Explain, in a (complete) sentence or two, your reasoning.

1. **True** or **False**: As x increases to 1000, the function $f(x) = \frac{1}{x}$ gets closer to 0, so the limit as x goes to 1000 is 0. Justify your answer.

   Solution: False. It's true that $\frac{1}{x}$ does get smaller, i.e. closer to 0, as x gets bigger, but it also gets closer to $\frac{1}{1000}$ as x gets close to 1000, and it gets much closer to this number than it does to 0. We should also think about what happens when x approaches from the right. That is, if we start with x really big, say $10,000$, and make it smaller to approach $x = 1000$. Then the values are getting further from 0, but still closer to $\frac{1}{1000}$

2. **True** or **False**: If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a}[f(x) - g(x)] = 0$.

   Solution: False. There is not enough information to tell what happens. For instance, if $f(x) = x^2 + 3$ and $g(x) = x^2$, then $f(x) - g(x) = 3$ for any x, so the limit as x goes to infinity will be 3. We can also think about $f(x) = x^2$ and $g(x) = x$, or $f(x) = x$ and $g(x) = x^2$, to get limits which go off to ∞ or $-\infty$.

3. **True** or **False**: If $\lim_{x\to a} f(x) = L$, then that means if x_1 is closer to a than x_2 is, then $f(x_1)$ will be closer to L than $f(x_2)$ is.

   Solution: False. Think about $x \cdot \sin(\frac{1}{x})$. As x approaches $a = 0$, the limit L is equal to 0. In fact, for lots of inputs, like $x = \frac{1}{1000\pi}$, the value is actually 0. But, there are inputs x smaller than this which are not 0 (so, are further away from 0). The point is, going to the limit does not have to be a 'monotonic' process, the function could still oscillate up and down.

4. Consider the function

$$k(x) = \begin{cases} x^2 & x \text{ is rational} \\ -x^2 & x \text{ is irrational} \end{cases}$$

   Then

   A. there is no a for which $\lim_{x\to a} f(x)$ exists

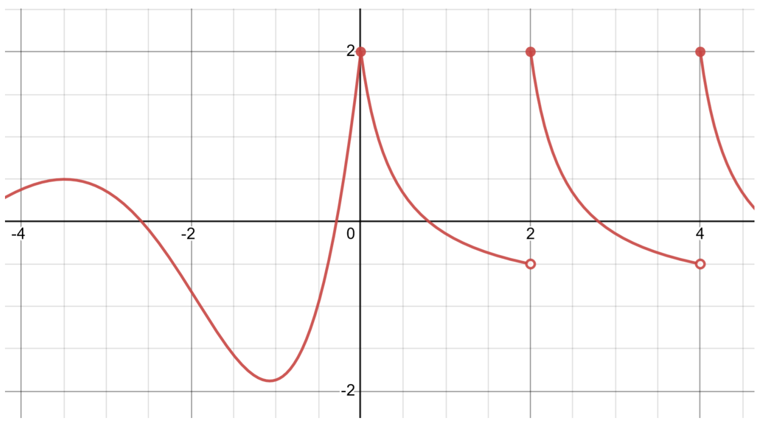
   B. there might be some a for which $\lim_{x\to a} f(x)$ exists, but it is impossible to say for sure without more information

   C. The number $a = 0$ is the only number for which $\lim_{x\to a} f(x)$ exists

D. The limit $\lim_{x\to a} f(x)$ exists for infinitely may a.

Solution: This one is a bit tricky. If a is not equal to 0, then there are some inputs x near a which are fractions, so the output is positive, and some which are not fractions, so the output is negative. So, nearby $x = a$, there is no single value the outputs are getting close to. The only exception is at $a = 0$, where for nearby inputs the outputs are still positive and negative, but they get smaller and smaller in absolute value as we get closer to 0, so the limiting value does exist and is equal to 0.

5. Consider the following graph of the function $m(x)$:



A. $\lim_{x\to 0} m \circ m(x) = 2$

B. $\lim_{x\to 0} m \circ m(x) = -\frac{1}{2}$

C. $\lim_{x\to 0} m \circ m(x) = 0$

D. $\lim_{x\to 0} m \circ m(x)$ does not exist

Solution: We talked about an example like this. When x is near 0, $m(x)$ is near 2. In fact, $m(x)$ is actually continuous near $x = 0$, so $m(0) = 2$. But, the outputs of $m(x)$ are always less than or equal to 2. This means, if we look at $m(m(x))$, when x is near 0 we have $m(m(x)) = m(u)$ with u a number near, but less than, 2.

All in all, this means

$$\lim_{x\to 0} m(m(x)) = \lim_{u\to 2^-} m(u) = -\frac{1}{2}$$

6. Without using a calculator, estimate how many solutions there are to the equation
$$\sin(x) = \frac{x}{100}$$
Explain your reasoning. (Hint: try graphing)

Solution: Remember that $\sin(x)$ is always between -1 and 1. So, if $x < -100$ or $x > 100$, we have $\frac{x}{100} < -1$ or $\frac{x}{100} > 1$, respectively. This means, any solution to the equation will be between $x = -100$ and $x = 100$. The function $\sin(x)$ repeats itself every 2π interval. That is, $\sin(x + 2\pi) = \sin(x)$. On each of these intervals (x in $[0, 2\pi]$, x in $[2\pi, 4\pi]$, ...), the value of $\frac{x}{100}$ does not change much. So, when the values of $\sin(x)$ go up to 1, it crosses the line $\frac{x}{100}$ once, and when it goes down it crosses again. A similar discussion holds for negative x. So, on each interval of length 2π, we get two intersection points. Since the total length of the interval we need to look at, $[-100, 100]$, is 200, we should have $2 \cdot \frac{200}{2\pi} \approx 63.66$. Of course, there can only be a whole number of solutions, so we could estimate 63. In fact, if we think about our discussion above, you might notice that $x = 0$ is a solution, and we count it in both the $[0, 2\pi]$ interval and the $[-2\pi, 0]$ interval. So 62 solutions is probably a better estimate.

7. Is it true that for any functions f, g, and h, we have the equation
$$(f + g) \circ h(x) = [f \circ h + g \circ h](x)$$
Is it true that
$$f \circ (g + h)(x) = [f \circ g + f \circ h](x)$$
If it is true, try to explain why, if it is false, give an example of three functions where the equation fails.

Solution: The first equation is true, the second equation is false.

Recall, $[f + g](u) = f(u) + g(u)$, by definition. That is, the function "$f + g$" takes an input, gives it to both f and g, then adds the result. In this case, the input I give happens to be the output of a different function, $u = h(x)$.

On the other hand,
$$f \circ (g + h)(x) = f(g(x) + h(x))$$
It is not clear how to relate this to $f(g(x)) + f(h(x))$, since the g and h are together inside the input of f. For instance, if $f(x) = x^2$, then

$$\begin{aligned} f \circ (g + h)(x) &= f(g(x) + h(x)) \\ &= (g(x) + h(x))^2 \\ &= g(x)^2 + 2g(x)(h(x) + h(x)^2 \end{aligned}$$

which is not the same as

$$f(g(x)) + f(h(x)) = g(x)^2 + h(x)^2$$

You should try picking functions for $g(x)$ and $h(x)$ and checking that these don't match up.

8. Consider the function $q(t) = \frac{t-2}{2t+3}$.

   (a) What are the domain and range of $q(t)$?

   Solution: The domain of $q(t)$ is the set of all inputs for which the formula makes sense. In particular, we cannot divide by zero, so we need

   $$\begin{aligned} 2t + 3 &\neq 0 \\ t &\neq -\frac{3}{2} \end{aligned}$$

   That is, the domain is all $t \neq -\frac{3}{2}$. You could write this as $(-\infty, -\frac{3}{2}) \cup (-\frac{3}{2}, \infty)$. How about the range? Well, if t is very close to $-\frac{3}{2}$, then the numerator is close to $-\frac{3}{2} - 2 \approx -\frac{7}{2}$. In particular, it is some number away from 0. If t is a little smaller than this number, the denominator is negative and small in absolute value, so $q(t)$ will be large and positive. Similarly, if t is slightly bigger, the values of $q(t)$ will be extremely negative. So, you might think everything is in the range. In fact, this is not true. The number $\frac{1}{2}$ is not in the range. I realize now you need to do the next part in order to see this...

   (b) Find a formula for the inverse function $q^{-1}(t)$. What are its domain and range?

   Solution: Let's suppose we want the function $q(a)$ to have the value t. That is, set $q(a) = t$, and solve for t

   $$\begin{aligned} q(a) &= t \\ \frac{a-2}{2a+3} &= t \\ a - 2 &= 2ta + 3t \\ a(1-2t) &= 3t + 2 \\ a &= \frac{3t+2}{1-2t} \end{aligned}$$

   The function $q^{-1}(t)$ is defined to be the number a so that $q(a) = t$. This is what we just solved for. That is,

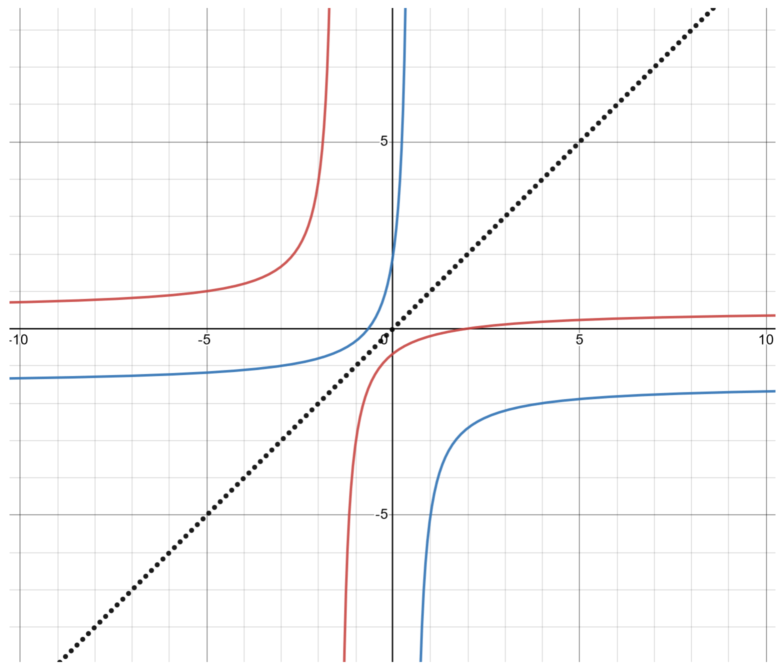
   $$q^{-1}(t) = \frac{3t+2}{1-2t}$$

   Notice the domain here is $(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$, since the denominator is not defined at $t = \frac{1}{2}$ This corresponds to the range in part (i). Similarly, the range of $q^{-1}(t)$ is the domain of $q(t)$.

   The fact that these match up is kind of special. We don't have to do the thing we did by saying $\sqrt{}$ means the **positive** solution, because the function $q(t)$ satisfies the 'horizontal line test' as well as the 'vertical line test'. A fancier way of saying this is that $q(t)$ is 'one-to-one', or 'injective'. This means that if $q(a) = q(b)$, then $a = b$. That is, no two different inputs are sent to the same output (this is not true for $f(x) = x^2$.

(c) Graph $q(t)$ and $q^{-1}(t)$ in the same plot. Can you see any relationship between the two graphs?

Solution:



In red is $q(t)$, in blue is $q^{-1}(t)$. The dotted line is $y = t$.

If you flip the red graph over the dotted line, you will get the blue graph. This is the geometric analog of replacing y with t and t with y: if $y = q(t)$ is replaced with $t = q(y)$, then using the inverse function gives $q^{-1}(t) = q^{-1}(q(y)) = y$.

(d) Find a formula for $f \circ f(x)$. What are the domain and range of $f \circ f(x)$?

Solution: This should be $q \circ q(t)$... it's already a bit of a trick question without the typo

We can just compute

$$\begin{aligned}
q \circ q(t) &= q(q(t)) \\
&= q(\frac{t-2}{2t+3}) \\
&= \frac{(\frac{t-2}{2t+3}) - 2}{2(\frac{t-2}{2t+3}) + 3} \\
&= \frac{\frac{t-2}{2t+3} - \frac{2(2t+3)}{2t+3}}{\frac{2(t-2)}{2t+3} + \frac{3(2t+3)}{2t+3}} \\
&= \frac{\frac{t-2-4t-6}{2t+3}}{\frac{2t-4+6t+9}{2t+3}} \\
&= \frac{t-2-4t-6}{2t-4+6t+9} \\
&= \frac{-3t-8}{8t+5}
\end{aligned}$$

but there's a bit of a trick here. You might say "the domain is anywhere this expression is defined, so the domain is all numbers $t \neq -\frac{5}{8}$", but there's something we missed. What if we evaluate at $t = -\frac{3}{2}$? You might say $\frac{-3(-\frac{3}{2})-8}{8(-\frac{3}{2})+5} = \frac{\frac{9}{2}-8}{-12+5} = \frac{-\frac{7}{2}}{-7} = \frac{1}{2}$, but remember the function:

$$q \circ q(-\frac{3}{2}) = q(q(-\frac{3}{2}) = q(?????)$$

Since $t = -\frac{3}{2}$ is not a valid input for q, it is not a valid input for the literal definition of $q \circ q$ as "do q, then do q again". So in fact

$$q \circ q(t) = \begin{cases} \frac{-3t-8}{8t+5} & t \neq -\frac{3}{2}, -\frac{5}{8} \\ \text{undefined} & t = -\frac{3}{2}, -\frac{5}{8} \end{cases}$$

As for the range, we can ask whether, given some target output y, there is some t with $q \circ q(t) = y$. That is, we solve

$$y = q \circ q(t) = \frac{-3t-8}{8t+5}$$

or

$$(8t+5)y = -3t-8$$

$$8ty + 5y + 3t + 8 = 0$$

$$(8y+3)t + 5y + 8 = 0$$

$$t = \frac{-5y-8}{8y+3}$$

So, it seems like we have a formula for what t we need to get whatever y we want, except for $y = -\frac{3}{8}$. But, unfortunately, there is another trick. There is a certain y which will make $t = -\frac{3}{2}$. We saw earlier that when $t = -\frac{3}{2}$, $q \circ q(t)$ "wants to be" $\frac{1}{2}$, but since this input is not allowed, this value is never actually achieved.

So, the range of $q \circ q(t)$ is all real numbers except $-\frac{3}{8}$ and $\frac{1}{2}$. In interval notation, you could say the range is $(-\infty, -\frac{3}{8}) \cup (-\frac{3}{8}, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$.