differentiation to multiplication:*

$$\mathscr{L}(f') = s\mathscr{L}(f) - f(0)$$

Proof.

$$\begin{aligned}
\mathscr{L}(f'(t)) &= \int_0^\infty f'(t)e^{-st}dt \\
&= f(t)e^{-st}|_{t=0}^{\infty} - \int_0^\infty f(t)(-se^{-st})dt \\
&= -f(0) + s\int_0^\infty f(t)e^{-st}dt \\
&= s\mathscr{L}(f(t)) - f(0)
\end{aligned}$$

In the third line, we use the fact that $f(t)e^{-st}$ must go to 0 as s goes to infinity, in order for the Laplace transform of $f(t)$ to exist. $\square$

We also do the second derivative:

$$\begin{aligned}
\mathscr{L}(f''(t)) &= s\mathscr{L}(f'(t)) - f'(0) \\
&= s(s\mathscr{L}(f(t)) - f(0)) - f'(0) \\
&= s^2\mathscr{L}(f(t)) - sf(0) - f'(0)
\end{aligned}$$

Thus, the Laplace transform turns differential equations into integral equations: for instance, if we want to solve the equation

$$y'' + 2y' + y = e^{2t}, \quad y(0) = y'(0) = 0$$

we can apply the Laplace transform to get

$$\begin{aligned}
y'' + 3y' + 2y &= e^{2t} \\
\mathscr{L}(y'' + 3y' + 2y) &= \mathscr{L}(e^{2t}) \\
\mathscr{L}(y'') + 3\mathscr{L}(y') + 2\mathscr{L}(y) &= \frac{1}{s-2} \\
s^2\mathscr{L}(y) - sy(0) - y'(0) + 3s\mathscr{L}(y) - 3y(0) + 2\mathscr{L}(y) &= \frac{1}{s-2} \\
s^2\mathscr{L}(y) + 3s\mathscr{L}(y) + 2\mathscr{L}(y) &= \frac{1}{s-2} \\
(s^2 + 3s + 2)\mathscr{L}(y) &= \frac{1}{s-2} \\
\mathscr{L}(y) &= \frac{1}{(s-2)(s^2+3s+2)} \\
\mathscr{L}(y) &= \frac{1}{(s-2)(s+1)(s+2)} \\
\mathscr{L}(y) &= \frac{1}{12} \cdot \frac{1}{s-2} - \frac{1}{3} \cdot \frac{1}{s+1} + \frac{1}{4} \cdot \frac{1}{s+2} \\
\mathscr{L}(y) &= \frac{1}{12} \cdot \mathscr{L}(e^{2t}) - \frac{1}{3} \cdot \mathscr{L}(e^{-t}) + \frac{1}{4} \cdot \mathscr{L}(e^{-2t}) \\
\mathscr{L}(y) &= \mathscr{L}\left(\frac{1}{12} \cdot e^{2t} - \frac{1}{3} \cdot e^{-t} + \frac{1}{4} \cdot e^{-2t}\right)
\end{aligned}$$

We used partial fraction decomposition near the end, to get to the point. We now have the Laplace transform of the function $y(t)$ is equal to the Laplace transform of a known function, a combination of exponentials. If we knew that the Laplace transform was invertible (i.e. one-to-one/injective), then we could undo the Laplace transform, and say that

$$y(t) = \frac{1}{12} \cdot e^{2t} - \frac{1}{3} \cdot e^{-t} + \frac{1}{4} \cdot e^{-2t}$$

You can check that this is in fact a solution.

We state without proof the

Theorem 1 (Lerch's Theorem)**.** *If f and g are continuous with exponential order, and there is some s_0 such that for all $s \geq s_0$ we have*

$$\mathscr{L}(f)(s) = \mathscr{L}(g)(s)$$

then $f(t) = g(t)$.

The Laplace transform seems fairly unmotivated, so we offer an analogy: some functions can be represented by power series:

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + ...$$

Here, the coefficient sequence a_n can be thought of as a function of the discrete variable n. If we instead consider a continuous function, of the continuous variable t, we should have

$$F(x) = \int_0^{\infty} f(t) x^t dt = \int_0^{\infty} f(t) e^{t \ln(x)} dt$$

or, with a change of variables $s = -\ln(x)$:

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

So, instead of the function $A(x)$ being represented in terms of monomials, the function $F(s)$ is represented in terms of exponentials of different frequencies.