EGMO Chapter 8: Inversion

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§1 Problem 8.25

Problem 1.1. Let A, B, C be three collinear points and P be a point not on this line. Prove that the circumcenters of $\triangle PAB, \triangle PBC$, and $\triangle PCA$ lie on a circle passing through P.

Solution. Let X, Y, Z be the centers of $\triangle PAB, \triangle PBC$, and $\triangle PCA$. We invert around P with radius 1.

Thus, due to lemma 8.10, we have that X' is the reflection of P over $A'B'$.

However, as a property of inversion we will have that $ABC \to A'B'C'$ where $A'B'C'$ is a circle which passes through P. Thus, by Simson's the feet from P to each side are collinear, and a homothety of factor $\times 2$ gives us that X', Y', Z' are collinear.

Inverting back, this clearly gives us that $P \in (XYZ)$ and we are done. □

§2 Problem 8.26 - BAMO 2008/6

Invert about D with a circle of radius 1. Note that $(BDCA_1)$ cyclic implies $B'A_1'C'$ are all collinear. Similar results by symmetry on other sides.

Then,

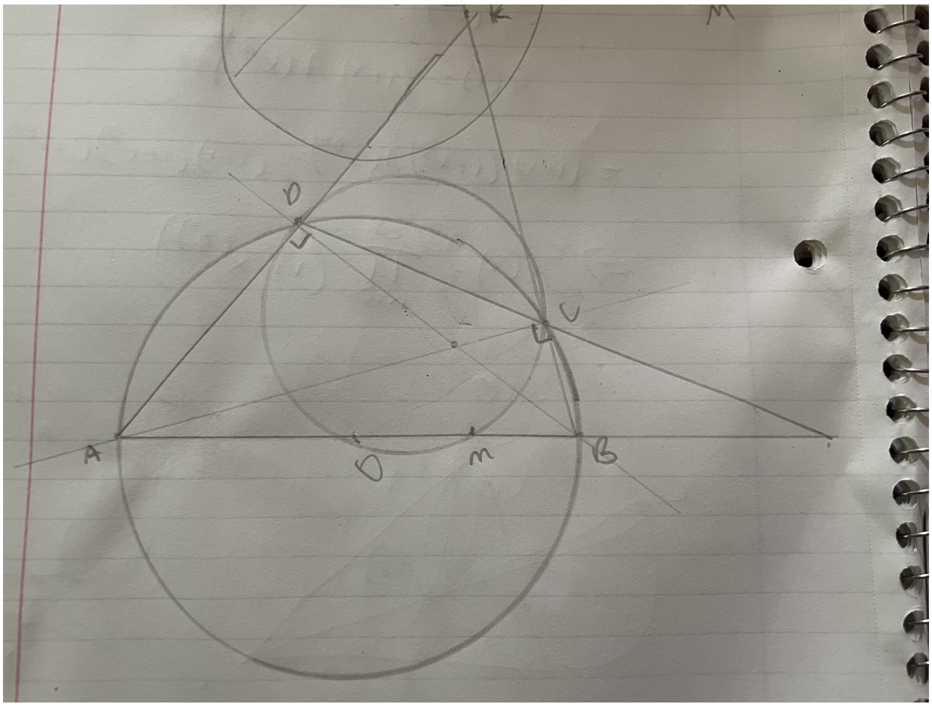
$$\frac{AD}{AA_1} = \frac{AD}{AD + DA_1} = \frac{\frac{1}{AD'}}{\frac{1}{AD'} + \frac{1}{DA_1'}} = \frac{DA_1'}{DA_1' + A'D} = \frac{DA_1'}{AA_1'}$$

Thus sum of this becomes, with second equality due to Area ratio's.

$$\frac{AD}{AA_1}+\frac{BD}{BB_1}+\frac{CD}{CC_1}=\frac{DA_1'}{AA_1'}+\frac{DB_1'}{BB_1'}+\frac{DC_1'}{CC_1'}=\frac{[B'D'C']+[C'DA']+[A'DB']}{[A'B'C']}=1$$

and we are done $\blacksquare$.

§3 Problem 8.27 - Iran Olympiad 1996

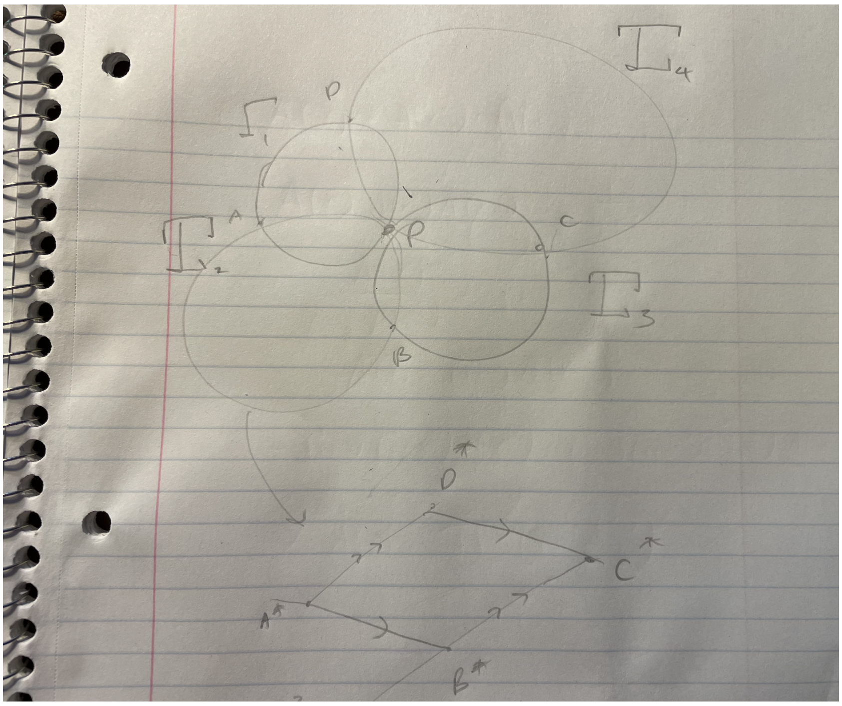


Invert about ω, the semicircle. Note that A, B, C, D were free in the original diagram, and will be sent to themselves post inversion. Next, since $M = AB \cap CD$, we have that $M^* = AB \cap (COD)$.

Next, $K = (AOC) \cap (BOD)$, since (AOC) passes through the center of inversion, it will get sent to a line through AC. Similarly $(BOD) \to \overline{BD}$. Thus, we have $K^* = AD \cap BC$.

Now, simply note that (COD) is the 9-point circle of $\triangle KAB$, thus M must be the foot of the altitude from K so we have $\angle KMO = 90$ and we are done $\blacksquare$.

§4 Problem 8.28 - Shortlist 2003/G4



We rename P to O. Invert about O with radius 1. Note that since Γ_1 is tangent to Γ_3, we will have $A^*D^* \parallel B^*C^*$, and similarly Γ_2 tangent to Γ_4 implies that $A^*B^* \parallel C^*D^*$.

Thus, $A^*B^*C^*D^*$ is a parallelogram, so $A^*B^* = C^*D^*$ and $A^*D^* = B^*C^*$.

Now, we evaluate using Inversion Distance Formula,

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{\frac{A^*B^*}{OA^*OB^*} \cdot \frac{B^*C^*}{OB^*OC^*}}{\frac{AD^*}{OA^*OD^*} \cdot \frac{D^*C^*}{OD^*OC^*}} = \frac{OD^{*2}}{OB^{*2}} \cdot \frac{A^*B^* \cdot B^*C^* \cdot OA^* \cdot OC^*}{C^*D^* \cdot A^*B^* \cdot OA^* \cdot OC^*} = \frac{OD^{*2}}{OB^{*2}} = \frac{OB^2}{OD^2}$$

and we are done $\blacksquare$.

§5 Problem 8.29 - IO passes through the centroid of the contact triangle

Note that due to Lemma 8.11, the inversion with respect to the incircle sends Γ, the circumcircle, to the nine-point circle of $\triangle DEF$. Let the nine-point circle have center N_9.

Now, by proposition 8.6, we have that the centers of the two circles are collinear with the center of inversion, so I, O, N_9 are collinear.

However, the Euler Line of $\triangle DEF$ is I(circumcenter), G_1(median) and N(nine-point center of $\triangle DEF$. Thus, combining ION and IG_1N, we have IOG_1 collinear and we are done.