In Exercise 5.4.83 we see how to use electric power consumption data and an integral to compute the amount of electric energy used in a typical day in the New England states.

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5 Integrals

IN CHAPTER 2 WE USED the tangent and velocity problems to introduce the derivative. In this chapter we use the area and distance problems to introduce the other central idea in calculus—the integral. The all-important relationship between the derivative and the integral is expressed in the Fundamental Theorem of Calculus, which says that differentiation and integration are in a sense inverse processes. We learn in this chapter, and in Chapters 6 and 8, how integration can be used to solve problems involving volumes, length of curves, population predictions, cardiac output, forces on a dam, work, consumer surplus, and baseball, among many others.

5.1 The Area and Distance Problems

Now is a good time to read (or reread) A Preview of Calculus, which discusses the unifying ideas of calculus and helps put in perspective where we have been and where we are going.

In this section we discover that in trying to find the area under a curve or the distance traveled by a car, we end up with the same special type of limit.

The Area Problem

We begin by attempting to solve the *area problem:* find the area of the region S that lies under the curve $y = f(x)$ from a to b. This means that S, illustrated in Figure 1, is bounded by the graph of a continuous function f [where $f(x) \geqslant 0$], the vertical lines $x = a$ and $x = b$, and the x-axis.

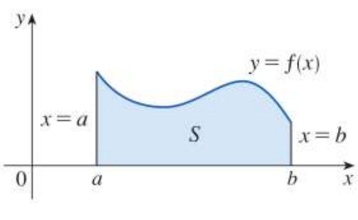


FIGURE 1
$S = \{(x, y) \mid a \leqslant x \leqslant b, 0 \leqslant y \leqslant f(x)\}$

In trying to solve the area problem we have to ask ourselves: what is the meaning of the word *area*? This question is easy to answer for regions with straight sides. For a rectangle, the area is defined as the product of the length and the width. The area of a triangle is half the base times the height. The area of a polygon is found by dividing it into triangles (as in Figure 2) and adding the areas of the triangles.

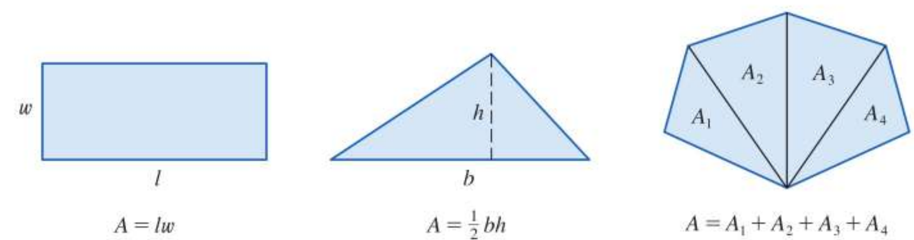


FIGURE 2

It isn't so easy, however, to find the area of a region with curved sides. We all have an intuitive idea of what the area of a region is. But part of the area problem is to make this intuitive idea precise by giving an exact definition of area.

Recall that in defining a tangent we first approximated the slope of the tangent line by slopes of secant lines and then we took the limit of these approximations. We pursue a similar idea for areas. We first approximate the region S by rectangles and then we take the limit of the sum of the areas of the approximating rectangles as we increase the number of rectangles. The following example illustrates the procedure.

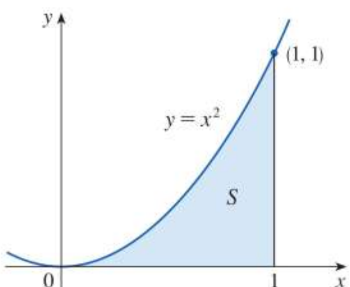


FIGURE 3

EXAMPLE 1 Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1 (the parabolic region S illustrated in Figure 3).

SOLUTION We first notice that the area of S must be somewhere between 0 and 1 because S is contained in a square with side length 1, but we can certainly do better than that. Suppose we divide S into four strips S_1, S_2, S_3, and S_4 by drawing the vertical lines $x = \frac{1}{4}$, $x = \frac{1}{2}$, and $x = \frac{3}{4}$ as in Figure 4(a).

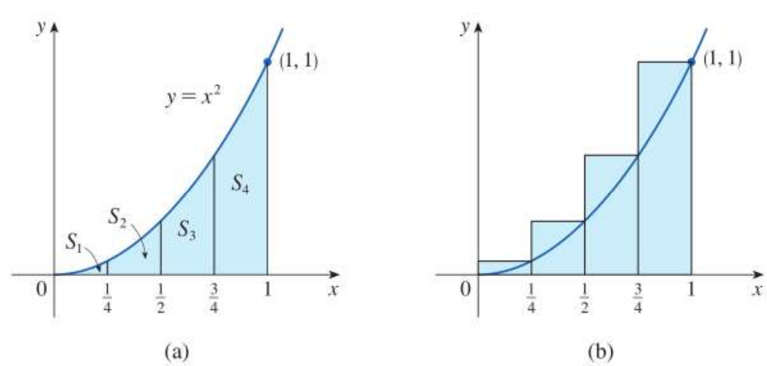


FIGURE 4

We can approximate each strip by a rectangle that has the same base as the strip and whose height is the same as the right edge of the strip [see Figure 4(b)]. In other words, the heights of these rectangles are the values of the function $f(x) = x^2$ at the *right* endpoints of the subintervals $\left[0, \frac{1}{4}\right]$, $\left[\frac{1}{4}, \frac{1}{2}\right]$, $\left[\frac{1}{2}, \frac{3}{4}\right]$, and $\left[\frac{3}{4}, 1\right]$.

Each rectangle has width $\frac{1}{4}$ and the heights are $\left(\frac{1}{4}\right)^2$, $\left(\frac{1}{2}\right)^2$, $\left(\frac{3}{4}\right)^2$, and 1^2. If we let R_4 be the sum of the areas of these approximating rectangles, we get

$$R_4 = \tfrac{1}{4} \cdot \left(\tfrac{1}{4}\right)^2 + \tfrac{1}{4} \cdot \left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4} \cdot \left(\tfrac{3}{4}\right)^2 + \tfrac{1}{4} \cdot 1^2 = \tfrac{15}{32} = 0.46875$$

From Figure 4(b) we see that the area A of S is less than R_4, so

$$A < 0.46875$$

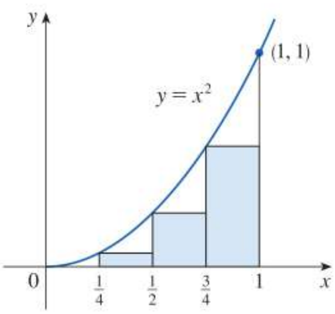


FIGURE 5

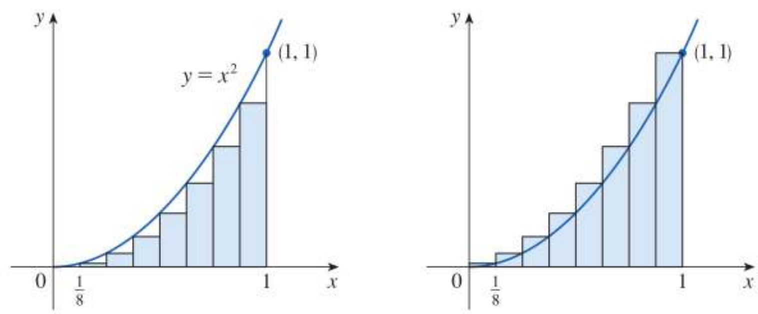
Instead of using the rectangles in Figure 4(b) we could use the smaller rectangles in Figure 5 whose heights are the values of f at the *left* endpoints of the subintervals. (The leftmost rectangle has collapsed because its height is 0.) The sum of the areas of these approximating rectangles is

$$L_4 = \tfrac{1}{4} \cdot 0^2 + \tfrac{1}{4} \cdot \left(\tfrac{1}{4}\right)^2 + \tfrac{1}{4} \cdot \left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4} \cdot \left(\tfrac{3}{4}\right)^2 = \tfrac{7}{32} = 0.21875$$

We see that the area of S is larger than L_4, so we have lower and upper estimates for A:

$$0.21875 < A < 0.46875$$

We can repeat this procedure with a larger number of strips. Figure 6 shows what happens when we divide the region S into eight strips of equal width.



(a) Using left endpoints

(b) Using right endpoints

FIGURE 6
Approximating S with eight rectangles

By computing the sum of the areas of the smaller rectangles (L_8) and the sum of the areas of the larger rectangles (R_8), we obtain better lower and upper estimates for A:

$$0.2734375 < A < 0.3984375$$

So one possible answer to the question is to say that the true area of S lies somewhere between 0.2734375 and 0.3984375.

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

We could obtain better estimates by increasing the number of strips. The table at the left shows the results of similar calculations (with a computer) using n rectangles whose heights are found with left endpoints (L_n) or right endpoints (R_n). In particular, we see by using 50 strips that the area lies between 0.3234 and 0.3434. With 1000 strips we narrow it down even more: A lies between 0.3328335 and 0.3338335. A good estimate is obtained by averaging these numbers: $A \approx 0.3333335$. ■

From the values listed in the table in Example 1, it looks as if R_n is approaching $\frac{1}{3}$ as n increases. We confirm this in the next example.

EXAMPLE 2 For the region S in Example 1, show that the approximating sums R_n approach $\frac{1}{3}$, that is,

$$\lim_{n\to\infty} R_n = \tfrac{1}{3}$$

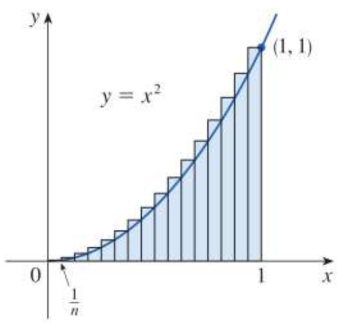


FIGURE 7

SOLUTION R_n is the sum of the areas of the n rectangles in Figure 7. Each rectangle has width $1/n$ and the heights are the values of the function $f(x) = x^2$ at the points $1/n, 2/n, 3/n, \ldots, n/n$; that is, the heights are $(1/n)^2, (2/n)^2, (3/n)^2, \ldots, (n/n)^2$. Thus

$$\begin{aligned}
R_n &= \frac{1}{n} f\left(\frac{1}{n}\right) + \frac{1}{n} f\left(\frac{2}{n}\right) + \frac{1}{n} f\left(\frac{3}{n}\right) + \cdots + \frac{1}{n} f\left(\frac{n}{n}\right) \\
&= \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \frac{1}{n}\left(\frac{3}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2 \\
&= \frac{1}{n} \cdot \frac{1}{n^2}(1^2 + 2^2 + 3^2 + \cdots + n^2) \\
&= \frac{1}{n^3}(1^2 + 2^2 + 3^2 + \cdots + n^2)
\end{aligned}$$

Here we need the formula for the sum of the squares of the first n positive integers:

$$\boxed{1} \qquad 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$$

Perhaps you have seen this formula before. It is proved in Example 5 in Appendix E.

Putting Formula 1 into our expression for R_n, we get

$$R_n = \frac{1}{n^3} \cdot \frac{n(n + 1)(2n + 1)}{6} = \frac{(n + 1)(2n + 1)}{6n^2}$$

Here we are computing the limit of the sequence $\{R_n\}$. Sequences and their limits will be studied in detail in Section 11.1. The idea is very similar to a limit at infinity (Section 2.6) except that in writing $\lim_{n\to\infty}$ we restrict n to be a positive integer. In particular, we know that

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

When we write $\lim_{n\to\infty} R_n = \frac{1}{3}$ we mean that we can make R_n as close to $\frac{1}{3}$ as we like by taking n sufficiently large.

Thus we have

$$\begin{aligned}
\lim_{n\to\infty} R_n &= \lim_{n\to\infty} \frac{(n + 1)(2n + 1)}{6n^2} \\
&= \lim_{n\to\infty} \frac{1}{6}\left(\frac{n + 1}{n}\right)\left(\frac{2n + 1}{n}\right) \\
&= \lim_{n\to\infty} \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) \\
&= \frac{1}{6} \cdot 1 \cdot 2 = \frac{1}{3}
\end{aligned}$$

■

It can be shown that the approximating sums L_n in Example 2 also approach $\frac{1}{3}$, that is,

$$\lim_{n\to\infty} L_n = \tfrac{1}{3}$$

From Figures 8 and 9 it appears that as n increases, both L_n and R_n become better and better approximations to the area of S. Therefore we *define* the area A to be the limit of the sums of the areas of the approximating rectangles, that is,

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} L_n = \tfrac{1}{3}$$

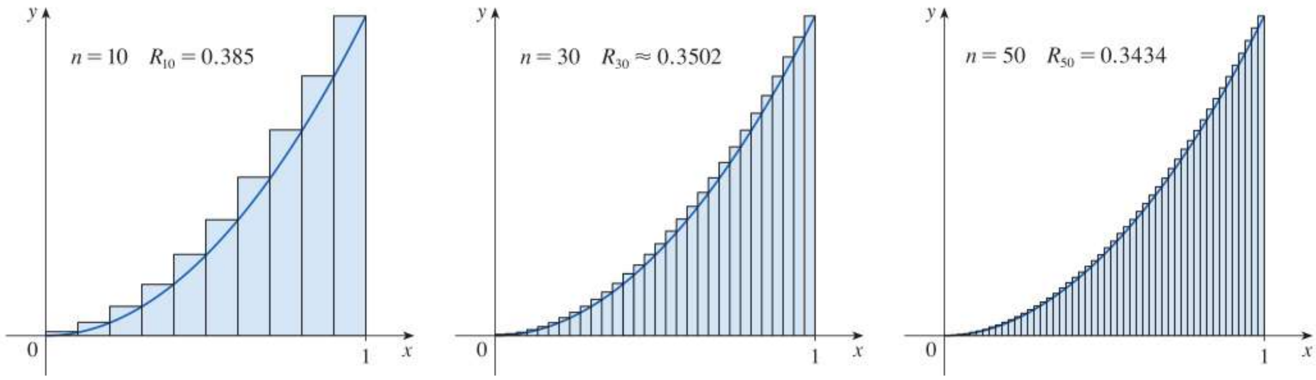


FIGURE 8 Right endpoints produce upper estimates because $f(x) = x^2$ is increasing.

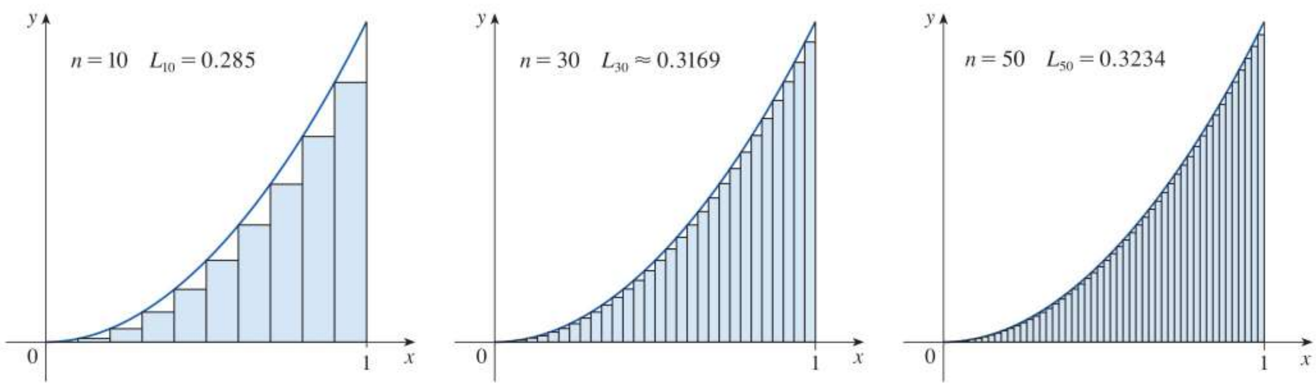


FIGURE 9 Left endpoints produce lower estimates because $f(x) = x^2$ is increasing.

Let's apply the idea of Examples 1 and 2 to the more general region S of Figure 1. We start by subdividing S into n strips $S_1, S_2, \ldots, S_n$ of equal width as in Figure 10.

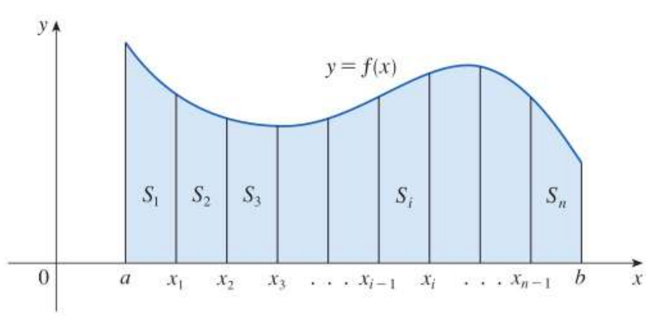


FIGURE 10

The width of the interval $[a, b]$ is $b - a$, so the width of each of the n strips is

$$\Delta x = \frac{b - a}{n}$$

These strips divide the interval $[a, b]$ into n subintervals

$$[x_0, x_1], \quad [x_1, x_2], \quad [x_2, x_3], \quad \ldots, \quad [x_{n-1}, x_n]$$

where $x_0 = a$ and $x_n = b$. The right endpoints of the subintervals are

$$\begin{aligned} x_1 &= a + \Delta x, \\ x_2 &= a + 2\,\Delta x, \\ x_3 &= a + 3\,\Delta x, \\ &\vdots \end{aligned}$$

and, in general, $x_i = a + i\,\Delta x$. Now let's approximate the ith strip S_i by a rectangle with width Δx and height $f(x_i)$, which is the value of f at the right endpoint (see Figure 11). Then the area of the ith rectangle is $f(x_i)\,\Delta x$. What we think of intuitively as the area of S is approximated by the sum of the areas of these rectangles, which is

$$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

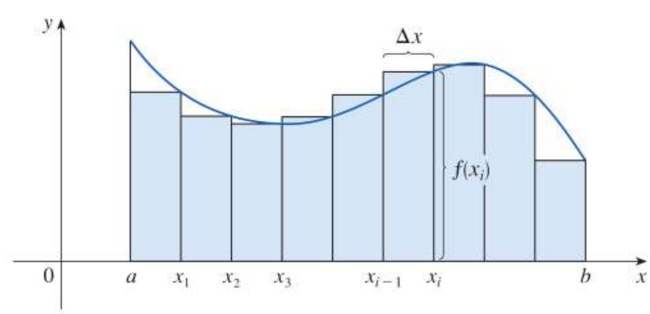


FIGURE 11

Figure 12 shows this approximation for $n = 2$, 4, 8, and 12. Notice that this approximation appears to become better and better as the number of strips increases, that is, as $n \to \infty$. Therefore we define the area A of the region S in the following way.

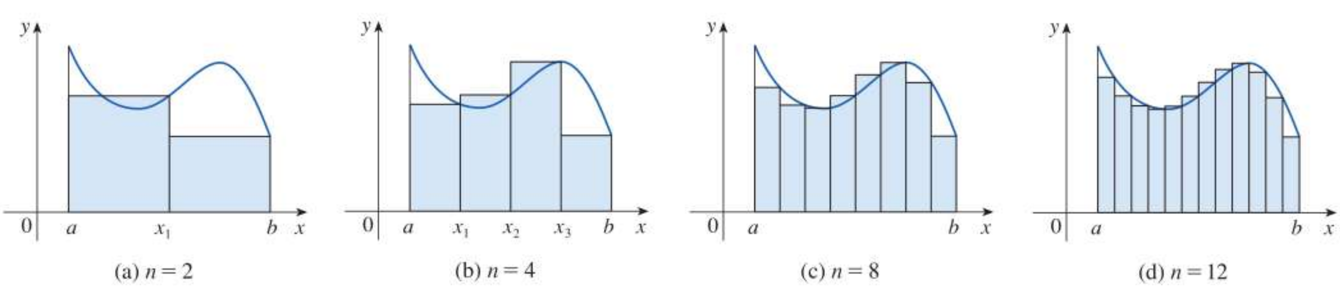


FIGURE 12

2 Definition The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

It can be proved that the limit in Definition 2 always exists, since we are assuming that f is continuous. It can also be shown that we get the same value if we use left endpoints:

3 $$A = \lim_{n \to \infty} L_n = \lim_{n \to \infty} [f(x_0)\,\Delta x + f(x_1)\,\Delta x + \cdots + f(x_{n-1})\,\Delta x]$$

In fact, instead of using left endpoints or right endpoints, we could take the height of the ith rectangle to be the value of f at *any* number x_i^* in the ith subinterval $[x_{i-1}, x_i]$. We call the numbers $x_1^*, x_2^*, \ldots, x_n^*$ the **sample points**. Figure 13 shows approximating rectangles when the sample points are not chosen to be endpoints. So a more general expression for the area of S is

4 $$A = \lim_{n \to \infty} [f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x]$$

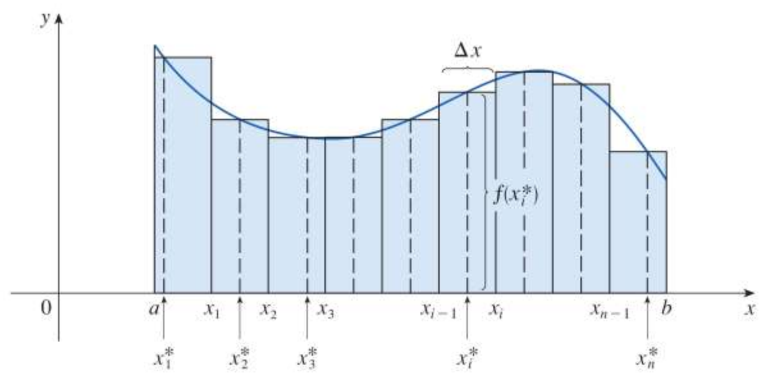
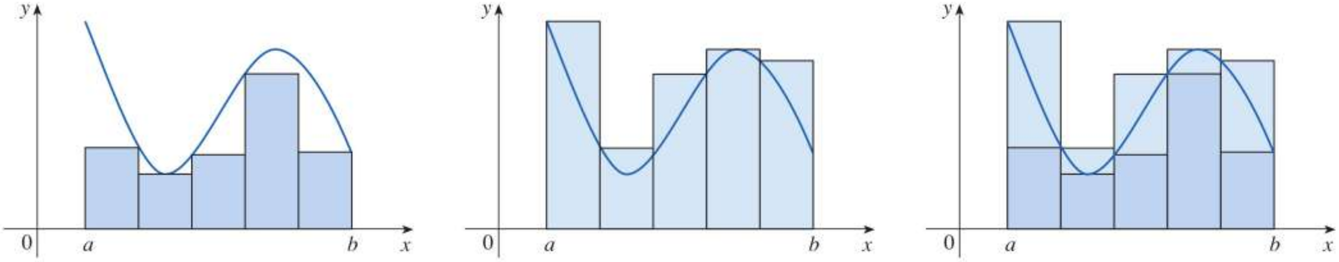


FIGURE 13

NOTE To approximate the area under the graph of f we can form **lower sums** (or **upper sums**) by choosing the sample points x_i^* so that $f(x_i^*)$ is the minimum (or maximum) value of f on the ith subinterval (see Figure 14). [Since f is continuous, we know that the minimum and maximum values of f exist on each subinterval by the Extreme Value Theorem.] It can be shown that an equivalent definition of area is the following: *A is the unique number that is smaller than all the upper sums and bigger than all the lower sums.*



(a) Lower sums (b) Upper sums (c) Upper and lower sums

FIGURE 14

We saw in Examples 1 and 2, for instance, that the area $\left(A = \frac{1}{3}\right)$ is trapped between all the left approximating sums L_n and all the right approximating sums R_n. The function in those examples, $f(x) = x^2$, happens to be increasing on $[0, 1]$ and so the lower sums arise from left endpoints and the upper sums from right endpoints. (See Figures 8 and 9.)

We often use **sigma notation** to write sums with many terms more compactly. For instance,

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

This tells us to end with $i = n$.

This tells us to add.

$$\sum_{i=m}^{n} f(x_i)\,\Delta x$$

This tells us to start with $i = m$.

If you need practice with sigma notation, look at the examples and try some of the exercises in Appendix E.

So the expressions for area in Equations 2, 3, and 4 can be written as follows:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

We can also rewrite Formula 1 in the following way:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

EXAMPLE 3 Let A be the area of the region that lies under the graph of $f(x) = e^{-x}$ between $x = 0$ and $x = 2$.

(a) Using right endpoints, find an expression for A as a limit. Do not evaluate the limit.
(b) Estimate the area by taking the sample points to be midpoints and using four subintervals and then ten subintervals.

SOLUTION

(a) Since $a = 0$ and $b = 2$, the width of a subinterval is

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

So $x_1 = 2/n$, $x_2 = 4/n$, $x_3 = 6/n$, $x_i = 2i/n$, and $x_n = 2n/n$. The sum of the areas of the approximating rectangles is

$$\begin{aligned} R_n &= f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x \\ &= e^{-x_1}\,\Delta x + e^{-x_2}\,\Delta x + \cdots + e^{-x_n}\,\Delta x \\ &= e^{-2/n}\left(\frac{2}{n}\right) + e^{-4/n}\left(\frac{2}{n}\right) + \cdots + e^{-2n/n}\left(\frac{2}{n}\right) \end{aligned}$$

According to Definition 2, the area is

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{2}{n}\left(e^{-2/n} + e^{-4/n} + e^{-6/n} + \cdots + e^{-2n/n}\right)$$

Using sigma notation we could write

$$A = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} e^{-2i/n}$$

It is difficult to evaluate this limit directly by hand, but with the aid of a computer algebra system it isn't hard (see Exercise 32). In Section 5.3 we will be able to find A more easily using a different method.

(b) With $n = 4$ the subintervals of equal width $\Delta x = 0.5$ are $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, and $[1.5, 2]$. The midpoints of these subintervals are $x_1^* = 0.25$, $x_2^* = 0.75$, $x_3^* = 1.25$, and $x_4^* = 1.75$, and the sum M_4 of the areas of the four approximating rectangles (see Figure 15) is

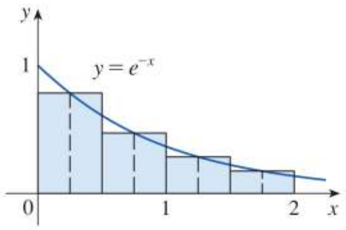


FIGURE 15

$$\begin{aligned} M_4 &= \sum_{i=1}^{4} f(x_i^*)\,\Delta x \\ &= f(0.25)\,\Delta x + f(0.75)\,\Delta x + f(1.25)\,\Delta x + f(1.75)\,\Delta x \\ &= e^{-0.25}(0.5) + e^{-0.75}(0.5) + e^{-1.25}(0.5) + e^{-1.75}(0.5) \\ &= 0.5\,(e^{-0.25} + e^{-0.75} + e^{-1.25} + e^{-1.75}) \approx 0.8557 \end{aligned}$$

So an estimate for the area is

$$A \approx 0.8557$$

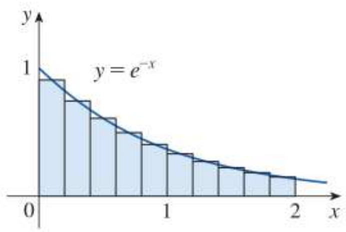


FIGURE 16

With $n = 10$ the subintervals are $[0, 0.2]$, $[0.2, 0.4]$, $\ldots$, $[1.8, 2]$ and the midpoints are $x_1^* = 0.1$, $x_2^* = 0.3$, $x_3^* = 0.5$, $\ldots$, $x_{10}^* = 1.9$. Thus

$$\begin{aligned} A \approx M_{10} &= f(0.1)\,\Delta x + f(0.3)\,\Delta x + f(0.5)\,\Delta x + \cdots + f(1.9)\,\Delta x \\ &= 0.2(e^{-0.1} + e^{-0.3} + e^{-0.5} + \cdots + e^{-1.9}) \approx 0.8632 \end{aligned}$$

From Figure 16 it appears that this estimate is better than the estimate with $n = 4$. ■

The Distance Problem

In Section 2.1 we considered the *velocity problem:* find the velocity of a moving object at a given instant if the distance of the object (from a starting point) is known at all times. Now let's consider the *distance problem:* find the distance traveled by an object during a certain time period if the velocity of the object is known at all times. (In a sense this is the inverse problem of the velocity problem.) If the velocity remains constant, then the distance problem is easy to solve by means of the formula

$$\text{distance} = \text{velocity} \times \text{time}$$

But if the velocity varies, it's not so easy to find the distance traveled. We investigate the problem in the following example.

EXAMPLE 4 Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table:

Time (s)	0	5	10	15	20	25	30
Velocity (mi/h)	17	21	24	29	32	31	28

In order to have the time and the velocity in consistent units, let's convert the velocity readings to feet per second (1 mi/h = 5280/3600 ft/s):

Time (s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	45	41

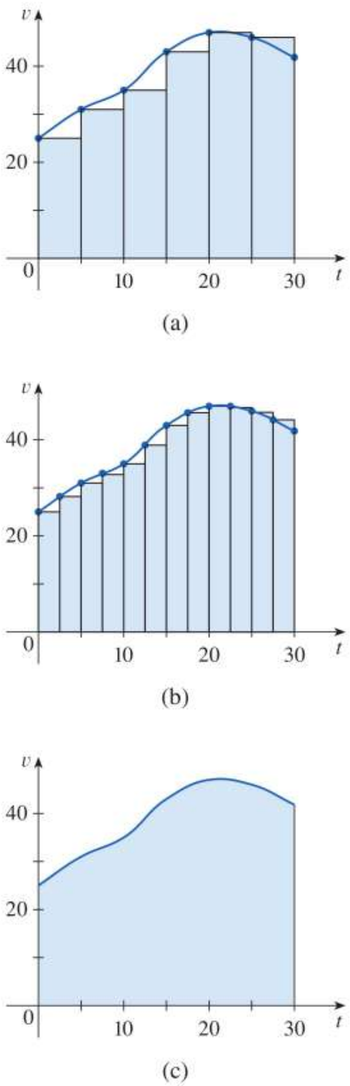


FIGURE 17

During the first five seconds the velocity doesn't change very much, so we can estimate the distance traveled during that time by assuming that the velocity is constant. If we take the velocity during that time interval to be the initial velocity (25 ft/s), then we obtain the approximate distance traveled during the first five seconds:

$$25 \text{ ft/s} \times 5 \text{ s} = 125 \text{ ft}$$

Similarly, during the second time interval the velocity is approximately constant and we take it to be the velocity when $t = 5$ s. So our estimate for the distance traveled from $t = 5$ s to $t = 10$ s is

$$31 \text{ ft/s} \times 5 \text{ s} = 155 \text{ ft}$$

If we add similar estimates for the other time intervals, we obtain an estimate for the total distance traveled:

$$(25 \times 5) + (31 \times 5) + (35 \times 5) + (43 \times 5) + (47 \times 5) + (45 \times 5) = 1130 \text{ ft}$$

We could just as well have used the velocity at the *end* of each time period instead of the velocity at the beginning as our assumed constant velocity. Then our estimate becomes

$$(31 \times 5) + (35 \times 5) + (43 \times 5) + (47 \times 5) + (45 \times 5) + (41 \times 5) = 1210 \text{ ft}$$

Now let's sketch an approximate graph of the velocity function of the car along with rectangles whose heights are the initial velocities for each time interval [see Figure 17(a)]. The area of the first rectangle is $25 \times 5 = 125$, which is also our estimate for the distance traveled in the first five seconds. In fact, the area of each rectangle can be interpreted as a distance because the height represents velocity and the width represents time. The sum of the areas of the rectangles in Figure 17(a) is $L_6 = 1130$, which is our initial estimate for the total distance traveled.

If we want a more accurate estimate, we could take velocity readings more often, as illustrated in Figure 17(b). You can see that the more velocity readings we take, the closer the sum of the areas of the rectangles gets to the exact area under the velocity curve [see Figure 17(c)]. This suggests that the total distance traveled is equal to the area under the velocity graph. ■

In general, suppose an object moves with velocity $v = f(t)$, where $a \leqslant t \leqslant b$ and $f(t) \geqslant 0$ (so the object always moves in the positive direction). We take velocity readings at times $t_0\ (= a), t_1, t_2, \ldots, t_n\ (= b)$ so that the velocity is approximately constant on each subinterval. If these times are equally spaced, then the time between consecutive readings is $\Delta t = (b - a)/n$. During the first time interval the velocity is approximately $f(t_0)$ and so the distance traveled is approximately $f(t_0)\,\Delta t$. Similarly, the distance traveled during the second time interval is about $f(t_1)\,\Delta t$ and the total distance traveled during the time interval $[a, b]$ is approximately

$$f(t_0)\,\Delta t + f(t_1)\,\Delta t + \cdots + f(t_{n-1})\,\Delta t = \sum_{i=1}^{n} f(t_{i-1})\,\Delta t$$

If we use the velocity at right endpoints instead of left endpoints, our estimate for the total distance becomes

$$f(t_1)\,\Delta t + f(t_2)\,\Delta t + \cdots + f(t_n)\,\Delta t = \sum_{i=1}^{n} f(t_i)\,\Delta t$$

The more frequently we measure the velocity, the more accurate our estimates become, so it seems plausible that the *exact* distance d traveled is the *limit* of such expressions:

$$\boxed{5} \qquad d = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1})\,\Delta t = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\,\Delta t$$

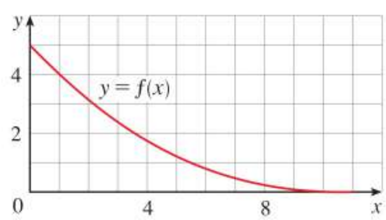
We will see in Section 5.4 that this is indeed true.

Because Equation 5 has the same form as our expressions for area in Equations 2 and 3, it follows that the distance traveled is equal to the area under the graph of the velocity function. In Chapters 6 and 8 we will see that other quantities of interest in the natural and social sciences—such as the work done by a variable force or the cardiac output of the heart—can also be interpreted as the area under a curve. So when we compute areas in this chapter, bear in mind that they can be interpreted in a variety of practical ways.

5.1 Exercises

1. (a) By reading values from the given graph of f, use five rectangles to find a lower estimate and an upper estimate for the area under the given graph of f from $x = 0$ to $x = 10$. In each case sketch the rectangles that you use.

(b) Find new estimates using ten rectangles in each case.



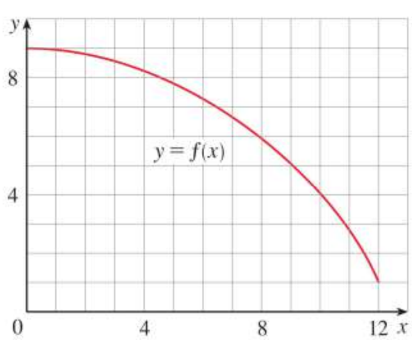
2. (a) Use six rectangles to find estimates of each type for the area under the given graph of f from $x = 0$ to $x = 12$.

(i) L_6 (sample points are left endpoints)
(ii) R_6 (sample points are right endpoints)
(iii) M_6 (sample points are midpoints)

(b) Is L_6 an underestimate or overestimate of the true area?

(c) Is R_6 an underestimate or overestimate of the true area?

(d) Which of the numbers L_6, R_6, or M_6 gives the best estimate? Explain.



3. (a) Estimate the area under the graph of $f(x) = 1/x$ from $x = 1$ to $x = 2$ using four approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?

(b) Repeat part (a) using left endpoints.

4. (a) Estimate the area under the graph of $f(x) = \sin x$ from $x = 0$ to $x = \pi/2$ using four approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?

(b) Repeat part (a) using left endpoints.

5. (a) Estimate the area under the graph of $f(x) = 1 + x^2$ from $x = -1$ to $x = 2$ using three rectangles and right endpoints. Then improve your estimate by using six rectangles. Sketch the curve and the approximating rectangles.

(b) Repeat part (a) using left endpoints.

(c) Repeat part (a) using midpoints.

(d) From your sketches in parts (a)–(c), which estimate appears to be the most accurate?

6. (a) Graph the function

$$f(x) = e^{x-x^2} \qquad 0 \leqslant x \leqslant 2$$

(b) Estimate the area under the graph of f using four approximating rectangles and taking the sample points to be (i) right endpoints and (ii) midpoints. In each case sketch the curve and the rectangles.

(c) Improve your estimates in part (b) by using eight rectangles.

7. Evaluate the upper and lower sums for $f(x) = 6 - x^2$, $-2 \leqslant x \leqslant 2$, with $n = 2$, 4, and 8. Illustrate with diagrams like Figure 14.

8. Evaluate the upper and lower sums for

$$f(x) = 1 + \cos(x/2) \qquad -\pi \leqslant x \leqslant \pi$$

with $n = 3$, 4, and 6. Illustrate with diagrams like Figure 14.

9. The speed of a runner increased steadily during the first three seconds of a race. Her speed at half-second intervals is given in the table. Find lower and upper estimates for the distance that she traveled during these three seconds.

t (s)	0	0.5	1.0	1.5	2.0	2.5	3.0
v (ft/s)	0	6.2	10.8	14.9	18.1	19.4	20.2

10. The table shows speedometer readings at 10-second intervals during a 1-minute period for a car racing at the Daytona International Speedway in Florida.

(a) Estimate the distance the race car traveled during this time period using the velocities at the beginning of the time intervals.

(b) Give another estimate using the velocities at the end of the time periods.

(c) Are your estimates in parts (a) and (b) upper and lower estimates? Explain.

Time(s)	Velocity (mi/h)
0	182.9
10	168.0
20	106.6
30	99.8
40	124.5
50	176.1
60	175.6

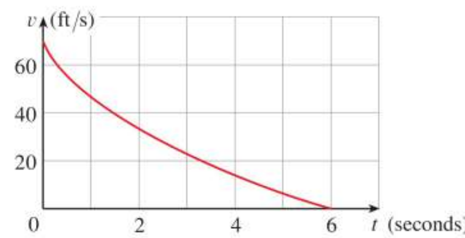
11. Oil leaked from a tank at a rate of $r(t)$ liters per hour. The rate decreased as time passed and values of the rate at two-hour time intervals are shown in the table. Find lower and upper estimates for the total amount of oil that leaked out.

t (h)	0	2	4	6	8	10
$r(t)$ (L/h)	8.7	7.6	6.8	6.2	5.7	5.3

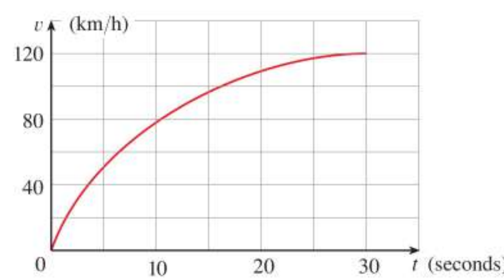
12. When we estimate distances from velocity data, it is sometimes necessary to use times $t_0, t_1, t_2, t_3, \ldots$ that are not equally spaced. We can still estimate distances using the time periods $\Delta t_i = t_i - t_{i-1}$. For example, in 1992 the space shuttle *Endeavour* was launched on mission STS-49 in order to install a new perigee kick motor in an Intelsat communications satellite. The table, provided by NASA, gives the velocity data for the shuttle between liftoff and the jettisoning of the solid rocket boosters. Use these data to estimate the height above the earth's surface of the *Endeavour*, 62 seconds after liftoff.

Event	Time (s)	Velocity (ft/s)
Launch	0	0
Begin roll maneuver	10	185
End roll maneuver	15	319
Throttle to 89%	20	447
Throttle to 67%	32	742
Throttle to 104%	59	1325
Maximum dynamic pressure	62	1445
Solid rocket booster separation	125	4151

13. The velocity graph of a braking car is shown. Use it to estimate the distance traveled by the car while the brakes are applied.



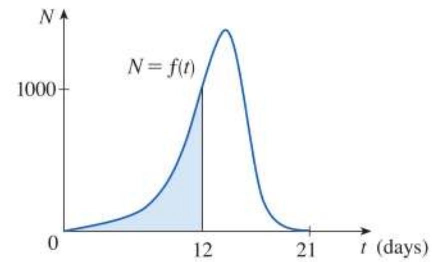
14. The velocity graph of a car accelerating from rest to a speed of 120 km/h over a period of 30 seconds is shown. Estimate the distance traveled during this period.



15. In a person infected with measles, the virus level N (measured in number of infected cells per mL of blood plasma) reaches a peak density at about $t = 12$ days (when a rash appears) and then decreases fairly rapidly as a result of immune response. The area under the graph of $N(t)$ from $t = 0$ to $t = 12$ (as shown in the figure) is equal to the total amount of infection needed to develop symptoms (measured in density of infected cells $\times$ time). The function N has been modeled by the function

$$f(t) = -t(t - 21)(t + 1)$$

Use this model with six subintervals and their midpoints to estimate the total amount of infection needed to develop symptoms of measles.



Source: J. M. Heffernan et al., "An In-Host Model of Acute Infection: Measles as a Case Study," *Theoretical Population Biology* 73 (2006): 134–47.

16–19 Use Definition 2 to find an expression for the area under the graph of f as a limit. Do not evaluate the limit.

16. $f(x) = x^2e^x, \quad 0 \leqslant x \leqslant 4$

17. $f(x) = 2 + \sin^2 x, \quad 0 \leqslant x \leqslant \pi$

18. $f(x) = x + \ln x, \quad 3 \leqslant x \leqslant 8$

19. $f(x) = x\sqrt{x^3 + 8}, \quad 1 \leqslant x \leqslant 5$

20–23 Determine a region whose area is equal to the given limit. Do not evaluate the limit.

20. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^3$

21. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\,\frac{1}{1 + (2i/n)}$

22. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{1 + \frac{3i}{n}}$

23. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan\frac{i\pi}{4n}$

24. (a) Use Definition 2 to express the area under the curve $y = x^3$ from 0 to 1 as a limit.
(b) The following formula for the sum of the cubes of the first n integers is proved in Appendix E. Use it to evaluate the limit in part (a).

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

25. Let A be the area under the graph of an increasing continuous function f from a to b, and let L_n and R_n be the approximations to A with n subintervals using left and right endpoints, respectively.
(a) How are A, L_n, and R_n related?
(b) Show that

$$R_n - L_n = \frac{b-a}{n}[f(b) - f(a)]$$

Then draw a diagram to illustrate this equation by showing that the n rectangles representing $R_n - L_n$ can be reassembled to form a single rectangle whose area is the right-hand side of the equation.
(c) Deduce that

$$R_n - A < \frac{b-a}{n}[f(b) - f(a)]$$

26. If A is the area under the curve $y = e^x$ from 1 to 3, use Exercise 25 to find a value of n such that $R_n - A < 0.0001$.

T **27–28** With a programmable calculator (or a computer), it is possible to evaluate the expressions for the sums of areas of approximating rectangles, even for large values of n, using looping. (On a TI use the Is> command or a For-EndFor loop, on a Casio use Isz, on an HP or in BASIC use a FOR-NEXT loop.) Compute the sum of the areas of approximating rectangles using equal subintervals and right endpoints for $n = 10$, 30, 50, and 100. Then guess the value of the exact area.

27. The region under $y = x^4$ from 0 to 1

28. The region under $y = \cos x$ from 0 to $\pi/2$

T **29–30** Some computer algebra systems have commands that will draw approximating rectangles and evaluate the sums of their areas, at least if x_i^* is a left or right endpoint. (For instance, in Maple use `leftbox`, `rightbox`, `leftsum`, and `rightsum`.)

29. Let $f(x) = 1/(x^2 + 1)$, $0 \leqslant x \leqslant 1$.
(a) Find the left and right sums for $n = 10$, 30, and 50.
(b) Illustrate by graphing the rectangles in part (a).
(c) Show that the exact area under f lies between 0.780 and 0.791.

30. Let $f(x) = \ln x$, $1 \leqslant x \leqslant 4$.
(a) Find the left and right sums for $n = 10$, 30, and 50.
(b) Illustrate by graphing the rectangles in part (a).
(c) Show that the exact area under f lies between 2.50 and 2.59.

T **31.** (a) Express the area under the curve $y = x^5$ from 0 to 2 as a limit.
(b) Use a computer algebra system to find the sum in your expression from part (a).
(c) Evaluate the limit in part (a).

T **32.** Find the exact area of the region under the graph of $y = e^{-x}$ from 0 to 2 by using a computer algebra system to evaluate the sum and then the limit in Example 3(a). Compare your answer with the estimate obtained in Example 3(b).

T **33.** Find the exact area under the cosine curve $y = \cos x$ from $x = 0$ to $x = b$, where $0 \leqslant b \leqslant \pi/2$. (Use a computer algebra system both to evaluate the sum and compute the limit.) In particular, what is the area if $b = \pi/2$?

34. (a) Let A_n be the area of a polygon with n equal sides inscribed in a circle with radius r. By dividing the polygon into n congruent triangles with central angle $2\pi/n$, show that

$$A_n = \tfrac{1}{2}nr^2 \sin \frac{2\pi}{n}$$

(b) Show that $\lim_{n\to\infty} A_n = \pi r^2$. [*Hint:* Use Equation 3.3.5.]

5.2 The Definite Integral

We saw in Section 5.1 that a limit of the form

$$\boxed{1} \qquad \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} [f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x]$$

arises when we compute an area. We also saw that it arises when we try to find the distance traveled by an object. It turns out that this same type of limit occurs in a wide variety of situations even when f is not necessarily a positive function. In Chapters 6 and 8 we will see that limits of this type also arise in finding lengths of curves, volumes of solids, centers of mass, force due to water pressure, and work, as well as other quantities.

The Definite Integral

We give limits of the form (1) a special name and notation.

$\boxed{2}$ **Definition of a Definite Integral** If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0\,(= a), x_1, x_2, \ldots, x_n\,(= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of *f* from *a* to *b*** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

The precise meaning of the limit that defines the integral is as follows:

For every number $\varepsilon > 0$ there is an integer N such that

$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f(x_i^*)\,\Delta x \right| < \varepsilon$$

for every integer $n > N$ and for every choice of x_i^* in $[x_{i-1}, x_i]$.

NOTE 1 The symbol $\int$ was introduced by Leibniz and is called an **integral sign**. It is an elongated S and was chosen because an integral is a limit of sums. In the notation $\int_a^b f(x)\,dx$, $f(x)$ is called the **integrand** and a and b are called the **limits of integration**; a is the **lower limit** and b is the **upper limit**. For now, the symbol dx has no meaning by itself; $\int_a^b f(x)\,dx$ is all one symbol. The dx simply indicates that the independent variable is x. The procedure of calculating an integral is called **integration**.

NOTE 2 The definite integral $\int_a^b f(x)\,dx$ is a number; it does not depend on x. In fact, we could use any letter in place of x without changing the value of the integral:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr$$

NOTE 3 The sum

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

that occurs in Definition 2 is called a **Riemann sum** after the German mathematician Bernhard Riemann (1826–1866). So Definition 2 says that the definite integral of an integrable function can be approximated to within any desired degree of accuracy by a Riemann sum.

We know that if f happens to be positive, then the Riemann sum can be interpreted as a sum of areas of approximating rectangles (see Figure 1). By comparing Definition 2 with the definition of area in Section 5.1, we see that the definite integral $\int_a^b f(x)\,dx$ can be interpreted as the area under the curve $y = f(x)$ from a to b. (See Figure 2.)

Riemann

Bernhard Riemann received his Ph.D. under the direction of the legendary Gauss at the University of Göttingen and remained there to teach. Gauss, who was not in the habit of praising other mathematicians, spoke of Riemann's "creative, active, truly mathematical mind and gloriously fertile originality." The definition (2) of an integral that we use is due to Riemann. He also made major contributions to the theory of functions of a complex variable, mathematical physics, number theory, and the foundations of geometry. Riemann's broad concept of space and geometry turned out to be the right setting, 50 years later, for Einstein's general relativity theory. Riemann's health was poor throughout his life, and he died of tuberculosis at the age of 39.

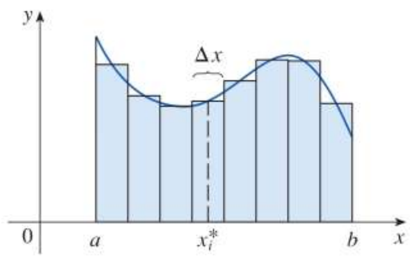


FIGURE 1
If $f(x) \geqslant 0$, the Riemann sum $\sum f(x_i^*)\,\Delta x$ is the sum of areas of rectangles.

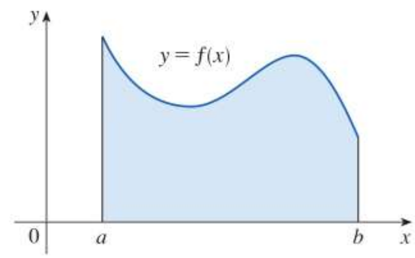


FIGURE 2
If $f(x) \geqslant 0$, the integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from a to b.

If f takes on both positive and negative values, as in Figure 3, then the Riemann sum is the sum of the areas of the rectangles that lie above the x-axis and the *negatives* of the areas of the rectangles that lie below the x-axis (the areas of the blue rectangles *minus* the areas of the gold rectangles). When we take the limit of such Riemann sums, we get the situation illustrated in Figure 4. A definite integral can be interpreted as a **net area**, that is, a difference of areas:

$$\int_a^b f(x)\,dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the x-axis and above the graph of f.

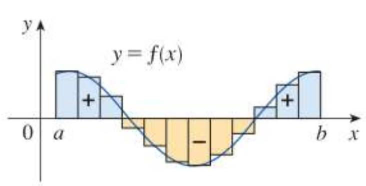


FIGURE 3
$\sum f(x_i^*)\,\Delta x$ is an approximation to the net area.

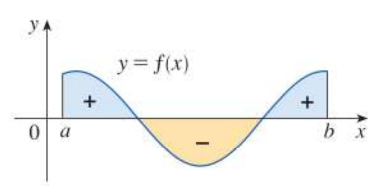


FIGURE 4
$\int_a^b f(x)\,dx$ is the net area.

EXAMPLE 1 Evaluate the Riemann sum for $f(x) = x^3 - 6x$, $0 \leqslant x \leqslant 3$, with $n = 6$ subintervals and taking the sample endpoints to be right endpoints.

SOLUTION
With $n = 6$ subintervals, the interval width is $\Delta x = (3 - 0)/6 = \frac{1}{2}$ and the right endpoints are

$$x_1 = 0.5 \quad x_2 = 1.0 \quad x_3 = 1.5 \quad x_4 = 2.0 \quad x_5 = 2.5 \quad x_6 = 3.0$$

So the Riemann sum is

$$\begin{aligned} R_6 &= \sum_{i=1}^{6} f(x_i)\,\Delta x \\ &= f(0.5)\,\Delta x + f(1.0)\,\Delta x + f(1.5)\,\Delta x + f(2.0)\,\Delta x + f(2.5)\,\Delta x + f(3.0)\,\Delta x \\ &= \tfrac{1}{2}(-2.875 - 5 - 5.625 - 4 + 0.625 + 9) \\ &= -3.9375 \end{aligned}$$

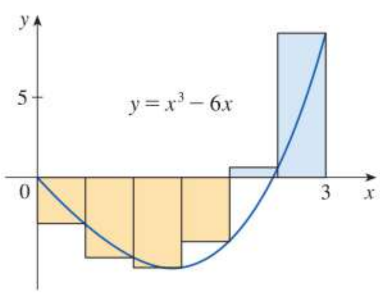


FIGURE 5

Notice that f is not a positive function and so the Riemann sum does not represent a sum of areas of rectangles. But it does represent the sum of the areas of the blue rectangles (above the x-axis) minus the sum of the areas of the gold rectangles (below the x-axis) in Figure 5. ■

NOTE 4 Although we have defined $\int_a^b f(x)\,dx$ by dividing $[a, b]$ into subintervals of equal width, there are situations in which it is advantageous to work with subintervals of unequal width. For instance, in Exercise 5.1.12, NASA provided velocity data at times that were not equally spaced, but we were still able to estimate the distance traveled. And there are methods for numerical integration that take advantage of unequal subintervals. If the subinterval widths are $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$, we have to ensure that all these widths approach 0 in the limiting process. This happens if the largest width, $\max \Delta x_i$, approaches 0. So in this case the definition of a definite integral becomes

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

We have defined the definite integral for an integrable function, but not all functions are integrable (see Exercises 81–82). The following theorem shows that the most commonly occurring functions are in fact integrable. The theorem is proved in more advanced courses.

3 Theorem If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

If f is integrable on $[a, b]$, then the limit in Definition 2 exists and gives the same value no matter how we choose the sample points x_i^*. To simplify the calculation of the integral we often take the sample points to be right endpoints. Then $x_i^* = x_i$ and the definition of an integral simplifies as follows.

4 Theorem If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $$\Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\,\Delta x$$

EXAMPLE 2 Express

$$\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\,\Delta x$$

as an integral on the interval $[0, \pi]$.

SOLUTION Comparing the given limit with the limit in Theorem 4, we see that they will be identical if we choose $f(x) = x^3 + x \sin x$. We are given that $a = 0$ and $b = \pi$. Therefore, by Theorem 4 we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\,\Delta x = \int_0^{\pi} (x^3 + x \sin x)\,dx$$ ■

Later, when we apply the definite integral to physical situations, it will be important to recognize limits of sums as integrals, as we did in Example 2. When Leibniz chose the notation for an integral, he chose the ingredients as reminders of the limiting process. In general, when we write

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \int_a^b f(x)\,dx$$

we replace $\lim \Sigma$ by $\int$, x_i^* by x, and Δx by dx.

Evaluating Definite Integrals

In order to use a limit to evaluate a definite integral, we need to know how to work with sums. The following four equations give formulas for sums of powers of positive integers. Equation 6 may be familiar to you from a course in algebra. Equations 7 and 8 were discussed in Section 5.1 and are proved in Appendix E.

Sums of Powers

5 $$\sum_{i=1}^{n} 1 = n$$

6 $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

7 $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

8 $$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

The remaining formulas are simple rules for working with sigma notation:

Properties of Sums

$\boxed{9}$ $$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$\boxed{10}$ $$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$\boxed{11}$ $$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

Formulas 9–11 are proved by writing out each side in expanded form. The left side of Equation 9 is

$$ca_1 + ca_2 + \cdots + ca_n$$

The right side is

$$c(a_1 + a_2 + \cdots + a_n)$$

These are equal by the distributive property. The other formulas are discussed in Appendix E.

In the next example we calculate a definite integral of the function f from Example 1.

EXAMPLE 3 Evaluate $\int_0^3 (x^3 - 6x)\,dx$.

SOLUTION We use Theorem 4. We have $f(x) = x^3 - 6x$, $a = 0$, $b = 3$, and

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

Then the endpoints of the subintervals are $x_0 = 0$, $x_1 = 0 + 1(3/n) = 3/n$, $x_2 = 0 + 2(3/n) = 6/n$, $x_3 = 0 + 3(3/n) = 9/n$, and in general,

$$x_i = 0 + i\left(\frac{3}{n}\right) = \frac{3i}{n}$$

Thus

$$\begin{aligned}
\int_0^3 (x^3 - 6x)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n} \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right] && \text{(Equation 9 with } c = 3/n) \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{27}{n^3} i^3 - \frac{18}{n} i\right] \\
&= \lim_{n\to\infty} \left[\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i\right] && \text{(Equations 11 and 9)} \\
&= \lim_{n\to\infty} \left\{\frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\,\frac{n(n+1)}{2}\right\} && \text{(Equations 8 and 6)} \\
&= \lim_{n\to\infty} \left[\frac{81}{4}\left(1 + \frac{1}{n}\right)^2 - 27\left(1 + \frac{1}{n}\right)\right] \\
&= \frac{81}{4} - 27 = -\frac{27}{4} = -6.75
\end{aligned}$$

In the sum, n is a constant (unlike i), so we can move $3/n$ in front of the $\sum$ sign.

This integral can't be interpreted as an area because f takes on both positive and negative values. But it can be interpreted as the difference of areas $A_1 - A_2$, where A_1 and A_2 are shown in Figure 6. ■

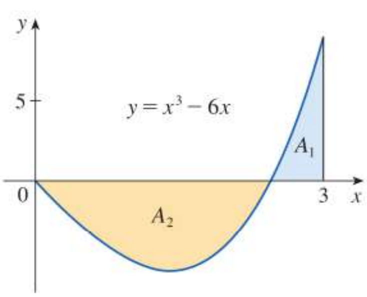


FIGURE 6
$\int_0^3 (x^3 - 6x)\,dx = A_1 - A_2 = -6.75$

Figure 7 illustrates the calculation in Example 3 by showing the positive and negative terms in the right Riemann sum R_n for $n = 40$. The values in the table show the Riemann sums approaching the exact value of the integral, -6.75, as $n \to \infty$.

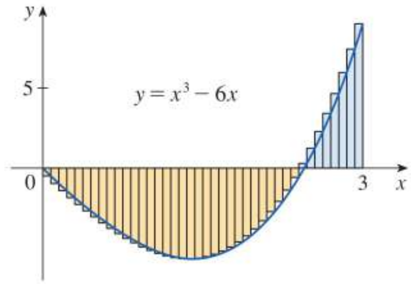


FIGURE 7
$R_{40} \approx -6.3998$

n	R_n
40	-6.3998
100	-6.6130
500	-6.7229
1000	-6.7365
5000	-6.7473

A much simpler method (made possible by the Fundamental Theorem of Calculus) for evaluating integrals like the one in Example 3 will be given in Section 5.3.

Because $f(x) = e^x$ is positive, the integral in Example 4 represents the area shown in Figure 8.

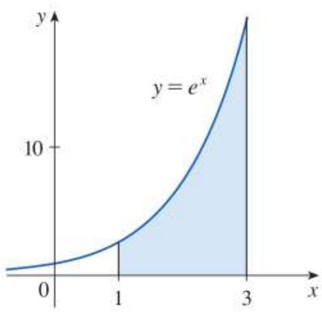


FIGURE 8

EXAMPLE 4

(a) Set up an expression for $\int_1^3 e^x\,dx$ as a limit of sums.
(b) Use a computer algebra system to evaluate the expression.

SOLUTION

(a) Here we have $f(x) = e^x$, $a = 1$, $b = 3$, and

$$\Delta x = \frac{b-a}{n} = \frac{2}{n}$$

So $x_0 = 1$, $x_1 = 1 + 2/n$, $x_2 = 1 + 4/n$, $x_3 = 1 + 6/n$, and

$$x_i = 1 + \frac{2i}{n}$$

From Theorem 4, we get

$$\begin{aligned}\int_1^3 e^x\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right)\frac{2}{n} \\ &= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} e^{1+2i/n}\end{aligned}$$

(b) If we ask a computer algebra system to evaluate the sum and simplify, we obtain

$$\sum_{i=1}^{n} e^{1+2i/n} = \frac{e^{(3n+2)/n} - e^{(n+2)/n}}{e^{2/n} - 1}$$

A computer algebra system is able to find an explicit expression for this sum because it is a geometric series. The limit could be found using l'Hospital's Rule.

Now we ask the computer algebra system to evaluate the limit:

$$\int_1^3 e^x\,dx = \lim_{n\to\infty} \frac{2}{n} \cdot \frac{e^{(3n+2)/n} - e^{(n+2)/n}}{e^{2/n} - 1} = e^3 - e$$ ■

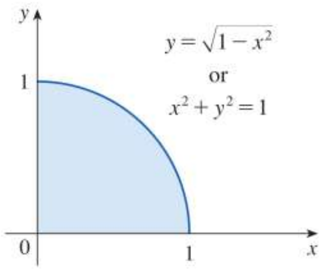


FIGURE 9

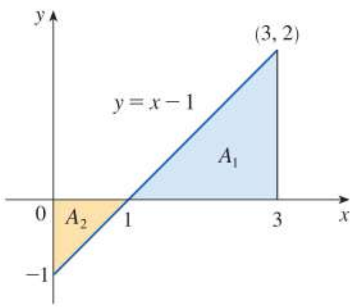


FIGURE 10

EXAMPLE 5 Evaluate the following integrals by interpreting each in terms of areas.

(a) $\int_0^1 \sqrt{1-x^2}\,dx$ (b) $\int_0^3 (x-1)\,dx$

SOLUTION

(a) Since $f(x) = \sqrt{1-x^2} \geqslant 0$, we can interpret this integral as the area under the curve $y = \sqrt{1-x^2}$ from 0 to 1. But, because $y^2 = 1 - x^2$, we get $x^2 + y^2 = 1$, which shows that the graph of f is the quarter-circle with radius 1 in Figure 9. Therefore

$$\int_0^1 \sqrt{1-x^2}\,dx = \tfrac{1}{4}\pi(1)^2 = \frac{\pi}{4}$$

(In Section 7.3 we will be able to *prove* that the area of a circle of radius r is πr^2.)

(b) The graph of $y = x - 1$ is the line with slope 1 shown in Figure 10. We compute the integral as the difference of the areas of the two triangles:

$$\int_0^3 (x-1)\,dx = A_1 - A_2 = \tfrac{1}{2}(2 \cdot 2) - \tfrac{1}{2}(1 \cdot 1) = 1.5$$ ■

The Midpoint Rule

We often choose the sample point x_i^* to be the right endpoint of the ith subinterval because it is convenient for computing the limit. But if the purpose is to find an *approximation* to an integral, it is usually better to choose x_i^* to be the midpoint of the interval, which we denote by $\bar{x}_i$. Any Riemann sum is an approximation to an integral, but if we use midpoints we get the following approximation.

Midpoint Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x = \Delta x\,[f(\bar{x}_1) + \cdots + f(\bar{x}_n)]$$

where $$\Delta x = \frac{b-a}{n}$$

and $$\bar{x}_i = \tfrac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$$

EXAMPLE 6 Use the Midpoint Rule with $n = 5$ to approximate $\int_1^2 \frac{1}{x}\,dx$.

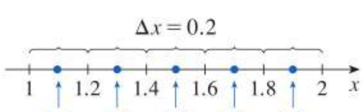


FIGURE 11
The endpoints and midpoints of the subintervals used in Example 6

SOLUTION The endpoints of the five subintervals are 1, 1.2, 1.4, 1.6, 1.8, and 2.0, so the midpoints are 1.1, 1.3, 1.5, 1.7, and 1.9. (See Figure 11.) The width of the subintervals is $\Delta x = (2-1)/5 = \frac{1}{5}$, so the Midpoint Rule gives

$$\begin{aligned} \int_1^2 \frac{1}{x}\,dx &\approx \Delta x\,[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)] \\ &= \frac{1}{5}\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right) \\ &\approx 0.691908 \end{aligned}$$

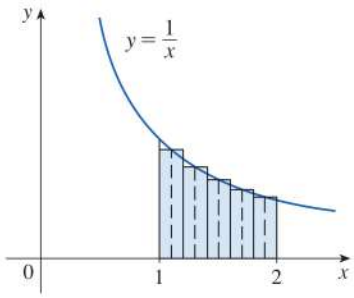


FIGURE 12

Since $f(x) = 1/x > 0$ for $1 \leqslant x \leqslant 2$, the integral represents an area, and the approximation given by the Midpoint Rule is the sum of the areas of the rectangles shown in Figure 12. ■

At the moment we don't know how accurate the approximation in Example 6 is, but in Section 7.7 we will learn a method for estimating the error involved in using the Midpoint Rule. At that time we will discuss other methods for approximating definite integrals.

If we apply the Midpoint Rule to the integral in Example 3, we get the picture in Figure 13. The approximation $M_{40} \approx -6.7563$ is much closer to the true value -6.75 than the right endpoint approximation, $R_{40} \approx -6.3998$, shown in Figure 7.

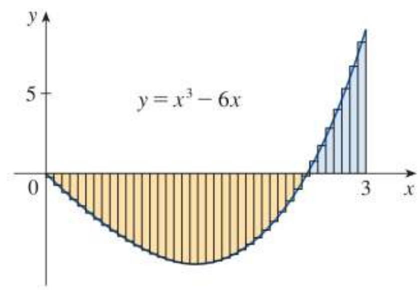


FIGURE 13
$M_{40} \approx -6.7563$

Properties of the Definite Integral

When we defined the definite integral $\int_a^b f(x)\,dx$, we implicitly assumed that $a < b$. But the definition as a limit of Riemann sums makes sense even if $a > b$. Notice that if we interchange a and b, then Δx changes from $(b - a)/n$ to $(a - b)/n$. Therefore

$$\boxed{\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx}$$

If $a = b$, then $\Delta x = 0$ and so

$$\boxed{\int_a^a f(x)\,dx = 0}$$

We now develop some basic properties of integrals that will help us to evaluate integrals in a simple manner. We assume that f and g are continuous functions.

Properties of the Integral

1. $\int_a^b c\,dx = c(b - a)$, where c is any constant

2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant

4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Property 1 says that the integral of a constant function $f(x) = c$ is the constant times the length of the interval. If $c > 0$ and $a < b$, this is to be expected because $c(b - a)$ is the area of the shaded rectangle in Figure 14.

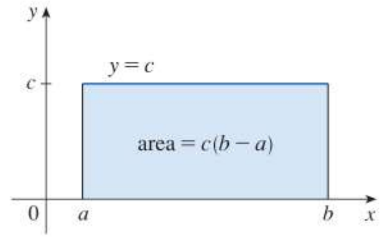


FIGURE 14
$\int_a^b c\,dx = c(b - a)$

Property 2 says that the integral of a sum is the sum of the integrals. For positive functions it says that the area under $f + g$ is the area under f plus the area under g. Figure 15 helps us understand why this is true: in view of how graphical addition works, the corresponding vertical line segments have equal height.

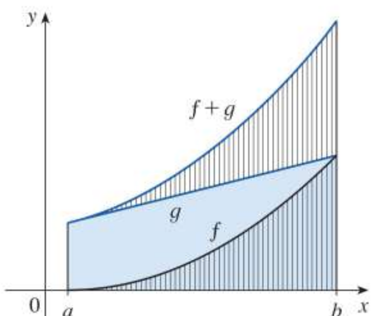


FIGURE 15
$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

In general, Property 2 follows from Theorem 4 and the fact that the limit of a sum is the sum of the limits:

$$\begin{aligned}
\int_a^b [f(x) + g(x)]\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) + g(x_i)]\,\Delta x \\
&= \lim_{n\to\infty} \left[\sum_{i=1}^{n} f(x_i)\,\Delta x + \sum_{i=1}^{n} g(x_i)\,\Delta x \right] \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x + \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i)\,\Delta x \\
&= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx
\end{aligned}$$

Property 3 seems intuitively reasonable because we know that multiplying a function by a positive number c stretches or shrinks its graph vertically by a factor of c. So it stretches or shrinks each approximating rectangle by a factor of c and therefore it has the effect of multiplying the area by c.

Property 3 can be proved in a similar manner and says that the integral of a constant times a function is the constant times the integral of the function. In other words, a constant (but *only* a constant) can be taken in front of an integral sign. Property 4 is proved by writing $f - g = f + (-g)$ and using Properties 2 and 3 with $c = -1$.

EXAMPLE 7 Use the properties of integrals to evaluate $\int_0^1 (4 + 3x^2)\,dx$.

SOLUTION Using Properties 2 and 3 of integrals, we have

$$\int_0^1 (4 + 3x^2)\,dx = \int_0^1 4\,dx + \int_0^1 3x^2\,dx = \int_0^1 4\,dx + 3\int_0^1 x^2\,dx$$

We know from Property 1 that

$$\int_0^1 4\,dx = 4(1 - 0) = 4$$

and we found in Example 5.1.2 that $\int_0^1 x^2\,dx = \frac{1}{3}$. So

$$\begin{aligned}
\int_0^1 (4 + 3x^2)\,dx &= \int_0^1 4\,dx + 3\int_0^1 x^2\,dx \\
&= 4 + 3 \cdot \tfrac{1}{3} = 5
\end{aligned}$$

■

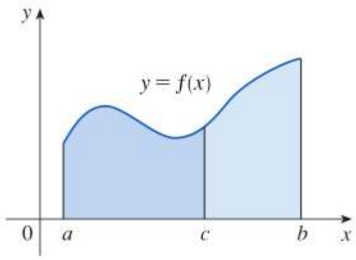


FIGURE 16

The next property tells us how to combine integrals of the same function over adjacent intervals.

5. $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

This is not easy to prove in general, but for the case where $f(x) \geqslant 0$ and $a < c < b$ Property 5 can be seen from the geometric interpretation in Figure 16: the area under $y = f(x)$ from a to c plus the area from c to b is equal to the total area from a to b.

EXAMPLE 8 If it is known that $\int_0^{10} f(x)\,dx = 17$ and $\int_0^8 f(x)\,dx = 12$, find $\int_8^{10} f(x)\,dx$.

SOLUTION By Property 5, we have

$$\int_0^8 f(x)\,dx + \int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx$$

so

$$\int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx - \int_0^8 f(x)\,dx = 17 - 12 = 5$$

Properties 1–5 are true whether $a < b$, $a = b$, or $a > b$. The following properties, in which we compare sizes of functions and sizes of integrals, are true only if $a \leqslant b$.

Comparison Properties of the Integral

6. If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant 0$.

7. If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$.

8. If $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then

$$m(b - a) \leqslant \int_a^b f(x)\,dx \leqslant M(b - a)$$

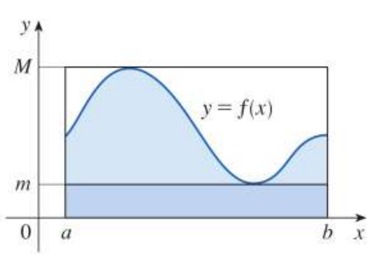


FIGURE 17

If $f(x) \geqslant 0$, then $\int_a^b f(x)\,dx$ represents the area under the graph of f, so the geometric interpretation of Property 6 is simply that areas are positive. (It also follows directly from the definition because all the quantities involved are positive.) Property 7 says that a bigger function has a bigger integral. It follows from Properties 6 and 4 because $f - g \geqslant 0$.

Property 8 is illustrated by Figure 17 for the case where $f(x) \geqslant 0$. If f is continuous, we could take m and M to be the absolute minimum and maximum values of f on the interval $[a, b]$. In this case Property 8 says that the area under the graph of f is greater than the area of the rectangle with height m and less than the area of the rectangle with height M.

PROOF OF PROPERTY 8 Since $m \leqslant f(x) \leqslant M$, Property 7 gives

$$\int_a^b m\,dx \leqslant \int_a^b f(x)\,dx \leqslant \int_a^b M\,dx$$

Using Property 1 to evaluate the integrals on the left and right sides, we obtain

$$m(b - a) \leqslant \int_a^b f(x)\,dx \leqslant M(b - a)$$

Property 8 is useful when all we want is a rough estimate of the size of an integral without going to the bother of using the Midpoint Rule.

EXAMPLE 9 Use Property 8 to estimate $\int_0^1 e^{-x^2}\,dx$.

SOLUTION Because $f(x) = e^{-x^2}$ is a decreasing function on $[0, 1]$, its absolute maximum value is $M = f(0) = 1$ and its absolute minimum value is $m = f(1) = e^{-1}$. Thus, by Property 8,

$$e^{-1}(1 - 0) \leqslant \int_0^1 e^{-x^2}\,dx \leqslant 1(1 - 0)$$

or

$$e^{-1} \leqslant \int_0^1 e^{-x^2}\,dx \leqslant 1$$

Since $e^{-1} \approx 0.3679$, we can write

$$0.367 \leqslant \int_0^1 e^{-x^2}\,dx \leqslant 1$$

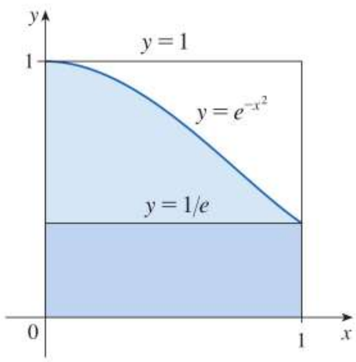


FIGURE 18

The result of Example 9 is illustrated in Figure 18. The integral is greater than the area of the lower rectangle and less than the area of the square.

5.2 Exercises

1. Evaluate the Riemann sum for $f(x) = x - 1$, $-6 \leqslant x \leqslant 4$, with five subintervals, taking the sample points to be right endpoints. Explain, with the aid of a diagram, what the Riemann sum represents.

2. If

$$f(x) = \cos x \qquad 0 \leqslant x \leqslant 3\pi/4$$

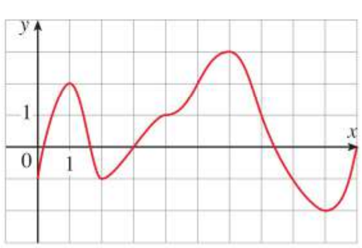
evaluate the Riemann sum with $n = 6$, taking the sample points to be left endpoints. (Give your answer correct to six decimal places.) What does the Riemann sum represent? Illustrate with a diagram.

3. If $f(x) = x^2 - 4$, $0 \leqslant x \leqslant 3$, evaluate the Riemann sum with $n = 6$, taking the sample points to be midpoints. What does the Riemann sum represent? Illustrate with a diagram.

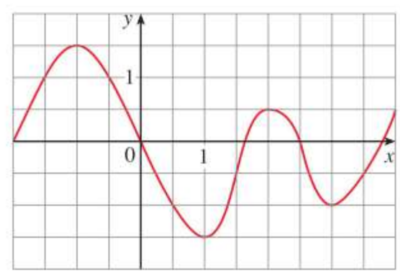
4. (a) Evaluate the Riemann sum for $f(x) = 1/x$, $1 \leqslant x \leqslant 2$, with four terms, taking the sample points to be right endpoints. (Give your answer correct to six decimal places.) Explain what the Riemann sum represents with the aid of a sketch.

(b) Repeat part (a) with midpoints as the sample points.

5. The graph of a function f is given. Estimate $\int_0^{10} f(x)\,dx$ using five subintervals with (a) right endpoints, (b) left endpoints, and (c) midpoints.



6. The graph of a function g is shown. Estimate $\int_{-2}^{4} g(x)\,dx$ with six subintervals using (a) right endpoints, (b) left endpoints, and (c) midpoints.



7. A table of values of an increasing function f is shown. Use the table to find lower and upper estimates for $\int_{10}^{30} f(x)\,dx$.

x	10	14	18	22	26	30
$f(x)$	-12	-6	-2	1	3	8

8. The table gives the values of a function obtained from an experiment. Use them to estimate $\int_3^9 f(x)\,dx$ using three equal subintervals with (a) right endpoints, (b) left endpoints, and (c) midpoints. If the function is known to be an increasing function, can you say whether your estimates are less than or greater than the exact value of the integral?

x	3	4	5	6	7	8	9
$f(x)$	-3.4	-2.1	-0.6	0.3	0.9	1.4	1.8

9–10 Use the Midpoint Rule with $n = 4$ to approximate the integral.

9. $\int_0^8 x^2\,dx$ **10.** $\int_0^2 (8x + 3)\,dx$

11–14 Use the Midpoint Rule with the given value of n to approximate the integral. Round the answer to four decimal places.

11. $\int_0^3 e^{\sqrt{x}}\,dx,\quad n = 6$ **12.** $\int_0^1 \sqrt{x^3 + 1}\,dx,\quad n = 5$

13. $\int_1^3 \frac{x}{x^2 + 8}\,dx,\quad n = 5$ **14.** $\int_0^\pi x\sin^2 x\,dx,\quad n = 4$

T **15.** Use a computer algebra system that evaluates midpoint approximations and graphs the corresponding rectangles (use `RiemannSum` or `middlesum` and `middlebox` commands in Maple) to check the answer to Exercise 13 and illustrate with a graph. Then repeat with $n = 10$ and $n = 20$.

T **16.** Use a computer algebra system to compute the left and right Riemann sums for the function $f(x) = x/(x + 1)$ on the interval $[0, 2]$ with $n = 100$. Explain why these estimates show that

$$0.8946 < \int_0^2 \frac{x}{x + 1}\,dx < 0.9081$$

T **17.** Use a calculator or computer to make a table of values of right Riemann sums R_n for the integral $\int_0^\pi \sin x\,dx$ with $n = 5$, 10, 50, and 100. What value do these numbers appear to be approaching?

T **18.** Use a calculator or computer to make a table of values of left and right Riemann sums L_n and R_n for the integral $\int_0^2 e^{-x^2}\,dx$ with $n = 5$, 10, 50, and 100. Between what two numbers must the value of the integral lie? Can you make a similar statement for the integral $\int_{-1}^2 e^{-x^2}\,dx$? Explain.

19–22 Express the limit as a definite integral on the given interval.

19. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{e^{x_i}}{1 + x_i}\,\Delta x,\quad [0, 1]$

20. $\lim_{n\to\infty} \sum_{i=1}^{n} x_i\sqrt{1 + x_i^3}\,\Delta x,\quad [2, 5]$

21. $\lim_{n\to\infty} \sum_{i=1}^{n} [5(x_i^*)^3 - 4x_i^*]\,\Delta x,\quad [2, 7]$

22. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{x_i^*}{(x_i^*)^2 + 4}\,\Delta x,\quad [1, 3]$

23–24 Show that the definite integral is equal to $\lim_{n\to\infty} R_n$ and then evaluate the limit.

23. $\int_0^4 (x - x^2)\,dx,\quad R_n = \frac{4}{n}\sum_{i=1}^{n}\left[\frac{4i}{n} - \frac{16i^2}{n^2}\right]$

24. $\int_1^3 (x^3 + 5x^2)\,dx,\quad R_n = \frac{2}{n}\sum_{i=1}^{n}\left[6 + \frac{26i}{n} + \frac{32i^2}{n^2} + \frac{8i^3}{n^3}\right]$

25–26 Express the integral as a limit of Riemann sums using right endpoints. Do not evaluate the limit.

25. $\int_1^3 \sqrt{4 + x^2}\,dx$

26. $\int_2^5 \left(x^2 + \frac{1}{x}\right) dx$

27–34 Use the form of the definition of the integral given in Theorem 4 to evaluate the integral.

27. $\int_0^2 3x\,dx$ **28.** $\int_0^3 x^2\,dx$

29. $\int_0^3 (5x + 2)\,dx$ **30.** $\int_0^4 (6 - x^2)\,dx$

31. $\int_1^5 (3x^2 + 7x)\,dx$ **32.** $\int_{-1}^2 (4x^2 + x + 2)\,dx$

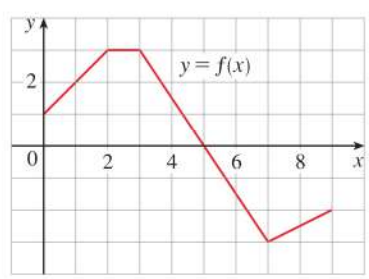
33. $\int_0^1 (x^3 - 3x^2)\,dx$ **34.** $\int_0^2 (2x - x^3)\,dx$

35. The graph of f is shown. Evaluate each integral by interpreting it in terms of areas.

(a) $\int_0^2 f(x)\,dx$ (b) $\int_0^5 f(x)\,dx$

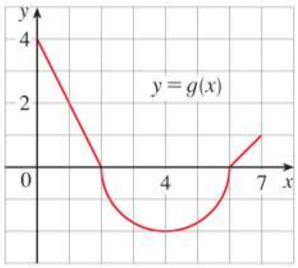
(c) $\int_5^7 f(x)\,dx$ (d) $\int_3^7 f(x)\,dx$

(e) $\int_3^7 |f(x)|\,dx$ (f) $\int_2^0 f(x)\,dx$



36. The graph of g consists of two straight lines and a semicircle. Evaluate each integral by interpreting it in terms of areas.

(a) $\int_0^2 g(x)\,dx$ (b) $\int_2^6 g(x)\,dx$ (c) $\int_0^7 g(x)\,dx$



37–38
(a) Use the form of the definition of the integral given in Theorem 4 to evaluate the given integral.
(b) Confirm your answer to part (a) graphically by interpreting the integral in terms of areas.

37. $\int_0^3 4x\,dx$ **38.** $\int_{-1}^4 \left(2 - \frac{1}{2}x\right) dx$

39–40
(a) Find an approximation to the integral using a Riemann sum with right endpoints and $n = 8$.
(b) Draw a diagram like Figure 3 to illustrate the approximation in part (a).
(c) Use Theorem 4 to evaluate the integral.
(d) Interpret the integral in part (c) as a difference of areas and illustrate with a diagram like Figure 4.

39. $\int_0^8 (3 - 2x)\,dx$ **40.** $\int_0^4 (x^2 - 3x)\,dx$

41–46 Evaluate the integral by interpreting it in terms of areas.

41. $\int_{-2}^5 (10 - 5x)\,dx$ **42.** $\int_{-1}^3 (2x - 1)\,dx$

43. $\int_{-4}^3 \left|\frac{1}{2}x\right| dx$ **44.** $\int_0^1 |2x - 1|\,dx$

45. $\int_{-3}^0 \left(1 + \sqrt{9 - x^2}\right) dx$ **46.** $\int_{-4}^4 \left(2x - \sqrt{16 - x^2}\right) dx$

47. Prove that $\displaystyle\int_a^b x\,dx = \frac{b^2 - a^2}{2}$.

48. Prove that $\displaystyle\int_a^b x^2\,dx = \frac{b^3 - a^3}{3}$.

T **49–50** Express the integral as a limit of sums. Then evaluate, using a computer algebra system to find both the sum and the limit.

49. $\int_0^\pi \sin 5x\,dx$ **50.** $\int_2^{10} x^6\,dx$

51. Evaluate $\int_1^1 \sqrt{1 + x^4}\,dx$.

52. Given that $\int_0^\pi \sin^4 x\,dx = \frac{3}{8}\pi$, what is $\int_\pi^0 \sin^4 \theta\,d\theta$?

53. In Example 5.1.2 we showed that $\int_0^1 x^2\,dx = \frac{1}{3}$. Use this fact and the properties of integrals to evaluate $\int_0^1 (5 - 6x^2)\,dx$.

54. Use the properties of integrals and the result of Example 4 to evaluate $\int_1^3 (2e^x - 1)\,dx$.

55. Use the result of Example 4 to evaluate $\int_1^3 e^{x+2}\,dx$.

56. Use the result of Exercise 47 and the fact that $\int_0^{\pi/2} \cos x\,dx = 1$ (from Exercise 5.1.33), together with the properties of integrals, to evaluate $\int_0^{\pi/2} (2\cos x - 5x)\,dx$.

57. Write as a single integral in the form $\int_a^b f(x)\,dx$:

$$\int_{-2}^2 f(x)\,dx + \int_2^5 f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$$

58. If $\int_2^8 f(x)\,dx = 7.3$ and $\int_2^4 f(x)\,dx = 5.9$, find $\int_4^8 f(x)\,dx$.

59. If $\int_0^9 f(x)\,dx = 37$ and $\int_0^9 g(x)\,dx = 16$, find

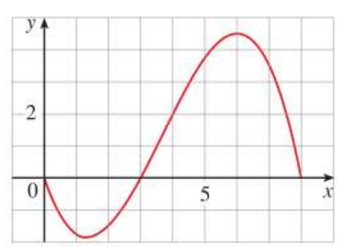
$$\int_0^9 [2f(x) + 3g(x)]\,dx$$

60. Find $\int_0^5 f(x)\,dx$ if

$$f(x) = \begin{cases} 3 & \text{for } x < 3 \\ x & \text{for } x \geqslant 3 \end{cases}$$

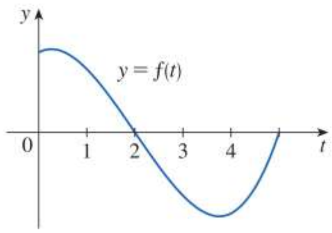
61. For the function f whose graph is shown, list the following quantities in increasing order, from smallest to largest, and explain your reasoning.

(A) $\int_0^8 f(x)\,dx$ (B) $\int_0^3 f(x)\,dx$ (C) $\int_3^8 f(x)\,dx$
(D) $\int_4^8 f(x)\,dx$ (E) $f'(1)$



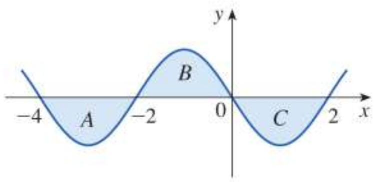
62. If $F(x) = \int_2^x f(t)\,dt$, where f is the function whose graph is given, which of the following values is largest?

(A) $F(0)$ (B) $F(1)$ (C) $F(2)$
(D) $F(3)$ (E) $F(4)$



63. Each of the regions A, B, and C bounded by the graph of f and the x-axis has area 3. Find the value of

$$\int_{-4}^{2} [f(x) + 2x + 5]\,dx$$



64. Suppose f has absolute minimum value m and absolute maximum value M. Between what two values must $\int_0^2 f(x)\,dx$ lie? Which property of integrals allows you to make your conclusion?

65–68 Use the properties of integrals to verify the inequality without evaluating the integrals.

65. $\int_0^4 (x^2 - 4x + 4)\,dx \geqslant 0$

66. $\int_0^1 \sqrt{1 + x^2}\,dx \leqslant \int_0^1 \sqrt{1 + x}\,dx$

67. $2 \leqslant \int_{-1}^1 \sqrt{1 + x^2}\,dx \leqslant 2\sqrt{2}$

68. $\dfrac{\pi}{12} \leqslant \int_{\pi/6}^{\pi/3} \sin x\,dx \leqslant \dfrac{\sqrt{3}\,\pi}{12}$

69–74 Use Property 8 of integrals to estimate the value of the integral.

69. $\int_0^1 x^3\,dx$

70. $\int_0^3 \dfrac{1}{x + 4}\,dx$

71. $\int_{\pi/4}^{\pi/3} \tan x\,dx$

72. $\int_0^2 (x^3 - 3x + 3)\,dx$

73. $\int_0^2 xe^{-x}\,dx$

74. $\int_{\pi}^{2\pi} (x - 2\sin x)\,dx$

75–76 Use properties of integrals, together with Exercises 47 and 48, to prove the inequality.

75. $\int_1^3 \sqrt{x^4 + 1}\,dx \geqslant \dfrac{26}{3}$

76. $\int_0^{\pi/2} x \sin x\,dx \leqslant \dfrac{\pi^2}{8}$

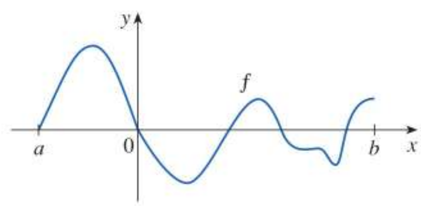
77. Which of the integrals $\int_1^2 \arctan x\,dx$, $\int_1^2 \arctan \sqrt{x}\,dx$, and $\int_1^2 \arctan(\sin x)\,dx$ has the largest value? Why?

78. Which of the integrals $\int_0^{0.5} \cos(x^2)\,dx$, $\int_0^{0.5} \cos \sqrt{x}\,dx$ is larger? Why?

79. Prove Property 3 of integrals.

80. (a) For the function f shown in the graph, verify graphically that the following inequality holds:

$$\left| \int_a^b f(x)\,dx \right| \leqslant \int_a^b |f(x)|\,dx$$



(b) Prove that the inequality from part (a) holds for any function f that is continuous on $[a, b]$.

(c) Show that

$$\left| \int_a^b f(x) \sin 2x\,dx \right| \leqslant \int_a^b |f(x)|\,dx$$

81. Let $f(x) = 0$ if x is any rational number and $f(x) = 1$ if x is any irrational number. Show that f is not integrable on $[0, 1]$.

82. Let $f(0) = 0$ and $f(x) = 1/x$ if $0 < x \leqslant 1$. Show that f is not integrable on $[0, 1]$. [*Hint:* Show that the first term in the Riemann sum, $f(x_1^*)\,\Delta x$, can be made arbitrarily large.]

83–84 Express the limit as a definite integral.

83. $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \dfrac{i^4}{n^5}$ [*Hint:* Consider $f(x) = x^4$.]

84. $\lim\limits_{n \to \infty} \dfrac{1}{n} \sum\limits_{i=1}^{n} \dfrac{1}{1 + (i/n)^2}$

85. Find $\int_1^2 x^{-2}\,dx$. *Hint:* Choose x_i^* to be the geometric mean of x_{i-1} and x_i (that is, $x_i^* = \sqrt{x_{i-1}x_i}$) and use the identity

$$\frac{1}{m(m + 1)} = \frac{1}{m} - \frac{1}{m + 1}$$

DISCOVERY PROJECT | AREA FUNCTIONS

1. (a) Draw the line $y = 2t + 1$ and use geometry to find the area under this line, above the t-axis, and between the vertical lines $t = 1$ and $t = 3$.

(b) If $x > 1$, let $A(x)$ be the area of the region that lies under the line $y = 2t + 1$ between $t = 1$ and $t = x$. Sketch this region and use geometry to find an expression for $A(x)$.

(c) Differentiate the area function $A(x)$. What do you notice?

2. (a) If $x \geqslant -1$, let

$$A(x) = \int_{-1}^{x} (1 + t^2)\, dt$$

$A(x)$ represents the area of a region. Sketch that region.

(b) Use the result of Exercise 5.2.48 to find an expression for $A(x)$.

(c) Find $A'(x)$. What do you notice?

(d) If $x \geqslant -1$ and h is a small positive number, then $A(x + h) - A(x)$ represents the area of a region. Describe and sketch the region.

(e) Draw a rectangle that approximates the region in part (d). By comparing the areas of these two regions, show that

$$\frac{A(x + h) - A(x)}{h} \approx 1 + x^2$$

(f) Use part (e) to give an intuitive explanation for the result of part (c).

3. (a) Draw the graph of the function $f(x) = \cos(x^2)$ in the viewing rectangle $[0, 2]$ by $[-1.25, 1.25]$.

(b) If we define a new function g by

$$g(x) = \int_{0}^{x} \cos(t^2)\, dt$$

then $g(x)$ is the area under the graph of f from 0 to x [until $f(x)$ becomes negative, at which point $g(x)$ becomes a difference of areas]. Use part (a) to determine the value of x at which $g(x)$ starts to decrease. [Unlike the integral in Problem 2, it is impossible to evaluate the integral defining g to obtain an explicit expression for $g(x)$.]

(c) Use the integration command on a calculator or computer to estimate $g(0.2)$, $g(0.4)$, $g(0.6)$, . . . , $g(1.8)$, $g(2)$. Then use these values to sketch a graph of g.

(d) Use your graph of g from part (c) to sketch the graph of g' using the interpretation of $g'(x)$ as the slope of a tangent line. How does the graph of g' compare with the graph of f?

4. Suppose f is a continuous function on the interval $[a, b]$ and we define a new function g by the equation

$$g(x) = \int_{a}^{x} f(t)\, dt$$

Based on your results in Problems 1–3, conjecture an expression for $g'(x)$.

5.3 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus is appropriately named because it establishes a connection between the two branches of calculus: differential calculus and integral calculus. Differential calculus arose from the tangent problem, whereas integral calculus arose from a seemingly unrelated problem, the area problem. Newton's mentor at Cambridge, Isaac Barrow (1630–1677), discovered that these two problems are actually closely related. In fact, he realized that differentiation and integration are inverse processes. The Fundamental Theorem of Calculus gives the precise inverse relationship between the derivative and the integral. It was Newton and Leibniz who exploited this relationship and used it to develop calculus into a systematic mathematical method. In particular, they saw that the Fundamental Theorem enabled them to compute areas and integrals very easily without having to compute them as limits of sums as we did in Sections 5.1 and 5.2.

The Fundamental Theorem of Calculus, Part 1

The first part of the Fundamental Theorem deals with functions defined by an equation of the form

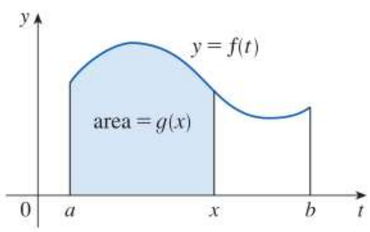


FIGURE 1

1 $$g(x) = \int_a^x f(t)\,dt$$

where f is a continuous function on $[a, b]$ and x varies between a and b. Observe that g depends only on x, which appears as the variable upper limit in the integral. If x is a fixed number, then the integral $\int_a^x f(t)\,dt$ is a definite number. If we then let x vary, the number $\int_a^x f(t)\,dt$ also varies and defines a function of x denoted by $g(x)$.

If f happens to be a positive function, then $g(x)$ can be interpreted as the area under the graph of f from a to x, where x can vary from a to b. (Think of g as the "area so far" function; see Figure 1.)

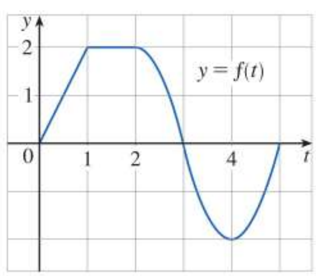


FIGURE 2

EXAMPLE 1 If f is the function whose graph is shown in Figure 2 and $g(x) = \int_0^x f(t)\,dt$, find the values of $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$, and $g(5)$. Then sketch a rough graph of g.

SOLUTION First we notice that $g(0) = \int_0^0 f(t)\,dt = 0$. From Figure 3 we see that $g(1)$ is the area of a triangle:

$$g(1) = \int_0^1 f(t)\,dt = \tfrac{1}{2}(1 \cdot 2) = 1$$

To find $g(2)$ we add to $g(1)$ the area of a rectangle:

$$g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt = 1 + (1 \cdot 2) = 3$$

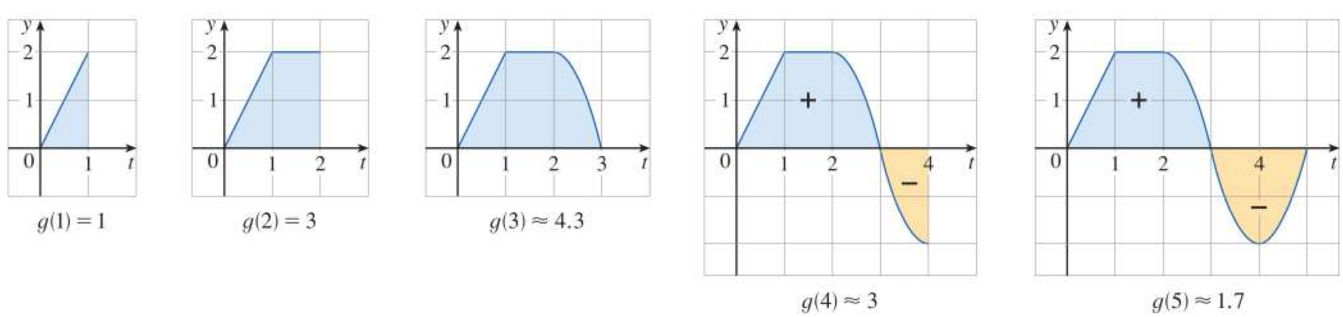


FIGURE 3

We estimate that the area under f from 2 to 3 is about 1.3, so

$$g(3) = g(2) + \int_2^3 f(t)\,dt \approx 3 + 1.3 = 4.3$$

For $t > 3$, $f(t)$ is negative and so we start subtracting areas:

$$g(4) = g(3) + \int_3^4 f(t)\,dt \approx 4.3 + (-1.3) = 3.0$$

$$g(5) = g(4) + \int_4^5 f(t)\,dt \approx 3 + (-1.3) = 1.7$$

We use these values to sketch the graph of g in Figure 4. Notice that, because $f(t)$ is positive for $t < 3$, we keep adding area for $t < 3$ and so g is increasing up to $x = 3$, where it attains a maximum value. For $x > 3$, g decreases because $f(t)$ is negative. ■

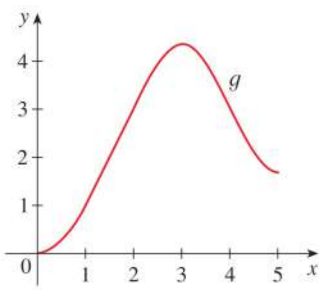


FIGURE 4
$g(x) = \int_0^x f(t)\,dt$

If we take $f(t) = t$ and $a = 0$, then, using Exercise 5.2.47, we have

$$g(x) = \int_0^x t\,dt = \frac{x^2}{2}$$

Notice that $g'(x) = x$, that is, $g' = f$. In other words, if g is defined as the integral of f by Equation 1, then g turns out to be an antiderivative of f, at least in this case. And if we sketch the derivative of the function g shown in Figure 4 by estimating slopes of tangents, we get a graph like that of f in Figure 2. So we suspect that $g' = f$ in Example 1 too.

To see why this might be generally true we consider any continuous function f with $f(x) \geqslant 0$. Then $g(x) = \int_a^x f(t)\,dt$ can be interpreted as the area under the graph of f from a to x, as in Figure 1.

In order to compute $g'(x)$ from the definition of a derivative we first observe that, for $h > 0$, $g(x + h) - g(x)$ is obtained by subtracting areas, so it is the area under the graph of f from x to $x + h$ (the blue area in Figure 5). For small h you can see from the figure that this area is approximately equal to the area of the rectangle with height $f(x)$ and width h:

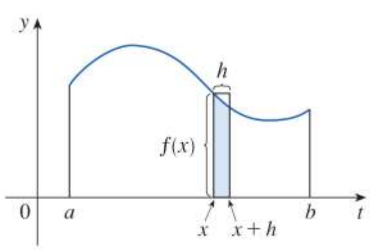


FIGURE 5

$$g(x + h) - g(x) \approx hf(x)$$

so

$$\frac{g(x + h) - g(x)}{h} \approx f(x)$$

Intuitively, we therefore expect that

$$g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} = f(x)$$

The fact that this is true, even when f is not necessarily positive, is the first part of the Fundamental Theorem of Calculus.

We abbreviate the name of this theorem as FTC1. In words, it says that the derivative of a definite integral with respect to its upper limit is the integrand evaluated at the upper limit.

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

PROOF If x and $x + h$ are in (a, b), then

$$\begin{aligned} g(x+h) - g(x) &= \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \\ &= \left(\int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt\right) - \int_a^x f(t)\,dt \qquad \text{(by Property 5 of integrals)} \\ &= \int_x^{x+h} f(t)\,dt \end{aligned}$$

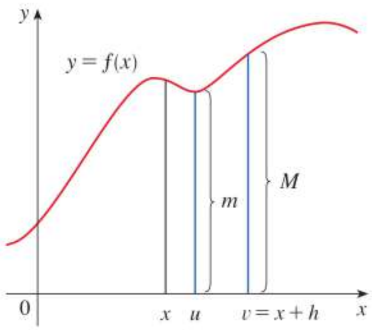


FIGURE 6

and so, for $h \neq 0$,

2 $$\frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

For now let's assume that $h > 0$. Since f is continuous on $[x, x + h]$, the Extreme Value Theorem says that there are numbers u and v in $[x, x + h]$ such that $f(u) = m$ and $f(v) = M$, where m and M are the absolute minimum and maximum values of f on $[x, x + h]$. (See Figure 6.)

By Property 8 of integrals, we have

$$mh \leqslant \int_x^{x+h} f(t)\,dt \leqslant Mh$$

that is,
$$f(u)h \leqslant \int_x^{x+h} f(t)\,dt \leqslant f(v)h$$

Since $h > 0$, we can divide this inequality by h:

$$f(u) \leqslant \frac{1}{h}\int_x^{x+h} f(t)\,dt \leqslant f(v)$$

Now we use Equation 2 to replace the middle part of this inequality:

3 $$f(u) \leqslant \frac{g(x+h) - g(x)}{h} \leqslant f(v)$$

Inequality 3 can be proved in a similar manner for the case where $h < 0$. (See Exercise 87.)

Now we let $h \to 0$. Then $u \to x$ and $v \to x$ because u and v lie between x and $x + h$. Therefore

$$\lim_{h\to 0} f(u) = \lim_{u\to x} f(u) = f(x) \qquad \text{and} \qquad \lim_{h\to 0} f(v) = \lim_{v\to x} f(v) = f(x)$$

because f is continuous at x. We conclude, from (3) and the Squeeze Theorem, that

4 $$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f(x)$$

If $x = a$ or b, then Equation 4 can be interpreted as a one-sided limit. Then Theorem 2.8.4 (modified for one-sided limits) shows that g is continuous on $[a, b]$. ■

Using Leibniz notation for derivatives, we can write FTC1 as

5 $$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

when f is continuous. Roughly speaking, Equation 5 says that if we first integrate f and then differentiate the result, we get back to the original function f.

EXAMPLE 2 Find the derivative of the function $g(x) = \int_0^x \sqrt{1 + t^2}\, dt$.

SOLUTION Since $f(t) = \sqrt{1 + t^2}$ is continuous, Part 1 of the Fundamental Theorem of Calculus gives

$$g'(x) = \sqrt{1 + x^2}$$

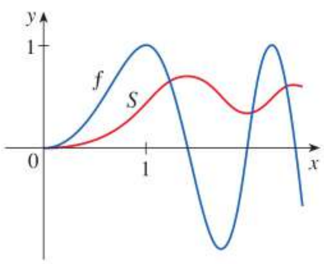


FIGURE 7
$f(x) = \sin(\pi x^2/2)$
$S(x) = \int_0^x \sin(\pi t^2/2)\, dt$

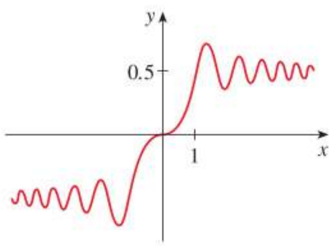


FIGURE 8
The Fresnel function
$S(x) = \int_0^x \sin(\pi t^2/2)\, dt$

EXAMPLE 3 Although a formula of the form $g(x) = \int_a^x f(t)\, dt$ may seem like a strange way of defining a function, books on physics, chemistry, and statistics are full of such functions. For instance, the **Fresnel function**

$$S(x) = \int_0^x \sin(\pi t^2/2)\, dt$$

is named after the French physicist Augustin Fresnel (1788–1827), who is famous for his works in optics. This function first appeared in Fresnel's theory of the diffraction of light waves, but more recently it has been applied to the design of highways.

Part 1 of the Fundamental Theorem tells us how to differentiate the Fresnel function:

$$S'(x) = \sin(\pi x^2/2)$$

This means that we can apply all the methods of differential calculus to analyze S (see Exercise 81).

Figure 7 shows the graphs of $f(x) = \sin(\pi x^2/2)$ and the Fresnel function $S(x) = \int_0^x f(t)\, dt$. A computer was used to graph S by computing the value of this integral for many values of x. It does indeed look as if $S(x)$ is the area under the graph of f from 0 to x [until $x \approx 1.4$ when $S(x)$ becomes a difference of areas]. Figure 8 shows a larger part of the graph of S.

If we now start with the graph of S in Figure 7 and think about what its derivative should look like, it seems reasonable that $S'(x) = f(x)$. [For instance, S is increasing when $f(x) > 0$ and decreasing when $f(x) < 0$.] So this gives a visual confirmation of Part 1 of the Fundamental Theorem of Calculus.

EXAMPLE 4 Find $\dfrac{d}{dx} \displaystyle\int_1^{x^4} \sec t\, dt$.

SOLUTION Here we have to be careful to use the Chain Rule in conjunction with FTC1. Let $u = x^4$. Then

$$\begin{aligned}
\frac{d}{dx} \int_1^{x^4} \sec t\, dt &= \frac{d}{dx} \int_1^{u} \sec t\, dt \\
&= \frac{d}{du} \left[\int_1^u \sec t\, dt \right] \frac{du}{dx} && \text{(by the Chain Rule)} \\
&= \sec u \, \frac{du}{dx} && \text{(by FTC1)} \\
&= \sec(x^4) \cdot 4x^3
\end{aligned}$$

The Fundamental Theorem of Calculus, Part 2

In Section 5.2 we computed integrals from the definition as a limit of Riemann sums and we saw that this procedure is sometimes long and difficult. The second part of the Fundamental Theorem of Calculus, which follows easily from the first part, provides us with a much simpler method for the evaluation of integrals.

We abbreviate this theorem as FTC2.

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function F such that $F' = f$.

PROOF Let $g(x) = \int_a^x f(t)\,dt$. We know from Part 1 that $g'(x) = f(x)$; that is, g is an antiderivative of f. If F is any other antiderivative of f on $[a, b]$, then we know from Corollary 4.2.7 that F and g differ by a constant:

$$\boxed{6} \qquad F(x) = g(x) + C$$

for $a < x < b$. But both F and g are continuous on $[a, b]$ and so, by taking limits of both sides of Equation 6 (as $x \to a^+$ and $x \to b^-$), we see that it also holds when $x = a$ and $x = b$. So $F(x) = g(x) + C$ for all x in $[a, b]$.

If we put $x = a$ in the formula for $g(x)$, we get

$$g(a) = \int_a^a f(t)\,dt = 0$$

So, using Equation 6 with $x = b$ and $x = a$, we have

$$\begin{aligned} F(b) - F(a) &= [g(b) + C] - [g(a) + C] \\ &= g(b) - g(a) = g(b) = \int_a^b f(t)\,dt \end{aligned}$$

■

Part 2 of the Fundamental Theorem states that if we know an antiderivative F of f, then we can evaluate $\int_a^b f(x)\,dx$ simply by subtracting the values of F at the endpoints of the interval $[a, b]$. It's very surprising that $\int_a^b f(x)\,dx$, which was defined by a complicated procedure involving all of the values of $f(x)$ for $a \leqslant x \leqslant b$, can be found by knowing the values of $F(x)$ at only two points, a and b.

Although the theorem may be surprising at first glance, it becomes plausible if we interpret it in physical terms. If $v(t)$ is the velocity of an object and $s(t)$ is its position at time t, then $v(t) = s'(t)$, so s is an antiderivative of v. In Section 5.1 we considered an object that always moves in the positive direction and made the observation that the area under the velocity curve is equal to the distance traveled. In symbols:

$$\int_a^b v(t)\,dt = s(b) - s(a)$$

That is exactly what FTC2 says in this context.

EXAMPLE 5 Evaluate the integral $\int_1^3 e^x\,dx$.

SOLUTION The function $f(x) = e^x$ is continuous everywhere and we know that an antiderivative is $F(x) = e^x$, so Part 2 of the Fundamental Theorem gives

Compare the calculation in Example 5 with the much harder one in Example 5.2.4.

$$\int_1^3 e^x\,dx = F(3) - F(1) = e^3 - e$$

Notice that FTC2 says we can use *any* antiderivative F of f. So we may as well use the simplest one, namely $F(x) = e^x$, instead of $e^x + 7$ or $e^x + C$. ■

Notation We often use the notation

$$F(x)\Big]_a^b = F(b) - F(a)$$

So the equation of FTC2 can be written as

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b \qquad \text{where} \qquad F' = f$$

Other common notations are $F(x)\,|_a^b$ and $[F(x)]_a^b$.

In applying the Fundamental Theorem we use a particular antiderivative F of f. It is not necessary to use the most general antiderivative.

EXAMPLE 6 Find the area under the parabola $y = x^2$ from 0 to 1.

SOLUTION An antiderivative of $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3$. The required area A is found using Part 2 of the Fundamental Theorem:

$$A = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$ ■

If you compare the calculation in Example 6 with the one in Example 5.1.2, you will see that the Fundamental Theorem gives a *much* shorter method.

EXAMPLE 7 Evaluate $\displaystyle\int_3^6 \frac{dx}{x}$.

SOLUTION The given integral is another way of writing

$$\int_3^6 \frac{1}{x}\,dx$$

An antiderivative of $f(x) = 1/x$ is $F(x) = \ln|x|$ and, because $3 \leqslant x \leqslant 6$, we can write $F(x) = \ln x$. So

$$\int_3^6 \frac{1}{x}\,dx = \ln x\Big]_3^6 = \ln 6 - \ln 3 = \ln\frac{6}{3} = \ln 2$$ ■

EXAMPLE 8 Find the area under the cosine curve from 0 to b, where $0 \leqslant b \leqslant \pi/2$.

SOLUTION Since an antiderivative of $f(x) = \cos x$ is $F(x) = \sin x$, we have

$$A = \int_0^b \cos x\,dx = \sin x\Big]_0^b = \sin b - \sin 0 = \sin b$$

In particular, taking $b = \pi/2$, we have proved that the area under the cosine curve from 0 to $\pi/2$ is $\sin(\pi/2) = 1$. (See Figure 9.) ■

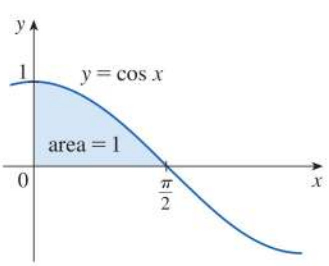


FIGURE 9

When the French mathematician Gilles de Roberval first found the area under the sine and cosine curves in 1635, this was a very challenging problem that required a great deal of ingenuity. If we didn't have the benefit of the Fundamental Theorem, we would have to compute a difficult limit of sums using obscure trigonometric identities (or use a computer algebra system as in Exercise 5.1.33). It was even more difficult for Roberval because the apparatus of limits had not been invented in 1635. But in the 1660s and 1670s, when the Fundamental Theorem was discovered by Barrow and exploited by Newton and Leibniz, such problems became very easy, as you can see from Example 8.

EXAMPLE 9 What is wrong with the following calculation?

$$\int_{-1}^{3} \frac{1}{x^2}\,dx = \frac{x^{-1}}{-1}\bigg]_{-1}^{3} = -\frac{1}{3} - 1 = -\frac{4}{3}$$

SOLUTION To start, we notice that this calculation must be wrong because the answer is negative but $f(x) = 1/x^2 \geqslant 0$ and Property 6 of integrals says that $\int_a^b f(x)\,dx \geqslant 0$ when $f \geqslant 0$. The Fundamental Theorem of Calculus applies to continuous functions. It can't be applied here because $f(x) = 1/x^2$ is not continuous on $[-1, 3]$. In fact, f has an infinite discontinuity at $x = 0$, and we will see in Section 7.8 that

$$\int_{-1}^{3} \frac{1}{x^2}\,dx \qquad \text{does not exist.}$$

Differentiation and Integration as Inverse Processes

We end this section by bringing together the two parts of the Fundamental Theorem.

> **The Fundamental Theorem of Calculus** Suppose f is continuous on $[a, b]$.
>
> **1.** If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.
>
> **2.** $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

We noted that Part 1 can be rewritten as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

This says that if we integrate a continuous function f and then differentiate the result, we arrive back at the original function f. We could use Part 2 to write

$$\int_a^x F'(t)\,dt = F(x) - F(a)$$

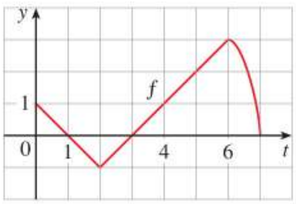
which says that if we differentiate a function F and then integrate the result, we arrive back at the original function F, except for the constant $F(a)$. So taken together, the two parts of the Fundamental Theorem of Calculus say that integration and differentiation are inverse processes.

The Fundamental Theorem of Calculus is unquestionably the most important theorem in calculus and, indeed, it ranks as one of the great accomplishments of the human mind. Before it was discovered, from the time of Eudoxus and Archimedes to the time of Galileo and Fermat, problems of finding areas, volumes, and lengths of curves were so difficult that only a genius could meet the challenge, and even then, only for very special cases. But now, armed with the systematic method that Newton and Leibniz fashioned out of the Fundamental Theorem, we will see in the chapters to come that these challenging problems are accessible to all of us.

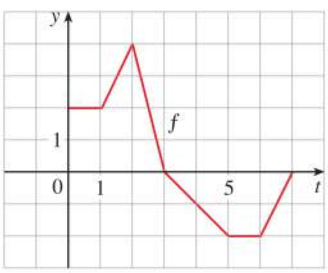
5.3 Exercises

1. Explain exactly what is meant by the statement that "differentiation and integration are inverse processes."

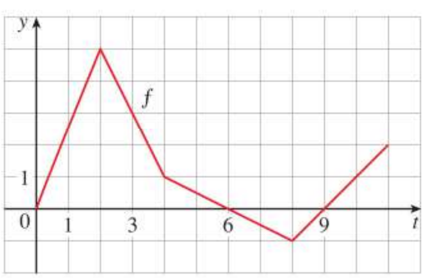
2. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.
(a) Evaluate $g(x)$ for $x = 0, 1, 2, 3, 4, 5$, and 6.
(b) Estimate $g(7)$.
(c) Where does g have a maximum value? Where does it have a minimum value?
(d) Sketch a rough graph of g.



3. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.
(a) Evaluate $g(0)$, $g(1)$, $g(2)$, $g(3)$, and $g(6)$.
(b) On what interval is g increasing?
(c) Where does g have a maximum value?
(d) Sketch a rough graph of g.

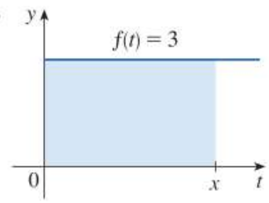


4. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.
(a) Use Part 1 of the Fundamental Theorem of Calculus to graph g'.
(b) Find $g(3)$, $g'(3)$, and $g''(3)$.
(c) Does g have a local maximum, a local minimum, or neither at $x = 6$?
(d) Does g have a local maximum, a local minimum, or neither at $x = 9$?

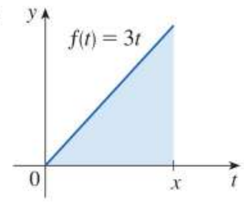


5–6 The graph of a function f is shown. Let g be the function that represents the area under the graph of f between 0 and x.
(a) Use geometry to find a formula for $g(x)$.
(b) Verify that g is an antiderivative of f and explain how this confirms Part 1 of the Fundamental Theorem of Calculus for the function f.

5.



6.



7–8 Sketch the area represented by $g(x)$. Then find $g'(x)$ in two ways: (a) by using Part 1 of the Fundamental Theorem and (b) by evaluating the integral using Part 2 and then differentiating.

7. $g(x) = \int_1^x t^2\,dt$

8. $g(x) = \int_0^x (2 + \sin t)\,dt$

9–20 Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of the function.

9. $g(x) = \int_0^x \sqrt{t + t^3}\,dt$

10. $g(x) = \int_1^x \ln(1 + t^2)\,dt$

11. $g(w) = \int_0^w \sin(1 + t^3)\,dt$

12. $h(u) = \int_0^u \dfrac{\sqrt{t}}{t + 1}\,dt$

13. $F(x) = \int_x^0 \sqrt{1 + \sec t}\,dt$

$$\left[\textit{Hint: } \int_x^0 \sqrt{1 + \sec t}\,dt = -\int_0^x \sqrt{1 + \sec t}\,dt \right]$$

14. $A(w) = \int_w^{-1} e^{t + t^2}\,dt$

15. $h(x) = \int_1^{e^x} \ln t\,dt$

16. $h(x) = \int_1^{\sqrt{x}} \dfrac{z^2}{z^4 + 1}\,dz$

17. $y = \int_1^{3x+2} \dfrac{t}{1 + t^3}\,dt$

18. $y = \int_0^{\tan x} e^{-t^2}\,dt$

19. $y = \int_{\sqrt{x}}^{\pi/4} \theta \tan \theta\,d\theta$

20. $y = \int_{1/x}^{4} \sqrt{1 + \dfrac{1}{t}}\,dt$

21–24 Use Part 2 of the Fundamental Theorem of Calculus to evaluate the integral and interpret the result as an area or a difference of areas. Illustrate with a sketch.

21. $\int_{-1}^{2} x^3\,dx$

22. $\int_0^4 (x^2 - 4x)\,dx$

23. $\int_{\pi/2}^{2\pi} (2 \sin x)\, dx$ **24.** $\int_{-1}^{2} (e^x + 2)\, dx$

25–54 Evaluate the integral.

25. $\int_1^3 (x^2 + 2x - 4)\, dx$ **26.** $\int_{-1}^1 x^{100}\, dx$

27. $\int_0^2 \left(\frac{4}{5}t^3 - \frac{3}{4}t^2 + \frac{2}{5}t\right) dt$ **28.** $\int_0^1 \left(1 - 8v^3 + 16v^7\right) dv$

29. $\int_1^9 \sqrt{x}\, dx$ **30.** $\int_1^8 x^{-2/3}\, dx$

31. $\int_0^4 (t^2 + t^{3/2})\, dt$ **32.** $\int_1^3 \left(\frac{1}{z^2} + \frac{1}{z^3}\right) dz$

33. $\int_{\pi/2}^0 \cos\theta\, d\theta$ **34.** $\int_{-5}^5 e\, dx$

35. $\int_0^1 (u + 2)(u - 3)\, du$ **36.** $\int_0^4 (4 - t)\sqrt{t}\, dt$

37. $\int_1^4 \frac{2 + x^2}{\sqrt{x}}\, dx$ **38.** $\int_{-1}^2 (3u - 2)(u + 1)\, du$

39. $\int_1^3 \left(2x + \frac{1}{x}\right) dx$ **40.** $\int_5^5 \sqrt{t^2 + \sin t}\, dt$

41. $\int_0^{\pi/3} \sec\theta \tan\theta\, d\theta$ **42.** $\int_1^3 \frac{y^3 - 2y^2 - y}{y^2}\, dy$

43. $\int_0^1 (1 + r)^3\, dr$ **44.** $\int_0^3 (2 \sin x - e^x)\, dx$

45. $\int_1^2 \frac{v^3 + 3v^6}{v^4}\, dv$ **46.** $\int_1^{18} \sqrt{\frac{3}{z}}\, dz$

47. $\int_0^1 (x^e + e^x)\, dx$ **48.** $\int_0^1 \cosh t\, dt$

49. $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1 + x^2}\, dx$ **50.** $\int_1^3 \frac{(3x + 1)^2}{x^3}\, dx$

51. $\int_0^4 2^s\, ds$ **52.** $\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1 - x^2}}\, dx$

53. $\int_0^\pi f(x)\, dx$ where $f(x) = \begin{cases} \sin x & \text{if } 0 \leqslant x < \pi/2 \\ \cos x & \text{if } \pi/2 \leqslant x \leqslant \pi \end{cases}$

54. $\int_{-2}^2 f(x)\, dx$ where $f(x) = \begin{cases} 2 & \text{if } -2 \leqslant x \leqslant 0 \\ 4 - x^2 & \text{if } 0 < x \leqslant 2 \end{cases}$

55–58 Sketch the region enclosed by the given curves and calculate its area.

55. $y = \sqrt{x}$, $y = 0$, $x = 4$ **56.** $y = x^3$, $y = 0$, $x = 1$

57. $y = 4 - x^2$, $y = 0$ **58.** $y = 2x - x^2$, $y = 0$

59–62 Use a graph to give a rough estimate of the area of the region that lies beneath the given curve. Then find the exact area.

59. $y = \sqrt[3]{x}$, $0 \leqslant x \leqslant 27$ **60.** $y = x^{-4}$, $1 \leqslant x \leqslant 6$

61. $y = \sin x$, $0 \leqslant x \leqslant \pi$ **62.** $y = \sec^2 x$, $0 \leqslant x \leqslant \pi/3$

63–66 What is wrong with the equation?

63. $\int_{-2}^1 x^{-4}\, dx = \left.\frac{x^{-3}}{-3}\right]_{-2}^1 = -\frac{3}{8}$

64. $\int_{-1}^2 \frac{4}{x^3}\, dx = \left.-\frac{2}{x^2}\right]_{-1}^2 = \frac{3}{2}$

65. $\int_{\pi/3}^\pi \sec\theta \tan\theta\, d\theta = \sec\theta\Big]_{\pi/3}^\pi = -3$

66. $\int_0^\pi \sec^2 x\, dx = \tan x\Big]_0^\pi = 0$

67–71 Find the derivative of the function.

67. $g(x) = \int_{2x}^{3x} \frac{u^2 - 1}{u^2 + 1}\, du$

$\left[\textit{Hint: } \int_{2x}^{3x} f(u)\, du = \int_{2x}^0 f(u)\, du + \int_0^{3x} f(u)\, du\right]$

68. $g(x) = \int_{1-2x}^{1+2x} t \sin t\, dt$

69. $F(x) = \int_x^{x^2} e^{t^2}\, dt$

70. $F(x) = \int_{\sqrt{x}}^{2x} \arctan t\, dt$

71. $y = \int_{\cos x}^{\sin x} \ln(1 + 2v)\, dv$

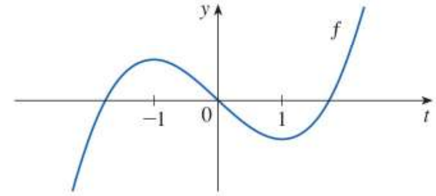
72. If $f(x) = \int_0^x (1 - t^2)e^{t^2}\, dt$, on what interval is f increasing?

73. On what interval is the curve

$$y = \int_0^x \frac{t^2}{t^2 + t + 2}\, dt$$

concave downward?

74. Let $F(x) = \int_1^x f(t)\, dt$, where f is the function whose graph is shown. Where is F concave downward?



75. Let $F(x) = \int_2^x e^{t^2}\,dt$. Find an equation of the tangent line to the curve $y = F(x)$ at the point with x-coordinate 2.

76. If $f(x) = \int_0^{\sin x} \sqrt{1 + t^2}\,dt$ and $g(y) = \int_3^y f(x)\,dx$, find $g''(\pi/6)$.

77–78 Use l'Hospital's Rule to evaluate the limit.

77. $\displaystyle \lim_{x \to 0} \frac{1}{x^2} \int_0^x \frac{2t}{\sqrt{t^3 + 1}}\,dt$

78. $\displaystyle \lim_{x \to \infty} \frac{1}{x^2} \int_0^x \ln(1 + e^t)\,dt$

79. If $f(1) = 12$, f' is continuous, and $\int_1^4 f'(x)\,dx = 17$, what is the value of $f(4)$?

80. The Error Function The *error function*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

is used in probability, statistics, and engineering.

(a) Show that $\int_a^b e^{-t^2}\,dt = \frac{1}{2}\sqrt{\pi}\,[\operatorname{erf}(b) - \operatorname{erf}(a)]$.

(b) Show that the function $y = e^{x^2}\operatorname{erf}(x)$ satisfies the differential equation $y' = 2xy + 2/\sqrt{\pi}$.

81. The Fresnel Function The Fresnel function S was defined in Example 3 and graphed in Figures 7 and 8.

(a) At what values of x does this function have local maximum values?

(b) On what intervals is the function concave upward?

T (c) Use a graph to solve the following equation correct to two decimal places:

$$\int_0^x \sin(\pi t^2/2)\,dt = 0.2$$

T **82. The Sine Integral Function** The *sine integral function*

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt$$

is important in electrical engineering. [The integrand $f(t) = (\sin t)/t$ is not defined when $t = 0$, but we know that its limit is 1 when $t \to 0$. So we define $f(0) = 1$ and this makes f a continuous function everywhere.]

(a) Draw the graph of Si.

(b) At what values of x does this function have local maximum values?

(c) Find the coordinates of the first inflection point to the right of the origin.

(d) Does this function have horizontal asymptotes?

(e) Solve the following equation correct to one decimal place:

$$\int_0^x \frac{\sin t}{t}\,dt = 1$$

83–84 Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.

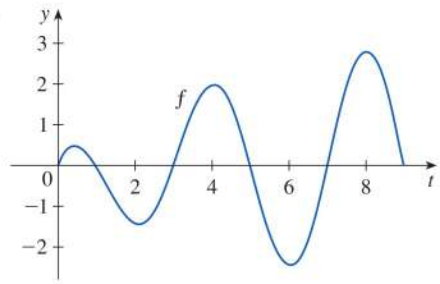
(a) At what values of x do the local maximum and minimum values of g occur?

(b) Where does g attain its absolute maximum value?

(c) On what intervals is g concave downward?

(d) Sketch the graph of g.

83.



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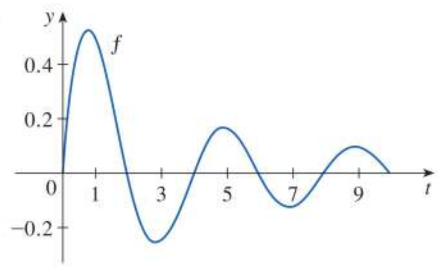


85–86 Evaluate the limit by first recognizing the sum as a Riemann sum for a function defined on $[0, 1]$.

85. $\displaystyle \lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{i^4}{n^5} + \frac{i}{n^2} \right)$

86. $\displaystyle \lim_{n \to \infty} \frac{1}{n} \left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \sqrt{\frac{3}{n}} + \cdots + \sqrt{\frac{n}{n}} \right)$

87. Justify (3) for the case $h < 0$.

88. If f is continuous and g and h are differentiable functions, show that

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))\,h'(x) - f(g(x))\,g'(x)$$

89. (a) Show that $1 \leqslant \sqrt{1 + x^3} \leqslant 1 + x^3$ for $x \geqslant 0$.

(b) Show that $1 \leqslant \int_0^1 \sqrt{1 + x^3}\,dx \leqslant 1.25$.

90. (a) Show that $\cos(x^2) \geqslant \cos x$ for $0 \leqslant x \leqslant 1$.

(b) Deduce that $\int_0^{\pi/6} \cos(x^2)\,dx \geqslant \frac{1}{2}$.

91. Show that

$$0 \leqslant \int_5^{10} \frac{x^2}{x^4 + x^2 + 1}\,dx \leqslant 0.1$$

by comparing the integrand to a simpler function.

92. Let

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leqslant x \leqslant 1 \\ 2 - x & \text{if } 1 < x \leqslant 2 \\ 0 & \text{if } x > 2 \end{cases}$$

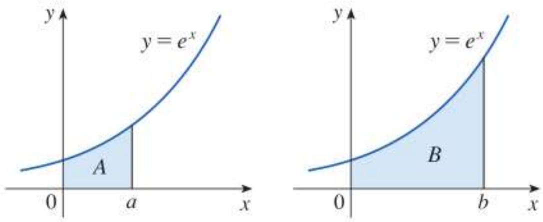
and $$g(x) = \int_0^x f(t)\,dt$$

(a) Find an expression for $g(x)$ similar to the one for $f(x)$.
(b) Sketch the graphs of f and g.
(c) Where is f differentiable? Where is g differentiable?

93. Find a function f and a number a such that

$$6 + \int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x} \qquad \text{for all } x > 0$$

94. The area labeled B is three times the area labeled A. Express b in terms of a.



95. A manufacturing company owns a major piece of equipment that depreciates at the (continuous) rate $f(t)$, where t is the time measured in months since its last overhaul. Because a fixed cost A is incurred each time the machine is overhauled, the company wants to determine the optimal time T (in months) between overhauls.

(a) Explain why $\int_0^t f(s)\,ds$ represents the loss in value of the machine over the period of time t since the last overhaul.

(b) Let $C = C(t)$ be given by

$$C(t) = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right]$$

What does C represent and why would the company want to minimize C?

(c) Show that C has a minimum value at the numbers $t = T$ where $C(T) = f(T)$.

5.4 Indefinite Integrals and the Net Change Theorem

We saw in Section 5.3 that the second part of the Fundamental Theorem of Calculus provides a very powerful method for evaluating the definite integral of a function, assuming that we can find an antiderivative of the function. In this section we introduce a notation for antiderivatives, review the formulas for antiderivatives, and use them to evaluate definite integrals. We also reformulate FTC2 in a way that makes it easier to apply to science and engineering problems.

Indefinite Integrals

Both parts of the Fundamental Theorem establish connections between antiderivatives and definite integrals. Part 1 says that if f is continuous, then $\int_a^x f(t)\,dt$ is an antiderivative of f. Part 2 says that $\int_a^b f(x)\,dx$ can be found by evaluating $F(b) - F(a)$, where F is an antiderivative of f.

We need a convenient notation for antiderivatives that makes them easy to work with. Because of the relation between antiderivatives and integrals given by the Fundamental Theorem, the notation $\int f(x)\,dx$ is traditionally used for an antiderivative of f and is called an **indefinite integral**. Thus

$$\boxed{\int f(x)\,dx = F(x) \qquad \text{means} \qquad F'(x) = f(x)}$$

For example, we can write

$$\int x^2\,dx = \frac{x^3}{3} + C \qquad \text{because} \qquad \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$

So we can regard an indefinite integral as representing an entire *family* of functions (one antiderivative for each value of the constant C).

You should distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)\,dx$ is a *number*, whereas an indefinite integral $\int f(x)\,dx$ is a *function* (or family of functions). The connection between them is given by Part 2 of the Fundamental Theorem: if f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big]_a^b$$

The effectiveness of the Fundamental Theorem depends on having a supply of antiderivatives of functions. We therefore restate the Table of Antidifferentiation Formulas from Section 4.9, together with a few others, in the notation of indefinite integrals. Any formula can be verified by differentiating the function on the right side and obtaining the integrand. For instance,

$$\int \sec^2 x\,dx = \tan x + C \qquad \text{because} \qquad \frac{d}{dx}(\tan x + C) = \sec^2 x$$

1 Table of Indefinite Integrals

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

Recall from Theorem 4.9.1 that the most general antiderivative *on a given interval* is obtained by adding a constant to a particular antiderivative. **We adopt the convention that when a formula for a general indefinite integral is given, it is valid only on an interval.** Thus we write

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$$

with the understanding that it is valid on the interval $(0, \infty)$ or on the interval $(-\infty, 0)$. This is true despite the fact that the general antiderivative of the function $f(x) = 1/x^2$, $x \neq 0$, is

$$F(x) = \begin{cases} -\dfrac{1}{x} + C_1 & \text{if } x < 0 \\[2ex] -\dfrac{1}{x} + C_2 & \text{if } x > 0 \end{cases}$$

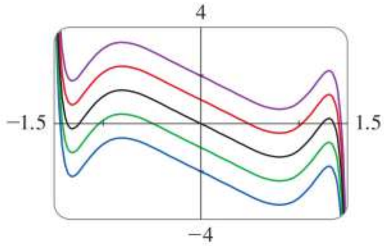


FIGURE 1

The indefinite integral in Example 1 is graphed in Figure 1 for several values of C. Here the value of C is the y-intercept.

EXAMPLE 1 Find the general indefinite integral

$$\int (10x^4 - 2\sec^2 x)\,dx$$

SOLUTION Using our convention and Table 1, we have

$$\begin{aligned}\int (10x^4 - 2\sec^2 x)\,dx &= 10\int x^4\,dx - 2\int \sec^2 x\,dx \\ &= 10\frac{x^5}{5} - 2\tan x + C \\ &= 2x^5 - 2\tan x + C\end{aligned}$$

You should check this answer by differentiating it. ■

EXAMPLE 2 Evaluate $\displaystyle\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$.

SOLUTION This indefinite integral isn't immediately apparent from Table 1, so we use trigonometric identities to rewrite the function before integrating:

$$\begin{aligned}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta &= \int \left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)d\theta \\ &= \int \csc\theta\cot\theta\,d\theta = -\csc\theta + C\end{aligned}$$ ■

EXAMPLE 3 Evaluate $\displaystyle\int_0^3 (x^3 - 6x)\,dx$.

SOLUTION Using FTC2 and Table 1, we have

$$\begin{aligned}\int_0^3 (x^3 - 6x)\,dx &= \frac{x^4}{4} - 6\frac{x^2}{2}\Big]_0^3 \\ &= \left(\tfrac{1}{4}\cdot 3^4 - 3\cdot 3^2\right) - \left(\tfrac{1}{4}\cdot 0^4 - 3\cdot 0^2\right) \\ &= \tfrac{81}{4} - 27 - 0 + 0 = -6.75\end{aligned}$$

Compare this calculation with Example 5.2.3. ■

Figure 2 shows the graph of the integrand in Example 4. We know from Section 5.2 that the value of the integral can be interpreted as a net area: the sum of the areas labeled with a plus sign minus the area labeled with a minus sign.

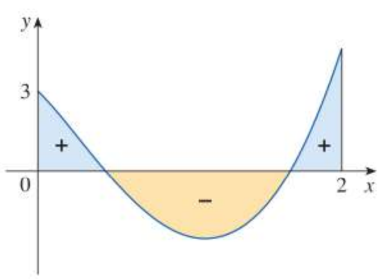


FIGURE 2

EXAMPLE 4 Find $\displaystyle\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right)dx$ and interpret the result in terms of areas.

SOLUTION The Fundamental Theorem gives

$$\begin{aligned}\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right)dx &= 2\frac{x^4}{4} - 6\frac{x^2}{2} + 3\tan^{-1}x\Big]_0^2 \\ &= \tfrac{1}{2}x^4 - 3x^2 + 3\tan^{-1}x\Big]_0^2 \\ &= \tfrac{1}{2}(2^4) - 3(2^2) + 3\tan^{-1}2 - 0 \\ &= -4 + 3\tan^{-1}2\end{aligned}$$

This is the exact value of the integral. If a decimal approximation is desired, we can use a calculator to approximate $\tan^{-1} 2$. Doing so, we get

$$\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2 + 1}\right) dx \approx -0.67855$$

EXAMPLE 5 Evaluate $\displaystyle\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt$.

SOLUTION First we need to write the integrand in a simpler form by carrying out the division:

$$\begin{aligned}
\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt &= \int_1^9 (2 + t^{1/2} - t^{-2})\, dt \\
&= 2t + \frac{t^{3/2}}{\frac{3}{2}} - \frac{t^{-1}}{-1}\Bigg]_1^9 = 2t + \tfrac{2}{3}t^{3/2} + \frac{1}{t}\Bigg]_1^9 \\
&= \left(2 \cdot 9 + \tfrac{2}{3} \cdot 9^{3/2} + \tfrac{1}{9}\right) - \left(2 \cdot 1 + \tfrac{2}{3} \cdot 1^{3/2} + \tfrac{1}{1}\right) \\
&= 18 + 18 + \tfrac{1}{9} - 2 - \tfrac{2}{3} - 1 = 32\tfrac{4}{9}
\end{aligned}$$

The Net Change Theorem

Part 2 of the Fundamental Theorem says that if f is continuous on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

where F is any antiderivative of f. This means that $F' = f$, so the equation can be rewritten as

$$\int_a^b F'(x)\, dx = F(b) - F(a)$$

We know that $F'(x)$ represents the rate of change of $y = F(x)$ with respect to x and $F(b) - F(a)$ is the change in y when x changes from a to b. [Note that y could, for instance, increase, then decrease, then increase again. Although y might change in both directions, $F(b) - F(a)$ represents the *net* change in y.] So we can reformulate FTC2 in words as follows.

Net Change Theorem The integral of a rate of change is the net change:

$$\int_a^b F'(x)\, dx = F(b) - F(a)$$

The principle expressed in the Net Change Theorem applies to all of the rates of change in the natural and social sciences that we discussed in Section 3.7. These applications show that part of the power of mathematics is in its abstractness. A single abstract idea (in this case the integral) can have many different interpretations. Here are a few instances of how the Net Change Theorem can be applied.

- If $V(t)$ is the volume of water in a reservoir at time t, then its derivative $V'(t)$ is the rate at which water flows into the reservoir at time t. So

  $$\int_{t_1}^{t_2} V'(t)\, dt = V(t_2) - V(t_1)$$

  is the change in the amount of water in the reservoir between time t_1 and time t_2.

- If $[\mathrm{C}](t)$ is the concentration of the product of a chemical reaction at time t, then the rate of reaction is the derivative $d[\mathrm{C}]/dt$. So

$$\int_{t_1}^{t_2} \frac{d[\mathrm{C}]}{dt}\,dt = [\mathrm{C}](t_2) - [\mathrm{C}](t_1)$$

is the change in the concentration of C from time t_1 to time t_2.

- If the mass of a rod measured from the left end to a point x is $m(x)$, then the linear density is $\rho(x) = m'(x)$. So

$$\int_a^b \rho(x)\,dx = m(b) - m(a)$$

is the mass of the segment of the rod that lies between $x = a$ and $x = b$.

- If the rate of growth of a population is dn/dt, then

$$\int_{t_1}^{t_2} \frac{dn}{dt}\,dt = n(t_2) - n(t_1)$$

is the net change in population during the time period from t_1 to t_2. (The population increases when births happen and decreases when deaths occur. The net change takes into account both births and deaths.)

- If $C(x)$ is the cost of producing x units of a commodity, then the marginal cost is the derivative $C'(x)$. So

$$\int_{x_1}^{x_2} C'(x)\,dx = C(x_2) - C(x_1)$$

is the increase in cost when production is increased from x_1 units to x_2 units.

- If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

2
$$\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$$

is the net change of position, or *displacement*, of the object during the time period from t_1 to t_2. In Section 5.1 we guessed that this was true for the case where the object moves in the positive direction, but now we have proved that it is always true.

- If we want to calculate the distance the object travels during a time interval, we have to consider the intervals when $v(t) \geqslant 0$ (the object moves to the right) and also the intervals when $v(t) \leqslant 0$ (the object moves to the left). In both cases the distance is computed by integrating $|v(t)|$, the speed. Therefore

3
$$\int_{t_1}^{t_2} |v(t)|\,dt = \text{total distance traveled}$$

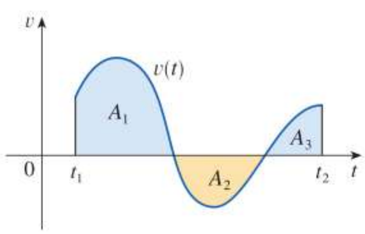


FIGURE 3

Figure 3 shows how both displacement and distance traveled can be interpreted in terms of areas under a velocity curve.

$$\text{displacement} = \int_{t_1}^{t_2} v(t)\,dt = A_1 - A_2 + A_3$$

$$\text{distance} = \int_{t_1}^{t_2} |v(t)|\,dt = A_1 + A_2 + A_3$$

- The acceleration of the object is $a(t) = v'(t)$, so

$$\int_{t_1}^{t_2} a(t)\,dt = v(t_2) - v(t_1)$$

is the change in velocity from time t_1 to time t_2.

EXAMPLE 6 A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).
(a) Find the displacement of the particle during the time period $1 \leqslant t \leqslant 4$.
(b) Find the distance traveled during this time period.

SOLUTION
(a) By Equation 2, the displacement is

$$s(4) - s(1) = \int_1^4 v(t)\,dt = \int_1^4 (t^2 - t - 6)\,dt$$
$$= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4 = -\frac{9}{2}$$

This means that the particle moved 4.5 m toward the left.

(b) Note that $v(t) = t^2 - t - 6 = (t - 3)(t + 2)$ and so $v(t) \leqslant 0$ on the interval $[1, 3]$ and $v(t) \geqslant 0$ on $[3, 4]$. Thus, from Equation 3, the distance traveled is

To integrate the absolute value of $v(t)$, we use Property 5 of integrals from Section 5.2 to split the integral into two parts, one where $v(t) \leqslant 0$ and one where $v(t) \geqslant 0$.

$$\int_1^4 |v(t)|\,dt = \int_1^3 [-v(t)]\,dt + \int_3^4 v(t)\,dt$$
$$= \int_1^3 (-t^2 + t + 6)\,dt + \int_3^4 (t^2 - t - 6)\,dt$$
$$= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4$$
$$= \frac{61}{6} \approx 10.17 \text{ m}$$

■

EXAMPLE 7 Figure 4 shows the power consumption in the city of San Francisco for a day in September (P is measured in megawatts; t is measured in hours starting at midnight). Estimate the energy used on that day.

FIGURE 4

SOLUTION Power is the rate of change of energy: $P(t) = E'(t)$. So, by the Net Change Theorem,

$$\int_0^{24} P(t)\,dt = \int_0^{24} E'(t)\,dt = E(24) - E(0)$$

is the total amount of energy used on that day. We approximate the value of the integral using the Midpoint Rule with 12 subintervals and $\Delta t = 2$:

$$\begin{aligned}\int_0^{24} P(t)\,dt &\approx [P(1) + P(3) + P(5) + \cdots + P(21) + P(23)]\,\Delta t \\ &\approx (440 + 400 + 420 + 620 + 790 + 840 + 850 \\ &\qquad + 840 + 810 + 690 + 670 + 550)(2) \\ &= 15{,}840\end{aligned}$$

The energy used was approximately 15,840 megawatt-hours. ■

A note on units

How did we know what units to use for energy in Example 7? The integral $\int_0^{24} P(t)\,dt$ is defined as the limit of sums of terms of the form $P(t_i^*)\,\Delta t$. Now $P(t_i^*)$ is measured in megawatts and Δt is measured in hours, so their product is measured in megawatt-hours. The same is true of the limit. In general, the unit of measurement for $\int_a^b f(x)\,dx$ is the product of the unit for $f(x)$ and the unit for x.

5.4 Exercises

1–4 Verify by differentiation that the formula is correct.

1. $\int \ln x\,dx = x\ln x - x + C$

2. $\int \tan^2 x\,dx = \tan x - x + C$

3. $\int \dfrac{1}{x^2\sqrt{1+x^2}}\,dx = -\dfrac{\sqrt{1+x^2}}{x} + C$

4. $\int x\sqrt{a+bx}\,dx = \dfrac{2}{15b^2}(3bx - 2a)(a+bx)^{3/2} + C$

5–24 Find the general indefinite integral.

5. $\int (3x^2 + 4x + 1)\,dx$

6. $\int (5 + 2\sqrt{x})\,dx$

7. $\int (x + \cos x)\,dx$

8. $\int \left(\sqrt[3]{x} + \dfrac{1}{\sqrt[3]{x}}\right) dx$

9. $\int (x^{1.3} + 7x^{2.5})\,dx$

10. $\int \sqrt[4]{x^5}\,dx$

11. $\int \left(5 + \frac{2}{3}x^2 + \frac{3}{4}x^3\right) dx$

12. $\int \left(u^6 - 2u^5 - u^3 + \frac{2}{7}\right) du$

13. $\int (u+4)(2u+1)\,du$

14. $\int \sqrt{t}(t^2 + 3t + 2)\,dt$

15. $\int \dfrac{1 + \sqrt{x} + x}{x}\,dx$

16. $\int \left(x^2 + 1 + \dfrac{1}{x^2+1}\right) dx$

17. $\int \left(e^x + \dfrac{1}{x}\right) dx$

18. $\int (2 + 3^x)\,dx$

19. $\int (\sin x + \sinh x)\,dx$

20. $\int \left(\dfrac{1+r}{r}\right)^2 dr$

21. $\int (2 + \tan^2\theta)\,d\theta$

22. $\int \sec t\,(\sec t + \tan t)\,dt$

23. $\int 3\csc^2 t\,dt$

24. $\int \dfrac{\sin 2x}{\sin x}\,dx$

25–26 Find the general indefinite integral. Illustrate by graphing several members of the family on the same screen.

25. $\int \left(\cos x + \frac{1}{2}x\right) dx$

26. $\int (e^x - 2x^2)\,dx$

27–54 Evaluate the definite integral.

27. $\int_{-2}^{3} (x^2 - 3)\,dx$

28. $\int_1^2 (4x^3 - 3x^2 + 2x)\,dx$

29. $\int_1^4 (8t^3 - 6t^{-2})\,dt$

30. $\int_0^8 \left(\frac{1}{8} + \frac{1}{2}w + \frac{1}{3}w^{1/3}\right) dw$

31. $\int_0^2 (2x-3)(4x^2+1)\,dx$

32. $\int_1^2 \left(\dfrac{1}{x^2} - \dfrac{4}{x^3}\right) dx$

33. $\int_1^3 \left(\dfrac{3x^2 + 4x + 1}{x}\right) dx$

34. $\int_{-1}^{1} t(1-t)^2\,dt$

35. $\int_1^4 \left(\dfrac{4 + 6u}{\sqrt{u}}\right) du$

36. $\int_0^1 \dfrac{4}{1+p^2}\,dp$

37. $\int_{\pi/6}^{\pi/3} (4\sec^2 y)\,dy$

38. $\int_0^{\pi/2} (\sqrt{t} - 3\cos t)\,dt$

39. $\int_0^1 x(\sqrt[3]{x} + \sqrt[4]{x})\, dx$

40. $\int_1^4 \frac{\sqrt{y} - y}{y^2}\, dy$

41. $\int_1^2 \left(\frac{x}{2} - \frac{2}{x}\right) dx$

42. $\int_0^1 (5x - 5^x)\, dx$

43. $\int_{-2}^2 (\sinh x + \cosh x)\, dx$

44. $\int_0^{\pi/4} (3e^x - 4 \sec x \tan x)\, dx$

45. $\int_0^{\pi/4} \frac{1 + \cos^2\theta}{\cos^2\theta}\, d\theta$

46. $\int_0^{\pi/3} \frac{\sin\theta + \sin\theta \tan^2\theta}{\sec^2\theta}\, d\theta$

47. $\int_3^4 \sqrt{\frac{3}{x}}\, dx$

48. $\int_{-10}^{10} \frac{2e^x}{\sinh x + \cosh x}\, dx$

49. $\int_0^{\sqrt{3}/2} \frac{dr}{\sqrt{1 - r^2}}$

50. $\int_{\pi/6}^{\pi/2} \csc t \cot t\, dt$

51. $\int_0^{1/\sqrt{3}} \frac{t^2 - 1}{t^4 - 1}\, dt$

52. $\int_0^2 |2x - 1|\, dx$

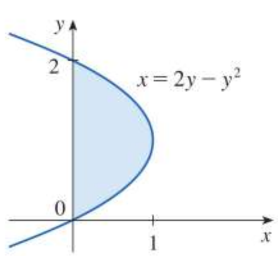
53. $\int_{-1}^2 (x - 2|x|)\, dx$

54. $\int_0^{3\pi/2} |\sin x|\, dx$

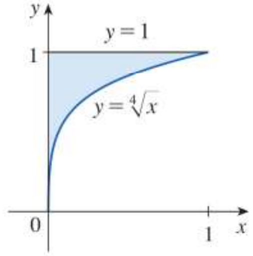
55. Use a graph to estimate the x-intercepts of the curve $y = 1 - 2x - 5x^4$. Then use this information to estimate the area of the region that lies under the curve and above the x-axis.

56. Repeat Exercise 55 for the curve $y = (x^2 + 1)^{-1} - x^4$.

57. The area of the region that lies to the right of the y-axis and to the left of the parabola $x = 2y - y^2$ (the shaded region in the figure) is given by the integral $\int_0^2 (2y - y^2)\, dy$. (Turn your head clockwise and think of the region as lying below the curve $x = 2y - y^2$ from $y = 0$ to $y = 2$.) Find the area of the region.



58. The boundaries of the shaded region are the y-axis, the line $y = 1$, and the curve $y = \sqrt[4]{x}$. Find the area of this region by writing x as a function of y and integrating with respect to y (as in Exercise 57).



59. If $w'(t)$ is the rate of growth of a child in pounds per year, what does $\int_5^{10} w'(t)\, dt$ represent?

60. The current in a wire is defined as the derivative of the charge: $I(t) = Q'(t)$. (See Example 3.7.3.) What does $\int_a^b I(t)\, dt$ represent?

61. If oil leaks from a tank at a rate of $r(t)$ gallons per minute at time t, what does $\int_0^{120} r(t)\, dt$ represent?

62. A honeybee population starts with 100 bees and increases at a rate of $n'(t)$ bees per week. What does $100 + \int_0^{15} n'(t)\, dt$ represent?

63. In Section 4.7 we defined the marginal revenue function $R'(x)$ as the derivative of the revenue function $R(x)$, where x is the number of units sold. What does $\int_{1000}^{5000} R'(x)\, dx$ represent?

64. If $f(x)$ is the slope of a trail at a distance of x miles from the start of the trail, what does $\int_3^5 f(x)\, dx$ represent?

65. If $h(t)$ is a person's heart rate in beats per minute t minutes into an exercise session, what does $\int_0^{30} h(t)\, dt$ represent?

66. If the units for x are feet and the units for $a(x)$ are pounds per foot, what are the units for da/dx? What units does $\int_2^8 a(x)\, dx$ have?

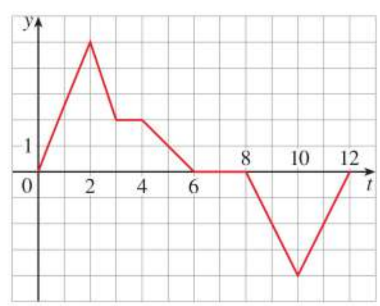
67. If x is measured in meters and $f(x)$ is measured in newtons, what are the units for $\int_0^{100} f(x)\, dx$?

68. The graph shows the velocity (in m/s) of an electric autonomous vehicle moving along a straight track. At $t = 0$ the vehicle is at the charging station.

(a) How far is the vehicle from the charging station when $t = 2, 4, 6, 8, 10,$ and 12?

(b) At what times is the vehicle farthest from the charging station?

(c) What is the total distance traveled by the vehicle?



69–70 The velocity function (in m/s) is given for a particle moving along a line. Find (a) the displacement and (b) the distance traveled by the particle during the given time interval.

69. $v(t) = 3t - 5, \quad 0 \leqslant t \leqslant 3$

70. $v(t) = t^2 - 2t - 3, \quad 2 \leqslant t \leqslant 4$

71–72 The acceleration function (in m/s^2) and the initial velocity are given for a particle moving along a line. Find (a) the velocity at time t and (b) the distance traveled during the given time interval.

71. $a(t) = t + 4, \quad v(0) = 5, \quad 0 \leq t \leq 10$

72. $a(t) = 2t + 3, \quad v(0) = -4, \quad 0 \leq t \leq 3$

73. The linear density of a rod of length 4 m is given by $\rho(x) = 9 + 2\sqrt{x}$ measured in kilograms per meter, where x is measured in meters from one end of the rod. Find the total mass of the rod.

74. Water flows from the bottom of a storage tank at a rate of $r(t) = 200 - 4t$ liters per minute, where $0 \leq t \leq 50$. Find the amount of water that flows from the tank during the first 10 minutes.

75. The velocity of a car was read from its speedometer at 10-second intervals and recorded in the table. Use the Midpoint Rule to estimate the distance traveled by the car.

t (s)	v (mi/h)	t (s)	v (mi/h)
0	0	60	56
10	38	70	53
20	52	80	50
30	58	90	47
40	55	100	45
50	51		

76. Suppose that a volcano is erupting and readings of the rate $r(t)$ at which solid materials are spewed into the atmosphere are given in the table. The time t is measured in seconds and the units for $r(t)$ are tonnes (metric tons) per second.

t	0	1	2	3	4	5	6
$r(t)$	2	10	24	36	46	54	60

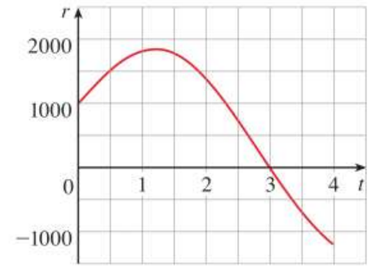
(a) Give upper and lower estimates for the total quantity $Q(6)$ of erupted materials after six seconds.
(b) Use the Midpoint Rule to estimate $Q(6)$.

77. The marginal cost of manufacturing x yards of a certain fabric is

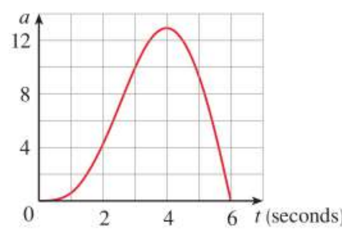
$$C'(x) = 3 - 0.01x + 0.000006x^2$$

(in dollars per yard). Find the increase in cost if the production level is raised from 2000 yards to 4000 yards.

78. Water flows into and out of a storage tank. A graph of the rate of change $r(t)$ of the volume of water in the tank, in liters per day, is shown. If the amount of water in the tank at time $t = 0$ is 25,000 L, use the Midpoint Rule to estimate the amount of water in the tank four days later.



79. The graph of a car's acceleration $a(t)$, measured in ft/s^2, is shown. Use the Midpoint Rule to estimate the increase in the velocity of the car during the six-second time interval.

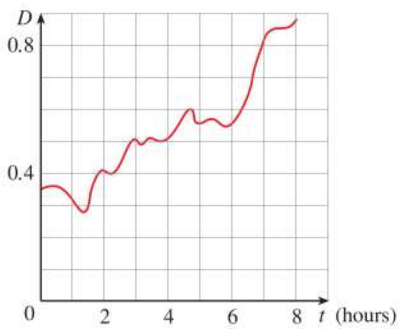


80. Lake Lanier in Georgia, USA, is a reservoir created by Buford Dam on the Chattahoochee River. The table shows the rate of inflow of water, in cubic feet per second, as measured every morning at 7:30 AM by the US Army Corps of Engineers. Use the Midpoint Rule to estimate the amount of water that flowed into Lake Lanier from July 18th, 2013, at 7:30 AM to July 26th at 7:30 AM.

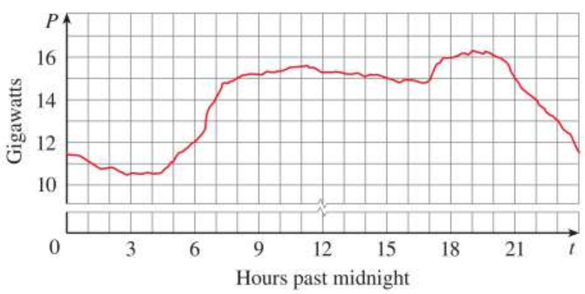
Day	Inflow rate (ft^3/s)
July 18	5275
July 19	6401
July 20	2554
July 21	4249
July 22	3016
July 23	3821
July 24	2462
July 25	2628
July 26	3003

81. A bacteria population is 4000 at time $t = 0$ and its rate of growth is $1000 \cdot 2^t$ bacteria per hour after t hours. What is the population after one hour?

82. Shown is the graph of traffic on an Internet service provider's T1 data line from midnight to 8:00 AM. D is the data throughput, measured in megabits per second. Use the Midpoint Rule to estimate the total amount of data transmitted during that time period.



83. Shown is a graph of the electric power consumption in the New England states (Connecticut, Maine, Massachusetts, New Hampshire, Rhode Island, and Vermont) for October 22, 2010 (P is measured in gigawatts and t is measured in hours starting at midnight). Use the fact that power is the rate of change of energy to estimate the electric energy used on that day.



Source: US Energy Information Administration

T 84. In 1992 the space shuttle *Endeavour* was launched on mission STS-49 in order to install a new perigee kick motor in an Intelsat communications satellite. The table gives the velocity data for the shuttle between liftoff and the jettisoning of the solid rocket boosters.

(a) Use a graphing calculator or computer to model these data by a third-degree polynomial.

(b) Use the model in part (a) to estimate the height reached by the *Endeavour*, 125 seconds after liftoff.

Event	Time (s)	Velocity (ft/s)
Launch	0	0
Begin roll maneuver	10	185
End roll maneuver	15	319
Throttle to 89%	20	447
Throttle to 67%	32	742
Throttle to 104%	59	1325
Maximum dynamic pressure	62	1445
Solid rocket booster separation	125	4151

WRITING PROJECT | NEWTON, LEIBNIZ, AND THE INVENTION OF CALCULUS

We sometimes read that the inventors of calculus were Sir Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716). But we know that the basic ideas behind integration were investigated 2500 years ago by ancient Greeks such as Eudoxus and Archimedes, and methods for finding tangents were pioneered by Pierre Fermat (1601–1665), Isaac Barrow (1630–1677), and others. Barrow—who taught at Cambridge and was a major influence on Newton—was the first to understand the inverse relationship between differentiation and integration. What Newton and Leibniz did was to use this relationship, in the form of the Fundamental Theorem of Calculus, in order to develop calculus into a systematic mathematical discipline. It is in this sense that both Newton and Leibniz are credited with the invention of calculus.

Search the Internet to find out more about the contributions of these men, and consult one or more of the given references. Write an essay on one of the following three topics. You can include biographical details, but the main thrust of your report should be a description, in some detail, of their methods and notations. In particular, you should consult one of the source books, which give excerpts from the original publications of Newton and Leibniz, translated from Latin to English.

- The Role of Newton in the Development of Calculus
- The Role of Leibniz in the Development of Calculus
- The Controversy between the Followers of Newton and Leibniz over Priority in the Invention of Calculus

References

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3. C. H. Edwards, *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979), Chapters 8 and 9.
4. Howard Eves, *An Introduction to the History of Mathematics,* 6th ed. (New York: Saunders, 1990), Chapter 11.
5. C. C. Gillispie, ed., *Dictionary of Scientific Biography* (New York: Scribner's, 1974). See the article on Leibniz by Joseph Hofmann in Volume VIII and the article on Newton by I. B. Cohen in Volume X.
6. Victor Katz, *A History of Mathematics: An Introduction* (New York: HarperCollins, 1993), Chapter 12.
7. Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1972), Chapter 17.

Source Books

1. John Fauvel and Jeremy Gray, eds., *The History of Mathematics: A Reader* (London: MacMillan Press, 1987), Chapters 12 and 13.
2. D. E. Smith, ed., *A Sourcebook in Mathematics* (New York: Dover, 1959), Chapter V.
3. D. J. Struik, ed., *A Sourcebook in Mathematics,* 1200–1800 (Princeton, NJ: Princeton University Press, 1969), Chapter V.

5.5 The Substitution Rule

Because of the Fundamental Theorem, it's important to be able to find antiderivatives. But our antidifferentiation formulas don't tell us how to evaluate integrals such as

$$\boxed{1} \qquad \int 2x\sqrt{1+x^2}\,dx$$

PS To find this integral we use the problem-solving strategy of *introducing something extra.* Here the "something extra" is a new variable; we change from the variable x to a new variable u.

Substitution: Indefinite Integrals

Differentials were defined in Section 3.10. If $u = f(x)$, then
$$du = f'(x)\,dx$$

Suppose that we let u be the quantity under the root sign in (1), $u = 1 + x^2$. Then the differential of u is $du = 2x\,dx$. Notice that if the dx in the notation for an integral were to be interpreted as a differential, then the differential $2x\,dx$ would occur in (1) and so, formally, without justifying our calculation, we could write

$$\boxed{2} \qquad \begin{aligned}\int 2x\sqrt{1+x^2}\,dx &= \int \sqrt{1+x^2}\,2x\,dx = \int \sqrt{u}\,du \\ &= \tfrac{2}{3}u^{3/2} + C = \tfrac{2}{3}(1+x^2)^{3/2} + C\end{aligned}$$

But now we can check that we have the correct answer by using the Chain Rule to differentiate the final function of Equation 2:

$$\frac{d}{dx}\left[\tfrac{2}{3}(1+x^2)^{3/2}+C\right]=\tfrac{2}{3}\cdot\tfrac{3}{2}(1+x^2)^{1/2}\cdot 2x=2x\sqrt{1+x^2}$$

In general, this method works whenever we have an integral that we can write in the form $\int f(g(x))\,g'(x)\,dx$. Observe that if $F'=f$, then

$$\boxed{3} \qquad \int F'(g(x))\,g'(x)\,dx=F(g(x))+C$$

because, by the Chain Rule,

$$\frac{d}{dx}[F(g(x))]=F'(g(x))\,g'(x)$$

If we make the "change of variable" or "substitution" $u=g(x)$, then from Equation 3 we have

$$\int F'(g(x))\,g'(x)\,dx=F(g(x))+C=F(u)+C=\int F'(u)\,du$$

or, writing $F'=f$, we get

$$\int f(g(x))\,g'(x)\,dx=\int f(u)\,du$$

Thus we have proved the following rule.

4 The Substitution Rule If $u=g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\,g'(x)\,dx=\int f(u)\,du$$

Notice that the Substitution Rule for integration was proved using the Chain Rule for differentiation. Notice also that if $u=g(x)$, then $du=g'(x)\,dx$, so a way to remember the Substitution Rule is to think of dx and du in (4) as differentials.

Thus the Substitution Rule says: **it is permissible to operate with dx and du after integral signs as if they were differentials.**

EXAMPLE 1 Find $\int x^3\cos(x^4+2)\,dx$.

SOLUTION We make the substitution $u=x^4+2$ because its differential is $du=4x^3\,dx$, which, apart from the constant factor 4, occurs in the integral. Thus, using $x^3\,dx=\frac{1}{4}\,du$ and the Substitution Rule, we have

$$\begin{aligned}\int x^3\cos(x^4+2)\,dx&=\int \cos u\cdot\tfrac{1}{4}\,du=\tfrac{1}{4}\int\cos u\,du\\&=\tfrac{1}{4}\sin u+C\\&=\tfrac{1}{4}\sin(x^4+2)+C\end{aligned}$$

Check the answer by differentiating it.

Notice that at the final stage we had to return to the original variable x. ■

The idea behind the Substitution Rule is to replace a relatively complicated integral by a simpler integral. This is accomplished by changing from the original variable x to a new variable u that is a function of x. Thus in Example 1 we replaced the integral $\int x^3 \cos(x^4 + 2)\,dx$ by the simpler integral $\frac{1}{4}\int \cos u\,du$.

The main challenge in using the Substitution Rule is to think of an appropriate substitution. You should try to choose u to be some function in the integrand whose differential also occurs (except for a constant factor). This was the case in Example 1. If that is not possible, try choosing u to be some complicated part of the integrand (perhaps the inner function in a composite function). Finding the right substitution is a bit of an art. It's not unusual to guess wrong; if your first guess doesn't work, try another substitution.

EXAMPLE 2 Evaluate $\int \sqrt{2x+1}\,dx$.

SOLUTION 1 Let $u = 2x + 1$. Then $du = 2\,dx$, so $dx = \frac{1}{2}\,du$. Thus the Substitution Rule gives

$$\begin{aligned}\int \sqrt{2x+1}\,dx &= \int \sqrt{u}\cdot \tfrac{1}{2}\,du = \tfrac{1}{2}\int u^{1/2}\,du \\ &= \frac{1}{2}\cdot\frac{u^{3/2}}{3/2} + C = \tfrac{1}{3}u^{3/2} + C \\ &= \tfrac{1}{3}(2x+1)^{3/2} + C\end{aligned}$$

SOLUTION 2 Another possible substitution is $u = \sqrt{2x+1}$. Then

$$du = \frac{dx}{\sqrt{2x+1}} \qquad \text{so} \qquad dx = \sqrt{2x+1}\,du = u\,du$$

(Or observe that $u^2 = 2x + 1$, so $2u\,du = 2\,dx$.) Therefore

$$\begin{aligned}\int \sqrt{2x+1}\,dx &= \int u\cdot u\,du = \int u^2\,du \\ &= \frac{u^3}{3} + C = \tfrac{1}{3}(2x+1)^{3/2} + C\end{aligned}$$

■

EXAMPLE 3 Find $\int \dfrac{x}{\sqrt{1-4x^2}}\,dx$.

SOLUTION Let $u = 1 - 4x^2$. Then $du = -8x\,dx$, so $x\,dx = -\frac{1}{8}\,du$ and

$$\begin{aligned}\int \frac{x}{\sqrt{1-4x^2}}\,dx &= -\tfrac{1}{8}\int \frac{1}{\sqrt{u}}\,du = -\tfrac{1}{8}\int u^{-1/2}\,du \\ &= -\tfrac{1}{8}\left(2\sqrt{u}\right) + C = -\tfrac{1}{4}\sqrt{1-4x^2} + C\end{aligned}$$

■

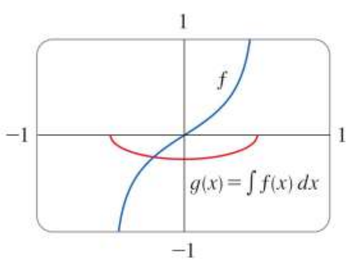


FIGURE 1

$f(x) = \dfrac{x}{\sqrt{1-4x^2}}$

$g(x) = \int f(x)\,dx = -\frac{1}{4}\sqrt{1-4x^2}$

The answer to Example 3 could be checked by differentiation, but instead let's check it with a graph. In Figure 1 we have used a computer to graph both the integrand $f(x) = x/\sqrt{1-4x^2}$ and its indefinite integral $g(x) = -\frac{1}{4}\sqrt{1-4x^2}$ (we take the case $C = 0$). Notice that $g(x)$ decreases when $f(x)$ is negative, increases when $f(x)$ is positive, and has its minimum value when $f(x) = 0$. So it seems reasonable, from the graphical evidence, that g is an antiderivative of f.

EXAMPLE 4 Evaluate $\int e^{5x}\,dx$.

SOLUTION If we let $u = 5x$, then $du = 5\,dx$, so $dx = \frac{1}{5}\,du$. Therefore

$$\int e^{5x}\,dx = \tfrac{1}{5}\int e^u\,du = \tfrac{1}{5}e^u + C = \tfrac{1}{5}e^{5x} + C$$

■

NOTE With some experience, you might be able to evaluate integrals like those in Examples 1–4 without going to the trouble of making an explicit substitution. By recognizing the pattern in Equation 3, where the integrand on the left side is the product of the derivative of an outer function and the derivative of the inner function, we could work Example 1 as follows:

$$\int x^3\cos(x^4+2)\,dx = \int \cos(x^4+2)\cdot x^3\,dx = \tfrac{1}{4}\int \cos(x^4+2)\cdot(4x^3)\,dx$$

$$= \tfrac{1}{4}\int \cos(x^4+2)\cdot\frac{d}{dx}(x^4+2)\,dx = \tfrac{1}{4}\sin(x^4+2) + C$$

Similarly, the solution to Example 4 could be written like this:

$$\int e^{5x}\,dx = \tfrac{1}{5}\int 5e^{5x}\,dx = \tfrac{1}{5}\int \frac{d}{dx}(e^{5x})\,dx = \tfrac{1}{5}e^{5x} + C$$

The following example, however, is more complicated and so an explicit substitution is advisable.

EXAMPLE 5 Find $\int \sqrt{1+x^2}\,x^5\,dx$.

SOLUTION An appropriate substitution becomes more apparent if we factor x^5 as $x^4 \cdot x$. Let $u = 1 + x^2$. Then $du = 2x\,dx$, so $x\,dx = \frac{1}{2}\,du$. Also $x^2 = u - 1$, so $x^4 = (u-1)^2$:

$$\begin{aligned}\int \sqrt{1+x^2}\,x^5\,dx &= \int \sqrt{1+x^2}\,x^4\cdot x\,dx\\ &= \int \sqrt{u}\,(u-1)^2\cdot\tfrac{1}{2}\,du = \tfrac{1}{2}\int \sqrt{u}\,(u^2-2u+1)\,du\\ &= \tfrac{1}{2}\int (u^{5/2} - 2u^{3/2} + u^{1/2})\,du\\ &= \tfrac{1}{2}\left(\tfrac{2}{7}u^{7/2} - 2\cdot\tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2}\right) + C\\ &= \tfrac{1}{7}(1+x^2)^{7/2} - \tfrac{2}{5}(1+x^2)^{5/2} + \tfrac{1}{3}(1+x^2)^{3/2} + C\end{aligned}$$

■

EXAMPLE 6 Evaluate $\int \tan x\,dx$.

SOLUTION First we write tangent in terms of sine and cosine:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

This suggests that we should substitute $u = \cos x$, since then $du = -\sin x\,dx$ and so $\sin x\,dx = -du$:

$$\begin{aligned}\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{u}\,du\\ &= -\ln|u| + C = -\ln|\cos x| + C\end{aligned}$$

■

Notice that $-\ln|\cos x| = \ln(|\cos x|^{-1}) = \ln(1/|\cos x|) = \ln|\sec x|$, so the result of Example 6 can also be written as

5 $$\boxed{\int \tan x\, dx = \ln|\sec x| + C}$$

Substitution: Definite Integrals

When evaluating a *definite* integral by substitution, two methods are possible. One method is to evaluate the indefinite integral first and then use the Fundamental Theorem. For instance, using the result of Example 2, we have

$$\begin{aligned}\int_0^4 \sqrt{2x+1}\, dx &= \int \sqrt{2x+1}\, dx\Big]_0^4 \\ &= \tfrac{1}{3}(2x+1)^{3/2}\Big]_0^4 = \tfrac{1}{3}(9)^{3/2} - \tfrac{1}{3}(1)^{3/2} \\ &= \tfrac{1}{3}(27-1) = \tfrac{26}{3}\end{aligned}$$

Another method, which is usually preferable, is to change the limits of integration when the variable is changed.

This rule says that when using a substitution in a definite integral, we must put everything in terms of the new variable u, not only x and dx but also the limits of integration. The new limits of integration are the values of u that correspond to $x = a$ and $x = b$.

6 The Substitution Rule for Definite Integrals If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

PROOF Let F be an antiderivative of f. Then, by (3), $F(g(x))$ is an antiderivative of $f(g(x))\,g'(x)$, so by Part 2 of the Fundamental Theorem, we have

$$\int_a^b f(g(x))\, g'(x)\, dx = F(g(x))\Big]_a^b = F(g(b)) - F(g(a))$$

But, applying FTC2 a second time, we also have

$$\int_{g(a)}^{g(b)} f(u)\, du = F(u)\Big]_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$$ ■

EXAMPLE 7 Evaluate $\int_0^4 \sqrt{2x+1}\, dx$ using (6).

SOLUTION Using the substitution from Solution 1 of Example 2, we have $u = 2x + 1$ and $dx = \frac{1}{2}\, du$. To find the new limits of integration we note that

when $x = 0$, $u = 2(0) + 1 = 1$ and when $x = 4$, $u = 2(4) + 1 = 9$

Therefore

$$\begin{aligned}\int_0^4 \sqrt{2x+1}\, dx &= \int_1^9 \tfrac{1}{2}\sqrt{u}\, du \\ &= \tfrac{1}{2}\cdot\tfrac{2}{3}u^{3/2}\Big]_1^9 \\ &= \tfrac{1}{3}(9^{3/2} - 1^{3/2}) = \tfrac{26}{3}\end{aligned}$$ ■

Observe that when using (6) we do *not* return to the variable x after integrating. We simply evaluate the expression in u between the appropriate values of u.

Another way of writing the integral given in Example 8 is

$$\int_1^2 \frac{1}{(3-5x)^2}\,dx$$

EXAMPLE 8 Evaluate $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2}$.

SOLUTION Let $u = 3 - 5x$. Then $du = -5\,dx$, so $dx = -\frac{1}{5}\,du$. When $x = 1$, $u = -2$ and when $x = 2$, $u = -7$. Thus

$$\int_1^2 \frac{dx}{(3-5x)^2} = -\frac{1}{5}\int_{-2}^{-7} \frac{du}{u^2} = -\frac{1}{5}\left[-\frac{1}{u}\right]_{-2}^{-7} = \frac{1}{5u}\Bigg]_{-2}^{-7}$$

$$= \frac{1}{5}\left(-\frac{1}{7} + \frac{1}{2}\right) = \frac{1}{14}$$

■

Because the function $f(x) = (\ln x)/x$ in Example 9 is positive for $x > 1$, the integral represents the area of the shaded region in Figure 2.

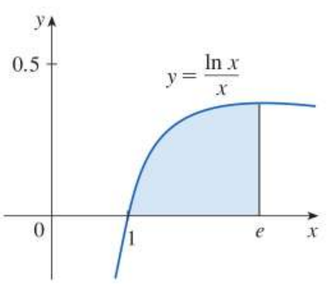


FIGURE 2

EXAMPLE 9 Evaluate $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$.

SOLUTION We let $u = \ln x$ because its differential $du = (1/x)\,dx$ occurs in the integral. When $x = 1$, $u = \ln 1 = 0$; when $x = e$, $u = \ln e = 1$. Thus

$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \frac{u^2}{2}\Bigg]_0^1 = \frac{1}{2}$$

■

Symmetry

The next theorem uses the Substitution Rule for Definite Integrals (6) to simplify the calculation of integrals of functions that possess symmetry properties.

7 Integrals of Symmetric Functions Suppose f is continuous on $[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

(b) If f is odd $[f(-x) = -f(x)]$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

PROOF We split the integral in two:

$$\boxed{8} \qquad \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx$$

In the first integral on the far right side we make the substitution $u = -x$. Then $du = -dx$ and when $x = -a$, $u = a$. Therefore

$$-\int_0^{-a} f(x)\,dx = -\int_0^a f(-u)\,(-du) = \int_0^a f(-u)\,du$$

and so Equation 8 becomes

$$\boxed{9} \qquad \int_{-a}^{a} f(x)\,dx = \int_0^a f(-u)\,du + \int_0^a f(x)\,dx$$

(a) If f is even, then $f(-u) = f(u)$ so Equation 9 gives

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(u)\,du + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

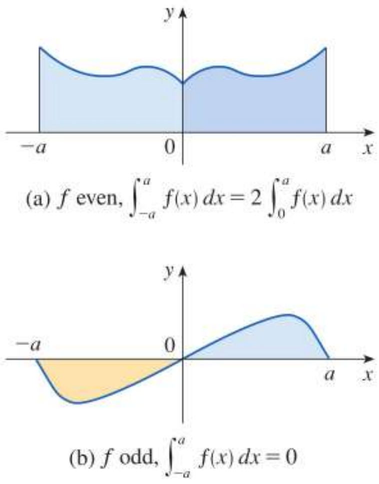


FIGURE 3

(b) If f is odd, then $f(-u) = -f(u)$ and so Equation 9 gives

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = 0 \qquad \blacksquare$$

Theorem 7 is illustrated by Figure 3. For the case where f is positive and even, part (a) says that the area under $y = f(x)$ from $-a$ to a is twice the area from 0 to a because of symmetry. Recall that an integral $\int_a^b f(x)\,dx$ can be expressed as the area above the x-axis and below $y = f(x)$ minus the area below the axis and above the curve. So part (b) says the integral is 0 because the areas cancel.

EXAMPLE 10 Because $f(x) = x^6 + 1$ satisfies $f(-x) = f(x)$, it is even and so

$$\int_{-2}^{2} (x^6 + 1)\,dx = 2\int_{0}^{2} (x^6 + 1)\,dx$$

$$= 2\Big[\tfrac{1}{7}x^7 + x\Big]_0^2 = 2\big(\tfrac{128}{7} + 2\big) = \tfrac{284}{7} \qquad \blacksquare$$

EXAMPLE 11 Because $f(x) = (\tan x)/(1 + x^2 + x^4)$ satisfies $f(-x) = -f(x)$, it is odd and so

$$\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\,dx = 0 \qquad \blacksquare$$

5.5 Exercises

1–8 Evaluate the integral by making the given substitution.

1. $\int \cos 2x\,dx, \quad u = 2x$

2. $\int xe^{-x^2}\,dx, \quad u = -x^2$

3. $\int x^2\sqrt{x^3 + 1}\,dx, \quad u = x^3 + 1$

4. $\int \sin^2\theta \cos\theta\,d\theta, \quad u = \sin\theta$

5. $\int \frac{x^3}{x^4 - 5}\,dx, \quad u = x^4 - 5$

6. $\int \frac{1}{x^2}\sqrt{1 + \frac{1}{x}}\,dx, \quad u = 1 + \frac{1}{x}$

7. $\int \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt, \quad u = \sqrt{t}$

8. $\int z\sqrt{z - 1}\,dz, \quad u = z - 1$

9–54 Evaluate the indefinite integral.

9. $\int x\sqrt{1 - x^2}\,dx$

10. $\int (5 - 3x)^{10}\,dx$

11. $\int t^3 e^{-t^4}\,dt$

12. $\int \sin t\sqrt{1 + \cos t}\,dt$

13. $\int \sin(\pi t/3)\,dt$

14. $\int \sec^2 2\theta\,d\theta$

15. $\int \frac{dx}{4x + 7}$

16. $\int y^2(4 - y^3)^{2/3}\,dy$

17. $\int \frac{\cos\theta}{1 + \sin\theta}\,d\theta$

18. $\int \frac{z^2}{z^3 + 1}\,dz$

19. $\int \cos^3\theta \sin\theta\,d\theta$

20. $\int e^{-5r}\,dr$

21. $\int \frac{e^u}{(1 - e^u)^2}\,du$

22. $\int \frac{\sin(1/x)}{x^2}\,dx$

23. $\int \frac{a + bx^2}{\sqrt{3ax + bx^3}}\,dx$

24. $\int \frac{t + 1}{3t^2 + 6t - 5}\,dt$

25. $\int \frac{(\ln x)^2}{x}\,dx$

26. $\int \sin x \sin(\cos x)\,dx$

27. $\int \sec^2\theta \tan^3\theta \, d\theta$

28. $\int x\sqrt{x+2}\, dx$

29. $\int \left(x - \frac{1}{x^2}\right)\left(x^2 + \frac{2}{x}\right)^5 dx$

30. $\int \frac{dx}{ax+b} \quad (a \neq 0)$

31. $\int e^r(2+3e^r)^{3/2}\, dr$

32. $\int \frac{e^{\arcsin x}}{\sqrt{1-x^2}}\, dx$

33. $\int \frac{\sec^2\theta}{\tan\theta}\, d\theta$

34. $\int \frac{\sec^2 x}{\tan^2 x}\, dx$

35. $\int \frac{(\arctan x)^2}{x^2+1}\, dx$

36. $\int \frac{1}{(x^2+1)\arctan x}\, dx$

37. $\int 5^t \sin(5^t)\, dt$

38. $\int \frac{\sin\theta\cos\theta}{1+\sin^2\theta}\, d\theta$

39. $\int \cos(1+5t)\, dt$

40. $\int \frac{\cos(\pi/x)}{x^2}\, dx$

41. $\int \sqrt{\cot x}\, \csc^2 x\, dx$

42. $\int \frac{2^t}{2^t+3}\, dt$

43. $\int \sinh^2 x \cosh x\, dx$

44. $\int \frac{dt}{\cos^2 t\sqrt{1+\tan t}}$

45. $\int \frac{\sin 2x}{1+\cos^2 x}\, dx$

46. $\int \frac{\sin x}{1+\cos^2 x}\, dx$

47. $\int \cot x\, dx$

48. $\int \frac{\cos(\ln t)}{t}\, dt$

49. $\int \frac{dx}{\sqrt{1-x^2}\,\sin^{-1}x}$

50. $\int \frac{x}{1+x^4}\, dx$

51. $\int \frac{1+x}{1+x^2}\, dx$

52. $\int x^2\sqrt{2+x}\, dx$

53. $\int x(2x+5)^8\, dx$

54. $\int x^3\sqrt{x^2+1}\, dx$

55–58 Evaluate the indefinite integral. Illustrate and check that your answer is reasonable by graphing both the function and its antiderivative (take $C = 0$).

55. $\int x(x^2-1)^3\, dx$

56. $\int \tan^2\theta \sec^2\theta\, d\theta$

57. $\int e^{\cos x}\sin x\, dx$

58. $\int \sin x \cos^4 x\, dx$

59–80 Evaluate the definite integral.

59. $\int_0^1 \cos(\pi t/2)\, dt$

60. $\int_0^1 (3t-1)^{50}\, dt$

61. $\int_0^1 \sqrt[3]{1+7x}\, dx$

62. $\int_{\pi/3}^{2\pi/3} \csc^2\left(\tfrac{1}{2}t\right) dt$

63. $\int_0^{\pi/6} \frac{\sin t}{\cos^2 t}\, dt$

64. $\int_1^4 \frac{\sqrt{2+\sqrt{x}}}{\sqrt{x}}\, dx$

65. $\int_1^2 \frac{e^{1/x}}{x^2}\, dx$

66. $\int_0^1 \frac{e^x}{1+e^{2x}}\, dx$

67. $\int_{-\pi/4}^{\pi/4} (x^3 + x^4\tan x)\, dx$

68. $\int_0^{\pi/2} \cos x \sin(\sin x)\, dx$

69. $\int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}}$

70. $\int_0^a x\sqrt{a^2-x^2}\, dx$

71. $\int_0^a x\sqrt{x^2+a^2}\, dx \quad (a > 0)$

72. $\int_{-\pi/3}^{\pi/3} x^4 \sin x\, dx$

73. $\int_1^2 x\sqrt{x-1}\, dx$

74. $\int_0^4 \frac{x}{\sqrt{1+2x}}\, dx$

75. $\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

76. $\int_0^2 (x-1)e^{(x-1)^2}\, dx$

77. $\int_0^1 \frac{e^z+1}{e^z+z}\, dz$

78. $\int_1^4 \frac{1}{(x+1)\sqrt{x}}\, dx$

79. $\int_0^1 \frac{dx}{(1+\sqrt{x})^4}$

80. $\int_1^{16} \frac{x^{1/2}}{1+x^{3/4}}\, dx$

81–82 Use a graph to give a rough estimate of the area of the region that lies under the given curve. Then find the exact area.

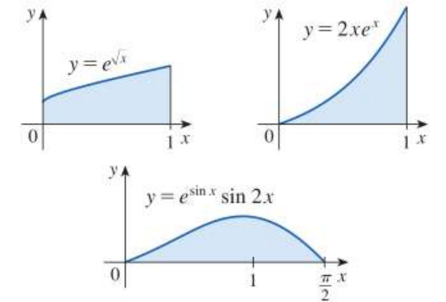
81. $y = \sqrt{2x+1}, \quad 0 \leqslant x \leqslant 1$

82. $y = 2\sin x - \sin 2x, \quad 0 \leqslant x \leqslant \pi$

83. Evaluate $\int_{-2}^{2} (x+3)\sqrt{4-x^2}\, dx$ by writing it as a sum of two integrals and interpreting one of those integrals in terms of an area.

84. Evaluate $\int_0^1 x\sqrt{1-x^4}\, dx$ by making a substitution and interpreting the resulting integral in terms of an area.

85. Which of the following areas are equal? Why?



86. A model for the basal metabolism rate, in kcal/h, of a young man is $R(t) = 85 - 0.18\cos(\pi t/12)$, where t is the time in hours measured from 5:00 AM. What is the total basal metabolism of this man, $\int_0^{24} R(t)\,dt$, over a 24-hour time period?

87. An oil storage tank ruptures at time $t = 0$ and oil leaks from the tank at a rate of $r(t) = 100e^{-0.01t}$ liters per minute. How much oil leaks out during the first hour?

88. A bacteria population starts with 400 bacteria and grows at a rate of $r(t) = (450.268)e^{1.12567t}$ bacteria per hour. How many bacteria will there be after three hours?

89. Breathing is cyclic and a full respiratory cycle from the beginning of inhalation to the end of exhalation takes about 5 seconds. The maximum rate of air flow into the lungs is about 0.5 L/s. This explains, in part, why the function $f(t) = \frac{1}{2}\sin(2\pi t/5)$ has often been used to model the rate of air flow into the lungs. Use this model to find the volume of inhaled air in the lungs at time t.

90. The rate of growth of a fish population was modeled by the equation

$$G(t) = \frac{60{,}000e^{-0.6t}}{(1 + 5e^{-0.6t})^2}$$

where t is the number of years since 2000 and G is measured in kilograms per year. If the biomass was 25,000 kg in the year 2000, what is the predicted biomass for the year 2020?

91. Dialysis treatment removes urea and other waste products from a patient's blood by diverting some of the bloodflow externally through a machine called a dialyzer. The rate at which urea is removed from the blood (in mg/min) is often well described by the equation

$$u(t) = \frac{r}{V} C_0 e^{-rt/V}$$

where r is the rate of flow of blood through the dialyzer (in mL/min), V is the volume of the patient's blood (in mL), and C_0 is the amount of urea in the blood (in mg) at time $t = 0$. Evaluate the integral $\int_0^{30} u(t)\,dt$ and interpret it.

92. Alabama Instruments Company has set up a production line to manufacture a new calculator. The rate of production of these calculators after t weeks is

$$\frac{dx}{dt} = 5000\left(1 - \frac{100}{(t + 10)^2}\right) \text{ calculators/week}$$

(Notice that production approaches 5000 per week as time goes on, but the initial production is lower because of the workers' unfamiliarity with the new techniques.) Find the number of calculators produced from the beginning of the third week to the end of the fourth week.

93. If f is continuous and $\int_0^4 f(x)\,dx = 10$, find $\int_0^2 f(2x)\,dx$.

94. If f is continuous and $\int_0^9 f(x)\,dx = 4$, find $\int_0^3 xf(x^2)\,dx$.

95. If f is continuous on $\mathbb{R}$, prove that

$$\int_a^b f(-x)\,dx = \int_{-b}^{-a} f(x)\,dx$$

For the case where $f(x) \geqslant 0$ and $0 < a < b$, draw a diagram to interpret this equation geometrically as an equality of areas.

96. If f is continuous on $\mathbb{R}$, prove that

$$\int_a^b f(x + c)\,dx = \int_{a+c}^{b+c} f(x)\,dx$$

For the case where $f(x) \geqslant 0$, draw a diagram to interpret this equation geometrically as an equality of areas.

97. If a and b are positive numbers, show that

$$\int_0^1 x^a(1 - x)^b\,dx = \int_0^1 x^b(1 - x)^a\,dx$$

98. If f is continuous on $[0, \pi]$, use the substitution $u = \pi - x$ to show that

$$\int_0^\pi xf(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$$

99. Use Exercise 98 to evaluate the integral

$$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x}\,dx$$

100. (a) If f is continuous, prove that

$$\int_0^{\pi/2} f(\cos x)\,dx = \int_0^{\pi/2} f(\sin x)\,dx$$

(b) Use part (a) to evaluate

$$\int_0^{\pi/2} \cos^2 x\,dx \quad \text{and} \quad \int_0^{\pi/2} \sin^2 x\,dx$$

5 REVIEW

CONCEPT CHECK

Answers to the Concept Check are available at StewartCalculus.com.

1. (a) Write an expression for a Riemann sum of a function f. Explain the meaning of the notation that you use.
(b) If $f(x) \geqslant 0$, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.
(c) If $f(x)$ takes on both positive and negative values, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.

2. (a) Write the definition of the definite integral of a continuous function from a to b.
(b) What is the geometric interpretation of $\int_a^b f(x)\,dx$ if $f(x) \geqslant 0$?
(c) What is the geometric interpretation of $\int_a^b f(x)\,dx$ if $f(x)$ takes on both positive and negative values? Illustrate with a diagram.

3. State the Midpoint Rule.

4. State both parts of the Fundamental Theorem of Calculus.

5. (a) State the Net Change Theorem.
(b) If $r(t)$ is the rate at which water flows into a reservoir, what does $\int_{t_1}^{t_2} r(t)\,dt$ represent?

6. Suppose a particle moves back and forth along a straight line with velocity $v(t)$, measured in feet per second, and acceleration $a(t)$.
(a) What is the meaning of $\int_{60}^{120} v(t)\,dt$?
(b) What is the meaning of $\int_{60}^{120} |\,v(t)\,|\,dt$?
(c) What is the meaning of $\int_{60}^{120} a(t)\,dt$?

7. (a) Explain the meaning of the indefinite integral $\int f(x)\,dx$.
(b) What is the connection between the definite integral $\int_a^b f(x)\,dx$ and the indefinite integral $\int f(x)\,dx$?

8. Explain exactly what is meant by the statement that "differentiation and integration are inverse processes."

9. State the Substitution Rule. In practice, how do you use it?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If f and g are continuous on $[a, b]$, then

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

2. If f and g are continuous on $[a, b]$, then

$$\int_a^b [f(x)g(x)]\,dx = \left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$$

3. If f is continuous on $[a, b]$, then

$$\int_a^b 5f(x)\,dx = 5\int_a^b f(x)\,dx$$

4. If f is continuous on $[a, b]$, then

$$\int_a^b xf(x)\,dx = x\int_a^b f(x)\,dx$$

5. If f is continuous on $[a, b]$ and $f(x) \geqslant 0$, then

$$\int_a^b \sqrt{f(x)}\,dx = \sqrt{\int_a^b f(x)\,dx}$$

6. $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(z)\,dz$

7. If f' is continuous on $[1, 3]$, then $\displaystyle\int_1^3 f'(v)\,dv = f(3) - f(1)$.

8. If $v(t)$ is the velocity at time t of a particle moving along a line, then $\int_a^b v(t)\,dt$ is the distance traveled during the time period $a \leqslant t \leqslant b$.

9. $\displaystyle\int_a^b f'(x)\,[f(x)]^4\,dx = \tfrac{1}{5}[f(x)]^5 + C$

10. If f and g are differentiable and $f(x) \geqslant g(x)$ for $a < x < b$, then $f'(x) \geqslant g'(x)$ for $a < x < b$.

11. If f and g are continuous and $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then

$$\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$$

12. $\displaystyle\int_{-5}^{5} (ax^2 + bx + c)\,dx = 2\int_0^5 (ax^2 + c)\,dx$

13. All continuous functions have derivatives.

14. All continuous functions have antiderivatives.

15. $\displaystyle\int_0^3 e^{x^2}\,dx = \int_0^5 e^{x^2}\,dx + \int_5^3 e^{x^2}\,dx$

16. If $\int_0^1 f(x)\,dx = 0$, then $f(x) = 0$ for $0 \leqslant x \leqslant 1$.

17. If f is continuous on $[a, b]$, then

$$\frac{d}{dx}\left(\int_a^b f(x)\,dx\right) = f(x)$$

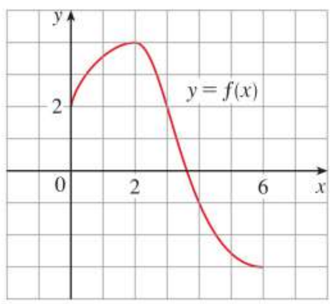
18. $\int_0^2 (x - x^3)\,dx$ represents the area under the curve $y = x - x^3$ from 0 to 2.

19. $\displaystyle\int_{-2}^{1} \frac{1}{x^4}\,dx = -\frac{3}{8}$

20. If f has a discontinuity at 0, then $\displaystyle\int_{-1}^{1} f(x)\,dx$ does not exist.

EXERCISES

1. Use the given graph of f to find the Riemann sum with six subintervals. Take the sample points to be (a) left endpoints and (b) midpoints. In each case draw a diagram and explain what the Riemann sum represents.



2. (a) Evaluate the Riemann sum for

$$f(x) = x^2 - x \qquad 0 \leq x \leq 2$$

with four subintervals, taking the sample points to be right endpoints. Explain, with the aid of a diagram, what the Riemann sum represents.

(b) Use the definition of a definite integral (with right endpoints) to calculate the value of the integral

$$\int_0^2 (x^2 - x)\,dx$$

(c) Use the Fundamental Theorem to check your answer to part (b).

(d) Draw a diagram to explain the geometric meaning of the integral in part (b).

3. Evaluate

$$\int_0^1 \left(x + \sqrt{1 - x^2}\right) dx$$

by interpreting it in terms of areas.

4. Express

$$\lim_{n \to \infty} \sum_{i=1}^{n} \sin x_i \, \Delta x$$

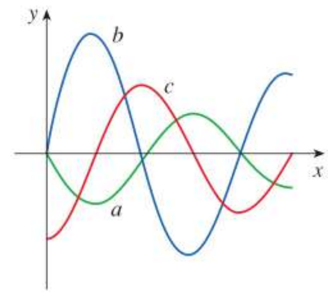
as a definite integral on the interval $[0, \pi]$ and then evaluate the integral.

5. If $\int_0^6 f(x)\,dx = 10$ and $\int_0^4 f(x)\,dx = 7$, find $\int_4^6 f(x)\,dx$.

T **6.** (a) Write $\int_1^5 (x + 2x^5)\,dx$ as a limit of Riemann sums, taking the sample points to be right endpoints. Use a computer algebra system to evaluate the sum and to compute the limit.

(b) Use the Fundamental Theorem to check your answer to part (a).

7. The figure shows the graphs of f, f', and $\int_0^x f(t)\,dt$. Identify each graph, and explain your choices.

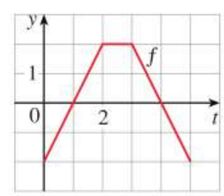


8. Evaluate:

(a) $\displaystyle\int_0^1 \frac{d}{dx}\left(e^{\arctan x}\right) dx$ (b) $\displaystyle\frac{d}{dx}\int_0^1 e^{\arctan x}\,dx$

(c) $\displaystyle\frac{d}{dx}\int_0^x e^{\arctan t}\,dt$

9. The graph of f consists of the three line segments shown. If $g(x) = \int_0^x f(t)\,dt$, find $g(4)$ and $g'(4)$.



10. If f is the function in Exercise 9, find $g''(4)$.

11–42 Evaluate the integral, if it exists.

11. $\displaystyle\int_{-1}^{0} (x^2 + 5x)\,dx$ **12.** $\displaystyle\int_0^T (x^4 - 8x + 7)\,dx$

13. $\displaystyle\int_0^1 (1 - x^9)\,dx$ **14.** $\displaystyle\int_0^1 (1 - x)^9\,dx$

15. $\displaystyle\int_1^9 \frac{\sqrt{u} - 2u^2}{u}\,du$ **16.** $\displaystyle\int_0^1 \left(\sqrt[4]{u} + 1\right)^2 du$

17. $\displaystyle\int_0^1 y(y^2 + 1)^5\,dy$ **18.** $\displaystyle\int_0^2 y^2\sqrt{1 + y^3}\,dy$

19. $\displaystyle\int_1^5 \frac{dt}{(t - 4)^2}$ **20.** $\displaystyle\int_0^1 \sin(3\pi t)\,dt$

21. $\displaystyle\int_0^1 v^2 \cos(v^3)\,dv$ **22.** $\displaystyle\int_{-1}^{1} \frac{\sin x}{1 + x^2}\,dx$

23. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{t^4 \tan t}{2 + \cos t}\,dt$ **24.** $\displaystyle\int_{-2}^{-1} \frac{z^2 + 1}{z}\,dz$

25. $\displaystyle\int \frac{x}{x^2 + 1}\,dx$ **26.** $\displaystyle\int \frac{dx}{x^2 + 1}$

27. $\displaystyle\int \frac{x+2}{\sqrt{x^2+4x}}\,dx$

28. $\displaystyle\int \frac{\csc^2 x}{1+\cot x}\,dx$

29. $\displaystyle\int \sin \pi t \cos \pi t\,dt$

30. $\displaystyle\int \sin x \cos(\cos x)\,dx$

31. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

32. $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx$

33. $\displaystyle\int \tan x \ln(\cos x)\,dx$

34. $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$

35. $\displaystyle\int \frac{x^3}{1+x^4}\,dx$

36. $\displaystyle\int \sinh(1+4x)\,dx$

37. $\displaystyle\int \frac{\sec\theta\tan\theta}{1+\sec\theta}\,d\theta$

38. $\displaystyle\int_0^{\pi/4} (1+\tan t)^3 \sec^2 t\,dt$

39. $\displaystyle\int x(1-x)^{2/3}\,dx$

40. $\displaystyle\int \frac{x}{x-3}\,dx$

41. $\displaystyle\int_0^3 |x^2-4|\,dx$

42. $\displaystyle\int_0^4 |\sqrt{x}-1|\,dx$

43–44 Evaluate the indefinite integral. Illustrate and check that your answer is reasonable by graphing both the function and its antiderivative (take $C = 0$).

43. $\displaystyle\int \frac{\cos x}{\sqrt{1+\sin x}}\,dx$

44. $\displaystyle\int \frac{x^3}{\sqrt{x^2+1}}\,dx$

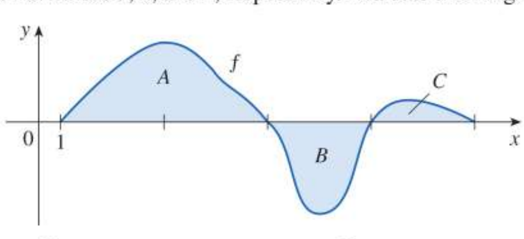
45. Use a graph to give a rough estimate of the area of the region that lies under the curve $y = x\sqrt{x}$, $0 \leqslant x \leqslant 4$. Then find the exact area.

46. Graph the function $f(x) = \cos^2 x \sin x$ and use the graph to guess the value of the integral $\int_0^{2\pi} f(x)\,dx$. Then evaluate the integral to confirm your guess.

47. Find the area under the graph of $y = x^2 + 5$ and above the x-axis, between $x = 0$ and $x = 4$.

48. Find the area under the graph of $y = \sin x$ and above the x-axis, between $x = 0$ and $x = \pi/2$.

49–50 The regions A, B, and C bounded by the graph of f and the x-axis have areas 3, 2, and 1, respectively. Evaluate the integral.



49. (a) $\displaystyle\int_1^5 f(x)\,dx$ (b) $\displaystyle\int_1^5 |f(x)|\,dx$

50. (a) $\displaystyle\int_1^4 f(x)\,dx + \int_3^5 f(x)\,dx$ (b) $\displaystyle\int_1^3 2f(x)\,dx + \int_3^5 6f(x)\,dx$

51–56 Find the derivative of the function.

51. $\displaystyle F(x) = \int_0^x \frac{t^2}{1+t^3}\,dt$

52. $\displaystyle F(x) = \int_x^1 \sqrt{t+\sin t}\,dt$

53. $\displaystyle g(x) = \int_0^{x^4} \cos(t^2)\,dt$

54. $\displaystyle g(x) = \int_1^{\sin x} \frac{1-t^2}{1+t^4}\,dt$

55. $\displaystyle y = \int_{\sqrt{x}}^x \frac{e^t}{t}\,dt$

56. $\displaystyle y = \int_{2x}^{3x+1} \sin(t^4)\,dt$

57–58 Use Property 8 of integrals to estimate the value of the integral.

57. $\displaystyle\int_1^3 \sqrt{x^2+3}\,dx$

58. $\displaystyle\int_2^4 \frac{1}{x^3+2}\,dx$

59–62 Use the properties of integrals to verify the inequality.

59. $\displaystyle\int_0^1 x^2 \cos x\,dx \leqslant \frac{1}{3}$

60. $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\sin x}{x}\,dx \leqslant \frac{\sqrt{2}}{2}$

61. $\displaystyle\int_0^1 e^x \cos x\,dx \leqslant e - 1$

62. $\displaystyle\int_0^1 x \sin^{-1} x\,dx \leqslant \pi/4$

63. Use the Midpoint Rule with $n = 6$ to approximate $\int_0^3 \sin(x^3)\,dx$. Round to four decimal places.

64. A particle moves along a line with velocity function $v(t) = t^2 - t$, where v is measured in meters per second. Find (a) the displacement and (b) the distance traveled by the particle during the time interval $[0, 5]$.

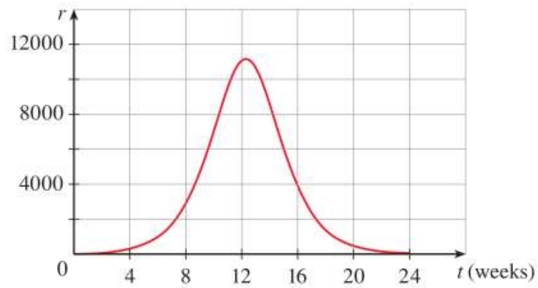
65. Let $r(t)$ be the rate at which the world's oil is consumed, where t is measured in years starting at $t = 0$ on January 1, 2000, and $r(t)$ is measured in barrels per year. What does $\int_{15}^{20} r(t)\,dt$ represent?

66. A radar gun was used to record the speed of a runner at the times given in the table. Use the Midpoint Rule to estimate the distance the runner covered during those 5 seconds.

t (s)	v (m/s)	t (s)	v (m/s)
0	0	3.0	10.51
0.5	4.67	3.5	10.67
1.0	7.34	4.0	10.76
1.5	8.86	4.5	10.81
2.0	9.73	5.0	10.81
2.5	10.22		

67. A population of honeybees increased at a rate of $r(t)$ bees per week, where the graph of r is as shown. Use the

Midpoint Rule with six subintervals to estimate the increase in the bee population during the first 24 weeks.



68. Let

$$f(x) = \begin{cases} -x - 1 & \text{if } -3 \leqslant x \leqslant 0 \\ -\sqrt{1 - x^2} & \text{if } 0 \leqslant x \leqslant 1 \end{cases}$$

Evaluate $\int_{-3}^{1} f(x)\,dx$ by interpreting the integral as a difference of areas.

69. If f is continuous and $\int_0^2 f(x)\,dx = 6$, evaluate

$$\int_0^{\pi/2} f(2 \sin \theta) \cos \theta \, d\theta$$

70. The Fresnel function $S(x) = \int_0^x \sin\left(\frac{1}{2}\pi t^2\right) dt$ was introduced in Section 5.3. Fresnel also used the function

$$C(x) = \int_0^x \cos\left(\tfrac{1}{2}\pi t^2\right) dt$$

in his theory of the diffraction of light waves.

(a) On what intervals is C increasing?

(b) On what intervals is C concave upward?

(c) Use a graph to solve the following equation correct to two decimal places:

$$\int_0^x \cos\left(\tfrac{1}{2}\pi t^2\right) dt = 0.7$$

(d) Plot the graphs of C and S on the same screen. How are these graphs related?

71. Estimate the value of the number c such that the area under the curve $y = \sinh cx$ between $x = 0$ and $x = 1$ is equal to 1.

72. Suppose that the temperature in a long, thin rod placed along the x-axis is initially $C/(2a)$ if $|x| \leqslant a$ and 0 if $|x| > a$. It can be shown that if the heat diffusivity of the rod is k, then the temperature of the rod at the point x at time t is

$$T(x, t) = \frac{C}{a\sqrt{4\pi kt}} \int_0^a e^{-(x-u)^2/(4kt)}\, du$$

To find the temperature distribution that results from an initial hot spot concentrated at the origin, we need to compute $\lim_{a \to 0} T(x, t)$. Use l'Hospital's Rule to find this limit.

73. If f is a continuous function such that

$$\int_1^x f(t)\,dt = (x - 1)e^{2x} + \int_1^x e^{-t} f(t)\,dt$$

for all x, find an explicit formula for $f(x)$.

74. Suppose h is a function such that $h(1) = -2$, $h'(1) = 2$, $h''(1) = 3$, $h(2) = 6$, $h'(2) = 5$, $h''(2) = 13$, and h'' is continuous everywhere. Evaluate $\int_1^2 h''(u)\,du$.

75. If f' is continuous on $[a, b]$, show that

$$2\int_a^b f(x) f'(x)\,dx = [f(b)]^2 - [f(a)]^2$$

76. Find

$$\lim_{h \to 0} \frac{1}{h} \int_2^{2+h} \sqrt{1 + t^3}\, dt$$

77. If f is continuous on $[0, 1]$, prove that

$$\int_0^1 f(x)\,dx = \int_0^1 f(1 - x)\,dx$$

78. Evaluate

$$\lim_{n \to \infty} \frac{1}{n} \left[\left(\frac{1}{n}\right)^9 + \left(\frac{2}{n}\right)^9 + \left(\frac{3}{n}\right)^9 + \cdots + \left(\frac{n}{n}\right)^9 \right]$$

Problems Plus

Before you look at the solution of the following example, cover it up and first try to solve the problem yourself.

EXAMPLE Evaluate $\displaystyle\lim_{x\to 3}\left(\frac{x}{x-3}\int_3^x \frac{\sin t}{t}\,dt\right)$.

SOLUTION Let's start by having a preliminary look at the ingredients of the function. What happens to the first factor, $x/(x-3)$, when x approaches 3? The numerator approaches 3 and the denominator approaches 0, so we have

$$\frac{x}{x-3}\to\infty \quad \text{as} \quad x\to 3^+ \qquad \text{and} \qquad \frac{x}{x-3}\to -\infty \quad \text{as} \quad x\to 3^-$$

The second factor approaches $\int_3^3 (\sin t)/t\,dt$, which is 0. It's not clear what happens to the function as a whole. (One factor is becoming large while the other is becoming small.) So how do we proceed?

PS Review the Principles of Problem Solving following Chapter 1.

One of the principles of problem solving is *recognizing something familiar.* Is there a part of the function that reminds us of something we've seen before? Well, the integral

$$\int_3^x \frac{\sin t}{t}\,dt$$

has x as its upper limit of integration and that type of integral occurs in Part 1 of the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

This suggests that differentiation might be involved.

Once we start thinking about differentiation, the denominator $(x-3)$ reminds us of something else that should be familiar: one of the forms of the definition of the derivative in Chapter 2 is

$$F'(a) = \lim_{x\to a}\frac{F(x)-F(a)}{x-a}$$

and with $a = 3$ this becomes

$$F'(3) = \lim_{x\to 3}\frac{F(x)-F(3)}{x-3}$$

So what is the function F in our situation? Notice that if we define

$$F(x) = \int_3^x \frac{\sin t}{t}\,dt$$

then $F(3) = 0$. What about the factor x in the numerator? That's just a red herring, so let's factor it out and put together the calculation:

Another approach is to use l'Hospital's Rule.

$$\lim_{x\to 3}\left(\frac{x}{x-3}\int_3^x \frac{\sin t}{t}\,dt\right) = \lim_{x\to 3} x\cdot\lim_{x\to 3}\frac{\displaystyle\int_3^x \frac{\sin t}{t}\,dt}{x-3} = 3\lim_{x\to 3}\frac{F(x)-F(3)}{x-3}$$

$$= 3F'(3) = 3\,\frac{\sin 3}{3} = \sin 3 \qquad \text{(FTC1)}$$

Problems

1. If $x \sin \pi x = \int_0^{x^2} f(t)\,dt$, where f is a continuous function, find $f(4)$.

2. Suppose f is continuous, $f(0) = 0$, $f(1) = 1$, $f'(x) > 0$, and $\int_0^1 f(x)\,dx = \frac{1}{3}$. Find the value of the integral $\int_0^1 f^{-1}(y)\,dy$.

3. If $\int_0^4 e^{(x-2)^4}\,dx = k$, find the value of $\int_0^4 x e^{(x-2)^4}\,dx$.

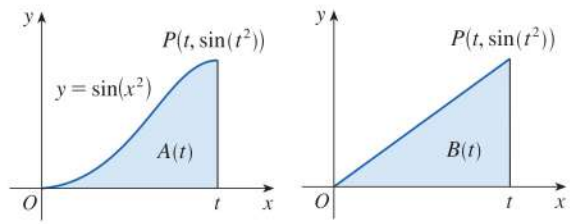
4. (a) Graph several members of the family of functions $f(x) = (2cx - x^2)/c^3$ for $c > 0$ and look at the regions enclosed by these curves and the x-axis. Make a conjecture about how the areas of these regions are related.
   (b) Prove your conjecture in part (a).
   (c) Take another look at the graphs in part (a) and use them to sketch the curve traced out by the vertices (highest points) of the family of functions. Can you guess what kind of curve this is?
   (d) Find an equation of the curve you sketched in part (c).

5. If $f(x) = \int_0^{g(x)} \frac{1}{\sqrt{1+t^3}}\,dt$, where $g(x) = \int_0^{\cos x} [1 + \sin(t^2)]\,dt$, find $f'(\pi/2)$.

6. If $f(x) = \int_0^x x^2 \sin(t^2)\,dt$, find $f'(x)$.

7. Evaluate $\lim_{x\to 0} (1/x) \int_0^x (1 - \tan 2t)^{1/t}\,dt$. [Assume that the integrand is defined and continuous at $t = 0$; see Exercise 5.3.82.]

8. The figure shows two regions in the first quadrant: $A(t)$ is the area under the curve $y = \sin(x^2)$ from 0 to t, and $B(t)$ is the area of the triangle with vertices O, P, and $(t, 0)$. Find $\lim_{t\to 0^+} [A(t)/B(t)]$.



9. Find the interval $[a, b]$ for which the value of the integral $\int_a^b (2 + x - x^2)\,dx$ is a maximum.

10. Use an integral to estimate the sum $\sum_{i=1}^{10{,}000} \sqrt{i}$.

11. (a) Evaluate $\int_0^n [\![x]\!]\,dx$, where n is a positive integer.
    (b) Evaluate $\int_a^b [\![x]\!]\,dx$, where a and b are real numbers with $0 \leqslant a < b$.

12. Find $\frac{d^2}{dx^2} \int_0^x \left(\int_1^{\sin t} \sqrt{1 + u^4}\,du \right) dt$.

13. Suppose the coefficients of the cubic polynomial $P(x) = a + bx + cx^2 + dx^3$ satisfy the equation

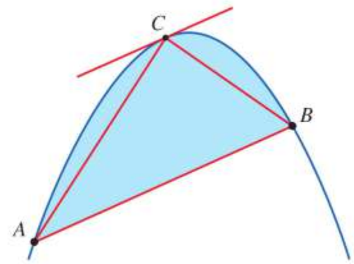
$$a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4} = 0$$

Show that the equation $P(x) = 0$ has a solution between 0 and 1. Can you generalize this result for an nth-degree polynomial?

14. A circular disk of radius r is used in an evaporator and is rotated in a vertical plane. If it is to be partially submerged in the liquid so as to maximize the exposed wetted area of the disk, show that the center of the disk should be positioned at a height $r/\sqrt{1 + \pi^2}$ above the surface of the liquid.

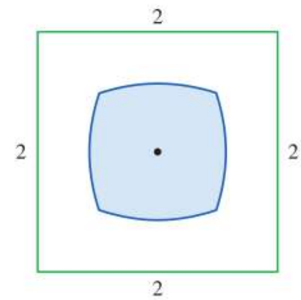
15. Prove that if f is continuous, then $\int_0^x f(u)(x - u)\,du = \int_0^x \left(\int_0^u f(t)\,dt \right) du$.

16. The figure shows a parabolic segment, that is, a portion of a parabola cut off by a chord AB. It also shows a point C on the parabola with the property that the tangent line at C is parallel to the chord AB. Archimedes proved that the area of the parabolic segment is $\frac{4}{3}$ times the area of the inscribed triangle ABC. Verify Archimedes' result for the parabola $y = 4 - x^2$ and the line $y = x + 2$.



17. Given the point (a, b) in the first quadrant, find the downward-opening parabola that passes through the point (a, b) and the origin such that the area under the parabola is a minimum.

18. The figure shows a region consisting of all points inside a square that are closer to the center than to the sides of the square. Find the area of the region.



19. Evaluate

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n}\,\sqrt{n+1}} + \frac{1}{\sqrt{n}\,\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{n}\,\sqrt{n+n}} \right)$$

20. For any number c, we let $f_c(x)$ be the smaller of the two numbers $(x - c)^2$ and $(x - c - 2)^2$. Then we define $g(c) = \int_0^1 f_c(x)\,dx$. Find the maximum and minimum values of $g(c)$ if $-2 \leqslant c \leqslant 2$.