First Definitions

An important part of modern mathematics is the concept of a 'function'. We think of a knot as an 'abstract' circle which we *put* into space. That is, a kind of function from a circle into three-dimensional space. If we wanted to be formal, we could say

Definition 1. *A* ***parametrized circle in*** $\mathbb{R}^3$ *is a function* $f : [0, 2\pi] \to \mathbb{R}^3$ *such that* $f(0) = f(2\pi)$*.*

Here, $\mathbb{R}$ means the set of real numbers and $\mathbb{R}^3 == \{(x, y, z) : x, y, z \in \mathbb{R}\}$ means the set of all (ordered) triples of real numbers.

As an example of a parametrized circle, we have the usual unit circle in xy-plane, given by

$$f(\theta) = (\cos(\theta), \sin(\theta), 0)$$

We should check that $f(2\pi) = f(0)$, so that this really fits our definition. It is easy to see that, e.g., $(\cos(\theta), 0, \sin(\theta))$ or $(\sin(\theta), 6, \cos(\theta))$ also give parametrized circles in $\mathbb{R}^3$.

It is traditional to use θ as the circle variable. Prasolov gives a parametrization of a trefoil knot in 3-dimensional space.

Here are some parametrizations which **are not** knots:

$$f(\theta) = (\cos(\theta) + \sin(2\theta), \sin(\theta) - \cos(2\theta), 0)$$

$$g(\theta) = (\cos(\theta) + \sin(2\theta), \sin(\theta) + \cos(2\theta), 0)$$

$$h(\theta) = (\sqrt{\cos(2\theta)}\cos(\theta), \sqrt{\cos(2\theta)}\sin(\theta), 0)$$

Notice these are all in the xy-plane, so you might say they aren't 'really' 3-dimensional, but that's not the issue I want to focus on. Rather, all of these curves have **self-intersection** (the last example is interesting: it looks like it 'passes through' itself, but that isn't what really happens. Try graphing it for different values of θ).

One way of saying what this 'self-intersection' means is

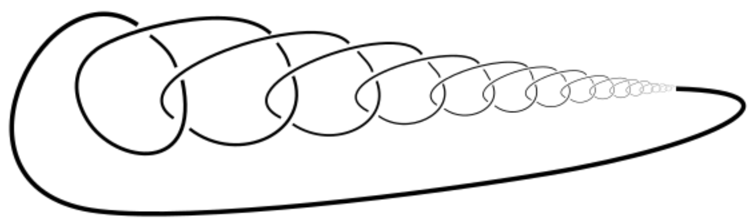
Definition 2. *A function* $f : X \to Y$ *is* ***injective*** *if, for any pair* a, b *of different elements of* X*, the values* $f(a)$ *and* $f(b)$ *are different.*

More symbolically:

$$\forall a, b \in X : a \neq b \implies f(a) \neq f(b)$$

The symbol $\forall$ means 'for all', so the symbols are read 'for all a, b in X, if $a \neq b$ then $f(a) \neq b$.'

Since knots are supposed to model real physical knots (ignoring things like thickness), we should not allow two points of the knot to be in literally the same place. So, we could define knots as injective functions of the circle. This almost works... except it allows things like this:



(image courtesy of Wikipedia)

It might be fun to study such things, but let's pretend we want a somewhat realistic model of physical knots, so we should exclude this. The kind of knots we want, we call 'tame', and knots like the above are called 'wild'. We want a good definition of 'tame knot'. Try to come up with one.

Here's one attempt, with the idea that 'real' knots have thickness:

Definition 3. *An injective function $f : \mathbb{S}^1 \to \mathbb{R}^3$ from the circle into 3-space is called **tame** if it can be thickened. More precisely, if $\mathbb{D}^2$ denotes the 2-dimensional disk, then $\mathbb{S}^1 \times \mathbb{D}^2$ denotes the solid torus, and we ask that f is able to be extended to an injective function $\hat{f} : \mathbb{S}^1 \times \mathbb{D}^2 \to \mathbb{R}^3$. Here, 'extended' means that $\hat{f}(\theta, c) = f(\theta)$, where c is the center of the disk $\mathbb{D}^2$.*

(In particular, tame functions must be injective.)

Definition 4. *A **knot** is a tame function from the circle into 3-dimensional space.*

So, a (mathematical) knot is tame if it can be represented by some actual knot in physical space.