a and b with $x_1 < x_2$, then $f(x_1) < f(x_2)$. We use this as the defining property of an increasing function.

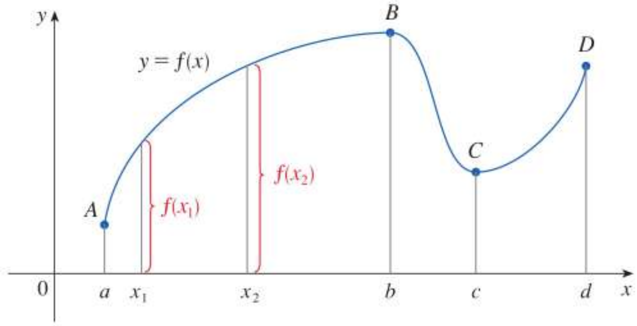


FIGURE 22

A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on I if

$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

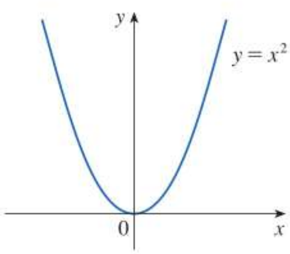


FIGURE 23

In the definition of an increasing function it is important to realize that the inequality $f(x_1) < f(x_2)$ must be satisfied for *every* pair of numbers x_1 and x_2 in I with $x_1 < x_2$.

You can see from Figure 23 that the function $f(x) = x^2$ is decreasing on the interval $(-\infty, 0]$ and increasing on the interval $[0, \infty)$.

1.1 Exercises

1. If $f(x) = x + \sqrt{2 - x}$ and $g(u) = u + \sqrt{2 - u}$, is it true that $f = g$?

2. If

$$f(x) = \frac{x^2 - x}{x - 1} \qquad \text{and} \qquad g(x) = x$$

is it true that $f = g$?

3. The graph of a function g is given.

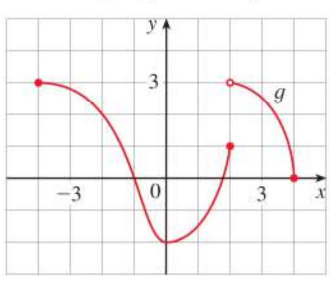
(a) State the values of $g(-2)$, $g(0)$, $g(2)$, and $g(3)$.

(b) For what value(s) of x is $g(x) = 3$?

(c) For what value(s) of x is $g(x) \leqslant 3$?

(d) State the domain and range of g.

(e) On what interval(s) is g increasing?



4. The graphs of f and g are given.

(a) State the values of $f(-4)$ and $g(3)$.

(b) Which is larger, $f(-3)$ or $g(-3)$?

(c) For what values of x is $f(x) = g(x)$?

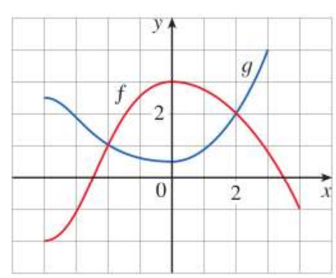
(d) On what interval(s) is $f(x) \leqslant g(x)$?

(e) State the solution of the equation $f(x) = -1$.

(f) On what interval(s) is g decreasing?

(g) State the domain and range of f.

(h) State the domain and range of g.



5. Figure 1 was recorded by an instrument operated by the California Department of Mines and Geology at the

University Hospital of the University of Southern California in Los Angeles. Use it to estimate the range of the vertical ground acceleration function at USC during the Northridge earthquake.

6. In this section we discussed examples of ordinary, everyday functions: population is a function of time, postage cost is a function of package weight, water temperature is a function of time. Give three other examples of functions from everyday life that are described verbally. What can you say about the domain and range of each of your functions? If possible, sketch a rough graph of each function.

7–14 Determine whether the equation or table defines y as a function of x.

7. $3x - 5y = 7$

8. $3x^2 - 2y = 5$

9. $x^2 + (y - 3)^2 = 5$

10. $2xy + 5y^2 = 4$

11. $(y + 3)^3 + 1 = 2x$

12. $2x - |y| = 0$

13.

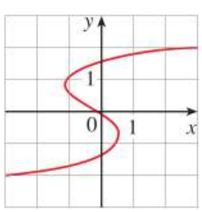
x Height (in)	y Shoe size
72	12
60	8
60	7
63	9
70	10

14.

x Year	y Tuition cost ($)
2016	10,900
2017	11,000
2018	11,200
2019	11,200
2020	11,300

15–18 Determine whether the curve is the graph of a function of x. If it is, state the domain and range of the function.

15.



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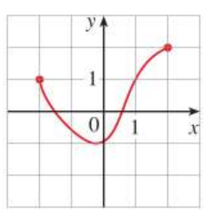

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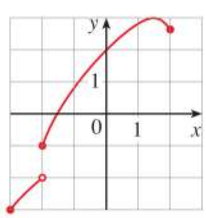

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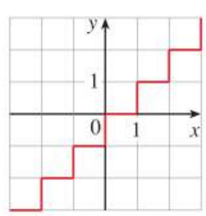


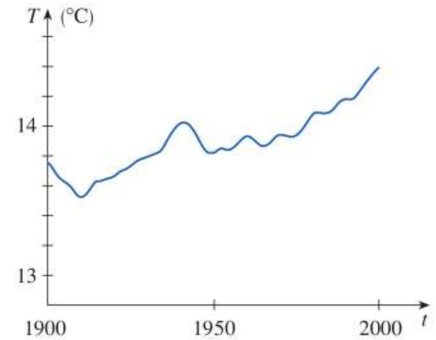
19. Shown is a graph of the global average temperature T during the 20th century. Estimate the following.

(a) The global average temperature in 1950

(b) The year when the average temperature was 14.2°C

(c) The years when the temperature was smallest and largest

(d) The range of T

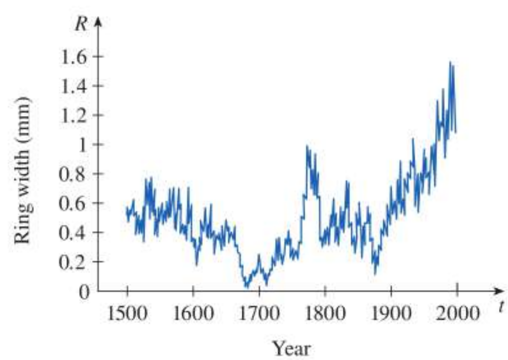


Source: Adapted from *Globe and Mail* [Toronto], 5 Dec. 2009. Print.

20. Trees grow faster and form wider rings in warm years and grow more slowly and form narrower rings in cooler years. The figure shows ring widths of a Siberian pine from 1500 to 2000.

(a) What is the range of the ring width function?

(b) What does the graph tend to say about the temperature of the earth? Does the graph reflect the volcanic eruptions of the mid-19th century?



Source: Adapted from G. Jacoby et al., "Mongolian Tree Rings and 20th-Century Warming," *Science* 273 (1996): 771–73.

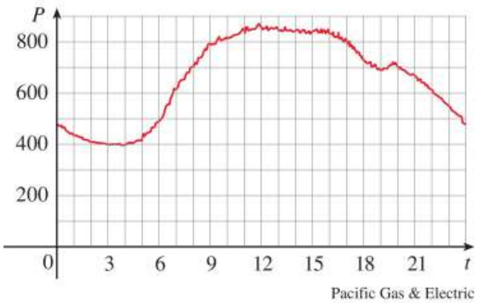
21. You put some ice cubes in a glass, fill the glass with cold water, and then let the glass sit on a table. Describe how the temperature of the water changes as time passes. Then sketch a rough graph of the temperature of the water as a function of the elapsed time.

22. You place a frozen pie in an oven and bake it for an hour. Then you take it out and let it cool. Describe how the temperature of the pie changes as time passes. Then sketch a rough graph of the temperature of the pie as a function of time.

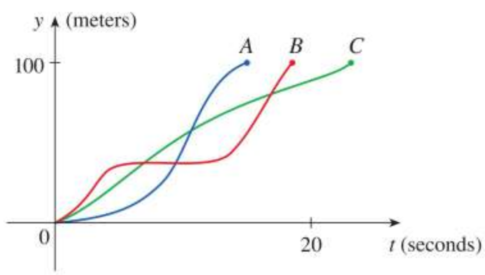
23. The graph shows the power consumption for a day in September in San Francisco. (P is measured in megawatts; t is measured in hours starting at midnight.)

(a) What was the power consumption at 6 AM? At 6 PM?

(b) When was the power consumption the lowest? When was it the highest? Do these times seem reasonable?



24. Three runners compete in a 100-meter race. The graph depicts the distance run as a function of time for each runner. Describe in words what the graph tells you about this race. Who won the race? Did each runner finish the race?



25. Sketch a rough graph of the outdoor temperature as a function of time during a typical spring day.

26. Sketch a rough graph of the number of hours of daylight as a function of the time of year.

27. Sketch a rough graph of the amount of a particular brand of coffee sold by a store as a function of the price of the coffee.

28. Sketch a rough graph of the market value of a new car as a function of time for a period of 20 years. Assume the car is well maintained.

29. A homeowner mows the lawn every Wednesday afternoon. Sketch a rough graph of the height of the grass as a function of time over the course of a four-week period.

30. An airplane takes off from an airport and lands an hour later at another airport, 400 miles away. If t represents the time in minutes since the plane has left the terminal building, let $x(t)$ be the horizontal distance traveled and $y(t)$ be the altitude of the plane.

(a) Sketch a possible graph of $x(t)$.
(b) Sketch a possible graph of $y(t)$.
(c) Sketch a possible graph of the ground speed.
(d) Sketch a possible graph of the vertical velocity.

31. Temperature readings T (in °F) were recorded every two hours from midnight to 2:00 PM in Atlanta on a day in June. The time t was measured in hours from midnight.

t	0	2	4	6	8	10	12	14
T	74	69	68	66	70	78	82	86

(a) Use the readings to sketch a rough graph of T as a function of t.
(b) Use your graph to estimate the temperature at 9:00 AM.

32. Researchers measured the blood alcohol concentration (BAC) of eight adult male subjects after rapid consumption of 30 mL of ethanol (corresponding to two standard alcoholic drinks). The table shows the data they obtained by averaging the BAC (in g/dL) of the eight men.

(a) Use the readings to sketch a graph of the BAC as a function of t.
(b) Use your graph to describe how the effect of alcohol varies with time.

t (hours)	BAC	t (hours)	BAC
0	0	1.75	0.022
0.2	0.025	2.0	0.018
0.5	0.041	2.25	0.015
0.75	0.040	2.5	0.012
1.0	0.033	3.0	0.007
1.25	0.029	3.5	0.003
1.5	0.024	4.0	0.001

Source: Adapted from P. Wilkinson et al., "Pharmacokinetics of Ethanol after Oral Administration in the Fasting State," *Journal of Pharmacokinetics and Biopharmaceutics* 5 (1977): 207–24.

33. If $f(x) = 3x^2 - x + 2$, find $f(2)$, $f(-2)$, $f(a)$, $f(-a)$, $f(a + 1)$, $2f(a)$, $f(2a)$, $f(a^2)$, $[f(a)]^2$, and $f(a + h)$.

34. If $g(x) = \dfrac{x}{\sqrt{x + 1}}$, find $g(0)$, $g(3)$, $5g(a)$, $\frac{1}{2}g(4a)$, $g(a^2)$, $[g(a)]^2$, $g(a + h)$, and $g(x - a)$.

35–38 Evaluate the difference quotient for the given function. Simplify your answer.

35. $f(x) = 4 + 3x - x^2$, $\quad \dfrac{f(3 + h) - f(3)}{h}$

36. $f(x) = x^3$, $\quad \dfrac{f(a + h) - f(a)}{h}$

37. $f(x) = \dfrac{1}{x}$, $\quad \dfrac{f(x) - f(a)}{x - a}$

38. $f(x) = \sqrt{x + 2}$, $\quad \dfrac{f(x) - f(1)}{x - 1}$

39–46 Find the domain of the function.

39. $f(x) = \dfrac{x + 4}{x^2 - 9}$

40. $f(x) = \dfrac{x^2 + 1}{x^2 + 4x - 21}$

41. $f(t) = \sqrt[3]{2t - 1}$

42. $g(t) = \sqrt{3 - t} - \sqrt{2 + t}$

43. $h(x) = \dfrac{1}{\sqrt[4]{x^2 - 5x}}$

44. $f(u) = \dfrac{u + 1}{1 + \dfrac{1}{u + 1}}$

45. $F(p) = \sqrt{2 - \sqrt{p}}$

46. $h(x) = \sqrt{x^2 - 4x - 5}$

47. Find the domain and range and sketch the graph of the function $h(x) = \sqrt{4 - x^2}$.

48. Find the domain and sketch the graph of the function

$$f(x) = \frac{x^2 - 4}{x - 2}$$

49–52 Evaluate $f(-3)$, $f(0)$, and $f(2)$ for the piecewise defined function. Then sketch the graph of the function.

49. $f(x) = \begin{cases} x^2 + 2 & \text{if } x < 0 \\ x & \text{if } x \geqslant 0 \end{cases}$

50. $f(x) = \begin{cases} 5 & \text{if } x < 2 \\ \frac{1}{2}x - 3 & \text{if } x \geqslant 2 \end{cases}$

51. $f(x) = \begin{cases} x + 1 & \text{if } x \leqslant -1 \\ x^2 & \text{if } x > -1 \end{cases}$

52. $f(x) = \begin{cases} -1 & \text{if } x \leqslant 1 \\ 7 - 2x & \text{if } x > 1 \end{cases}$

53–58 Sketch the graph of the function.

53. $f(x) = x + |x|$

54. $f(x) = |x + 2|$

55. $g(t) = |1 - 3t|$

56. $f(x) = \dfrac{|x|}{x}$

57. $f(x) = \begin{cases} |x| & \text{if } |x| \leqslant 1 \\ 1 & \text{if } |x| > 1 \end{cases}$

58. $g(x) = ||x| - 1|$

59–64 Find a formula for the function whose graph is the given curve.

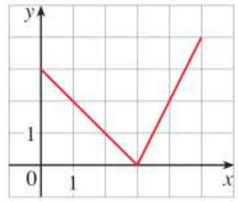
59. The line segment joining the points $(1, -3)$ and $(5, 7)$

60. The line segment joining the points $(-5, 10)$ and $(7, -10)$

61. The bottom half of the parabola $x + (y - 1)^2 = 0$

62. The top half of the circle $x^2 + (y - 2)^2 = 4$

63.



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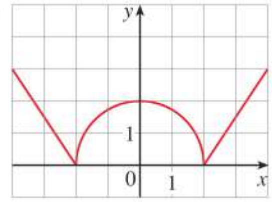


65–70 Find a formula for the described function and state its domain.

65. A rectangle has perimeter 20 m. Express the area of the rectangle as a function of the length of one of its sides.

66. A rectangle has area 16 m^2. Express the perimeter of the rectangle as a function of the length of one of its sides.

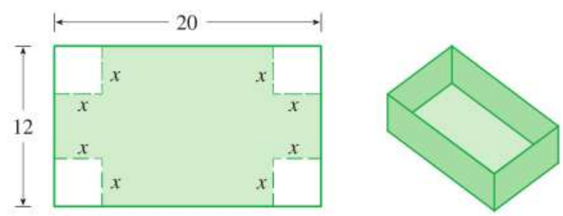
67. Express the area of an equilateral triangle as a function of the length of a side.

68. A closed rectangular box with volume 8 ft^3 has length twice the width. Express the height of the box as a function of the width.

69. An open rectangular box with volume 2 m^3 has a square base. Express the surface area of the box as a function of the length of a side of the base.

70. A right circular cylinder has volume 25 in^3. Express the radius of the cylinder as a function of the height.

71. A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 12 in. by 20 in. by cutting out equal squares of side x at each corner and then folding up the sides as in the figure. Express the volume V of the box as a function of x.



72. A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 30 ft, express the area A of the window as a function of the width x of the window.

73. In a certain state the maximum speed permitted on freeways is 65 mi/h and the minimum speed is 40 mi/h. The fine for violating these limits is \$15 for every mile per hour above the maximum speed or below the minimum speed. Express the amount of the fine F as a function of the driving speed x and graph $F(x)$ for $0 \leqslant x \leqslant 100$.

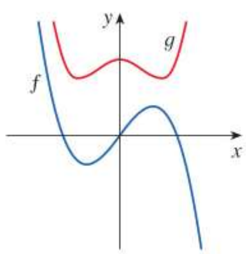
74. An electricity company charges its customers a base rate of \$10 a month, plus 6 cents per kilowatt-hour (kWh) for the first 1200 kWh and 7 cents per kWh for all usage over 1200 kWh. Express the monthly cost E as a function of the amount x of electricity used. Then graph the function E for $0 \leqslant x \leqslant 2000$.

75. In a certain country, income tax is assessed as follows. There is no tax on income up to \$10,000. Any income over \$10,000 is taxed at a rate of 10%, up to an income of \$20,000. Any income over \$20,000 is taxed at 15%.
(a) Sketch the graph of the tax rate R as a function of the income I.
(b) How much tax is assessed on an income of \$14,000? On \$26,000?
(c) Sketch the graph of the total assessed tax T as a function of the income I.

76. (a) If the point (5, 3) is on the graph of an even function, what other point must also be on the graph?
(b) If the point (5, 3) is on the graph of an odd function, what other point must also be on the graph?

77–78 Graphs of f and g are shown. Decide whether each function is even, odd, or neither. Explain your reasoning.

77.



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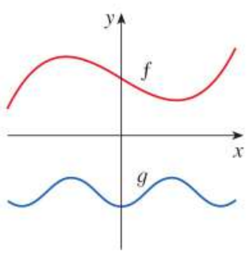
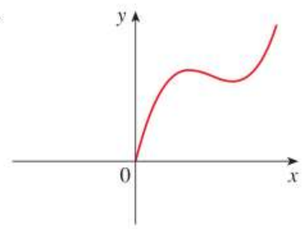


79–80 The graph of a function defined for $x \geqslant 0$ is given. Complete the graph for $x < 0$ to make (a) an even function and (b) an odd function.

79.



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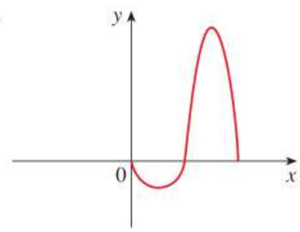


81–86 Determine whether f is even, odd, or neither. You may wish to use a graphing calculator or computer to check your answer visually.

81. $f(x) = \dfrac{x}{x^2 + 1}$

82. $f(x) = \dfrac{x^2}{x^4 + 1}$

83. $f(x) = \dfrac{x}{x + 1}$

84. $f(x) = x|x|$

85. $f(x) = 1 + 3x^2 - x^4$

86. $f(x) = 1 + 3x^3 - x^5$

87. If f and g are both even functions, is $f + g$ even? If f and g are both odd functions, is $f + g$ odd? What if f is even and g is odd? Justify your answers.

88. If f and g are both even functions, is the product fg even? If f and g are both odd functions, is fg odd? What if f is even and g is odd? Justify your answers.

1.2 Mathematical Models: A Catalog of Essential Functions

A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon such as the size of a population, the demand for a product, the speed of a falling object, the concentration of a product in a chemical reaction, the life expectancy of a person at birth, or the cost of emissions reductions. The purpose of the model is to understand the phenomenon and perhaps to make predictions about future behavior.

Given a real-world problem, our first task in the mathematical modeling process is to formulate a mathematical model by identifying and naming the independent and dependent variables and making assumptions that simplify the phenomenon enough to make it mathematically tractable. We use our knowledge of the physical situation and our

1.2 Exercises

1–2 Classify each function as a power function, root function, polynomial (state its degree), rational function, algebraic function, trigonometric function, exponential function, or logarithmic function.

1. (a) $f(x) = x^3 + 3x^2$ (b) $g(t) = \cos^2 t - \sin t$

(c) $r(t) = t^{\sqrt{3}}$ (d) $v(t) = 8^t$

(e) $y = \dfrac{\sqrt{x}}{x^2 + 1}$ (f) $g(u) = \log_{10} u$

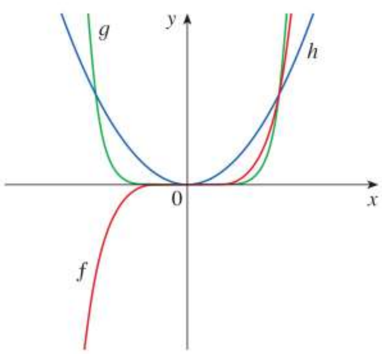
2. (a) $f(t) = \dfrac{3t^2 + 2}{t}$ (b) $h(r) = 2.3^r$

(c) $s(t) = \sqrt{t + 4}$ (d) $y = x^4 + 5$

(e) $g(x) = \sqrt[3]{x}$ (f) $y = \dfrac{1}{x^2}$

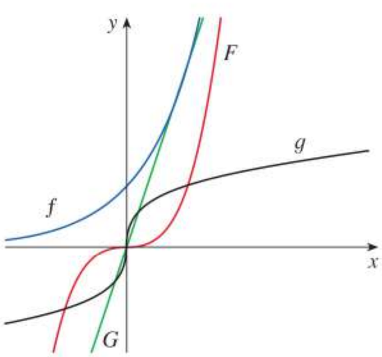
3–4 Match each equation with its graph. Explain your choices. (Don't use a computer or graphing calculator.)

3. (a) $y = x^2$ (b) $y = x^5$ (c) $y = x^8$



4. (a) $y = 3x$ (b) $y = 3^x$

(c) $y = x^3$ (d) $y = \sqrt[3]{x}$



5–6 Find the domain of the function.

5. $f(x) = \dfrac{\cos x}{1 - \sin x}$ **6.** $g(x) = \dfrac{1}{1 - \tan x}$

7. (a) Find an equation for the family of linear functions with slope 2 and sketch several members of the family.
(b) Find an equation for the family of linear functions such that $f(2) = 1$. Sketch several members of the family.
(c) Which function belongs to both families?

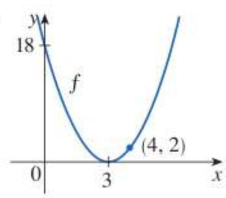
8. What do all members of the family of linear functions $f(x) = 1 + m(x + 3)$ have in common? Sketch several members of the family.

9. What do all members of the family of linear functions $f(x) = c - x$ have in common? Sketch several members of the family.

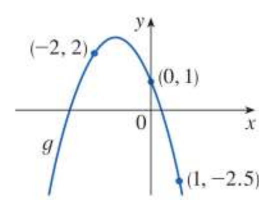
10. Sketch several members of the family of polynomials $P(x) = x^3 - cx^2$. How does the graph change when c changes?

11–12 Find a formula for the quadratic function whose graph is shown.

11.



12.



13. Find a formula for a cubic function f if $f(1) = 6$ and $f(-1) = f(0) = f(2) = 0$.

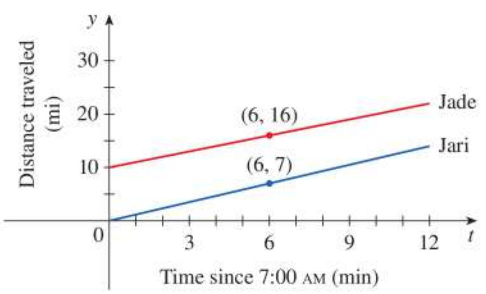
14. Recent studies indicate that the average surface temperature of the earth has been rising steadily. Some scientists have modeled the temperature by the linear function $T = 0.02t + 8.50$, where T is temperature in °C and t represents years since 1900.
(a) What do the slope and T-intercept represent?
(b) Use the equation to predict the earth's average surface temperature in 2100.

15. If the recommended adult dosage for a drug is D (in mg), then to determine the appropriate dosage c for a child of age a, pharmacists use the equation $c = 0.0417D(a + 1)$. Suppose the dosage for an adult is 200 mg.
(a) Find the slope of the graph of c. What does it represent?
(b) What is the dosage for a newborn?

16. The manager of a weekend flea market knows from past experience that if he charges x dollars for a rental space at the market, then the number y of spaces that will be rented is given by the equation $y = 200 - 4x$.
(a) Sketch a graph of this linear function. (Remember that the rental charge per space and the number of spaces rented can't be negative quantities.)
(b) What do the slope, the y-intercept, and the x-intercept of the graph represent?

17. The relationship between the Fahrenheit (F) and Celsius (C) temperature scales is given by the linear function $F = \frac{9}{5}C + 32$.
(a) Sketch a graph of this function.
(b) What is the slope of the graph and what does it represent? What is the F-intercept and what does it represent?

18. Jade and her roommate Jari commute to work each morning, traveling west on I-10. One morning Jade left for work at 6:50 AM, but Jari left 10 minutes later. Both drove at a constant speed. The graphs show the distance (in miles) each of them has traveled on I-10, t minutes after 7:00 AM.
(a) Use the graph to decide which driver is traveling faster.
(b) Find the speed (in mi/h) at which each of them is driving.
(c) Find linear functions f and g that model the distances traveled by Jade and Jari as functions of t (in minutes).



19. The manager of a furniture factory finds that it costs \$2200 to manufacture 100 chairs in one day and \$4800 to produce 300 chairs in one day.
(a) Express the cost as a function of the number of chairs produced, assuming that it is linear. Then sketch the graph.
(b) What is the slope of the graph and what does it represent?
(c) What is the y-intercept of the graph and what does it represent?

20. The monthly cost of driving a car depends on the number of miles driven. Lynn found that in May it cost her \$380 to drive 480 mi and in June it cost her \$460 to drive 800 mi.
(a) Express the monthly cost C as a function of the distance driven d, assuming that a linear relationship gives a suitable model.
(b) Use part (a) to predict the cost of driving 1500 miles per month.
(c) Draw the graph of the linear function. What does the slope represent?
(d) What does the C-intercept represent?
(e) Why does a linear function give a suitable model in this situation?

21. At the surface of the ocean, the water pressure is the same as the air pressure above the water, 15 lb/in^2. Below the surface, the water pressure increases by 4.34 lb/in^2 for every 10 ft of descent.
(a) Express the water pressure as a function of the depth below the ocean surface.
(b) At what depth is the pressure 100 lb/in^2?

22. The resistance R of a wire of fixed length is related to its diameter x by an inverse square law, that is, by a function of the form $R(x) = kx^{-2}$.
(a) A wire of fixed length and 0.005 meters in diameter has a resistance of 140 ohms. Find the value of k.
(b) Find the resistance of a wire made of the same material and of the same length as the wire in part (a) but with a diameter of 0.008 meters.

23. The illumination of an object by a light source is related to the distance from the source by an inverse square law. Suppose that after dark you are sitting in a room with just one lamp, trying to read a book. The light is too dim, so you move your chair halfway to the lamp. How much brighter is the light?

24. The pressure P of a sample of oxygen gas that is compressed at a constant temperature is related to the volume V of gas by a reciprocal function of the form $P = k/V$.
(a) A sample of oxygen gas that occupies 0.671 m^3 exerts a pressure of 39 kPa at a temperature of 293 K (absolute temperature measured on the Kelvin scale). Find the value of k in the given model.
(b) If the sample expands to a volume of 0.916 m^3, find the new pressure.

25. The power output of a wind turbine depends on many factors. It can be shown using physical principles that the power P generated by a wind turbine is modeled by

$$P = kAv^3$$

where v is the wind speed, A is the area swept out by the blades, and k is a constant that depends on air density, efficiency of the turbine, and the design of the wind turbine blades.
(a) If only wind speed is doubled, by what factor is the power output increased?
(b) If only the length of the blades is doubled, by what factor is the power output increased?
(c) For a particular wind turbine, the length of the blades is 30 m and $k = 0.214$ kg/m^3. Find the power output (in watts, W = m^2 · kg/s^3) when the wind speed is 10 m/s, 15 m/s, and 25 m/s.

26. Astronomers infer the radiant exitance (radiant flux emitted per unit area) of stars using the Stefan Boltzmann Law:

$$E(T) = (5.67 \times 10^{-8})T^4$$

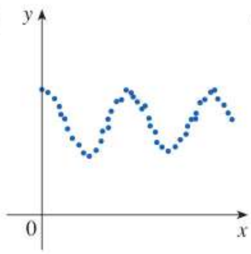
where E is the energy radiated per unit of surface area measured in watts (W) and T is the absolute temperature measured in kelvins (K).

(a) Graph the function E for temperatures T between 100 K and 300 K.

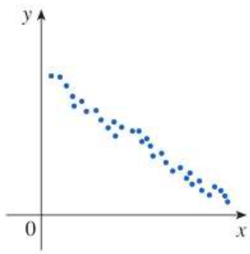
(b) Use the graph to describe the change in energy E as the temperature T increases.

27–28 For each scatter plot, decide what type of function you might choose as a model for the data. Explain your choices.

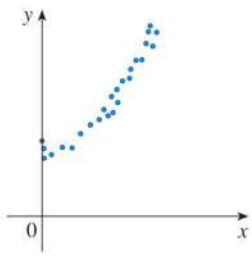
27. (a)



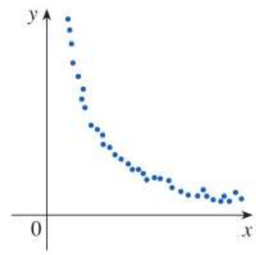
(b)



28. (a)



(b)



T 29. The table shows (lifetime) peptic ulcer rates (per 100 population) for various family incomes as reported by the National Health Interview Survey.

(a) Make a scatter plot of these data and decide whether a linear model is appropriate.

(b) Find and graph a linear model using the first and last data points.

(c) Find and graph the regression line.

(d) Use the linear model in part (c) to estimate the ulcer rate for people with an income of $25,000.

(e) According to the model, how likely is someone with an income of $80,000 to suffer from peptic ulcers?

(f) Do you think it would be reasonable to apply the model to someone with an income of $200,000?

Income	Ulcer rate (per 100 population)
$4,000	14.1
$6,000	13.0
$8,000	13.4
$12,000	12.5
$16,000	12.0
$20,000	12.4
$30,000	10.5
$45,000	9.4
$60,000	8.2

T 30. When laboratory rats are exposed to asbestos fibers, some of them develop lung tumors. The table lists the results of several experiments by different scientists.

(a) Find the regression line for the data.

(b) Make a scatter plot and graph the regression line. Does the regression line appear to be a suitable model for the data?

(c) What does the y-intercept of the regression line represent?

Asbestos exposure (fibers/mL)	Percent of mice that develop lung tumors	Asbestos exposure (fibers/mL)	Percent of mice that develop lung tumors
50	2	1600	42
400	6	1800	37
500	5	2000	38
900	10	3000	50
1100	26		

T 31. Anthropologists use a linear model that relates human femur (thighbone) length to height. The model allows an anthropologist to determine the height of an individual when only a partial skeleton (including the femur) is found. Here we find the model by analyzing the data on femur length and height for the eight males given in the table.

(a) Make a scatter plot of the data.

(b) Find and graph the regression line that models the data.

(c) An anthropologist finds a human femur of length 53 cm. How tall was the person?

Femur length (cm)	Height (cm)	Femur length (cm)	Height (cm)
50.1	178.5	44.5	168.3
48.3	173.6	42.7	165.0
45.2	164.8	39.5	155.4
44.7	163.7	38.0	155.8

T 32. The table shows average US retail residential prices of electricity from 2000 to 2016, measured in cents per kilowatt hour.

(a) Make a scatter plot. Is a linear model appropriate?

(b) Find and graph the regression line.

(c) Use your linear model from part (b) to estimate the average retail price of electricity in 2005 and 2017.

Years since 2000	Cents/kWh	Years since 2000	Cents/kWh
0	8.24	10	11.54
2	8.44	12	11.88
4	8.95	14	12.52
6	10.40	16	12.90
8	11.26		

Source: US Energy Information Administration

T **33.** The table shows world average daily oil consumption from 1985 to 2015, measured in thousands of barrels per day.
(a) Make a scatter plot and decide whether a linear model is appropriate.
(b) Find and graph the regression line.
(c) Use the linear model to estimate the oil consumption in 2002 and 2017.

Years since 1985	Thousands of barrels of oil per day
0	60,083
5	66,533
10	70,099
15	76,784
20	84,077
25	87,302
30	94,071

Source: US Energy Information Administration

T **34.** The table shows the mean (average) distances d of the planets from the sun (taking the unit of measurement to be the distance from the earth to the sun) and their periods T (time of revolution in years).
(a) Fit a power model to the data.
(b) Kepler's Third Law of Planetary Motion states that "The square of the period of revolution of a planet is proportional to the cube of its mean distance from the sun." Does your model corroborate Kepler's Third Law?

Planet	d	T
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.523	1.881
Jupiter	5.203	11.861
Saturn	9.541	29.457
Uranus	19.190	84.008
Neptune	30.086	164.784

35. It makes sense that the larger the area of a region, the larger the number of species that inhabit the region. Many ecologists have modeled the species-area relation with a power function. In particular, the number of species S of bats living in caves in central Mexico has been related to the surface area A of the caves by the equation $S = 0.7A^{0.3}$.
(a) The cave called *Misión Imposible* near Puebla, Mexico, has a surface area of $A = 60\ \text{m}^2$. How many species of bats would you expect to find in that cave?
(b) If you discover that four species of bats live in a cave, estimate the area of the cave.

T **36.** The table shows the number N of species of reptiles and amphibians inhabiting Caribbean islands and the area A of the island in square miles.
(a) Use a power function to model N as a function of A.
(b) The Caribbean island of Dominica has area 291 mi^2. How many species of reptiles and amphibians would you expect to find on Dominica?

Island	A	N
Saba	4	5
Monserrat	40	9
Puerto Rico	3,459	40
Jamaica	4,411	39
Hispaniola	29,418	84
Cuba	44,218	76

37. Suppose that a force or energy originates from a point source and spreads its influence equally in all directions, such as the light from a lightbulb or the gravitational force of a planet. So at a distance r from the source, the intensity I of the force or energy is equal to the source strength S divided by the surface area of a sphere of radius r. Show that I satisfies the inverse square law $I = k/r^2$, where k is a positive constant.

1.3 New Functions from Old Functions

In this section we start with the basic functions we discussed in Section 1.2 and obtain new functions by shifting, stretching, and reflecting their graphs. We also show how to combine pairs of functions by the standard arithmetic operations and by composition.

Transformations of Functions

By applying certain transformations to the graph of a given function we can obtain the graphs of related functions. This will give us the ability to sketch the graphs of many functions quickly by hand. It will also enable us to write equations for given graphs.

(d) $$(g \circ g)(x) = g(g(x)) = g(\sqrt{2-x}) = \sqrt{2 - \sqrt{2-x}}$$

This expression is defined when both $2 - x \geqslant 0$ and $2 - \sqrt{2-x} \geqslant 0$. The first inequality means $x \leqslant 2$, and the second is equivalent to $\sqrt{2-x} \leqslant 2$, or $2 - x \leqslant 4$, or $x \geqslant -2$. Thus $-2 \leqslant x \leqslant 2$, so the domain of $g \circ g$ is the closed interval $[-2, 2]$. ■

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h, then g, and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

EXAMPLE 8 Find $f \circ g \circ h$ if $f(x) = x/(x+1)$, $g(x) = x^{10}$, and $h(x) = x + 3$.

SOLUTION

$$\begin{aligned}(f \circ g \circ h)(x) &= f(g(h(x))) = f(g(x+3)) \\ &= f((x+3)^{10}) = \frac{(x+3)^{10}}{(x+3)^{10}+1}\end{aligned}$$ ■

So far we have used composition to build complicated functions from simpler ones. But in calculus it is often useful to be able to *decompose* a complicated function into simpler ones, as in the following example.

EXAMPLE 9 Given $F(x) = \cos^2(x + 9)$, find functions f, g, and h such that $F = f \circ g \circ h$.

SOLUTION Since $F(x) = [\cos(x+9)]^2$, the formula for F says: first add 9, then take the cosine of the result, and finally square. So we let

$$h(x) = x + 9 \qquad g(x) = \cos x \qquad f(x) = x^2$$

Then $$\begin{aligned}(f \circ g \circ h)(x) &= f(g(h(x))) = f(g(x+9)) = f(\cos(x+9)) \\ &= [\cos(x+9)]^2 = F(x)\end{aligned}$$ ■

1.3 Exercises

1. Suppose the graph of f is given. Write equations for the graphs that are obtained from the graph of f as follows.

(a) Shift 3 units upward.

(b) Shift 3 units downward.

(c) Shift 3 units to the right.

(d) Shift 3 units to the left.

(e) Reflect about the x-axis.

(f) Reflect about the y-axis.

(g) Stretch vertically by a factor of 3.

(h) Shrink vertically by a factor of 3.

2. Explain how each graph is obtained from the graph of $y = f(x)$.

(a) $y = f(x) + 8$ (b) $y = f(x + 8)$

(c) $y = 8f(x)$ (d) $y = f(8x)$

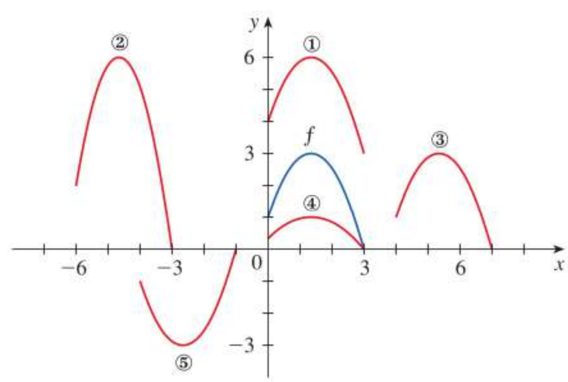
(e) $y = -f(x) - 1$ (f) $y = 8f\left(\frac{1}{8}x\right)$

3. The graph of $y = f(x)$ is given. Match each equation with its graph and give reasons for your choices.

(a) $y = f(x - 4)$ (b) $y = f(x) + 3$

(c) $y = \frac{1}{3}f(x)$ (d) $y = -f(x + 4)$

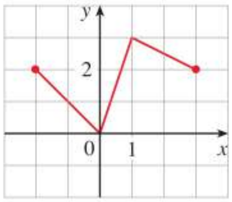
(e) $y = 2f(x + 6)$



4. The graph of f is given. Draw the graphs of the following functions.

(a) $y = f(x) - 3$ (b) $y = f(x + 1)$

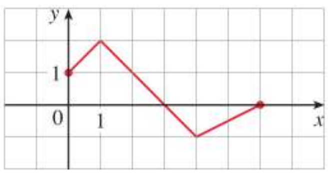
(c) $y = \frac{1}{2}f(x)$ (d) $y = -f(x)$



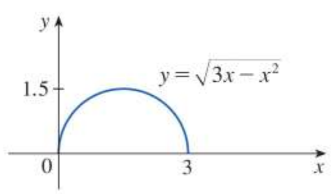
5. The graph of f is given. Use it to graph the following functions.

(a) $y = f(2x)$ (b) $y = f\left(\frac{1}{2}x\right)$

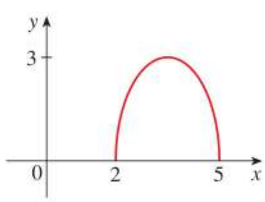
(c) $y = f(-x)$ (d) $y = -f(-x)$



6–7 The graph of $y = \sqrt{3x - x^2}$ is given. Use transformations to create a function whose graph is as shown.



6.



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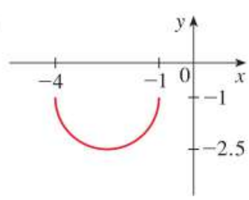


8. (a) How is the graph of $y = 1 + \sqrt{x}$ related to the graph of $y = \sqrt{x}$? Use your answer and Figure 4(a) to sketch the graph of $y = 1 + \sqrt{x}$.

(b) How is the graph of $y = 5 \sin \pi x$ related to the graph of $y = \sin x$? Use your answer and Figure 6 to sketch the graph of $y = 5 \sin \pi x$.

9–26 Graph the function by hand, not by plotting points, but by starting with the graph of one of the standard functions given in Table 1.2.3, and then applying the appropriate transformations.

9. $y = 1 + x^2$

10. $y = (x + 1)^2$

11. $y = |x + 2|$

12. $y = 1 - x^3$

13. $y = \dfrac{1}{x} + 2$

14. $y = -\sqrt{x} - 1$

15. $y = \sin 4x$

16. $y = 1 + \dfrac{1}{x^2}$

17. $y = 2 + \sqrt{x + 1}$

18. $y = -(x - 1)^2 + 3$

19. $y = x^2 - 2x + 5$

20. $y = (x + 1)^3 + 2$

21. $y = 2 - |x|$

22. $y = 2 - 2 \cos x$

23. $y = 3 \sin \frac{1}{2}x + 1$

24. $y = \dfrac{1}{4}\tan\left(x - \dfrac{\pi}{4}\right)$

25. $y = |\cos \pi x|$

26. $y = |\sqrt{x} - 1|$

27. The city of New Orleans is located at latitude 30°N. Use Figure 9 to find a function that models the number of hours of daylight at New Orleans as a function of the time of year. To check the accuracy of your model, use the fact that on March 31 the sun rises at 5:51 AM and sets at 6:18 PM in New Orleans.

28. A variable star is one whose brightness alternately increases and decreases. For the most visible variable star, Delta Cephei, the time between periods of maximum brightness is 5.4 days, the average brightness (or magnitude) of the star is 4.0, and its brightness varies by ± 0.35 magnitude. Find a function that models the brightness of Delta Cephei as a function of time.

29. Some of the highest tides in the world occur in the Bay of Fundy on the Atlantic Coast of Canada. At Hopewell Cape the water depth at low tide is about 2.0 m and at high tide it is about 12.0 m. The natural period of oscillation is about 12 hours and on a particular day, high tide occurred at 6:45 AM. Find a function involving the cosine function that models the water depth $D(t)$ (in meters) as a function of time t (in hours after midnight) on that day.

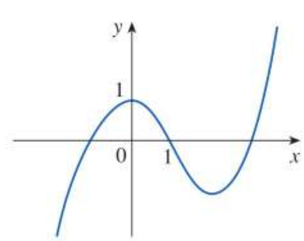
30. In a normal respiratory cycle the volume of air that moves into and out of the lungs is about 500 mL. The reserve and residue volumes of air that remain in the lungs occupy about 2000 mL and a single respiratory cycle for an average human takes about 4 seconds. Find a model for the total volume of air $V(t)$ in the lungs as a function of time.

31. (a) How is the graph of $y = f(|x|)$ related to the graph of f?

(b) Sketch the graph of $y = \sin |x|$.

(c) Sketch the graph of $y = \sqrt{|x|}$.

32. Use the given graph of f to sketch the graph of $y = 1/f(x)$. Which features of f are the most important in sketching $y = 1/f(x)$? Explain how they are used.



33–34 Find (a) $f+g$, (b) $f-g$, (c) fg, and (d) f/g and state their domains.

33. $f(x)=\sqrt{25-x^2}$, $g(x)=\sqrt{x+1}$

34. $f(x)=\dfrac{1}{x-1}$, $g(x)=\dfrac{1}{x}-2$

35–40 Find the functions (a) $f\circ g$, (b) $g\circ f$, (c) $f\circ f$, and (d) $g\circ g$ and their domains.

35. $f(x)=x^3+5$, $g(x)=\sqrt[3]{x}$

36. $f(x)=\dfrac{1}{x}$, $g(x)=2x+1$

37. $f(x)=\dfrac{1}{\sqrt{x}}$, $g(x)=x+1$

38. $f(x)=\dfrac{x}{x+1}$, $g(x)=2x-1$

39. $f(x)=\dfrac{2}{x}$, $g(x)=\sin x$

40. $f(x)=\sqrt{5-x}$, $g(x)=\sqrt{x-1}$

41–44 Find $f\circ g\circ h$.

41. $f(x)=3x-2$, $g(x)=\sin x$, $h(x)=x^2$

42. $f(x)=|x-4|$, $g(x)=2^x$, $h(x)=\sqrt{x}$

43. $f(x)=\sqrt{x-3}$, $g(x)=x^2$, $h(x)=x^3+2$

44. $f(x)=\tan x$, $g(x)=\dfrac{x}{x-1}$, $h(x)=\sqrt[3]{x}$

45–50 Express the function in the form $f\circ g$.

45. $F(x)=(2x+x^2)^4$

46. $F(x)=\cos^2 x$

47. $F(x)=\dfrac{\sqrt[3]{x}}{1+\sqrt[3]{x}}$

48. $G(x)=\sqrt[3]{\dfrac{x}{1+x}}$

49. $v(t)=\sec(t^2)\tan(t^2)$

50. $H(x)=\sqrt{1+\sqrt{x}}$

51–54 Express the function in the form $f\circ g\circ h$.

51. $R(x)=\sqrt{\sqrt{x}-1}$

52. $H(x)=\sqrt[8]{2+|x|}$

53. $S(t)=\sin^2(\cos t)$

54. $H(t)=\cos\left(\sqrt{\tan t}+1\right)$

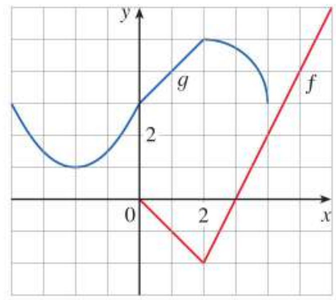
55–56 Use the table to evaluate each expression.

x	1	2	3	4	5	6
$f(x)$	3	1	5	6	2	4
$g(x)$	5	3	4	1	3	2

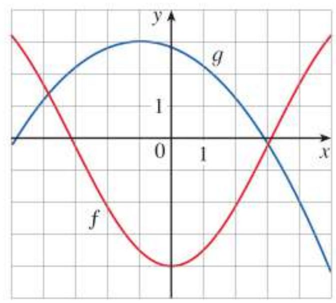
55. (a) $f(g(3))$ (b) $g(f(2))$
(c) $(f\circ g)(5)$ (d) $(g\circ f)(5)$

56. (a) $g(g(g(2)))$ (b) $(f\circ f\circ f)(1)$
(c) $(f\circ f\circ g)(1)$ (d) $(g\circ f\circ g)(3)$

57. Use the given graphs of f and g to evaluate each expression, or explain why it is undefined.
(a) $f(g(2))$ (b) $g(f(0))$ (c) $(f\circ g)(0)$
(d) $(g\circ f)(6)$ (e) $(g\circ g)(-2)$ (f) $(f\circ f)(4)$



58. Use the given graphs of f and g to estimate the value of $f(g(x))$ for $x=-5,-4,-3,\ldots,5$. Use these estimates to sketch a rough graph of $f\circ g$.



59. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s.
(a) Express the radius r of this circle as a function of the time t (in seconds).
(b) If A is the area of this circle as a function of the radius, find $A\circ r$ and interpret it.

60. A spherical balloon is being inflated and the radius of the balloon is increasing at a rate of 2 cm/s.
(a) Express the radius r of the balloon as a function of the time t (in seconds).
(b) If V is the volume of the balloon as a function of the radius, find $V\circ r$ and interpret it.

61. A ship is moving at a speed of 30 km/h parallel to a straight shoreline. The ship is 6 km from shore and it passes a lighthouse at noon.
(a) Express the distance s between the lighthouse and the ship as a function of d, the distance the ship has traveled since noon; that is, find f so that $s=f(d)$.
(b) Express d as a function of t, the time elapsed since noon; that is, find g so that $d=g(t)$.
(c) Find $f\circ g$. What does this function represent?

62. An airplane is flying at a speed of 350 mi/h at an altitude of one mile and passes directly over a radar station at time $t = 0$.

(a) Express the horizontal distance d (in miles) that the plane has flown as a function of t.

(b) Express the distance s between the plane and the radar station as a function of d.

(c) Use composition to express s as a function of t.

63. The Heaviside Function The *Heaviside function H* is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geqslant 0 \end{cases}$$

It is used in the study of electric circuits to represent the sudden surge of electric current, or voltage, when a switch is instantaneously turned on.

(a) Sketch the graph of the Heaviside function.

(b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and 120 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$.

(c) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 5$ seconds and 240 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$. (Note that starting at $t = 5$ corresponds to a translation.)

64. The Ramp Function The Heaviside function defined in Exercise 63 can also be used to define the *ramp function* $y = ctH(t)$, which represents a gradual increase in voltage or current in a circuit.

(a) Sketch the graph of the ramp function $y = tH(t)$.

(b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and the voltage is gradually increased to 120 volts over a 60-second time interval. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leqslant 60$.

(c) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 7$ seconds and the voltage is gradually increased to 100 volts over a period of 25 seconds. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leqslant 32$.

65. Let f and g be linear functions with equations $f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$. Is $f \circ g$ also a linear function? If so, what is the slope of its graph?

66. If you invest x dollars at 4% interest compounded annually, then the amount $A(x)$ of the investment after one year is $A(x) = 1.04x$. Find $A \circ A$, $A \circ A \circ A$, and $A \circ A \circ A \circ A$. What do these compositions represent? Find a formula for the composition of n copies of A.

67. (a) If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that $f \circ g = h$. (Think about what operations you would have to perform on the formula for g to end up with the formula for h.)

(b) If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, find a function g such that $f \circ g = h$.

68. If $f(x) = x + 4$ and $h(x) = 4x - 1$, find a function g such that $g \circ f = h$.

69. Suppose g is an even function and let $h = f \circ g$. Is h always an even function?

70. Suppose g is an odd function and let $h = f \circ g$. Is h always an odd function? What if f is odd? What if f is even?

71. Let $f(x)$ be a function with domain $\mathbb{R}$.

(a) Show that $E(x) = f(x) + f(-x)$ is an even function.

(b) Show that $O(x) = f(x) - f(-x)$ is an odd function.

(c) Prove that every function $f(x)$ can be written as a sum of an even function and an odd function.

(d) Express the function $f(x) = 2^x + (x - 3)^2$ as a sum of an even function and an odd function.

1.4 Exponential Functions

The function $f(x) = 2^x$ is called an *exponential function* because the variable, x, is the exponent. It should not be confused with the power function $g(x) = x^2$, in which the variable is the base.

Exponential Functions and Their Graphs

In general, an **exponential function** is a function of the form

$$f(x) = b^x$$

where b is a positive constant. Let's recall what this means.

If $x = n$, a positive integer, then

$$b^n = \underbrace{b \cdot b \cdot \cdots \cdot b}_{n \text{ factors}}$$

In Appendix G we present an alternative approach to the exponential and logarithmic functions using integral calculus.

1.4 Exercises

1–2 Use the Laws of Exponents to rewrite and simplify each expression.

1. (a) $\dfrac{-2^6}{4^3}$ (b) $\dfrac{(-3)^6}{9^6}$ (c) $\dfrac{1}{\sqrt[4]{x^5}}$

(d) $\dfrac{x^3 \cdot x^n}{x^{n+1}}$ (e) $b^3(3b^{-1})^{-2}$ (f) $\dfrac{2x^2y}{(3x^{-2}y)^2}$

2. (a) $\dfrac{\sqrt[3]{4}}{\sqrt[3]{108}}$ (b) $27^{2/3}$ (c) $2x^2(3x^5)^2$

(d) $(2x^{-2})^{-3}x^{-3}$ (e) $\dfrac{3a^{3/2} \cdot a^{1/2}}{a^{-1}}$ (f) $\dfrac{\sqrt{a\sqrt{b}}}{\sqrt[3]{ab}}$

3. (a) Write an equation that defines the exponential function with base $b > 0$.
(b) What is the domain of this function?
(c) If $b \neq 1$, what is the range of this function?
(d) Sketch the general shape of the graph of the exponential function for each of the following cases.
(i) $b > 1$
(ii) $b = 1$
(iii) $0 < b < 1$

4. (a) How is the number e defined?
(b) What is an approximate value for e?
(c) What is the natural exponential function?

5–8 Graph the given functions on a common screen. How are these graphs related?

5. $y = 2^x, \quad y = e^x, \quad y = 5^x, \quad y = 20^x$

6. $y = e^x, \quad y = e^{-x}, \quad y = 8^x, \quad y = 8^{-x}$

7. $y = 3^x, \quad y = 10^x, \quad y = \left(\frac{1}{3}\right)^x, \quad y = \left(\frac{1}{10}\right)^x$

8. $y = 0.9^x, \quad y = 0.6^x, \quad y = 0.3^x, \quad y = 0.1^x$

9–14 Make a rough sketch by hand of the graph of the function. Use the graphs given in Figures 3 and 15 and, if necessary, the transformations of Section 1.3.

9. $g(x) = 3^x + 1$

10. $h(x) = 2\left(\frac{1}{2}\right)^x - 3$

11. $y = -e^{-x}$

12. $y = 4^{x+2}$

13. $y = 1 - \frac{1}{2}e^{-x}$

14. $y = e^{|x|}$

15. Starting with the graph of $y = e^x$, write the equation of the graph that results from
(a) shifting 2 units downward.
(b) shifting 2 units to the right.
(c) reflecting about the x-axis.
(d) reflecting about the y-axis.
(e) reflecting about the x-axis and then about the y-axis.

16. Starting with the graph of $y = e^x$, find the equation of the graph that results from
(a) reflecting about the line $y = 4$.
(b) reflecting about the line $x = 2$.

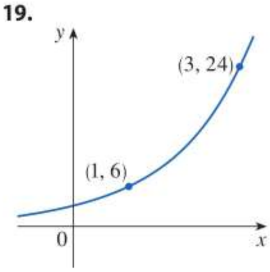
17–18 Find the domain of each function.

17. (a) $f(x) = \dfrac{1 - e^{x^2}}{1 - e^{1-x^2}}$ (b) $f(x) = \dfrac{1 + x}{e^{\cos x}}$

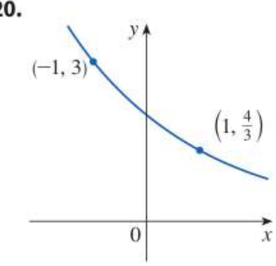
18. (a) $g(t) = \sqrt{10^t - 100}$ (b) $g(t) = \sin(e^t - 1)$

19–20 Find the exponential function $f(x) = Cb^x$ whose graph is given.

19.



20.



21. If $f(x) = 5^x$, show that

$$\frac{f(x+h) - f(x)}{h} = 5^x\left(\frac{5^h - 1}{h}\right)$$

22. Suppose you are offered a job that lasts one month. Which of the following methods of payment do you prefer?
I. One million dollars at the end of the month.
II. One cent on the first day of the month, two cents on the second day, four cents on the third day, and, in general, 2^{n-1} cents on the nth day.

23. Suppose the graphs of $f(x) = x^2$ and $g(x) = 2^x$ are drawn on a coordinate grid where the unit of measurement is 1 inch. Show that at a distance 2 ft to the right of the origin, the height of the graph of f is 48 ft but the height of the graph of g is about 265 mi.

24. Compare the functions $f(x) = x^5$ and $g(x) = 5^x$ by graphing both functions in several viewing rectangles. Find all points of intersection of the graphs correct to one decimal place. Which function grows more rapidly when x is large?

25. Compare the functions $f(x) = x^{10}$ and $g(x) = e^x$ by graphing both functions in several viewing rectangles. When does the graph of g finally surpass the graph of f?

26. Use a graph to estimate the values of x such that $e^x > 1{,}000{,}000{,}000$.

T 27. A researcher is trying to determine the doubling time for a population of the bacterium *Giardia lamblia*. He starts a culture in a nutrient solution and estimates the bacteria count every four hours. His data are shown in the table.

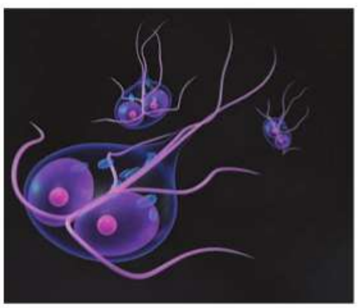
Time (hours)	0	4	8	12	16	20	24
Bacteria count (CFU/mL)	37	47	63	78	105	130	173

(a) Make a scatter plot of the data.

(b) Use a calculator or computer to find an exponential curve $f(t) = a \cdot b^t$ that models the bacteria population t hours later.

(c) Graph the model from part (b) together with the scatter plot in part (a). Use the graph to estimate how long it takes for the bacteria count to double.

Sebastian Kaulitzki / Shutterstock.com



G. lamblia

T 28. The table gives the population of the United States, in millions, for the years 1900–2010. Use a graphing calculator (or computer) with exponential regression capability to model the US population since 1900. Use the model to estimate the population in 1925 and to predict the population in the year 2020.

Year	Population
1900	76
1910	92
1920	106
1930	123
1940	131
1950	150
1960	179
1970	203
1980	227
1990	250
2000	281
2010	310

29. A bacteria culture starts with 500 bacteria and doubles in size every half hour.

(a) How many bacteria are there after 3 hours?

(b) How many bacteria are there after t hours?

(c) How many bacteria are there after 40 minutes?

(d) Graph the population function and estimate the time for the population to reach 100,000.

30. A gray squirrel population was introduced in a certain region 18 years ago. Biologists observe that the population doubles every six years, and now the population is 600.

(a) What was the initial squirrel population?

(b) What is the expected squirrel population t years after introduction?

(c) Estimate the expected squirrel population 10 years from now.

31. In Example 4, the patient's viral load V was 76.0 RNA copies per mL after one day of treatment. Use the graph of V in Figure 11 to estimate the additional time required for the viral load to decrease to half that amount.

32. After alcohol is fully absorbed into the body, it is metabolized. Suppose that after consuming several alcoholic drinks earlier in the evening, your blood alcohol concentration (BAC) at midnight is 0.14 g/dL. After 1.5 hours your BAC is half this amount.

(a) Find an exponential model for your BAC t hours after midnight.

(b) Graph your BAC and use the graph to determine when your BAC reaches the legal limit of 0.08 g/dL.

Source: Adapted from P. Wilkinson et al., "Pharmacokinetics of Ethanol after Oral Administration in the Fasting State," *Journal of Pharmacokinetics and Biopharmaceutics* 5 (1977): 207–24.

33. If you graph the function

$$f(x) = \frac{1 - e^{1/x}}{1 + e^{1/x}}$$

you'll see that f appears to be an odd function. Prove it.

34. Graph several members of the family of functions

$$f(x) = \frac{1}{1 + ae^{bx}}$$

where $a > 0$. How does the graph change when b changes? How does it change when a changes?

35. Graph several members of the family of functions

$$f(x) = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$$

where $a > 0$. How does the graph change as a increases?

We know that the lines $x = \pm\pi/2$ are vertical asymptotes of the graph of tan. Since the graph of $\tan^{-1}$ is obtained by reflecting the graph of the restricted tangent function about the line $y = x$, it follows that the lines $y = \pi/2$ and $y = -\pi/2$ are horizontal asymptotes of the graph of $\tan^{-1}$.

The remaining inverse trigonometric functions are not used as frequently and are summarized here.

12
$$\begin{aligned}
y = \csc^{-1}x \ (|x| \geqslant 1) &\iff \csc y = x \quad \text{and} \quad y \in (0, \pi/2] \cup (\pi, 3\pi/2] \\
y = \sec^{-1}x \ (|x| \geqslant 1) &\iff \sec y = x \quad \text{and} \quad y \in [0, \pi/2) \cup [\pi, 3\pi/2) \\
y = \cot^{-1}x \ (x \in \mathbb{R}) &\iff \cot y = x \quad \text{and} \quad y \in (0, \pi)
\end{aligned}$$

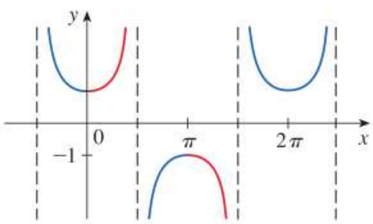


FIGURE 26
$y = \sec x$

The choice of intervals for y in the definitions of $\csc^{-1}$ and $\sec^{-1}$ is not universally agreed upon. For instance, some authors use $y \in [0, \pi/2) \cup (\pi/2, \pi]$ in the definition of $\sec^{-1}$. [You can see from the graph of the secant function in Figure 26 that both this choice and the one in (12) will work.]

1.5 Exercises

1. (a) What is a one-to-one function?
(b) How can you tell from the graph of a function whether it is one-to-one?

2. (a) Suppose f is a one-to-one function with domain A and range B. How is the inverse function f^{-1} defined? What is the domain of f^{-1}? What is the range of f^{-1}?
(b) If you are given a formula for f, how do you find a formula for f^{-1}?
(c) If you are given the graph of f, how do you find the graph of f^{-1}?

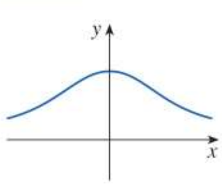
3–16 A function is given by a table of values, a graph, a formula, or a verbal description. Determine whether it is one-to-one.

3.

x	1	2	3	4	5	6
$f(x)$	1.5	2.0	3.6	5.3	2.8	2.0

4.

x	1	2	3	4	5	6
$f(x)$	1.0	1.9	2.8	3.5	3.1	2.9

5.


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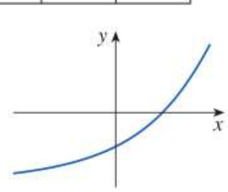

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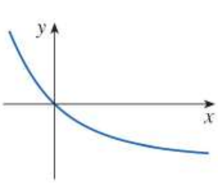

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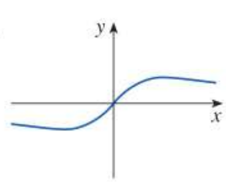


9. $f(x) = 2x - 3$

10. $f(x) = x^4 - 16$

11. $r(t) = t^3 + 4$

12. $g(x) = \sqrt[3]{x}$

13. $g(x) = 1 - \sin x$

14. $f(x) = x^4 - 1, \quad 0 \leqslant x \leqslant 10$

15. $f(t)$ is the height of a football t seconds after kickoff.

16. $f(t)$ is your height at age t.

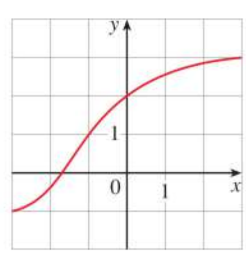
---

17. Assume that f is a one-to-one function.
(a) If $f(6) = 17$, what is $f^{-1}(17)$?
(b) If $f^{-1}(3) = 2$, what is $f(2)$?

18. If $f(x) = x^5 + x^3 + x$, find $f^{-1}(3)$ and $f(f^{-1}(2))$.

19. If $g(x) = 3 + x + e^x$, find $g^{-1}(4)$.

20. The graph of f is given.
(a) Why is f one-to-one?
(b) What are the domain and range of f^{-1}?
(c) What is the value of $f^{-1}(2)$?
(d) Estimate the value of $f^{-1}(0)$.



21. The formula $C = \frac{5}{9}(F - 32)$, where $F \geqslant -459.67$, expresses the Celsius temperature C as a function of the Fahrenheit temperature F. Find a formula for the inverse function and interpret it. What is the domain of the inverse function?

22. In the theory of relativity, the mass of a particle with speed v is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle and c is the speed of light in a vacuum. Find the inverse function of f and explain its meaning.

23–30 Find a formula for the inverse of the function.

23. $f(x) = 1 - x^2, \quad x \geqslant 0$

24. $g(x) = x^2 - 2x, \quad x \geqslant 1$

25. $g(x) = 2 + \sqrt{x + 1}$

26. $h(x) = \dfrac{6 - 3x}{5x + 7}$

27. $y = e^{1-x}$

28. $y = 3\ln(x - 2)$

29. $y = \left(2 + \sqrt[3]{x}\right)^5$

30. $y = \dfrac{1 - e^{-x}}{1 + e^{-x}}$

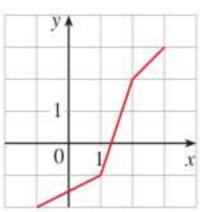
31–32 Find an explicit formula for f^{-1} and use it to graph f^{-1}, f, and the line $y = x$ on the same screen. To check your work, see whether the graphs of f and f^{-1} are reflections about the line.

31. $f(x) = \sqrt{4x + 3}$

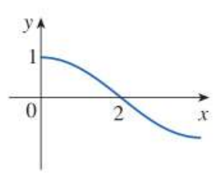
32. $f(x) = 1 + e^{-x}$

33–34 Use the given graph of f to sketch the graph of f^{-1}.

33.



34.



35. Let $f(x) = \sqrt{1 - x^2}$, $0 \leqslant x \leqslant 1$.
(a) Find f^{-1}. How is it related to f?
(b) Identify the graph of f and explain your answer to part (a).

36. Let $g(x) = \sqrt[3]{1 - x^3}$.
(a) Find g^{-1}. How is it related to g?
(b) Graph g. How do you explain your answer to part (a)?

37. (a) How is the logarithmic function $y = \log_b x$ defined?
(b) What is the domain of this function?
(c) What is the range of this function?
(d) Sketch the general shape of the graph of the function $y = \log_b x$ if $b > 1$.

38. (a) What is the natural logarithm?
(b) What is the common logarithm?
(c) Sketch the graphs of the natural logarithm function and the natural exponential function with a common set of axes.

39–42 Find the exact value of each expression.

39. (a) $\log_3 81$ (b) $\log_3\left(\frac{1}{81}\right)$ (c) $\log_9 3$

40. (a) $\ln \dfrac{1}{e^2}$ (b) $\ln \sqrt{e}$ (c) $\ln\left(\ln e^{e^{50}}\right)$

41. (a) $\log_2 30 - \log_2 15$
(b) $\log_3 10 - \log_3 5 - \log_3 18$
(c) $2\log_5 100 - 4\log_5 50$

42. (a) $e^{3\ln 2}$ (b) $e^{-2\ln 5}$ (c) $e^{\ln(\ln e^3)}$

43–44 Use the laws of logarithms to expand each expression.

43. (a) $\log_{10}(x^2 y^3 z)$ (b) $\ln\left(\dfrac{x^4}{\sqrt{x^2 - 4}}\right)$

44. (a) $\ln\sqrt{\dfrac{3x}{x - 3}}$ (b) $\log_2\left[(x^3 + 1)\sqrt[3]{(x - 3)^2}\right]$

45–46 Express as a single logarithm.

45. (a) $\log_{10} 20 - \frac{1}{3}\log_{10} 1000$ (b) $\ln a - 2\ln b + 3\ln c$

46. (a) $3\ln(x - 2) - \ln(x^2 - 5x + 6) + 2\ln(x - 3)$
(b) $c\log_a x - d\log_a y + \log_a z$

47–48 Use Formula 11 to evaluate each logarithm correct to six decimal places.

47. (a) $\log_5 10$ (b) $\log_{15} 12$

48. (a) $\log_3 12$ (b) $\log_{12} 6$

49–50 Use Formula 11 to graph the given functions on a common screen. How are these graphs related?

49. $y = \log_{1.5} x, \quad y = \ln x, \quad y = \log_{10} x, \quad y = \log_{50} x$

50. $y = \ln x, \quad y = \log_8 x, \quad y = e^x, \quad y = 8^x$

51. Suppose that the graph of $y = \log_2 x$ is drawn on a coordinate grid where the unit of measurement is an inch. How many miles to the right of the origin do we have to move before the height of the curve reaches 3 ft?

52. Compare the functions $f(x) = x^{0.1}$ and $g(x) = \ln x$ by graphing both functions in several viewing rectangles. When does the graph of f finally surpass the graph of g?

53–54 Make a rough sketch by hand of the graph of each function. Use the graphs given in Figures 12 and 13 and, if necessary, the transformations of Section 1.3.

53. (a) $y = \log_{10}(x + 5)$ (b) $y = -\ln x$

54. (a) $y = \ln(-x)$ (b) $y = \ln|x|$

55–56
(a) What are the domain and range of f?
(b) What is the x-intercept of the graph of f?
(c) Sketch the graph of f.

55. $f(x) = \ln x + 2$ **56.** $f(x) = \ln(x - 1) - 1$

57–60 Solve each equation for x. Give both an exact value and a decimal approximation, correct to three decimal places.

57. (a) $\ln(4x + 2) = 3$ (b) $e^{2x-3} = 12$

58. (a) $\log_2(x^2 - x - 1) = 2$ (b) $1 + e^{4x+1} = 20$

59. (a) $\ln x + \ln(x - 1) = 0$ (b) $5^{1-2x} = 9$

60. (a) $\ln(\ln x) = 0$ (b) $\dfrac{60}{1 + e^{-x}} = 4$

61–62 Solve each inequality for x.

61. (a) $\ln x < 0$ (b) $e^x > 5$

62. (a) $1 < e^{3x-1} < 2$ (b) $1 - 2\ln x < 3$

63. (a) Find the domain of $f(x) = \ln(e^x - 3)$.
(b) Find f^{-1} and its domain.

64. (a) What are the values of $e^{\ln 300}$ and $\ln(e^{300})$?
(b) Use your calculator to evaluate $e^{\ln 300}$ and $\ln(e^{300})$. What do you notice? Can you explain why the calculator has trouble?

T **65.** Graph the function $f(x) = \sqrt{x^3 + x^2 + x + 1}$ and explain why it is one-to-one. Then use a computer algebra system to find an explicit expression for $f^{-1}(x)$. (Your CAS will produce three possible expressions. Explain why two of them are irrelevant in this context.)

T **66.** (a) If $g(x) = x^6 + x^4, x \geqslant 0$, use a computer algebra system to find an expression for $g^{-1}(x)$.
(b) Use the expression in part (a) to graph $y = g(x)$, $y = x$, and $y = g^{-1}(x)$ on the same screen.

67. If a bacteria population starts with 100 bacteria and doubles every three hours, then the number of bacteria after t hours is $n = f(t) = 100 \cdot 2^{t/3}$.
(a) Find the inverse of this function and explain its meaning.
(b) When will the population reach 50,000?

68. The National Ignition Facility at the Lawrence Livermore National Laboratory maintains the world's largest laser facility. The lasers, which are used to start a nuclear fusion reaction, are powered by a capacitor bank that stores a total of about 400 megajoules of energy. When the lasers are fired the capacitors discharge completely and then immediately begin recharging. The charge Q of the capacitors t seconds after the discharge is given by

$$Q(t) = Q_0(1 - e^{-t/a})$$

(The maximum charge capacity is Q_0 and t is measured in seconds.)
(a) Find a formula for the inverse of this function and explain its meaning.
(b) How long does it take to recharge the capacitors to 90% of capacity if $a = 50$?

69–74 Find the exact value of each expression.

69. (a) $\cos^{-1}(-1)$ (b) $\sin^{-1}(0.5)$

70. (a) $\tan^{-1}\sqrt{3}$ (b) $\arctan(-1)$

71. (a) $\csc^{-1}\sqrt{2}$ (b) $\arcsin 1$

72. (a) $\sin^{-1}(-1/\sqrt{2})$ (b) $\cos^{-1}(\sqrt{3}/2)$

73. (a) $\cot^{-1}(-\sqrt{3})$ (b) $\sec^{-1} 2$

74. (a) $\arcsin(\sin(5\pi/4))$ (b) $\cos(2\sin^{-1}(\frac{5}{13}))$

75. Prove that $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$.

76–78 Simplify the expression.

76. $\tan(\sin^{-1}x)$ **77.** $\sin(\tan^{-1}x)$ **78.** $\sin(2\arccos x)$

79-80 Graph the given functions on the same screen. How are these graphs related?

79. $y = \sin x,\ -\pi/2 \leqslant x \leqslant \pi/2;\quad y = \sin^{-1}x;\quad y = x$

80. $y = \tan x,\ -\pi/2 < x < \pi/2;\quad y = \tan^{-1}x;\quad y = x$

81. Find the domain and range of the function

$$g(x) = \sin^{-1}(3x + 1)$$

82. (a) Graph the function $f(x) = \sin(\sin^{-1}x)$ and explain the appearance of the graph.
(b) Graph the function $g(x) = \sin^{-1}(\sin x)$. How do you explain the appearance of this graph?

83. (a) If we shift a curve to the left, what happens to its reflection about the line $y = x$? In view of this geometric principle, find an expression for the inverse of $g(x) = f(x + c)$, where f is a one-to-one function.
(b) Find an expression for the inverse of $h(x) = f(cx)$, where $c \neq 0$.

1 REVIEW

CONCEPT CHECK

Answers to the Concept Check are available at StewartCalculus.com.

1. (a) What is a function? What are its domain and range?
(b) What is the graph of a function?
(c) How can you tell whether a given curve is the graph of a function?

2. Discuss four ways of representing a function. Illustrate your discussion with examples.

3. (a) What is an even function? How can you tell if a function is even by looking at its graph? Give three examples of an even function.
(b) What is an odd function? How can you tell if a function is odd by looking at its graph? Give three examples of an odd function.

4. What is an increasing function?

5. What is a mathematical model?

6. Give an example of each type of function.
(a) Linear function (b) Power function
(c) Exponential function (d) Quadratic function
(e) Polynomial of degree 5 (f) Rational function

7. Sketch by hand, on the same axes, the graphs of the following functions.
(a) $f(x) = x$ (b) $g(x) = x^2$
(c) $h(x) = x^3$ (d) $j(x) = x^4$

8. Draw, by hand, a rough sketch of the graph of each function.
(a) $y = \sin x$ (b) $y = \tan x$ (c) $y = e^x$
(d) $y = \ln x$ (e) $y = 1/x$ (f) $y = |x|$
(g) $y = \sqrt{x}$ (h) $y = \tan^{-1}x$

9. Suppose that f has domain A and g has domain B.
(a) What is the domain of $f + g$?
(b) What is the domain of fg?
(c) What is the domain of f/g?

10. How is the composite function $f \circ g$ defined? What is its domain?

11. Suppose the graph of f is given. Write an equation for each of the graphs that are obtained from the graph of f as follows.
(a) Shift 2 units upward. (b) Shift 2 units downward.
(c) Shift 2 units to the right. (d) Shift 2 units to the left.
(e) Reflect about the x-axis.
(f) Reflect about the y-axis.
(g) Stretch vertically by a factor of 2.
(h) Shrink vertically by a factor of 2.
(i) Stretch horizontally by a factor of 2.
(j) Shrink horizontally by a factor of 2.

12. (a) What is a one-to-one function? How can you tell if a function is one-to-one by looking at its graph?
(b) If f is a one-to-one function, how is its inverse function f^{-1} defined? How do you obtain the graph of f^{-1} from the graph of f?

13. (a) How is the inverse sine function $f(x) = \sin^{-1}x$ defined? What are its domain and range?
(b) How is the inverse cosine function $f(x) = \cos^{-1}x$ defined? What are its domain and range?
(c) How is the inverse tangent function $f(x) = \tan^{-1}x$ defined? What are its domain and range?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If f is a function, then $f(s + t) = f(s) + f(t)$.

2. If $f(s) = f(t)$, then $s = t$.

3. If f is a function, then $f(3x) = 3f(x)$.

4. If the function f has an inverse and $f(2) = 3$, then $f^{-1}(3) = 2$.

5. A vertical line intersects the graph of a function at most once.

6. If f and g are functions, then $f \circ g = g \circ f$.

7. If f is one-to-one, then $f^{-1}(x) = \dfrac{1}{f(x)}$.

8. You can always divide by e^x.

9. If $0 < a < b$, then $\ln a < \ln b$.

10. If $x > 0$, then $(\ln x)^6 = 6 \ln x$.

11. If $x > 0$ and $a > 1$, then $\dfrac{\ln x}{\ln a} = \ln \dfrac{x}{a}$.

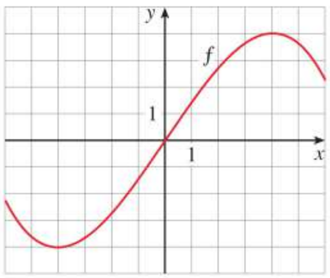
12. $\tan^{-1}(-1) = 3\pi/4$

13. $\tan^{-1}x = \dfrac{\sin^{-1}x}{\cos^{-1}x}$

14. If x is any real number, then $\sqrt{x^2} = x$.

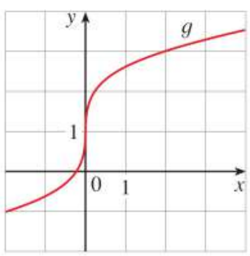
EXERCISES

1. Let f be the function whose graph is given.



(a) Estimate the value of $f(2)$.
(b) Estimate the values of x such that $f(x) = 3$.
(c) State the domain of f.
(d) State the range of f.
(e) On what interval is f increasing?
(f) Is f one-to-one? Explain.
(g) Is f even, odd, or neither even nor odd? Explain.

2. The graph of g is given.



(a) State the value of $g(2)$.
(b) Why is g one-to-one?
(c) Estimate the value of $g^{-1}(2)$.
(d) Estimate the domain of g^{-1}.
(e) Sketch the graph of g^{-1}.

3. If $f(x) = x^2 - 2x + 3$, evaluate the difference quotient

$$\frac{f(a+h) - f(a)}{h}$$

4. Sketch a rough graph of the yield of a crop as a function of the amount of fertilizer used.

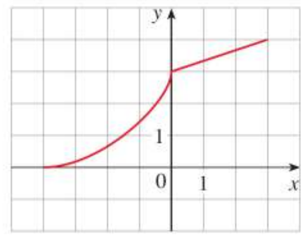
5–8 Find the domain and range of the function. Write your answer in interval notation.

5. $f(x) = 2/(3x - 1)$ **6.** $g(x) = \sqrt{16 - x^4}$

7. $h(x) = \ln(x + 6)$ **8.** $F(t) = 3 + \cos 2t$

9. Suppose that the graph of f is given. Describe how the graphs of the following functions can be obtained from the graph of f.
(a) $y = f(x) + 5$ (b) $y = f(x + 5)$
(c) $y = 1 + 2f(x)$ (d) $y = f(x - 2) - 2$
(e) $y = -f(x)$ (f) $y = f^{-1}(x)$

10. The graph of f is given. Draw the graphs of the following functions.
(a) $y = f(x - 8)$ (b) $y = -f(x)$
(c) $y = 2 - f(x)$ (d) $y = \frac{1}{2}f(x) - 1$
(e) $y = f^{-1}(x)$ (f) $y = f^{-1}(x + 3)$



11–18 Use transformations to sketch the graph of the function.

11. $f(x) = x^3 + 2$ **12.** $f(x) = (x - 3)^2$

13. $y = \sqrt{x + 2}$ **14.** $y = \ln(x + 5)$

15. $g(x) = 1 + \cos 2x$ **16.** $h(x) = -e^x + 2$

17. $s(x) = 1 + 0.5^x$ **18.** $f(x) = \begin{cases} -x & \text{if } x < 0 \\ e^x - 1 & \text{if } x \geq 0 \end{cases}$

19. Determine whether f is even, odd, or neither even nor odd.
(a) $f(x) = 2x^5 - 3x^2 + 2$ (b) $f(x) = x^3 - x^7$
(c) $f(x) = e^{-x^2}$ (d) $f(x) = 1 + \sin x$
(e) $f(x) = 1 - \cos 2x$ (f) $f(x) = (x + 1)^2$

20. Find an expression for the function whose graph consists of the line segment from the point $(-2, 2)$ to the point $(-1, 0)$ together with the top half of the circle with center the origin and radius 1.

21. If $f(x) = \ln x$ and $g(x) = x^2 - 9$, find the functions (a) $f \circ g$, (b) $g \circ f$, (c) $f \circ f$, (d) $g \circ g$, and their domains.

22. Express the function $F(x) = 1/\sqrt{x + \sqrt{x}}$ as a composition of three functions.

23. Life expectancy has improved dramatically in recent decades. The table gives the life expectancy at birth (in years) of males born in the United States. Use a scatter plot to choose an appropriate type of model. Use your model to predict the life span of a male born in the year 2030.

Birth year	Life expectancy	Birth year	Life expectancy
1900	48.3	1960	66.6
1910	51.1	1970	67.1
1920	55.2	1980	70.0
1930	57.4	1990	71.8
1940	62.5	2000	73.0
1950	65.6	2010	76.2

24. A small-appliance manufacturer finds that it costs \$9000 to produce 1000 toaster ovens a week and \$12,000 to produce 1500 toaster ovens a week.

(a) Express the cost as a function of the number of toaster ovens produced, assuming that it is linear. Then sketch the graph.

(b) What is the slope of the graph and what does it represent?

(c) What is the y-intercept of the graph and what does it represent?

25. If $f(x) = 2x + 4^x$, find $f^{-1}(6)$.

26. Find the inverse function of $f(x) = \dfrac{2x+3}{1-5x}$.

27. Use the laws of logarithms to expand each expression.

(a) $\ln x\sqrt{x+1}$ (b) $\log_2 \sqrt{\dfrac{x^2+1}{x-1}}$

28. Express as a single logarithm.

(a) $\frac{1}{2}\ln x - 2\ln(x^2+1)$

(b) $\ln(x-3) + \ln(x+3) - 2\ln(x^2-9)$

29–30 Find the exact value of each expression.

29. (a) $e^{2\ln 5}$ (b) $\log_6 4 + \log_6 54$ (c) $\tan\left(\arcsin \frac{4}{5}\right)$

30. (a) $\ln \dfrac{1}{e^3}$ (b) $\sin(\tan^{-1} 1)$ (c) $10^{-3\log 4}$

31–36 Solve the equation for x. Give both an exact value and a decimal approximation, correct to three decimal places.

31. $e^{2x} = 3$

32. $\ln x^2 = 5$

33. $e^{e^x} = 10$

34. $\cos^{-1}x = 2$

35. $\tan^{-1}(3x^2) = \dfrac{\pi}{4}$

36. $\ln x - 1 = \ln(5+x) - 4$

37. The viral load for an HIV patient is 52.0 RNA copies/mL before treatment begins. Eight days later the viral load is half of the initial amount.

(a) Find the viral load after 24 days.

(b) Find the viral load $V(t)$ that remains after t days.

(c) Find a formula for the inverse of the function V and explain its meaning.

(d) After how many days will the viral load be reduced to 2.0 RNA copies/mL?

38. The population of a certain species in a limited environment with initial population 100 and carrying capacity 1000 is

$$P(t) = \frac{100{,}000}{100 + 900e^{-t}}$$

where t is measured in years.

(a) Graph this function and estimate how long it takes for the population to reach 900.

(b) Find the inverse of this function and explain its meaning.

(c) Use the inverse function to find the time required for the population to reach 900. Compare with the result of part (a).