MATH 3002 - Practice Exam 1 Solutions

Name: ____

Instructions: Write cleanly, show all work. Explain any trick questions.

1. Find the general solution to the following differential equations:

   (a)
   $$2xy' + y = 0$$

   Solution:

   $$\begin{aligned} 2xy' + y &= 0 \\ 2xy' &= -y \\ \frac{y'}{y} &= -\frac{1}{2x} \\ \frac{d}{dx}\left(\ln|y|\right) &= -\frac{1}{2x} \\ \ln|y| &= -\frac{1}{2}\int \frac{1}{x}dx \\ \ln|y| &= -\frac{1}{2}\ln|x| + c \\ |y| &= \frac{C}{\sqrt{|x|}} \\ y(x) &= \frac{C}{\sqrt{|x|}} \end{aligned}$$

   Notice we got rid of the absolute value around y, since the constant C can be taken positive or negative, but the absolute value around x we should keep since it does give us a valid solution defined for x negative (you can check this by replacing $|x|$ with $-x$).

   (b)
   $$\cos(x)y' - \sin(x)y = \cos(x)$$

Solution:

$$
\begin{aligned}
\cos(x)y' - \sin(x)y &= \cos(x) \\
(\cos(x)y)' &= \cos(x) \\
\cos(x) \cdot y &= \int \cos(x)dx \\
\cos(x) \cdot y &= \sin(x) + c \\
y(x) &= \frac{\sin(x) + c}{\cos(x)}
\end{aligned}
$$

I would probably keep it like that, but this is also $y(x) = \tan(x) + c\sec(x)$ if you prefer.

(c)

$$y' = 3xy - 3x^2$$

Solution:

$$
\begin{aligned}
y' &= 3xy - 3x^2 \\
y' - 3xy &= -3x^2 \\
e^{-\frac{3}{2}x^2}y' - 3xe^{-\frac{3}{2}x^2}y &= -3x^2e^{-\frac{3}{2}x^2} \\
\left(e^{-\frac{3}{2}x^2}y\right)' &= -3x^2e^{-\frac{3}{2}x^2} \\
e^{-\frac{3}{2}x^2}y &= \int -3x^2e^{-\frac{3}{2}x^2} \\
y(x) &= e^{\frac{3}{2}x^2}\int -3x^2e^{-\frac{3}{2}x^2}
\end{aligned}
$$

There is an integrand involving an e^{x^2} term, so this cannot be expressed with elementary functions.

If the non-homogeneous term had been $-3x$, we would have $\int -3xe^{-\frac{3}{2}x^2}dx = \int e^u du$, where $u = -\frac{3}{2}x^2$, which we could then express in simple terms.

2. Solve the initial value problem

$$
\begin{cases}
\frac{dx}{dt} = 1 + x^2 \\
x(0) = 1
\end{cases}
$$

What is the maximum domain where this solution is defined and continuous?

Solution: Treating this as a separable equation, we rearrange

$$\begin{aligned}\frac{dx}{dt} &= 1 + x^2 \\ \frac{dx}{1+x^2} &= dt \\ \int \frac{dx}{1+x^2} &= \int dt \\ \arctan(x) &= t + c \\ x(t) &= \tan(t+c)\end{aligned}$$

Here, you are supposed to know the antiderivative of $\frac{1}{1+x^2}$. You can do this via substitution, $x = \tan(u)$, $dx = \sec^2(u)du$, or if you're dumb like me you vaguely recall this is one of the arc-trig functions and just differentiate them all until you find the right one.

Anyway, we want our function to satisfy the initial condition $x(0) = 1$, so $\tan(0+c) = \tan(c) = 1$, so $c = \frac{\pi}{4}$ (or shifted by any multiple of π). Since tan has discontinuities at any $(k + \frac{1}{2})\pi$, for $k \in \mathbb{Z}$, we see that the maximal domain of $\tan(x + \frac{\pi}{4})$ containing 0 without any discontinuities is $(-\frac{3\pi}{4}, \frac{\pi}{4})$

3. Find the general form of the solution to

$$\frac{dy}{dx} = -\frac{1 + (xy+1)e^{xy}}{x^2 e^{xy}}$$

Write the solution so that y is a function of x. What is the domain of this function (this will depend on a parameter)?

Solution: This equation is neither separable nor linear, so we better hope it's exact. Rearranging:

$$\begin{aligned}\frac{dy}{dx} &= -\frac{1 + (xy+1)e^{xy}}{x^2 e^{xy}} \\ x^2 e^{xy} dy &= -\left(1 + (xy+1)e^{xy}\right) dx \\ \left(1 + (xy+1)e^{xy}\right) dx + x^2 e^{xy} dy &= 0\end{aligned}$$

Let's check exactness here: we want the y-derivative of the dx coefficient to equal the x-derivative of the dy coefficient.

$$\begin{aligned}\frac{\partial}{\partial y}\left(1+(xy+1)e^{xy}\right) &= \frac{\partial}{\partial y}1+\frac{\partial}{\partial y}xye^{xy}+\frac{\partial}{\partial y}e^{xy}\\ &= xe^{xy}+x^2ye^{xy}+xe^{xy}\\ &= 2xe^{xy}+x^2ye^{xy}\end{aligned}$$

Similarly,

$$\frac{\partial}{\partial x}x^2e^{xy} = 2xe^{xy}+x^2ye^{xy}$$

by the product rule. These are equal, so our original equation is exact, and we look for a potential function $\Psi(x, y)$.

We must have

$$\frac{\partial}{\partial x}\Psi(x,y) = 1+xye^{xy}+e^{xy}$$

so integrating with respect to x gives

$$\begin{aligned}\Psi(x,y) &= \int\left(1+xye^{xy}+e^{xy}\right)dx\\ &= x+\int xye^{xy}dx+\frac{e^{xy}}{y}\\ &= x+xe^{xy}-\frac{e^{xy}}{y}+y\cdot f(y)+\frac{e^{xy}}{y}\\ &= x+xe^{xy}+y\cdot f(y)\end{aligned}$$

On the other hand,

$$\frac{\partial}{\partial y}\Psi(x,y) = x^2e^{xy}$$

so

$$\begin{aligned}\Psi(x,y) &= \int x^2e^{xy}dy\\ &= x^2\left(\frac{e^{xy}}{x}+g(x)\right)\\ &= xe^{xy}+x^2g(x)\end{aligned}$$

If we set $f(y) = 0$ and $g(x) = \frac{1}{x}$, we get

$$\Psi(x,y) = x+xe^{xy},$$

so solutions to our differential equation are of the form

$$\Psi(x,y) = k$$

for some constant $k \in \mathbb{R}$.

Solving for y as a function of x, we get

$$\begin{aligned} x + xe^{xy} &= k \\ xe^{xy} &= k - x \\ e^{xy} &= \frac{k-x}{x} \\ xy &= \ln(\frac{k-x}{x}) \\ y(x) &= \frac{\ln(\frac{k-x}{x})}{x} \end{aligned}$$

The domain of this $y(x)$ is then $(0, k)$ (if k is negative, then this is $(k, 0)$. Warning: if you split $\ln(\frac{k-x}{x})$ into $\ln(k-x) - \ln(x)$, you may not see this, but $k < 0$ does lead to valid solutions).

4. Solve the following initial value problems

(a)

$$\begin{cases} y'' + 4y' - 12y = 0 \\ y(0) = 1 \\ y'(0) = -2 \end{cases}$$

Solution: We write the characteristic equation:

$$\lambda^2 + 4\lambda - 12 = 0$$

which factors as

$$(\lambda - 2)(\lambda + 6) = 0$$

So the general solution of the differential equation is

$$y(x) = c_1 e^{2x} + c_2 e^{-6x}$$

Evaluating

$$y(0) = c_1 + c_2 = 1$$

and

$$y'(0) = 2c_1 - 6c_2 = -2$$

we can solve e.g. by substituting $c_1 = 1 - c_2$ to obtain

$$\begin{aligned} 2(1-c_2) - 6c_2 &= -2 \\ 2 - 2c_2 - 6c_2 &= -2 \\ 4 &= 8c_2 \\ c_2 &= \frac{1}{2} \end{aligned}$$

which then gives $c_1 = 1 - \frac{1}{2} = \frac{1}{2}$, so our particular solution solving these initial conditions is

$$y(x) = \frac{1}{2}e^{2x} + \frac{1}{2}e^{-6x}$$

(b)

$$\begin{cases} y'' + 4y' + 4y = 0 \\ y(0) = 3 \\ y'(0) = 0 \end{cases}$$

Solution: Again we write the characteristic equation

$$\lambda^2 + 4\lambda + 4 = 0$$

and notice it factors

$$(\lambda + 2)^2 = 0$$

Thus, our general solution (in the case of repeated roots) is

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x}$$

Here, evaluating at 0 gives

$$y(0) = c_1 = 3$$

Since the derivative is slightly complicated, I'll show it first:

$$y'(x) = -2c_1 e^{-2x} + c_2 e^{-2x} - c_2 x e^{-2x}$$

Then

$$y'(0) = -2c_1 + c_2 = 0$$

or

$$c_2 = 2c_1 = 6$$

and our particular solution is

$$y(x) = 3e^{-2x} + 6xe^{-2x}$$

(c)

$$\begin{cases} y'' - 2y' + 3y = 0 \\ y(0) = 1 \\ y'(0) = \sqrt{2} \end{cases}$$

Solution: Again, we write the characteristic equation

$$\lambda^2 - 2\lambda + 3 = 0$$

which does not factor over $\mathbb{R}$:

$$(\lambda - (1 - \sqrt{2}i))(\lambda - (1 + \sqrt{2}i)) = 0$$

Instead of writing $e^{(1\pm\sqrt{2}i)x}$, I'll just cut to the chase: our general solution is

$$y(x) = c_1 e^x \cos(\sqrt{2}x) + c_2 e^x \sin(\sqrt{2}x)$$

which has derivative

$$y'(x) = c_1 e^x \cos(\sqrt{2}x) - \sqrt{2}c_1 e^x \sin(\sqrt{2}x) + c_2 e^x \sin(\sqrt{2}x) + \sqrt{2}c_2 e^x \cos(\sqrt{2}x)$$

Evaluating at $x = 0$ gives

$$y(0) = c_1 = 1$$

and

$$y'(0) = c_1 + \sqrt{2}c_2 = \sqrt{2}$$

Solving this gives $c_2 = 1 - \frac{\sqrt{2}}{2}$, so our particular solution is

$$y(x) = e^x \cos(\sqrt{2}x) + (1 - \frac{\sqrt{2}}{2})e^x \sin(\sqrt{2}x)$$