Math 3002: Problem Set 1

1. Just to practice some integrals, compute the following:

   (a) $\int x\cos(x)dx$

   Solution: Integrate by parts: set $u = x$ and $dv = \cos(x)dx$, so that $v = \sin(x)$ and $du = dx$. This gives

   $$\begin{aligned}\int x\cos(x)dx &= x\sin(x) - \int \sin(x)dx \\ &= x\sin(x) - (-\cos(x) + c) \\ &= x\sin(x) + \cos(x) - c\end{aligned}$$

   for some constant $c \in \mathbb{R}$.

   (b) $\int (2x+1)^2 dx$

   Solution: My prefence here is to multiply out, so

   $$\begin{aligned}\int (2x+1)^2 dx &= \int 4x^2 + 4x + 1dx \\ &= \frac{4}{3}x^3 + 2x^2 + x + c\end{aligned}$$

   for some constant $c \in \mathbb{R}$

   (c) $\int \arctan(x)dx$

   Solution: I think I put this on as a joke, because someone suggested arctan for an integration problem in class. How do you do this?

   I guess integrate by parts, since arctan is confusing but the derivative isn't so bad: set $u = \arctan(x)$ and $dv = dx$, so $v = x$ and $du = \frac{1}{1+x^2}$ (derivative of an inverse function is easy, using the chain rule). Then

$$\begin{aligned}\int \arctan(x)dx &= x\arctan(x) - \int \frac{x}{1+x^2}dx \\ &= x\arctan(x) - \int \frac{x}{1+x^2}dx\end{aligned}$$

Setting $w = 1 + x^2$, so $dw = 2dx$, we get

$$\begin{aligned} &= x\arctan(x) - \int \frac{1}{w} \cdot \frac{1}{2}dw \\ &= x\arctan(x) - \frac{1}{2}\ln(|w|) + c \\ \int \arctan(x)dx &= x\arctan(x) - \frac{1}{2}\ln(|1+x^2|) + c\end{aligned}$$

2. Compute

$$\int xe^x dx,$$

(hint: integrate by parts) then compute

$$\int x^2 e^x dx$$

Can you write a general formula for

$$\int x^n e^x$$

?

Solution: Let $u = x$ and $dv = e^x dx$, so $v = e^x$ and $du = dx$. Then integration by parts gives

$$\begin{aligned}\int xe^x dx &= xe^x - \int e^x dx \\ &= xe^x - e^x + c \\ &= (x-1)e^x + c\end{aligned}$$

for some constant $c \in \mathbb{R}$.

For the second integral, we again integrate by parts. Set ux^2, so $du = 2xdx$, and keep $dv = e^x dx$ so $v = e^x$. Then

$$\begin{aligned}\int x^2 e^x &= x^2 e^x - \int 2x e^x dx \\ &= x^2 e^x - 2\left((x-1)e^x + c\right) \\ &= (x^2 - 2x + 2)e^x + c\end{aligned}$$

(notice I relabelled the constant $2c$ back to c, because its still just some constant).

Thinking about this, we can guess that the integral of $x^n e^x$ is some degree n polynomial times e^x, plus a constant.

So, let's get at it:

$$\begin{aligned}\int x^n e^x &= x^n e^x - \int n x^{n-1} e^x dx \\ &= x^n e^x - n \int x^{n-1} e^x dx\end{aligned}$$

If we know what $\int x^{n-1} e^x dx$ is, then we can put it into this formula. If we know it is a polynomial times e^x, plus a constant, then we can factor out the e^x and our result will be a polynomial times e^x (plus a constant).

For instance, we see

$$\begin{aligned}\int x^3 e^x &= x^3 e^x - 3 \int x^2 e^x dx \\ &= x^3 e^x - 3\big((x^2 - 2x + 2)e^x + c\big) \\ &= (x^3 - 3x^2 + 6x - 6)e^x + c\end{aligned}$$

(again absorbing constants).

Meditating a bit on this, we find an inductive formula

$$\begin{aligned}\int x^n e^x = (&x^n - nx^{n-1} + n(n-1)x^{n-2} - n(n-1)(n-2)x^{n-3} \\ &+ ... + (-1)^{n-1} n(n-1)(n-2)... \cdot 2x \\ &+ (-1)^n n(n-1)(n-2)... \cdot 2 \cdot 1)e^x + c \\ = &\left(\sum_{i=0}^{n} (-1)^{n-i} \frac{n!}{i!} x^i\right) e^x + c\end{aligned}$$

3. We've mentioned in class a couple of times that the indefinite integral(/antiderivative)

$\int e^{x^2}$ cannot be written in terms of simpler function. Let $E(x) = \int_0^x e^{t^2} dt$, so $E(x)$ is the antiderivative of e^{x^2} such that $E(0) = 0$.

Find

$$\int xe^{x^2}$$

in terms of $E(x)$, and then

$$\int x^2 e^{x^2}$$

Would you want to write a general form for

$$\int x^n e^{x^2} dx$$

?

Assuming you said no, what is different between this question and the last question? If you said yes, write out a general form.

Solution: Again, integrate by parts. Set $u = x$ and $dv = e^{x^2}$, so that $du = dx$ and $v = E(x)$. Then

$$\int xe^{x^2} = xE(x) - \int E(x)dx$$

Similarly, integrate the second by parts - with $u = x^2$ and $dv = e^{x^2}$ so $du = 2xdx$ and $v = E(x)$ - to obtain

$$\begin{aligned}\int x^2 e^{x^2} &= x^2 E(x) - \int 2xE(x)dx \\ &= x^2 E(x) - 2\int xE(x)dx\end{aligned}$$

Let's do another: with $u = x^3$ and $dv = e^{x^2}$ so $du = 3x^2 dx$ and $v = E(x)$, we get

$$\begin{aligned}\int x^3 e^{x^2} &= x^3 E(x) - \int 3x^2 E(x)dx \\ &= x^3 E(x) - \int 3x^2 E(x)dx \\ &= x^3 E(x) - 3\left(\int x^2 E(x)dx\right)\end{aligned}$$

If we integrate by parts again, with $u = x^2$ and $dv = E(x)dx$ so $du = 2xdx$ and $v = \int E(x)dx$, we get

$$\begin{aligned} &= x^3E(x) - 3\left(x^2\int E(x)dx - \int 2x\int E(x)dx\right) \\ &= x^3E(x) - 3x^2\int E(x)dx + 6\int x\left(\int E(x)dx\right)dx \end{aligned}$$

We can integrate by parts yet again, for $u = x$ and $dv = \left(\int E(x)dx\right)dx$, giving $du = dx$ and $v = \int\left(\int E(x)dx\right)dx$, which gives as

$$\begin{aligned} &= x^3E(x) - 3x^2\int E(x)dx \\ &\quad + 6\left(x\int\left(\int E(x)dx\right)dx - \int\left(\int\left(\int E(x)dx\right)dx\right)dx\right) \end{aligned}$$

Let's ignore parentheses and the 'dx':

$$\int x^3e^{x^2} = x^3E(x) - 3x^2\int E(x) + 6x\iint E(x) - 6\iiint E(x)$$

Well, okay... a general formula is given by

$$\int x^ne^{x^2} = \sum_{i=0}^{n}(-1)^i\frac{n!}{(n-i)!}x^{n-i}\int^{[i]}E(x)$$

where $\int^{[i]}E(x)$ is the i-th iterated (indefinite) integral, (with $\int^{[0]}E(x) = E(x)$).

The correct answer is 'no', I would not want to write a general form. A difference in the last question is that taking integrals of e^x 'stabilizes', in the sense that we keep getting the same result (so the integral 'goes away'). Even if we we give a special name $E(x)$ to the integral of e^{x^2}, as we keep doing integration by parts we keep getting more and more integrals. We can never hope to make up some finite number of 'basic' functions in which we can express this general form

4. Show all work (i.e., do not just cite a formula) (integrate when possible):

   (a) Find a function $y(x)$ such that $y' = x^2y$ and $y(1) = 3$.

Solution: Divide both sides by y to obtain

$$\frac{y'}{y} = x^2$$

and recognize $\frac{y'}{y} = (\ln(|y|))'$, so that

$$(\ln(|y|))' = x^2$$

and we can integrate:

$$\ln(|y|) = \int x^2 dx = \frac{x^3}{3} + c$$

and exponentiate:

$$|y| = e^{\frac{x^3}{3}+c} = Ce^{\frac{x^3}{3}}$$

Here C must be positive, but you should check that we can allow negatives, so

$$y(x) = Ce^{\frac{x^3}{3}}$$

To satisfy the initial condition, we need $y(1) = 3$, or

$$3 = y(1) = Ce^{\frac{1^3}{3}} = Ce^{\frac{1}{3}}$$

so that

$$C = 3e^{-\frac{1}{3}}$$

and

$$y(x) = 3e^{\frac{x^3-1}{3}}$$

Ideally we should differentiate to check this really does satisfy the ODE.

(b) Find a function $y(x)$ such that $x \cdot y' + y = 0$ and $y(1) = 2$. Show the solution is unique, or show it is not.

Solution: First, let's divide by x to get a more familiar format:

$$y' + \frac{1}{x}y = 0$$

or

$$y' = -\frac{1}{x}y$$

Then divide by way to get another familiar format:

$$\frac{y'}{y} = -\frac{1}{x}$$

or

$$(\ln(|y|))' = -\frac{1}{x}$$

so integrating gives

$$\ln(|y|) = -\int \frac{1}{x}dx + c = -\ln(|x|) + c$$

and exponentiating gives

$$|y| = Ce^{-\ln(|x|)}$$

or

$$y(x) = -\frac{C}{|x|}$$

if we allow C to be negative.

Notice that I was very careful about absolute values in my logarithms, even though in class I've been quite lazy. If we go about with my usual laziness, we would integrate

$$\frac{y'}{y} = -\frac{1}{x}$$

and get

$$\ln(y) = -\ln(x) + c$$

so

$$y = Ce^{-\ln(x)}$$

and finally

$$y(x) = -\frac{C}{x}$$

Let's take some derivatives...

Consider $C = -2$, so both of these functions have $y(1) = 2$. To distinguish between the two cases, write

$$r(x) = -\frac{-2}{|x|} = \begin{cases} -\frac{2}{x} & x < 0 \\ \frac{2}{x} & x > 0 \end{cases}$$

and

$$s(x) = \frac{2}{x}$$

You can take the derivative

$$r'(x) = \begin{cases} \frac{1}{x^2} & x < 0 \\ -\frac{1}{x^2} & x > 0 \end{cases}$$

and check

$$x \cdot r'(x) + r(x) = \begin{cases} x \cdot \frac{2}{x^2} + \left(-\frac{2}{x}\right) & x < 0 \\ x \cdot \left(-\frac{2}{x^2}\right) + \frac{2}{x} & x > 0 \end{cases} = \begin{cases} 0 & x < 0 \\ 0 & x > 0 \end{cases}$$

Similarly (easier notationally);

$$s'(x) = -\frac{2}{x^2}$$

so

$$x \cdot s'(x) + s(x) = x \cdot \left(-\frac{2}{x^2}\right) + \frac{2}{x} = 0$$

Therefore, both $r(x)$ and $s(x)$ satisfy this differential equation, with $r(1) = s(1) = 2$. What happened to unique solutions?

The problem is already present in our mess of notation above, namely domain issues. For $x = 0$, neither $r(x)$, $s(x)$, nor the first line of our solution make sense. Notice that both of our solutions are equal on the domain $(0, \infty)$. You should really be thinking that the general form of solution is

$$y(x) = \begin{cases} \frac{c_1}{x} & x < 0 \\ \frac{c_2}{x} & x > 0 \end{cases}$$

where c_1 and c_2 are arbitrary real constants. The initial condition tells us one of these constants, but the other is free.

5. Recall that we solved any first order linear homogeneous differential equation by noticing the derivative relation $\frac{d}{dx}\log(y) = \frac{y'}{y}$

   Compute $\frac{d}{dx}\left(y^2\right)$

   Use this to solve the equation

$$y \cdot y' - x^3 = 0$$

   Warning: here by 'solve' I mean find the general form of solutions. Make a remark about the constant of integration.

Solution: First we compute

$$\frac{d}{dx}\left(y^2\right) = 2y \cdot \frac{d}{dx} y = 2y \cdot y'$$

or for convenience

$$\frac{d}{dx}\left(\frac{1}{2}y^2\right) = y \cdot y'$$

Then our equation becomes

$$\begin{aligned} y \cdot y' - x^3 &= 0 \\ \left(\frac{1}{2}y^2\right)' - x^3 &= 0 \\ \left(\frac{1}{2}y^2\right)' &= x^3 \\ \frac{1}{2}y^2 &= \int x^3 dx \\ \frac{1}{2}y^2 &= \frac{1}{4}x^4 + c \\ y^2 &= \frac{1}{2}x^4 + C \\ y(x) &= \sqrt{\frac{1}{2}x^4 + C} \end{aligned}$$

Observe that the domain on which $y(x)$ is defined depends on the constant of integration c.