Finance 5350: Derivatives Markets

Binomial Model Derivation

Tyler J. Brough

Department of Finance and Economics

UtahStateUniversity

Introduction

What follows is a derivation of the single–period Binomial option pricing formula. This derivation is slightly different than the one found in your textbook. I use different variable names than the text in order to be more consistent with the Black–Scholes model.

One-Period Trees

To fix ideas, recall that our simple assumption of binomial prices leads to two binomial trees: one for the stock price, and one for the option price:

Stock Price Tree

$$
\begin{tikzcd}
 & uS_0 \\
S_0 \arrow[ur] \arrow[dr] & \\
 & dS_0
\end{tikzcd}
$$

Option Price Tree

$$
\begin{tikzcd}
 & C_u \\
C_0 \arrow[ur] \arrow[dr] & \\
 & C_d
\end{tikzcd}
$$

The Replicating Portfolio Concept

The basic idea of the Binomial Option Pricing Model is to set up a replicating portfolio to synthetically replicate the European call option payoff. This lead to a simple equation:

$$C_0 = \Delta S + B$$

where Δ and B are chosen with care so as to perfectly replicate the call option[1] This begs the question: just how are Δ and B chosen? We can solve for these parameters by noting that the following must hold:

$$C_u = \Delta uS + Be^{rh}$$
$$C_d = \Delta dS + Be^{rh}$$

[1] The same logic applies for put options, so we can talk only about call options without loss of generality.

Solving for B

We can now see how to solve for these parameters. First we will solve for Be^{rh} in the second equation as follows:

$$Be^{rh} = C_d - \Delta dS$$

and plug it into the first for Be^{rh} as follows:

$$C_u = \Delta uS + C_d - \Delta dS$$

Solving for Δ

We notice that B has now disappeared from the first equation and we can solve for Δ as follows:

$$\Delta S(u-d) = C_u - C_d$$

which leads to:

$$\Delta = \frac{C_u - C_d}{S(u-d)}$$

So now we have solved for the correct value of Δ that gives us the number of shares we need to hold in our portfolio to synthetically replicate the call option. We can now plug this Δ into $Be^{rh} = C_d - \Delta dS$ to get an equation, for which the only unknown is B and solve for it. We do this as follows:

$$Be^{rh} = C_d - \left(\frac{C_u - C_d}{S(u-d)} \right) dS$$

This we can rearrange as:

$$\begin{aligned} Be^{rh} = & \quad C_d \frac{(u-d)}{(u-d)} - \left(\frac{dC_u - dC_d}{u-d} \right) \\ = & \qquad \frac{uC_d - dC_d - dC_u + dC_d}{u-d} \\ = & \qquad \frac{uC_d - dC_u}{u-d} \end{aligned}$$

Finally, we can multiply both sides of the equation by e^{-rh} to get the following:

$$B = e^{-rh} \left(\frac{uC_d - dC_u}{u-d} \right)$$

The No-Arbitrage Solution

We now know what the values of Δ and B need to be to perfectly replicate the call option. Since we can observe these quantities, we can figure out by applying the **law of one price** (or in other words by assuming no arbitrage opportunities exist) the equilibrium price of the call option now, or C_0.
We simply plug in for Δ and B in the following:

$$\begin{aligned} C_0 &= \Delta S + B \\ &= \left(\frac{C_u - C_d}{S(u-d)}\right) S + e^{-rh}\left(\frac{uC_d - dC_u}{u-d}\right) \end{aligned}$$

The Risk-Neutral Representation

Essentially we could stop here. We are done. We have derived the single–period Binomial Option Pricing Model. But we will keep working to rearrange this equation to express it in such a manner to get even more deep intuition from it.

We can rewrite the model as follows:

$$
\begin{aligned}
C_0 = & \left(\frac{C_u - C_d}{S(u-d)}\right) S + e^{-rh}\left(\frac{uC_d - dC_u}{u-d}\right) \\
= & \left(\frac{C_u - C_d}{(u-d)}\right) + e^{-rh}\left(\frac{uC_d - dC_u}{u-d}\right) \\
= & e^{-rh}\left(\frac{e^{rh}C_u - e^{rh}C_d + uC_d - dC_u}{u-d}\right) \\
= & e^{-rh}\left(C_u\frac{e^{rh} - d}{u-d} + C_d\frac{u - e^{rh}}{u-d}\right)
\end{aligned}
$$

Finally, we can let $p_u^* = \frac{e^{rh} - d}{u - d}$ and $p_d^* = \frac{u - e^{rh}}{u - d}$. Now we can write the model simply as:

$$C_0 = e^{-rh} \left[C_u p_u^* + C_d p_d^* \right]$$