1. Prove that the following languages are not regular. You may use the pumping lemma and the closure of the class of regular languages under union, intersection, and complement.

(a) $L = \{ w \mid w \in \{0,1\}^* \text{ is not a palindrome} \}$

- We assume that L is regular.

- In this case let's attempt to prove $\bar{L}$ is regular as it's easier to work with
$\bar{L} = \{ w \mid w \in \{0,1\}^* \text{ is a palindrome} \}$

- Using the pumping lemma we have the following properties

1. $w = xyz$
2. $|xy| \leq p$
3. $y \neq \varepsilon$

- I choose $w = 0^p 1 0^p$ as a palindrome string

let $x = 0^{p-i}$
$y = 0^i$
$z = 1 0^p$

$xy = 0^p$
↳ since $y \neq \varepsilon$ then $y = 0^i$
and $x = 0^{p-i}$

↳ $xyz = 0^{p-i} 0^{i} 1 0^{p} = 0^{p} 1 0^{p}$

↳ because w is a member of L and w has length $> p$, the pumping lemma guarantees that w can be split in three pieces xyz.

↳ $w = xy^{K}z$

$$= \underbrace{0^{p-i}}_{x} \underbrace{(0^{i})^{K}}_{y} \underbrace{1 0^{p}}_{z}$$

↳ we set the pumping length to 2

$$0^{p-i} (0^{i})^{2} 1 0^{p}$$

$$= 0^{p-i} \cdot 0^{2i} 1 0^{p}$$

$$= 0^{p+i} 1 0^{p}$$

↳ $w = 0^{p+i} 1 0^{p} \notin \overline{L}$ since $0^{p+i} \neq 0^{p}$ and therefore it isn't a palindrome. Thus $\overline{L}$ is not regular, and by the closure laws the complement of $\overline{L}$ is not regular. So L is not regular [b.) $L = \{wtw \mid w, t \in \{0,1\}^+\}$

- This is similar to (a) in that wtw is a palindrome, but there has to be at least one occurence of w and t

- We assume that L is regular

- let $w_2 = wtw = 1^p 0 1^p$

↳ let $x = 1^{p-i}$

$y = 1^i$

$z = 0 1^p$

- so by the pumping lemma we have

$xy^k z$

$= 1^{p-i} (1^i)^k 0 1^p$

↳ let the pumping length $k = 2$

$= 1^{p-i} (1^i)^2 0 1^p$

$= 1^{p-i} \cdot 1^{2i} 0 1^p$

$= 1^{p+i} \cdot 0 \cdot 1^p$

↳ however in this instance we see that $1^{p+i} \neq 1^p$, therefore the pumping lemma proves that $w_2 = ww = 1^p 0 1^p$ is not regular.

2. Give context-free grammars that generate the following languages.

(a) The complement of the language $\{a^n b^n \mid n \geq 0\}$

- If $L = \{a^n b^n \mid n \geq 0\}$ then $\bar{L}$ can have no instance where there are the same number of a's and b's

$L_1 = \{a^i b^j \mid i \neq j\}$

$S_1 \to aIb \mid A \mid B$
$I \to aIb \mid A \mid B$
$A \to aA \mid a$
$B \to bB \mid b$

Construction notes

(1) cannot be ε because that would mean $i = j = \varnothing$

(2) have to account for initial infinite build

$S \to aIb$, $I \to aIb$

(3) If more a's than b's then break from I and substitute $A \to aA \mid a$

(4) If more b's than a's then break from I and substitute $B \to bB \mid b$

(5) Note there could be 0 a's or 0 b's so the start state could just go to A or B

- Also have to account if there is ever an instance of ba at any point in a string

$$L_2 = \{ w \mid w = (a \cup b)^* ba (a \cup b)^* \}$$

- Construction Notes

(1) The start state will always have ba in it

$S \to \_\ ba\ \_$

(2) Need to account for $(a \cup b)^*$ on either side of the ba.

$E \to aE \mid bE \mid \varepsilon$

(3) Final Build

$S_2 \to E\ ba\ E$
$E \to aE \mid bE \mid \varepsilon$

- Now we must combine the CFG for both cases,

$L = L_1 \cup L_2$

$$\boxed{\begin{aligned}
S &\to S_1 \mid S_2 \\
S_1 &\to aIb \mid A \mid B \\
I &\to aIb \mid A \mid B \\
A &\to aA \mid a \\
B &\to bB \mid b \\
S_2 &\to E\ ba\ E \\
E &\to aE \mid bE \mid \varepsilon
\end{aligned}}$$

(b) $\{x_1 \# x_2 \# \cdots \# x_k \mid k \geq 1$, each $x_i \in \{a,b\}^*$, and for some i and j, $x_i = x_j^R \}$

- If any $x_i \in \{a,b\}^*$ then for any $x_1, x_2, \ldots, x_k$ each of them have the form $(a \cup b)^*$

- Therefore the language starts to look like

$$(a \cup b)^* \# (a \cup b)^* \# \ldots \# (a \cup b)^*$$

↳ For any $i \in [1,k]$ and any $j \in [1,k]$, $x_i = x_j^R$

↳ I take this meaning that for any selection, including $i = j$, $x_i = x_j^R$. In the grand scheme this would mean that every x_i and x_j are palindromes of one another.

CFG Construction:

① The string will always have at least one # as the suffix regardless of x_i

$$S \to X \#$$

② $(a \cup b)^*$ could be represented by

$L \to aL \mid bL \mid \varepsilon$

(3) However we need to be aware of adding more #'s and ending with an X. So modify

$S \to L\#L$
$E \to \qquad\qquad \mid \# L$
$L \to aL \mid bL \mid \varepsilon$

(4) But any string needs to be mirrored so

$$\boxed{\begin{aligned} &S \to EL \\ &E \to 0E0 \mid 1E1 \mid \#X \\ &L \to aL \mid bL \mid \varepsilon \end{aligned}}$$

3. Give informal descriptions of PDAs for the languages in the previous problem.

(a) The complement of the language $\{a^n b^n \mid n \geq 0\}$

↳ From the previous question we know that this is the unions of

$L_1 = \{a^m b^n \mid m \neq n\}$ and
$L_2 = \{(a \cup b)^* ba (a \cup b)^*\}$

The PDA will
1. Read symbols from the input
2. As each a is read push it onto the stack
   - If there is a b on top of the stack accept the input. This is due to a b entering as input before an a.
3. As each b is read
   - If there was an a on top of the stack, pop it
   - If the stack is empty then accept, this means more b's were in the string than a's.
4. If the input is complete and the stack is empty, then reject as there were an equal number of a's followed by b's.

(b) $\{x_1 \# x_2 \# \ldots \# x_k \mid k \geq 1$, each $x_i \{a, b\}^*$, and for some i and j, $x_i = x_j^R\}$

- So we established that every $x_i = x_j$ string must be the same as each must be a palindrome of one another.

- The informal description of the PDA is

1. Read symbols from the input
2. If an a or b is read push those symbols to the stack until a # is read
   - ↳ If the input finishes and no # is read then reject.

2a) Once a # is read we

- ↳ If an a is read and an a is on top of the stack then pop it, otherwise reject
- ↳ If a b is read and a b is on top of the stack then pop it, otherwise reject.
- ↳ If the input ends and the stack is empty then accept

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4. Convert the following CFG into an equivalent CFG in Chomsky Normal Form

$A \to BAB \mid B \mid \varepsilon$
$B \to 00 \mid \varepsilon$

- By definition, a context free grammar is in Chomsky Normal form if every rule is of the form

$$A \to BC$$
$$A \to a$$

where a is any terminal and A, B, C are any variables — except $B C$ may not be the start variable. We permit the rule $S \to \varepsilon$ where S is the start variable.

- $S \to A$
  $A \to BAB \mid B \mid \varepsilon$
  $B \to 00 \mid \varepsilon$

• add new start variable

• eliminate all ε-variables

eliminate $A \to \varepsilon$

$S \to A \mid \varepsilon$
$A \to BAB \mid B \mid BB$
$B \to 00 \mid \varepsilon$

eliminate $B \to \varepsilon$

$S \to A \mid \varepsilon$

$A \to BAB \mid B \mid BB \mid AB \mid BA \mid A$

$B \to 00$

eliminate unit rule $S \to A$

$S \to BAB \mid B \mid BB \mid AB \mid BA \mid A \mid \varepsilon$

$A \to BAB \mid B \mid BB \mid AB \mid BA \mid A$

$B \to 00$

eliminate unit rule $A \to A$

$S \to BAB \mid B \mid BB \mid AB \mid BA \mid \varepsilon$

$A \to BAB \mid B \mid BB \mid AB \mid BA \mid$

$B \to 00$

eliminate $S \to B$

$S \to BAB \mid 00 \mid BB \mid AB \mid BA \mid \varepsilon$

$A \to BAB \mid 00 \mid BB \mid AB \mid BA$

$B \to 00$

- Now we need to substitute a variable for the terminals.

$S \to BAB \mid 00 \mid BB \mid AB \mid BA \mid \varepsilon$
$A \to BAB \mid 00 \mid BB \mid AB \mid BA$
$B \to TT$
$T \to 0$

- Now address the BAB since it's a 3 element, with another substitution

$$\boxed{\begin{array}{l} S \to CB \mid 00 \mid BB \mid AB \mid BA \mid \varepsilon \\ A \to CB \mid 00 \mid BB \mid AB \mid BA \\ C \to BA \\ B \to TT \\ T \to 0 \end{array}}$$

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5. Let $A/B = \{w \mid wx \in A \text{ for some } x \in B\}$. Show that if A is context free and B is regular, then A/B is context free

- So we know A is context free and B is regular.

↳ We know a language is context free if a PDA recognizes it
↳ We know that a language is regular if an NFA recognizes it.
↳ A/B refers to the language of A

- Constructing the PDA for A we get

$$A = (Q, \Sigma, \Gamma, \delta, q_0, F)$$

- We assume that A/B is also context free so

$$A/B = (Q^{A/B}, \Sigma^{A/B}, \Gamma^{A/B}, \delta^{A/B}, q_0^{A/B}, F^{A/B})$$