Math 3002: Problem Set 8

1. Solve the following systems of differential equations, and verify your solutions are in fact solutions.

   (a)
   $$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

   (b)
   $$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} -10 & -18 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

   (c) (compare to problem 1(e) from Problem Set 7)
   $$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

   (d) Describe the behavior of solutions to the above systems as t goes to ∞. Warning: there may be some special cases.

2. Solve the differential equation

$$\begin{cases} y'' - 11y' + 30y = 0 \\ y(0) = 1 \\ y'(0) = 4 \end{cases}$$

by writing it as a system of first order differential equations.

3. Consider a linear homogeneous third order differential equation with constant coefficients:

$$y''' + ay'' + by' + cy = 0$$

(for convenience, the coefficient of y''' is 1)

Write this as a 3×3 system of first-order differential equations. Find the characteristic polynomial of the coefficient matrix, and verify that it gives the same characteristic equation from our discussion of constant coefficient differential equations.

Supposing there are three real distinct eigenvalues, find an eigenvector for the eigenvalue λ (hint: you know $y(t) = e^{\lambda t}$ is a solution to the original differential equation. Taking inspiration from this, what *should* the eigenvector be? Prove this is really an eigenvector with eigenvalue λ, using the fact that λ solves the characteristic equation.)

4. Consider a matrix A, with (real) eigenvalue λ. A **Jordan chain** is a sequence of vectors $v_1, v_2, ..., v_k$ such that

$$\begin{aligned} Av_1 &= \lambda v_1 \\ Av_2 &= \ \ v_2 + v_1 \\ \vdots &= \vdots \\ Av_k &= \lambda v_k + v_{k-1} \end{aligned}$$

That is, v_1 is an eigenvector, v_2 is a generalized eigenvector of rank 2 related to v_1, v_3 is a generalized eigenvector of rank 3 related to v_2, and so on.

Show that $\{v_1, ..., v_k\}$ is a linearly independent set of vectors. (For notational convenience, you can take $k = 3$).