Analyzing Competitive Gerrymandering with Combinatorial Game Theory

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April 27, 2023

Abstract

In this paper we investigate the game of competitive gerrymandering, played on an $a \times b$ grid where players take turns creating districts, a set of d contiguous squares, under normal playing conditions. We begin by review the theory of impartial games. We characterize most triples (a, b, d) as either a P1 win or a P2 win. We study the game of linear gerrymandering ($d = 1$). We investigate boards $a \times b$ for which the game is a first player win for all d. We investigate and conjecture that the nim value of the game (n, n, n) is n. We show that for $d \geq O(\sqrt[3]{n^4})$, P1 can automatically win. All results are motivated by code, from which we have the nim-value of thousands of (a, b, d) triples, as well as a highly efficient computation method using dynamic programming and a depth first search.

Contents

1 Background

Gerrymandering is the process by which the electoral district boundaries of a certain area are redrawn by a party legislature in an effort to maximize their control. For instance, a party legislature may redraw the boundaries of a district such that their party holds a plurality within the majority districts and concentrating as much of the opposition party into as few districts as possible, diluting the opposition's overall influence. In the context of the United States, partisan gerrymandering, a process where district boundaries are drawn unfairly to benefit one party over the other, has become an increasingly contentious issue, and has been the subject of Supreme Court cases such as *Vieth v. Jubelirer (2004), Gill v. Whitford (2018),* and *Rucho v. Common Clause (2019).*

In this work, we approach competitive gerrymandering as a process where each of the two competing sides take turns to draw districts. Each move consists of a specific number of contiguous squares within the pre-defined game board and the side that is unable to draw first loses.

1.1 Impartial Games

A type of combinatorial game, an impartial game is one where each of the players take turns placing their moves and all players have access to the same range of moves. For instance, games like chess are not impartial as the player can move only either the white or black pieces, while a game like green Hackenbush is as both players can make identical moves. Thus, these games have perfect information and the winner is determined solely by which player starts first, assuming optimal moves are made. According to the Sprague-Grundy theorem, every impartial game can be represented by a single Sprague-Grundy value, also called a nimber. This nimber n represents an equivalent game of one stack of n nim sticks and is represented with a $*$ symbol followed by an integer. The nimbers of multiple games can be summed using nimber arithmetic (XOR), where if the nim-sum $= 0$, then the game is a second-player win, while if the nim-sum $\neq 0$, the game is a first-player win.

1.2 Octal Games

A generalization of combinatorial games such as Nim and Kayles, Octal games are a type of game where multiple players take turns to remove items (such as tokens or nim-sticks) from a pile of items. These games are also impartial. In normal Octal games, the objective is to be the last person to remove items from the pile, while the misère version of this game is the opposite, with the first person being unable to remove items winning.

Each octal game is assigned an octal code starting with "0." followed by d_i for each subsequent digit which represents the sum of whether zero, one, or two heaps can be left in the game after the removal of n tokens. For calculating each sum for d_i, 1 is added if players can leave zero heaps, 2 is added if players can leave one heap, and 4 is added if players can leave two heaps. Examples of Octal games and their codes include Nim, which is 0.3333..., Kayles, which is 0.77, and Dawson's Kayles, which is 0.07. Additional Octal games can be found on the On-Line Encyclopedia of Integer Sequences (OEIS).

1.3 Partisan Games vs. Impartial Games

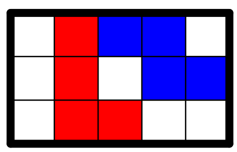
Partisan games are a type of combinatorial games which are not impartial, meaning that some moves are only available to one of the players and not the other. As aforementioned, this contrasts with impartial games where both players have access to the same series of moves. The Sprague-Grundy theorem applies only to impartial games.

A representative example of a partisan game is Domineering or Crosscram[1], where one player can only use horizontal dominoes and the other can only use horizontal dominoes to fill a predefined rectangular board. The impartial version of Crosscram is called Cram[1], where both players can use both horizontal and vertical dominoes.

2 Problem Statement

The titular "competitive gerrymandering" refers to a novel game that attempts to generalize Cram ($d = 2$ case), inspired by the district drawing process in real-life gerrymandering.

The game is played on $a \times b$ grid, as pictured below



Players take turns marking d contiguous squares (each of the d squares must share an edge with one of the other squares). The first player who is unable to mark such a contiguous set of d squares loses.

In the rest of this paper, we analyze special cases and make conjectures for various (a, b, d) triples.

3 The First-Player Wins (Usually)

For most triples (a, b, d), the game is a first player win. We by no means claim to have characterized which triples are a first player win, but offer many results giving many classes of games which are a first player win.

This means that the nim-value is almost always positive, and this allows us to analyze nim values in the rest of the paper. (P1 win if and only if nim value is $\neq 0$)

Proposition 3.1. *For $d = 1$, it's a second player win if and only if ab is even.*

Proof. There are a total of ab available moves, and nothing can change that. In fact, the nim value of the positive is $ab\%2$. (remainder when ab is divided by 2) □

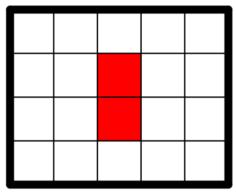
Proposition 3.2. *For $d = 2$, $\times$ boards are P2 win, and odd $\times$ even are P1 win.*

.	even	odd
even	P2	P1
odd	P1	.

Table 1: $d = 2$

Proof. For Part 1, note that the second player can adopt a mirroring strategy, where they take P1's move and reflect it across the central vertex of the *even* $\times$ *even*.

For Part 2, P1 can place a domino on the *odd* $\times$ *even* to make a symmetric game. The opener should look like:



□

Proposition 3.3. *For odd d*

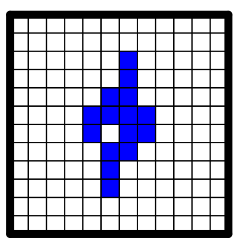
.	even	odd
even	P1 win for $d \geq 7 + \min(a, b)$	P1 win for $d \geq 6 + \min(a, b)$
odd	P1 win for $d \geq 6 + \min(a, b)$	P1

Table 2: $v_2(d) = 0$

Odd $\times$ odd boards are a first player win. On their first move, player 1 marks the center square and pairs of $(\mathscr{C}, \mathscr{C}')$, where $\mathscr{C}'$ is the reflection of $\mathscr{C}$ over the center. Priority goes towards pairs along the central row, to prevent P2 from being able to wrap around.

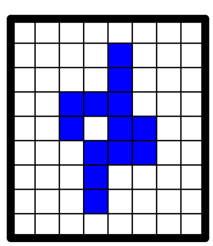
After doing this, P1's strategy is to mirror, whenever P2 marks some squares, P1 marks the reflections of those squares. This clearly gives P1 a win as long as P2 can't mark two squares which are reflections of one another, and since P1 has filled the central row, P2 cannot wrap around and hit two squares which are reflections of each other. □

even $\times$ even. For $d \geq 7 + \min(a, b)$, we can do the following



□

odd$\times$even. We do this for $d \geq 6 + odd$.



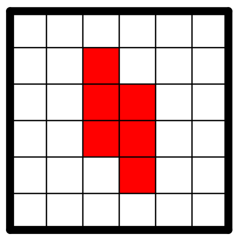
□

Proposition 3.4. *For all $d \equiv 2 \pmod 4$, odd $\times$ even boards are a first player win.*

.	even	odd
even	P1 win for $d \geq 6$	P1
odd	P1	$d \geq 10$ (See Proposition 3.5 for construction)

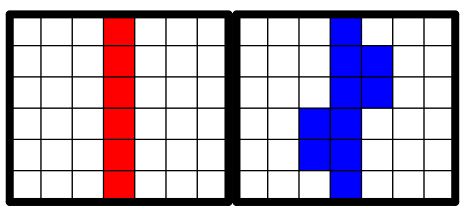
Table 3: $v_2(d) = 1$

even×even. P1 can open with the following, and like before finish the win by mirroring.



□

odd×even. The player chooses the central column so that there are the same number of columns on each side. Then, expanding from the center it places pairs. Here are two example openers (for $d = 6$ and $d = 10$) that create symmetry, and thus a win:



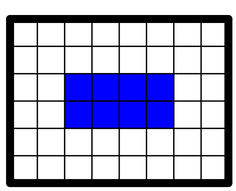
□

Proposition 3.5. *General $d \equiv 0 \pmod 4$.*

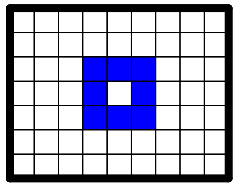
.	even	odd
even	P1	P1
odd	P1	P1 win for $d \geq 8$

Table 4: $v_2(d) \geq 2$

Proof. Firstly, for *even* $\times$ *even*



For *even* $\times$ *odd*, the argument in Proposition 3.4 works. Finally, for *odd* $\times$ *odd*, with $d \geq 8$, we can form a ring



For $d = 8$, one would need a chain of $d = 9$ to get two squares which are the reflections of each other across the center. For $d \geq 12$, extending out towards the edges works. Thus, for $d \geq 8$, one can create an opening from which mirroring finishes. $\square$

4 Linear Gerrymandering

The case of games where $a = 1$ are especially easy to analyze. They are in fact Octal games of the form $0.0^{d-1}7$.

$d = 2$ is placed with dominoes, has octal value of 0.07, and is completely solved.[2] In fact, it "turns out to be periodic from $n = 53$ onwards, with a period of 34." [4]

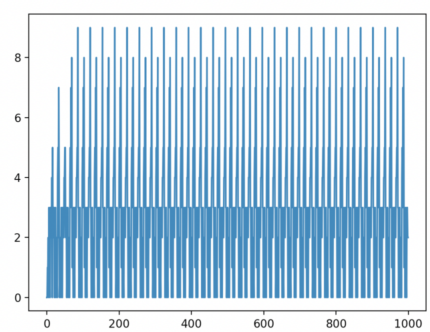
Proposition 4.1. *If $d \equiv n \pmod 2$, the game is a first player win.*

Proof. By splitting into two symmetric $1 \times \frac{n-d}{2}$ boards, P1 clearly wins. $\square$

Question 4.2. *For $d \geq 3$, is the nim-value eventually periodic?*

Based on the fact that $d = 2$ is periodic, one would expect that $d \geq 3$ will also be periodic because they share basically the same structure. However, based on experimental data, this doesn't seem true.

Compare $d = 2$ data (which is periodic)



with

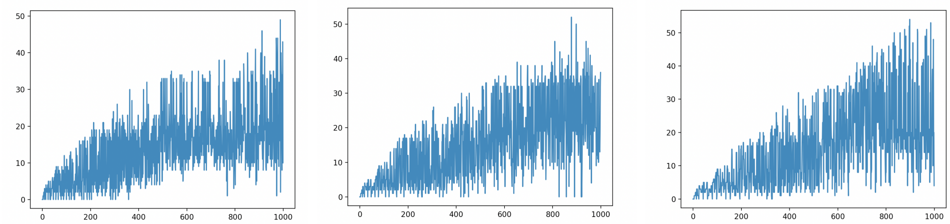
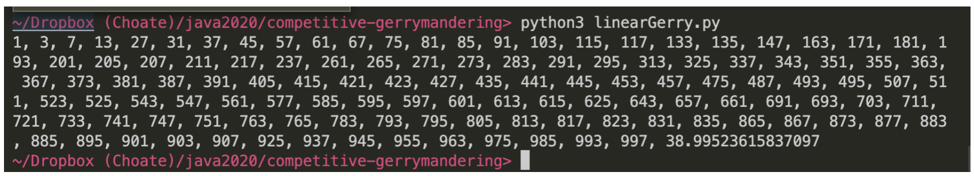


Figure 1: $d = 3, 4, 5$ (Left to Right)

4.1 All P1 Win

For $n < 1000$, a $1 \times n$ board is always a first player win for the following values:



```
~/Dropbox (Choate)/java2020/competitive-gerrymandering> python3 linearGerry.py
1, 3, 7, 13, 27, 31, 37, 45, 57, 61, 67, 75, 81, 85, 91, 103, 115, 117, 133, 135, 147, 163, 171, 181, 1
93, 201, 205, 207, 211, 217, 237, 261, 265, 271, 273, 283, 291, 295, 313, 325, 337, 343, 351, 355, 363,
 367, 373, 381, 387, 391, 405, 415, 421, 423, 427, 435, 441, 445, 453, 457, 475, 487, 493, 495, 507, 51
1, 523, 525, 543, 547, 561, 577, 585, 595, 597, 601, 613, 615, 625, 643, 657, 661, 691, 693, 703, 711,
721, 733, 741, 747, 751, 763, 765, 783, 793, 795, 805, 813, 817, 823, 831, 835, 865, 867, 873, 877, 883
, 885, 895, 901, 903, 907, 925, 937, 945, 955, 963, 975, 985, 993, 997, 38.99523615837097
~/Dropbox (Choate)/java2020/competitive-gerrymandering>
```

We also observed that 3×5 and 3×7 are first player wins for all values of d.

5 $a = b = d$ case

We also consider the case when $a = b = d$. We let this game be G_n. Then we have the following values:

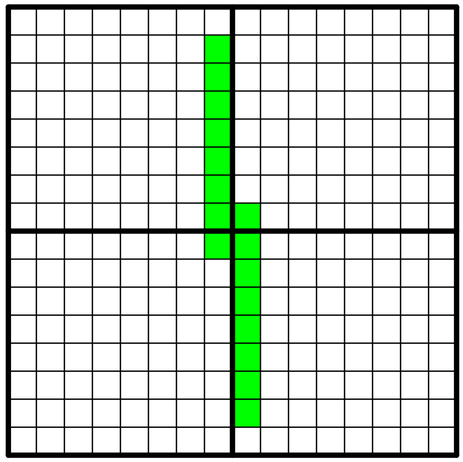
n	Value
1	1
2	0
3	3
4	1
5	5

Table 5: n values

Proposition 5.1. *For all $n \neq 2$, G_n is a first player win.*

Proof. If n is odd, place a dividing line straight down the middle, creating two $\frac{n-1}{2} \times n$ rectangles, at which point a mirroring strategy wins.

For even n, write $n = 2k$ where $k \geq 2$. Then, we make a move like the following:



At which point the board is symmetric, and whenever P2 marks some set $\mathscr{S}$, we mark the reflection of $\mathscr{S}$ over the center of the grid. $\square$

5.1 Conjectures

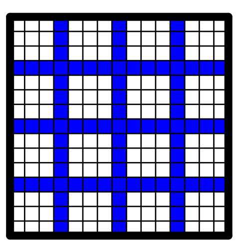
Conjecture 5.2. *The value of G_n for odd n is n.*

This is reasonable since the upper bound on the nim-value is n, as at most n pieces can be placed down. Additionally, each move has lots of freedom, which means that we should, in theory, be able to hit lots of nim-values, and thus the mex will always be large.

6 Games with Large d

Theorem 6.1. *On an $n \times n$ board, the minimum value of d for which the first player can win on move 1 is $O(\sqrt[3]{n^4})$. Specifically, it is $\approx \sqrt[3]{4n^4}$.*

Construction: This works by drawing $\frac{d}{2n} = \sqrt[3]{\frac{n}{2}}$ lines both vertically and horizontally. This splits the board into a bunch of squares with side length $\frac{n}{\sqrt[3]{\frac{n}{2}}} = \sqrt[3]{2n^2}$, so each square has area $\sqrt[3]{4n^4} < d$. $\square$



Bound: Note that if you split the board into regions of area d, then each of these has side length at least $O(\sqrt{d})$. Since there are $\frac{n}{d}$ regions, we need at least $\frac{n^2}{d} \cdot \frac{1}{2} O(\sqrt{d}) = n^2 \cdot O(\frac{1}{\sqrt{d}})$ squares to win immediately as player 1. Thus, $n^2 \cdot O(\frac{1}{\sqrt{d}}) = d$, so $d = O((n^2)^{\frac{3}{2}}) = O(n^{\frac{4}{3}})$. $\square$

Conjecture 6.2. *For all $d \geq n$, it's a first player win.*

We believe that this is likely true because the first player has lots of freedom in how to split the $n \times n$ game into 2 halves, and presumably P1 will be able to make both games have the same nim-value.

References

[1] Martin Gardner. Mathematical games: Cram, crosscram and quadraphage: new games having elusive winning strategies. *Scientific American*, 230(2):106–108, 1974.

[2] Richard K. Guy and Cedric A. B. Smith. The G-values of various games. *Proceedings of the Cambridge Philosophical Society*, 52(3):514–526, July 1956.

[3] Paul Ottaway. Analysis of three new combinatorial games. 2003.

[4] Jos Uiterwijk. *Solving Cram Using Combinatorial Game Theory*, pages 91–105. 12 2020.

7 Appendix

7.1 Pseudocode

The code uses Dynamic Programming (DP) to save past values, meaning it calculates the nim-values of $\frac{1}{d}2^{ab}$ possible board positions, and then uses DFS to find and evaluate all possible moves. The number of move pieces (`https://oeis.org/A001168`) is exponential, which means that we can't speed it up very much. An upper bound on the time complexity of $dp(a, b, d)$ is

$$O(2^{ab} \cdot ab \cdot d \cdot 4.6496^d / 2^d)$$

7.2 Github Code Repository

All code used for this project can be found at:

`https://github.com/YunruiRyanYang/competitive-gerrymandering`.

The most relevant code files are dpWithDFS.java, which has the function $dp(a, b, d)$ which returns the nim value of the game played on $a \times b$ with piece size d, and linearGerry which can compute nim values of $1 \times n$ games played with pieces of size d very quickly.

8 Specific values

Nim-values that we have computed for tetrominoes. Bolded numbers refer to games where the nim-value is $\lfloor \frac{ab}{d} \rfloor$, *i.e.*the maximum possible value. To see the values listed for each $a \times b$ board, for $d = 1, 1, \ldots, ab$, see:

`https://github.com/YunruiRyanYang/competitive-gerrymandering/blob/main/infinite_gerry3a.txt`

8.1 Domino Cramming(Cram) - $d = 2$

This first table is supplemented by previous work. Unfortunately our computational model, while more generalizable, is not as efficient as the graph-based representation others use to compute large values.

	1	2	3	4	5	6	7	8
1	0	1	1	2	0	3	1	1
2	.	0	1	0	1	0	1	0
3	.	.	0	1	1	4	1	3
4	.	.	.	0	2	0	3	0
5	.	.	.	.	0	2	1	1
6	.	.	.			0	5	0
7	.	.					1	
8	.	.						

Table 6: Domino Values. Large Values from Wikipedia.

8.2 Trimino Cramming - d=3

Here are the values for $d = 3$ for each board size:

	1	2	3	4	5	6	7	8
1	**0**	**0**	**1**	**1**	**1**	**2**	**2**	0
2	.	**1**	**2**	0	**3**	1	0	3
3	.	.	**3**	0	**5**	0	1	0
4	.	.	.	2	0	1		
5	.	.	.	.	1			
6	.	.	.					
7	.	.						
8	.	.						

Table 7: Trimino Values

8.3 Tetromino Cramming - d=4

Here are the values for $d = 4$ for each board size:

	1	2	3	4	5	6	7	8
1	0	0	0	1	1	1	1	2
2	.	1	1	2	2	3	3	1
3	.	.	2	3	1	1		
4	.	.	.	1	5			
5	.	.	.	.				
6	.	.	.					
7	.	.						
8	.	.						

8.4 Pentomino Cramming - d=5

Here are the values for $d = 5$ for each board size:

	1	2	3	4	5	6	7	8
1	0	0	0	0	1	1	1	1
2	.	0	1	1	2	2	2	3
3	.	.	1	2	3	3		
4	.	.	.	3	1			
5	.	.	.	.	5			
6	.	.	.					
7	.	.						
8	.	.						

Table 8: Pentomino Values

8.5 Hexamino Cramming - d=6

Here are the values for $d = 6$ for each board size:

For $d = 6$, we note that for all $a \times b$ table sizes we have computed, the game is a first player win. We conjecture that the larger d is compared to (a, b), the more likely it is that the game is a first player win.

	1	2	3	4	5	6	7	8
1	0	0	0	0	0	1	1	1
2	.	0	1	1	1	2	2	2
3	.	.	1	2	2	3		
4	.	.	.	2	3			
5	.	.	.	.				
6	.	.	.					
7	.	.						
8	.	.						