Math 61 Practice Problems

[With Solutions]

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FAQ

1. *Are these questions representative of the content/difficulty/format/etc. of the midterm?*

   I make no such guarantees and had no such intentions when making this worksheet. These questions are only designed to be instructive, i.e. so that solving them requires understanding some important concept from the course.

2. *What does* (($\star$)) *mean?*

   I've used the symbol (($\star$)) to mark problems which are particularly challenging or somewhat beyond the scope of what we've covered. I recommend at least thinking about how you could solve them, but don't stress if you find it difficult to tackle those problems.

3. *Are your proofs the only correct proofs for their corresponding claims?*

   Definitely not.

4. *Would all of your proofs be good enough to get full credit if the question was asked on a midterm?*

   Certainly not all; I'm not infallible and may have made mistakes (if so, email me!), and more importantly some of the solutions below are just proof sketches, or just answers without justification. However, I've tried to make most of the proofs below of reasonably high quality.

Problems

1. Let A, B, and C be sets that $A \subseteq B$, $B \subseteq C$, and $C \subseteq A$. Show that $A = B = C$.

   Solution: Since $A \subseteq B$ and $B \subseteq C$, $A \subseteq C$. We also know that $C \subseteq A$, so $A = C$. Likewise, $B \subseteq C \subseteq A$ so $B \subseteq A$, while also $A \subseteq B$. Thus, we have $A = B$. By transitivity of equality, $A = B = C$.

2. Prove that any set X with at least two elements has a subset Y such that $Y \neq \varnothing$ and $Y \neq X$.

   Solution: Since X has at least two elements, there is some x such that $x \in X$. Now $\{x\} \subseteq X$ is nonempty and $\{x\} \neq X$ since $\{x\}$ has one element while X has at least 2.

3. How many elements does the set $\mathbb{Z} \cap \{x \in \mathbb{Q} : -10 < x < 10\}$ have?

   Solution: This intersection is simply $\{x \in \mathbb{Z} : -10 < x < 10\}$, or in other words $\{-9, -8, \ldots, 8, 9\}$, which has 19 elements.

4. If X and Y are sets, $X \triangle Y$ (called the *symmetric difference* of X and Y) is defined to be the set $(X \cup Y) \setminus (X \cap Y)$.

   (a) Prove that $X \triangle X = \varnothing$ for any set X.

   Solution: $X \triangle X = (X \cup X) \setminus (X \cap X) = X \setminus X = \varnothing$.

   (b) Prove that $(X \cup Y) \setminus (X \triangle Y) = X \cap Y$ for any sets X and Y.

   Solution:

$$
\begin{aligned}
(X \cup Y) \setminus (X \triangle Y) &= \{a \in X \cup Y : a \notin X \triangle Y\} \\
&= \{a \in X \cup Y : a \notin \{a' \in X \cup Y : a' \notin X \cap Y\}\} \\
&= \{a \in X \cup Y : a \in X \cap Y\} = X \cap Y.
\end{aligned}
$$

   (c) Prove that $(X \triangle Y) \cap Z = (X \cap Z) \triangle (Y \cap Z)$ for any sets X, Y, and Z.

   Solution: Let $a \in (X \triangle Y) \cap Z$ be arbitrary. Then $a \in X \cup Y$ so $a \in X$ or $a \in Y$. Since $a \in Z$, we have $a \in X \cap Z$ or $a \in Y \cap Z$, thus $a \in (X \cap Z) \cup (Y \cap Z)$. Suppose for contradiction that $a \in (X \cap Z) \cap (Y \cap Z) =$

$X \cap Y \cap Z$. Then $a \in X \cap Y$ so $a \notin X \triangle Y$, which contradicts the assumption to that $a \in (X \triangle Y) \cap Z$. Thus, $a \notin (X \cap Z) \cap (Y \cap Z)$, i.e. we can conclude $a \in (X \cap Z) \triangle (Y \cap Z)$. Since a was arbitrary, $(X \triangle Y) \cap Z \subseteq (X \cap Z) \triangle (Y \cap Z)$.

Now let $b \in (X \cap Z) \triangle (Y \cap Z)$ be arbitrary. Then $b \in (X \cap Z) \cup (Y \cap Z) = (X \cup Y) \cap Z$ so $b \in X \cup Y$ and $b \in Z$. Suppose for contradiction that $b \in X \cap Y$. Then $b \in X \cap Y \cap Z = (X \cap Z) \cap (Y \cap Z)$, which contradicts our assumption that $b \in (X \cap Z) \triangle (Y \cap Z)$. Thus, $b \notin X \cap Y$, so $b \in X \triangle Y$. We now conclude that $b \in (X \triangle Y) \cap Z$, and since b was arbitrary we have that $(X \cap Z) \triangle (Y \cap Z) \subseteq (X \triangle Y) \cap Z$.

We have shown both directions of inclusion, so

$$(X \triangle Y) \cap Z = (X \cap Z) \triangle (Y \cap Z).$$

(d) Prove that $(X \triangle Y) \triangle Z = X \triangle (Y \triangle Z)$ for any sets X, Y, and Z.

Solution: I claim that

$(X \triangle Y) \triangle Z = \{a \in X \cup Y \cup Z : a \text{ is in all or exactly one of the sets } X, Y, Z\}$.

To see this, suppose $a \in (X \triangle Y) \cup Z$. Then $a \in X \triangle Y$ or $a \in Z$. In the former case, $a \in X$ or $a \in Y$, so $a \in X \cup Y \subseteq X \cup Y \cup Z$. In the latter case, $a \in Z \subseteq X \cup Y \cup Z$. Thus, $a \in X \cup Y \cup Z$, telling us that a is in at least one of the sets X, Y, Z. Suppose for contradiction that a is in exactly two of the sets X, Y, Z. Without loss of generality (since $X \triangle Y = Y \triangle X$), we have either $a \in (X \cap Z) \setminus Y$ or $a \in (X \cap Y) \setminus Z$. In the former case, $a \in X \setminus Y$ tells us that $a \in X \triangle Y$ so $a \in Z$ tells us that $a \notin (X \triangle Y) \triangle Z$. In the latter case, $a \in X \cap Y$ tells us that $a \notin X \triangle Y$ so $a \notin Z$ tells us that $a \notin (X \triangle Y) \triangle Z$. This contradicts the fact that $a \in (X \triangle Y) \triangle Z$, so it is not the case that a is in exactly two of the sets X, Y, Z. Thus, a is in all or exactly one of the sets X, Y, Z.
In the other direction, suppose a is an element of all of or exactly one of the sets X, Y, Z. If a is in all of the sets then $a \in Z$ and $a \in X \cap Y$, so $a \notin X \triangle Y$, whence $a \in (X \triangle Y) \triangle Z$. Now suppose a is in exactly one of the sets X, Y, Z. If $a \in X$ then since $a \notin Y$ we have $a \in X \triangle Y$, so since $a \notin Z$ we have $a \in (X \triangle Y) \triangle Z$. If $a \in Y$ then since $a \notin X$ we have $a \in X \triangle Y$, so since $a \notin Z$ we have $a \in (X \triangle Y) \triangle Z$. If $a \in Z$ then since $a \notin X$ and $a \notin Y$ we have $a \notin X \cup Y \supseteq X \triangle Y$, so $a \in (X \triangle Y) \triangle Z$. We have now shown the desired inclusions, so

$(X \triangle Y) \triangle Z = \{a \in X \cup Y \cup Z : a \text{ is in all or exactly one of the sets } X, Y, Z\}$.

Finally, since the above holds for any sets X, Y, Z, we have

$$\begin{aligned} X \triangle (Y \triangle Z) &= (Y \triangle Z) \triangle X \\ &= \{a \in Y \cup Z \cup X : a \text{ is in all or exactly one of the sets } Y, Z, X\} \\ &= \{a \in X \cup Y \cup Z : a \text{ is in all or exactly one of the sets } X, Y, Z\} \\ &= (X \triangle Y) \triangle Z. \end{aligned}$$

5. Show that every natural number greater than 23 is of the form $5x + 7y$ for some $x, y \in \mathbb{N}$.

Solution: We will use strong induction.

Base Cases $24 = 5 \cdot 2 + 7 \cdot 2$, $25 = 5 \cdot 5 + 7 \cdot 0$, $26 = 5 \cdot 1 + 7 \cdot 3$, $27 = 5 \cdot 4 + 7 \cdot 1$, and $28 = 5 \cdot 0 + 7 \cdot 4$.

Inductive Step Let n be a natural number greater than 27 and suppose every $k \in \{24, 25, \ldots, n\}$ is of the form $5x+7y$ for some $x, y \in \mathbb{N}$. Then $n-4 > 23$ so $n-4 = 5x+7y$ for some $x, y \in \mathbb{N}$. Now $n+1 = n-4+5 = 5x+7y+5 = 5(x+1)+7y$. Since $x \in \mathbb{N}$, $x+1 \in \mathbb{N}$, so $n+1$ is of the desired form.

By mathematical induction, every natural number greater than 23 is of the form $5x + 7y$ for some $x, y \in \mathbb{N}$.

6. Let n and k be positive integers. Show that there exist non-negative integers a and b such that $n = ak + b$, $b < k$, and $a + b \neq 0$.

Solution: If $k = 1$ then we can always choose $a = n$ and $b = 0$. Thus, we will assume $k > 1$ and proceed by strong induction on n.

Base Cases Suppose n is a positive integer such that $n < k$. Setting $a = 0$ and $b = n$, we see that $n = ak + b$, that $b = n < k$, and that $a + b = n \neq 0$, as desired. Now suppose $n = k$. Setting $a = 1$ and $b = 0$, we see that $n = ak + b$, $b = 0 < k$, and that $a + b = 1 \neq 0$, as desired. We have now shown the desired result whenever $n \leq k$.

Inductive Step Suppose $n_0 > k$ and that the desired result holds for all $n < n_0$. Then $0 < n_0 - k < n_0$, so there are non-negative integers a and b such that $n_0 - k = ak + b$, $b < k$, and $a + b \neq 0$. This shows that $n_0 = (a+1)k + b$, and again we have that $a + 1$ and b are non-negative integers such that $b < k$ and $a + 1 + b \neq 0$.

By mathematical induction, the desired result holds for all positive integers n.

7. Let $R = \{(x, y) \in \mathbb{R} : \sin(xy) = 0\}$. As a relation on $\mathbb{R}$, which of the following properties does R have? Select all that apply.

   A. Reflexive

   B. Antireflexive

   C. Symmetric

   D. Antisymmetric

   E. Transitive

   F. Total

> **Solution:** R is not reflexive because $(\sqrt{\pi/2}, \sqrt{\pi/2}) \notin R$.
>
> R is not antireflexive because $(0, 0) \in R$.
>
> R is symmetric because if $(x, y) \in R$ then $\sin(xy) = 0$ so $\sin(yx) = 0$ so $(y, x) \in R$.
>
> R is not antisymmetric because $(0, 1) \in R$ and $(1, 0) \in R$ but $0 \neq 1$.
>
> R us not transitive because $(1, 0) \in R$ and $(0, \pi/2) \in R$ but $(1, \pi/2) \notin R$.
>
> R is not total because $(\sqrt{\pi/2}, \sqrt{\pi/2}) \notin R$.

8. Let X be the set of functions from $\mathbb{R}$ to $\mathbb{R}$. Let $R = \{(f, g) \in X \times X : \forall t \in \mathbb{R}(f(t) < g(t))\}$. As a relation on X, which of the following properties does R have? Select all that apply.

   A. Reflexive

   B. Antireflexive

   C. Symmetric

   D. Antisymmetric

   E. Transitive

   F. Total

> **Solution:** R is not reflexive because $(x \mapsto 0, x \mapsto 0) \notin R$.
>
> R is antireflexive because $f(t) < f(t)$ is never true for any $t \in \mathbb{R}$ and any $f \in X$.
>
> R is not symmetric because $(x \mapsto 0, x \mapsto 1) \in R$ while $(x \mapsto 1, x \mapsto 0) \notin R$.
>
> R is antisymmetric because if $(f, g) \in R$ then $(g, f) \notin R$ (so there is nothing to check).
>
> R is transitive because if $(f, g) \in R$ and $(g, h) \in R$ then $f(t) < g(t) < h(t)$ for all $t \in \mathbb{R}$, so $(f, h) \in R$.
>
> R is not total because $(x \mapsto 0, x \mapsto 0) \notin R$.

9. Determine if sine is injective, surjective, both, or neither, when viewed as a function with following domains and codomains:

(a) $\sin : \{0\} \to \{0\}$
A. Neither B. Just injective C. Just surjective **D. Both**

> **Solution:** Every function $\{0\} \to \{0\}$ is bijective.

(b) $\sin : \mathbb{R} \to \mathbb{R}$
A. Neither B. Just injective C. Just surjective D. Both

> **Solution:** It's not injective because $\sin(0) = 0 = \sin(\pi)$ and it's not surjective because for all $x \in \mathbb{R}$, $\sin(x) \neq 2$.

(c) $\sin : [0, \pi/2] \to [0, 1]$
A. Neither B. Just injective C. Just surjective **D. Both**

> **Solution:** $\sin(\arcsin(t)) = t$ for all $t \in [0, 1]$, and $\arcsin(\sin(\theta)) = \theta$ for all $\theta \in [0, \pi/2]$, so this function has an inverse.

(d) (($\star$)) $\sin : \mathbb{Q} \to [-1, 1]$
A. Neither **B. Just injective** C. Just surjective D. Both

> **Solution:** First we will show this function is not surjective. Suppose $\sin(x) = 1$ for some $x \in \mathbb{Q}$. Then $x = (2k+1)\pi$ for some $k \in \mathbb{Z}$. Thus, $\pi = x/(2k+1)$. This shows that π is rational, but this is not the case. Thus, we cannot have $\sin(x) = 1$ for any $x \in \mathbb{Q}$, so the function is not surjective.
>
> Now we will show this function is injective. Suppose $\sin(x) = \sin(y)$ for some $x, y \in \mathbb{Q}$. Then $0 = \sin(x) - \sin(y) = 2\sin((x-y)/2)\cos((x+y)/2)$, so $\sin((x-y)/2) = 0$ or $\cos((x+y)/2) = 0$. Thus, there is some $k \in \mathbb{Z}$ such that $(x - y)/2 = k\pi$ or $(x + y)/2 = k\pi/2$. If $k \neq 0$ then $\pi = (x - y)/(2k) \in \mathbb{Q}$ or $\pi = (x + y)/k \in \mathbb{Q}$, but π is irrational so this cannot happen. Thus, $k = 0$, so $(x - y)/2 = 0$ or $(x + y)/2 = 0$. This shows that $x = y$ or $x = -y$. In the latter case, we will have had $\sin(y) = \sin(x) = \sin(-y) = -\sin(y)$, so $\sin(y) = 0$. This tells us that $y = \pi\ell$ for some $\ell \in \mathbb{Z}$, whence $\ell = 0$ (or else $\pi = y/\ell$ would be rational). Thus, in the case where $x = -y$, we must have $y = 0$, and thus also $x = y$.

10. Let X be a set and let $m : X \times X \to X$ be a function such that $m(m(x, y), z) = m(x, m(y, z))$ for all $x, y, z \in X$. Let $L = \{e \in X : \forall x \in X(m(e, x) = m(x, e) = x)\}$. Prove that $|L| \leq 1$.

> **Solution:** If $e, e' \in L$ then $m(e, e') = e$ and $m(e, e') = e'$ so $e = e'$.

11. Let $f : X \to X$ be a bijection and suppose that $f \circ f = f$. Must it be the case that $f = \mathrm{id}_X$?

A. Yes B. No

Solution:

$$f = \mathrm{id}_X \circ f = (f^{-1} \circ f) \circ f = f^{-1} \circ (f \circ f) = f^{-1} \circ f = \mathrm{id}_X .$$

12. Let $f : X \to X$ be a bijection and suppose that $f \circ f = \mathrm{id}_X$. Must it be the case that $f = \mathrm{id}_X$?

    A. Yes **B. No**

Solution: If $X = \{1, 2\}$ and $f = \{(1, 2), (2, 1)\}$ then $f : X \to X$ is a bijection and $f \circ f = \mathrm{id}_X$ but $f \neq \mathrm{id}_X$.

13. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be a function such that $f((\alpha x + x', \alpha y + y')) = \alpha f((x, y)) + f((x', y'))$ for all $x, y, x', y', \alpha \in \mathbb{R}$.

    (a) Prove that $f((0, 0)) = (0, 0)$.

    Solution: $f((0, 0)) = f((0 \cdot 1, 0 \cdot 1)) = 0 \cdot f((1, 1)) = (0, 0)$.

    (b) Suppose $(0, 0)$ is the only element of $\mathbb{R} \times \mathbb{R}$ which gets mapped to $(0, 0)$ by f. Prove that f is injective.

    Solution: Suppose $f((x, y)) = f((x', y'))$ for some $x, y, x', y' \in \mathbb{R}$. Then

    $$f(((-1)x + x', (-1)y + y')) = (-1)f((x, y)) + f((x', y')) = 0$$

    so by assumption $((-1)x + x', (-1)y + y') = (0, 0)$. Thus, $x = x'$ and $y = y'$ so $(x, y) = (x', y')$.

    (c) (($\star$)) Suppose f is injective. Must f be surjective?

    A. Yes B. No

    Solution: Yes, because then f is an injective linear map between finite-dimensional vector spaces of the same dimension.
    This explanation won't make sense if you haven't taken linear algebra (it can still be solved directly, it just takes some work).

14. How many equivalence relations are there on the set $\{1, 2, 3, 4\}$?

Solution: Since equivalence relations on $\{1, 2, 3, 4\}$ are in bijection with partitions of $\{1, 2, 3, 4\}$, we will count the number of partitions of $\{1, 2, 3, 4\}$. Any partition of $\{1,2,3,4\}$ has at least one part and at most four parts. There is 1 partition with one part (namely $\{\{1,2,3,4\}\}$) and 1 partition with four parts (namely $\{\{1\},\{2\},\{3\},\{4\}\}$). A partition with two parts either consists of a singleton set and a set with three elements (there are 4 such partitions) or two sets with two elements (there are $\binom{4}{2}/2 = 3$ such partitions), so there are 7 partitions with two parts. A parition with three parts consists of a set with two elements and two singleton sets, so there are $\binom{4}{2} = 6$ partitions with three parts. Thus, there are $1+7+6+1 = 15$ partitions of the set $\{1,2,3,4\}$, whence there are 15 equivalence relations on $\{1,2,3,4\}$.

15. How many partial orders are there on the set $\{1,2,3\}$?

Solution: We'll do this one by brute force. Every partial order on $\{1,2,3\}$, by virtue of being reflexive, is a superset of $\mathrm{id}_{\{1,2,3\}} = \{(1,1),(2,2),(3,3)\}$. Let P be a partial order on $\{1,2,3\}$ and set $P' = P \setminus \mathrm{id}_{\{1,2,3\}}$. We proceed in cases:

Case 0 ($|P'| = 0$) In this case, $P = \mathrm{id}_{\{1,2,3\}}$ (there's exactly one partial order P could be).

Case 1 ($|P'| = 1$) In this case, $P = \mathrm{id}_{\{1,2,3\}} \cup \{(i,j)\}$ for some $i,j \in \{1,2,3\}$ with $i \neq j$. In fact, for any $i,j \in \{1,2,3\}$ with $i \neq j$, $\mathrm{id}_{\{1,2,3\}} \cup \{(i,j)\}$ is a partial order on $\{1,2,3\}$, so there are exactly $3 \cdot 2 = 6$ partial orders P could be.

Case 2 ($|P'| = 2$) In this case, $P = \mathrm{id}_{\{1,2,3\}} \cup \{(i,j),(i',j')\}$ for some $i,j,i',j' \in \{1,2,3\}$ such that $i \neq j$ and $i' \neq j'$ and $(i,j) \neq (i',j')$. By the pigeonhole principle, at least two of the numbers i,j,i',j' must be equal (since there are 4 numbers each having one of 3 possible values). Therefore, $i = i'$ or $i = j'$ or $i' = j$ or $j = j'$. We split into subcases:

Case 2a ($|P'| = 2$, $i = i'$) In this case, $P = \mathrm{id}_{\{1,2,3\}} \cup \{(i,j),(i,j')\}$ for some $i,j,j' \in \{1,2,3\}$ such that $i \neq j$ and $i \neq j'$ and $j \neq j'$, i.e. such that $\{i,j,j'\} = \{1,2,3\}$. Indeed, for any i,j,j' such that $\{i,j,j'\} = \{1,2,3\}$, $\mathrm{id}_{\{1,2,3\}} \cup \{(i,j),(i,j')\}$ is a partial order. We can obtain 3 partial orders in this way (one for each choice of i), so there are exactly 3 partial orders P could be.

Case 2b ($|P'| = 2$, $i = j'$) By transitivity of P, $(i',j) \in P$. Then $i' = j$ or $(i',j) = (i,j)$ or $(i',j) = (i',j')$. If $i' = j$ then $(j',i') = (i,j) \in P$ so (by antisymmetry of P) $i' = j'$, a contradiction. If $(i',j) = (i,j)$ then $i' = i = j'$, again a contradiction. If $(i',j) = (i',j')$ then $i = j' = j$, also a contradiction. Thus, this case is impossible.

Case 2c ($|P'| = 2$, $i' = j$) Swap (i,j) with (i',j') at their introduction and repeat the proof, where Case 2b tells us this is impossible.

Case 2d ($|P'| = 2$, $i' = j'$) Let $P^{-1} = \{(x, y) \in \{1, 2, 3\} \times \{1, 2, 3\} : (y, x) \in P\}$. Then P^{-1} is a partial order on $\{1, 2, 3\}$ such that repeating the proof with P^{-1} in place of P would bring us to case 2a. Thus, there are exactly 3 partial orders P^{-1} could be. Since $P \mapsto P^{-1}$ is a bijection on the set of partial orders on $\{1, 2, 3\}$, there are exactly 3 partial orders P could be. Notably, the partials orders P could be in this case are disjoint from the partial orders constructed in Case 2a.

Thus, in Case 2 there are exactly 6 partial orders P could be.

Case 3 ($|P'| = 3$) Let $Q = \{(x, y) \in \{1, 2, 3\} \times \{1, 2, 3\} : (y, x) \in P'\}$. By antisymmetry of P, $Q \cap P' = \varnothing$. This tells us that $|Q \cup P'| = 6$, so $Q \cup P' = (\{1, 2, 3\} \times \{1, 2, 3\}) \setminus \mathrm{id}_{\{1,2,3\}}$. Thus, for any $i, j \in \{1, 2, 3\}$ such that $i \neq j$, we have $(i, j) \in P' \subseteq P$ or $(i, j) \in Q$, whence $(j, i) \in P' \subseteq P$. This shows that P is a total order on $\{1, 2, 3\}$, of which there are exactly 6 (one for each permutation of $\{1, 2, 3\}$). Indeed, it can be verified that all total orders on $\{1, 2, 3\}$ have exactly three non-reflexive pairs, so there are exactly 6 partial orders P could be.

Case 4 ($|P'| > 3$) Let S be the set of subsets of $\{1, 2, 3\}$ of size 2. Let $f : P' \to S$ be the map $f((i, j)) = \{i, j\}$. Since $|S| = \binom{3}{2} = 3 < |P'|$, the pigeonhole principle tells us that f is not injective, meaning we must have $(i, j), (j, i) \in P'$ for some $i, j \in \{1, 2, 3\}$ with $i \neq j$. Now antisymmetry of P tells us that $i = j$, a contradiction. Thus, this case is impossible.

Therefore, there are $1 + 6 + 6 + 6 = 19$ partial orders on the set $\{1, 2, 3\}$.

16. Define the sequence of natural numbers $(F_n)_{n\in\mathbb{N}}$ by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \in \mathbb{N}$ (you've probably seen this sequence before, it's the Fibonacci numbers). Show that

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

for all $n \in \mathbb{N}$.

Solution: Define

$$\alpha = \frac{1+\sqrt{5}}{2}$$
$$\beta = \frac{1-\sqrt{5}}{2}$$

so that (by the quadratic formula) α and β are both roots of $x^2 - x - 1$. This tells us that $1 + \alpha = \alpha^2$ and $1 + \beta = \beta^2$. We proceed by strong induction on n.

First Base Case By definition, $F_0 = 0 = \frac{1}{\sqrt{5}}(\alpha^0 - \beta^0)$.

Second Base Case $\alpha - \beta = \sqrt{5}$, so by definition $F_1 = 1 = \frac{1}{\sqrt{5}}(\alpha^1 - \beta^1)$.

Inductive Step Suppose $n \geq 1$ and $F_k = \frac{1}{\sqrt{5}}(\alpha^k - \beta^k)$ for all $k \leq n$. Then

$$\begin{aligned} F_n + F_{n-1} &= \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) + \frac{1}{\sqrt{5}}(\alpha^{n-1} - \beta^{n-1}) \\ &= \frac{1}{\sqrt{5}}\big(\alpha^{n-1}(\alpha+1) - \beta^{n-1}(\beta+1)\big) = \frac{1}{\sqrt{5}}(\alpha^{n+1} - \beta^{n+1}) = F_{n+1}, \end{aligned}$$

as desired.

By mathematical induction, we have the desired conclusion for all $n \in \mathbb{N}$.

17. Define the sequence of natural numbers $(d_n)_{n\in\mathbb{N}}$ by $d_0 = 0$, $d_1 = 1$, $d_2 = 2$, and $d_{n+3} = d_{n+2} + d_{n+1} + d_n$ for all $n \in \mathbb{N}$. Show that $d_n < 2^n$ for all $n \in \mathbb{N}$.

Solution: For any $n \in \mathbb{N}$, note that

$$d_{n+4} - d_{n+3} = (d_{n+3} + d_{n+2} + d_{n+1}) - (d_{n+2} + d_{n+1} + d_n) = d_{n+3} - d_n,$$

or in other words $d_{n+4} = 2d_{n+3} - d_n$. Thus, $d_{n+4} \leq 2d_{n+3}$ for all $n \in \mathbb{N}$. We proceed by strong induction on n.

Base Cases $d_n < 2^n$ for $n \in \{0, 1, 2, 3\}$ as can be checked directly.

Inductive Step Let $n \geq 3$ and suppose $d_k < 2^k$ for all $k \leq n$. Then

$$d_{n+1} = d_{(n-3)+4} \leq 2d_{(n-3)+3} = 2d_n < 2 \cdot 2^n = 2^{n+1},$$

as desired.

By mathematical induction, we are done.

18. How many functions are there $\{1, \ldots, n\} \to \{1, \ldots, m\}$?

Solution: There are m^n such functions: each element of the domain (of which there are n) can be sent to any element of the codomain (of which there are m). This can be proven rigorously by induction on n.

19. How many injections are there $\{1, \ldots, n\} \to \{1, \ldots, m\}$?

Solution: If $n > m$, there are no such injections by the pigeonhole principle. If $n \leq m$, the number of such injections is $P(m, n) = m!/(m-n)!$

20. Let A, B, and C be finite sets. Express the size of $A \cup B \cup C$ in terms of $|A|$, $|B|$, $|C|$, $|A \cap B|$, $|A \cap C|$, $|B \cap C|$, and $|A \cap B \cap C|$.

Solution: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$, as can be seen by applying inclusion-exclusion to $(A \cup B) \cup C$ then applying inclusion-exclusion again to $A \cup B$.

21. Let $X \subseteq \mathbb{N}$. If $|X| \geq 5$, show that there are some $a, b, c \in X$ such that 3 divides $a + b + c$.

Solution: Let x_1, x_2, x_3, x_4, x_5 be distinct elements of X. For each $i \in \{1, \ldots, 5\}$, we have (by Problem 6) that $x_i = 3a_i + b_i$ for some $a_i \in \mathbb{N}$ and $b_i \in \{0, 1, 2\}$. Let $f : \{x_1, \ldots, x_5\} \to \{0, 1, 2\}$ be the function $f(x_i) = b_i$. By the pigeonhole principle, f is not injective. Without loss of generality, $b_1 = b_2$. There are two cases: either $b_i = b_1$ for some $i > 2$ or not. In the former case,

$$x_1 + x_2 + x_i = 3a_1 + b_1 + 3a_2 + b_2 + 3a_i + b_i = 3(a_1 + a_2 + a_i + b_1)$$

is divisible by 3 and we are done. In the latter case, consider the restriction $f|_{\{x_3, x_4, x_5\}} : \{x_3, x_4, x_5\} \to \{0, 1, 2\} \setminus \{b_1\}$. By the pigeonhole principle again, this function is not injective, so (without loss of generality) $b_3 = b_4$. By the same argument as before, if $b_5 = b_i$ for any $i < 5$ we are done, so $\{b_1, b_3, b_5\} = \{0, 1, 2\}$. This means that

$$x_1 + x_3 + x_5 = 3a_1 + b_1 + 3a_3 + b_3 + 3a_5 + b_5 = 3(a_1 + a_3 + a_5) + 3$$

is disivisible by 3, as desired.

22. Show that n^7 divides $8!$ for some integer $n > 1$.

Solution:

$$8! = 2 \cdot 3 \cdot 4 \cdot 5 \cdot (2 \cdot 3) \cdot 7 \cdot 8 = (2 \cdot 4 \cdot 2 \cdot 8) \cdot 3 \cdot 5 \cdot 3 \cdot 7 = 2^7(3 \cdot 5 \cdot 3 \cdot 7)$$

is divisible by 2^7.

23. How many 5-card poker hands are flushes?

Solution: $4\binom{13}{5} = 5148$

Idea: Choose a suit and then choice which five cards to include from that suit.

24. How many 5-card poker hands have 3 of a kind?

Solution: $13 \cdot 48 + 52 \cdot \binom{48}{2} = 59280$

Idea: Split into cases of having 4 of a kind and having 3 of a kind but not 4 of a kind. In the former case, choose which value to include all 4 of, then choose which card from the remaining cards in the deck to put in the hand. In the latter case, choose one card in the deck, set it aside, and put the three other cards with that value in the hand, then pick any two cards from the remaining cards in the deck.

25. How many 5-card poker hands have two pairs?

Solution: $13 \cdot 48 + 52 \cdot 12 \cdot \binom{4}{2} + \binom{13}{2}\binom{4}{2}\binom{4}{2} \cdot 44 = 7176$

Idea: Split into cases of having 4 of a kind, having a full house, and having two pairs with different values but not a full house. The first case is the same as in Problem 24. For the second case, choose a card from the deck, set it aside, and put the three other cards with that value in the hand, then pick one of the 12 remaining values, and choose 2 suits from which to put the cards of that value into the hand. In the final case, we pick two values to have pairs of, then two suits for each pair to be in, then one of the remaining cards in the deck to complete the hand.

26. How many 5-card poker hands have 3 of a kind or two pairs?

Solution: $59280 + 7176 - 52 \cdot 12 \cdot \binom{4}{2} = 62712$

Idea: Inclusion-exclusion using Problems 24 and 25. The time you can have 3 of a kind and two pairs is when you have a full house.

27. My favorite band, "Half-Nicked", has 12 members. 6 of them are named Nick and the rest are not named Nick.

    (a) How many ways can the band members be lined up so that no two Nicks are next to each other?

Solution: $6! \cdot \binom{7}{6} \cdot 6! = 3628800$
Idea: First choose the order in which the non-Nicks should line up. Then choose 6 of the 7 spots adjacent to a non-Nick where the Nicks should go. Then choose an order for the Nicks to go into those spots.

(b) How many ways can the band members be lined up so that no five Nicks appear in a row?

Solution: $12! - 6! \cdot 7! - 6! \cdot 6! \cdot 7 \cdot 6 = 453600000$
Idea: Subtract the number of ways the band members can be lined up so that five Nicks do appear in a row from the total number of ways to line up the band members. Split into cases of six Nicks appearing in a row and five Nicks appearing in a row but not six. In the former case, choose an ordering of the Nicks then treat them as a single entity and order the 7 entities. In the latter case, choose an ordering of 5 Nicks, then choose an ordering of the 6 non-Nicks, then choose where (between two non-Nicks) the group of 5 Nicks should go, then (from the remaining spaces between non-Nicks) choose where the lone Nick should go.

28. Let $X \subseteq \{1, \ldots, 2n\}$ for some positive integer n and suppose $|X| = n + 1$.

(a) Show that there exist some $a, b \in X$ such that a and b have no prime factors in common.

Solution: Suppose for contradiction that every two elements $a, b \in X$ have a common prime factor. Then for any $a, b \in X$, we have that $a - b$ is divisible by a common prime factor of a and b, so $a - b = 0$ (in which case $a = b$) or $|a - b| \geq 2$. Write $X = \{x_0, \ldots, x_n\}$ where $x_0 < x_1 < \cdots < x_n$. Then $x_{i+1} \geq x_i + 2$ for all $i \in \{0, \ldots, n-1\}$ so $x_n \geq x_0 + 2n > 2n$, contradicting the assumption that $X \subseteq \{1, \ldots, 2n\}$.

(b) Show that there exists some $a \in X$ such that $a + 1 \in X$.

Solution: Essentially the same proof as Part (a).

(c) (($\star$)) Show that there exist some $a, b \in X$ such that a divides b.

Solution: Use induction on n.

29. Let $X \subseteq \{1, \ldots, 2018\}$ such that $|X| = 1010$. Prove that there exist $a, b \in X$ such that $a + b = 2019$.

Solution: For each $i \in \{1, \ldots, 1009\}$, define $S_i = \{i, 2019 - i\}$. Let $\mathcal{S}$ be the set $\{S_i : i \in \{1, \ldots, 1009\}\}$. $\mathcal{S}$ is a partition of $\{1, \ldots, 2018\}$ with 1009 parts since every $x \in \{1, \ldots, 2018\}$ is of the form i or $2019 - i$ for a unique $i \in \{1, \ldots, 1009\}$. Let $f : X \to \mathcal{S}$ be the function sending an element $x \in X$ to the unique element of $\mathcal{S}$ containing it. By the pigeonhole principle, f is not injective, so we have some $x, y \in X, i \in \{1, \ldots, 1009\}$ with $x \neq y$ and $x, y \in S_i$. This means that $\{x, y\} = \{i, 2019 - i\}$, so $x + y = i + 2019 - i = 2019$.

30. α places 5 dots inside an equilateral triangle of side length 2. Show that there must be two distinct dots of distance at most 1 from each other.

Solution: Divide the triangle into 4 equilateral triangles of side length 1 (a la Sierpiński). By the Pigeonhole Principle, one of these smaller triangles contains at least 2 dots. The distance between any two points in an equilateral triangle with side length 1 is at most 1, so we must have two dots of distance at most 1 from each other.

31. I have 17 kinds of socks and 200 pairs of each kind. If I blindly pull socks out of my (enormous) sock drawer in the morning, how many socks will I have to pull out to guarantee I have a pair? My socks are ambidextrous, i.e. each can be worn on either foot.

Solution: 18 by the pigeonhole principle.

32. (a) Find a sequence $(a_n)_{n \in \mathbb{N}}$ of real numbers such that $a_0 = 17$ and $a_{n+2} = 3a_n$ for all $n \in \mathbb{N}$.

    Solution: $a_n = 17 \cdot 3^{n/2}$.

    (b) How many sequences of real numbers $(a_n)_{n \in \mathbb{N}}$ satisfy $a_0 = 1$ and $a_{n+2} = 3a_n + 3$ for all $n \in \mathbb{N}$?

    A. 1

    B. 2

    C. More than 2 but finitely many

    D. Infinitely many

    Solution: We know there are infinitely many sequences $(b_n)_{n \in \mathbb{N}}$ of real numbers such that $b_0 = 5/2$ and $b_{n+2} = 3b_n$ for all $n \in \mathbb{N}$ (as this is a second-order linear homogeneous recurrence relation with positive discriminant). Given such a sequence $(b_n)_{n \in \mathbb{N}}$, define $a_n = b_n - 3/2$ and note that we have

$a_0 = 1$ and $3a_n + 3 = 3(b_n - 3/2) + 3 = 3b_n - 3/2 = b_{n+2} - 3/2 = a_{n+2}$ for all $n \in \mathbb{N}$, so $(a_n)_{n\in\mathbb{N}}$ is one of the sequences in question. The function from the set of sequences of real numbers to itself defined by mapping a sequence $(b_n)_{n\in\mathbb{N}}$ to $(b_n - 3/2)_{n\in\mathbb{N}}$ is a bijection, so there are infinitely many sequences satisfying the conditions from part (a).

33. Let $a_0 = 1$, $a_1 = 0$, and $a_{n+2} = (n+1)(a_{n+1} + a_n)$ for all $n \in \mathbb{N}$.

(a) Is this recurrence relation homogeneous?

A. Yes B. No

(b) Is this recurrence relation linear?

A. Yes B. No

(c) What is the order of this recurrence relation?

Solution: 2

(d) Find the value of a_5.

Solution: 44

(e) Prove that $a_{n+1} = (n+1)a_n + (-1)^{n+1}$ for all $n \in \mathbb{N}$.

Solution: We use induction on n.

Base Case ($n = 0$) $(0+1)a_0 + (-1)^{0+1} = 1 - 1 = 0 = a_1$, as desired.

Inductive Step Suppose $a_{n+1} = (n+1)a_n + (-1)^{n+1}$ for some $n \in \mathbb{N}$. Then $(n+1)a_n = a_{n+1} + (-1)^{n+2}$ so we have

$$\begin{aligned} a_{n+2} &= (n+1)(a_{n+1} + a_n) = (n+1)a_{n+1} + (n+1)a_n \\ &= (n+1)a_{n+1} + a_{n+1} + (-1)^{n+2} = (n+2)a_{n+1} + (-1)^{n+2}. \end{aligned}$$

By mathematical induction, the desired result holds for all $n \in \mathbb{N}$.

(f) Prove that

$$a_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

for all $n \in \mathbb{N}$.

Solution: We leverage Part (e) by induction on n.

Base Case ($n = 0$) $0! \sum_{i=0}^{0} (-1)^i / i! = 1 = a_0$.

Inductive Step Suppose $a_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$ for some $n \in \mathbb{N}$. Then

$$a_{n+1} = (n+1)a_n + (-1)^{n+1} = (n+1)n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} + (-1)^{n+1}$$
$$= (n+1)! \sum_{i=0}^{n} \frac{(-1)^i}{i!} + (n+1)! \frac{(-1)^{n+1}}{(n+1)!} = (n+1)! \sum_{i=0}^{n+1} \frac{(-1)^i}{i!},$$

as desired.

(g) (($\star$)) Determine the convergence of

$$\lim_{n\to\infty} \frac{a_n}{n!}$$

and, if it converges, find its value.

Solution: This won't make sense if you haven't seen Maclaurin/Taylor series (and can't be done without that knowledge without a lot of work). By Part (f),

$$\lim_{n\to\infty} \frac{a_n}{n!} = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{(-1)^i}{i!} = e^{-1}.$$

34. (($\star$)) Let $a_0 = 1$, $a_1 = e$, $a_2 = \pi$, and

$$a_{n+3} = (\pi + e + 1)a_{n+2} - (\pi + e + \pi e)a_{n+1} + \pi e a_n$$

for all $n \in \mathbb{N}$. Find a formula for a_n. *Hint: 1 is a root of the characteristic polynomial.*

Solution:

$$a_n = \alpha + \beta\pi^n + \gamma e^n$$

where

$$\alpha = \frac{\pi - e^2}{(e-1)(\pi-1)},$$
$$\beta = \frac{e^2 - \pi}{(e-\pi)(\pi-1)},$$
$$\gamma = \frac{e - 2\pi + e\pi}{e - e^2 - \pi + e\pi}.$$

35. Let $a_0 = 1$, $a_1 = 2$, and $a_{n+2} = 4a_{n+1} - 4a_n$. Find a formula for a_n.

Solution: $a_n = 2^n$.

36. (($\star$)) Let $a_0 = 1$, $a_1 = 1$, and $a_{n+2} = 4a_{n+1} - 4a_n$. Find a formula for a_n.

Solution: $a_n = -2^{n-1}(n-2)$.

37. Let $a_0 = 42$, $a_1 = 0$, and $3a_{n+2} = 11a_{n+1} + 4a_n$. Find a formula for a_n.

Solution:

$$a_n = \frac{56}{13 \cdot 3^{n-2}}((-1)^n + 12^{n-1}).$$

38. Let G be a simple graph with at least two vertices (and having only finitely many in total). Show that there must be two vertices in G which have the same degree.

Solution: Let V be the set of vertices in G and let $f : V \to \mathbb{N}$ be the function sending a vertex to its degree. The range of f is a subset of $\{0, \ldots, n-1\}$, and the range of f cannot contain both 0 and $n-1$ since then there would be a vertex adjacent to every vertex (including the following vertex) and a vertex which is adjacent to no vertex (including the previous vertex). Thus, the range of f is a subset of either $\{1, \ldots, n-1\}$ or $\{0, \ldots, n-2\}$, each of which is a set of size $n - 1 < n = |V|$. By the pigeonhole principle, f is not injective, so G has two distinct vertices of the same degree.

39. Let G be a connected graph with $n > 0$ vertices. What's the least number of edges G could have?

Solution: $n - 1$.

Idea: Suppose G is a connected graph with $n > 0$ vertices having the minimum possible number of edges among all such graphs. It can be shown that G has a vertex of degree 1, so that by removing it (and the incident edge) you get a connected graph with $n - 1$ vertices. Use this to formulate an inductive proof that $n - 1$ is a lower bound for the number of edges in a connected graph with $n > 0$ vertices. Finally, construct a connected graph with n vertices and $n - 1$ edges for each $n > 0$.

40. How many nonempty subgraphs of K_4 have an Euler cycle?

Solution: 11 by direct counting.

41. For which positive integers n, m does $K_{n,m}$ have an Euler cycle?

Solution: $K_{n,m}$ has $n > 0$ vertices of degree m and $m > 0$ vertices of degree n. Thus, $K_{n,m}$ has an Euler cycle if and only if n and m are even.

42. How many simple graphs have a subset of $\{1, 2, 3, 4, 5\}$ as their set of vertices?

Solution: The number of simple graphs with a fixed set of i vertices is $2^{\binom{i}{2}}$. There are $\binom{5}{i}$ subsets of size i in $\{1, \ldots, 5\}$. Thus, the number of simple graphs whose set of vertices is a subset of $\{1, \ldots, 5\}$ is

$$\sum_{i=0}^{5} \binom{5}{i} 2^{\binom{i}{2}} = 1450.$$

This is also the number of subgraphs of K_5.

43. Does every graph have a complete, connected, simple subgraph?

A. Yes B. No

Solution: The empty graph is complete, connected, simple, and a subgraph of every graph. Even if we only consider nonempty graphs, any vertex of a nonempty graph (taken on its own without any edges) forms a complete, connected, simple, nonempty subgraph.

44. Let G be the simple graph whose vertices are all n-bit binary strings, where there's an edge between two binary strings whenever the first bit of each string is the last bit of the other. For which positive integers n does G have an Euler cycle?

Solution: Let s be the n-bit string of all 0's. By a simple inductive proof, any vertex connected to s has 0 as its first and last bit, so the n-bit string of all 1's is not connected to s. This is means G is never connected for any positive integer n, so G never has an Euler cycle.

45. Let G be a simple graph with 18 vertices and 109 edges such that the degree of each vertex is at least 2. Prove that G is connected.

Solution: Suppose for contradiction that G is not connected. Then G has at least two components. Since each vertex in G has degree at least 2, each component of G has at least 3 vertices. Let N be the number of vertices in the largest component of G, so that $N \leq 15$. Then the number of edges in G is at most

$$\binom{N}{2} + \binom{18-N}{2} = \frac{N(N-1) + (18-N)(17-N)}{2} = N^2 - 18N + 108.$$

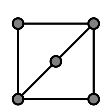
Since $N \leq 15 < 18$, $N^2 < 18N$, so the number of edges in G is less than 108, a contradiction.

46. Let G be a subgraph of K_5 having 7 edges. Show that G has a complete subgraph with 3 vertices. Also, find a subgraph of K_5 having 6 edges and no complete subgraph with 3 vertices.

Solution:

Proof. G must have 5 vertices since any simple graph with 4 or fewer vertices has at most 6 edges. Suppose for the sake of contradiction that G has no complete subgraph with 3 vertices. Consider first the possibility that G has a vertex v of degree at least 3; say v is adjacent to the distinct vertices w_1, w_2, w_3. Then there cannot be any edges between the w_i's in G or we would have a complete subgraph with vertices v and those two w_i's. There are three edges between the w_i's in K_5 so the edges of G must come from the remaining 7 edges of K_5. But G has 7 edges, so G must contain all the remaining edges in K_5. This means that G has a complete subgraph with three vertices (namely v, the remaining unnamed vertex of G, and w_1) and we have a contradiction. Now we must consider the case where every vertex of G has degree at most 2. Let $d_1, d_2, \ldots, d_5$ be the degrees of the vertices of G. Since G has 7 edges, $14 = d_1+d_2+d_3+d_4+d_5 \leq 2+2+2+2+2 = 10$, again a contradiction. Thus, it cannot be the case that G has no complete subgraph with 3 vertices; i.e. G has a complete subgraph with 3 vertices. $\square$

An example of a subgraph of K_5 with 6 edges and no complete subgraph with 3 vertices is depicted below:



47. How many connected simple graphs have $\{1, 2, 3, 4\}$ as their vertex set?

Solution: 38 by direct counting.