1. • 9 programming textbooks

↳ 5 C++

↳ 4 Java

a) $n! = 9! = \boxed{362880}$

b) If the languages should alternate

C J C J C J C J

$C's = 5!$

$J's = 4!$

• $5! \cdot 4! = \boxed{2880}$

c) If all C's next to each other

• $\underbrace{C\ C\ C\ C\ C}$ $= 5!$

is now object C_1

• C_1 J J J J $= 5!$

$= 5! \cdot 5! = \boxed{14400}$

d) If all C's next to each other and all J's next to each other

$\underline{C}\,\underline{C}\,\underline{C}\,\underline{C}\,\underline{C}$ $\quad$ $\underline{J}\,\underline{J}\,\underline{J}\,\underline{J}$

$5!$ $\quad$ $4!$

↳ multiply by 2 since they can be on either side

$= 5! \cdot 4! \cdot 2 \; \boxed{= 5760}$

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2. Suppose that you draw 5 cards from a standard deck of 52.

a) How many ways to draw exactly 2 spades

- $\binom{13}{2}$ ⇒ choose 2 spades out of 13
- $\binom{39}{3}$ ⇒ choose 3 cards from remaining 39 cards that aren't spades

$= \binom{13}{2}\binom{39}{3} \; \boxed{= 712842}$

b) How many ways can you draw at most 2 spades?

- Case 1: 2 spades

$$= \binom{13}{2} \cdot \binom{39}{3} = 712842$$

- Case 2: 1 spade

$$= \binom{13}{1} \cdot \binom{39}{4} = 82251$$

- Case 3: 0 spades

$$= \binom{13}{0} \cdot \binom{39}{5} = 575757$$

- case 1 + case 2 + case 3 $\boxed{= 1,378,650}$

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c) How many ways can you draw 3 spades and 2 hearts

- $\binom{13}{3} \cdot \binom{13}{2} \boxed{= 22308}$

d) How many ways can you draw exactly 1 spade, 2 hearts, and 2 kings?

Case 1: 1 spade is a king
1 heart is a king,

King of Spades = 1
King of Hearts = 1
one more heart out of 12 = $\binom{12}{1}$

2 more cards, no spades, no hearts, no kings = $\binom{24}{2}$

$= \binom{12}{1}\binom{24}{2} = 3312$

Case 2: No spade is king
1 heart is king,

Spade = $\binom{12}{1}$
King of herts = 1
remaining heart = $\binom{12}{1}$

remaining two cards
no kings = $\binom{24}{2}$

$$= \binom{12}{1}\binom{12}{1}\binom{24}{2} = 39744$$

Case 3: No spade is king
Neither heart is king
King of diamonds & King of Clubs

- Spade = $\binom{12}{1}$
- Heart = $\binom{12}{2}$
- King of Diamonds & Clubs = 1

$$\binom{12}{1}\binom{12}{2} = 792$$

Case 1 + case 2 + case 3

$$\boxed{= 43848}$$

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3. Show that if n is a positive integer and $n > 2$, then $\binom{n}{2} + \binom{n-1}{2}$

is a <u>perfect square</u>

- $\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n\cdot(n-1)\cdot(n-2)!}{2\ (n-2)!}$

$$= \frac{n(n-1)}{2}$$

- $\binom{n-1}{2} = \frac{(n-1)!}{2!\ (n-1-2)!} = \frac{(n-1)(n-2)(n-3)!}{2\ (n-3)!}$

$$= \frac{(n-1)(n-2)}{2}$$

- $\binom{n}{2} + \binom{n-1}{2} = \frac{n(n-1)}{2} + \frac{(n-1)(n-2)}{2}$

$$= \frac{n(n-1)+(n-1)(n-2)}{2} = \frac{(n-1)[n+n-2]}{2}$$

$$= \frac{(n-1)(2n-2)}{2} = \frac{2n^2-2n-2n+2}{2}$$

$$= \frac{2(n^2-2n+1)}{2} = (n-1)^2$$

↳ Therefore if n is a positive integer and greater than 2 then $(n-1)$ will always be positive and give $(n-1)^2$. Therefore since the only solutions are squared numbers and it is proven that $\binom{n}{2} + \binom{n-1}{2}$ is a perfect square.

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b) For x a real number and n a positive integer, show that,

$$1 = (1+x)^n - \binom{n}{1} x (1+x)^{n-1} + \binom{n}{2} x^2 (1+x)^{n-2} - \dots + (-1)^n \binom{n}{n} x^n$$

- This looks similar to the Binomial Theorem

$$(x+y)^n = \binom{n}{0} x^0 y^n + \binom{n}{1} x^1 y^{n-1} + \dots + \binom{n}{n} x^n y^0$$

- $$(1+x)^n = \binom{n}{0} x^0 y^n$$
$$= (1)(1) y^n$$

$$(1+x)^n = y^n$$
$$1+x = y$$

- $-\cancel{\binom{n}{1}} x (1+x)^{n-1} = \cancel{\binom{n}{1}} x^1 y^{n-1}$

$-x(y)^{n-1} = x^1 y^{n-1}$

$x = -x \quad , \quad y = (1+x)$

- $(x+y)^n = (-x+1+x)^n = 1^n \boxed{= 1}$

↳ Therefore it is proven that

$1 = (1+x)^n - \binom{n}{1} x (1+x)^{n-1} + \binom{n}{2} x^2 (1+x)^{n-2} - \cdots + (-1)^n \binom{n}{n} x^n$

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4. Determine the number of integer solutions of
$x_1 + x_2 + x_3 + x_4 = 32$, where

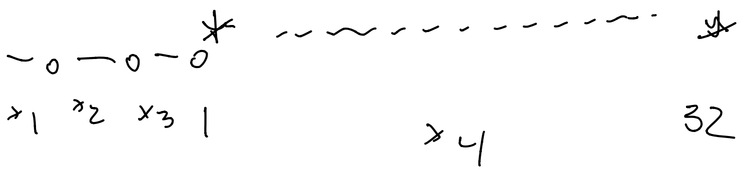
a) $x_i \geq 0 \quad , \quad 1 \leq i \leq 4$

- This looks like the stars and bars method.

$\ast \mid \ast \quad \ast \mid \cdots\cdots \mid \ast \quad \ast \quad \rightarrow (1, 2, 27, 2)$

$1 \quad 2 \quad 3 \qquad\qquad 31 \quad 32$

↳ 31 gaps + 3 gaps to accomodate for zeroes.
= 34 gaps and 3 sticks



$$= \frac{35!}{32!\,3!} = \frac{35 \cdot 34 \cdot 33 \cdot \cancel{32!}}{3 \cdot 2 \cdot \cancel{32!}} \boxed{= 6545}$$

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b) $x_1 \geq 0$, $x_2 > 1$, $x_3 > 2$, $x_4 > 3$

- we need to determine the number of gaps available in stars, to do so we will reduce the restrictions to be greater than zero.

$x_2 > 1$	$x_3 > 2$	$x_4 > 3$
$x_2 - 1 \geq 0$	$x_3 - 2 > 0$	$x_4 - 3 > 0$
↳ let $a = x_2 - 1$	↳ let $b = x_3 - 2$	↳ let $c = x_4 - 3$

$$x_1 + x_2 + x_3 + x_4 = 32$$
$$x_1 + (x_2 - 1 + 1) + (x_3 - 2 + 2) + (x_4 - 3 + 3) = 32$$
$$x_1 + a + b + c + 6 = 32$$
$$x_1 + a + b + c = 26$$

- Now we know we have 26 gaps and 4 sticks, we know we have 4 sticks since

$$\underset{x_1}{**} \mid \underset{b}{*} \mid \underset{c}{**} \mid \underset{d}{**} \cdots\cdots * \mid$$ f $*$'s (from the substitution)

$$\binom{(n-1)!}{(k-1)!} = \binom{25!}{3!} = \frac{25!}{(25-3)! \cdot 3!}$$

$$= \frac{25 \cdot 24 \cdot 23 \cdot \cancel{22!}}{6 \cdot \cancel{22!}} \quad \boxed{= 2300}$$

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5. - (A, R_1), (B, R_2) are 2 posets

- Consider set derived from cross product of sets A and B

$\hookrightarrow A \times B = \{(a,b) : a \in A, b \in B\}$

- Define relation R on $A \times B$ by $((a,b),(x,y)) \in R$ if $(a,x) \in R_1$, $(b,y) \in R_2$

- Prove that $\mathcal{R}$ is partial order
  1) Reflexive
  2) Antisymmetric
  3) Transitive

1)