Second Day

I started by listing a bunch of definitions, so I could say I said them.

Definition 1. *An* ***ordinary differential equation*** *(ODE for short) is an equation written in the variables x, y, y', ..., $y^{(n)}$ (stopping at some finite n).*

Unless otherwise noted, all differential equations in this course are ordinary differential equations.

Definition 2. *A* ***solution*** *to an ordinary differential equation is a function $y = y(x)$ such that the differential equation is true when evaluated at $y = y(x)$, $y' = \frac{d}{dx}y(x)$, ..., $y^{(n)} = \frac{d^n}{dx^n}y(x)$.*

i.e., plug in the function y and its derivatives to the equation.

Definition 3. *The* ***order*** *of a differential equation $F(x, y, y', ..., y^{(n)}) = 0$ is n, the largest derivative involved.*

So

$$xy' \cdot y - \cos(x)y^2 = 12x^3$$

is a first order differential equation, while

$$y''' - y' + 6yy' = 0$$

is third order (this is a simplified version of the Korteweg-de Vries (partial differential) equation, so it is an 'actual' equation)

For today, we restrict to first order equations.

On the first day, we discussed how the ODE

$$y' = y$$

has infinitely many solutions, all of the form

$$y(x) = ce^x$$

for some number $c \in \mathbb{R}$.

Notice none of the graphs of these functions intersect each other. If we want to pick out some particular solution, we can say the graph passes through the point $(0, c)$, or equivalently we can ask for a specific value of the function, say $y(0) = c$. There is nothing special about 0 here: if we ask for a solution of $y' = y$ which satisfies $y(2) = 5$, we can do some algebra to conclude that the only possible solution is $y(x) = \frac{5}{e^2}e^x$.

Definition 4. *An* ***initial condition****, or initial value, is a prescribed value of the unknown function in the differential equation which lets us pick out a specific solution.*

Recalling our example $y'' = 0$ from the first day, we see that giving a single value, say $y(0) = 2$, still allows for infinitely many solutions any $y(x) = cx+2$. We could give the value of y at a different point, or the value of y' at the same point (or I suppose a different point), and that would get us down to a single solution. Let's save this discussion for later.

Definition 5. *An* ***Initial Value Problem*** *(or IVP) is a differential equation with enough initial conditions to have a single solution.*

Usually people call the family of solutions to a differential equation the 'general solution' and the solution satisfying some particular initial conditions the 'particular solution' or 'specific solution'.

Definition 6. *A* ***linear*** *(n-th order) differential equation is one of the form*

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0(x)y = b(x)$$

for some functions $a_0(x), ..., a_n(x), b(x)$.

Definition 7. *A* ***homogeneous*** *linear differential equation is one where* $b(x) = 0$ *(so there are no 'only x' terms).*

Now we give a general solution to equations with all the adjectives.

Consider a first order homogeneous linear ODE:

$$a_1(x)y' + a_0(x)y = 0$$

Ignore what happens when $a_1(x) = 0$, and divide to get

$$y' + \frac{a_0(x)}{a_1(x)}y = 0,$$

and set $p(x) = \frac{a_0(x)}{a_1(x)}$, so

$$y' + p(x)y = 0$$

or

$$y' = -p(x)y$$

If the y term were not on the right, we would be able to integrate both sides and get a solution. Let's rearrange so that y is not on the right:

$$\frac{y'}{y} = -p(x)$$

Now we can integrate the right, but not the left. Or can we? Consider $u = \ln(y)$ and differentiate. We get

$$\begin{aligned} \frac{d}{dx}(u) &= \frac{d}{dx}\ln(y) \\ &= \frac{1}{y}\frac{d}{dx}(y) \\ &= \frac{y'}{y} \end{aligned}$$

Our above equation is then

$$u' = -p(x)$$

which we can integrate both sides to obtain

$$u + c = -\int p(x)dx$$

or

$$\ln(y) = -\int p(x)dx - c$$

or

$$y(x) = Ce^{-\int p(x)dx}$$

(where $C = e^{-c}$).

Consider the differential equation

$$y' = xy$$

Notice this is equivalent to

$$y' + (-x)y = 0$$

so take $p(x) = -x$.

Using the above formula (watch the signs!)) gives

$$y(x) = ce^{-\int -x dx} = ce^{\frac{1}{2}x^2}$$

One can check that

$$\frac{d}{dx}y = \frac{d}{dx}\left(ce^{\frac{1}{2}x^2}\right) = ce^{\frac{1}{2}x^2}\frac{d}{dx}\left(\frac{1}{2}x^2\right) = y(x)\cdot x$$

so this $y(x)$ is indeed a solution.

Another example:

Consider

$$y' = e^x y$$

This is equivalent to

$$y' + (-e^x)y = 0$$

so take $p(x) = -e^x$.

Using the above formula (watch the signs!)) gives

$$y(x) = ce^{-\int -e^x dx} = ce^{e^x}$$

One can check that

$$\frac{d}{dx}y = \frac{d}{dx}\left(ce^{e^x}\right) = ce^{e^x}\frac{d}{dx}\left(e^x\right) = y(x)\cdot e^x$$

so this $y(x)$ is indeed a solution.

In both of these examples, I did not put the '$+c$' constant of integration in the exponent. This is because the exponent laws allow me to pull it out into a product with a constant e^c term, so I just put it in the constant c that is already in the formula.

Recap

On the first day, I commented that you already know differential equations of the form

$$y' = f(x)$$

for some given function $f(x)$. This is exactly saying 'y is an antiderivative of f', so the Fundamental Theorem of Calculus says that $y(x)$ is 'the' indefinite integral of $f(x)$. More generally, the Fundamental Theorem of Calculus says that integration is the inverse of differentiation, so if we want to solve differential equations we expect to be integrating.

In the simple setting of first order linear homogeneous differential equations, the situation is very similar; by rearranging the equation, we obtain $\frac{y'}{y} = p(x)$ for some known(/given) function $p(x)$. The left-hand side is not immediately the derivative of a function, but it is not too hard to recognize $\frac{d}{dx}\ln(y) = \frac{y'}{y}$ (an instance of the chain rule or u-substitution, depending on which direction you like going). Then, again, we can obtain (the logarithm of) a solution $y(x)$ (and then exponentiate).

I shouldn't put those comments in parentheses: this is also the beginning of a theme we will develop about the important of exponentials in solving differential equations. Speaking naively, if the derivative is a function of x, we just use the fundamental theorem of calculus, while if the derivative involve the function y, it must involve exponentials, since exponentials are exactly the functions whose derivatives are equal (or proportional) to themselves.