Math 61 Final Exam
Winter 2010

Name: ______________________________

Signature: ______________________________

UCLA ID Number: ______________________________

Instructions:

- There are 14 problems. Make sure you are not missing any problems.
- YOU MUST EXPLAIN YOUR ANSWERS TO RECEIVE CREDIT.
- No calculators, books, or notes are allowed.
- Do not use your own scratch paper.

Question	Points	Score
1	10	
2	10	
3	15	
4	15	
5	10	
6	20	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
13	10	
14	10	
Total:	160	

1. **(10 points) Does there exist a bipartite graph G whose adjacency matrix A satisfies the equation**

$$A^2 = \begin{pmatrix} 3 & 3 & 3 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 & 3 \end{pmatrix}?$$

Prove your answer. (WARNING: A "Yes" or "No" without justification will receive no credit.)

Solution:

Yes. If A is the adjacency matrix of the graph $K_{3,3}$, then A^2 is the matrix written above. We know this because

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Squaring this matrix gives the matrix above.

2. (10 points) Recall that $\mathbb{Z}$ is the set of integers. For any subset $A \subseteq \mathbb{Z}$, define the function

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$

Prove that for two sets of integers A and B,

$$f_A(x) + f_B(x) = f_{A\cup B}(x) \text{ for all } x \in \mathbb{Z}$$

if and only if $|A \cap B| = 0$.

Solution:

First assume $|A \cap B| \neq 0$. Then there is some $x \in \mathbb{Z}$ such that $x \in A$ and $x \in B$. This implies that $f_A(x) = 1 = f_B(x)$. But we also know that $x \in A \cup B$, which implies that $f_{A\cup B}(x) = 1$. Hence

$$f_A(x) + f_B(x) = 1 + 1 = 2 \neq 1 = f_{A\cup B}(x).$$

This means that if $|A \cap B| \neq 0$, then $f_A(x) + f_B(x) = f_{A\cup B}(x)$ must not be true for every x. This proves one direction.

Now assume $|A \cap B| = 0$. Consider the following cases:

CASE 1: $x \in A$ but $x \notin B$. This means that $x \in A \cup B$ also. Hence

$$f_A(x) + f_B(x) = 1 + 0 = 1 = f_{A\cup B}(x).$$

CASE 2: $x \in B$ but $x \notin A$. This means that $x \in A \cup B$ also. Hence

$$f_A(x) + f_B(x) = 0 + 1 = 1 = f_{A\cup B}(x).$$

CASE 3: $x \notin A$ and $x \notin B$. This means that $x \notin A \cup B$. Hence

$$f_A(x) + f_B(x) = 0 + 0 = 0 = f_{A\cup B}(x).$$

Since $|A \cap B| = 0$, these are the only cases to consider. This proves the other direction.

3. (15 points) Let G be a graph with vertex set V. Define a relation R on $V \times V$ by

$$v_1 R v_2 \text{ if and only if there is a path in } G \text{ from } v_1 \text{ to } v_2 \text{ .}$$

(10 points) (a): Prove R is an equivalence relation.
(5 points) (b): Consider this relation when the graph is the complete bipartite graph $K_{m,n}$. How many equivalence classes are there? Justify your answer.
Solution (a):
R is reflexive: Given any vertex v, there is a path of length 0 from v to itself, so vRv.

R is symmetric: Assume vRw. This means there is a path from v to w. But the path goes in both directions, which means there is also a path from w to v, which implies wRv.

R is transitive: Assume uRv and vRw. This means there are paths from u to v and from v to w. By connecting these paths (they meet at v), we obtain a path from u to w, which implies that uRw.

Proving the three things above shows that R is an equivalence relation.
Solution (b):

There is one equivalence class because the graph is connected.

4. (15 points) (a) (10 points) : Prove that a list of numbers a_1, a_2, a_3 can be put in order using only three comparisons.

(b) (5 points): Prove that there is no algorithm that orders these numbers with only two comparisons.

Solution :

There is a decision tree printed on page 307 (Section 7.7) of the textbook that sorts the numbers using only three comparisons. The discussion below the tree proves part (b).

5. (10 points) Recall that K_j is the complete graph on j vertices. For each $j = 1, 2, \ldots, n$, let T_j be a spanning tree of K_j. Let

$$G_n = T_1 \cup T_2 \cup \cdots \cup T_n.$$

How many edges does G_n have? Prove your answer. (Here I use the convention that if $G = (V, E)$ and $G' = (V', E')$ are graphs, then $G \cup G' = (V \cup V', E \cup E')$.)

Solution :

Since T_j is a spanning tree of K_j, we know T_j has j vertices. We also know that any tree with j vertices has $j - 1$ edges. Adding up all the edges from $T_1, T_2, \ldots, T_n$ gives us

$$0 + 1 + 2 + \ldots n - 1 = \sum_{k=1}^{n-1} k = \frac{1}{2}(n-1)n.$$

6. (20 points) (a) (10 points) : Let T be a binary tree with a root. Prove T does not contain a subgraph homeomorphic to $K_{3,3}$ or K_5.
(b) (5 points) : Prove T is planar.
(c) (5 points) : Let E be the set of edges in T and let V be the set of vertices in T. Prove $|E| = |V| - 1$ by using Euler's formula.
Solution (a):
Since T is a tree, it contains no cycles. This implies that no subgraph of T contains a cycle either. We know that both $K_{3,3}$ and K_5 contain cycles. Since it is not possible for a graph with cycles to be homeomorphic to a graph without cycles, we know T has no subgraph homeomorphic to $K_{3,3}$ or K_5.
Solution (b):
Kuratowski's theorem together with part (a) proves that T is planar.
Solution (c):
Euler's formula says

$$|F| = |E| - |V| + 2.$$

Since T is a tree it has no cycles and hence only one face. This implies

$$1 = |E| - |V| + 2,$$

which implies

$$|V| - 1 = |E|.$$

7. (10 points) Let $n \geq 1000$. Let G be a connected subgraph of K_n with at least half as many edges as K_n has. Prove G is not planar. (Of course by K_n I mean the complete simple graph on n vertices.)
Solution :
From the homework we know that in any simple connected planar graph we have

$$|E| \leq 3|V| - 6.$$

We will show that the edges and vertices of G do not satisfy this inequality; this will imply that G is not planar. First, we know that K_n has $\frac{1}{2}n(n-1)$ edges. Since $n \geq 1000$, we know that $\frac{1}{2}n(n-1) \geq \frac{1}{8}n^2$. Hence $|E| > \frac{1}{8}n^2$. But we know that $|V| \leq n$ since G is a subgraph if K_n, which has only n vertices. This implies that

$$|E| > \frac{1}{8}n^2 > 3n - 6 \geq 3|V| - 6;$$

the middle inequality holds because we assume n is large. This contradicts the inequality above, and shows that G is not planar.

8. (10 points) Prove

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \leq \frac{\sqrt{n}}{\sqrt{n+1}}.$$

for integers $n \geq 1$.

Solution :

Since $\frac{2n-1}{2n} < 1$ for all $n \geq 1$, we know that the left hand side above is always less than $\frac{1}{2}$. So it is enough to show that the right hand side is always greater than or equal to $\frac{1}{2}$. But this is true because

$$\frac{\sqrt{n}}{\sqrt{n+1}} \geq \frac{1}{2}$$

if

$$2\sqrt{n} \geq \sqrt{n+1},$$

which is true if

$$4n \geq n+1,$$

which is true if

$$3n \geq 1,$$

which is obviously true if $n \geq 1$.

9. (10 points) Let T be the full binary tree of height n with 2^n terminal vertices. Let r denote the root. Define a weight on the edges of T as follows: If an edge e connects a vertex with its left child, define $w(e) = 0$. If an edge connects a vertex with its right child, define $w(e) = 1$. Now let $\mathcal{P}$ be the set of simple paths from r to the terminal vertices. Note that $|\mathcal{P}| = 2^n$ since there is one simple path from r to each terminal vertex. Let $\mathcal{P}_k$ be the set of paths in $\mathcal{P}$ with total weight k. Show

$$|\mathcal{P}_3| \leq n^3.$$

(Recall that the weight of a path is the sum of the weights of the edges in the path.)

Solution :

Each path in $\mathcal{P}$ corresponds to a sequence of 0's and 1's, depending on exactly how the path got from the root to the bottom. So we can identify the set $\mathcal{P}$ with the set of binary strings of length n. The question is then, "How many binary strings of length n contain exactly k ones? The answer to this question is $C(n, k)$. Hence

$$|\mathcal{P}_3| = C(n, 3) = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{3 \cdot 2 \cdot 1} \leq n^3.$$

10. (10 points) Let G be a simple graph with vertex set V. Let $d(v)$ be the degree of the vertex v. Prove there are $v_1, v_2 \in V$ with $v_1 \neq v_2$ such that $d(v_1) = d(v_2)$.

Solution :

There are two cases:

CASE 1: There exists a vertex v such that $d(v) = 0$. In this case, we know that no vertex is connected to every other vertex (because no vertex is connected to v!). Further, since the graph is simple, no vertex is connected to itself. This means that for any vertex u, there are at least 2 vertices that u is NOT connected to, namely v and u itself. Hence $d(u) \leq |V| - 2$ for all $u \in V$. But now we have established that the function d defined on V has range $\{0, 1, 2, \ldots, |V| - 2\}$. Since the domain has $|V|$ elements and the range has $|V| - 1$ elements, the pigeonhole principle tells us there must be $v_1, v_2 \in V$ such that $d(v_1) = d(v_2)$, which is what we needed to show.

CASE 2: There is NO vertex v such that $d(v) = 0$. We argue as above, except that in this case the range of d is $\{1, 2, 3, \ldots, |V| - 1\}$. But this set still has size $|V| - 1$, so we may apply the pigeonhole principle as above.

11. (10 points) How many integers are there between 1 and 999,999 whose digits add up to 10? (For example: The digits of 91 add up to 10, but the digits of 10 add up to 1.)
Solution :

We need to divide ten objects among six categories, except we are not allowed to put all ten objects in one category. There are $C(15, 5)$ ways of dividing ten objects among six categories (5 walls, 15 walls and objects combined, so $C(15, 5)$ places to put the walls). Since there are 6 ways to put all ten objects in one category, we have a total of

$$C(15, 5) - 6$$

integers in the range above whose digits sum to 10.

12. (10 points) Let G be a simple graph with vertex set V. For any $v \in V$, define $N(v)$ to be the set of all vertices that are adjacent to v. (This does not include v itself.) Let $d(v)$ be the degree of v. Prove

$$\sum_{v \in V} d(v)^2 = \sum_{u \in V} \sum_{w \in V} |N(u) \cap N(w)|.$$

Solution :
Fix $u, w \in V$. What is the number $|N(u) \cap N(w)|$? It is the number of neighbors common to both u and w. For each of these common neighbors c, there is a unique path from u to w through c. Further, if v is some vertex that is not a neighbor of both u and w, then there can be no path of length two from u to w through v. Hence $|N(u) \cap N(w)|$ is the number of paths of length two from u to w. Summing over all pairs u and w shows that the right hand side above is the total number of paths of length two in the graph. But we saw in the second midterm that the left hand side is also equal to this number, which establishes the equality above.

13. (10 points) Let G be a simple graph with vertex set V and edge set E. Define

$$\overline{E} = \{(v, u) \in V \times V \colon (v, u) \notin E \text{ and } v \neq u\}.$$

Now define $\overline{G} = (V, \overline{E})$. (This is called the complement of G. In words: $\overline{G}$ is the graph with the same vertices as G, and with the set of edges $\overline{E}$. Note that $\overline{G}$ is also simple and contains precisely those edges that are not in E.) Prove that either G or $\overline{G}$ is connected.

Solution :

Let's assume that G is NOT connected, because if it is connected then we are already done. Since G is not connected, there are nonempty disjoint sets of vertices V_1 and V_2 such that $V_1 \cup V_2 = V$ and such that no vertex in V_1 is connected to any vertex in V_2. This implies that in $\overline{G}$, every vertex in V_1 is connected to every vertex in V_2. Now let v, u be any vertices in V. We will show there is a path between them.

CASE 1: $v \in V_1$ and $u \in V_2$. In this case, we already know v and u are connected directly by an edge.

CASE 2: v, u are both in V_1. (If v, u are both in V_2, apply the same argument.) In this case, both v and u are connected to some vertex $w \in V_2$, and hence there is a path of length 2 from v to u through w. (All we are using here is that V_2 is not empty, and this holds because otherwise G was connected in the first place.)

14. (10 points) Prove that tournament sort (as describe in the homework) requires $\leq 1000 \cdot 2^n \cdot n$ comparisons to order 2^n numbers. (Don't take the number 1000 too seriously; you can probably do better.)

Solution :

You proved in the homework that finding the largest number using tournament sort requires no more than 2^n comparisons: all 2^n numbers can be placed as the terminal vertices of a binary tree of height n, and there are $\leq 2^n$ internal vertices, each corresponding to a comparison of numbers. To find the second largest number, we replace the largest number with some number smaller than all numbers on the list. (As discussed in class, we can similarly find the smallest number in fewer than 2^n comparisons.) Then we follow that number up the tree, making a comparison at every vertex that is an ancestor of the new number, which requires n comparisons. The same procedure finds the third largest, fourth largest, ... etc., each time requiring only n new comparisons. After doing this at most 2^n times, we have used at most $n2^n$ comparisons. Adding in the number required to find the smallest at largest numbers gives us $n2^n + 2^n + 2^n \leq 3n2^n$ total comparisons.