Recall the definitions of (X, τ) a topological space, and $K \subseteq X$ compact (in terms of open covers).

Definition 1. *A topological space (X, τ) is* ***Hausdorff*** *if for any pair of distinct points $x \neq y$ there are open sets U, V such that*

1. *$x \in U$ and $y \in V$*
2. *$U \cap V = \varnothing$*

Proposition 1. *If (X, τ) is a Hausdorff topological space, then compact subsets are closed.*

Proof. Say $K \subseteq X$ is compact, and choose any point $x \notin K$. Now, for any point $y \in K$, we have disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$.

Consider all of the V_y formed this way, for $y \in K$. That is, consider $\mathcal{V} = \{V_y : y \in K\}$. Notice $\mathcal{V}$ is an open cover of K, since any $y \in K$ is in V_y. Since K is compact, there is a finite subset of $\mathcal{V}$ which also cover K. Choose such a subcover, so we have $y_1, ..., y_n$ with

$$K \subseteq V_{y_1} \cup V_{y_2} ... \cup V_{y_n}$$

Since $U_{y_i} \cap V_{y_i} = \varnothing$, we see that $U = U_{y_1} \cap U_{y_2} \cap ... \cap U_{y_n}$ is disjoint from K. By construction, $x \in U$, and since U is an intersection of finitely many open sets, U is open.

Thus, $X \setminus K$ is open, so we have shown that K is closed. □

Definition 2. *A topological space (X, τ) is* ***first countable*** *if, for any $x \in X$, there is a countable collection $U_1, U_2, U_3, ...$ of open sets containing x such that any other open set V containing x must contain U_k for some k.*

Claim 1. *If (X, τ) is a first countable topological space, and $A \subseteq X$ is any subset, then $x \in cl(A)$ if and only if there is a sequence $(x_i)_{i \in \mathbb{N}}$ with each $x_i \in A$ such that $\lim x_i = x$.*

Proof. Exercise (:P) □

Corollary 1. *A function $f : X \to Y$, with X first countable, is continuous if and only if for any sequence (x_i) with $\lim x_i = x$ we have $\lim f(x_i) = f(x)$.*

Claim 2. *Any metric space is Hausdorff and first countable.*

Proof. If $x \neq y$, then $d(x,y) = r > 0$, so the balls $B_{r/3}(x)$ and $B_{r/3}(y)$ are disjoint (why? which metric space axiom...).

On the other hand, for any x we have the balls $B_{1/n}(x)$. Any open set V containing x must contain some ball $B_r(x)$, by the definition of the metric topology. Here $r > 0$ is real, so there must be some large n with $\frac{1}{n} < r$ (Why? which axiom of the real numbers...). $\square$