UNIVERSITY OF VICTORIA

EXAMINATIONS August 2020

MATH 204: Calculus IV, Sections[A01]

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Duration: 1 hour

STUDENTS MUST COUNT THE NUMBER OF PAGES IN THIS EXAMINATION PAPER BEFORE BEGINNING TO WRITE, AND REPORT ANY DISCREPANCY IMMEDIATELY TO THE INSTRUCTOR.

THIS QUESTION PAPER HAS 7 PAGES plus COVER plus INSTRUCTIONS.

SCORE OUT OF 50: ______

INSTRUCTIONS:

1. OPTIONAL: **Print this .pdf.** If you do not have easy access to a printer, that is ok, but it is easier for me instructor to have a consistent format when marking so if you can print it out please do so. There are 6 questions and 9 pages (including this page).
2. For each problem, **write out a full solution**. Solutions should be clear, complete, and justified. *Final answers without supporting work will be graded as zero.*
3. **This exam is individual**. Communicating with anybody else during the test is a strict violation of Academic Integrity. Posting the test on the internet is a violation not just of academic integrity but of Canadian copyright law.
4. **This exam is open book**. You may consult your notes, the book, the videos, etc, but you must still write full solutions. We consider "googling" the problems to be unethical, and have written the problems aiming to minimize the usefulness of this.
5. **If you need help during the exam**, I will be available live at ibrahims@uvic.ca
6. **Announcements** such as to report any discovered typos will be found at the TOP of Coursespaces highlighted in Yellow. Please refresh periodically.
7. The normal time for test is **60 minutes for writing and 20 minutes for scanning, and uploading**. If you have a time multiplier through CAL then the upload link will be available for that longer time period. Please do not leave this to the last few minutes to scan in case of a technological issue. Use your phone or other scanning device. Apps such as Adobe Scan can make a clean pdf file. Make sure all your pages are oriented correctly and in the right order. It's ok to insert your own pages if needed.
8. **If something goes wrong** with scanning and uploading let me know ASAP. Take a clean photo of each page and email them to ibrahims@uvic.ca or mawais@uvic.ca by the end of the exam.
9. Please **keep your exam for at least two weeks** in case we need you to rescan.
10. Please **read and sign** the Confidentiality Agreement [1] before solving the quiz

[1] Confidentiality Agreement: I did not communicate with any other person or share this exam in any way. I followed all exam instructions.

SIGANUTURE: (MANDATORY)

Q1. [**8 marks**] Verify the tangential form of Green's theorem for the vector field

$$\vec{F} = (y^2 - 7y)\,\hat{\imath} + (2xy + 2x)\,\hat{\jmath}$$

and the region R bounded by the unit circle $x^2 + y^2 = 1$ centered at the origin. *Make sure you compute both sides of the Green's theorem.*

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C M\,dx + N\,dy$$

$$x^2+y^2 = 1 \Rightarrow \vec{r} = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath}$$

$$\vec{r}\,'(t) = -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath}$$

$$F(\vec{r}(t)) = (\sin^2 t - 7\sin t)\,\hat{\imath} + (2\cos t\sin t + 2\cos t)$$

$$F(\vec{r}(t))\cdot\vec{r}\,'(t) = -\sin^3 t + 7\sin^2 t\cdot 2\cos^2 t\sin t + 2\cos^2 t$$

$$\oint_C \vec{F}\,dr = \int_0^{2\pi} -\sin^3 t + 7\sin^2 t + 2\cos^2 t\sin t + 2\cos^2 t$$

2nd $\oint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_D (2y+2 - (2y-7))\,dx\,dy$

$= \iint_D 9\,dx\,dy \Rightarrow 9\iint_D dA.$ area of unit circle is π

$\boxed{= 9\pi}$

Q2. [**8 marks**] Use the Divergence theorem to calculate the flux of the vector field

$$\vec{F} = xy\,\hat{\imath} + yz\,\hat{\jmath} + z\,\hat{k}$$

across the cylinder

r cylinder

$$x^2 + y^2 \le 1, \quad 0 \le z \le 1$$

Hint: *You might find cylindrical polar coordinates $\vec{r} = < r\cos\theta, r\sin\theta, z >$ helpful here!*

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x} xy + \frac{\partial}{\partial y} yz + \frac{\partial}{\partial z} z$$

$$= y + z + 1$$

polar

$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

$0 \le r \le 1$

$$\iiint \operatorname{div} \vec{F} \cdot dV = \int_0^1 \int_0^{2\pi} \int_0^1 r\sin\theta + z + 1 \; dz\, d\theta\, dr$$

$$= \int_0^1 \int_0^{2\pi} z\, r\sin\theta + \frac{z^2}{2} + z \Big|_0^1 d\theta\, dr = \int_0^1 \int_0^{2\pi} r\sin\theta + \frac{1}{2} + 1\, d\theta\, dr$$

$$= \int_0^1 \int_0^{2\pi} r\sin\theta + \frac{3}{2}\, d\theta\, dr = \int_0^1 -r\cos\theta + \frac{3}{2}\theta \Big|_0^{2\pi} dr$$

$$= \int_0^1 [-r(1) + 3\pi] - [-r(1) + 0]\, dr = \int_0^1 -r + 3\pi + r + 0\, dr$$

$$= \int_0^1 3\pi\, dr = 3\pi r \Big|_0^1 = \boxed{3\pi}$$

Q3. [**9 marks**] A radioactive substance decays at a rate proportional to the amount present at time t (in hours). Initially, A_0 grams of the substance was present, and after 10 hours, the amount has decreased by 20% $Q(0) = A_0$

(a) How long will it take the substance to decay to $\frac{A_0}{5}$?

$$\frac{dQ}{dt} = -kQ \Rightarrow \frac{dQ}{Q} = -k\,dt \Rightarrow \int \frac{dQ}{Q} = \int -k\,dt$$

can just be C

$$\Rightarrow \ln|Q| = -kt + c \Rightarrow Q = e^{-kt+C} = e^{-kt} e^{C}$$

$$\Rightarrow Q(0) = (1) = C \quad , \quad A_0 = C$$

$$Q(t) = A_0 e^{-kt}$$

$$Q(10) = 0.8A_0 = A_0 e^{-k(10)}$$

$$0.8 = e^{-10k}$$

$$\ln|0.8| = -10k$$

$$k \approx 0.02231$$

$$0.2A_0 = A_0 e^{-0.02231t}$$

$$0.2 = e^{-0.2231t}$$

$$\ln|0.2| = -0.2231t$$

$$\boxed{t = 7.21}$$

(b) What is the half life of this substance?
(**Hint:** the half-life is the time required for half of the initial substance to decay).

Half Life $= 0.5A_0$

$$0.5A_0 = A_0 e^{-kt}$$

$$0.5 = e^{-0.2231t}$$

$$\frac{\ln|0.5|}{-0.2231} = t$$

$$\boxed{t = 3.17 \text{ hrs}}$$

half life $= 3.17$ hours.

Q4. [**8 marks**] Use the **Laplace transform** to find the solution $y(t)$ to the initial value problem

$$y'' - 4y' + 4y = 0, \qquad y(0) = 1, \quad y'(0) = 1$$

Hint: You might find the following expression useful: $\frac{s-3}{(s-2)^2} = \frac{1}{s-2} - \frac{1}{(s-2)^2}$.

$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = 0$

$s^2Y(s) - sy(0) - y'(0) - 4[sY(s) - y(0)] + 4Y(s) = 0$

- $y(0)=1,\ y'(0)=1$
- $s^2Y(s) - s - 1 - 4[sY(s) - 1] + 4Y(s) = 0$
- $s^2Y(s) - s - 1 - 4sY(s) + 4 + 4Y(s) = 0$
- $Y(s)[s^2 - 4s + 4] - s - 1 + 4 = 0$

$Y(s)[s^2 - 4s + 4] = s - 3$

$Y(s) = \frac{s-3}{s^2 - 4s + 4}$

partial fractions

$\frac{s-3}{(s-2)^2} = \frac{A}{s-2} + \frac{B}{(s-2)^2}$

$= \frac{(s-2)A + B}{(s-2)^2}$

$s - 3 = (s-2)A + B$

$s - 3 = sA - 2A + B$

$A = 1$

$-2A + B = -3$

$\Rightarrow B = 1$ $(A=1, B=-1)$

$Y(s) = \frac{s-3}{(s-2)^2}$

$Y(s) = \frac{1}{s-2} - \frac{1}{(s-2)^2}$

$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\}$

$$\boxed{y(t) = e^{2t} - te^{2t} = e^{2t}(1-t)}$$

Q5. [**9 marks**] Consider the differential equation

$$xy'' - (x+1)y' + y = 0. \tag{1}$$

(a) Check that $y_1(x) = e^x$ is a solution to (1).

$Y_1(x) = e^x$
$Y_1'(x) = e^x$
$Y_1''(x) = e^x$

$x \cdot e^x - (x+1)e^x + e^x = 0$
$xe^x - xe^x - e^x + e^x = 0$
$0 = 0$

Y_1 is verified sol^n

(b) Set $y_2(x) = e^x u(x)$, and find an ODE satisfied by $u(x)$ so that $y_2(x)$ is a second linearly independent solution of (1).

$Y_2(x) = e^x u(x)$
$Y_2'(x) = e^x u(x) + u'(x)e^x$
$Y_2''(x) = e^x u(x) + u'(x)e^x + u''(x)e^x + u'(x)e^x$
$= e^x(u(x) + 2u'(x) + u''(x))$

$x[u(x) + 2u'(x) + u''(x)]e^x - (x+1)[e^x u(x) + u'(x)e^x] + e^x(u(x)) = 0$
$\cancel{xe^x u(x)} + 2xe^x u'(x) + xe^x u''(x) - \cancel{xe^x u(x)} - xu'(x)e^x - e^x u(x) - u'(x)e^x + e^x u(x) = 0$
$xe^x u'(x) + xe^x u''(x)$

(c) Solve the ODE on u and thus give the general solution to (1).

Q6. [**8 marks**]

(a) Use the **Power Series method** to find the general solution $y(x)$ for the differential equation:

$$xy'' - (x+1)y' + y = 0.$$

$y = \sum_{n=0}^{\infty} C_n x^n$, $y' = \sum_{n=1}^{\infty} n C_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$

$x \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - (x+1) \sum_{n=1}^{\infty} n C_n x^{n-1} + \sum_{n=0}^{\infty} C_n x^n = 0$

$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-1} - \sum_{n=1}^{\infty} n C_n x^n - \sum_{n=1}^{\infty} C_n x^{n-1} + \sum_{n=0}^{\infty} C_n x^n = 0$

$n \geq 0$ $\quad$ $n \geq 0$.

$\sum_{n=1}^{\infty} (n+1) n C_{n+1} x^n - \sum_{n=1}^{\infty} n C_n x^n - \sum_{n=0}^{\infty} C_{n+1} x^n + \sum_{n=0}^{\infty} C_n x^n = 0$

$n \geq 0$ makes $n=0$ so we can just drop the index without changing function

$\sum_{n=0}^{\infty} (n+1) n C_{n+1} x^n - \sum_{n=0}^{\infty} n C_n x^n - \sum_{n=0}^{\infty} C_{n+1} x^n + \sum_{n=0}^{\infty} C_n x^n = 0$

$(n+1) n C_{n+1} - n C_n - C_{n+1} + C_n = 0$

(b) Specify the radius of convergence of the series solution, and identify it with familiar elementary functions

$(n+1) n C_{n+1} - C_{n+1} = n C_n - C_n$

$C_{n+1}(n^2+n-1) = (n-1) C_n$

$C_{n+1} = \frac{(n-1) C_n}{(n^2+n-1)}$

$n=0$: $2 = -C_0$ $\quad C_0$.

$n=1$

$n=2$: $C_3 = \frac{C_2}{5-1}$

$n=3$: $C_4 = \frac{C_3}{9} = \frac{C_2}{5 \cdot 9}$

$n=4$: $C_5 = C_4$

$y(x) = C_0 + C_1 x + C_2 x$

$y(x) = C_0$

radius of convergence

$\lim_{n \to \infty} \left| \frac{C_n}{C_{n+1}} \right|$

– END –