9.3.1 Generalization for Weighted Process Noise

We can modify the dynamics equation to weight the process noise:

$$x(k+1) = F(k)x(k) + G(k)u(k) + \Gamma(k)v(k) \tag{9.28}$$

Setting $\Gamma(k) = I$ recovers the previous (unweighted) form. With this generalization, the only change in the Kalman filter is in the covariance propagation:

$$\bar{P}(k+1) = F(k)P(k)F^T(k) + \Gamma(k)Q(k)\Gamma^T(k) \tag{9.29}$$

9.4 MAP-based Kalman Filter Derivation

For the linear Kalman filter under our discrete-time stochastic-linear-time-varying (SLTV) problem, the pdfs are all Gaussian. As a result, $\hat{x}_{\mathrm{MAP}}(k) = \hat{x}_{\mathrm{MMSE}}(k) = \mathbb{E}\left[x(k)|Z^k\right]$ (i.e., MAP and MMSE are equivalent). For nonlinear systems, MAP and MMSE are not necessarily equivalent, and MAP is generally a more practical approach.

MAP benefits:

- More practical than MMSE for nonlinear systems
- Aids interpretation of square-root filtering
- Aids in statistical hypothesis testing of the Kalman filter

We will derive the MAP estimator for SLTV systems.

9.4.1 Setting up the cost function

In the MAP derivation, we estimate the process noise $v(k)$ along with the state $x(k+1)$. Conditioning on Z^k is implied throughout the derivation but not shown explicitly for brevity. We begin with the conditional distribution

$$p[x(k+1), v(k)|z(k+1)] = \frac{p[z(k+1)|x(k+1), v(k)]p[x(k+1), v(k)]}{p[z(k+1)]}$$

The MAP approach tells us to find the $x(k+1)$ and $v(k)$ that maximize $p[x(k+1), v(k)|z(k+1)]$. We can ignore $p[z(k+1)]$, since it doesn't depend on $x(k+1)$ or $v(k)$, and just maximize the numerator. This is equivalent to minimizing the cost function

$$J[x(k+1), v(k)] = -\log\left(p[z(k+1)|x(k+1), v(k)]\right) - \log\left(p[x(k+1), v(k)]\right)$$

where log() denotes the natural logarithm.

In the next step, we'll exploit the fact that

$$p_{x(k+1),v(k)}[x(k+1), v(k)] = c \cdot p_{x(k),v(k)}[x(k), v(k)]$$

where c is a constant and the subscripts make clear that these are two distinct probability distributions.

Q: Why can we make the above transformation from $p[x(k+1), v(k)]$ to $p[x(k), v(k)]$?
A: Because $x(k+1)$ is a function of $x(k)$; specifically,

$$x(k+1) = F(k)x(k) + G(k)u(k) + \Gamma(k)v(k)$$

In general, for any invertible 1-to-1 scalar function $Y = g(X)$, it can be shown that

$$p_Y[y] = \frac{p_X[g^{-1}(y)]}{|\frac{dy}{dx}|}$$

For vector-valued functions, the scaling factor in the denominator is the determinant of the Jacobian $[\partial y/\partial x]$. In the case of $x(k+1)$ and $v(k)$ above, $c = |F^{-1}(k)|$ ■

Applying this transformation, the cost function becomes the sum of three components plus a constant:

$$\begin{aligned} J[x(k+1), v(k)] =& \tfrac{1}{2}[z(k+1) - H(k+1)x(k+1)]^T R^{-1}(k+1)[\dots] \quad \leftarrow \text{from } p[z(k+1)|x(k+1), v(k)] \\ &+\tfrac{1}{2}[x(k) - \hat{x}(k)]^T P^{-1}(k)[x(k) - \hat{x}(k)] + \tfrac{1}{2}v^T(k)Q^{-1}(k)v(k) \quad \leftarrow \text{from } p[x(k), v(k)] \\ &+\text{const} \end{aligned}$$

Substituting $x(k) = F^{-1}(k)[x(k+1) - G(k)u(k) - \Gamma(k)v(k)]$ and ignoring the constant yields

$$\begin{aligned} J[x(k+1), v(k)] =& \tfrac{1}{2}[z(k+1) - H(k+1)x(k+1)]^T R^{-1}(k+1)[\dots] \\ &+\tfrac{1}{2}\left\{F^{-1}(k)[x(k+1) - G(k)u(k) - \Gamma(k)v(k)] - \hat{x}(k)\right\}^T P^{-1}(k)\{\dots\} \\ &+\tfrac{1}{2}v^T(k)Q^{-1}(k)v(k) \end{aligned}$$

Recall that $\bar{x}(k+1) = F(k)\hat{x}(k) + G(k)u(k)$; then

$$\begin{aligned} J[x(k+1), v(k)] =& \tfrac{1}{2}[z(k+1) - H(k+1)x(k+1)]^T R^{-1}(k+1)[\dots] \\ &+\tfrac{1}{2}\left\{F^{-1}(k)[x(k+1) - \Gamma(k)v(k) - \bar{x}(k+1)]\right\}^T P^{-1}(k)\{\dots\} \\ &+\tfrac{1}{2}v^T(k)Q^{-1}(k)v(k) \end{aligned}$$

Minimizing this cost is equivalent to maximizing the *a posteriori* distribution $p[x(k+1), v(k)|Z^{k+1}]$.

9.4.2 Solving for the minimum-cost estimate

We minimize $J[x(k+1), v(k)]$ by finding the $x(k+1)$ and $v(k)$ that satisfy the first-order necessary conditions:

$$\begin{aligned} \left[\frac{\partial J}{\partial v(k)}\right]^T &= 0 = -\Gamma^T(k)F^{-T}(k)P^{-1}(k)F^{-1}(k)[x(k+1) - \Gamma(k)v(k) - \bar{x}(k+1)] + Q^{-1}(k)v(k) \\ \left[\frac{\partial J}{\partial x(k+1)}\right]^T &= 0 = -H^T(k+1)R^{-1}(k+1)[z(k+1) - H(k+1)x(k+1)] \\ &\qquad + F^{-T}(k)P^{-1}(k)F^{-1}(k)[x(k+1) - \Gamma(k)v(k) - \bar{x}(k+1)] \end{aligned}$$

We could put these two equations into the form

$$A\begin{bmatrix} x(k+1) \\ v(k) \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

and solve by taking the inverse of A. But instead we'll solve by subsitution. We begin by isolating $v(k)$ in the first equation:

$$v(k) = [\Gamma^T(k)F^{-T}(k)P^{-1}(k)F^{-1}(k)\Gamma(k) + Q^{-1}(k)]^{-1}\Gamma^T(k)F^{-T}(k)P^{-1}(k)F^{-1}(k)[x(k+1) - \bar{x}(k+1)]$$

Applying the matrix inversion lemma and the definition of $\bar{P}(k+1)$ we obtain

$$\begin{aligned} v(k) &= Q(k)\Gamma^T(k)[\underbrace{F(k)P(k)F^T(k) + \Gamma(k)Q(k)\Gamma^T(k)}_{\bar{P}(k+1)}]^{-1}[x(k+1) - \bar{x}(k+1)] \\ &= Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)[x(k+1) - \bar{x}(k+1)] \end{aligned}$$

We substitute this expression for $v(k)$ into the $[\partial J/\partial x(k+1)]^T = 0$ equation to get

$$\begin{aligned}
0 = &- H^T(k+1)R^{-1}(k+1)[z(k+1) - H(k+1)x(k+1)] \\
&+ F^{-T}(k)P^{-1}(k)F^{-1}(k)\left\{x(k+1) - \Gamma(k)Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)[x(k+1) - \bar{x}(k+1)] - \bar{x}(k+1)\right\} \\
0 = &- H^T(k+1)R^{-1}(k+1)[z(k+1) - H(k+1)x(k+1)] \\
&+ F^{-T}(k)P^{-1}(k)F^{-1}(k)[\underbrace{I}_{\bar{P}\bar{P}^{-1}} - \Gamma(k)Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)][x(k+1) - \bar{x}(k+1)] \\
0 = &- H^T(k+1)R^{-1}(k+1)[z(k+1) - H(k+1)x(k+1)] \\
&+ \underbrace{F^{-T}(k)P^{-1}(k)F^{-1}(k)[\bar{P}(k+1) - \Gamma(k)Q(k)\Gamma^T(k)]}_{I}\bar{P}^{-1}(k+1)[x(k+1) - \bar{x}(k+1)]
\end{aligned}$$

By replacing $\bar{P}(k+1)$ with its definition, the underbraced quantity can be recognized as the identity matrix. We can then solve the simplified equation for $x(k+1)$ to obtain $\hat{x}(k+1)$:

$$\begin{aligned}
\hat{x}(k+1) = &[\bar{P}^{-1}(k+1) + H^T(k+1)R^{-1}(k+1)H(k+1)]^{-1} \\
&\times [\bar{P}^{-1}(k+1)\bar{x}(k+1) + H^T(k+1)R^{-1}(k+1)z(k+1)]
\end{aligned}$$

This is identical to the recursive least squares form. Further manipulation yields

$$\hat{x}(k+1) = \bar{x}(k+1) + W(k+1)[z(k+1) - H(k+1)\bar{x}(k+1)]$$

with

$$W(k+1) = [\bar{P}^{-1}(k+1) + H^T(k+1)R^{-1}(k+1)H(k+1)]^{-1}H^T(k+1)R^{-1}(k+1)$$

which is equivalent to our MMSE formula for the *a posteriori* state estimate. Substituting this formula into the one for $v(k)$ gives

$$\begin{aligned}
\hat{v}(k|k+1) \triangleq \mathbb{E}\left[v(k)|Z^{k+1}\right] &= Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)[\hat{x}(k+1) - \bar{x}(k+1)] \\
&= Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)W(k+1)[z(k+1) - H(k+1)\bar{x}(k+1)]
\end{aligned}$$

This is the MAP estimate of the process noise "looking back" after having taken the measurement $z(k+1)$.

9.4.3 Summary

For the SLTV system, the MMSE and MAP estimates for $x(k+1)$ given Z^{k+1} are identical. But the MAP derivation yields as a by-product an estimate of the process noise that drove the system from k to $k+1$:

$$\begin{aligned}
\hat{x}(k+1) &= \bar{x}(k+1) + W(k+1)[z(k+1) - H(k+1)\bar{x}(k+1)] \\
\hat{v}(k|k+1) &= Q(k)\Gamma^T(k)\bar{P}^{-1}(k+1)W(k+1)[z(k+1) - H(k+1)\bar{x}(k+1)]
\end{aligned}$$

where $W(k+1)$ is the Kalman gain matrix. Note that, even with a perfect model, $\hat{v}(k|k+1)$ is not necessarily equal to $v(k)$ because the former is conditioned on $z(k+1)$. The conditioning also causes $\hat{v}(k|k+1)$ to be correlated with $\hat{v}(j|j+1)$ for $j \neq k$ so that, unlike $v(k)$, $\hat{v}(k|k+1)$ is not white.

9.5 Stability of Kalman Filter

The general question of the stability of a Kalman filter applied to a stochastic linear time-varying (SLTV) system has not yet been resolved. Our approach to KF stability analysis will be to "cheat" in two different ways. First, we'll examine KF in the "zero-input" case of no stochastic inputs to our dynamical system. Second, we'll examine the case of a linear time invariant (LTI) system with stochastic inputs.

9.5.1 Zero-Input Stability

Assume $\boldsymbol{v}(k) = \boldsymbol{w}(k) = 0\ \forall k$ (zero stochastic input). We want to show that the error vector

$$\boldsymbol{e}(k) = \boldsymbol{x}(k) - \hat{\boldsymbol{x}}(k)$$

decays to zero.

(Almost) proof:

$$\begin{aligned}\boldsymbol{e}(k+1) &= \boldsymbol{x}(k+1) - \hat{\boldsymbol{x}}(k+1)\\ &= F(k)\boldsymbol{x}(k) + G(k)\boldsymbol{u}(k) - [\bar{\boldsymbol{x}}(k+1) + W(k+1)\{\boldsymbol{z}(k+1) - H(k+1)\bar{\boldsymbol{x}}(k+1)\}]\end{aligned}$$

But

$$\begin{aligned}\boldsymbol{z}(k+1) &= H(k+1)\boldsymbol{x}(k+1) = H(k+1)[F(k)\boldsymbol{x}(k) + G(k)\boldsymbol{u}(k)]\\ \bar{\boldsymbol{x}}(k+1) &= F(k)\hat{\boldsymbol{x}}(k) + G(k)\boldsymbol{u}(k)\end{aligned}$$

Substituting these causes the $\boldsymbol{u}(k)$'s to cancel, giving the error dynamics for $\boldsymbol{v}(k) = \boldsymbol{w}(k) = 0\ \forall k$:

$$\boldsymbol{e}(k+1) = [I - W(k+1)H(k+1)]F(k)\boldsymbol{e}(k)$$

It is tricky to prove stability because the system implied by the error dynamics is time varying. If it were time invariant, it would be possible to look at the moduli of the system's eigenvalues. Instead, we will use a Lyapunov-type energy method to prove stability. Define an energy-like function

$$V[k, \boldsymbol{e}(k)] = \tfrac{1}{2}\ \boldsymbol{e}^T(k)P^{-1}(k)\boldsymbol{e}(k)$$

Note that $\sqrt{V}$ is a weighted 2-norm of $\boldsymbol{e}(k)$ because $P(k) > 0$. Therefore, $V \geq 0$, with equality if and only if $\boldsymbol{e}(k) = 0$. We wish to show that V always gets smaller as k increases.

$$\begin{aligned}V[k+1, \boldsymbol{e}(k+1)] &= \tfrac{1}{2}\boldsymbol{e}^T(k+1)P^{-1}(k+1)\boldsymbol{e}(k+1)\\ &= \tfrac{1}{2}\boldsymbol{e}^T(k)[P(k) + D(k)]^{-1}\boldsymbol{e}(k)\end{aligned}$$

where

$$D(k) \triangleq F^{-1}(k)[\Gamma(k)Q(k)\Gamma(k)^T + \bar{P}(k+1)H^T(k+1)R^{-1}(k+1)H(k+1)\bar{P}(k+1)]F^{-T}(k)$$

Note that $D(k) \geq 0$, which implies $[P(k)+D(k)]^{-1} \leq P^{-1}(k)$. Thus $V[k+1, e(k+1)] \leq V[k, e(k)]$. But the fact that V always gets smaller as k increases isn't quite enough for our proof because V could be getting smaller due to $P(k)$ increasing, not $e(k)$ decreasing.

It turns out that, under suitable conditions on $Q(k)$, $R(k)$, $F(k)$, and $H(k)$ (i.e. all unstable or neutrally stable subspaces of original system are observable, Q and R are not too big or small, system is controllable with respect to points of entry of process noise, or "stochastic controllability and observability") we can show that

1. $P^{-1}(k) \geq \epsilon > 0$ for some bound ϵ.

2. $V[k+N, \boldsymbol{e}(k+N)] < \alpha V[k, \boldsymbol{e}(k)]$ for some α: $0 \leq \alpha < 1$ and some N.

Thus, V is decreasing because $e(k)$ is decreasing, not because $P(k)$ is increasing. Also note that *nowhere in this proof did we assume the original system was stable.* Indeed, it need not be for the *estimation error dynamics* to be stable.

9.5.2 Matrix Riccati Equation

Recall that

$$\bar{P}(k+1) = F(k)P(k)F^T(k) + \Gamma(k)Q(k)\Gamma^T(k)$$

We substitute for $P(k)$ to get

$$\begin{aligned}\bar{P}(k+1) = F(k)&\left\{\bar{P}(k) - \bar{P}(k)H^T(k)[H(k)\bar{P}(k)H^T(k) + R(k)]^{-1}H(k)\bar{P}(k)\right\}F^T(k)\\ &+ \Gamma(k)Q(k)\Gamma^T(k)\end{aligned}$$

This dynamic model for $\bar{P}(k)$, nonlinear in $\bar{P}(k)$, is called the Matrix Riccati equation (MRE). Beware, a general-case analysis of MRE is not easy! We "cheat" by limiting ourselves to the special case of an LTI system: $F(k) = F$, $\Gamma(k) = \Gamma$, $H(k) = H$, $Q(k) = Q$, $R(k) = R$.

If the pair (F, H) is observable, and if (F, Γ) is controllable, and if Q, $R > 0$, and $\bar{P}(0) > 0$, then the MRE converges to a steady-state solution $\bar{P}_{ss} > 0$, which can be determined by solving the Algebraic Riccati Equation (ARE):

$$\bar{P}_{ss} = F\left\{\bar{P}_{ss} - \bar{P}_{ss}H^T[H\bar{P}_{ss}H^T + R]^{-1}H\bar{P}_{ss}\right\}F^T + \Gamma Q\Gamma^T$$

This form is called the ARE because it's algebraic, not dynamic like the MRE; but it remains nonlinear in $\bar{P}_{ss}$. Note that the form is similar to the discrete-time Lyapunov equation for the steady-state covariance of a linear dynamic system.

The MATLAB function `kalman` can be used to solve the ARE and provide the Kalman gain matrix W_{ss}:

$$\begin{aligned}\texttt{sys} &= \texttt{ss}(F, [B\ \Gamma], H, 0, -1);\\ [\texttt{km}, L, \bar{P}_{ss}, W_{ss}] &= \texttt{kalman}(\texttt{sys}, Q, R);\end{aligned}$$

Apart from $\bar{P}_{ss}$ and W_{ss}, this command returns the state-space model of the Kalman filter in `km` and a gain matrix L that can be used in an alternate form of the Kalman filter.

9.5.3 Steady-State KF Equations

The steady-state Kalman Filter equations are given by

$$\begin{aligned}\bar{\boldsymbol{x}}(k+1) &= F\hat{\boldsymbol{x}}(k) + G\boldsymbol{u}(k)\\ \hat{\boldsymbol{x}}(k+1) &= \bar{\boldsymbol{x}}(k+1) + W_{ss}[\boldsymbol{z}(k+1) - H(k+1)\bar{\boldsymbol{x}}(k+1)]\end{aligned}$$

9.5.4 Steady-State Error Dynamics

The steady-state error dynamics are given by

$$\boldsymbol{e}(k+1) = \underbrace{[I - W_{ss}H]F}_{A_{ss}}\,\boldsymbol{e}(k)$$

We know that the dynamics of $\boldsymbol{e}(k)$ are stable—they must be for a steady-state solution to exist. Thus, it must be the case that

$$\max_i |\lambda_i| < 1$$

where $\{\lambda_i\}_i^{n_x}$ are the eigenvalues of A_{ss}.

Again, keep in mind that this holds whether or not the original system dynamics are stable.