THE CITY COLLEGE OF NEW YORK
MATH 201 Exam 1

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Name (Last, First): ______________________________

Instructions: Write cleanly in complete sentences, and show all of your work. No calculators or other electronic devices are allowed during the examination. Good luck!

Question	Points	Score
1	30 (6×5)	
2	10 (2×5)	
3	10 (1×10)	
4	50 (5×10)	
Total	100	

1. Write 'True' or 'False', no need to justify your answer.

(a) If $\lim_{x\to 3} f(x) = 0$ and $\lim_{x\to 3} g(x) = 0$, then $\lim_{x\to 3}[\frac{f(x)}{g(x)}]$ does not exist.

Solution: False. For instance, if $f(x) = x - 3$ and $g(x) = x - 3$, then

$$\frac{f(x)}{g(x)} = \begin{cases} 1 & x \neq 3 \\ \text{undefined} & x = 3 \end{cases}$$

Here, $\lim_{x\to 3}[\frac{f(x)}{g(x)}] = 1$, i.e. the limit does exist. For a case where the limit does not exist, we could consider $g(x) = (x-3)^2$, with the same $f(x)$ as before.

(b) If the function $f(x)$ does not have a horizontal asymptote, it must be true that either $\lim_{x\to\infty} f(x) = \infty$ or $\lim_{x\to\infty} f(x) = -\infty$.

Solution: False. For $f(x)$ to have a horizontal asymptote means $\lim_{x\to\infty} f(x)$ exists as a number. There are ways for this limit to not exist besides just going to infinity. For instance, $f(x) = \sin(x)$ does not have a horizontal asymptote, but does not go to (positive or negative) infinity, since it is always between -1 and 1.

(c) You are trying to guess $\lim_{x\to 0} w(x)$ for some function $w(x)$. You don't have a formula, but I tell you that when you plug in $x = 0.1, 0, 01, 0, 001, ...$, then $w(x) = 0$ for all of these inputs. Since these numbers are getting closer and closer to 0, we can say that $\lim_{x\to 0} w(x) = 0$

Solution: False. Just because there are inputs x near 0 where the value of $w(x)$ is 0, does not mean the limit as x approaches 0 is 0. For that, we need *all* the inputs near 0 to give values near 0. As an example of a function that satisfies the question but does not have a limit, consider $w(x) = \sin(\frac{1}{\pi x})$.

(d) If $f(x)$ is a function such that $\lim_{x\to 2} f(x)$ does not exist, then for any function $g(x)$ we know $\lim_{x\to 2} f(x)\cdot g(x)$ also does not exist.

> **Solution: False.** As an example, consider $f(x) = \frac{1}{x-2}$ and $g(x) = x - 2$. Even though $f(x)$ does not have a limit as x approaches 2, the product $f(x)\cdot g(x)$ does. This is because the 'way' in which $f(x)$ does not have a limit (a 'division by zero') is cancelled by the $g(x)$.

(e) $\dfrac{x^2-16}{x+4} = x - 4$

> **Solution: False.** When we write an equation, we implicitly mean 'any x which works for one side works for the other, and gives the same answer'. Here, $x = -4$ makes sense for the right-hand side, but not the left.

(f) $\displaystyle\lim_{x\to -4} \frac{x^2-16}{x+4} = \lim_{x\to -4} x - 4$

> **Solution: True.** Even though the domains of these functions are not the same, the values are equal for every $x \neq -4$. The limit only says what is happening for inputs $x \neq 4$, as they approach 4. So the limits are the same, even though the functions are not quite.

2. Consider the function

$$f(x) = \begin{cases} \sqrt{4-x^2} & -2 \le x < 0 \\ x+1 & 0 \le x < 2 \\ 1 & x = 2 \end{cases}$$

(a) What are the domain and range of f?

Solution: The domain of f, often written $\mathsf{dom}(f)$, is all numbers x for which $f(x)$ is defined. Here, we are given the values of $f(x)$ for any number x between -2 and 2, including -2 and 2. As a 'formula' we write this as

$$\mathsf{dom}(f) = [-2, 2]$$

For the range, we think about the values of $f(x)$ on each piece of the definition: at -2, we get $f(-2) = \sqrt{4-(-2)^2} = \sqrt{4-4} = 0$. As x moves from -2 to 0, the x^2 term gets smaller, so the inside of the $\sqrt{}$ gets bigger, up until $\lim_{x\to 0^-} f(x) = 2$. Notice the definition of $f(x)$ is different at $x = 0$, and $f(0) = 1$. So, just looking at x in $[-2, 0)$, the range is $[0, 2)$. That is, every number between 0 and 2, but not including 2. Now, think about x between 0 and 2. Including 0 we have the value 1, and we get every value between $f(0) = 0 + 1 = 1$ and $\lim_{x\to 2^-} f(x) = 3$, but not including 3, because $f(x)$ has a different definition at $x = 2$, namely $f(2) = 1$. Even though $f(x)$ gets as close to 3 as we want, by getting x close to 2, the function never actually has the value 3 for any input. The range of $f(x)$ is all numbers between 0 and 3, including 0 but not including 3. In formula,

$$\mathsf{ran}(f) = [0, 3)$$

(b) For which points c does $\lim_{x\to c} f(x)$ exist? If the limit does not exist for some point c, describe what the function values look like near $x = c$.

Solution: On each piece of the domain, we have a nice definition of a continuous function, defined at every input, so the limits all exist. The only concern is when two pieces of the domain meet. That is, when $x = 0$ or $x = 2$.

We previously discussed that $\lim_{x\to 0^-} f(x) = 2$, but $f(0) = 1$. In fact, $f(x)$ is continuous on the right at $x = 0$. This means $\lim_{x\to 0^+} f(x) = f(0) = 1$, since the function is just a linear function there. Since the one-sided limits do not agree, the limit does not exist.

We also discussed $\lim_{x\to 2^-} f(x) = 3$, but $f(2) = 1$. Since the function is not defined for $x > 2$, there is only one side of the limit at $x = 2$. So I would say the limit at $x = 2$ does exist, but the function is not continuous there (in particular, not continuous from the left).

3. Find numbers a and b so that the function

$$f(x) = \begin{cases} ax + b & x \leq 0 \\ x^2 + a + 2b & 0 < x \leq 2 \\ 7 - x & 2 < x \end{cases}$$

has a limit which exists at every point. Sketch a graph of this function.

Solution: On the first and third parts of the domain, the function is linear, and on the second part, the function is quadratic. So, the function is continuous on the inside of these intervals, and we only need to check what happens when these intervals touch.

First, let's consider $x = 0$. From the left, the function is defined as $ax + b$. As x gets close to 0, the ax term becomes smaller and smaller, and $\lim_{x \to 0^-} f(x) = b$. On the other hand, from right side we have $f(x) = x^2 + a + 2b$, and as x approaches 0 the x^2 term gets smaller and smaller, with $\lim_{x \to 0^+} f(x) = a + 2b$. If we want the limit $\lim_{x \to 0} f(x)$ to exist, these need to be equal, so

$$b = a + 2b$$

or

$$a = -b$$

Next, we consider $x = 2$. For x approaching 2 from the left, we have $f(x) = x^2 + a + 2b = x^2 + b$, using the previous $a = -b$. This means $\lim_{x \to 2^-} f(x) = 4 + b$. On the other hand, if x approaches from the right we have $f(x) = 7 - x$, so $\lim_{x \to 2^+} f(x) = 5$. If we want the limit as x approaches 2 to exist, these must be equal, so

$$4 + b = 5$$

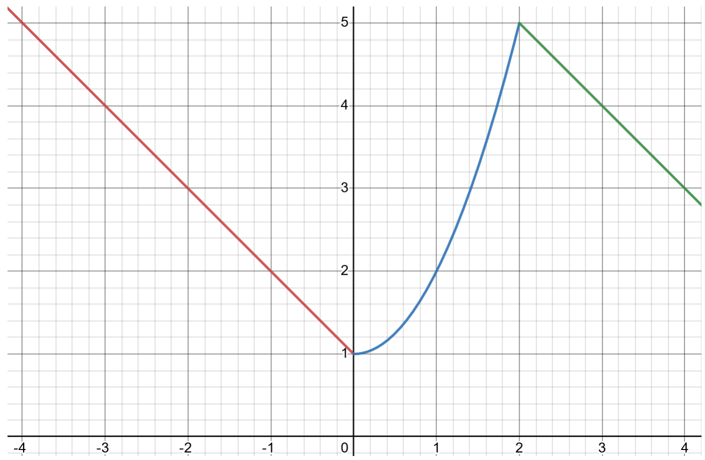
or

$$b = 1$$

Combining this with the above, we get $a = -1$, so

$$f(x) = \begin{cases} 1 - x & x \leq 0 \\ x^2 + 1 & 0 < x \leq 2 \\ 7 - x & 2 < x \end{cases}$$

You should be able to sketch graphs of linear and quadratic functions by hand, but I'll cheat:



Since we've made all the limits match up, and the function is defined everywhere, there are no holes or jumps or vertical asymptotes or anything.

4. Find the following limits, or explain why they do not exist.

(a) $\displaystyle\lim_{x\to 4} \frac{x^2-2x-8}{x-4}$

Solution: Factoring the numerator gives

$$\begin{aligned}\lim_{x\to 4} \frac{x^2-2x-8}{x-4} &= \lim_{x\to 4} \frac{(x-4)(x+2)}{x-4} \\ &= \lim_{x\to 4} x+2 \\ &= 4+2 \\ &= 6\end{aligned}$$

(b) $\displaystyle\lim_{x\to 1} \frac{5-\sqrt{x+3}}{x-1}$

Solution: Notice that we can evaluate the numerator at $x = 1$, and we get $5-\sqrt{1+3} = 5-2 = 3$, but the denominator will be 0 at $x = 1$. So, if x is close enough to 1, the numerator will be close to 3, but the denominator will be as small as we want. This means the limit as x approaches 1 cannot exist.

To be more precise, if x approaches 1 from the left, the denominator will be negative, and the quotient will be a negative number with large absolute value. If we want to be even more precise, say x is within $\frac{1}{100}$ of 1. Then $\sqrt{x+3} < \sqrt{9} = 3$, so $5-\sqrt{x+3} > 2$. But $|x-1|$ will be smaller than $\frac{1}{100}$, so the absolute value of the quotient $|\frac{5-\sqrt{x+3}}{x-1}|$ will be bigger than 200.

You can think about this as x is within $\frac{1}{1000}$ of 1, or 10^{-n} for any positive integer n, and conclude that

$$\lim_{x\to 1} \frac{5-\sqrt{x+3}}{x-1}$$

(c) $\displaystyle \lim_{x\to\infty} \frac{3x^3 - 2x + 9}{4x^3 - 3x^2 + 7x - 1}$

Solution: Factor the largest power of x to get

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x^3 - 2x + 9}{4x^3 - 3x^2 + 7x - 1} &= \lim_{x\to\infty} \frac{x^3}{x^3} \cdot \frac{3 - \frac{2}{x^2} + \frac{9}{x^3}}{4 - \frac{3}{x} + \frac{7}{x^2} - \frac{1}{x^3}} \\
&= \lim_{x\to\infty} \frac{3 - \frac{2}{x^2} + \frac{9}{x^3}}{4 - \frac{3}{x} + \frac{7}{x^2} - \frac{1}{x^3}}
\end{aligned}$$

For large x, the numerator and the denominator both make sense. Further, as x gets large, any constant number divided by x goes to 0 (that is, $\displaystyle \lim_{x\to\infty} \frac{1}{x^n} = 0$, also we can use the constant multiple and product limit laws). So

$$\begin{aligned}
\lim_{x\to\infty} \frac{3 - \frac{2}{x^2} + \frac{9}{x^3}}{4 - \frac{3}{x} + \frac{7}{x^2} - \frac{1}{x^3}} &= \frac{\displaystyle\lim_{x\to\infty} (3 - \frac{2}{x^2} + \frac{9}{x^3})}{\displaystyle\lim_{x\to\infty} (4 - \frac{3}{x} + \frac{7}{x^2} - \frac{1}{x^3})} \\
&= \frac{3 - \displaystyle\lim_{x\to\infty} \frac{2}{x^2} + \lim_{x\to\infty} \frac{9}{x^3}}{4 - \displaystyle\lim_{x\to\infty} \frac{3}{x} + \lim_{x\to\infty} \frac{7}{x^2} - \lim_{x\to\infty} \frac{1}{x^3}} \\
&= \frac{3 - 0 + 0}{4 - 0 + 0 - 0} \\
&= \frac{3}{4}
\end{aligned}$$

(d) $\lim_{x \to 0} \dfrac{\sin(3x)}{x^3 - 2x^2 + 5x}$

Solution: We must recall that $\lim_{x \to 0} \dfrac{\sin(x)}{x} = 1$, and in fact $\lim_{x \to 0} \dfrac{\sin(ax)}{ax} = 1$ for any real number a (you can justify this with a graph, or by thinking about composition with $f(x) = ax$).

Then, by factoring and using the product limit law we get

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin(3x)}{x^3 - 2x^2 + 5x} &= \lim_{x \to 0} \frac{\sin(3x)}{x \cdot (x^2 - 2x + 5)} \\
&= \lim_{x \to 0} 3 \cdot \frac{\sin(3x)}{3x \cdot (x^2 - 2x + 5)} \\
&= 3 \cdot \lim_{x \to 0} \frac{\sin(3x)}{3x} \cdot \lim_{x \to 0} \frac{1}{(x^2 - 2x + 5)} \\
&= 3 \cdot 1 \cdot \frac{1}{5} \\
&= \frac{3}{5}
\end{aligned}$$

(e) $\lim_{x \to \infty} \dfrac{1}{\sqrt{x^2 + 6} - x}$

Solution: Multiply both the numerator and denominator by the 'conjugate' (if you have $\sqrt{a} + b$, this is $\sqrt{a} - b$:

$$\begin{aligned}
\lim_{x \to \infty} \frac{1}{\sqrt{x^2 + 6} - x} &= \lim_{x \to \infty} \frac{1}{\sqrt{x^2 + 6} - x} \cdot \frac{\sqrt{x^2 + 6} + x}{\sqrt{x^2 + 6} + x} \\
&= \lim_{x \to \infty} \frac{\sqrt{x^2 + 6} + x}{x^2 + 6 - x^2} \\
&= \lim_{x \to \infty} \frac{\sqrt{x^2 + 6} + x}{6}
\end{aligned}$$

Now, notice $\sqrt{x^2 + 6}$ is always bigger than $\sqrt{x^2} = |x|$, and since x is going to positive infinity x will be positive, so $|x| = x$. So, the numerator is larger than $2x$, which goes to infinity as x goes to infinity. Thus

$$\lim_{x \to \infty} \frac{1}{\sqrt{x^2 + 6} - x} = \infty$$

Comment: even though 'multiplying by the conjugate' did not get rid of the square root, it made things easier by turning '$\infty - \infty = ???$' into '$\infty + \infty = \infty$'.