November 6

We consider some nonlinear systems of differential equations.

Recall we could convert a second order linear equation into a system of two first-order equations by introducing new variables.

Similarly, if we have a nonlinear equation like

$$y'' + (y')^3 + y = 0$$

we introduce $x_1(t) = y(t)$ and $x_2(t) = y'(t)$. Then

$$x_1' = y' = x_2$$

and

$$x_2' = y'' = -y - (y')^3 = -x_1 - x_2^2$$

so we have a nonlinear two-dimensional system

$$\begin{cases} x_1' = x_2 \\ x_2' = -x_1 - x_2^3 \end{cases}$$

If we want to write this as a 'vector equation', we would have

$$\vec{x}'(t) = F(\vec{x})$$

with the multivariable vector function $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F(x_1, x_2) = \begin{pmatrix} F_1(x_1, x_2) \\ F_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 - x_2^3 \end{pmatrix}$$

Notice this has a single equilibrium point, at $x_1 = x_2 = 0$. If we compute the Jacobian of F, we get

$$Jac_F = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -3x_2^2 \end{pmatrix}$$

Evaluating at $(x_1, x_2) = (0, 0)$, we get the matrix $\begin{pmatrix} 0 & 1 \\ -1 & -3x_2^2 \end{pmatrix}$. This is the **linearization** of the system at this equilibrium point, i.e. what we get when we ignore the higher-order term $-x_2^3$.

The trajectories of this linearization would be closed circles orbiting clockwise with a constant radius. What do trajectories of the nonlinear system look like?

Consider polar coordinates: if

$$r(t) = \sqrt{x_1(t)^2 + x_2(t)^2},$$

then

$$\begin{aligned}
\frac{d}{dt}r(t) &= \frac{d}{dt}\left(\sqrt{x_1(t)^2 + x_2(t)^2}\right) \\
&= \frac{1}{2}\frac{1}{\sqrt{x_1(t)^2 + x_2(t)^2}} \cdot (2x_1(t) \cdot x_1'(t) + 2x_2(t) \cdot x_2'(t)) \\
&= \frac{1}{\sqrt{x_1(t)^2 + x_2(t)^2}} \cdot (x_1(t) \cdot x_1'(t) + x_2(t) \cdot x_2'(t))
\end{aligned}$$

or, more compactly,

$$r' = \frac{1}{\sqrt{x_1^2 + x_2^2}} \cdot (x_1 x_1' + x_2 x_2')$$

For our system above, this gives

$$\begin{aligned}
\frac{dr}{dt} &= \frac{1}{\sqrt{x_1^2 + x_2^2}} \cdot \left(x_1 x_2 + x_2(-x_1 - x_2^3)\right) \\
&= \frac{1}{\sqrt{x_1^2 + x_2^2}} \cdot \left(x_1 x_2 - x_2 x_1 - x_2^4\right) \\
&= \frac{-x_2^4}{\sqrt{x_1^2 + x_2^2}}
\end{aligned}$$

Notice that $\frac{dr}{dt}$ is always non-positive.

Now, consider the angular component:

$$\theta(t) = \arctan(\frac{x_2(t)}{x_1(t)})$$

Differentiating gives

$$\begin{aligned}
\frac{d}{dt}\theta(t) &= \frac{d}{dt}\arctan\left(\frac{x_2(t)}{x_1(t)}\right) \\
&= \frac{1}{1+(\frac{x_2(t)}{x_1(t)})^2} \cdot \frac{d}{dt}\left(\frac{x_2(t)}{x_1(t)}\right) \\
&= \frac{1}{1+(\frac{x_2(t)}{x_1(t)})^2} \cdot \left(\frac{x_1(t)x_2'(t) - x_2(t)x_1'(t)}{x_1(t)^2}\right) \\
&= \frac{1}{x_1(t)^2 + x_2(t)^2} \cdot (x_1(t)x_2'(t) - x_2(t)x_1'(t))
\end{aligned}$$

or, more compactly:

$$\frac{d\theta}{dt} = \frac{1}{x_1^2 + x_2^2} \cdot (x_1x_2' - x_2x_1')$$

In our system, this gives

$$\begin{aligned}
\frac{d\theta}{dt} &= \frac{1}{x_1^2 + x_2^2} \cdot (x_1x_2' - x_2x_1') \\
&= \frac{1}{x_1^2 + x_2^2} \cdot \left(x_1(-x_1 - x_2^3) - x_2x_2\right) \\
&= \frac{1}{x_1^2 + x_2^2} \cdot \left(-x_1^2 - x_1x_2^3 - x_2^2\right) \\
&= -1 - \frac{x_1x_2^3}{x_1^2 + x_2^2}
\end{aligned}$$

Notice that $|x_1| = \sqrt{x_1^2} \leq \sqrt{x_1^2 + x_2^2}$, and similarly for x_2, so the second term here has absolute value no bigger than $x_1^2 + x_2^2 = r^2$. Since we said before the radius is non-increasing, this term is bounded by the initial radius of our trajectory. This means that trajectories which start close enough to the origin have $\frac{d\theta}{dt} < -\frac{1}{2}$ (the $-\frac{1}{2}$ is not important, we just want it to be smaller than some definite negative number).

This means that trajectories are spiraling clockwise at a fairly constant rate, and the radius is overall decreasing (a trajectory cannot stay in the $x_2 = 0$ region since it is always spiraling).

Thus, the equilibrium point here is a stable spiral.

Exercise: check that the system associated to $y'' - (y')^3 + y = 0$ has an unstable spiral.

Another example: consider the system

$$y'' + (y')^2 + y = 0$$

The system associated with this by setting $x_1(t) = y(t)$ and $x_2(t) = y'(t)$ is

$$\begin{cases} x_1' = x_2 \\ x_2' = -x_1 - x_2^2 \end{cases}$$

Notice again the only equilibrium point is at the origin, and the linearization here is the same as the linearization above.

Again, we analyze this system in polar coordinates: we have

$$r' = \frac{1}{\sqrt{x_1^2 + x_2^2}} \cdot (x_1 x_1' + x_2 x_2')$$

and

$$\theta' = \frac{1}{x_1^2 + x_2^2} \cdot (x_1 x_2' - x_2 x_1')$$

For our system, this gives

$$\begin{aligned} r' &= \frac{1}{\sqrt{x_1^2 + x_2^2}} \cdot (x_1 x_1' + x_2 x_2') \\ &= \frac{1}{\sqrt{x_1^2 + x_2^2}} \cdot \big(x_1 x_2 + x_2(-x_1 - x_2^2)\big) \\ &= \frac{-x_2^3}{\sqrt{x_1^2 + x_2^2}} \end{aligned}$$

and

$$\begin{aligned} \theta' &= \frac{1}{x_1^2 + x_2^2} \cdot (x_1 x_2' - x_2 x_1') \\ &= \frac{1}{x_1^2 + x_2^2} \cdot \big(x_1(-x_1 - x_2^2) - x_2 x_2\big) \\ &= -1 - \frac{x_1 x_2^2}{x_1^2 + x_2^2} \end{aligned}$$

Unfortunately, the radius of trajectories in this system both increases and decreases, depending on location. However, the change in radius is bounded by approximately the radius squared. It is plausible that if we start a trajectory with some very small radius, say less than $\frac{1}{100}$, then for the time interval $0 \leq t \leq 2\pi$ the radius will grow no bigger than $\frac{1}{4}$.

(Formally, we want to analyze the differential ***inequalities***

$$r' \leq r^2$$

and

$$r' \geq -r^2$$

Separating variables for the first one gives

$$\frac{dr}{r^2} \leq dt$$

and integrating from $t = 0$ to $t = s$ gives

$$\frac{1}{r(0)} - \frac{1}{r(s)} \leq s$$

which rearranges to

$$r(s) \leq \frac{r(0)}{1 - s \cdot r(0)}$$

so, if $r(0)$ is small enough, $r(s) < \frac{1}{4}$ for $s \in [0, 2\pi]$. One should be very careful here, I am using the fact that the radius is always positive.)

Once we have established this, then analyzing the change in angle gives

$$\theta' \leq -\frac{1}{2}$$

since the term $\frac{x_1 x_2^2}{x_1^2 + x_2^2}$ is bounded in absolute value by the radius.

The point of all of this, valid on the time interval $t \in [0, 2\pi]$, is that the angle must change by $-\pi$ in this interval. Thus, if a trajectory starts at $\begin{pmatrix} a \\ 0 \end{pmatrix}$ for a very small and positive, the trajectory has to hit the negative x_1-axis at some point $t_* \in [0, 2\pi]$

Now, we pull a clever trick out of our hats: consider the change of variables $v_1(t) = x_1(-t)$ and $v_2(t) = -x_2(-t)$. This gives the system

$$\begin{aligned} v_1'(t) &= \frac{d}{dt} v_1(t) \\ &= \frac{d}{dt}\left(x_1(-t)\right) \\ &= x_1'(-t) \cdot \frac{d}{dt}(-t) \\ &= -x_2(-t) \\ &= v_2(t) \end{aligned}$$

and

$$\begin{aligned} v_2'(t) &= \frac{d}{dt} v_2(t) \\ &= \frac{d}{dt} - x_2(-t) \\ &= -\frac{d}{dt} x_2(-t) \\ &= -x_2'(-t)\frac{d}{dt}(-t) \\ &= -\left(-x_1(-t) - x_2(-t)^2\right) \cdot (-1) \\ &= -x_1(-t) - (-x_2(-t))^2 \\ &= -v_1(t) - v_2(t)^2 \end{aligned}$$

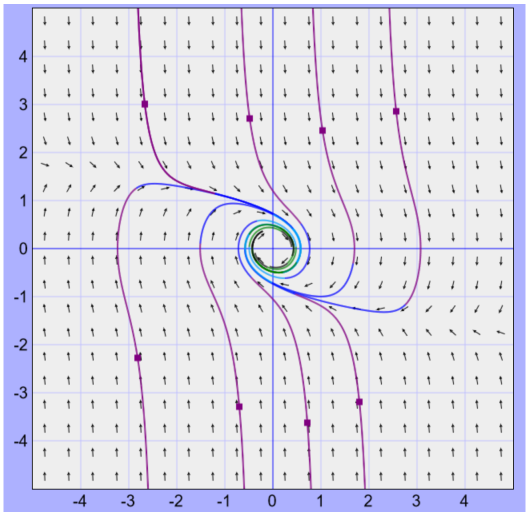
Notice this $v_1 - v - 2$ system is formally the same as the $x_1 - x_2$-system! In particular, all of our analysis above applies in exactly the same way, and a trajectory starting at $\begin{pmatrix} a \\ 0 \end{pmatrix}$ goes clockwise until time $t = t_*$, at which point it hits the negative v_1-axis. However, clockwise in the $v_1 - v_2$-plane corresponds to counterclockwise in the $x_1 - x_2$-plane, and going forward in time for the $v_1 - v_2$ system is the same as going backwards in time for the $x_1 - x_2$-system.

Thus, the trajectory starting at $\begin{pmatrix} a \\ 0 \end{pmatrix}$ is closed, and repeats itself every $2t_*$-length of time.

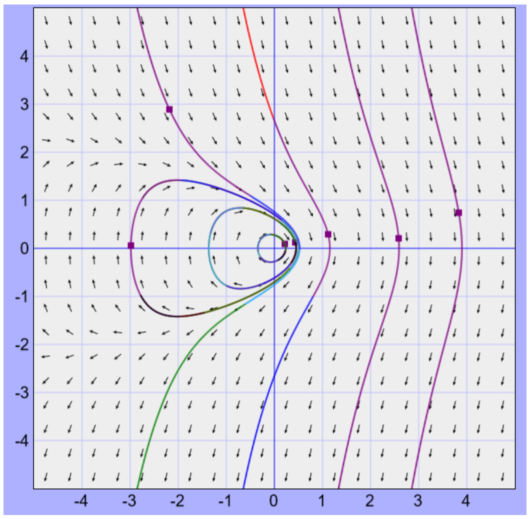
As a final interesting system, I showed

$$\begin{cases} x_1' = x_2 + x_1(x_1^2 + x_2^2)\sin(\frac{\pi}{\sqrt{x_1^2+x_2^2}}) \\ x_2' = -x_1 + x_2(x_1^2 + x_2^2)\sin(\frac{\pi}{\sqrt{x_1^2+x_2^2}}) \end{cases}$$

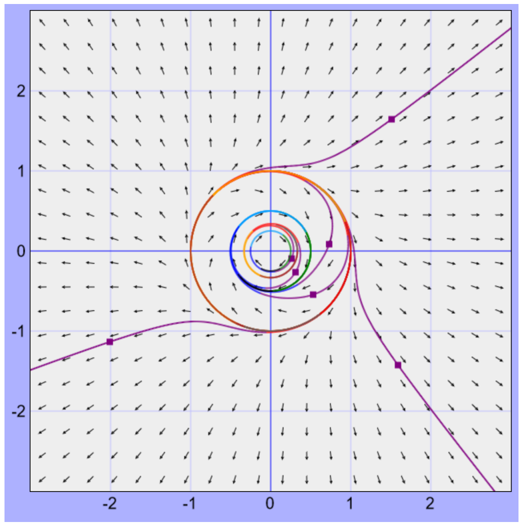
Here are some phase plots with some sample trajectories:



$$x_1' = x_2$$
$$x_2' = -x_1 - x_2^3$$



$$x_1' = x_2$$
$$x_2' = -x_1 - x_2^2$$



$$x_1' = x_2 + x_1(x_1^2 + x_2^2)\sin\left(\frac{\pi}{\sqrt{x_1^2 + x_2^2}}\right)$$
$$x_2' = -x_1 + x_2(x_1^2 + x_2^2)\sin\left(\frac{\pi}{\sqrt{x_1^2 + x_2^2}}\right)$$

Notice that each of these systems has the same linearization at the origin, but we get different behavior in the nonlinear system. In fact, this is only a problem for pure rotation, like in this example. If the linearization at an equilibrium point has eigenvalues with non-zero real parts, then the singularity type of the nonlinear system will be the same as that of the linear system.