October 4th

Recall last time, we encountered the false claim that if the Wronskian of two functions is zero on some interval, then the functions are linearly independent on that interval. We had a counterexample with $f(x) = x^3$ and $g(x) = |x|^3$, on the interval $\mathbb{R} = (-\infty, \infty)$.

Notice that on the interval $(0, \infty)$, these two functions f and g *are* linearly dependent (in fact, $f(x) = g(x)$ for $x > 0$, and on the interval $(-\infty, 0)$ the functions f and g are again linearly depndent (with $f(x) = -g(x)$ for $x < 0$).

What is happening at $x = 0$? These functions are equal to zero. This leads us to a correct

Theorem 1. *If $f(x)$ and $g(x)$ are never equal to 0 on some interval I, and $W(f, g)(x) = 0$ for all $x \in I$, then f and g are linearly dependent on I.*

Proof. By definition, we have

$$\begin{aligned} W(f,g)(x) &= f(x)g'(x) - g(x)f'(x) = 0 \\ g(x)f'(x) &= f(x)g'(x) \\ \frac{f'(x)}{f(x)} &= \frac{g'(x)}{g(x)} \\ (\ln(g(x)))' &= (\ln(f(x)))' \\ \ln|f(x)| &= \ln|g(x)| + c \\ f(x) &= Cg(x) \end{aligned}$$

Thus

$$f(x) + (-C)g(x) = 0$$

and we see that f and g are linearly dependent. □

Here I made a remark: for these second-order (homogeneous) equations, the vector space of solutions has dimension two, so linear dependence really means that one function is a number times the other function. All of our discussion also applies to third-order or higher equations, corresponding to three-or-more dimensions, where linear dependence is more complicated. We will explore this somewhat when we discuss systems of differential equations.

So far, we have only used the Wronskian to check if two solutions are independent, but we can go the other way: if we knew $W(f, g)(x) = w(x)$, and we knew a solution $f(x)$, then

$$\begin{aligned} f(x)g'(x) - f'(x)g(x) &= w(x) \\ g'(x) - \frac{f'(x)}{f(x)}g(x) &= \frac{w(x)}{f(x)} \end{aligned}$$

is a first order linear differential equation for the other solution $g(x)$, which we know how to solve.

Unfortunately this doesn't seem very helpful, since we need to know f *and* g in order to find w. Right?

Theorem 2 (Abel's Theorem)**.** *Given a 2nd order linear homogeneous equation*

$$y'' + p(x)y' + q(x)y = 0$$

(with p and q continuous on some interval I), the Wronskian of two fundamental solutions f and g satisfies

$$W(f,g)(x) = Ce^{-\int p(x)dx}$$

(for $x \in I$) for some constant $C \in \mathbb{R}$

Proof. Suppose we have two fundamental solutions f and g, so

$$W(f,g)(x) = f(x)g'(x) - g(x)f'(x)$$

and differentiate with respect to x:

$$\begin{aligned} W(f,g)'(x) &= (f(x)g'(x) - g(x)f'(x))' \\ &= f'(x)g'(x) + f(x)g''(x) - g'(x)f'(x) - g(x)f''(x) \\ &= f(x)g''(x) - g(x)f''(x) \\ &= f(x)\left(-p(x)g'(x) - q(x)g(x)\right) - g(x)\left(-p(x)f'(x) - q(x)f(x)\right) \\ &= -p(x)f(x)g'(x) + q(x)f(x)g(x) + p(x)g(x)f'(x) - q(x)f(x)g(x) \\ &= -p(x)W(f,g)(x) \end{aligned}$$

Suppressing f, g from the notation, this is

$$W'(x) = -p(x)W(x)$$

which we know how to solve. $\square$

As an example of what we discussed above, consider the equation

$$x^2y'' + xy' - y = 0$$

We check that $f(x) = \frac{1}{x}$ is a solution:

$$x^2 f'' + xf' - f = x^2 \cdot \frac{2}{x^3} + x \cdot \frac{-1}{x^2} - \frac{1}{x} = \frac{2}{x} - \frac{1}{x} - \frac{1}{x} = 0$$

so this $f(x)$ is infact a solution.

Dividing by x^2 to get into standard form, this differential equation is equivalent to

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0$$

Then Abel's theorem says the Wronskian of two independent solutions is

$$W(f,g)(x) = Ce^{-\int \frac{1}{x}dx} = \frac{C}{|x|}$$

For simplicity, let's only consider $x > 0$, so

$$W(f,g)(x) = \frac{C}{x}$$

Using $f(x) = x$ and the definition of the Wronskian, we have

$$\begin{aligned}
f(x)g'(x) - f'(x)g(x) &= W(f,g)(x) \\
\frac{1}{x}g'(x) + \frac{1}{x^2}g(x) &= \frac{1}{x} \\
xg'(x) + g(x) &= x \\
(x \cdot g(x)) &= x \\
x \cdot g(x)* = \frac{1}{2}x^2 + c & \\
g(x) &= \frac{x^2 + k}{2}
\end{aligned}$$

(to get from the second to third line, I multiplied by x^2, and later I replaced $2c$ with k).

Let's consider $k = 0$, so $g(x) = x$. It is easy to check that

$$x^2 g'' + xg' - g = x^2 \cdot 0 + x \cdot 1 - x = 0$$

(This an example of what is known as 'Euler's equation', not to be confused with a different 'Euler's equation' from fluid dynamics. Euler did at least two things.)