Math 201 - Quiz 2 (10 min)

Name: ____

Instructions: Write cleanly, show all work. Explain any trick questions.

1. Consider the function

$$q(x) = \frac{x^2 - 2x - 3}{x^2 - x - 6}$$

.

(a) Compute the limit

$$\lim_{x \to 3} q(x)$$

Solution: Factor both numerator and denominator to get

$$\begin{aligned}
\lim_{x \to 3} q(x) &= \lim_{x \to 3} \frac{x^2 - 2x - 3}{x^2 - x - 6} \\
&= \lim_{x \to 3} \frac{(x-3)(x+1)}{(x-3)(x+2)} \\
&= \lim_{x \to 3} \frac{(x+1)}{(x+2)} \\
&= \frac{\lim_{x \to 3}(x+1)}{\lim_{x \to 3}(x+2)} \\
&= \frac{4}{5}
\end{aligned}$$

In the second to last step, we used the quotient limit law, which is valid as long as the limit in the denominator is not 0. Notice $5 \neq 0$.

(b) Evaluate $q(-3)$ and $q(0)$

Solution: These are function evaluations, so there's no limit involved. Just plug in:

$$\begin{aligned}
q(-3) &= \frac{(-3)^2 - 2 \cdot (-3) - 3}{(-3)^2 - (-3) - 6} \\
&= \frac{9 + 6 - 3}{9 + 3 - 6} \\
&= \frac{12}{6} \\
&= 2
\end{aligned}$$

Similarly,

$$\begin{aligned} q(0) &= \frac{(0)^2 - 2\cdot(0) - 3}{(0)^2 - (0) - 6} \\ &= \frac{-3}{-6} \\ &= \frac{1}{2} \end{aligned}$$

(c) Little Johnny did the first two parts, and then said "Oh, by the Intermediate Value Theorem, there must be some number c between -3 and 0 where $q(c) = 1$".

This is not true! The function $q(x)$ is never equal to 1 (you can check this by trying to solve for $q(x) = 1$, but you don't have to here).

Where did Little Johnny go wrong?

Solution: The Intermediate Value Theorem says: "If $f(x)$ is a continuous function on a closed interval $[a, b]$, then for any number N between $f(a)$ and $f(b)$ there is a number c between a and b so that $f(c) = N$." Here, Little Johnny is trying to use this for $a = -3$, $b = 0$, and $N = 1$. Since $q(a) = 2$ and $q(b) = \frac{1}{2}$, it is true that the number 1 is in between $q(a)$ and $q(b)$. However, $q(x)$ is not continuous on the interval $[-3, 0]$. In particular, notice that $q(x)$ is not even defined at $x = -2$ (look at the factorization above!).

Since $q(x)$ is not continuous on the closed interval Little Johnny is talking about, he was mistaken in using the Intermediate Value Theorem. Not every requirement for the IVT was satisfied, and in fact this is an example where the IVT is not true.

As a bonus, let's see that this is true: is there every a number c with $q(c) = 1$? If there was, we would have

$$q(c) = \frac{c^2 - 2c - 3}{c^2 - c - 6} = 1$$

and multiplying both sides by the denominator we get

$$c^2 - 2c - 3 = c^2 - c - 6$$

or

$$c = 3$$

But, if we evaluate

$$q(3) = \frac{3^2 - 2\cdot 3 - 3}{3^2 - 3 - 6} = \frac{0}{0}$$

which is not a number. In fact, we see above that $q(x)$ is not defined at $x = 3$. More importantly this is also where we took the limit in the first question, so the answer is not even 'supposed to be 1'.

The equation 'falsely thinks that $\frac{0}{0} = 1$', when 'actually in this case $\frac{0}{0} = \frac{4}{5}$'. This is why we cannot treat $\frac{0}{0}$ (or ∞, or $\frac{\infty}{\infty}$) as numbers, they do not behave the same in different equations.