ILL Article Request - Rapid

6/8/2020 8:21:11 AM

Call #: HG6024.A3 D84 1989
Location: Main Collection

OSU ILLIAD TN#: 1177842

Journal Title: Futures markets
Volume:
Issue:
Month/Year: 1989
Pages: 201-278

Article Author: Darrell Duffie
Article Title: Hedging with Futures

ILL Number: -16225917

Borrower: RAPID:UUS

This item will be invoiced at the end of the month

Photocopy/Loan/Invoice/Postage Charges: **Exempt**

Odyssey: odyssey.rapid.exlibrisgroup.com

EMAIL: ILSRQST@USU.EDU

Email: ilsrqst@usu.edu

Pagers:

Initials: ______ NOS: ______ Lacking: ______

BC: Checked Table of Contents: _____ Checked Index: _____

Scanners:

Initials: ______ Date:

7

HEDGING WITH FUTURES

This chapter discusses some practical aspects of hedging with futures contracts. We apply the theoretical determination of optimal futures positions explained in Chapter 4, and the statistical estimation principles covered in Chapter 6, as well as some of the institutional details behind several hedging scenarios. A key technique is the estimation of risk-minimal hedges using linear regression and historical price data.

1. INTRODUCTION

By *futures hedging*, we mean taking a position in futures contracts that offsets some of the risk associated with some given market commitment. The essence of hedging is the adoption of a futures position that, on average, generates profits when the market value of the given commitment is lower than expected, and generates losses when the market value of the commitment is higher than expected. The notion of designing a futures strategy to generate losses under certain circumstances may seem quixotic to some. One must keep in mind the well-repeated adage: "There are no free lunches." One cannot expect trading profits as well as risk reduction (although that sometimes happens). The key is to coordinate losses in futures with gains elsewhere, and vice versa. How does one achieve that kind of coordination? That is the topic of this chapter.

A Simple Scenario: Hedging Foreign Exchange Risk

We begin our explanation of hedging with a simple scenario that involves a commitment by a firm to sell 25,000,000 West German marks on November 15, 1989, by taking a position in the IMM West German mark futures contract for December 1989 delivery. (This contract delivers 125,000 marks and is traded

Chapter Outline

on the Chicago Mercantile Exchange.) The spot market commitment could arise, for example, from an account receivable on foreign sales. The 25 million marks receivable will be sold immediately (on November 15) for U.S. dollars on the spot foreign exchange market. We will assume for simplicity that the value of a U.S. dollar is riskless; that is, we will ignore inflation risk. The hedger must make three basic decisions:

1. Whether to hedge short or long.
2. The size of position.
3. The timing of the hedge.

In this simple scenario, these three decisions are rather straightforward.

Short or Long? Generally, but not always, an increase in the spot exchange rate is accompanied by an increase in the nearby futures price. In statistical terms, spot price increments and futures price increments are positively correlated. Thus a short futures position, on average, generate profits if the dollar value of the marks receivable is lower than expected, and vice versa. The obvious hedging strategy is a short futures position.

Size of Position? How many contracts? If the spot commitment date were, instead, precisely the futures delivery date, the risk-minimizing futures position would be an equal and opposite position, or in this case, a short position of 200 contracts (25 million marks divided by 125,000 marks per contract). Since the futures and spot prices at the delivery date would be the same, every dollar lost on the futures position would be recouped by a corresponding dollar gain in the value of the committed marks receivable, and vice versa.

Because of the actual mismatch in spot commitment date and futures delivery date, however, there is no way to completely eliminate the spot risk. The risk due to the mismatch in dates is called *delivery basis risk.* Generally speaking, the delivery basis risk is smaller with smaller differences in time between the spot commitment date and the delivery date of the futures contract. This risk is minimized by choosing the futures position

$$h = -\,\text{size of spot commitment} \times \beta, \tag{1}$$

where β (beta), the *hedging coefficient*, is defined by the formula

$$\beta = \frac{\text{covariance of futures price change with spot price change}}{\text{variance of futures price change}}.$$

That is, h is the futures position that minimizes the variance of the dollar value of the total position value at the commitment date, which is made up of the spot commitment value plus the profits or losses on futures.

The key unknown is the hedging coefficient β. It is useful to interpret the optimal hedge by using the definition of correlation to rewrite the hedging coefficient β in the form

$$\beta = \text{corr(futures, spot)} \times \frac{\text{sdev(spot)}}{\text{sdev(futures)}}, \tag{2}$$

where "futures" and "spot" are of course shorthand for "futures price change" and "spot price change." The magnitude of the optimal futures hedging position is therefore increasing in the correlation between futures and spot, increasing in the standard deviation of the spot, and decreasing in the standard deviation of the futures.

The formula (1) for the risk-minimizing futures position h is justified in Chapter 4. Based on calculations to be explained later in this chapter, we will estimate β to be 0.935. The quantity commitment Q is 25,000,000 marks. We therefore estimate the risk-minimizing hedge $h = -Q\beta$ to be $-0.935 \times 25 = -23.37$ million marks. At 125,000 marks per contract, this is approximately -187 contracts, meaning a short position of 187 contracts. This is somewhat less than an equal and opposite (short) position of 25 million marks, or 200 contracts short. A short position of 200 contracts is in fact "over-hedged," and involves more risk than the minimum-variance hedge h.

Timing of Hedge? Normally, one would "put on the hedge," as is commonly said, as soon as the risk is perceived, and remove it (that is, offset the futures position) at the spot commitment date. A more sophisticated approach is *dynamic hedging*—continually adjusting the size of the hedge as the spot commitment date approaches. Dynamic hedges can compensate for changing volatility in the futures or spot markets, or for new information. Calculating an optimal dynamic hedging strategy is, in some cases, quite complicated.

This chapter reviews several methods for estimating h, the size of the optimal hedge, using simple principles and linear regression. We will also cover techniques for choosing the best futures contract for hedging purposes, for offsetting the effects of interest on margin accounts, for the simultaneous use of several different futures contracts, for hedging several different commitments at the same time, for obtaining dynamic hedging strategies, and for checking the quality of the hedge.

Alternate Meanings of "Hedging"

Among economists, the meaning of "to hedge" is not universally accepted as "to adopt a strategy designed to reduce risk," although that is the accepted English definition of the word. For some economists, to hedge with futures contracts may also mean to trade in order to profit from futures price changes. While there are certainly profit motives for trading futures, as discussed in Chapter 4, in this book we always prefer to treat the expression "futures hedging" as futures trading motivated solely by risk reduction. The same economic agent may take a futures position that is partly motivated by risk reduction (the hedge portion) and partly motivated by expected profits.

In a broad context, hedging also encompasses *synthetic insurance*, the adoption of a dynamic trading strategy that puts a lower bound on the value of a position at a given time in the future. We limit the discussion in this chapter to the selection of a minimum-variance futures position. Chapter 8 deals with synthetic insurance.

Alternate Hedging Markets

We cannot assert that the best hedge for a given commitment is always made with a futures position. There are alternatives. For instance, in order to hedge the commitment to sell 25 million marks on November 15, one could instead use the three-step strategy:

1. Borrow marks now from a German bank through the firm's German subsidiary, enough so that the amount to be repaid on November 15 is approximately 25 million marks.
2. Exchange the borrowed marks immediately for U.S. dollars and deposit the dollars in a U.S. bank.
3. On November 15, repay the 25 million marks due on the German bank loan with the German account receivable of 25 million marks.

This strategy essentially eliminates the spot commitment risk and, in some situations, may be more attractive than futures hedging. On the other hand, the transactions costs generated by the strategy may (or may not) be large relative to the futures transactions costs. In general, the main advantages of hedging in futures markets are the relatively low transactions costs, low default risk, and ease of execution relative to more customized hedging arrangements.

For protection against exchange rate risk, one can also turn to the interbank market for forward foreign exchange, outlined in Appendix 2B. In contrast with many forward markets, the interbank foreign exchange markets compare quite favorably with futures contracts in terms of their volume of trade and liquidity. In fact, forward foreign exchange markets are especially well suited to large trades, relative to the typical size of trades in currency futures. The principles of hedging in forward markets are quite similar to those of hedging in futures markets, and most of the techniques explained in this chapter apply to forwards with minor changes.

The options and futures options markets are also quite popular arenas for hedging. The idea of options hedging, however, is not to reduce the volatility of one's position, as it is with a futures hedge, but rather to obtain a floor under one's position value in return for the option premium. The distinction between these two approaches is illustrated in Chapter 8, along with a general review of options and futures options.

2. BASIS

Many situations call for hedging a fixed quantity of a spot market asset with a futures contract delivering the same or a related asset. The *basis* is the difference between the futures and spot prices at the spot commitment date. In general, the basis is random, and represents a risk that cannot be eliminated. In some cases, however, there is a formula for the futures price in terms of the spot price that, in principle, eliminates basis risk and makes the optimal hedge simple to choose and extremely effective. We will look at the two cases: zero basis and a random basis. (An exercise examines the case of a non-zero non-random basis.)

Hedging with Zero Basis

If the spot commitment date is the delivery date of the futures contract, the basis is zero! We illustrate the case of zero basis with the following example.

Example: Suppose Acme Mint, Inc. (a fictional firm), has commitments to buy 20,000 ounces of gold on December 14, 1989, in order to meet its production needs for a large order of gold coins. Assuming the Comex futures contract for December 1989 delivery also delivers on December 14, there is little question about basis risk. We know that, for any futures contract, the threat of arbitrage equates the futures price with the spot price on the contract's delivery date. Thus the cost of the gold can be effectively "locked in" by taking a long position of 200 contracts at 100 ounces each. The spot gold will cost Acme

Mint $20,000 \times s_T$, where s_T is the gold spot price per ounce at the delivery date T. The cumulative futures resettlement profit (or loss) from today's date t to the delivery date T is 200 contracts $\times$ 100 ounces per contract $\times (f_T - f_t)$, where f_t denotes the futures price per ounce at date t. Neglecting interest on margin, the cost of the gold, net of futures resettlement profits, is

$$\begin{aligned}\text{Net cost} &= 20{,}000 \times s_T - 200 \times 100 \times (f_T - f_t)\\ &= 20{,}000 \times s_T - 20{,}000 \times (s_T - f_t)\\ &= 20{,}000 \times f_t,\end{aligned}$$

using the fact that $s_T = f_T$. The net cost of the gold is therefore the cost at the current (known) futures price f_t, say \$500 per ounce, or \$10 million. (Of course, we are ignoring interest costs on margin payments, transactions costs, and so on.) Even though s_T and f_T are unknown, the fact that they are the same implies that an equal and opposite futures position of 20,000 ounces eliminates the risk of the hedged position.

Let's check our results against the "beta" hedging formula (1) given in the introduction. In order to calculate β, we divide $\text{cov}(f_T - f_t, s_T - s_t)$ by $\text{var}(f_T - f_t)$. Since f_t and s_t are known at time the hedge is calculated,

$$\text{cov}(f_T - f_t, s_T - s_t) = \text{cov}(f_T, s_T).$$

Since $s_T = f_T$, we know that $\text{cov}(f_T, s_T) = \text{var}(f_T)$, implying that

$$\beta = \frac{\text{cov}(f_T, s_T)}{\text{var}(f_T)} = \frac{\text{var}(f_T)}{\text{var}(f_T)} = 1.$$

The risk-minimizing futures position is $h = -Q\beta$, where Q is the size of the spot market commitment. In this case, $Q = -20,000$ ounces since Acme Mint has a commitment to buy 20,000 ounces of gold. (This is a short spot market position; had Acme committed instead to sell 20,000 ounces, Q would be $+20,000$ ounces.) Thus

$$h = -Q\beta = -(-20,000) \times 1 = 20,000 \text{ ounces},$$

or 200 contracts of 100 ounces each, confirming the "equal and opposite" rule of thumb for hedging with zero basis.

Hedging with a Random Basis

In general, the basis is random: a risk minimizing futures position exists, but does not reduce the total risk to zero.

Example: Consider a hedge for a spot market commitment to receive 10 million Dutch gulden, the currency of Holland, by USCX, Inc., a small U.S. exporter of computer equipment.[1] The payment is due to be deposited in USCX's account in one month and converted immediately into U.S. dollars, in return for a shipment of computers due to arrive in Amsterdam at the same time. Since there is currently no Dutch guilder futures contract, USCX must hedge in a different currency futures contract, say West German marks. Clearly, the basis in this case is random unless the German-Dutch exchange rate is fixed for the entire period of the hedge, and there is no evidence of fixed exchange rates. In fact, the monthly exchange rate data for 1984–1986 shown in Appendix 7A indicate that, although movements in the spot guilder exchange rate and the nearby futures price for marks are related, they are not perfectly correlated. (The OLS estimate of their correlation during this period is in fact 0.78.)

We have a formula, $h = -Q\beta$, for the risk-minimizing futures position h in deutsche marks, but before we estimate the hedging coefficient β, it may help USCX managers to better understand the hedging situation if we can show them the plot in Figure 7.1 of the estimated risk as a function of their futures position y in the nearby deutsche mark contract. There are two easy ways to produce this plot with a computer, as follows.

1. The first method requires one to calculate the value of the hedged position had the hedge been placed at some month t in the past for which data are available. Appendix 7A has data for 36 months during 1984–1986. The calculation is made separately for each month t from 1 to 36 as follows. For each futures position y, using the data in Appendix 7A, calculate the total dollar value, denoted $P_t(y)$, that USCX would receive at month $t+1$ if it were to be credited the change from month t to month $t+1$ in the cash market value of 10 million gulden plus the resettlement profits or losses on a futures position of y million deutsche marks established at month t and offset

[1] This example uses information provided by a former student, Joshua Sommer, and is based on his response to Exercise 7.2. Gulden is the plural of guilder.

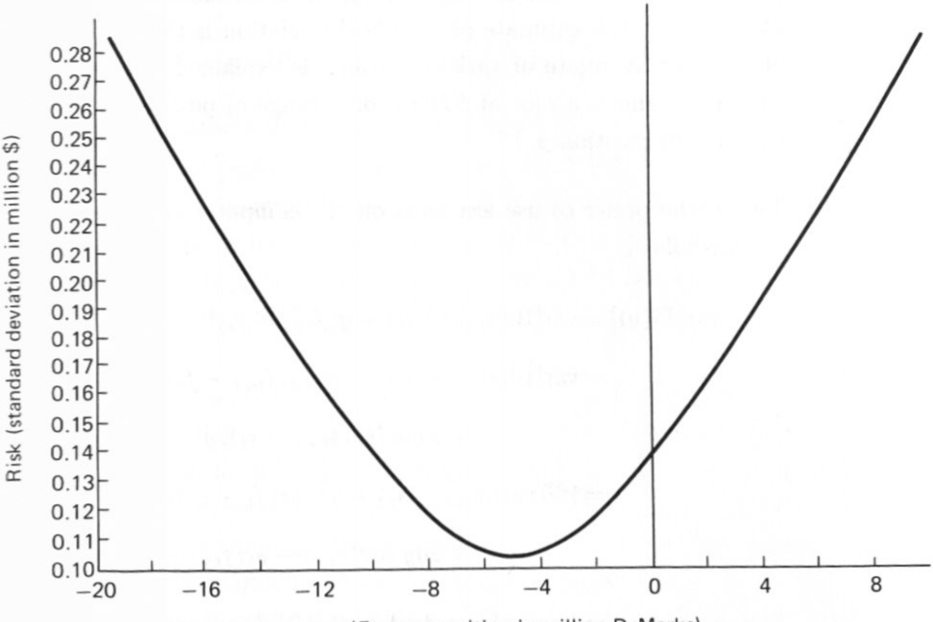


Figure 7.1 The Estimated Risk as a Function of the Futures Position

at month $t + 1$. This total, in millions of U.S. dollars, is

$$P_t(y) = 10(s_{t+1} - s_t) + y(f_{t+1} - f_t),$$

where s_t is the spot guilder exchange rate at month t (U.S. dollars per Dutch guilder), f_t is the deutsche mark futures price at month t (U.S. dollars per deutsche mark), and y is the futures position (in millions of marks). If USCX had faced its current hedging situation at month t and adopted the futures position y, $P_t(y)$ would have been the change in market value of its hedged position. For example, with $t = 1$ and a futures position of $y = 10$ contracts (10 short), the change in position value is

$$\begin{aligned} P_1(y) &= 10 \times (0.3397 - 0.3155) - 10 \times (0.3975 - 0.3670) \\ &= 0.063 \text{ million dollars.} \end{aligned}$$

Based on the 36 available monthly price observations for 1984–1986, this new series $P_1(y), P_2(y), \ldots, P_{35}(y)$ of monthly observations is put into a new column of data in the computer. At this stage, one can calculate the OLS estimate SD(y) of the standard deviation of $P_t(y)$. The OLS estimate of standard deviation is the square root of the OLS estimate of variance, which is explained in Chapter 6. Figure 7.1 shows a plot of $SD(y)$ for a range of possible values for the futures position y.

2. Those who prefer to use less time on the computer can instead use the formula

$$\begin{aligned}\operatorname{var}[P_t(y)] &= \operatorname{var}[10(s_{t+1} - s_t) + y(f_{t+1} - f_t)] \\ &= \operatorname{var}[10(s_{t+1} - s_t)] + \operatorname{var}[y(f_{t+1} - f_t)] \\ &\qquad + 2\operatorname{cov}[10(s_{t+1} - s_t), y(f_{t+1} - f_t)] \\ &= 100 \operatorname{var}(s_{t+1} - s_t) + y^2 \operatorname{var}(f_{t+1} - f_t) \\ &\qquad + 20y \operatorname{cov}(s_{t+1} - s_t, f_{t+1} - f_t).\end{aligned}$$

The computer can be used to calculate the OLS estimates V_s, V_f, and C_{sf} of the respective unknown statistics $\operatorname{var}(s_{t+1}-s_t)$, $\operatorname{var}(f_{t+1}-f_t)$, and $\operatorname{cov}(s_{t+1}-s_t, f_{t+1}-f_t)$, using the data in Appendix 7A. It turns out that $V_f = 0.0003376$, $V_s = 0.000189$, and $C_{sf} = 0.000167$. We then have an alternative derivation of the risk (standard deviation) estimate SD(y) via the equation

$$\begin{aligned}\mathrm{SD}(y) &= \sqrt{100V_s + 20y\, C_{sf} + y^2 V_f} \\ &= \sqrt{0.0189 + 0.00335\, y + 0.0003376\, y^2}.\end{aligned}$$

Using either method 1 or method 2, a plot of the estimated risk SD(y) for different values of y is shown in Figure 7.1. The managers of USCX may find this information quite useful in deciding whether to hedge, and if so, how much. From the plot, we can see that the minimum estimated risk is obtained at a short futures position of 5 million deutsche marks, or short 40 contracts of 125,000 deutsche marks each. The formula $h = -Q\beta$ produces the same result (as it must) if we substitute the OLS estimate for β, which is

$$\widehat{\beta} = \frac{C_{sf}}{V_f} = \frac{0.000167}{0.0003376} = 0.496.$$

Since the quantity to be hedged is $Q = 10$ million gulden, the futures position minimizing the estimated risk is

$$\widehat{h} = -Q\widehat{\beta} = -10 \times 0.496 = -4.96 \quad \text{(million deutsche marks)},$$

or about 40 contracts short, confirming the graphical solution.

To check the performance of the recommended hedge over the historical period, USCX management may also appreciate the plots of the change in value $P_t(y)$ of the hedged position shown in Figure 7.2 for the risk-minimizing futures position of $y = -5$ million deutsche marks, as well as for the zero hedging choice, $y = 0$. The plot of $P_t(0)$ clearly shows greater volatility than that of $P_t(-5)$.

At this writing, the spot exchange rates are 0.5319 U.S. dollars per guilder and 0.5968 U.S. dollars per deutsche mark. The mark-guilder exchange rate is therefore $0.5319/0.5968 = 0.8913$ deutsche marks per guilder. An "equal and opposite" strategy in deutsche mark futures would therefore call for a futures position of -10 million gulden $\times$ 0.8913 deutsche marks per guilder $= -8.9$ million deutsche marks, or 71 contracts (short). Comparing the performance of the equal-and-opposite strategy with the risk-minimizing strategy leaves no doubt as to the relative efficacy of the risk-minimizing strategy in controlling the volatility of the hedged position change during the three-year historical period. This can be verified graphically in both Figures 7.1 and 7.3. The equal-and-opposite strategy over-hedges by not accounting for the basis risk.

Exercise 7.3 supplies an additional 12 months of exchange rate data for the period following the initial 36-month historical period. This exercise asks for a check of the performance during the latter 12-month period of the hedge chosen using the first 36 months of data. Since there is no guarantee that exchange rate data come even close to satisfying the OLS conditions, this is a prudent check to make when using old data for statistical estimation of β.

3. PRINCIPLES OF STATISTICAL HEDGING

This section reviews several basic principles of the statistical approach to hedging illustrated in the previous two sections.

The Minimum-Variance Criterion

One should immediately question the minimum-variance criterion. Reusing the notation of Chapter 4, for example, suppose the commitment of a given

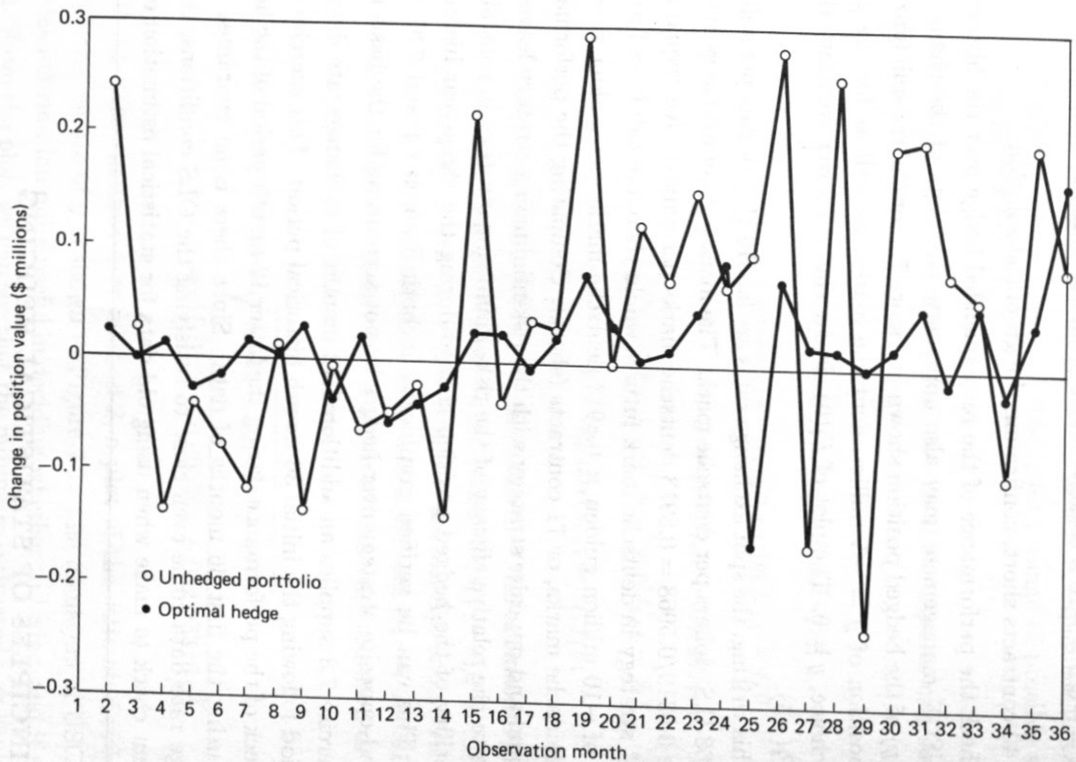


Figure 7.2 Historical Performance of Risk-Minimizing Hedge

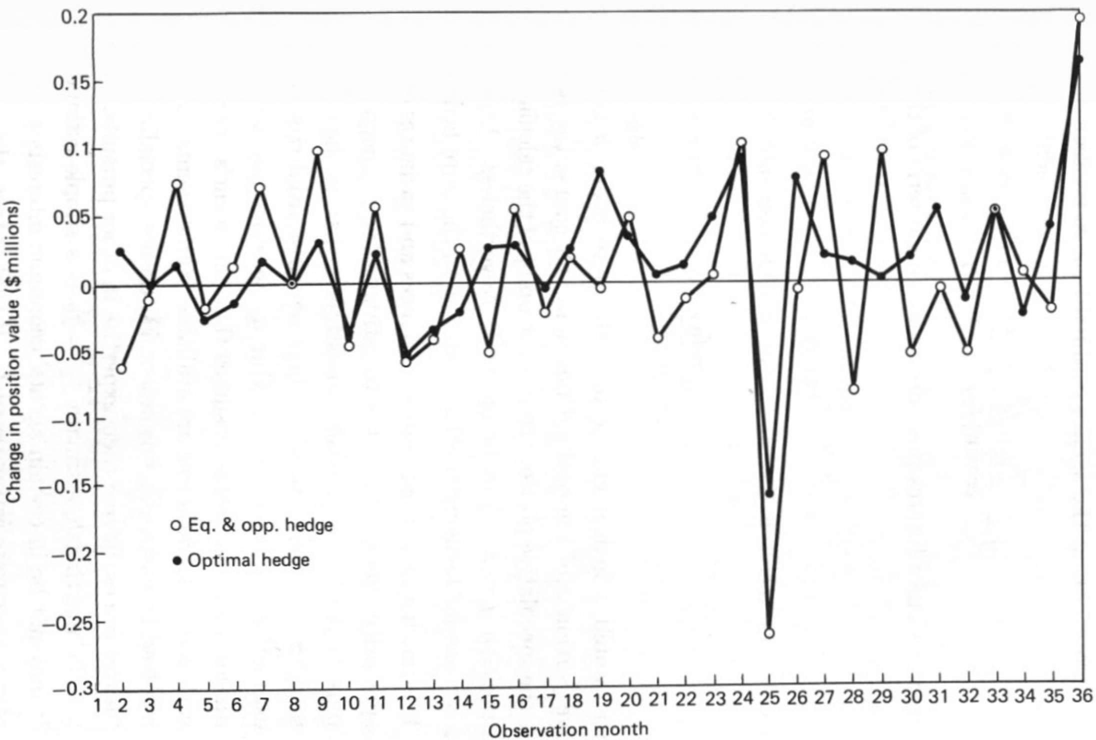


Figure 7.3 Risk-Minimizing and Equal-and-Opposite Hedging Performance

agent has an uncertain market value e (the *endowment*), while f_1 is the futures price at the date the endowment is received. The current futures price is f_0. If the given agent has mean-variance utility with risk-aversion coefficient r, we saw in Chapter 4 that the agent's optimal futures position is

$$y = \frac{\overline{f}_1 - f_0}{2r \operatorname{var}(f_1)} - \frac{\operatorname{cov}(e, f_1)}{\operatorname{var}(f_1)}.$$

The minimum-variance futures position is the second term of this expression, the *pure hedge*

$$h = -\frac{\operatorname{cov}(e, f_1)}{\operatorname{var}(f_1)}.$$

The remainder, we recall, is the purely speculative demand

$$z = \frac{\overline{f}_1 - f_0}{2r \operatorname{var}(f_1)}.$$

Why, then, would a hedger choose only the pure hedge h and ignore the speculative portion z of the optimal position? Of course the hedger should *not* ignore the speculative portion; that is the point of the calculations showing the total position $y = z + h$ to be optimal. Nevertheless, there are at least two good reasons for focusing special attention on the pure hedging term h:

1. It is much easier to estimate covariances and variances of futures and spot price changes than it is to estimate their expected values. The pure hedge h is thus easier to estimate than is the speculative demand z. Furthermore, there may be substantial risk involved from estimating $\overline{f}_1$ inaccurately. Our model does not account for estimation risk, but rather assumes that the agent's actual means, covariances, and variances are available. Some agents, especially those involved in particular business risks, do not specialize in projecting futures prices, having their expertise in other pursuits. Always making the martingale estimate $\overline{f}_1 = f_0$ is a simple estimation model. It may not be in certain agents' economic interests to pay the costs of more accurate estimates of $\overline{f}_1$. If that is true, the optimal futures position is h, since $\overline{f}_1 = f_0$ implies a speculative demand of $z = 0$.

2. The pure hedge (per unit of spot market position) is common to all mean-variance agents with commitments in the same spot market, while the speculative portion of a futures position is specific to

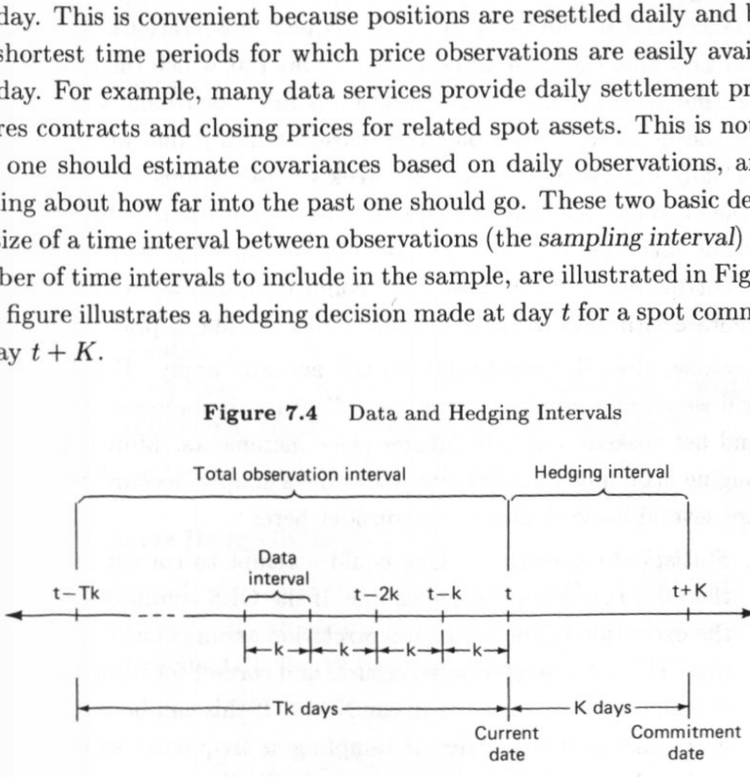
the individual agent, since the risk-aversion coefficient r depends on the agent. Furthermore, there are reasons to believe that, although different agents may estimate $\overline{f}_1$ quite differently on the basis of private information, their estimates of covariances and variances will not differ as much. Thus calculating the pure hedge (per unit of spot market commitment) can be a useful service to many different agents, who can individually multiply by the size of their spot market commitments and then add on their own speculative demands.

As far as the first reason is concerned, one could, in principle, extend the mean-variance model to account for estimation risk based on restrictive assumptions, but there are few available guidelines. For most of this chapter we focus on the "pure hedge," and naively proceed as though the least squares estimates of covariances are the agent's covariances. Possible corrections for estimation risk are suggested in sources cited in the Notes.

Data Intervals

For practical purposes, one often takes the length of a time interval to be one day. This is convenient because positions are resettled daily and because the shortest time periods for which price observations are easily available is one day. For example, many data services provide daily settlement prices for futures contracts and closing prices for related spot assets. This is not to say that one should estimate covariances based on daily observations, and says nothing about how far into the past one should go. These two basic decisions, the size of a time interval between observations (the *sampling interval*) and the number of time intervals to include in the sample, are illustrated in Figure 7.4. This figure illustrates a hedging decision made at day t for a spot commitment at day $t + K$.

Figure 7.4 Data and Hedging Intervals



For example, let $f_1, f_2, \ldots$ denote the stochastic process of the daily futures settlement prices of a given futures contract, and let $s_1, s_2, \ldots$ denote the stochastic process of daily closing spot prices for a related spot asset. For each day t, let $F_t = f_t - f_{t-1}$ and $S_t = s_t - s_{t-1}$ denote the corresponding price increments. Under the joint OLS conditions on F and S, we know that, for a time interval of k days,

$$\operatorname{var}(f_{t+k} - f_t) = \operatorname{var}(F_{t+1} + F_{t+2} + \cdots + F_{t+k}) = k \operatorname{var}(F_{t+1}), \tag{3}$$

and likewise that

$$\begin{aligned}
\operatorname{cov}(f_{t+k} - f_t, s_{t+k} - s_t) \\
&= \operatorname{cov}\left[(F_{t+1} + F_{t+2} \cdots + F_{t+k}), (S_{t+1} + S_{t+2} + \cdots + S_{t+k})\right] \\
&= k \operatorname{cov}(F_{t+1}, S_{t+1}).
\end{aligned} \tag{4}$$

Based on equations (3) and (4), one could use daily variance and covariance estimates to obtain, say, weekly ($k = 7$) covariance and variance estimates for hedging purposes. Furthermore, under the OLS conditions, better estimates of covariances can be obtained by more frequent observations. In this case, no matter how long the interval between the day t at which the hedge is "put on" (the time at which the futures position is first taken) and the day $t + K$ at which the hedge is "taken off" (the closeout date), one would be better off using daily observations than, say, weekly observations. Indeed, pushing this to the extreme, one might want to estimate covariances on the basis of observations taken as frequently as possible, say hour by hour or minute by minute. Furthermore, under these ideal conditions, one would always obtain more accurate estimates by including the earliest available price observations.

Of course, the OLS conditions do not actually apply. For instance, in Chapter 6 we cited a number of studies indicating the presence of autocorrelation and heteroskedasticity in futures price increments. Moreover, because of a changing economic environment, older data simply become less relevant. There are several basic strategies to consider here:

1. *Statistical Correction:* One could attempt to correct for failure of the OLS conditions. For example, if the OLS conditions apply with the exception of the zero autocorrelation assumption, one could estimate the autocorrelation structure, and correct for it, using methods explained in sources cited in the Notes. If this can be done properly, one would still be better off sampling as frequently as possible (ignoring data and computational costs). Similarly, one can correct for

heteroskedasticity problems such as the Samuelson effect or seasonality effects described in Chapter 6.

2. *Adjustment of Sampling Interval:* One could choose a sampling interval that accounts for the length of time the futures position is fixed. This is a vague prescription, and will be left mainly to the reader to sort out. There is still a trade-off between the number of observations and estimation bias. With only a year of relevant data and a proposed three-month hedging interval, for instance, one might be inclined to resort to weekly rather than quarterly observations if the data do not show severe autocorrelation. (Most statistical estimation packages include tests of significant autocorrelation.) In order to use all of the available data, one can also choose overlapping time intervals, for example, take monthly sampling intervals with weekly data, so that one month's interval includes three weeks from the previous month's interval. If the weekly data satisfies the OLS conditions, this procedure clearly generates a violation of the OLS conditions. As explained by Hansen and Hodrick (1980), however, by properly accounting for the induced autocorrelation, one can improve the quality of the estimates over those obtained by monthly non-overlapping sampling intervals, under certain conditions.

3. *Dynamic Hedging:* One can adjust the hedge as time transpires. For example, if the OLS conditions do not apply because of heteroskedasticity (along the lines, say, of the Samuelson effect or seasonality in variances), the appropriate hedge may change through time. If price increments are serially uncorrelated, for example, then the optimal hedge can be adjusted through time as though the hedge is to be taken off at the next opportunity, even if the spot commitment is far into the future. We will see a particular example of an optimal dynamic hedging strategy later in the chapter.

Strategies 1–3 can be combined when appropriate.

The Least Squares Hedge Estimator

Suppose it is currently day number t, and that one is committed to Q units of the spot asset at day $t+K$. (A commitment to buy the spot asset corresponds to a negative value for Q.) We have shown earlier that the risk-minimizing futures position is $h = -Q\beta$, where

$$\beta = \frac{\operatorname{cov}(s_{t+K} - s_t, f_{t+K} - f_t)}{\operatorname{var}(f_{t+K} - f_t)}.$$

Under the OLS conditions, equations (3) and (4) tell us that the hedging coefficient β is independent of the time interval over which the price change occurs. For example, we know that

$$\beta = \frac{\operatorname{cov}(s_{t+1} - s_t, f_{t+1} - f_t)}{\operatorname{var}(f_{t+1} - f_t)} = \frac{\operatorname{cov}(S_{t+1}, F_{t+1})}{\operatorname{var}(F_{t+1})}. \tag{5}$$

In fact, if we have price data at k-day intervals, the OLS conditions allow us to treat β as the ratio of covariance to variance over this sampling interval, or

$$\beta = \frac{\operatorname{cov}(s_{t+k} - s_t, f_{t+k} - f_t)}{\operatorname{var}(f_{t+k} - f_t)}. \tag{6}$$

How does one estimate β ? The obvious procedure is to calculate the sample covariance $C(S, F)$ for $\operatorname{cov}(s_{t+k} - s_t, f_{t+k} - f_t)$ and sample variance $V(F)$ for var $(f_{t+k} - f_t)$, and then to construct the *least squares beta estimator*

$$\widehat{\beta} = \frac{C(S, F)}{V(F)}. \tag{7}$$

(A more convenient way to estimate β is explained in Section 7.4.) The least squares estimator (7) is based on the observations:

$$[(f_k - f_0, s_k - s_0), (f_{2k} - f_k, s_{2k} - s_k), \ldots, (f_{kT} - f_{k(T-1)}, s_{kT} - s_{k(T-1)})],$$

or T observations over Tk days, as illustrated in Figure 7.4. Taking successive Wednesday settlement prices for T weeks, for example, would correspond to $k = 7$ days.

If the OLS conditions apply jointly to the spot and futures price increments, then the *least squares hedge estimator*

$$H = -Q\widehat{\beta} \tag{8}$$

is a reasonable estimator for the pure hedge $h = -Q\beta$. These qualifications, however, are rather large and important, and adjustments are often appropriate, as we have discussed. The least squares hedge estimator is widely used in practice.

Hedging Based on Percentage Price Changes

For many futures contracts, it is more palatable to assume that *percentage price changes* satisfy the OLS conditions than it is to assume that price changes themselves satisfy the OLS conditions. As pointed out in Chapter 6, there is an important distinction. For example, suppose the percentage price change from f_t to f_{t+1} is +50 percent or −50 percent, with equal probabilities, at any time t. If $f_t = 100$, then at time t, $\text{sdev}(f_{t+1} - f_t) = 50$. Suppose the futures price goes up from $f_t = 100$ to $f_{t+1} = 150$. Then at time $t+1$ an increase or decrease of 50 percent represents a price change of 75, and the standard deviation of $f_{t+2} - f_{t+1}$ at time $t+1$ goes up to 75, whereas the standard deviation of the percentage price change from f_{t+1} to f_{t+2} remains constant at 50 percent.

We can deal with OLS percentage price changes using a statistical hedging approach explained in Appendix 7D. It turns out that the optimal hedge depends on the current ratio of futures to spot prices, which changes with time. This may call for a dynamic hedging policy that readjusts the hedge as spot and futures prices change through time.

4. HEDGING ESTIMATES BY LINEAR REGRESSION

This section explains how to obtain least squares estimates of hedging coefficients from widely available linear regression software.

Simple Linear Regression

We recall the well-known model of *simple linear regression* of spot price increments S_{t+1} regressed on futures price increments F_{t+1}. (A more detailed review is given in Appendix 6B.) By *linear*, we mean that S_{t+1} is approximated by an expression of the form $a + bF_{t+1}$. The difference $\epsilon_{t+1} = S_{t+1} - a - bF_{t+1}$ is called the *residual*. The objective is to choose the coefficients a and b so that:

1. The approximation is unbiased, meaning $E(a + bF_{t+1}) = E(S_{t+1})$.
2. The residual ϵ_{t+1} has the minimum possible variance.

There is a unique solution to this problem: Choose $b = \beta$, as specified by equation (5), and choose $a = \alpha$, where $\alpha = E(S_{t+1}) - \beta E(F_{t+1})$. The fact that the best choice for b is β follows from the fact that, since a is a constant,

$$\text{var}(\epsilon_{t+1}) = \text{var}(S_{t+1} - bF_{t+1}),$$

and we have already solved this minimum-variance problem under the guise of hedging: we found in Chapter 4 that the variance of $S_{t+1} - bF_{t+1}$ is minimized by choosing $b = \text{cov}(S_{t+1}, F_{t+1})/\text{var}(F_{t+1})$, which is β. Of course, the choice of $a = \alpha$ is dictated by unbiasedness.

Hedging Coefficients from Regression Coefficients

We have discovered a natural coincidence: the optimal hedging coefficient β is the same as the coefficient of the simple linear regression of S_{t+1} on F_{t+1}. This coincidence can be put to use! The regression coefficient reported by standard statistical software packages is the least squares beta estimator $\widehat{\beta}$. Moreover, most such software packages also report several *regression diagnostics*, as shown below in an estimated regression of weekly spot price increments for deutsche marks on weekly nearby futures price increments for deutsche marks, based on 26 consecutive weeks of observations during 1983. We have

$$\begin{array}{lcccc} S_{t+1} = & -0.00005 + & 0.93468\, F_{t+1} & + \ \hat{\epsilon}_{t+1} & (R^2 = 0.937). \\ & (0.0003) & (0.0501^*) & (0.0013) & \end{array}$$

The key number for hedging purposes is $\widehat{\beta} = 0.93468$. We will not use the estimate of $\widehat{\alpha} = -0.00005$. The diagnostic reported as $R^2 = 0.937$ is the squared estimated correlation between S_{t+1} and $\widehat{\beta} F_{t+1}$, and is a useful measure of the quality of the hedge. An R^2 of 1.0 represents a perfect hedge (no risk); an R^2 of zero represents no hedging power at all. In fact, R^2 is the estimated fraction of variance in the spot deutsche mark position that is eliminated by the optimal hedging position, under the OLS conditions.

The diagnostics reported below the regression equation in parentheses are the least squares estimates of the standard deviations of $\widehat{\alpha}$, $\widehat{\beta}$, and $\hat{\epsilon}_{t+1}$, respectively. These are sometimes called *standard errors*, and indicate the precision of the respective estimates. The asterisk adjacent to the standard error 0.0501 of $\widehat{\beta}$ indicates "statistical significance." In this case, significance means merely that one can reject the hypothesis (with 95 percent probability under special assumptions) that F_{t+1} and S_{t+1} are uncorrelated. It appears, therefore, that the futures contract provides significant hedging power. The asterisk indicating significance appears whenever the ratio of $\widehat{\beta}$ to its standard error is sufficiently large; a rough rule of thumb for large samples under the OLS conditions is a ratio of 2.0 or more. In this example, the ratio, called the *t-statistic*, is actually 18.5, so there is little question that $\widehat{\beta}$ is statistically significant. On some software packages, t-statistics are reported in place of standard errors.

An estimated regression and the associated data points are illustrated in Figure 7.5. The *regression line* is fitted to the data points by choosing the *intercept* $\widehat{\alpha}$ and the *slope* $\widehat{\beta}$ so that the sum of squared vertical distances from the data points to the line is minimized.

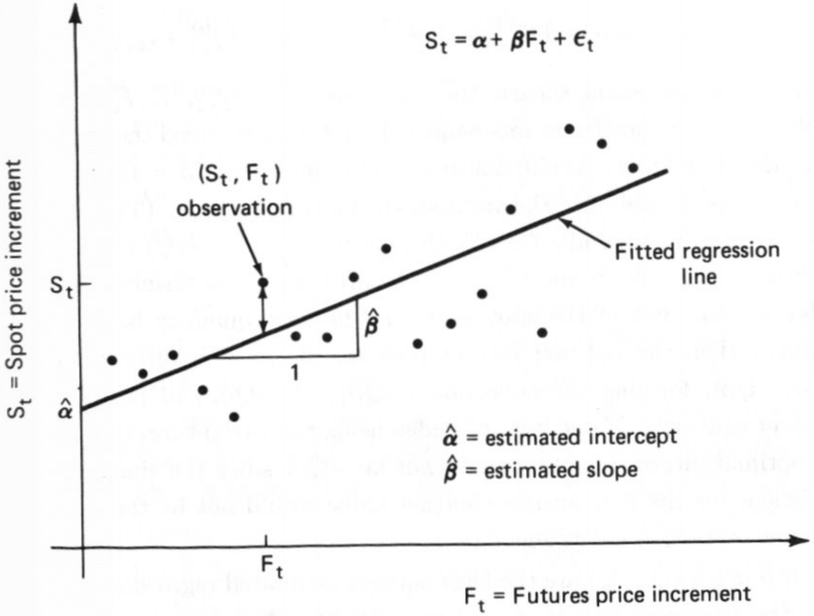


Figure 7.5 Estimated Hedging Regression Line and Data

According to our regression equation, the receivable of 25 million marks described in Section 7.1 has the least squares hedge estimate

$$H = -Q\widehat{\beta} = -25{,}000{,}000 \times 0.93464 = -23,410,000 \ \text{ marks.}$$

At 125,000 deutsche marks per contract, this is a short hedge position of 187 contracts, to the nearest whole contract, as reported at the beginning of the chapter.

Hedging by Multiple Linear Regression

In some situations there is an obvious case for hedging a given spot commitment with positions in several different types of futures contracts. As it turns out, hedging positions in each of the futures contracts can be estimated using *multiple linear regression*, the obvious extension of simple linear regression.

Suppose n different futures contracts are under consideration for hedging a spot market commitment. Let S_t denote the spot price increment at time t and let $F_t^{(i)}$ denote the corresponding price increment of the i-th futures contract under consideration. The *multiple linear regression* equation takes the form

$$S_{t+1} = \alpha + \beta_1 F_{t+1}^{(1)} + \beta_2 F_{t+1}^{(2)} + \cdots + \beta_n F_{t+1}^{(n)} + \epsilon_{t+1}.$$

In terms of regression theory, the variables $F_{t+1}^{(1)}, F_{t+1}^{(2)}, \ldots, F_{t+1}^{(n)}$ are called *explanatory* (or sometimes *independent*), and S_{t+1} is called the *explained* (or *dependent*) variable. Again, the regression coefficients $\beta = (\beta_1, \ldots, \beta_n)$ are chosen so as to minimize the variance of the residual ϵ_{t+1}. (The solution for β is reviewed in Appendix 6B.) Equivalently, $(-\beta_1, \ldots, -\beta_n)$ is the collection of futures positions in the respective futures contracts forming the optimal hedge for one unit of the spot asset. If the spot quantity being hedged is Q units, then the optimal futures position in the i-th futures contract is $h_i = -Q\beta_i$, forming the collection $(-Q\beta_1, \ldots, -Q\beta_n)$ of positions in the different contracts. If one were to hedge using the i-th futures contract alone, the optimal futures position would *not* be $-Q\beta_i$ since the simple regression coefficient for the i-th futures contract alone would not be the same as the multiple regression coefficient β_i.

If $\widehat{\beta} = (\widehat{\beta}_1, \ldots, \widehat{\beta}_n)$ are the least squares estimated regression coefficients, then the corresponding least squares estimator for the minimum variance hedging position is

$$(-Q\widehat{\beta}_1, \ldots, -Q\widehat{\beta}_n).$$

Most multiple regression software packages report the regression coefficients $\widehat{\beta}$ as well as the associated regression diagnostics.

Example: Consider the problem of hedging a portfolio of 10,000 shares of Coca-Cola common stock with a position in S&P 500 Index futures and Value Line Average futures. Let:

- S_t denote the change during week t in the share price of Coca-Cola
- $F_t^{(1)}$ denote the change in the nearby S&P 500 Futures Index of the Chicago Mercantile Exchange during week t
- $F_t^{(2)}$ denote the change in the nearby Value Line Futures Index of the Kansas City Board of Trade during week t

The above data were collected for 104 weeks, using Friday closing NYSE stock prices and Friday futures settlement prices for the first week of 1985 through the first week of 1987. A standard statistical software package reported the following estimated regression of S_{t+1} on $F^{(1)}_{t+1}$ and $F^{(2)}_{t+1}$:

$$\underset{}{S_{t+1}} = \widehat{\alpha} \underset{(0.04)}{- 0.038F^{(1)}_{t+1}} + \underset{(0.04^*)}{0.132F^{(2)}_{t+1}} + \underset{(0.72)}{\hat{\epsilon}_{t+1}}. \qquad (R^2 = 0.488)$$

The estimated hedging coefficients are thus $\widehat{\beta}_1 = -0.038$ and $\widehat{\beta}_2 = 0.132$. The numbers reported in parentheses below these coefficients are estimates of the standard errors of the regression coefficients, as discussed earlier. It appears (from the absence of an asterisk on the appropriate standard error) that the S&P 500 contract may not provide significant hedging ability. We will return to this issue shortly. The reported R^2 corresponds to an estimated 48.8 percent reduction in variance (risk) by hedging. All of these estimates are based, as usual, on the OLS conditions, a major assumption. The least squares hedge estimate is easily calculated as follows. Both the S&P 500 and Value Line contracts pay 500 times their indices at delivery. Since the quantity Q of Coca-Cola shares to hedge is 10,000, the estimated hedge is $H_1 = -Q\widehat{\beta}_1/500 = 0.76$ S&P 500 contracts and $H_2 = -Q\widehat{\beta}_2/500 = -2.64$ Value Line contracts.

The fact that $\widehat{\beta}_1$ fails the test of statistical significance, however, suggests that one may be better off dropping the S&P 500 contract from the hedge entirely. In that case, the Value Line hedging coefficient $\widehat{\beta}_2 = 0.132$ is *not* an appropriate estimate of the regression coefficient corresponding to a hedge in the Value Line contract alone. Instead, one should estimate a new regression of S_{t+1} on $F^{(2)}_{t+1}$. For our data, the results are

$$S_{t+1} = \widehat{\alpha} + \underset{(0.016^*)}{0.098\, F^{(2)}_{t+1}} + \underset{(0.72)}{\epsilon_{t+1}}. \qquad (R^2 = 0.483) \tag{9}$$

The corresponding estimated hedge is

$$H = -Q\widehat{\beta} = \frac{-10{,}000 \times 0.098}{500} = -1.96 \text{ contracts,}$$

or a short position of about two nearby Value Line contracts. Although the R^2 for this simple regression is reduced (from that of the multiple regression that includes the S&P 500 futures as an additional explanatory variable), the R^2 is not a valid measure for comparing the quality of two regressions with different numbers of explanatory variables.

Incidentally, the Durbin-Watson statistic 2.64 for this regression indicates a relatively high likelihood of autocorrelation in the price increments, as discussed in Appendix 6C. In some cases, advanced econometric techniques are called for in dealing with autocorrelation. We mention some references in the Notes.

5. COMMON HEDGING QUESTIONS

In this section we review some commonly encountered issues of hedging strategy.

Which Futures Contract?

Which type of futures contract provides the maximum risk reduction? When hedging the risk of spot price movements in a given asset, it is often, but not always, true that the optimal futures contract is that delivering the given asset. (An exception is the case of hedging Ginnie Mae mortgage risk, discussed in Appendix 9B.)

Of course, many spot assets are not delivered in any futures market, and one must *cross-hedge*, which means hedge in a futures contract delivering a different but related asset. The general rule for hedging a given quantity of an asset is:

> *Choose that futures contract whose price movements have the maximum possible correlation with price movements in the committed spot asset.*

Exercise 7.4 asks for a demonstration of the optimality of this rule.

How Many Different Futures Go in a Hedge?

Ignoring transactions costs, estimation risk, and the fact that one can only take positions in whole numbers of contracts, there are no theoretical limits on the number of different futures contracts one would use to hedge a particular commitment. The variance of the total spot plus futures position could always be reduced in principle by including more types of futures contracts in the hedge. Mathematically speaking, this is equivalent to the fact that the standard error of a linear regression can always be reduced by including more explanatory variables. Because one typically uses only statistical estimates of covariances, the estimated variance reduction achieved by including additional types of futures contracts can be illusory, just as can the reported reduction in the standard error of a regression achieved from including unrelated explanatory variables. A hedge based on estimated covariances can in fact be worsened by including unrelated or poorly related futures contracts. Statistical tests of the hypothesis of a significant improvement in the fit of one regression equation over another can be used; some of these tests can be found in sources cited in the Notes. Common sense and an understanding of the economics of the market are always useful in selecting a hedge.

Example: *(Junk Bond Hedge)* In practice, attention could be focused on those futures contracts that bear some clear economic relationship to the spot commitment. In hedging the purchase or sale price of a bond portfolio, for example, one would tend to look first at interest rate futures of the corresponding maturities. Superior understanding of the risks involved can lead to a better hedge. For example, low-grade corporate bonds have a return that is well correlated with stock returns, since default risk increases with unexpected decreases in share value. One might therefore improve a corporate bond hedge by including both stock index and interest rate futures. Grieves (1986), for example, compares the two regressions:

1. Monthly returns of industrial corporate bonds of Moody's Baa rating regressed on monthly log-price increments of the nearby T-bond futures contract of the Chicago Board of Trade.
2. Monthly returns of industrial corporate bonds of Moody's Baa rating regressed on monthly log-price increments of the nearby T-bond futures contract of the Chicago Board of Trade as well as monthly log-price increments of the nearby S&P 500 Index futures contract of the Chicago Mercantile Exchange.

Regression 2 merely adds S&P 500 futures log-price increments as an additional explanatory variable. Both regressions were based on 31 observations ending January 1985. The R^2 reported for Regression 1 is 0.36; the R^2 reported for Regression 2 is 0.51. Although these figures suggest that an optimal hedge in both T-bond futures and S&P 500 futures provides more variance reduction than a hedge in T-bonds alone, some warnings are in order. *First*, as mentioned earlier, a higher R^2 is not a test of significant reduction in variance. As it turns out, a standard statistical test reported in Grieves's paper shows that one cannot reject the hypothesis that Regression 2 offers a significantly better fit, at a given level of confidence. *Second*, in order to minimize the variance of the value of a position in corporate bonds, the minimum-variance hedging position is obtained from the regression of bond price increments on futures price increments, and not from the regression of returns on log-price increments. Hedging based on regression of log-price increments is discussed in Appendix 7D.

Of course, there is no reason to limit attention to futures contracts. Options and other assets can often improve a hedge. A corporate bond, for example, has many of the characteristics of an option on the value of the firm, and might be better hedged using both options and futures, if an appropriate option can be found.

Which Futures Delivery Date?

An important practical problem is the choice of delivery date. If the spot commitment date is precisely the delivery date of the nearby futures contract in the same commodity, the choice seems clear. Lack of arbitrage implies that the futures price and spot price are the same, or perfectly correlated, at delivery. In principle (ignoring delivery provisions and substitutions), one could therefore obtain a perfect hedge for a fixed quantity of the underlying spot commodity at delivery by taking an equal and opposite position in the futures contract. If these two dates do not coincide, the extent to which the futures price and spot price at the spot commitment date are not perfectly correlated is the delivery basis risk. The delivery basis risk usually rises with increases in the time between the spot commitment date and the futures delivery date. In principle, the futures contract providing the best hedge (maximum risk reduction) would be the one whose delivery date is closest to the spot commitment date, other things being equal.

Of course, "other things" are not equal. *First,* if the closest futures delivery date is before the spot commitment date, the hedge disappears at delivery, and might be replaced with a hedge in the next contract to deliver, or *rolled over*, as is often said. In some cases, it is more convenient to take a position initially in the following contract to deliver, thus eliminating the need to roll over the hedge. *Second*, liquidity often decreases dramatically with the successively later contracts to deliver. The cost of using the less liquid "distant" contracts may be large enough to make hedging in the nearby contract more attractive, and lead to periodic rolling of the hedge into the successive nearby contracts. This liquidity effect receives some discussion in Appendix 7C.

What Are the Effects of Interest on Margin?

So far in our calculations we have not accounted for the daily resettlement feature of futures contracts. The effect of \$1 of futures profits or losses today is magnified by the interest paid on that dollar if re-invested until the spot commitment date. This effect, if not accounted for properly, causes one to underestimate the effective standard deviation of futures profits or losses, and therefore can cause over-hedging. The effect turns out to be mild for short hedging periods and low interest rates, and can be corrected by *tailing the hedge*, as shown in Appendix 7B.

Is Integer Rounding OK?

Except in the case of hedging with one futures contract, it is not necessarily true that the best hedge in whole numbers (*integers*) of contracts is obtained by rounding to the nearest integers. It is possible, however, to check the relative quality of several possible integer position combinations. The hedging quality of a given position $y = (y_1, y_2)$ in two different futures contracts, for example, can be estimated as the least squares estimate $\widehat{V}(y)$ of the variance of $S_{t+1} + y_1 F_{t+1}^{(1)} + y_2 F_{t+1}^{(2)}$. One could try various integer combinations $y = (y_1, y_2)$ for a futures position, check $\widehat{V}(y)$ for each, and select the integer combination y^* with the lowest estimated variance $\widehat{V}(y^*)$.

What Is the Total Hedge for Several Different Risks?

Suppose, for example, one is hedging spot positions in both Italian lire *and* Dutch gulden. What is the optimal hedge (the risk-minimizing futures position) for the combined risk? The solution is quite easy, given the following general rule:

> *Calculate the optimal hedge for each risk separately and add the different hedging positions together.*

The optimality of this simple *additivity rule of hedging* is to be demonstrated in Exercise 7.7. The additivity rule works for any number of separate sources of risk, and for hedging each risk with any number of different types of futures contracts.

It is important in applying the additivity rule of hedging to choose an integer number of contracts *after* adding the individual hedges together, not before. Otherwise unnecessary risk can accumulate from rounding errors.

6. HEDGING BY CORPORATIONS

It is commonly believed that firms are better off if they reduce their risk by futures hedging. While this is true in many cases, a careful case for hedging by firms must be based on something more than mere risk aversion.

The Modigliani-Miller Irrelevance Principle

In 1958, Franco Modigliani and Merton Miller published a simple but significant fact: Under standard assumptions, a firm cannot increase its market value, nor better serve its shareholders, merely by changing its debt-equity ratio. Indeed, the reasoning behind this principle implies that any purely financial transaction is irrelevant in this sense. To be sure, the *Modigliani-Miller Irrelevance Principle* depends on the absence of taxes, transactions fees, differences in information, bankruptcy costs, and so on. Nevertheless, the idea forces one into subtle justifications for the financial policy of a firm. In particular, one must overhaul the usual simple explanation of why firms hedge in futures markets. This overly simple justification goes as follows: "By hedging, the firm reduces the riskiness of its total value. Since investors are risk averse, they are willing to pay more for the shares of a firm if its risk is reduced. Thus hedging increases the value of the firm." This statement, left on its own, is easy to contradict, as follows.

Suppose firms A and B are identical in every respect except that firm B hedges its risk using a futures position of y contracts per share. Suppose, in recognition of the risk reduction stemming from this futures position, that the share price p_B of firm B is higher than the share price p_A of firm A. An astute investor can then perform the following arbitrage:

1. Buy one share of firm A.
2. Take a futures position of y contracts.
3. Sell one share of firm B.

Since adopting a futures position requires no initial investment, the net initial cash flow of this strategy is $p_B - p_A > 0$. There is no further cash flow since a share of firm A plus a futures position of y contracts has the same payoff as a share of firm B. This strategy therefore generates profits with no investment or risk. As always, this form of arbitrage is (theoretically) impossible, so our assumption that $p_B > p_A$ must be wrong! Hedging does not necessarily improve the value of firm B. A more careful demonstration of the result is taken up in sources indicated in the Notes.

The Modigliani-Miller Irrelevance Principle is built on the premise that anything a firm can do in futures markets, its shareholders can also do. So if it is advantageous to shareholders to have the value of their shares hedged by a futures position, then the shareholders themselves can hedge. Why should an investor pay a premium for the shares of a hedged firm if that investor can hedge on his or her own at no cost?

The following example illustrates the Modigliani-Miller Irrelevance Principle, and may also serve to limit its scope of application.

Example: The Electric Motor Corporation (EMC), Inc. (a fictional firm), makes significant purchases of copper for the windings and other components of its electric motor products. EMC earns profits, on average, because it has fostered the development of a skilled labor force and has a reputation for incorporating the latest technological advances in motor design. Unfortunately, EMC is also subject to the risk of copper price volatility; a dramatic runup in copper prices can quickly darken the company's profits picture.

The head of EMC's purchasing department, Carol Hausmann, suggests in a memo that EMC reduce its risk of copper costs by taking long positions in Comex copper futures contracts as the company periodically sets its needs for copper during the following quarters. If copper prices increase dramatically, EMC's cost increases will be at least partially offset by profits on its long futures positions. (Likewise, if copper prices decline, EMC's futures losses will be offset by cost savings on its copper purchases.) Hausmann argues that in this way EMC can concentrate on its core manufacturing business, and not be concerned with fluctuations in the copper market that it cannot control.

The chief financial officer of EMC, Kevin Cooper, has authority over the firm's financial policy, and is familiar with the Modigliani-Miller Irrelevance Principle. When he received Hausmann's memo concerning hedging with copper futures, Cooper suggested that they discuss the issue.

Cooper begins their meeting by saying, "In principle, any futures position we take will have no effect on EMC's share price. Assuming we are in business to maximize our shareholders' wealth, should we really get involved in the expense of hiring someone to do our futures trading? Not to mention the brokerage fees and so on."

Hausmann, at first uncertain about Cooper's arguments, asks for an explanation. In only a minute or two, Cooper finishes the standard arguments by saying, "So, you see, anything we can do with futures contracts, our shareholders can do equally well on their own." After acknowledging that her initial motives for futures hedging were overly simple, Hausmann points out a flaw in Cooper's reasoning.

"Kevin, your argument only makes sense if shareholders always know our purchasing plans. Unless we're continually informing every shareholder about the sizes of our copper commitments, how will they know how much to hedge? Anyway, each Comex contract is so large that only our largest shareholders would want to take a position."

Cooper's face reddens slightly with embarrassment as he mentally takes one of his business school professors down a notch.

"I guess you're right," he concedes. "On top of everything you say, there's the fact that the total brokerage fees our shareholders would have to pay, not to mention their time and bother, more than justify the cost of doing the hedging here at EMC."

Motives for Hedging by Firms

The Electric Motor Corporation example illustrates that the Modigliani-Miller Irrelevance Principle must be applied carefully. The scenario included three possible justifications for futures hedging by firms:

1. Firms may have more information than their shareholders concerning the risks involved. Moreover, for strategic reasons, they may not wish to publicize their market commitments for their competitors to learn.
2. Futures contracts tend to be sized for firms and large investors, not for the smaller investor.
3. The transactions costs for hedging within the firm may be smaller than the total transactions costs that would otherwise be incurred by shareholders.

The last justification may not be borne out if investors already hold diversified portfolios of securities. The copper price risk faced by EMC, for example, may already be offset by those EMC shareholders who also own stocks of other firms that produce copper and would therefore profit from an increase in copper prices.

In addition to the three possible justifications for hedging by firms already noted, the following motives are also commonly mentioned:

4. The managers of a firm may not act in shareholders' best interests if their performance in operating the "core business" of the firm is measured by the total profits of the firm. For example, EMC officers (being risk averse) might choose less than the optimal number of motors to produce during periods of high copper price volatility if their performance is being measured by the total profits of EMC. This is known as the *principal-agent* effect. The owner of the firm is the principal; the manager is the agent of the owner.
5. The Modigliani-Miller Irrelevance Principle does not account for bankruptcy costs. For example, if EMC is forced into financial reorganization because of unexpectedly high copper costs (despite the long-run profitability of its core business), then EMC shareholders will bear corresponding reorganizational costs that might not have been necessary had EMC hedged with a futures position.
6. There may be tax-related justifications for hedging. For example, if corporate losses are treated differently for tax purposes than are profits, it may be profitable, on an after-tax basis, for the firm to engage in futures trading.

This list of potential reasons for financial hedging by firms is not exhaustive. Our main purpose here is to replace the naive motive for hedging, "less risk is better," with more careful economic reasoning.

Corporate Futures Trading Programs

Just as the Electric Motor Company example illustrated that futures hedging may improve the welfare of shareholders, the opposite can also occur. A firm engaging in a futures trading program should institute carefully considered policies, procedures, and controls. One possible set of guidelines is:

1. Prepare a policy stating the goals of the firm's futures trading, whether based on risk reduction or profit motives or both, and also providing arguments suggesting how futures trading can accomplish these goals.
2. Institute a study of legal, accounting, financial, and tax implications of futures trading for the firm.
3. Prepare a futures trading strategy, however broadly or narrowly defined, describing how the timing and sizing of contract positions are to be decided and executed. The strategy might be as specific as a set of mathematical formulas and statistical procedures determining

all trades automatically, or as vague as delegation of all trading decisions to one or more employees or consultants. Clearly, the strategy should be consistent with the policy (1) and the study (2). In particular, the strategy should clearly delineate responsibility for setting, approving, and transmitting futures orders. It should be ensured that the FCM handling the firm's account will accept futures orders only from authorized employees of the firm.

4. Establish a program of oversight and periodic performance evaluation. Oversight should attempt to ensure that the trading strategy will be carried out properly, and in particular, that it will not place the firm in undue jeopardy. Performance evaluation should check the quality of trading decisions and brokerage, and measure the extent to which the trading program is accomplishing its stated policy goals. Specific numerical performance measures should be tied to each of the specific policy goals whenever possible.
5. Shareholders should be advised of the firm's hedging policy, since they may otherwise incorrectly assess the risks they face and inappropriately hedge on their own, unnecessarily adding to their risks. Double hedging is not safer, it is riskier.

EXERCISES

7.1 *(Hedging with Heteroskedasticty)* Suppose that the increments of a futures price process $f_1, f_2, \ldots$ and a spot price process $s_1, s_2, \ldots$ satisfy the OLS conditions, except for the fact that $c_t = \text{cov}(f_{t+1} - f_t, s_{t+1} - s_t)$ and $v_t = \text{var}(f_{t+1} - f_t)$ can both change with time. (For a frame of reference, take a time period to be one day.) Suppose that c_t and v_t depend only on the month of the year. For example, c_t is constant during June, and the same in June 1984 as in June 1985, in June 1986, and so on; and likewise for v_t.

(a) Assuming that you know c_t and v_t for all t, calculate the futures position on May 27, 1989, that provides a variance-minimizing hedge against a commitment to buy 400 units of the spot asset on August 21, 1989. *Hint: Remember the zero autocorrelation condition and what it implies about the variance of a sum of price increments.*

(b) Assuming that you do not know c_t and v_t, but have access to any historical spot and futures price data that you would like, outline a statistical procedure that would give you a hedge based on estimated covariances and variances.

7.2 *(Hedging Project)* This exercise calls for the preparation of a case study of a hedging problem of your own choice. The various parts of the exercise below will ask you to pose a simple, perhaps imaginary, hedging scenario, to collect appropriate data, to analyze the data statistically, and to recommend a hedge.

(a) *Description of Scenario:* In 100 words or less, describe the scenario: "Who is hedging how much of what at what date, and why?" Suggest the futures contract or contracts that you will consider, and why.

(b) *Data Collection:* Collect your own data for the hedging problem. Supply, with your assignment, at least 26 observations on each of the prices or other quantities of interest. You will probably decide to analyze the data with a computer. Sources for data include the statistical annuals of the various futures exchanges. These annuals include related spot market data.

(c) *Data Analysis:* Using statistical methods, analyze the data for a reasonable estimate of the risk-minimizing futures position(s). You may wish to analyze both price increments and (based on Appendix 7D) log-price increments, suggesting which of these approaches may lead to a better hedging estimate (if either), and why you think so. Be extremely brief! Supply as an appendix any computer output to which you have referred, such as regression equations and diagnostics.

(d) *Hedge Recommendation:* Based on your analysis, and on any other (brief) reasoning that you may wish to supply, recommend a hedge. State precisely the timing and quantities involved in your hedging strategy, as though they are instructions to a broker.

7.3 Table 7.1 gives the German and Dutch exchange rate data for months 37 through 48, extending from Appendix 7A the data used for the guilder hedging example of Section 7.2.

(a) Estimate the hedging coefficient for this latter 12-month period.

(b) Prepare a plot similar to that shown in Figure 7.2 showing the performance of the hedging position $y = -5$ million deutsche marks (chosen from the data for months 1 through 36) during months 37 through 48.

TABLE 7.1 Guilder-Deutsche Mark Additional Hedging Data

Month No.	Guilder Spot	DM[a] Futures
37	0.4895	0.5468
38	0.4843	0.5479
39	0.4907	0.5572
40	0.4924	0.5610
41	0.4873	0.5489
42	0.4857	0.5522
43	0.4787	0.5395
44	0.4891	0.5516
45	0.4838	0.5458
46	0.5112	0.5811
47	0.5438	0.6122
48	0.5626	0.6421

[a]Nearby contract, end-of-month.

7.4 Consider a hedge at time t in one of two different futures contracts. There are Q units of the asset to be hedged, which has a spot price at time $t+1$ of s_{t+1}. Suppose that $\text{corr}(f_{t+1}^{(1)}, s_{t+1})$, the correlation of the first futures price with the spot asset price, is higher than $\text{corr}(f_{t+1}^{(2)}, s_{t+1})$, the corresponding correlation for the second type of futures contract.

(a) Show that an optimal hedge in the first futures contract achieves a lower total variance than does an optimal hedge in the second futures contract.

(b) Show that an optimal hedge in both futures contracts simultaneously achieves lower total variance than an optimal hedge in either alone. (This part of the exercise requires some skill in statistics.)

7.5 Based on the data in Appendix 7A, which futures contract, that delivering British pounds, West German marks, or Swiss Francs, would you recommend as a hedge against a commitment to Dutch gulden?

7.6 Based on the data in Appendix 7A, recommend a hedge in one or more of the foreign currency futures for a commitment to buy 20 million Dutch gulden in one month.

7.7 Show the additivity rule of hedging in the following context. Consider a position of Q_1 units of asset number 1 and Q_2 units of asset number 2. Their respective spot prices, $s_{t+1}^{(1)}$ and $s_{t+1}^{(2)}$ are sources of risk to be hedged. Let y_1 be the variance-minimizing futures position in a given futures contract for the first risk $e_1 = Q_1 s_{t+1}^{(1)}$, and likewise let y_2 be the variance-minimizing futures position for the other spot risk $e_2 = Q_2 s_{t+1}^{(2)}$.

(a) Show that $y_1 + y_2$ is the risk-mimimizing futures position for the combined spot commitment $e = e_1 + e_2 = Q_1 s_{t+1}^{(1)} + Q_2 s_{t+1}^{(2)}$.

(b) Extend Part (a) to the case of futures positions in several different futures contracts. (This part of the question requires more advanced statistical skills.)

7.8 (Hedging with a Known Basis) Suppose, changing the example given in Section 7.2, that that Acme Mint's commitment to buy gold is, instead, on October 16, two months before the delivery date of the contract. Calling October 16 "date k," the basis $s_k - f_k$ is not generally zero, and the "equal and opposite" hedging strategy for a zero basis is incorrect. Suppose, however, that the arbitrage formula $f_k = s_k B_{k,T}$ applies, where $B_{k,T}$ is the amount due at date $T =$ December 14 on a riskless loan of \$1 made at date k (October 16). (Corrections and extensions of this formula are discussed in Chapter 5.) Suppose also that, at the current date $t =$ September 15, $B_{k,T}$ can already be accurately estimated at 1.0101 from current Treasury bill prices, assuming that interest rates do not change between t and k.

(a) Show that, if Acme Mint takes a long position of

$$200/B_{k,T} = 200/1.0101 = 198 \text{ contracts},$$

then its net cost for the gold (spot market cost less futures resettlement profit or loss) is $19{,}800 \times f_t$. Since f_t is known at the date t on which the hedge is established, the total cost of the gold net of the futures profits or losses therefore has no risk at all.

(b) Show that general hedging formula (1) produces the hedge $h = 198$ contracts, by showing that the hedging coefficient is $\beta = 1/B_{k,T}$. In practice, it is impossible to know $B_{k,T}$ exactly in advance, and in any case the cost-of-carry formula does not apply exactly.

7.9 Verify the tailing calculations given in equations (11) and (12) of Appendix 7B.

7.10 (Random Quantity Constraints) Suppose the quantity Q of an asset to be hedged, the spot price s_{t+1} of the asset at the commitment date, and the price f_{t+1} of a futures contract are joint log-normally distributed. That is, their logarithms, denoted X, Y, and Z respectively, are joint normally distributed, as defined in Appendix 4C. Based on Equation (25) from Appendix 7E, calculate the optimal (variance-minimizing) futures position in terms of the means and covariances of X, Y, and Z. Warning: This exercise is a bit advanced mathematically, relative to the others.

7.11 This exercise asks for a verification that the two bond portfolios described in an example in Appendix 7F have the same duration and market value. Portfolio 1 consists of \$10 million (face value) worth of 4 year zero-coupon bonds. Portfolio 2 consists of \$7.57 million of 2 year zero-coupon bonds and \$4 million of 20 year zeros. All bonds are priced to yield a continuously compounding interest rate of 8 percent.

NOTES

On the effect of estimation risk on portfolio choice, the reader should see, for example, the paper by Klein and Bawa (1976). There are many services providing settlement prices for futures contracts. Without endorsing either, two examples are: *Interactive Data Services*, 22 Courtlandt Street, New York, NY 10007-3172; and *Commodity Systems Inc.*, 200 Palmetto Park Road, Suite 200, Boca Raton, FL 33432-9947. Grieves (1986) is the source of the example of corporate bond hedging. Hedging for institutional investment managers is treated by Figlewski (1986).

Johnston (1984) is a typical introductory reference for the regression theory discussed in this chapter. The F-test is a standard test of the relative fit of two different regression equations. The Cochrane-Orcutt method of correcting for serial correlation is also discussed by Johnston (1984). More advanced techniques, including the incorporation of overlapping sampling intervals, are treated in Hansen and Hodrick (1980). For an advanced treatment of regression with autocorrelation and heteroskedasticity, see Amemiya (1985). There are conditions on the log-price increment process (X, Y) leading to the OLS conditions on the price-growth factor process $(\mathcal{F}, \mathcal{S})$ discussed in Appendix 7D; these are the *iid*, or *identically and independently distributed* conditions, which are defined, for example, in Amemiya (1985). A solution for an optimal dynamic hedging strategy under log-normal price assumptions may be found in Duffie and Jackson (1986).

Modigliani and Miller (1958) are of course responsible for the original form of the Modigliani-Miller Irrelevance Principle. The result is extended, under general conditions, to show the irrelevance of corporate hedging policy for the value of the firm in a general multi-period model by Duffie and Shafer (1986), and in an improved form showing irrelevance of shareholder utility by DeMarzo (1987). A recent survey of the topic and literature has been prepared by Miller and McCormick (1988). DeMarzo (1987) gives additional references.

Textbook treatments of international financial markets include Grabbe (1986) and Solnik (1988).

Appendix 7F, on hedging interest rate risk, is based on a wide variety of literature, including Breeden and Giarla (1987), Kidder, Peabody and Company (1987), Klotz (1985), Kopprasch (1985), Jacobs (1982), Toevs and Jacobs (1984), and Yawitz (1986). Additional information on hedging interest rate risk using the immunization approach can be found in the book by Figlewski (1986). The concept in Figure 7.8 of illustrating duration as the center of gravity of the present value of cash flows is drawn from Kopprasch (1985). Smith, Smithson, and Wakeman (1987) contains additional material on swaps.

Appendix 7A: Example Hedging Data

TABLE 7.2 Spot and Futures Price Data
(U.S. $ per unit of foreign currency)

	Spot[a]	Nearby Futures[b]		
Month	Dutch Guilder	British £ Sterling	German DM	Swiss Franc
1	0.3155	1.4095	0.3670	0.4672
2	0.3397	1.5030	0.3975	0.4836
3	0.3423	1.4615	0.4013	0.4887
4	0.3288	1.4190	0.3805	0.4655
5	0.3248	1.4015	0.3786	0.4621
6	0.3171	1.3750	0.3698	0.4440
7	0.3055	1.3040	0.3512	0.4159
8	0.3068	1.3140	0.3524	0.4237
9	0.2933	1.2380	0.3292	0.4002
10	0.2927	1.2250	0.3334	0.4058
11	0.2864	1.1930	0.3216	0.3927
12	0.2817	1.1550	0.3228	0.3905
13	0.2793	1.1050	0.3247	0.3847
14	0.2654	1.0575	0.3084	0.3606
15	0.2872	1.2265	0.3355	0.3970
16	0.2835	1.2110	0.3266	0.3911
17	0.2872	1.2660	0.3326	0.3965
18	0.2901	1.2840	0.3337	0.3995
19	0.3192	1.3995	0.3633	0.4443
20	0.3193	1.3765	0.3588	0.4368
21	0.3315	1.3850	0.3753	0.4580
22	0.3388	1.4365	0.3839	0.4680
23	0.3539	1.4885	0.3986	0.4819
24	0.3608	1.4370	0.3954	0.4722
25	0.3706	1.3650	0.4315	0.5110
26	0.3986	1.3980	0.4602	0.5480
27	0.3827	1.4385	0.4352	0.5265
28	0.4081	1.5135	0.4689	0.5570
29	0.3848	1.4540	0.4361	0.5240
30	0.4040	1.5215	0.4608	0.5671
31	0.4238	1.4750	0.4813	0.6028
32	0.4319	1.4755	0.4948	0.6144
33	0.4377	1.4295	0.4954	0.6112
34	0.4281	1.3980	0.4854	0.5863
35	0.4475	1.4315	0.5072	0.6099
36	0.4562	1.4325	0.4965	0.5883

[a]Source: *The Wall Street Journal.*

[b]Source: *International Monetary Market Yearbook,* Chicago Mercantile Exchange.

Appendix 7B: Tailing the Hedge

One cannot always ignore the effect on the optimal hedge of daily resettlement and interest on margin accounts. To *tail the hedge* is to correct the size of the hedge for these effects, as described in this appendix. The discussion here will presume for simplicity that:

1. Margin is deposited in the form of interest-bearing assets, with interest credited daily.
2. Interest rates do not change during the hedging period.
3. Initial and maintenance margin are set at the same level.
4. Excess margin is not withdrawn (or is transferred to a reserve fund bearing interest at the same rate), while margin calls are met by borrowing at the same interest rate.

We still presume that the objective is to calculate the futures position that minimizes the variance of the total final position, and adopt the same scenario leading to the hedge shown in equation (8).

Let B denote the value after one day of \$1 invested risklessly at the interest rate paid on margin deposits. Then the value of the initially deposited \$1 after two days is B^2, after three days is B^3, and so on. For example, if the annual interest rate on a continually compounding basis is r, then

$$B = \exp\left(\frac{r}{365.25}\right), \tag{10}$$

which is a number that is typically close to 1.0, but not so close that large money fund managers are willing to wait overnight to invest cash. If $r = 10$ percent, for instance, then $B = 1.000274$, which represents almost 3 basis points. (A *basis point*, in this context, is 0.01 percent.)

Suppose the futures position y is adopted for K days at day t. Under our assumptions, the futures gains (or losses) yF_{t+1} over the first day of the hedge are invested in the margin account for $K-1$ days, yielding $B^{K-1}yF_{t+1}$ when the final position is calculated at day $t+K$. Similarly, the second day's gains yield $B^{K-2}yF_{t+2}$ at day $t+K$, and so on. The variance of the final position determined by Q units of the spot asset whose price changes from s_t to s_{t+K} is thus

$$V(y) = \mathrm{var}[Q(s_{K+t} - s_t) + B^{K-1}yF_{t+1} + B^{K-2}yF_{t+2} + \cdots + yF_{t+K}].$$

Under the OLS conditions on spot and futures price increments, this reduces to

$$V(y) = \mathrm{var}[KQS_{t+1} + (B^{K-1} + B^{K-2} + \cdots + B + 1)yF_{t+1}], \tag{11}$$

which is instructive to verify as an exercise. Once again doing the calculations leading to $V'(y) = 0$, we have the minimum variance futures position

$$y = -a(B, K)Q\beta,$$

where

$$a(B, K) = \frac{K}{1 + B + B^2 + \cdots + B^{K-1}}. \tag{12}$$

The coefficient $a(B, K)$ is called the *tailing factor*. The *tailed optimal hedge* is therefore $a(B, K)h$, where $h = -Q\beta$ is the "un-tailed" hedge shown earlier. The least squares estimator for the tailed hedge is $a(B, K)H$, where $H = -Q\widehat{\beta}$.

Sample values of the tailing factor $a(B, K)$ are given in Table 7.4, based on different scenarios for interest rates and hedging periods. Although the effect of tailing only becomes significant with long hedging periods or high interest rates, the effect is always to reduce the magnitude of the optimal hedging position. Of course, one could achieve even lower variance, under our assumptions, by periodically "re-tailing" the hedge, rather than maintaining a fixed position for the entire hedging period. If re-tailing daily (which may be excessive given transactions costs), the tailed optimal hedge with K days remaining is $B^{-K}h$, as can easily be shown in an exercise. Monthly re-tailing, for example, is also easily calculated.

TABLE 7.3 Tailing Factors

Interest	Hedging Period (Days)			
Rate (%)	10	100	500	1000
10	0.999	0.987	0.933	0.869
20	0.998	0.973	0.870	0.751

With continuous payments and continuous-compounding at the interest rate r, the denominator of (12) can be recognized as the future value of an annuity factor

$$A_{t,T} = \frac{\exp[r(T-t)] - 1}{r},$$

as defined in Section 5.2, where the length $T - t$ of the hedging period is measured in years. With that convention, the tailing factor is

$$a(r, T - t) = \frac{r(T - t)}{\exp[r(T - t)] - 1}.$$

Appendix 7C: Liquidity and Time to Delivery

Liquidity, measured perhaps by daily volume of trade, affects the cost of making a trade. With a low volume of trade, a market order to sell, for example, might have little chance of arriving at the market at about the same time as a corresponding market order to buy, and might have to be executed at a price low enough to induce a trader to hold the position on the trader's own account until an offsetting order arrives. The lower the volume of trade, the greater are the risk and opportunity cost for the accommodating trader, and the lower will be the price required to induce the trader to take the long position. Similarly, a market order to buy would typically be executed at a premium during inactive trading periods. The *bid-ask spread*, a term usually reserved for specialist markets such as the New York Stock Exchange, is the difference between the price at which a market order to buy would be executed and the price at which a market order to sell would be executed. Unlike the bid-ask spread in specialist markets, the bid-ask spread on a futures market is not quoted, and is impossible to measure directly. It nevertheless represents part of the total costs of transactions services. In Section 5.3, we called this the "market impact" portion of transactions costs.

Now, let us suppose that there is some natural tendency for trade to be more active in the contracts sooner to deliver. (This is true in many markets.) With somewhat more liquidity in the nearby contracts, the higher market impact costs facing hedge-motivated trades in the later-to-deliver contracts will cause some of these trades to be replaced with trades in contracts sooner to deliver. Of course, this would further skew liquidity in favor of nearby contracts, and the effect could feed back on itself to the point of having a high fraction of futures trades in a particular commodity concentrated in the nearby contract. The actual distribution of liquidity (measured by annual volume of trade) segregated by the nearness of the delivery date is shown in Figure 7.6 for S&P 500 Stock Index Futures Contracts (Chicago Mercantile Exchange), U.S. Treasury Bond Futures Contracts (Chicago Board of Trade), and wheat futures contracts (Chicago Board of Trade). The two financial futures contracts show a dramatic increase in liquidity with the nearness of the contract delivery date, while the effect is less pronounced with the wheat contract. Presumably, those hedging distant commitments with the financial futures contracts find that the nearby contract offers a reasonable hedging alternative to the contract whose delivery date is closest to the commitment date; the liquidity of the nearby contract is far superior. In the wheat market, however, perhaps hedgers facing spot commitments far into the future find that the nearby contract is not a good hedge. Perhaps this is because the difficulty of short selling wheat (in the way that financial securities are short sold) prevents a close arbitrage price relationship from holding between the nearby and distant futures prices. As shown in an exercise, an arbitrage price relationship (coupled with low interest rate uncertainty) allows most of the basis risk to be eliminated with an appropriately sized hedge.

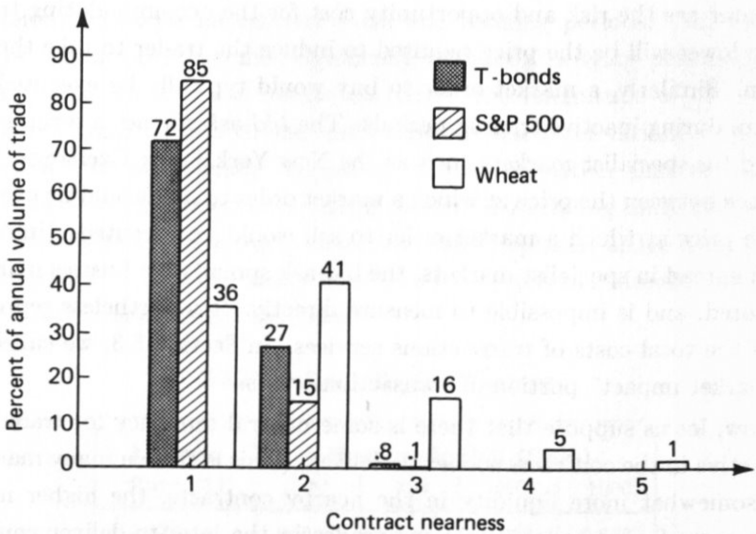


Figure 7.6 The Relationship Between Volume of Trade and Time to Delivery

There may also be speculative reasons (related to differences in information among traders) to believe that liquidity may be concentrated in a similar fashion. [See Admati and Pfleiderer (1988) for a model with a related property.]

These comments on liquidity are casually presented, but are supported at least by the observed distribution of trading volume across the various delivery dates of most types of futures. A conclusion is that there may be grounds for hedging with a futures contract delivering much sooner than the spot commitment date, even if contracts delivering closer to the commitment date are available. For example, one might hedge always in the nearby contract and *roll over* the hedge as the delivery date on the nearby contract approaches. For reasons also related to liquidity, a large hedge would be rolled over gradually, not as a single order.

Appendix 7D: Hedging Percentage Price Growth

If percentage price changes, rather than price increments, satisfy the OLS conditions, then linear regression based on price increment data will not generate the best estimators for the hedging coefficient β. As the current price becomes larger, so will the variance of the subsequent price increment increase, as illustrated in Chapter 6. We now review methods for estimating β based on OLS price growth. In reality, of course, neither price increments nor price growth satisfy the OLS conditions exactly, and judgment or more sophisticated approaches are often called for.

Price Growth Betas

Given stochastic futures and spot price processes, $f = (f_1, f_2, \ldots)$ and $s = (s_1, s_2, \ldots)$ respectively, let $\mathcal{S}_t = s_t/s_{t-1}$ and $\mathcal{F}_t = f_t/f_{t-1}$ denote the *price growth factors* for spot and futures, respectively. Since $s_{t+1} = s_t\mathcal{S}_{t+1}$ and $f_{t+1} = f_t\mathcal{F}_{t+1}$, we know at time t that

$$\operatorname{cov}(f_{t+1} - f_t, s_{t+1} - s_t) = s_t f_t \operatorname{cov}(\mathcal{F}_{t+1}, \mathcal{S}_{t+1})$$

and that $\operatorname{var}(f_{t+1} - f_t) = f_t^2 \operatorname{var}(\mathcal{F}_t)$. It follows that the hedging coefficient is

$$\beta = \frac{\operatorname{cov}(f_{t+1} - f_t, s_{t+1} - s_t)}{\operatorname{var}(f_{t+1} - f_t)} = \frac{s_t}{f_t}\,\beta^G, \tag{13}$$

where

$$\beta^G = \frac{\operatorname{cov}(\mathcal{F}_{t+1}, \mathcal{S}_{t+1})}{\operatorname{var}(\mathcal{F}_{t+1})}$$

is the regression coefficient for spot price growth $\mathcal{S}_{t+1}$ regressed on futures price growth $\mathcal{F}_{t+1}$. Given data, one can obtain the least squares estimator

$\widehat{\beta}^G$ for β^G. For a commitment of Q units of the spot asset at time $t+1$, the corresponding least squares estimator of the optimal hedge at time t is

$$H_t = -Q\frac{s_t}{f_t}\widehat{\beta}^G. \tag{14}$$

This calculation only applies to spot commitments one period into the future, so one must either obtain an estimate of β^G with data intervals equal to the time period of the hedge or adopt a dynamic hedging strategy. Both of these alternatives will be discussed shortly.

Hedging with Normally Distributed Price Growth Rates

A common, although sometimes unjustified, assumption is that the price growths $\mathcal{F}_{t+1}$ and $\mathcal{S}_{t+1}$ are log-normally distributed, which is to say that the log-price increments $X_{t+1} = \log(\mathcal{F}_{t+1})$ and $Y_{t+1} = \log(\mathcal{S}_{t+1})$ are normally distributed, a property explained in Appendix 4C. (Empirical tests of the log-normality assumption are discussed in Section 6.6.) The log-price increment X_{t+1} is the continuously compounding rate of growth in prices represented by the change from f_t to f_{t+1} during one time period (and likewise for Y_{t+1}). A calculation made in Appendix 7E shows that, with log-normality,

$$\beta^G = Jb^G, \tag{15}$$

where

$$J = \exp\left(E(Y_{t+1}) - E(X_{t+1}) + \frac{[\mathrm{var}(Y_{t+1}) - \mathrm{var}(X_{t+1})]}{2}\right) \tag{16}$$

and

$$b^G = \frac{\exp[\mathrm{cov}(X_{t+1}, Y_{t+1})] - 1}{\exp[\mathrm{var}(X_{t+1})] - 1}. \tag{17}$$

The variance-minimizing hedge for a commitment to Q units of the spot asset one period hence is therefore

$$h_t = -Q\frac{s_t}{f_t}Jb^G. \tag{18}$$

Once again, we point out that the hedging coefficient β^G in the case of OLS price growth depends on the length of the time interval, as opposed to the case of OLS price increments.

Example: Suppose that $E(X_{t+1}) = E(Y_{t+1}) = 0.3$, that $\text{sdev}(X_{t+1}) = \text{sdev}(Y_{t+1}) = 0.20$, and that $\text{corr}(X_{t+1}, Y_{t+1}) = 0.5$. These could be typical figures for annual periods. In this case, $J = 1$, and

$$\beta^G = b^G = \frac{\exp(0.02) - 1}{\exp(0.04) - 1} = 0.495.$$

If one incorrectly substituted for β^G the log-price increment regression coefficient

$$\beta^L = \frac{\text{cov}(X_{t+1}, Y_{t+1})}{\text{var}(X_{t+1})} = 0.500, \tag{19}$$

the damage would not be too severe in this example. Taking another case, if $\text{corr}(X_{t+1}, Y_{t+1}) = 0.9$ and all other parameters are the same, then $\beta^G = b^G = 0.899$, while $\beta^L = 0.9$.

As suggested by the example, for typical estimates of variances of log-price increments, b^G and β^L indeed do not differ by a great deal. Furthermore, when hedging with a futures contract delivering the same asset being hedged, it is often the case that J need not depart from 1 by much. In such a situation, the hedging position

$$h_t^L = -Q\frac{s_t}{f_t}\beta^L, \tag{20}$$

substituting the log-price increment regression coefficient β^L for the price growth regression coefficient β^G, will not result in a hedge differing drastically from the optimal hedge h_t given by equation (18), at least for small time intervals. The hedging position h_t^L has in fact often been suggested in practice, although we can see that it is not strictly speaking optimal.

Dynamic Hedging in the Log-Normal Case

It can be shown that as the length of the time interval under consideration becomes smaller and smaller, the optimal hedge h_t given by (18) converges to the hedging position h_t^L given by equation (20). Furthermore, if log-price increments satisfy the OLS conditions, then the regression coefficient β^L does not depend on the length of the time interval, analogously with the case of OLS price increments discussed in Section 7.3. These facts suggest that, if the hedger has the ability to adjust the hedging position continually through time, then the optimal position at any time t is merely the position h_t^L given by equation (20). Although the calculations are too extensive to review here, Duffie and Jackson (1987) show that this is indeed the case if the futures price process is a martingale (meaning no expected futures profits or losses). It is not true, in general, if the futures price is not a martingale. Generally speaking, the opportunity to adjust the hedge dynamically through time in response to changing prices and new information results in superior risk reduction.

Alternative Estimators for Price Growth Betas

We have discussed the risk-minimizing hedge in the case of OLS price growth. It is not immediately clear, however, how one should estimate the coefficient β^G determining the optimal hedge. There are three obvious possibilities:

1. Obtain the OLS estimator $\widehat{\beta}^G$, based on the observed growth factors: $(\mathcal{F}_1, \mathcal{S}_1), (\mathcal{F}_2, \mathcal{S}_2), \ldots$. This leads to the hedge

$$H_t = -Q \frac{s_t}{f_t} \widehat{\beta}^G. \tag{21}$$

2. Obtain the OLS estimator $\widehat{\beta}^L$ for β^L, the log-price increment regression coefficient. Under our presumption that β^G and β^L do not differ drastically for practical purposes, one could consider the hedge

$$H_t^L = -Q \frac{s_t}{f_t} \widehat{\beta}^L. \tag{22}$$

3. Let $C(X, Y)$ denote the least squares estimator for $\mathrm{cov}(X_{t+1}, Y_{t+1})$, let $V(X)$ denote the least squares estimator for $\mathrm{var}(X_{t+1})$, let $V(Y)$ denote the least squares estimator for $\mathrm{var}(Y_{t+1})$, and finally let $\overline{X}$ and $\overline{Y}$ denote the respective least squares estimators for $E(X_{t+1})$ and $E(Y_{t+1})$. If we substitute everywhere in equations (16) and (17) these estimators for the true (unknown) parameters, we obtain

$$\overline{J} = \exp\left(\overline{X} - \overline{Y} + \frac{[V(X) - V(Y)]}{2}\right)$$

and

$$\overline{b}^G = \frac{\exp[C(X, Y)] - 1}{\exp[V(X)] - 1}.$$

The estimator

$$\overline{\beta}^G = \overline{J}\,\overline{b}^G \tag{23}$$

can be shown to be a biased estimator for β^G. On the other hand, under the OLS conditions, the bias is not generally large, and shrinks to zero as the number of observations gets large. Moreover, this estimator $\overline{\beta}^G$ has certain attractive efficiency properties when the price growth factors are OLS and log-normal, although we will not make the effort to describe these efficiency properties here. In short, one can also make a case for the estimated hedge

$$\overline{H}_t = -Q\frac{s_t}{f_t}\overline{\beta}^G. \tag{24}$$

Now, which hedging estimate should one take, that given by (21), (22), or (24)? This is not easily resolved, although we now review evidence that it doesn't matter a great deal in some cases.

Monte Carlo Simulation Check of Hedging Estimates

We can check the performance of the three hedges suggested above by Monte Carlo simulation, referring to Tables 7.5, 7.6, and 7.7. Each table shows four trials for each of several choices for T, the number of simulated time periods of data. The simulations are based on pseudo-normal pseudo-randomly generated observations of X_t and Y_t, all with the means, standard deviations, and correlations indicated in the tables. The particular random number generator used for these tables is that available with the GAUSS programming language; however, many software packages support their own random number generators.

The key comparisons to make in these three tables are the differences between the trial outcomes of the estimators and the true underlying beta shown at the top of the table. [We recall that $\widehat{\beta}^L$ is the OLS estimator for β^L, that $\widehat{\beta}^G$ is the OLS estimator for β^G, and that $\overline{\beta}^G$ is a special biased estimator for β^G given by equation (23).] The estimators improve, on average, as the number of observations increases. In principle, each of these estimators will converge under fairly general assumptions to the true underlying betas, but their rates of convergence may vary. This Monte Carlo simulation approach can be applied under other statistical models for price increments.

Example: We return to the simple scenario presented in Section 7.1 of a commitment to sell 25 million deutsche marks, hedging with the nearby deutsche mark futures contract. Assuming OLS price growth, we might be inclined, on the basis of our previous discussion, to regress the spot log-price increment $Y_{t+1} = \log(s_{t+1}) - \log(s_t)$ on the futures log-price increment

TABLE 7.4 Monte Carlo Hedging Estimates (Case I)

Case I: sdev(X_t) = sdev(Y_t) = 0.20; corr(X_t, Y_t) = 0.5; $\overline{X}_t = \overline{Y}_t = 0.3$; $\beta^G = 0.505$ $\beta^L = 0.500$				
Observations	**Trial**	$\overline{\beta}^G$	$\widehat{\beta}^L$	$\widehat{\beta}^G$
$T = 10$	1	1.025	1.095	1.058
	2	0.292	0.308	0.403
	3	0.438	0.463	0.434
	4	1.115	1.101	1.314
$T = 100$	1	0.473	0.475	0.467
	2	0.590	0.594	0.594
	3	0.451	0.456	0.485
	4	0.544	0.540	0.536
$T = 1000$	1	0.486	0.480	0.484
	2	0.490	0.484	0.479
	3	0.515	0.509	0.503
	4	0.525	0.518	0.523

TABLE 7.5 Monte Carlo Hedging Estimates (Case II)

Case II: sdev(X_t) = sdev(Y_t) = 0.20; corr(X_t, Y_t) = 0.9; $\overline{X}_t = \overline{Y}_t = 0.3$; $\beta^G = 0.902$ $\beta^L = 0.900$				
Observations	**Trial**	$\overline{\beta}^G$	$\widehat{\beta}^L$	$\widehat{\beta}^G$
$T = 10$	1	0.970	0.956	0.964
	2	1.142	1.094	1.077
	3	0.812	0.819	0.836
	4	1.052	1.013	0.958
$T = 100$	1	0.873	0.870	0.872
	2	0.926	0.927	0.934
	3	0.963	0.954	0.938
	4	0.869	0.857	0.852
$T = 1000$	1	0.897	0.870	0.872
	2	0.876	0.876	0.884
	3	0.924	0.923	0.918
	4	0.902	0.900	0.891

$X_{t+1} = \log(f_{t+1}) - \log(f_t)$. For the same data used earlier to obtain the estimated price increment regression equation shown in Section 7.4, we obtain

TABLE 7.6 Monte Carlo Hedging Estimates (Case III)

Case III: sdev$(X_t) = 0.20$; sdev$(Y_t) = 0.40$; corr$(X_t, Y_t) = 0.5$, $\overline{X}_t = \overline{Y}_t = 0.3$; $\beta^G = 1.062$; $\beta^L = 1.000$				
Observations	**Trial**	$\overline{\beta}^G$	$\widehat{\beta}^L$	$\widehat{\beta}^G$
$T = 10$	1	0.564	0.642	0.876
	2	0.819	0.814	0.581
	3	0.929	0.934	0.870
	4	1.300	1.526	2.063
$T = 100$	1	0.967	0.933	0.954
	2	1.005	0.987	0.994
	3	1.071	1.000	1.162
	4	1.530	1.414	1.420
$T = 1000$	1	1.070	1.021	1.082
	2	1.014	0.969	1.033
	3	1.026	0.960	1.027
	4	0.992	0.944	1.050

the following estimated log-price increment regression:

$$Y_{t+1} = \underset{(0.052^*)}{\widehat{\alpha}} + \underset{(0.0034)}{0.940}\ X_{t+1} + \epsilon_{t+1}. \qquad (R^2 = 0.935)$$

The "fit," measured say by R^2, seems to be about the same as that of the price-increment regression estimated in Section 7.4. The estimate $\widehat{\beta}^L = 0.940$ for β^L is statistically significant (in the usual sense), judging from its relatively low standard error estimate of 0.052.

Suppose the current exchange rate is $s_t = \$0.60$ per deutsche mark and the current futures price for nearby delivery is $f_t = \$0.62$ per guilder. Then, referring to equation (22), we have the hedging estimate

$$H_t^L = -25{,}000{,}000 \times \frac{0.60}{0.62} \times \frac{0.940}{125{,}000} = -182 \ \text{(contracts)}.$$

Since β^L is not equal to β^G, this hedging estimate is biased. On the other hand, the net effect in this particular situation is not likely to be severe, especially considering the effect of estimation error. It is indeed common practice to base hedging calculations on the approach taken in this example.

Once again, we point out that, even ignoring estimation error, this hedging calculation is only accurate for short hedging periods. If the ratio of spot to futures price changes significantly during the period of hedging, it is generally advisable to readjust the hedge with changes in the spot-futures price ratio. This dynamic hedging strategy is optimal if the futures price is a martingale, as shown in Duffie and Jackson (1986).

The Multi-Contract Case

Suppose one is choosing the variance-minimizing hedging position in n different futures contracts for a commitment to Q units of a spot asset with price growth factor $\mathcal{S}_{t+1}$ from date t to date $t+1$. Given the respective price growths factors $\mathcal{F}_{t+1}^{(1)}, \mathcal{F}_{t+1}^{(2)}, \ldots, \mathcal{F}_{t+1}^{(n)}$ of the n futures contracts, we can estimate the multiple regression equation

$$\mathcal{S}_{t+1} = \alpha + \beta_1^G \mathcal{F}_{t+1}^{(1)} + \beta_2^G \mathcal{F}_{t+1}^{(2)} + \cdots + \beta_n^G \mathcal{F}_{t+1}^{(n)} + \epsilon_{t+1}.$$

The corresponding estimates $\widehat{\beta}_1^G, \ldots, \widehat{\beta}_n^G$ of the multiple regression coefficients then imply the futures position

$$H_t^i = -Q \frac{s_t}{f_t^{(i)}} \widehat{\beta}_i^G$$

in the i-th contract (whose current futures price is $f_t^{(i)}$) for each i. It is common, however, to substitute for $\widehat{\beta}_i^G$ the estimated regression coefficient $\widehat{\beta}_i^L$ from the corresponding log-price increment regression

$$Y_{t+1} = \alpha + \beta_1^L X_{t+1}^{(1)} + \cdots + \beta_n^L X_{t+1}^{(n)} + \epsilon_{t+1},$$

where $Y_{t+1} = \log(\mathcal{S}_{t+1})$ and $X_{t+1}^{(i)} = \log\left(\mathcal{F}_{t+1}^{(i)}\right)$.

An application of this approach to Ginnie Mae mortgage hedging is shown in Appendix 7F.

Appendix 7E: Log-Normal Hedging Calculations

This appendix provides the calculation of the hedging coefficient β under the assumption that the futures and spot log-price increments, $\log(f_{t+1}) - \log(f_t)$ and $\log(s_{t+1}) - \log(s_t)$, are normally distributed. The calculation is needed for equations (15)–(18).

First, suppose that W and Z are jointly normally distributed random variables. It can be shown as an exercise [using the formula for $E(e^W)$ given in Appendix 4C] that

$$\begin{aligned}&\text{cov}[\exp(W), \exp(Z)]\\&= \exp\left(\overline{W} + \overline{Z} + \frac{[\text{var}(W) + \text{var}(Z)]}{2}\right)(\exp[\text{cov}(W, Z)] - 1)\,. \quad (25)\end{aligned}$$

Applying this to the case $W = Z$, we also have the formula

$$\text{var}[\exp(W)] = \text{cov}[\exp(W), \exp(W)] = \exp[2\overline{W} + \text{var}(W)]\,(\exp[\text{var}(W)] - 1)\,.$$

If $X_{t+1} = \log(f_{t+1}/f_t)$ and $Y_{t+1} = \log(s_{t+1}/s_t)$ are jointly normally distributed, we then have

$$\text{cov}(f_{t+1} - f_t, s_{t+1} - s_t) = f_t s_t UV,$$

where

$$U = \exp\left(\overline{X}_{t+1} + \overline{Y}_{t+1} + \frac{[\text{var}(X_{t+1}) + \text{var}(Y_{t+1})]}{2}\right)$$

and

$$V = \exp[\text{cov}(X_{t+1}, Y_{t+1})] - 1.$$

Likewise,

$$\text{var}(f_{t+1} - f_t) = f_t^2 \exp\left[2\overline{X}_{t+1} + \text{var}(X_{t+1})\right]\,(\exp[\text{var}(X_{t+1})] - 1)\,.$$

This leaves

$$\beta = \frac{\text{cov}(f_{t+1} - f_t, s_{t+1} - s_t)}{\text{var}(f_{t+1} - f_t)} = \frac{s_t}{f_t}\beta^G = \frac{s_t}{f_t} J b^G, \quad (26)$$

where J and b^G are given by equations (16) and (17) respectively.

Appendix 7F: Hedging Interest Rate Risk

A potential change in interest rates is a risk faced by almost any investor or firm in the economy. A homeowner with a variable rate mortgage worries about a rise in interest rates; a pension fund manager about to reinvest the fund in T-bills worries about a decline in interest rates; and the manager of a savings and loan association, which typically pays short term rates to its depositors and earns long term rates on mortgages, is concerned about a decline in long term rates relative to short term rates.

Although the general principles of hedging discussed in Chapter 7 apply in particular to interest rate commitments, the problem of interest rate risk is sufficiently widespread to warrant special treatment here.

We will begin with a discussion of several important measures of interest rate risk: price elasticity, duration, and convexity. These concepts are useful guideposts in setting and adjusting a hedge against interest rate changes.

Bond Elasticity

Although there are exceptions to the rule, the sensitivity of bond prices to interest rate changes is greater for bonds of longer maturity. Consider, for example, the effect of interest rate changes on 1-year and 10-year zero-coupon bonds. Let $P_t(r)$ denote the price per dollar of face value of a zero-coupon bond with t years to maturity yielding a continuously compounding interest rate of r. We know that $P_t(r) = e^{-rt}$. (See Section 5.1 for details.) Assuming the 1-year and 10-year bonds both yield a continuously compounding rate of 11 percent, their prices are

$$\begin{aligned} P_{10}(0.11) &= e^{-0.11\times 10} = 0.3329 \\ P_1(0.11) &= e^{-0.11\times 1} = 0.8958. \end{aligned}$$

An increase in the continuously compounding yield to 12 percent reduces the respective bond prices to

$$\begin{aligned} P_{10}(0.12) &= e^{-0.12\times 10} = 0.3012 \\ P_1(0.11) &= e^{-0.12\times 1} = 0.8869. \end{aligned}$$

The price of the 10-year bond price has suffered more dramatically than that of the 1-year bond. Figure 7.7 shows this general effect in a plot of $P_1(r)$ and $P_{10}(r)$ at various levels of r. The first derivative $P'_{10}(r)$ gives the rate of change in the price of the 10-year bond per percentage point increase in r. In other words, $P'_{10}(r)$ is the slope of the function $P_{10}(r)$. Dividing $P'_{10}(r)$ by $P_{10}(r)$, we get the percentage rate of price decline per unit of interest rate change, which is called the *bond elasticity*

$$\mathcal{E}_{10}(r) = \frac{P'_{10}(r)}{P_{10}(r)}.$$

Since the derivative of $P_t(r) = e^{-rt}$ with respect to r is merely $P_t'(r) = -te^{-rt}$, we can easily calculate the elasticity

$$\mathcal{E}_t(r) = \frac{-te^{-rt}}{e^{-rt}} = -t.$$

In other words, for zero-coupon bonds, (minus) elasticity and maturity are the same thing! The price of the 10-year bond changes at a rate of $\mathcal{E}_{10}(r) = -10$ percent per unit increase in its continuously compounding interest rate r, at any level of r. The 1-year bond price elasticity is always $\mathcal{E}_1(r) = -1$ percent.

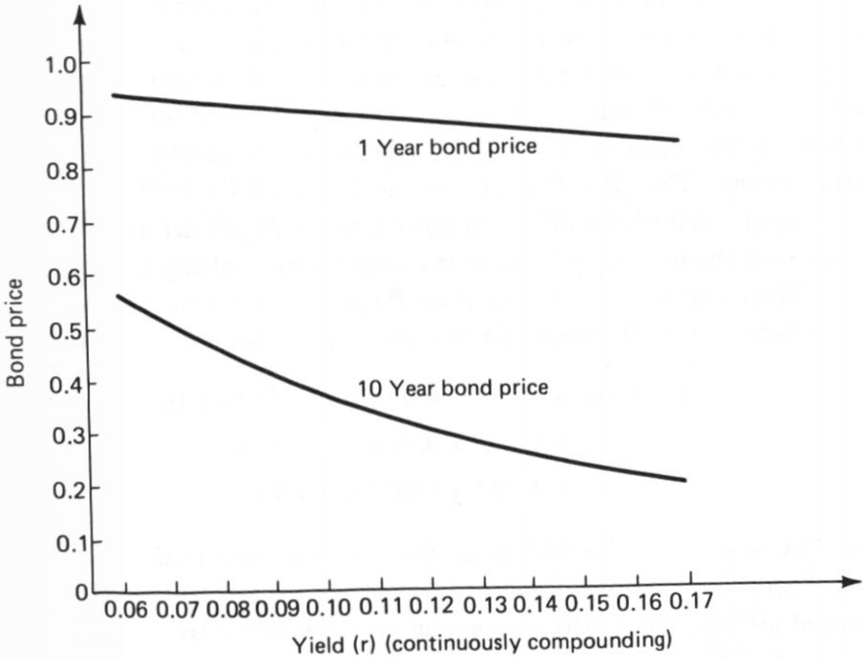


Figure 7.7 Interest Rate Sensitivity of Short versus Long Term Bonds

Bond elasticity, being a marginal rate of change, is not exactly equal to the total percentage price change for a sudden jump in interest rates of 1 percent. For example, the change from 11 percent to 12 percent in continuously compounding yield generates a total percentage change in the 10-year bond price of

$$\frac{P_{10}(0.12) - P_{10}(0.11)}{P_{10}(0.11)} = \frac{0.3012 - 0.3329}{0.3329} = -0.0952 = -9.52 \text{ percent.}$$

This is not exactly equal to the price elasticity $\mathcal{E}_{10}(r) = -10$ percent since the slope $P'_{10}(r)$ changes slightly between $r = 11$ percent and $r = 12$ percent. Nevertheless, for small changes in interest rates, the bond price elasticity and total percentage price decline divided by the interest rate change are approximately the same.

Elasticity and Duration of Coupon-Bearing Bonds

We have just seen that a zero-coupon bond's maturity and its price sensitivity to interest rate changes (elasticity) are the same thing. This useful fact is extended to coupon-bearing bonds as follows.

First of all, the maturity (time to repayment of principal) of a coupon-bearing bond is only a vague measure of its interest rate sensitivity since it ignores the effect of interim coupon payments, which tend to reduce the bond's interest rate sensitivity. Consider a 2-year "zero" and a 2-year 15-percent coupon bond, both priced to yield 10 percent interest, continuously compounding. The price $P_{0,2}(0.10)$ of the 2-year zero is $e^{-0.10\times 2} = 0.8187$ (dollars per dollar of face value). In general, we let $P_{c,t}(r)$ denote the price of a c-percent annual coupon bond of maturity t years yielding a continuously compounding interest rate r. The price $P_{15,2}(0.10)$ of the 15-percent coupon 2-year bond at $r = 10$ percent, for example, is given by

$$\begin{aligned} P_{15,2}(0.10) &= e^{-0.10\times 1} \times 0.15 + e^{-0.10\times 2} \times 1.15 \\ &= 0.9048 \times 0.15 + 0.8187 \times 1.15 \\ &= 0.1357 + 0.9415 = 1.0772. \end{aligned}$$

The first term, $e^{-0.10\times 1} \times 0.15$, is the discounted presented value of the first-year coupon of 15 cents per dollar of face value; the second term is the discounted present value of the coupon plus principal in 2 years.

An increase in r to 11 percent gives the new prices $P_{0,2}(0.11) = 0.8025$ and $P_{15,2}(0.11) = 1.0573$. The zero coupon price is reduced 1.98 percent by the rate increase; the 15-percent coupon bond price is reduced 1.85 percent. The coupon bond is less seriously affected because the present value of the first-year coupon is proportionately less affected by rate changes than the present value of the principal repayment in 2 years. We can account for this effect by an adjusted measure of a bond's maturity known as its *duration*.

The duration of a bond is the weighted average maturity of all cash flows stemming from the bond. The weight applied to the maturity of each bond payment is the present discounted value of that payment as a fraction of the total price of the bond. The duration of the 2-year 15-percent coupon bond at a continuously compounding interest rate of r is therefore given by the formula

$$D_{15,2}(r) = \frac{e^{-r\times 1} \times 0.15}{P_{15,2}(r)} \times 1 \text{ year } + \frac{e^{-r\times 2} \times 1.15}{P_{15,2}(r)} \times 2 \text{ years.}$$

At $r = 10$ percent,

$$D_{15,2}(0.10) = \frac{0.1357}{1.0772} \times 1 \text{ year } + \frac{0.9415}{1.0772} \times 2 \text{ years}$$

$$= 0.1260 \text{ years } + 1.7481 \text{ years } = 1.8741 \text{ years.}$$

This is less than the duration $D_{0,2}(r)$ of the zero-coupon 2-year bond, which is obviously equal to 2 years. Using an idea presented in Kopprasch (1985), Figure 7.8 illustrates the duration of a coupon-bearing bond as the center of gravity of the maturities of its discounted cash flows.

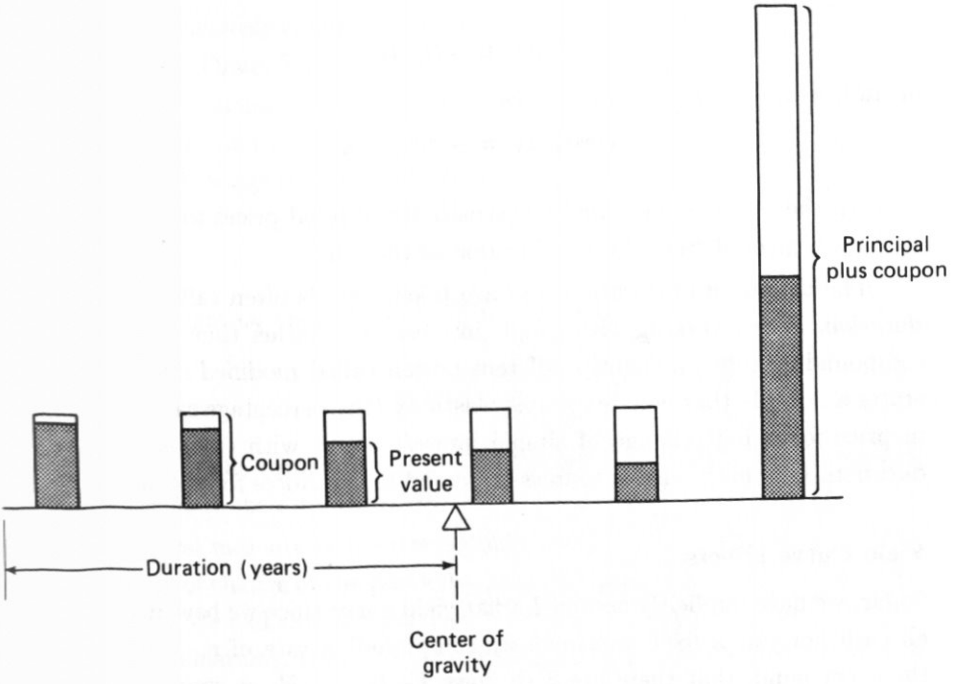


Figure 7.8 Duration as Center of Gravity of Payment Dates

The duration $D_{c,t}(r)$ of a t-year c-percent coupon bond yielding r (continuously compounding) is given by the weighted sum

$$D_{c,t}(r) = \frac{e^{-r\times 1} \times c \times 1 + e^{-r\times 2} \times c \times 2 + \cdots + e^{-r\times t}(100 + c) \times t}{100 \times P_{c,t}(r)},$$

where $P_{c,t}(r)$ is the bond's price. As with zero coupon bonds, the price elasticity $\mathcal{E}_{c,t}(r)$ of a t-year c-percent coupon bond yielding a continuously compounding interest rate of r is given by the formula

$$\mathcal{E}_{c,t}(r) = \frac{P'_{c,t}(r)}{P_{c,t}(r)},$$

the percentage rate of change in the bond price per unit change in the interest rate r. Using the rule for derivatives,

$$\frac{d}{dr}\left(e^{-rt}c\right) = -te^{-rt}c,$$

it is easy to calculate the derivative $P'_{c,t}(r)$ and check (from the preceding three equations) that

$$\mathcal{E}_{c,t}(r) = -D_{c,t}(r),$$

or, in words,

$$\text{elasticity} = -\,\text{duration}.$$

As with the zero coupon bond, the sensitivity of bond prices to interest rate changes is given directly by the duration of the bond.

The definition of duration we have taken here is often called *Macaulay duration*. When working with simple interest rates rather than continuously compounding rates, a slightly different notion called *modified duration* generates a formula that equates simple elasticity (the percentage rate of change in price for a unit change of simple interest rates) with (minus) modified duration. One may consult sources indicated in the Notes for the details.

Yield Curve Effects

So far, we have implicitly assumed a flat yield curve since we have discounted all cash flows at a fixed continuously compounding rate of r. Suppose, on the other hand, that there are cash flows $F_1, F_2, \ldots, F_n$ at respective times $t_1, t_2, \ldots, t_n$ (which need not be equally spaced), and that the yield curve

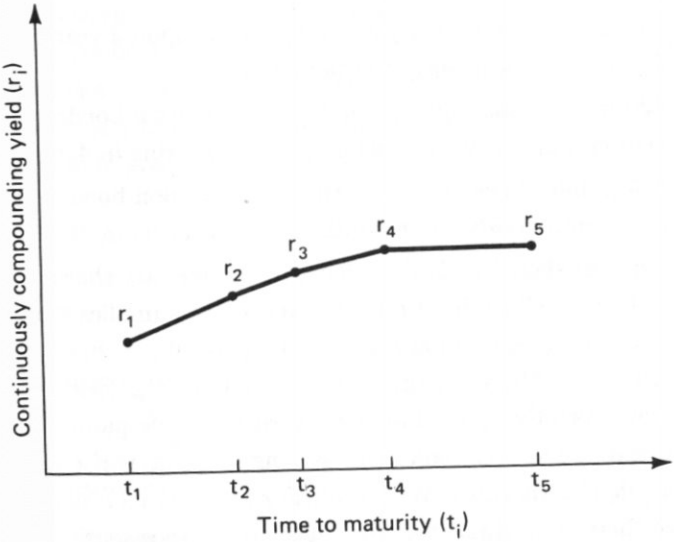


Figure 7.9 Yield Curve Data

applies a continuously compounding interest rate of r_i to payments of maturity t_i as shown in Figure 7.9.

Consider a bond (or portfolio of bonds) whose price is P with a dollar cash flow of F_i at time t_i for each i. The *duration* of this bond (or bond portfolio) is once again its present value weighted sum of maturities

$$D = w_1 t_1 + w_2 t_2 + \cdots + w_n t_n, \tag{27}$$

where the weight w_i applied to maturity t_i is given by

$$w_i = \frac{e^{-r_i t_i} F_i}{P},$$

the fraction of the total price P of the portfolio represented by the present value $e^{-r_i t_i} F_i$ of the i–th cash flow.

A natural measure of price sensitivity for the bond portfolio is the percentage rate of change of the portfolio price per unit upward shift of the entire yield curve. We call this the *portfolio elasticity*. Although we shall not produce the calculations, they are easy and show that the portfolio elasticity $\mathcal{E}$ is once again (minus) the duration D.

Example: Consider a portfolio made up of

- \$10 million (face value) worth of zero-coupon 2-year bonds yielding 6 percent continuously compounding.
- \$20 million (face value) worth of zero-coupon bonds yielding 7 percent continuously compounding and maturing in 4.25 years.
- \$25 million (face value) worth of zero-coupon bonds yielding 9 percent continuously compounding and maturing in 21.3 years.

The "spread sheet" style duration calculations are shown in Table 7.7. As shown, the portfolio's duration is 5.8 years. This implies that the price of the bond portfolio goes down at a rate of 5.8 percent per unit upward shift in all interest rates on the yield curve. Unfortunately, the yield curve does not generally shift directly upward or downward to a new parallel curve; it can also "tilt" and "bend" unpredictably as long term rates rise relatively more or less than short term rates. With arbitrary (non-parallel) movements in the yield curve there is no single accurate measure of price sensitivity to interest rates. For the greatest possible accuracy, one must measure sensitivity relative to interest rates at each of the various maturities.

TABLE 7.7 Example Duration Calulations

Maturity t_i (years)	Yield[a] r_i	Discount $e^{-r_i t_i}$	Pay F_i (\$m.)	Value $e^{-r_i t_i} F_i$ (\$m.)	Weight[b] w_i	Weighted t_i (years)
2.00	6%	0.887	10	8.87	0.324	0.648
4.25	7%	0.743	20	14.86	0.542	2.304
21.30	9%	0.147	25	3.67	0.134	2.854
TOTAL	—	—	—	27.40	1.000	5.806

[a]The yield is shown on a continuously compounding basis.
[b]The weight w_i is $e^{-r_t t_i} F_i/27.40$.

Convexity

An examination of the duration (or equivalently, as we have seen, elasticity) of a bond does not tell the full story of its percentage price response to yield changes. Figure 7.10 shows how the prices of two different bond portfolios with the same price and the same duration (or equivalently, the same elasticity) depend on yield changes. The market value of Portfolio 2 declines more slowly with increases in yield, and rises more quickly with decreases in yield, than that of Portfolio 1. Of the two curves defining price response, that of Portfolio 2 has more *convexity*, even though both have the same *slope* (duration). *Convexity* is a measure of the degree of inward curvature of the price curve. We will soon give a formula for convexity.

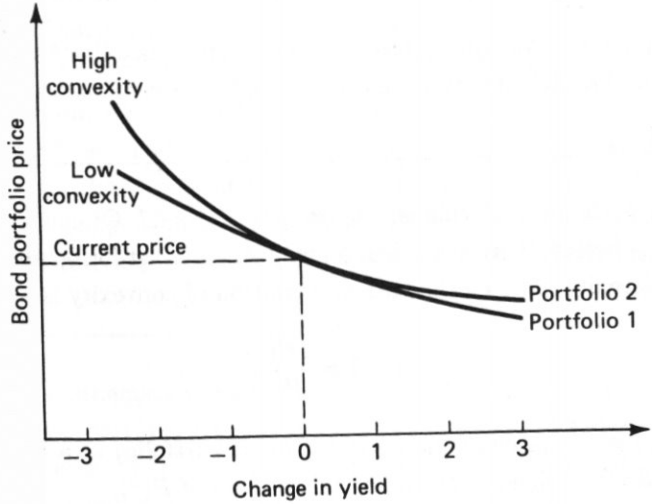


Figure 7.10 Low and High Convexity Bond Portfolios

Example: Suppose Portfolio 1 consists of \$10*m*. (\$10 million, face value) of 4-year zero-coupon bonds, while Portfolio 2 consists of \$7.57*m*. (face value) of 2-year zeros and \$4*m*. of 20-year zeros. Suppose, for simplicity, that the yield curve is flat at a continuously compounding rate of 8 percent.

The elasticity of Portfolio 1 is −4 (since it's duration is clearly 4 years). The price of Portfolio 1 is $e^{-0.08\times 4} \times \$10m. = \$7.26m.$ Portfolio 2 was constructed to have the same duration and price, respectively, as those of Portfolio 1. (This should be checked by the reader as an exercise.)

An increase in continuously compounding yields to 10 percent causes the price of Portfolio 1 (the four-year bond) to drop from \$7.26*m*. (that is,

$e^{-0.08\times 4} \times \$10m.$) to $\$6.70m.$, a decline of 7.7 percent in market value. The value of Portfolio 2, however, declines to

$$e^{-0.10\times 2} \times \$7.57m. + e^{-0.10\times 20} \times \$4m. = \$6.74m.,$$

which represents a smaller decline of 7.16 percent in value.

Likewise, a drop in yields from 8 percent to 6 percent causes the value of Portfolio 1 to increase to $\$7.87m.$, a rise in value of 8.4 percent. The move down to a 6-percent yield improves the value of Portfolio 2 relatively more; its new market value is $\$7.91m.$, which is 9.0 percent higher than its value at an 8-percent yield.

In summary, Portfolio 2 has more convexity than Portfolio 1: its value rises more sharply and drops more gradually in response to interest rate changes.

Clearly, the general rule is: "Convexity is good." Of course, since more convexity is better, there is a price to pay in the market for bonds of relatively greater convexity. The mathematical definition of convexity is

$$C(r) = \frac{P''(r)}{P(r)},$$

where $P(r)$ is the price of the bond (or bond portfolio) as a function of the yield r and $P''(r)$ denotes the second derivative of $P(r)$.

For a zero-coupon bond of maturity t, the price per dollar of face value is $P(r) = e^{-rt}$, which has the second derivative $P''(r) = t^2 e^{-rt}$. The convexity is therefore

$$C(r) = \frac{P''(r)}{P(r)} = t^2.$$

The convexity of a bond portfolio with cash flow of F_i at time t_i (with an associated continuously compounding interest rate of r_i), for times $t_1, \ldots, t_n$ (as shown in Figure 7.9), is the weighted sum of squared maturities:

$$C = w_1 t_1^2 + w_2 t_2^2 + \cdots + w_n t_n^2,$$

where the weight $w_i = e^{-r_i t_i} F_i / P$ is the same as that used in the formula (27) for the duration of the portfolio.

Example: Consider once again the bond portfolio described in Table 7.7. The weights w_1, w_2, and w_3 used to calculate duration can be re-used for an easy calculation of convexity. We have

$$\begin{aligned} C &= w_1 \times 2^2 + w_2 \times 4.25^2 + w_3 \times 21.3^2 \\ &= 0.324 \times 4 + 0.542 \times 18.06 + 0.134 \times 453.69 \\ &= 71.88 \text{ years}^2. \end{aligned}$$

The units of convexity, years squared, are not immediately of interest; it is best to think of convexity merely as the curvature of the portfolio's price-yield curve, with higher curvature preferred for a given duration.

Unless two bond portfolios have the same duration, a comparison of their convexities need not lead one portfolio to be favored over the other. For instance, the calculated portfolio convexity of 71.88 years^2 is inferior to that of a 9-year zero-coupon bond (since $9^2 > 71.88$), but the zero-coupon bond has a higher duration (9 years > 5.81 years), implying greater interest rate sensitivity.

Due to a mathematical rule known as *Jensen's Inequality*, it is always true that a bond portfolio (involving more than a single payment) with the same duration as that of a zero-coupon bond always has higher convexity than that of the zero-coupon bond.

FORTRAN code for the duration and convexity of coupon bonds as well as general bond portfolios is provided at the end of this appendix.

Duration and Convexity of Callable Bonds

The calculations of duration and convexity can be extended to cover bonds with a *call provision*, a clause in the bond contract allowing the seller of the bond to cancel the obligation at certain times by immediately repaying the principal. Call provisions can be complicated by terms restricting the timing or nature of the repayment provision.

Example: To take an unrealistically simple case, suppose that a 2-year 15-percent-coupon bond has a call provision allowing repayment (at the option of the borrower) 1 year before maturity. Clearly, the bond will be called (by an astute borrower) if and only if the 1-year simple rate of interest prevailing at the call date is less than 15 percent, since the borrower has the option of paying the prevailing 1-year rate for the last year of the loan rather than the 15-percent coupon rate. The price (per dollar of face value) of the callable bond at the call date is therefore 1.00 if the 1-year interest rate is below 15 percent [which is a continuously compounding rate of $r = \log(1.15) = 0.14 = 14$ percent], and otherwise the bond will sell at its discounted value $P(r) = e^{-r\times 1}\times 1.15$ at any continuously compounding rate r above 14 percent. The curve defining bond price at the call date, shown in Figure 7.11, therefore indicates *negative convexity*, at least in the vicinity of the coupon rate.

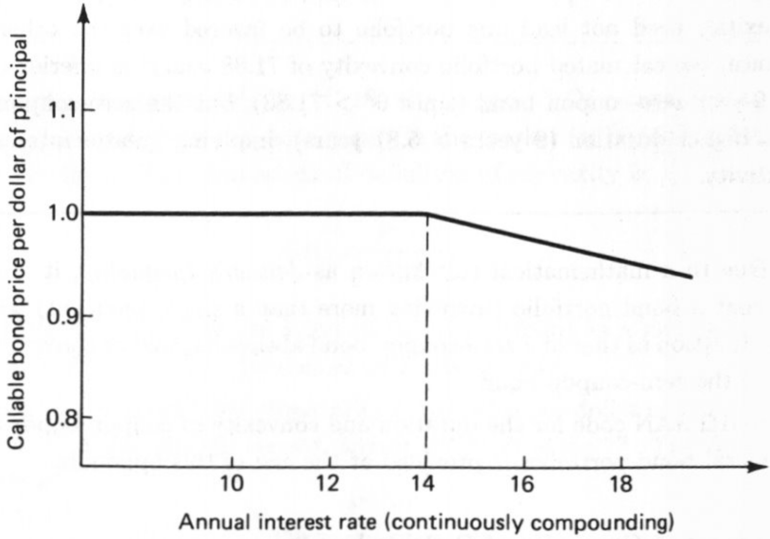


Figure 7.11 Negative Convexity of a Callable Bond

Pass-through mortgages such as Government National Mortgage Association (GNMA or "Ginnie Mae") certificates form a major class of callable bonds which generally exhibit negative convexity. (Appendix 9B discusses the role that the call provision played in the demise in popularity of the GNMA futures contract traded on the Chicago Board of Trade.) Transactions costs and individual factors often prevent GNMA mortgages from being called when it might otherwise seem optimal for homeowners to replace high interest mortgages with lower interest loans. Based on the estimated fraction of mortgages of a given coupon rate that are called at a given interest rate (such figures have been made available, for example, by Salomon Brothers), one can ultimately construct for a given portfolio of pass-through mortgages an estimate of the duration and convexity of the mortgage portfolio. Later in this appendix we estimate a futures hedge for Ginnie Mae mortgage commitments. Because of the call provision, the hedge is improved by including short-term interest rate futures.

Immunization and Duration Matching

Immunization means the adjustment of a portfolio of fixed income liabilities and assets so that the net interest rate risk is reduced or, in the ideal case, eliminated. *Duration matching* is an immunization strategy that calls for adjusting one's fixed income asset portfolio so that it has the same duration and the same market value as one's fixed income liabilities. In this way, a rise in interest rates causes the same marginal gain from reduced liabilities as the marginal loss from reduced asset value, and the portfolio is thereby effectively hedged against small interest rate changes.

Example: Suppose the yield curve is flat at 8 percent continuously compounding, and consider assets made up of \$10*m*. face value of 4-year zero-coupon bonds, and liabilities comprised of \$7.57*m*. face value of 2-year zero coupon notes and \$4*m*. of 20-year zero coupon notes. This situation was reviewed in an earlier example. The asset and liability portfolios have the same market value (\$7.26*m*.) and the same duration (4 years). The rate of change of the total market value of assets less liabilities with shifts in the yield curve up or down from 8 percent is therefore zero.

Immunization is only effective in this scenario for small shifts in the yield curve. As the yield curve moves down from 8 percent to 6 percent, the duration of the liability portfolio increases from 4 years to 4.75 years, while the duration of the asset portfolio remains fixed at 4 years. In other words, the value of the liability portfolio becomes more sensitive to rate changes as rates go down. The total effect of a decline in rates from 8 percent to 6 percent is an increase in the value of the asset portfolio from \$7.26*m*. to \$7.87*m*., while the liability portfolio moves up in value from \$7.26*m*. to \$7.91*m*. The \$0.61*m*. improvement on the asset side is more than offset, unfortunately, by the \$0.65*m*. loss on the liability side, for a net loss of approximately \$40,000. The loss is due to a mismatch in convexity. If the asset and liability portfolios had been constructed with the same convexity, the net effect on the total portfolio value due to changes in rates of this kind would be approximately zero. Of course, duration matching alone was enough to eliminate a large fraction of the interest risk. (Had the assets been held in cash rather than 4-year bonds, the total loss would have been the increase in liability value, \$650,000.)

This example shows that duration matching is not necessarily effective unless care is taken to match convexity as well. Alternatively, one can match duration dynamically as rates change. That is, as the yield curve moves down from 8 percent to 6 percent, the portfolio of liabilities might be continuously rebalanced by increasing short term liabilities relative to long term liabilities so as to always maintain a liability duration of 4 years. Alternatively, one could adjust the asset duration, gradually increasing it so as to always match the liability duration.

Immunization with Futures Positions

In the previous example, the liability portfolio of 2-year and 20-year bonds could have been hedged with futures positions. Since there is both short and long term interest rate risk, it is best to consider both short and long term interest rate futures. If it could be guaranteed, however, that the yield curve would move up and down in a parallel fashion, then one could use a single position in, say, 90-day Treasury bill futures to offset the risks. (This is unrealistic, and we will shortly offer a better recommendation.) The recommended position in T-bill futures could be calculated from the general formula:

$$\text{futures resettlement profit} = -\,\text{change in portfolio value}.$$

Let

- Δf denote the change in T-bill futures price per unit upward shift in the yield curve.
- y denote the futures position.
- ΔV denote the change in the value of the portfolio per unit upward shift in the yield curve.

We can then re-write the above formula more specifically as

$$y\,\Delta f = -\Delta V.$$

Solving, the recommended futures position is $y = -\Delta V/\Delta f$. This is the futures position that, in principle, generates resettlement profits (or losses) that exactly offset changes in the value of a portfolio being hedged.

Example: Consider once again our simple example of a liability portfolio consisting of \$7.57m. in 2-year zeros and \$4m. in 20-year zeros. Again assuming the yield curve only makes parallel movements, a downward shift in the yield curve from 8 percent to 7 percent increases the value of the liabilities from \$7.26m. to \$7.57m., or $\Delta V = -\$310,000$. The spot price of \$1m. (face value) worth of 90-day (0.25-year) Treasury bills (the deliverable on one futures contract) would change from $e^{-0.25\times0.08} \times \$1m. = \$980,200$ to $e^{-0.25\times0.09} \times \$1m. = \$977,750$. Assuming for simplicity that the futures contract is about to deliver, the futures and spot T-bill prices coincide, so $\Delta f = \$977,750 - \$980,200 = -\$24,500$. The recommended futures position is therefore

$$y = -\frac{\Delta V}{\Delta f} = -\frac{-\$310,000}{-\$24,500} = -12.66 \text{ contracts},$$

or about 13 T-bill contracts short.

One would need a great deal of confidence in the assumption of parallel movements of the yield curve in order to trust this hedge. Given the fact that the portfolio being hedged is made up of 2-year and 20-year notes, unpredictable relative movements of short and long term rates would normally make the calculated T-bill futures hedge a relatively poor means of reducing interest rate risk.

A more prudent futures hedging strategy calls for positions in both short and long term interest rate futures contracts in order to independently hedge the effects of interest rate movements on the short and long ends of the yield curve. For example, the U.S. Treasury bond futures contract could be effective against changes in the market value of the 20-year notes, while the U.S. T-bill contract, the Eurodollar contract, or the 5-year Treasury note contract could be used to hedge against changes in the value of the 2-year notes making up the short term portion of the liability portfolio in this example.

As to how to select the composition of the hedge, in order to account for basis risk and for correlation between short and long term interest rate changes, it seems conservative and appropriate to adopt the straightforward minimum variance procedures outlined in the body of Chapter 7. For instance, one can create a data file containing the market value of the liability portfolio (or a suitable proxy) for each of a number of historical time periods. With corresponding futures price data for short and long term interest rate futures prices, one can estimate the minimum variance hedging coefficients by estimating the appropriate multiple regression coefficients, as explained in Section 7.4 and Appendix 7D.

The following example illustrates the importance of both long and short term interest rate futures in hedging the value of pass-through mortgage loans. The example uses (with permission) data and calculations provided by John Wyche, a former student, as part of his response to Exercise 7.2.

Example: The Palo Alto Savings and Loan, a (fictional) lender, has contracted to sell 30 days from now \$30 million (face value) of 15-year callable 9.5-percent mortgages to the Government National Mortgage Association (GNMA, or Ginnie Mae). Since the value of these mortgages represents a significant fraction of Palo Alto Savings and Loan's market value, management has decided to hedge the price to be received for the mortgages with an interest rate futures position. Palo Alto S&L has decided to hedge the effects of long term interest rate risk with U.S. Treasury bond futures (traded on the Chicago Board of Trade). Since the mortages are callable, their value may also bear some relationship to short term rates. For this reason, a position in 3-month Eurodollar futures (traded on the Chicago Mercantile Exchange) will also be considered.

Following the prescription outlined in Appendix 7D, data for 97 weeks (May 15, 1986 to March 24, 1988) on the log-price increments (Y_t) of Ginnie Mae mortgage-backed 9.5-percent coupon securities were regressed on the corresponding time series for U.S. Treasury bond futures ($X_t^{(1)}$) and Eurodollar futures ($X_t^{(2)}$), using log-price increments for the nearby contract in each case. The estimated regression equation is

$$Y_t = 0.0001 + 0.204\, X_t^{(1)} + 1.51\, X_t^{(2)} + \epsilon_t. \qquad (R^2 = 0.711)$$

$$(0.034^*) \qquad (0.245^*) \qquad (0.005)$$

The regression coefficients easily pass (at standard confidence levels, under standard asssumptions) tests of statistical significance. The spot price of 9.5-percent coupon GNMA mortgage coefficients is currently 99.5625 percent of face value. The current T-bond futures price for nearby delivery is 90.50. The current 3-month Eurodollar nearby futures index is 92.67. As with the calculations of T-bill futures prices shown in Section 6.2, the resttlement payments on a Eurodollar position are actually one quarter of the change in the Eurodollar futures index. Following the analysis in Appendix 7D, the estimated risk-minimizing hedge is therefore made of the following futures positions:

$$\text{T-bond position} = 0.2042 \times \frac{99.5625}{90.50} \times \$10 \text{ million} = -\$2.246 \text{ million},$$

or a short position of about 22 contracts of $100,000 (face value) each, and

$$\text{Eurodollar position} = -1.51 \times \frac{99.5625}{92.67} \times \frac{1}{4} \times \$10 \text{ million} = -\$4.06 \text{ million},$$

or a short position of about 4 contracts.

The R^2 of 0.711 indicates an estimated reduction in risk (variance of the hedged value of the mortgages) of about 71 percent can be achieved by using this combined futures position. This is superior to the estimated risk reduction that can be achieved solely with T-bond futures, or solely with Eurodollar futures (as can be shown by estimating the corresponding simple linear regressions).

Based on additional regressions not reported here, Palo Alto S&L determined that Eurodollar and T-bond futures together provided a better hedge than a combined position in T-bill and T-bond futures. The Eurodollar contract, moreover, has recently been far more liquid than the T-bill contract, and therefore seems superior to the T-bill futures contract for the short term portion of the hedge. Further tests also showed that the U.S. Treasury note futures contract (delivering 6–10 year U.S. T-notes and traded on the CBOT) does not appear to give a significantly better fit when added to the above regression equation as an additional explanatory variable (again using log-price increment data).

Swaps

A recent innovation in capital markets, *swaps* comprise a class of asset exchanges between two parties that typically involve swapping one set of cash flows for another. The cash flows are usually coupon payments on a bond, or more generally, interest payments due on loans for which there may not be a secondary market. With a *currency swap*, the cash flows received by one party are in a different currency than the cash flows paid by that party.

With an *interest rate swap*, the cash flows paid by one party are fixed in advance, for example as the coupon payments on a fixed rate loan, while the cash flows received by that party are variable, for example based on a *floating interest rate* such as the London Interbank Offered Rate (LIBOR). There are combinations of currency and interest rate swaps, and many variations such as optional payment periods or staged drawdowns in the principal underlying the loan payments.

In some cases, a swap is negotiated on the basis of additional initial payments to one party or the intermediary. For example, an *off-market-rate swap* involves some initial payment from the fixed-rate payer to the floating-rate payer if the fixed rate is below currently set fixed rates for comparable loans, or a payment from the floating rate payer to the fixed rate payer under the opposite condition. Interest rate swaps, denominated in U.S. dollars (and therefore occasionally referred to as *dollar swaps*), appeared later than currency swaps, but have since come to be the more prevalent of these two basic forms of swaps. A variation is the *basis rate swap*, an exchange of cash flows based on different floating rates, such as a swap of the commercial paper rate for LIBOR. The bulk of swaps are intermediated by the large U.S. commercial and (to a lesser extent) investment banks. Prominent intermediaries include Citicorp, Chemical Bank, Bankers Trust, Chase, and Morgan Stanley. It has been increasingly common for a bank to take one side of a swap as its own position.

One normally thinks of a forward contract as an agreement to exchange assets at a particular date. From an abstract point of view, a swap is merely a forward agreement to exchange assets at a number of dates in the future. One could easily imagine a new type of futures contract, one calling for a series of cash settled payments from the short to the long in return for a series of fixed payments from the long to the short. The usual notion of a futures price would in this case be the amount paid by the long to the short at delivery, that is, at the date the series of payments begins, as in the manner of an off-the-market interest rate swap. Such a futures market does not yet exist.

The recent growth of swap transactions seems to be due, in large part, to the fact that swaps allow firms to design various cash flow hedges that cannot be easily constructed by using previously available security markets. In particular, the futures and forward markets do not typically deliver more than 1 or at most 2 years ahead, whereas interest rate or foreign currency payments and receivables may be staged over much longer periods. Interest rate swaps, for example, can be arranged for up to 15-year terms. The growth of the swap market, moreover, has promoted yet further growth by allowing reduced transactions fees accompanied by ever greater liquidity and convenience.

Example: (An Interest Rate Swap) This sort of example can be found in advertising paid for by swap intermediaries, who have an obvious incentive to portray swaps as hedges as well as profit sources. Two companies, say A and B, must each raise $100 million in debt financing for five years. For reasons such as relative credit ratings, tax considerations, access to different markets, or other institutional considerations, company A has a cost advantage in fixed rate borrowing over floating rate borrowing, relative to company B. Specifically, Company A can float 5-year commercial paper at fixed 8.80-percent coupon payments, and borrow at one quarter of a percent over the 6-month LIBOR. (The interest is stated on a semi-annual basis; interest payments on the commercial paper of 4.4 percent of the principal would be due each 6 months.) Company B, on the other hand, pays a fixed rate of 10.00 percent or a floating rate of 0.75 percent over LIBOR. Now suppose Company A indeed issues the commercial paper, and pays the 6-month LIBOR rate payments on the principal to Company B, via a bank. Company B actually borrows at LIBOR plus 0.75 percent, and makes payments at a fixed rate of 8.90 percent to Company A (via the same intermediary bank). The series of cash flows is illustrated in Figure 7.12 The net effect is that Company A pays at LIBOR less 0.10 percent, while Company B pays a fixed rate of 9.65 percent.

One way of looking at this example is that the two parties have engaged in an arbitrage: both are strictly better off, have a "saving" of 35 basis points from the swap. Some of these basis points go to the bank for its services. Of course, the fact that Company B must pay more than company B for fixed rate funding indicates that it may have a higher default risk, which must be borne by the bank or perhaps by Company A, depending on the contractual arrangements. Such a swap may or may not be motivated by the suggested cost advantage to both parties. A simpler justification for swap transactions is hedging. Company B, for example, may wish to hedge the uncertainty inherent in changes in LIBOR over the loan period; Company A may be relatively more willing to bear that risk. From this point of view, it may have been better for Company B to swap away only a fraction of its original floating rate commitment.

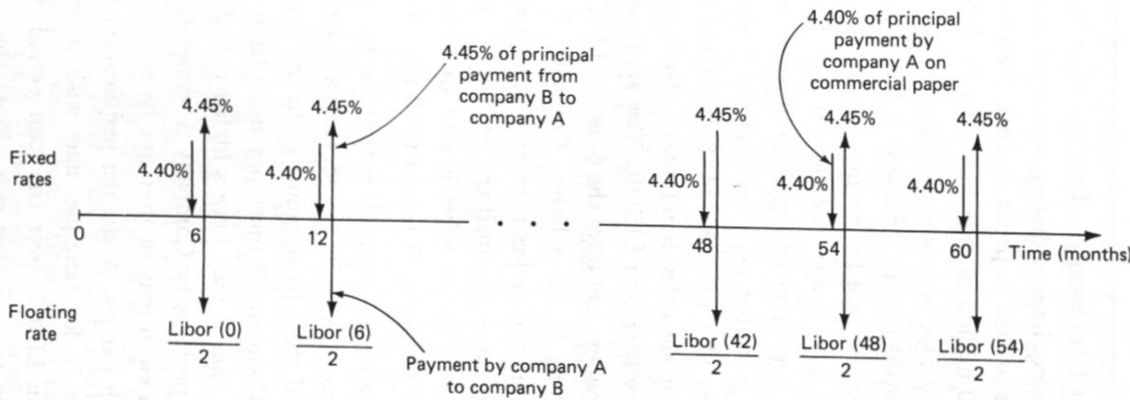


Figure 7.12 Interest Rate Swap Cash Flow

Example: (Currency Swap) A currency swap may be designed to reduce foreign exchange risk. For example, one could hedge the market value of foreign income streams by swapping domestic interest expenses for foreign interest expenses. Another reason for currency swaps may be a relative cost (interest rate) advantage of firms borrowing in their home countries. A cost advantage could be due to legal barriers separating lenders from potential borrowers in foreign countries, differential tax considerations, or perhaps governmental restrictions on international capital flows. Figure 7.13 portrays a currency swap between a U.S. firm that wishes to create Swiss Franc borrowing and a Swiss firm that wishes to create U.S. dollar borrowing. The U.S. firm can borrow at 10.50 percent in the U.S. dollar market and 4.2 percent in the Swiss Franc market, while the Swiss firm can borrow at 10.80 percent in the U.S. dollar market and 4.0 percent in the Swiss Franc market. An intermediary bank has arranged for them to swap their home currency borrowing costs, at a saving of 15 basis points to each, the remainder going to the bank.

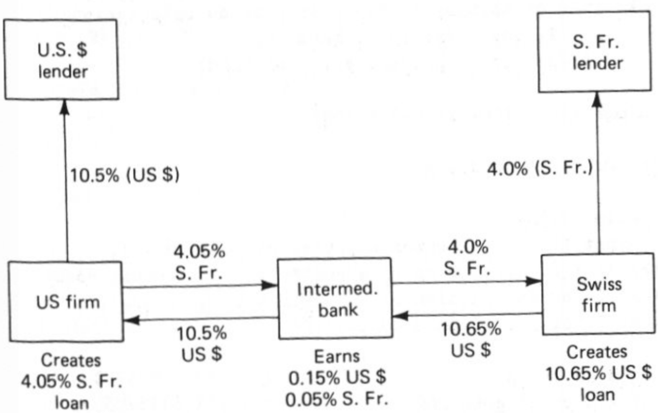


Figure 7.13 A Currency Swap

FORTRAN Code for the Duration and Convexity of Bond Portfolios

The calculations explained earlier in this appendix for the duration and convexity of bond portfolios are easily left to the computer. The following FORTRAN code, prepared by Stephen Fan under the direction of the author, has been debugged and used.

```
      Program MAIN
c
c  *****************************************************************
c  * This program calculates DURATIION and CONVEXITY for:          *
c  *   1 - a portfolio of zero coupon bonds (ZDRATN, ZCNVEX)       *
c  *     2 - a coupon bearing bond (CDRATN, CCNVEX)                *
c  * Coding by Stephen Fan at the direction of the author          *
c  *****************************************************************
c
c Variable Definitions
c ** For zero coupon bond portfolio
c
c   t(i): maturity of i-th bond from now
c   r(i): continuous compounding interest rate of i-th bond
c   f(i): amount of payment of i-th bond
c      n: total number of bonds
c
c ** For coupon bearing bond
c
c      c: coupon rate
c     ff: first coupon payment from now
c     tt: regular interval of coupon payment
c     xm: time to maturity from now
c      y: yield to maturity
c      p: present value of bonds (per dollar face value)
c      k: flag of whether to use y or p to do calculations
c         k = 1, use y (yield to maturity)
c           o/w, use p (present value of bond)
c
      dimension t(100),r(100),f(100)
c
c Choose calculation desired
c
  100 write (*,500)
  500 format(1x,'Please enter calculation desired:',/,
     *3x,'1 for calculation of a portfolio of Z-Coupon Bonds',/,
     *3x,'2 for calculation of a coupon bearing bonds',/,
     *3x,'3 for end of calculation',/)
c
      read (*,*) ik
      if (ik.eq.2) goto 200
      if (ik.eq.3) goto 900
c
c Zero Coupon Bond Calculation
c Input for zero coupon bond portfolio
c
      write(*,1000)
      read (*,*) n
      do 1 i=1,n
      write(*,1050) i
      read (*,*) t(i),r(i),f(i)
c
c convert interest rate to fraction
c
```

```
      r(i) = r(i)/100.
    1 continue
c
c call function routine
c
      ZDURA = ZDRATN(t,r,f,n)
      ZCONV = ZCNVEX(t,r,f,n)
c
c Write result for the portfolio of zero coupon bonds
c
      write (*,2000) ZDURA,ZCONV
      goto 100
c
c Coupon Bearing Bond Calculation
c
  200 continue
c
c Input for coupon bearing bond
c
      write (*,1100)
      read(*,*) c,ff,tt,xm,k,x
c
c convert coupon rate to fraction
c
      c = c/100.
c
      if(k.eq.2) goto 205
c
c convert yield to maturity to fraction
c
      y = x/100.
      p = 0.0
      goto 220
c
  205 y=0.0
      p = x
c
c call calculation subroutines
c
  220 CDURA = CDRATN(c,ff,tt,xm,y,p,k)
      CCONV = CCNVEX(c,ff,tt,xm,y,p,k)
c
c Write result for the coupon bearing bond
c
      write (*,2050) CDURA,CCONV
      goto 100
c
  900 stop
c
 1000 format(1x,'Please enter total no. of zero coupon bonds ',
     *          'in the portfolio:',/)
 1050 format(1x,'Please enter following parameters for Z-Bond # ',
     *       i3,':',/,
     *       3x,'t: time to maturity from now',/,
```

```
     *        3x,'r: continuous compound rate (%)',/,
     *        3x,'f: final payment of bond',/)
 2000 format(1x,'The Duration of Z-Bond Portfolio is: ',f10.3,/,
     *        1x,'The Convexity of Z-Bond Portfolio is: ',f10.3,/)
 1100 format(1x,'Please enter following parameters for the coupon ',
     *           'bearing bond:',/,
     *        3x,'c: coupon rate (%)',/,
     *        3x,'f: time to first coupon payment from now',/,
     *        3x,'t: regular coupon paying interval',/,
     *        3x,'m: time to maturity from now',/,
     *        3x,'k: whether inputting p or y next 1 for y;',
     *                                         ' 2 for p',/,
     *        3x,'y or p:',/,
     *        3x,'   where    y: yield to maturity (%)',/,
     *        3x,'            p: present value of bond ',
     *                              '(per dollar face value)',/)
 2050 format(1x,'The Duration of Coupon Bond is: ',f10.3,/,
     *        1x,'The Convexity of Coupon Bond is: ',f10.3,/)
c
      end
c
c
c
      FUNCTION ZDRATN(t,r,f,n)
c
c ******************************************************************
c *   This function calculates the DURATION for a portfolio of     *
c * zero coupon bonds with different interest rates and maturity   *
c ******************************************************************
c Variable Definitions
c   t(i): maturity of i-th bond from now
c   r(i): continuous compound interest rate of i-th bond (fraction)
c   f(i): amount of payment of i-th bond
c      n: total number of bonds
c
      dimension t(n),r(n),f(n)
c
c calculate overall present value of bond portfolio
c
      p = 0.0
      do 1 i = 1,n
      p = p + exp(-r(i)*t(i)) * f(i)
    1 continue
c
c calculate duration of portfolio
c
      d = 0.0
      do 2 i = 1,n
      d = d + exp(-r(i)*t(i))*f(i)/p * t(i)
    2 continue
c
      ZDRATN = d
c
      return
```

```
      end
c
c
c
      FUNCTION ZCNVEX(t,r,f,n)
c
c ***************************************************************
c *   This function calculates the CONVEXITY for a portfolio of   *
c * zero coupon bonds with different interest rates and maturities*
c ***************************************************************
c Variable Definitions
c   t(i): maturity of i-th bond from now
c   r(i): continuous compound interest rate of i-th bond (fraction)
c   f(i): amount of payment of i-th bond
c      n: total number of bonds
c
      dimension t(n),r(n),f(n)
c
c calculate overall present value of bond portfolio
c
      p = 0.0
      do 1 i = 1,n
      p = p + exp(-r(i)*t(i)) * f(i)
    1 continue
c
c calculate CONVEXITY of portfolio
c
      c = 0.0
      do 2 i = 1,n
      c = c + exp(-r(i)*t(i))*f(i)/p * t(i)**2
    2 continue
c
      ZCNVEX = c
c
      return
      end
c
c
c
      FUNCTION CDRATN(c,f,t,xm,y,p,k)
c
c ***************************************************************
c *   This function calculates the DURATION of  coupon bearing   *
c *   bonds with known yield to maturity or present bond value   *
c ***************************************************************
c Variable Definitions
c      c: coupon rate (fraction)
c      f: first coupon payment from now
c      t: regular interval of coupon payment
c     xm: time to maturity from now
c      y: yield to maturity (fraction)
c      p: present value of bonds (per dollar face value)
c      k: flag of whether to use y or p to do calculations
c         k = 1, use y (yield to maturity)
```

```
c           o/w, use p (present value of bond)
c
c
c calculate number of payments excluding first one
c
        n = int((xm-f)/t)
c
c check consistency of user specified t, f, xm
c
        e = t*n+f
        if(abs(e-xm).gt.0.001) goto 999
c
        nn = n - 1
        if (k.eq.1) goto 100
c
c calculate equivalent continuously compounding rate from p
c by Newton-Raphson Method
c
        r0=.01
c
  111 x = exp(-f*r0) * c*t
        xp = -f * exp(-f*r0) * c*t
        do 1 i = 1,nn
        x = x + exp(r0*(-f-i*t)) * c*t
        xp = xp - (f+i*t) * exp(r0*(-f-i*t)) * c*t
    1 continue
        x = x + exp(-r0*xm) * (1.+c*t) - p
        xp = xp - xm * exp(-r0*xm) * (1.+c*t)
        r1 = r0 - x/xp
        if(abs(r0-r1).le..00001) goto 222
        r0 = r1
        goto 111
c
  222 r = r0
c
c convert continuously compounding
c interest rate to yield to maturiry
c
        y = (exp(r*t)-1.)/t
        write (*,*) 'equivalent yield to maturity = ',y
        goto 200
c
  100 continue
c
c calculate present value of bond from input of y
c
        p = (1.+y*t)**(-f/t) * c*t
        do 333 i=1,nn
        p = p + (1+y*t)**(-f/t-i) *c*t
  333   continue
        p = p + (1+y*t)**(-xm/t) * (1.+c*t)
        write (*,*) 'equivalent present value = ',p
c
  200 continue
```

```
c
c calculate DURATION
c
      d = (1+y*t)**(-f/t) *c*t/p * f
      do 2 i = 1,nn
      d = d + (1+y*t)**(-f/t-i) *c*t/p * (f+i*t)
    2 continue
      d = d + (1+y*t)**(-xm/t) *(1.+c*t)/p * xm
c
      CDRATN = d
c
      return
c
  999 write(*,*) '*** inconsistent input of "f", "t", "m" ***'
      stop
      end
c
c
c
      FUNCTION CCNVEX(c,f,t,xm,y,p,k)
c
c  ***************************************************************
c  *  This function calculates the CONVEXITY for coupon bearing  *
c  *  bonds with known yield to maturity or present bond value   *
c  ***************************************************************
c Variable Definitions
c      c: coupon rate (fraction)
c      f: first coupon payment from now
c      t: regular interval of coupon payment
c     xm: time to maturity from now
c      y: yield to maturity (fraction)
c      p: present value of bonds (per dollar face value)
c      k: flag of whether to use y or p to do calculations
c         k = 1, use y (yield to maturity)
c           o/w, use p (present value of bond)
c
c
c calculate number of payments excluding first one
c
      n = int((xm-f)/t)
c
c check consistency of user specified t, f, xm
c
      e = t*n+f
      if(abs(e-xm).gt.0.001) goto 999
c
      nn = n - 1
      if (k.eq.1) goto 100
c
c calculate equivalent continuous compound rate from p
c by Newton-Raphson Method
c
      r0=.01
c
```

```
  111 x = exp(-f*r0) * c*t
      xp = -f * exp(-f*r0) * c*t
      do 1 i = 1,nn
      x = x + exp(r0*(-f-i*t)) * c*t
      xp = xp - (f+i*t) * exp(r0*(-f-i*t)) * c*t
    1 continue
      x = x + exp(-r0*xm) * (1.+c*t) - p
      xp = xp - xm * exp(-r0*xm) * (1.+c*t)
      r1 = r0 - x/xp
      if(abs(r0-r1).le..00001) goto 222
      r0 = r1
      goto 111
c
  222 r = r0
c
c convert continuously compounding
c interest rate to yield to maturity y
c
      y = (exp(r*t)-1.)/t
      goto 200
c
  100 continue
c
c calculate present value of bond from input of y
c
      p = (1.+y*t)**(-f/t) * c*t
      do 333 i=1,nn
      p = p + (1+y*t)**(-f/t-i) *c*t
 333  continue
      p = p + (1+y*t)**(-xm/t) * (1.+c*t)
c
  200 continue
c
c calculate CONVEXITY
c
      c = (1+y*t)**(-f/t) *c*t/p * f**2
      do 2 i = 1,nn
      c = c + (1+y*t)**(-f/t-i) *c*t/p * (f+i*t)**2
    2 continue
      c = c + (1+y*t)**(-xm/t) *(1.+c*t)/p * xm**2
c
      CCNVEX = c
c
      return
c
  999 write(*,*) '*** inconsistent input of "f", "t", "m" ***'
c
      return
      end
```