First Day

Q: What is a differential equation?
A: An equation that involves derivatives ('differentials').

For instance,

$$\frac{\partial}{\partial t}u = \frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u + \frac{\partial^2}{\partial z^2}u$$

is a differential equation. Here, the 'unknown' in the equation is the function u, which is implicitly a function of t, x, y, and z (to be a little fancy, $u = u(t, x, y, z) : \mathbb{R}^4 \to \mathbb{R}$, or maybe $u : D \to \mathbb{R}$ for some domain $D \subseteq \mathbb{R}^4$). This equation is known as the **heat equation**, and is very important throughout math and science. We could spend a semester on this one equation.

But, partial derivatives are too scary. This course will focus on single variable functions, and 'ordinary' (single variable) derivatives.

One example is

$$y'(x) + y(x) - 1 = 0$$

What does it mean to 'solve' a differential equation? We want to find a function, $y(x)$, which makes the given equation true.

For instance, check that if we set $y(x) = 1 - e^{-x}$, we have $y'(x) = e^{-x}$. Then

$$\begin{aligned} y'(x) + y(x) - 1 &= e^{-x} + (1 - e^{-x}) - 1 \\ &= e^{-x} + 1 - e^{-x} - 1 \\ &= 0 \end{aligned}$$

so the function $y(x) = 1 - e^{-x}$ does solve this differential equation.

Other examples of differential equations:

$$y' = \frac{y^3 - y}{y^2 + 1}$$

$$y' = \frac{xy^2 + x^2}{3xy + x^2}$$

$$y' = \frac{xy}{x^2 + y^2}$$

The first example is easily solvable with knowledge we will develop. The second example, I can't solve. The third example is not too bad, if you know the trick.

Of course, even these are relatively simple. Some other differential equations:

$$x^4y^{(4)} + x^3y^{(3)} + x^2y'' + xy' + y = 0$$

$$y'' + y' \cdot y = y$$

$$\cos(x^2)y^{(3)} - x^2e^xy' + 12xy = 44$$

A first course on differential equations has a bit of a reputation as being a 'bag of tricks', a bunch of unconnected methods to solve a bunch of different looking equations, with no overarching ideas. This reputation is not untrue, but I'd like to skip a few of the tricks and talk a bit about some concepts.

Before doing that though, let's discuss some differential equations you already know. The simplest might be

$$y' = 0$$

The solutions to this are $y(x) = c$ for some constant real number $c \in \mathbb{R}$. Actually, we know these functions are all solutions, but are they the only solutions?

Exercise 1. *Prove that the only solutions to $y' = 0$ are the constant functions $y(x) = c$, $c \in \mathbb{R}$.*

(Hint: Since we are doing differential equations, we will usually assume $y(x)$ is differentiable, i.e., actually has derivatives. Then prove by contradiction: if $y(x)$ is a solution to $y' = 0$ but $y(x)$ is not constant, we must have some numbers a, b with $y(a) \neq y(b)$. Apply a theorem from first semester calculus.)

Another simple example would be

$$y'' = 0$$

that is, the second derivative of a function is zero, what is the function? By thinking about the previous example a bit, you might realize that $y(x) =$

$ax+b$ is a solution for any real numbers a and b, and in fact these are the only solutions (again, since our differential equation has a second derivative, we assume $y(x)$ is supposed to be twice-differentiable).

It is not too hard to discuss solutions to

$$y^{(n)} = 0$$

One could also start to complicate the right hand side:

$$y' = 3$$

We recognize one solution is $y(x) = 3x$, or more generally $y(x) = 3x + c$ for some real number c, and these are the only solutions. Why? Because $y' = 3$, or $\frac{d}{dx}(y) = 3$, implies that

$$\begin{aligned} \frac{d}{dx}(y - 3x) &= \frac{d}{dx}(y) - \frac{d}{dx}(3x) \\ &= 3 - 3 \\ &= 0, \end{aligned}$$

so by exercise 1 we have $y(x) - 3x = c$ for some real number c.

A more interesting complication would be to consider

$$y' = x$$

so we want a function (of x) whose derivative is x (a function whose derivative at any input is equal to that input). You might recognize that

$$y(x) = \frac{1}{2}x^2$$

is a solution, and in fact

$$y(x) = \frac{1}{2}x^2 + c$$

is a solution for any real number c. (Why? Same trick: think about $y - \frac{1}{2}x^2$, and take the derivative.)

We could put all sorts of things on the right side, for instance

$$y' = x\sin(x)$$

or

$$y' = e^{-x^2}$$

Wait a minute... Find a function whose derivative is $f(x)$... That's just asking for an anti-derivative! The Fundamental Theorem of Calculus tells us that to solve this, we integrate.

Integration by parts in the first example gives

$$y(x) = \int x \sin(x) dx = \sin(x) - x\cos(x) + c$$

for some real number c. (I would prefer to write, as I did in class, that $\int_a^x t\sin(t)dt$ is a solution, and changing the a is the same as changing the c. This isn't that important, but you might try thinking about it a bit. Is there any potential danger about this way of thinking?)

The second example gives

$$y(x) = \int e^{-x^2} dx = ??? + c$$

You may have heard that this integral **cannot** be written in terms of 'elementary functions', whatever that means. Really, it means the same thing as making up $\sqrt{2}$ as a symbol for a specific number. The indefinite integral is a symbol for a certain function, but you can't always write it in terms of 'simple' functions, just like you can't write $\sqrt{2}$ as a fraction of integers. Saying precisely what this means is outside the scope of this course, but we may touch upon it when we discuss general linear equations.

The point of this discussion is, you've already seen a lot of differential equations. Any differential equation of the form

$$y'(x) = f(x)$$

for some given function $f(x)$ was part of your calculus education, and the solution was to integrate. Since you all know that integration is pretty hard (sometimes nigh impossible?!?), it should come as no surprise that general differential equations can be extremely difficult.

In a differential equation, we are allowed to write derivatives of our unknown function $y(x)$, and also the function $y(x)$ itself (we could think of $y(x)$ as the '0-th derivative' of itself). The simples differential equation where we ask for the derivative of y to be something other than a function of x would then be

$$y' = y$$

This is asking us to find a function whose derivative is itself. You remember from calculus that e^x is such a function (that's kind of the whole point of $e \approx 2.71828....$, its the base that makes the exponential function its own derivative).

So $y(x) = e^x$ is a solution. Notice another solution: if $y(x)$ is a constant, then $y'(x) = 0$. Since 0 is a constant, taking $y(x) = 0$ means that $y'(x) = 0 = y(x)$, so the constant 0 function is a solution as well.

Are these the only solutions? A bit of poking around will reveal $2e^x$ also works, as does $-e^x$ or $13e^x$. It's not hard to show ce^x works for any real number c ('by linearity of the derivative').

Are these the only solutions? Let us suppose $y(x)$ is a solution to this differential, so $y' = y$. Consider the new function $y \cdot e^{-x}$, and take its derivative:

$$\begin{aligned}
\frac{d}{dx}\left(ye^{-x}\right) &= \frac{d}{dx}(y)\, e^{-x} + y\frac{d}{dx}\left(e^{-x}\right) \\
&= y'e^{-x} + y(-e^{-x}) \\
&= y'e^{-x} - ye^{-x} \\
&= (y' - y)e^{-x} \\
&= 0 \cdot e^{-x} \\
&= 0
\end{aligned}$$

So, by exercise 1, we must have $ye^{-x} = c$ for some real number c, which simplifies to $y(x) = ce^{-x}$. So every solution of the differential equation $y' = y$ is of the form $y(x) = ce^x$.

Recap

We (supposedly) raised three questions one might ask about differential equations:

1. Is there a solution?
2. How many solutions are there?
3. How can we find them? (Can we find them? Can we write them down? ...?)

(I say supposedly because we didn't really see an example that didn't have any solutions, so for all we know the first answer is always 'yes')

In the examples above, we've had infinitely many solutions. Mostly one solution for each real number, but we also say the equation $y'' = 0$ has a solution for every **pair** of real numbers (trick question: are there more real numbers or more pairs of real numbers?). But they aren't just random functions, the solutions come in a nice family. This is to be discussed.

As I alluded to above when talking about integration, the question of whether we can find a solution is not as clear as it seems... does $\int e^{-x^2} dx$ count as a solution? It's literally a symbol that means 'the thing that solves the equation' (compare to $\sqrt{2}$ for numbers). Ideally, what we write as a 'solution' should be nice enough that we can study it as a function and describe some properties. Again, this is not so easy for integration, which is solving a specific kind of differential equation. So, in general, we expect difficulties.