Sept 18

Consider the 'sinc' function:

$$sinc(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Notice $\frac{\sin(x)}{x}$ is not defined for $x = 0$, but we decided to put the value of 1 in for the 'sinc' function ('sinc' is short for 'sine cardinal', I don't know why). Does this seem justified?

What does $\frac{\sin(x)}{x}$ look like as x gets close to 0? Let's collect some facts about the sine function first:

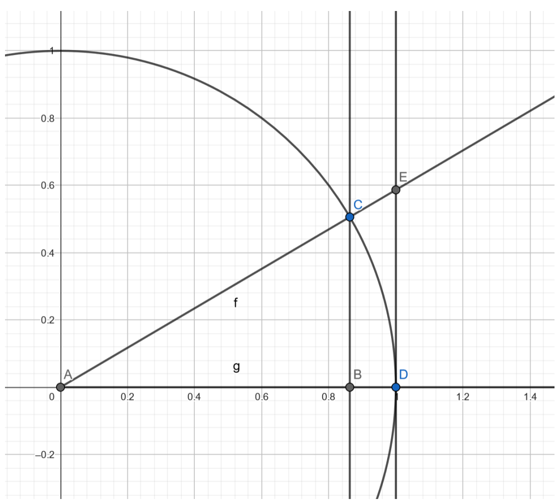
Claim 1. *For $x > 0$, we have*

$$\sin(x) \leq x$$

Claim 2. *For $x < 0$, we have*

$$\sin(x) \geq x$$

These are true by the definition of the sine function.



Namely, the height of the point C (which is $\sin(x)$) is less than the length of the curved path from D to C (which is x).

Then, we recall the

Fact 1. *Suppose we have numbers a and b, and a is less than or equal to b:*

$$a \leq b$$

If $k > 0$ is some positive number, then

$$k \cdot a \leq k \cdot b$$

If $k < 0$ is some negative number, then

$$k \cdot a \geq k \cdot b$$

(Note: This is also true with $<, >$ instead of $\leq, \geq$.)

If we divide the equation in Claim 1 by x on each side, since we are looking at $x > 0$, the inequalities stay the same direction. On the other hand, if we divide the equation in Claim 2 by x on each side, since we are looking at $x < 0$, the inequalities switch direction. Putting these together, we get

Claim 3. *For any $x \neq 0$, we have*

$$\frac{\sin(x)}{x} \leq 1$$

So, if we are looking at the 'sinc' function near $x = 0$, we know everything is at most 1.

Can we say anything else? We need to dig into what $\sin(x)$ means a little bit. Going back to the picture, we have similar triangles ABC and ADE. The first has side lengths $\sin(x)$ and $\cos(x)$, the other with side lengths $\tan(x)$ and 1.

Warning! This picture is assuming that x is near 0, otherwise it would wrap around and the curves wouldn't intersect in the same way.

It is plausible from the picture that the length of x is at most $\tan(x)$.

To really prove this, we should move to talking about areas instead of length. Namely, the circular sector is definitely smaller than the big triangle (since it fits inside). From geometry, we know the area of a sector with angle x is equal to $\frac{1}{2}x$ (quick check: if $x = 2\pi$, that's the whole circle, so the area should be $\pi r^2 = \pi$). On the other hand, the area of the triangle is $\frac{1}{2}1 \cdot \tan(x)$.

So we have

$$\frac{1}{2}x \leq \frac{1}{2}\tan(x)$$

or

$$x \leq \tan(x)$$

Warning! I'm using the picture a lot, which I've drawn for $x > 0$ a positive number. If we have a negative x, we should go around the circle in the other direction, and we should think of the heights (/areas) as negative. Then, the circular sector being inside the triangle means the area is less negative, so 'bigger'.

Collecting this together,

Claim 4. *If $x > 0$, then*

$$x \leq \tan(x)$$

and if $x < 0$, then

$$x \geq \tan(x)$$

Recall $\tan(x) = \frac{\sin(x)}{\cos(x)}$, and remember the fact above, how multiplying by a negative number flips the less-than sign, and we get for $x > 0$

$$x \leq \tan(x)$$
$$x \leq \frac{\sin(x)}{\cos(x)}$$
$$x \cdot \cos(x) \leq \sin(x)$$
$$\cos(x) \leq \frac{\sin(x)}{x}$$

and for $x < 0$

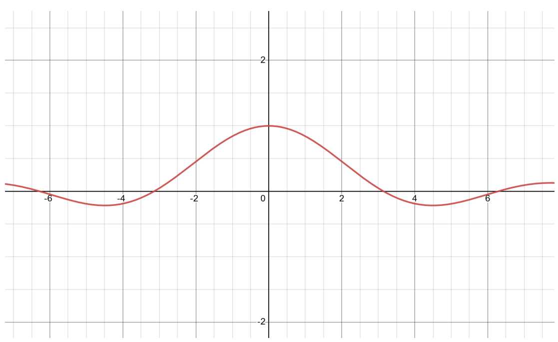
$$x \geq \tan(x)$$
$$x \geq \frac{\sin(x)}{\cos(x)}$$
$$x \cdot \cos(x) \geq \sin(x)$$
$$\cos(x) \leq \frac{\sin(x)}{x}$$

Together with the previous claims, we have

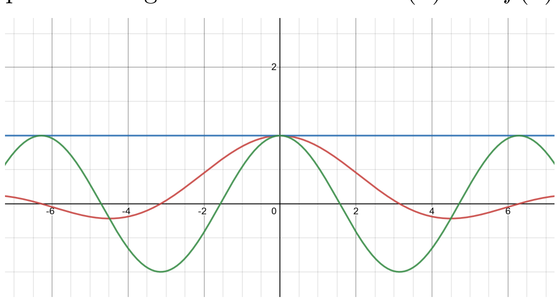
Claim 5. *For any x close to, but not equal to 0, we have*

$$\cos(x) \leq \frac{\sin(x)}{x} \leq 1$$

Now, I will allow us to graph the function $\frac{\sin(x)}{x}$:



Now, we graph this alongside the functions $\cos(x)$ and $f(x) = 1$.



All together, we arrive at the

Proposition 1.

$$\lim_{x\to a}\frac{\sin(x)}{x} = 1$$

Why? Well, as x gets closer and closer to 0, the function $\cos(x)$ gets closer and closer to 1. Since $\frac{\sin(x)}{x}$ has to be between $\cos(x)$ and 1, it has no room to go anywhere else: it has to get closer and closer to 1.

We state this more formally as the

Theorem 1 (Squeeze Theorem)**.** *If we have three functions f, g, and h, such that for all x near $x = a$ we have*

$$f(x) \leq g(x) \leq h(x),$$

and the limits

$$\lim_{x\to a} f(x) = L$$

and

$$\lim_{x\to a} h(x) = L$$

exist and are equal to the same number L, then the limit of $g(x)$ as x approaches a also exists and is equal to L. That is,

$$\lim_{x\to a} g(x) = L$$

As another example, consider the function

$$f(x) = x \cdot \sin(\frac{1}{x})$$

What is $\lim_{x\to 0} f(x)$? First, notice that the value of $\sin(\frac{1}{x})$ is between -1 and 1. This is because the values of sine are always between -1 and 1. If x is a number (not equal to 0), then $\frac{1}{x}$ is some other number, so when we put it in sin the output will be between -1 and 1. In symbols,

$$-1 \leq \sin(\frac{1}{x}) \leq 1$$

Now, we multiply by x. Remember, different things happen depending on the sign of x (that is, is it positive or negative).

In particular: if $x > 0$ is positive, we get

$$-x \leq x\sin(\frac{1}{x}) \leq x,$$

and if $x < 0$ is negative we get

$$-x \geq x\sin(\frac{1}{x}) \geq x$$

We can figure out the limit from here, actually: looking at $x > 0$ gives the 'right-side limit' of $f(x)$, and looking at $x < 0$ gives the 'left-side limit' of $f(x)$. If these both exist and are equal, then the full limit also exists and is equal to the same thing. Convince yourself that these inequalities mean the one-sided limits exist, and are equal to 0.

There is another way to get this: by using the function

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

If we put this in the second inequality above, we get

$$|x| \geq x \cdot \sin(\frac{1}{x}) \geq -|x|$$

(for $x < 0$) and the first equality becomes

$$-|x| \leq x \cdot \sin(\frac{1}{x}) \leq |x|$$

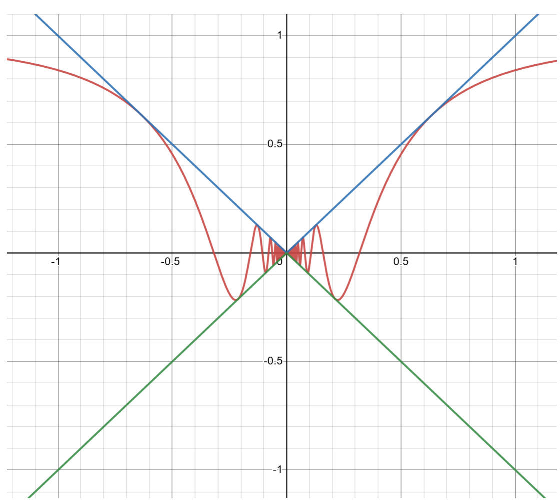
(for $x > 0$).

Now we have

$$-|x| \leq x \sin(\frac{1}{x}) \leq |x|$$

for all $x \neq 0$. Thus, we can apply the Squeeze Theorem. It is clear that the limit of $|x|$ as x approaches 0 is equal to 0, and the same for $-|x|$. So we arrive at

$$\lim_{x \to 0} x \sin(\frac{1}{x}) = 0$$



($x \sin(\frac{1}{x})$ graphed with $|x|$, $-|x|$)

I also made a remark: If you have a function $f(x)$, how does the graph of $f(x)$ compare to the graph of $f(2x)$, or $f(10x)$, or $f(\frac{1}{2}x)$, or $f(\frac{1}{10}x)$? The graph stretches or shrinks horizontally. By playing around with this a little, we arrive at the

Proposition 2. *For any number $k \neq 0$, we have*

$$\lim_{x \to 0} \frac{\sin(kx)}{kx} = 1$$

I made this remark because I was hoping to get to the example: Find

$$\lim_{x \to 0} \frac{\sin(3x)}{\sin(8x)}$$

The idea here is to 'multiply by 1', and turn it into limits we know. In particular,

$$\begin{aligned}
\frac{\sin(3x)}{\sin(8x)} &= \frac{\sin(3x)}{x} \frac{x}{\sin(8x)} \\
&= \frac{3}{8} \frac{\sin(3x)}{3x} \frac{8x}{\sin(8x)} \\
&= \frac{3}{8} \cdot \frac{\frac{\sin(3x)}{3x}}{\frac{\sin(8x)}{8x}}
\end{aligned}$$

Then, some of the limit laws give us

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin(3x)}{\sin(8x)} &= \lim_{x \to 0} \frac{3}{8} \cdot \frac{\frac{\sin(3x)}{3x}}{\frac{\sin(8x)}{8x}} \\
&= \frac{3}{8} \frac{\lim_{x \to 0} \frac{\sin(3x)}{3x}}{\lim_{x \to 0} \frac{\sin(8x)}{8x}} \\
&= \frac{3}{8} \cdot \frac{1}{1} \\
\lim_{x \to 0} \frac{\sin(3x)}{\sin(8x)} &= \frac{3}{8}
\end{aligned}$$

Finally, I briefly talked about 'How does a function behave as the input x gets bigger and bigger?'

As an example, think about

$$r(x) = \frac{1023x - 17}{3x^2 - 7x + 7}$$

Here are some values:

x	$r(x)$
1	335.3333333...
10	43.0928270....
100	3.49005357...
1000	0.034107901...
10000	0.003410079...
100000	0.00034100079...

Even though the 1023 in the numerator is big, and the values of $r(x)$ are pretty big for $x = 1$ or 10, as x gets bigger and bigger, the x^2 term on the bottom is *much* bigger than $1023x$. Namely, if $x = 10000$, then the x^2 term is about 10 times bigger. If $x = 100000$, then it's a hundred times bigger.

If x is really large, then x^2 is much larger than x, and both of these are larger than any given number. Then, the function $r(x)$ is almost like $\frac{1023x}{3x^2} \approx \frac{1023}{3} \cdot \frac{1}{x}$. Namely, the biggest terms in the numerator and the denominator are the biggest part of the value: the smaller pieces we add or subtract are so much smaller, they don't change the values by much.

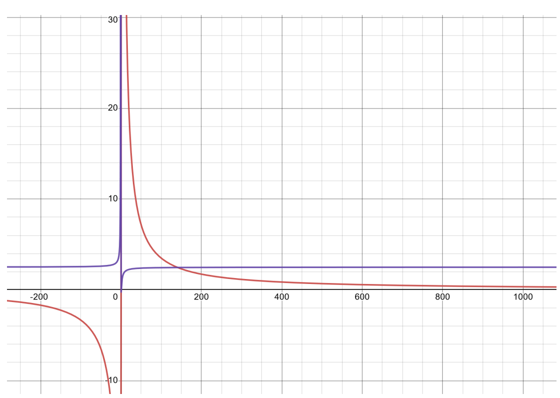
Of course, it's important here that I'm talking about 'what happens as x gets bigger and bigger',

A more interesting example might be

$$q(x) = \frac{5x^2 - 3x}{2x^2 + 3x + 1}$$

If $x = 1000000$, then $x^2 = 1000000000000$. So, the numerator is 4999997000000, and the denominator is 2000003000001

This is basically '5 with twelve 0s after it', and '2 with twelve 0s after it'. That means, the value of the function $q(1000000)$ is very close to $\frac{5}{2}$



Here is $r(x)$ in red and $q(x)$ in purple. Notice the scales on the graph.