CS 156
Intro to AI

Inference in Bayes Nets

Probabilistic Reasoning Over Time

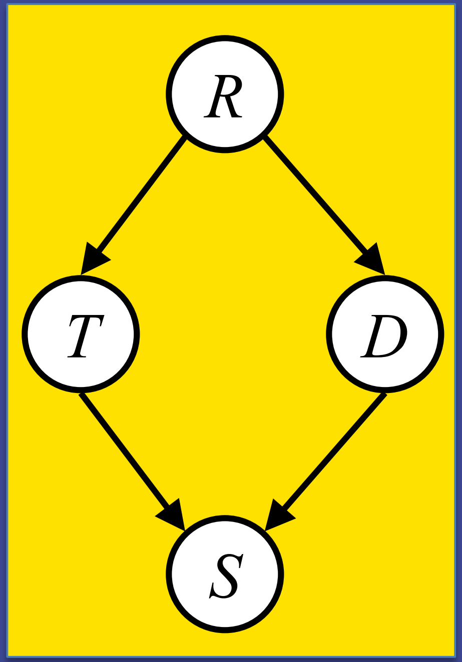
These slides are based on the slides created by Dan Klein and Pieter Abbeel for CS188 Intro to AI at UC Berkeley.
The artwork is by Ketrina Yim

Today

- More Independence Examples
- Inference by Enumeration in Bayes Nets
- Homework 6
- Markov Chains

Independence

- Variables:
  - R: Raining
  - T: Traffic
  - D: Roof drips
  - S: I'm sad

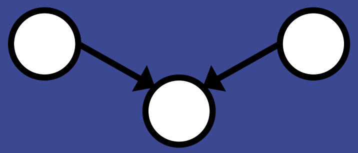


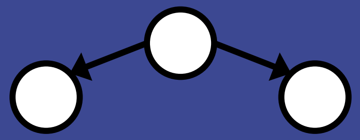
Independence

$T \perp\!\!\!\perp D$?

Consider all (undirected) paths from T to D

There are two paths: TSD and TRD

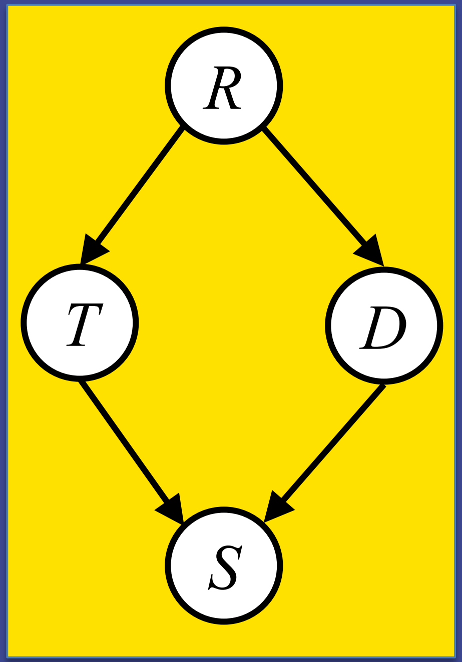




TSD inactive

TRD active =>

No independence



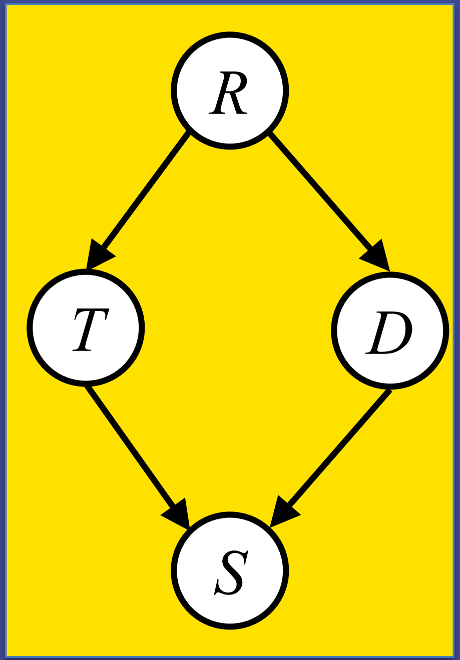
iClicker: Independence

$T \perp\!\!\!\perp D \mid R$?

iClicker:

A. Yes, we can guarantee independence

B. No, we cannot guarantee independence

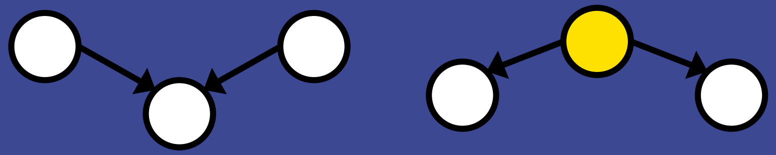


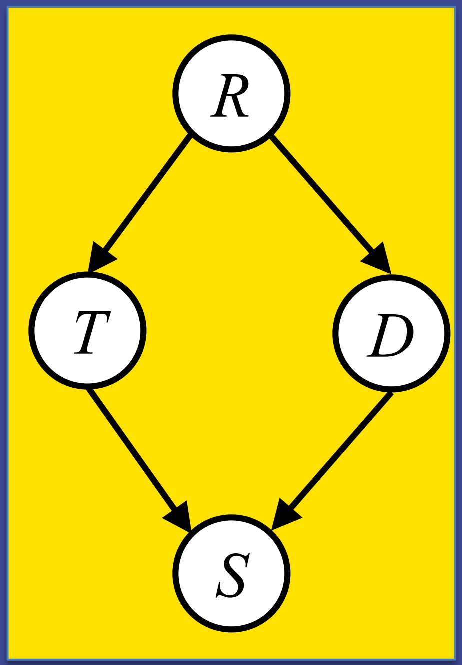
Independence

$T \perp\!\!\!\perp D \mid R$?

Consider all (undirected) paths from T to D

There are two paths: TSD and TRD





TSD inactive

TRD inactive

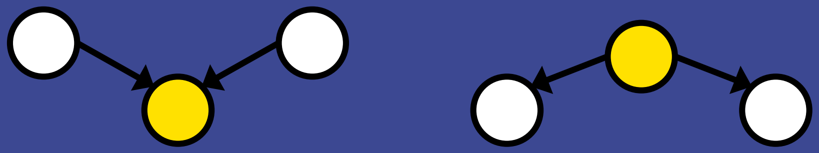
No active paths => independence

Independence

$T \perp\!\!\!\perp D \mid R, S$?

Consider all (undirected) paths from T to D

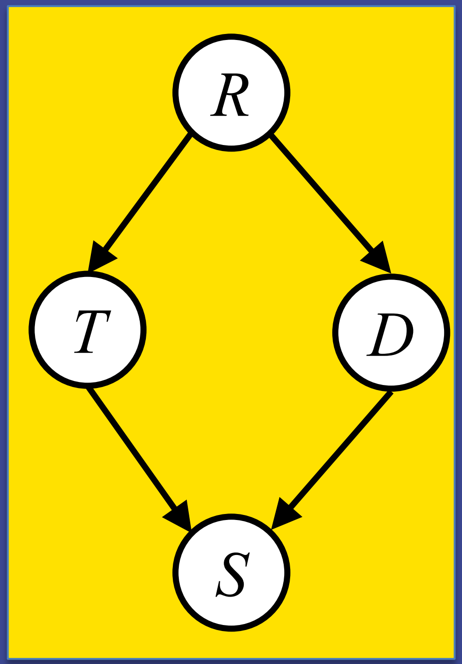
There are two paths: TSD and TRD



TSD active => *No independence*

TRD inactive

One active path => No independence



Inference by Enumeration

- General case:
  - Evidence variables: $E_1 \ldots E_k = e_{1 \ldots} e_k$
  - Query variable(s) : Q
  - Hidden variables: $H_1 \ldots H_k$

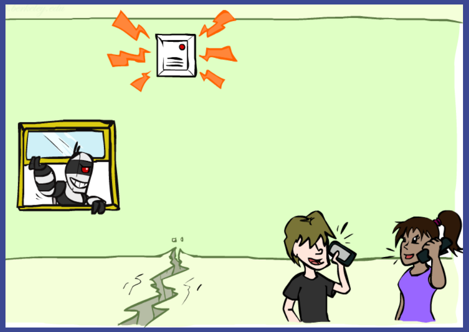
  - We want: $P(\mathrm{Q}|\mathrm{E}_1 = e_1, \ldots, \mathrm{Ek} = \mathrm{e}_k)$

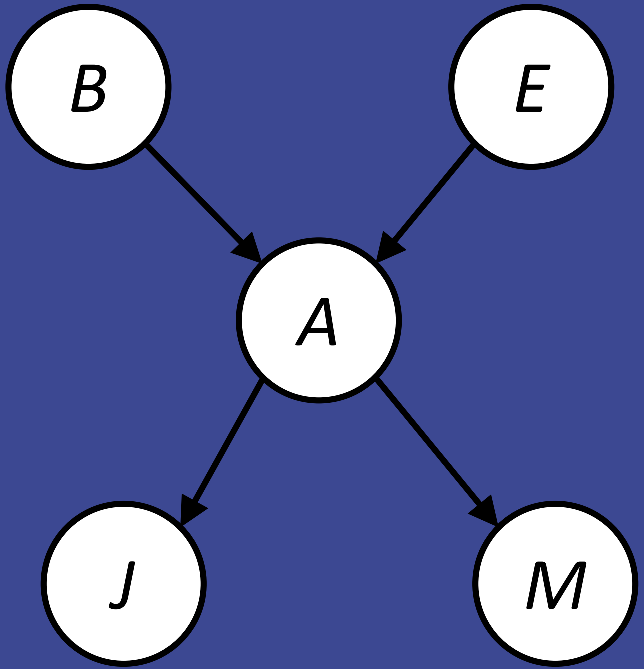
Inference by Enumeration in Bayes' Net

- John and Mary called.
- Was there a burglary?
- Query Variabl
  - B
- Evidence Varia
  - J, M
- Hidden Variab
  - E, A

What probability are we looking for?

A. P(+b, +j, +m)
B. P(+b | +j, +m)
C. P(+b, +j, +m, -e, +a)
D. P(+b, +j, +m| -e, +a)

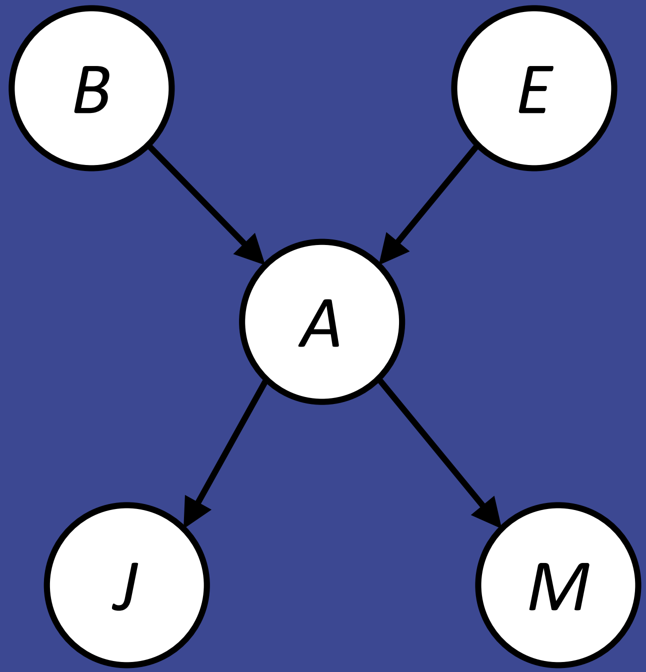




Inference by Enumeration in Bayes' Net

By definition of conditional probability:

$$P(+b \mid +j, +m) = \frac{P(+b, +j, +m)}{P(+j, +m)}$$



We also know that P(X) = P(X, +y) + P(X, -y):

P(+b, +j, +m) = ?

P(+b, +j, +m) = P(+b, +j, +m, +e) + P(+b, +j, +m, -e)

= P(+b, +j, +m, +e, +a) + P(+b, +j, +m, +e, -a) +

P(+b, +j, +m, -e, +a) +P(+b, +j, +m, -e, -a)

Example: Alarm Network

What is P(+b, +j, +m, +e, +a)?

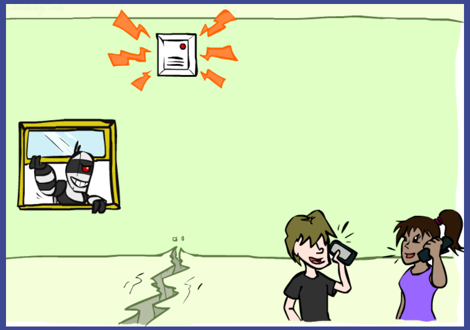
A. P(+b) P(+j) P(+m) P(+e) P(+a)

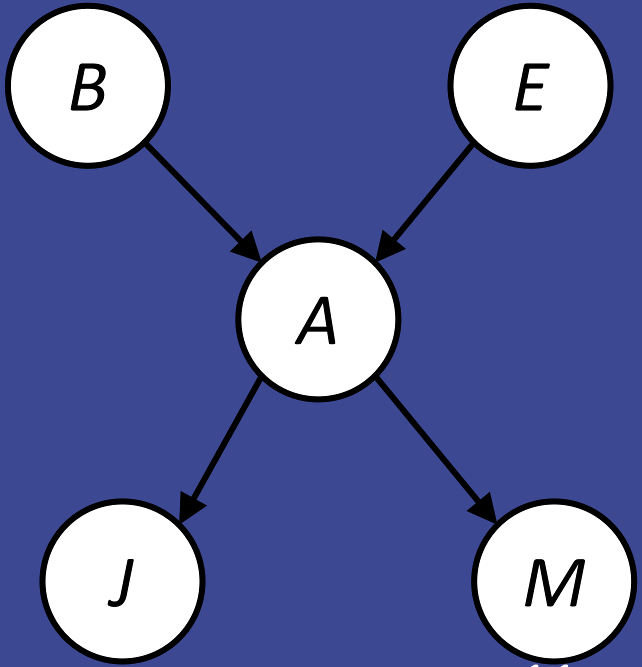
B. P(+b) P(+e) P(+a|+b,+e) P(+j|+a) P(+m|+a)

C. P(+j) P(+m) (+a|+j,+m) P(+b|+a) P(+e|+a)

D. P(+b) P(+e) P(+a|+b) P(+a|+e) P(+j|+a) P(+m|+a)

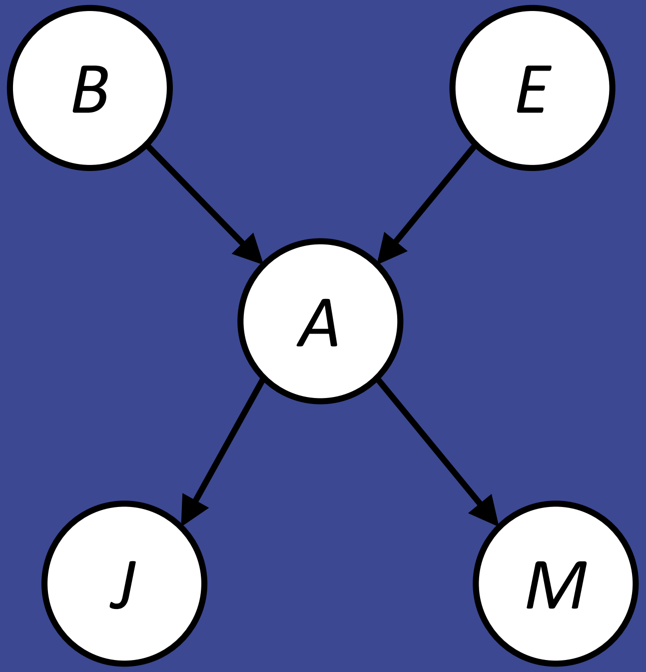
E. P(+b) P(+j) P(+m) P(+e) P(+a|+b,+e,+j,+m)



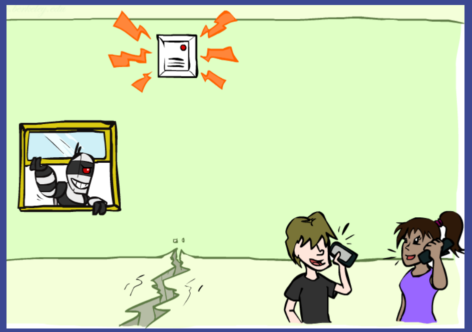


Example: Alarm Network

B	P(B)
+b	0.001
-b	0.999



E	P(E)
+e	0.002
-e	0.998



A	J	P(J|A)
+a	+j	0.9
+a	-j	0.1
-a	+j	0.05
-a	-j	0.95

A	M	P(M|A)
+a	+m	0.7
+a	-m	0.3
-a	+m	0.01
-a	-m	0.99

B	E	A	P(A|B,E)
+b	+e	+a	0.95
+b	+e	-a	0.05
+b	-e	+a	0.94
+b	-e	-a	0.06
-b	+e	+a	0.29
-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

$P(+b, +j, +m, +e, +a) =$

$P(+b)\, P(+e)\, P(+a \mid +b, +e)\, P(+j \mid +a)\, P(+m \mid +a) =$

Example: Alarm Network

B	P(B)
+b	0.001
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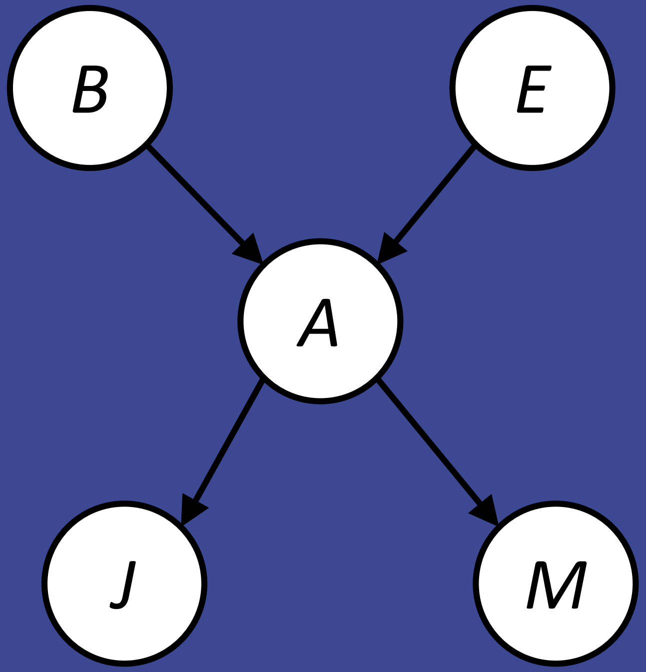

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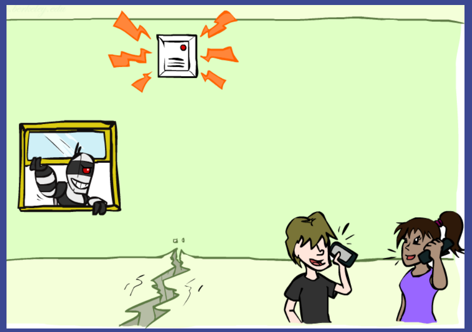

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-b	-e	+a	0.001
-b	-e	-a	0.999

$P(+b, +j, +m, +e, +a) =$

$P(+b)\ P(+e)\ P(+a|+b, +e)\ P(+j|+a)\ P(+m|+a) =$

0.001

Example: Alarm Network

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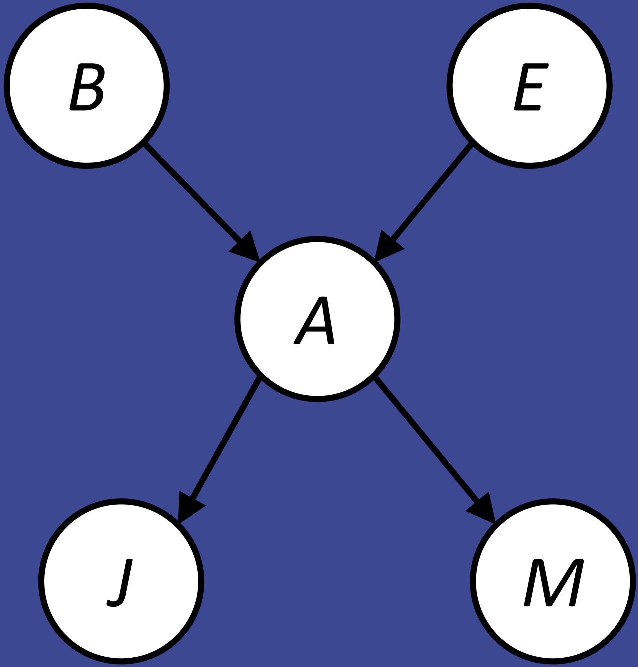

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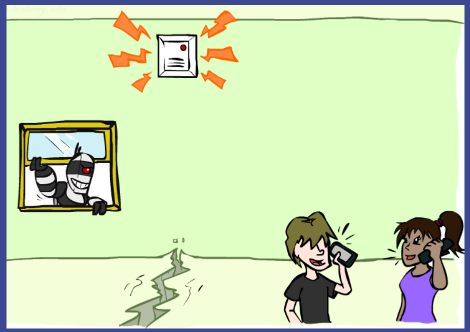

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$P(+b, +j, +m, +e, +a) =$

$P(+b)\ P(+e)\ P(+a|+b, +e)\ P(+j|+a)\ P(+m|+a) =$

0.001×0.002

Example: Alarm Network

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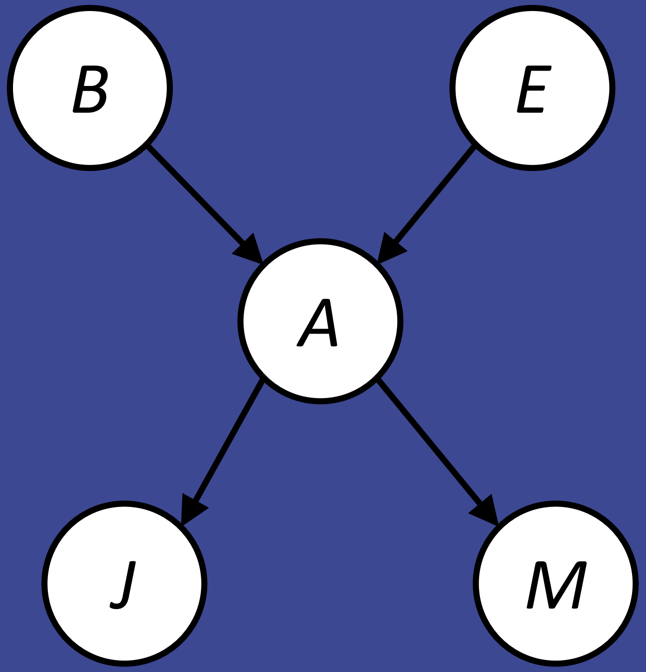

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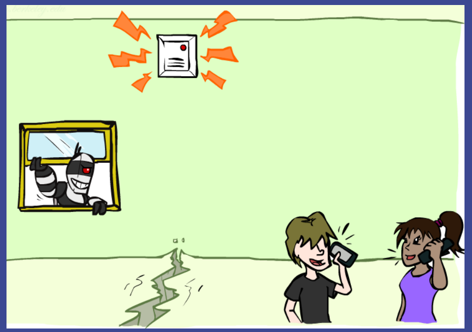

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$P(+b)\ P(+e)\ P(+a \mid +b, +e)$

0.001×0.002

iClicker:

What is $P(+a \mid +b, +e)$?

Example: Alarm Network

B	P(B)
+b	0.001
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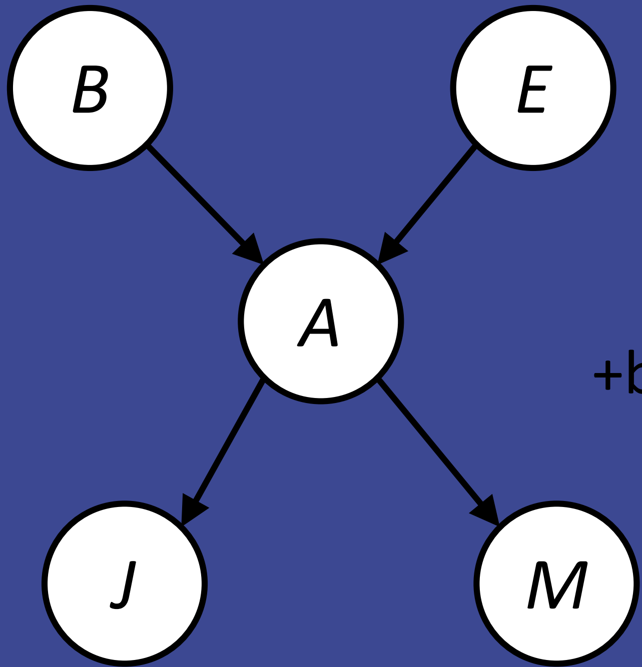

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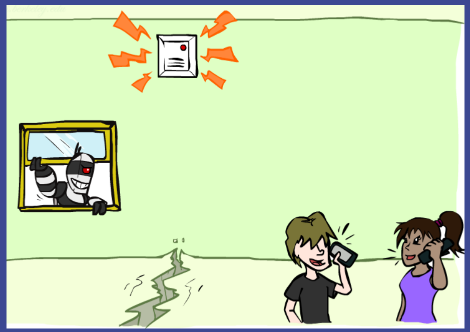

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$0.001 \times 0.002 \times 0.95$

Example: Alarm Network

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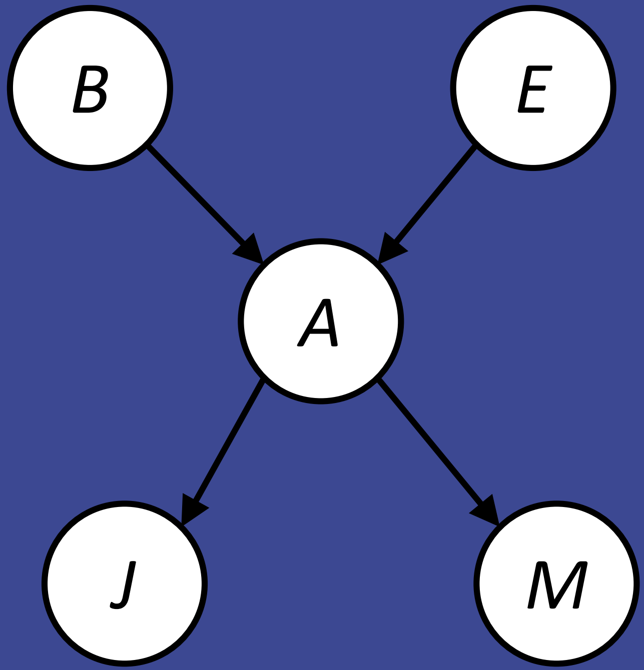

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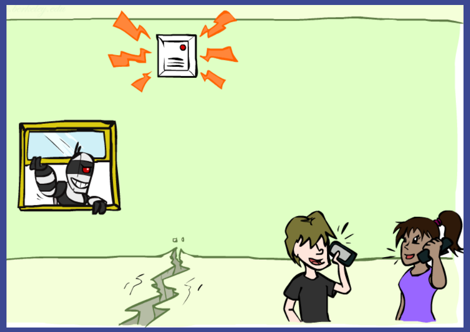

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$$P(+b, +j, +m, +e, +a) =$$

$$P(+b)\ P(+e)\ P(+a \mid +b, +e)\ P(+j \mid +a)\ P(+m \mid +a) =$$

$$0.001 \times 0.002 \times 0.95 \times 0.9 \times 0.7 = 1.2 \times 10^{-06}$$

Example: Alarm Network

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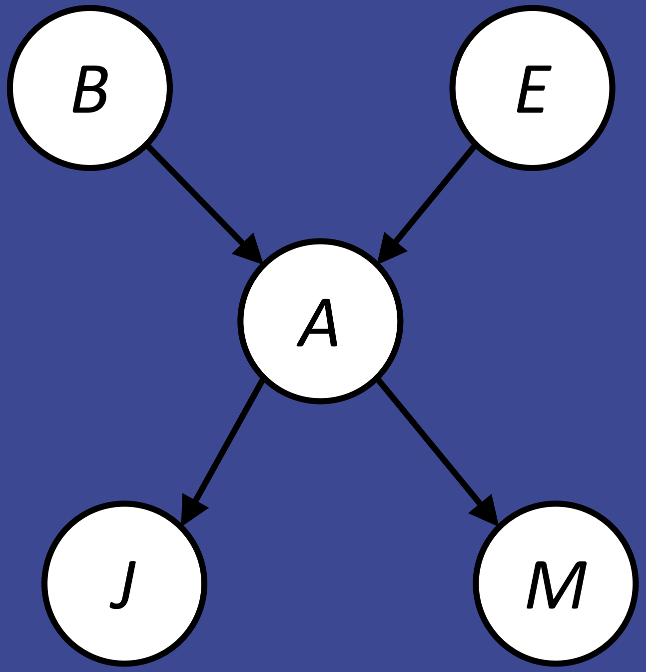

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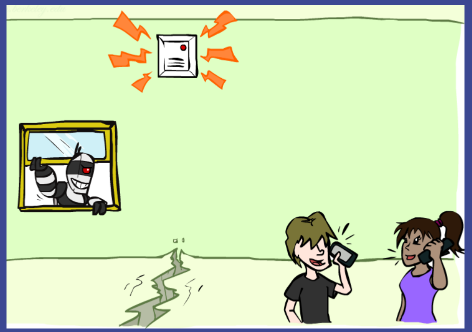

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$P(+b, +j, +m, +e, -a) =$

$P(+b)\, P(+e)\, P(-a|+b, +e)\, P(+j|-a)\, P(+m|-a) =$

Example: Alarm Network

B	P(B)
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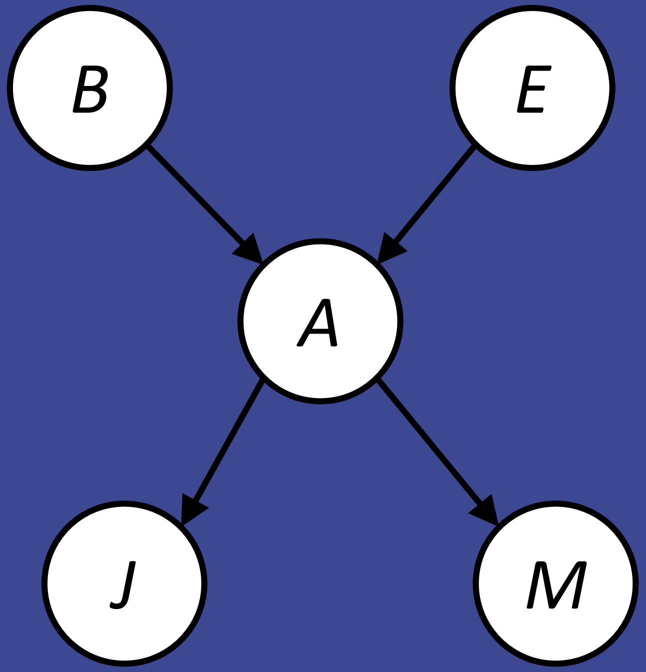
-b	0.999

E	P(E)
+e	0.002
-e	0.998



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-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

$P(+b, +j, +m, +e, -a) =$

$P(+b)\ P(+e)\ P(-a|+b, +e)\ P(+j|-a)$

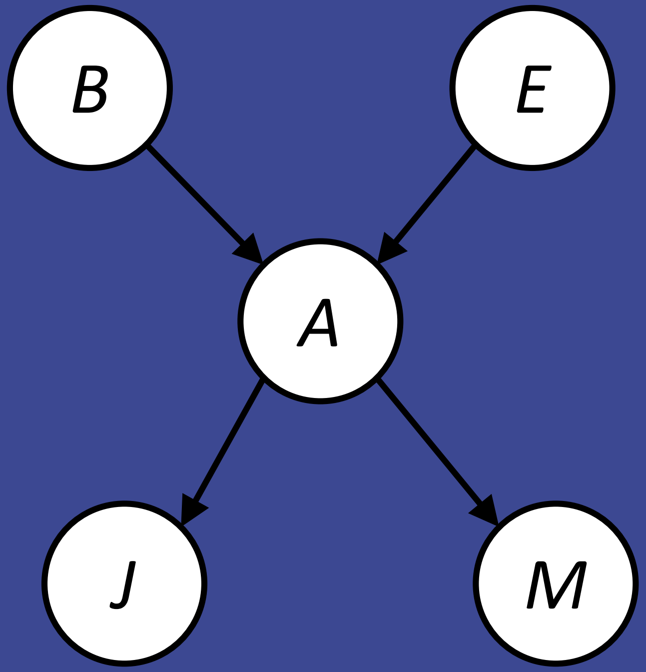
$0.001 \times 0.002 \times 0.05$

iClicker:

What is $P(+j|-a)$?

Example: Alarm Network

B	P(B)
+b	0.001
-b	0.999



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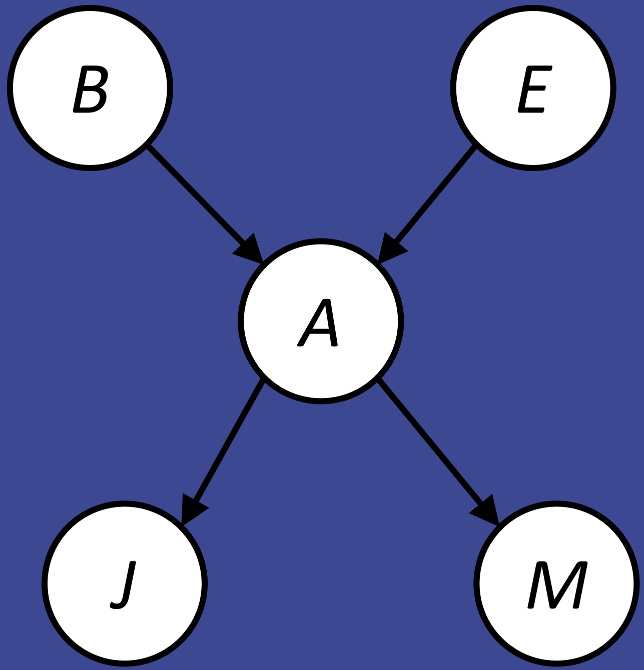
$P(+b, +j, +m, +e, -a) =$

$P(+b)\, P(+e)\, P(-a|+b, +e)\, P(+j|-a)\, P(+m|-a) =$

$0.001 \times 0.002 \times 0.05 \times 0.05 \times 0.01 = 5 \times 10^{-11}$

Example: Alarm Network

B	P(B)
+b	0.001
-b	0.999



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-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

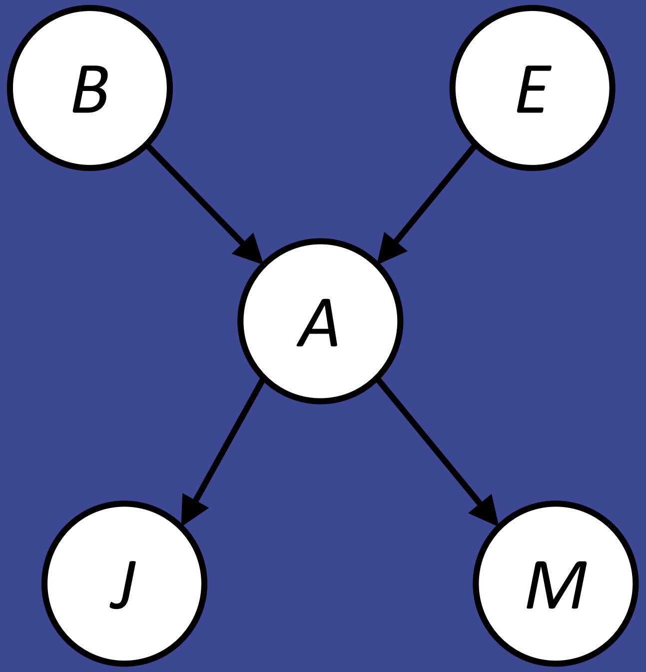
P(+b, +j, +m, **-e**, +a) =

P(+b) P(-e) P(+a | +b, -e) P(+j|+a) P(+m|+a) =

0.001 x 0.998 x 0.94 x 0.9 x 0.7 = 0.000591

Example: Alarm Network

B	P(B)
+b	0.001
-b	0.999



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+b	-e	+a	0.94
+b	-e	-a	0.06
-b	+e	+a	0.29
-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

A	J	P(J|A)
+a	+j	0.9
+a	-j	0.1
-a	+j	0.05
-a	-j	0.95

A	M	P(M|A)
+a	+m	0.7
+a	-m	0.3
-a	+m	0.01
-a	-m	0.99

$P(+b, +j, +m, -e, -a) =$

$P(+b)\, P(-e)\, P(-a \mid +b, -e)\, P(+j \mid -a)\, P(+m \mid -a) =$

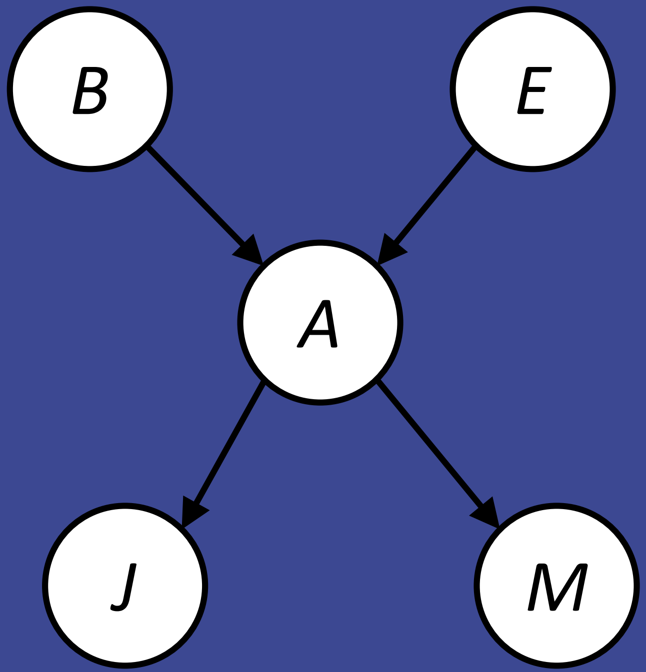
$0.001 \times 0.998 \times 0.06 \times 0.05 \times 0.01 = 3 \times 10^{-08}$

Inference by Enumeration in Bayes' Net

Putting it all together:

$$P(+b \mid +j, +m) = \frac{P(+b,+j,+m)}{P(+j,+m)}$$

$$\begin{aligned} P(+b, +j, +m) = {} & P(+b, +j, +m, +e, +a) + \\ & P(+b, +j, +m, +e, -a) + \\ & P(+b, +j, +m, -e, +a) + \\ & P(+b, +j, +m, -e, -a) \\ = {} & 0.000592 \end{aligned}$$

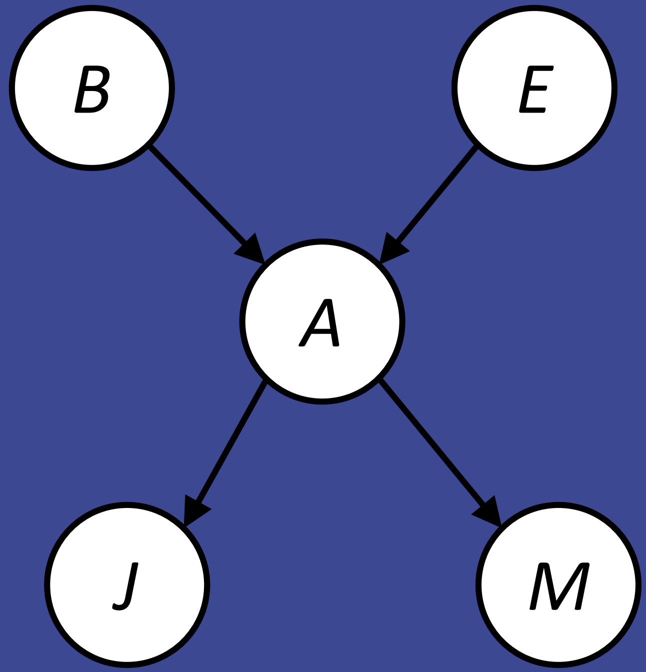


Inference by Enumeration in Bayes' Net

Similarly, we can calculate: P(-b | +j, +m)

$$P(-b \mid +j, +m) = \frac{P(-b,+j,+m)}{P(+j,+m)}$$

$$\begin{aligned} P(-b, +j, +m) = \; & P(-b, +j, +m, +e, +a) + \\ & P(-b, +j, +m, +e, -a) + \\ & P(-b, +j, +m, -e, +a) + \\ & P(-b, +j, +m, -e, -a) \\ = \; & 0.001492 \end{aligned}$$



Inference by Enumeration in Bayes' Net

P(-b, +j, +m) and P(+b, +j, +m) are the joint probabilities.

We still need to compute the conditional probabilities.

P(-b | +j, +m) = $\frac{P(-b,+j,+m)}{P(+j,+m)} = \frac{0.001492}{P(+j,+m)}$

P(+b | +j, +m) = $\frac{P(+b,+j,+m)}{P(+j,+m)} = \frac{0.000592}{P(+j,+m)}$

We normalize:

$Z = P(-b,+j,+m) + P(+b,+j,+m)$

= $0.001492 + 0.000592 = 0.002084$

P(-b | +j, +m) = 0.716

P(+b | +j, +m) = 0.284

Bayes Nets Summary

- Bayes nets compactly encode joint distributions
- Guaranteed independences of distributions can be deduced from BN graph structure
- Inference by enumeration is the basic exact inference method for Bayes nets. There are techniques (variable elimination) to make it more efficient
- There are also some approximate inference methods that include randomized sampling

Homework 6: beliefs.py

```
class Belief(object):
    """
    Belief class used to track the belief distribution based on the sensing evidence we have so far.
    Arguments:
    size (int): the number of rows/columns in the grid

    Attributes:
    open (set of tuples): set containing all the positions that have not been observed so far.
    current_distribution (dictionary): probability distribution based on the evidence observed so far.
        The keys of the dictionary are the possible grid positions
        The values represent the (conditional) probability that the treasure is found at that position
        given the evidence (sensor data) observed so far.
    """
```

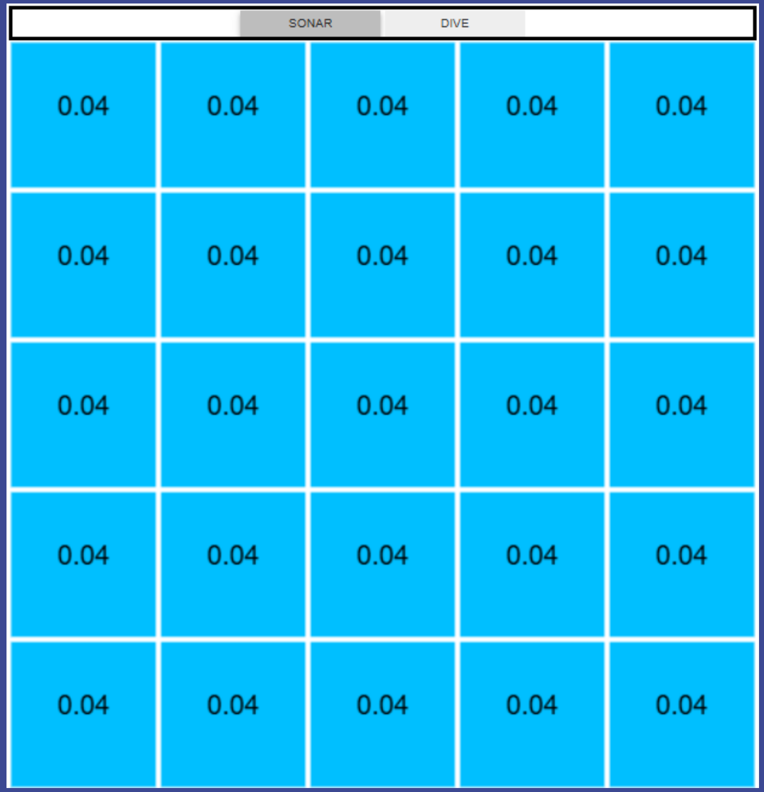
Homework 6: beliefs.py

```
class Belief(object):
    def __init__(self, size):
        # Initially all positions are open - have not been observed
        self.open = {(x, y) for x in range(size) for y in range(size)}
```

```
{(4, 0), (3, 4), (4, 3), (3, 1), (0, 2), (2, 2), (1, 0), (1, 3), (4, 2),
(3, 0), (3, 3), (0, 1), (2, 4), (1, 2), (0, 4), (2, 1), (3, 2), (4, 1),
(4, 4), (0, 0), (1, 1), (0, 3), (2, 0), (1, 4), (2, 3)}
```

```
    # Initialize to a uniform distribution
        self.current_distribution = {pos: 1 / (size ** 2) for pos in self.open}
```

```
{(4, 0): 0.04, (3, 4): 0.04, (4, 3): 0.04, (3, 1): 0.04, (0, 2): 0.04, (2,
2): 0.04, (1, 0): 0.04, (1, 3): 0.04, (4, 2): 0.04, (3, 0): 0.04, (3, 3):
0.04, (0, 1): 0.04, (2, 4): 0.04, (1, 2): 0.04, (0, 4): 0.04, (2, 1):
0.04, (3, 2): 0.04, (4, 1): 0.04, (4, 4): 0.04, (0, 0): 0.04, (1, 1):
0.04, (0, 3): 0.04, (2, 0): 0.04, (1, 4): 0.04, (2, 3): 0.04}
```



Inference in Homework 6: update

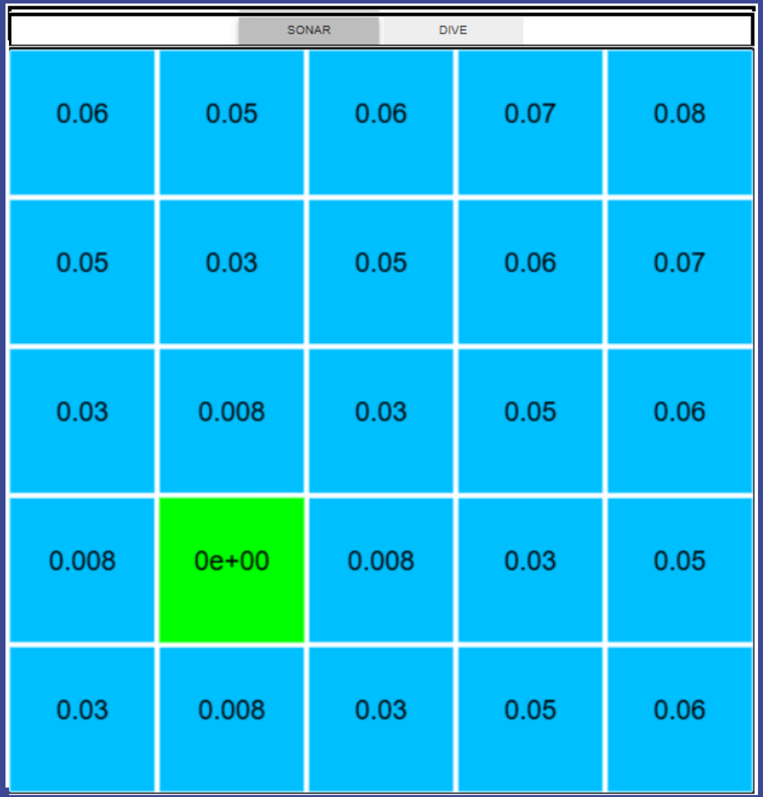
Prior distribution over treasure location: P(T)

- Initially uniform

- Sonar reading model: P(C|T)
  - Given: we know what our sensors do (use model.*pcolorgivendist*)
- C = sonar color measured at (1, 3)
- P(C = green | T for all treasure locations) from model
- We can calculate the posterior distribution P(T|C) over treasure locations given a sonar reading using Bayes' rule:

$$P(T|C) = \frac{P(T)\,P(C|T)}{P(C)} \;\alpha\; P(T)\,P(C|T)$$

α : is proportional

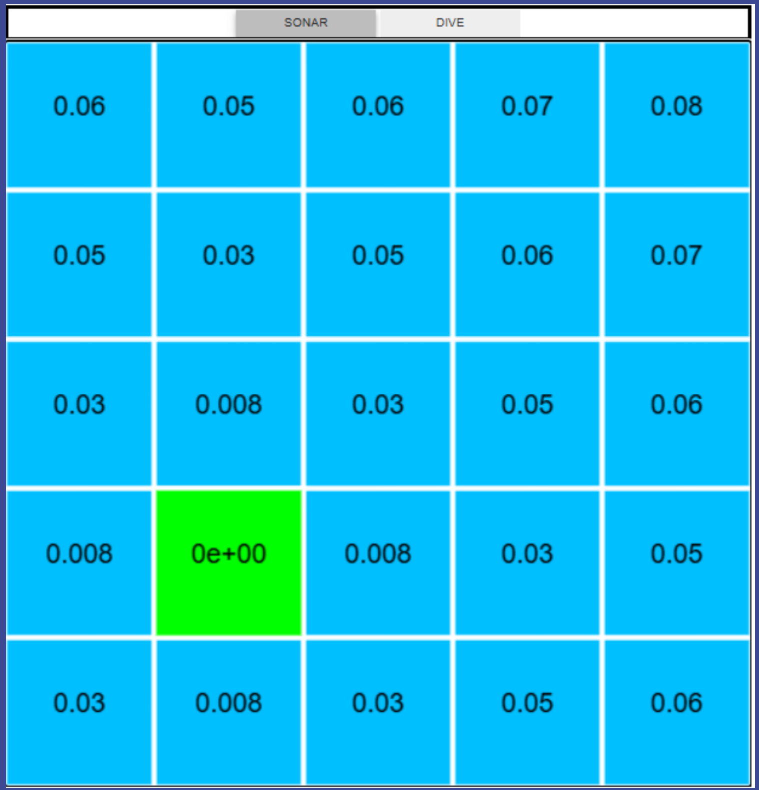


Inference in Homework 6: update

To compute the posterior distribution

$$P(T|C) = \frac{P(T)\,P(C|T)}{P(C)} \propto P(T)\,P(C|T)$$

1. Compute P(T) . P(C|T) for all possible treasure locations
2. Hint: store in self.current_distribution
3. Normalize (we have a single treasure and it must be in one location)
   - Add the products computed in 1 for all (x, y)
   - Hint: use *sum* and self.current_distribution.values()
   - Divide each value by the sum computed above



Inference in Homework 6: update

Prior distribution over treasure location: P(T)

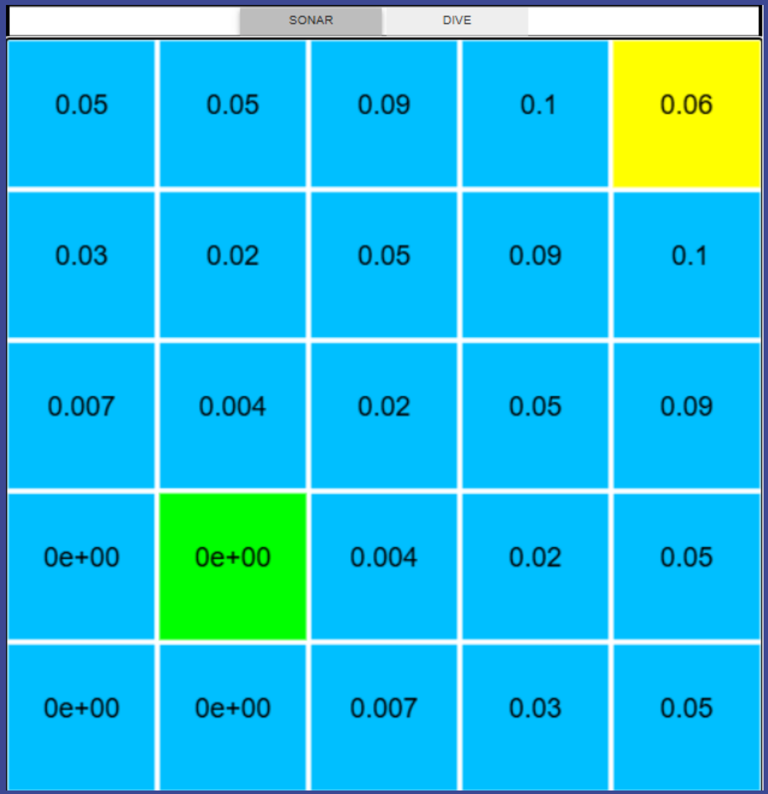
- no longer uniform

▸ C = sonar color measured at (4, 0)

▸ P(C = yellow | T for all treasure locations) from model

▸ We can calculate the posterior distribution P(T|C) over treasure locations given a sonar reading using Bayes' rule:

$$P(T|C) = \frac{P(T)\,P(C|T)}{P(C)} \quad \alpha \; P(T)\,P(C|T)$$

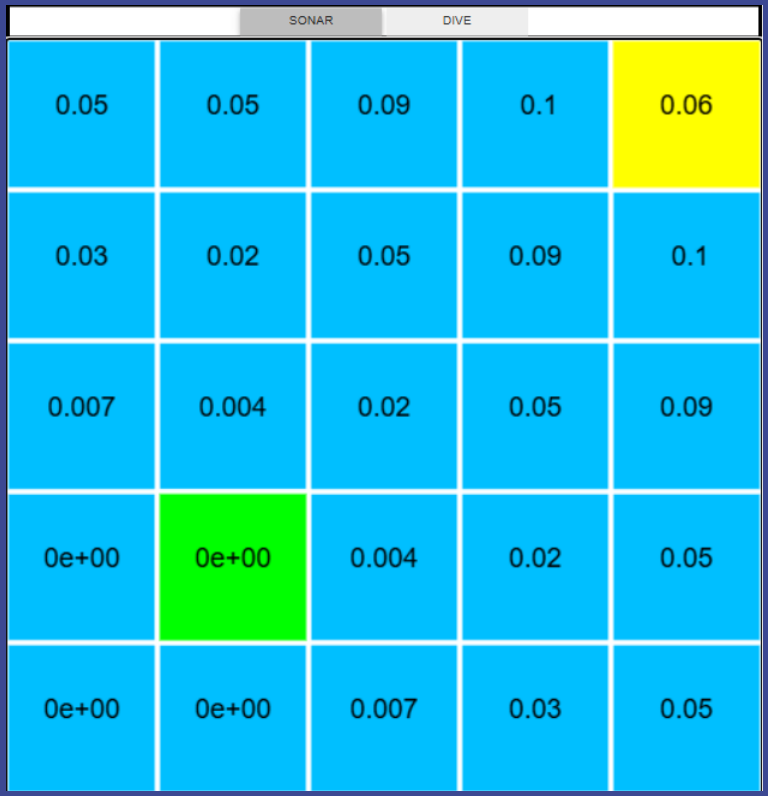


Inference in Homework 6: update

To compute the posterior distribution

$$P(T|C) = \frac{P(T)\,P(C|T)}{P(C)} \;\alpha\; P(T)\,P(C|T)$$

1. Compute P(T) . P(C|T) for all possible treasure locations
2. Hint: store in self.current_distribution
3. Normalize (we have a single treasure and it must be in one location)
   - Add the products computed in 1 for all (x, y)
   - Hint: use *sum* and self.current_distribution.values()
   - Divide each value by the sum computed above



Homework 6: recommend_sensing

Return the position where we should take the next sonar reading in the grid.

1. the most promising (highest probability) unobserved location: use *max* on the set of unobserved locations (self.open) with a lambda key
2. If all remaining unobserved locations have a probability of 0, return the unobserved location that is closest to the location with the highest probability: use the *closest_point* function in utils.
3. If there are no remaining unobserved locations, return the location with the highest probability: use *max* on the dictionary self.current_distribution with a lambda key.

Time and Uncertainty

- The world changes, we need to track it and predict it
- Often, we want to reason about a sequence of observations
  - Robot localization
  - Medical monitoring
  - Speech recognition…
- Need to introduce time into our models

Basic Idea

State random variables for each time step

- X_t = set of state random variables at time t

  State variables may be observable or unobservable

This assumes discrete time.

Bayes Net?

- Construct a Bayes net from these state variables?

$$X_1 \to X_2 \to X_3 \to \cdots \to X_{t-1} \to X_t$$

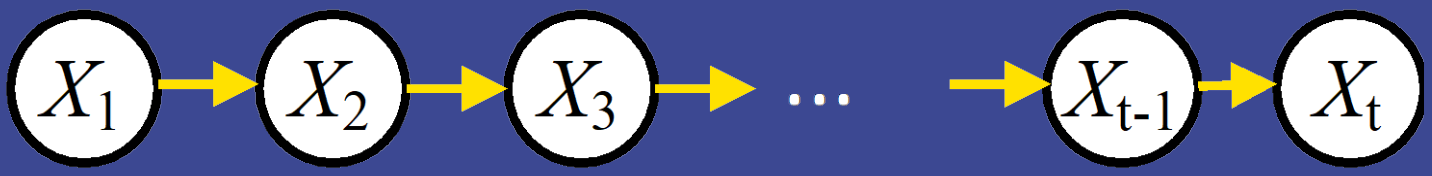
- Arcs / parents?
- Markov assumption: X_t depends only on a subset of $X_1, X_2, \ldots, X_{t-1}$
  - First-order Markov: state X_t at each time step only depends on the previous state X_{t-1}

$$X_t \perp\!\!\!\perp X_1, \ldots, X_{t-2} \mid X_{t-1}$$

$$P(X_t \mid X_{1:t-1}) = P(X_t \mid X_{t-1})$$

Notation: $X_{1:t-1} = X_1, X_2, \ldots, X_{t-1}$

Markov Chain



$P(X_1)$ $P(X_t|X_{t-1})$

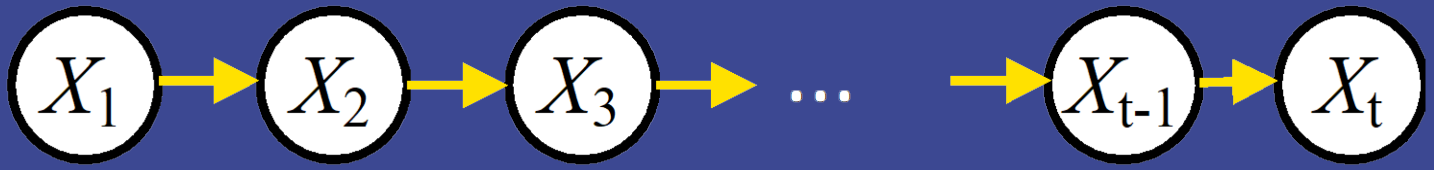
Parameters:

- initial state probabilities
- transition probabilities or dynamics, specify how the state evolves over time

Stationarity assumption: transition probabilities are the same at all times

$P(X_t \mid X_{t-1})$ is the same for all t

A Markov Chain is a Bayes Net



$P(X_1)$

$P(X_t|X_{t-1})$

$P(X_1, X_2, X_3) = ?$

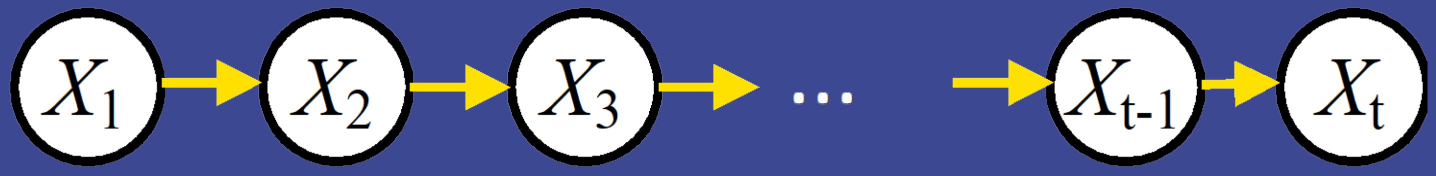
A. $P(X_1)\,P(X_2)\,P(X_3)$

B. $P(X_1|X_2)P(X_2\,|X_3)\,P(X_3)$

C. $P(X_1)P(X_2\,|X_1)\,P(X_3|X_2)$

D. $P(X_1) + P(X_2) + P(X_3)$

A Markov Chain is a Bayes Net



$P(X_1)$ $P(X_t|X_{t-1})$

$P(X_1, X_2, X_3) = P(X_1)P(X_2\,|X_1)\,P(X_3|X_2)$

More generally:

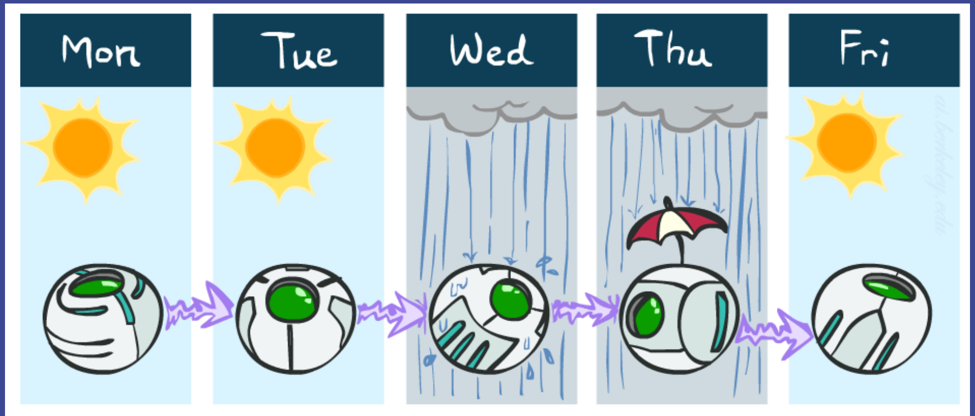
$P(X_1, X_{2\ldots}, X_t) = P(X_1)P(X_2\,|X_1)\ldots P(X_t|X_{t-1})$

Example Markov Chain: Weather

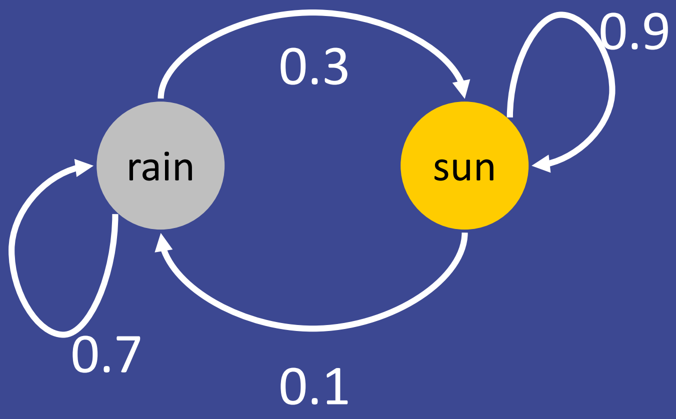
- States: X = {rain, sun}
- Initial distribution: 1.0 sun

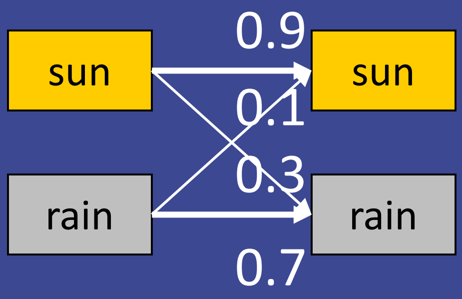
- CPT $P(X_t \mid X_{t-1})$:

X_{t-1}	X_t	$P(X_t \mid X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7



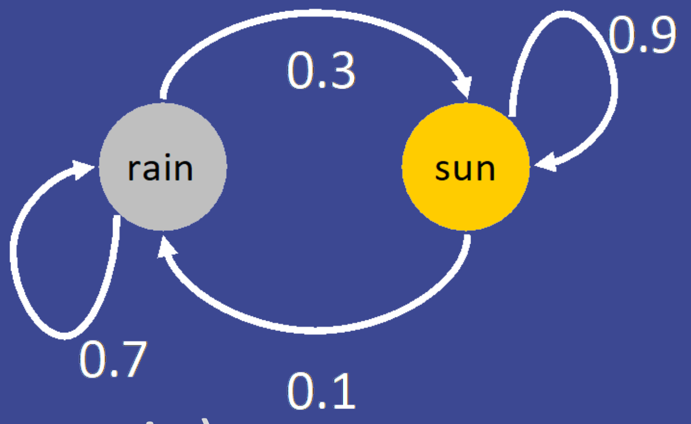
Two new ways of representing the same CPT





Example Markov Chain: Weather

- Initial distribution: $P(X_1 = \text{sun}) = 1.0$
- What is the probability distribution after one step/time unit?



$$P(X_2 = \text{sun}) = P(X_2 = \text{sun}, X_1 = \text{sun}) + P(X_2 = \text{sun}, X_1 = \text{rain})$$

$$= P(X_2 = \text{sun} \mid X_1 = \text{sun})\, P(X_1 = \text{sun}) +$$

$$P(X_2 = \text{sun} \mid X_1 = \text{rain})\ \ P(X_1 = \text{rain})$$

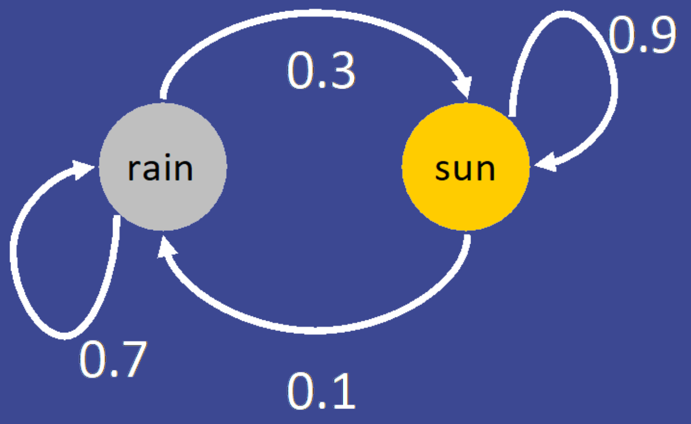
$$= 0.9 \,.\, 1.0 + 0.3 \,.\, 0.0 = 0.9$$

iClicker:

What is $P(X_2 = \text{rain})$?

Example Markov Chain: Weather

- Initial distribution: $P(X_1 = \text{sun}) = 1.0$
- What is the probability distribution after one step/time unit?



$$P(X_2 = \text{sun}) = P(X_2 = \text{sun}, X_1 = \text{sun}) + P(X_2 = \text{sun}, X_1 = \text{rain})$$

$$= P(X_2 = \text{sun} \mid X_1 = \text{sun})\, P(X_1 = \text{sun}) +$$

$$P(X_2 = \text{sun} \mid X_1 = \text{rain})\ \ P(X_1 = \text{rain})$$

$$= 0.9 \,.\, 1.0 \ + 0.3 \,.\, 0.0 = 0.9$$

$$P(X_2 = \text{rain}) = 1 - 0.9 = 0.1$$

Reminders

- Homework 6: due April 22