Linear Algebra I

In this chapter, we will introduce linear systems and the method of row reduction to solve them.

A linear equation with n variables $x_1, x_2, \ldots, x_n$ is an equation that can be expressed in the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

where $a_1, a_2, a_3, \ldots, a_n$ and b are real constants.

A system of m linear equations in n unknown variables $x_1, x_2, \ldots, x_n$ is a collection of m equations of the form

$$\begin{aligned}
a_{11}x_1 &+ a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 &+ a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\
a_{31}x_1 &+ a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\
\vdots & \qquad \vdots \qquad\quad \vdots \\
a_{m1}x_1 &+ a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m
\end{aligned}$$

The numbers a_{ij} are called the coefficients of the linear system; Because there are m equations and n unknown variables there are therefore $m \times n$ coefficients. The main problem with a linear system is of course to solve it.

NOTE!!! (a) A linear system may not have a solution at all. If this is the case, we say that the linear system is INCONSISTENT.

(b) A linear system with a solution is called CONSISTENT. The solution might be [illegible] equation.

1. No Solution
2. Exactly one Solution (Unique)
3. Infinitely Many Solution.

We will use matrices to develop systematic method to solve system of linear equations and to study the properties of the solution set. Matrix is an array of number (objects) in rows and columns. For example

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

A is a 3×3 matrix. ~~The concepts of matrices should have been dealt~~ For details about matrices see MTH 101. However, we are still going to discuss this going forward.

We can associate to a linear system three matrices (1) the coefficient matrix (2) the output column vector and (3) the augmented matrix. For example, for the linear system

$$\begin{aligned} 5x_1 - 3x_2 + 8x_3 &= -1 \\ x_1 + 4x_2 - 6x_3 &= 0 \\ 2x_2 + 4x_3 &= 3 \end{aligned}$$

the coefficient matrix A, the output vector b and the augmented matrix $[A \vdots b]$ are

$$A = \begin{bmatrix} 5 & -3 & 8 \\ 1 & 4 & -6 \\ 0 & 2 & 4 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$

$$[A \vdots b] = \left[\begin{array}{ccc|c} 5 & -3 & 8 & -1 \\ 1 & 4 & -6 & 0 \\ 0 & 2 & 4 & 3 \end{array}\right]$$

NOTE!!! If given an augmented matrix we can write down the associated linear system in an easy way. For example, the linear system associated to the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 4 & -2 & 8 & 12 \\ 0 & 1 & -7 & 2 & -4 \\ 0 & 0 & 5 & -1 & 7 \end{array}\right]$$

is

$$x_1 + 4x_2 - 2x_3 + 8x_4 = 12$$
$$x_2 - 7x_3 + 2x_4 = -4$$
$$5x_3 - x_4 = 7$$

There are three basic operations, called ELEMENTARY OPERATION, that can be performed:

1. Interchange two equations
2. Multiply an equation by a nonzero constant
3. Add a multiple of one equation to another

<u>Example</u>: Solve the linear system using elementary row operations.

$$-3x_1 + 2x_2 + 4x_3 = 12$$
$$x_1 - 2x_3 = -4$$
$$2x_1 - 3x_2 + 4x_3 = -3$$

<u>Solution</u>

Our goal is to perform elementary row operations to obtain a triangular structure and then use the back substitution to solve. The augmented matrix is

$$\left[\begin{array}{ccc|c} -3 & 2 & 4 & 12 \\ 1 & 0 & -2 & -4 \end{array}\right.$$

$$\left[\begin{array}{ccc|c} -3 & 2 & 4 & 12 \\ 1 & 0 & -2 & -4 \\ 2 & -3 & 4 & -3 \end{array}\right] \xleftrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -4 \\ -3 & 2 & 4 & 12 \\ 2 & -3 & 4 & -3 \end{array}\right]$$

NOTE!!! The Aim to get something of the form

$$\left[\begin{array}{ccc|c} 1 & b & a & e \\ 0 & 1 & c & f \\ 0 & 0 & 1 & g \end{array}\right]$$

We will discuss in detail in class.

So we have to make -3 zero. Way to

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -4 \\ -3 & 2 & 4 & 12 \\ 2 & -3 & 4 & -3 \end{array}\right] \xleftrightarrow{R_2 = R_2 + 3R_1} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -4 \\ 0 & 2 & -2 & 0 \\ 2 & -3 & 4 & -3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -4 \\ 0 & 2 & -2 & 0 \\ 2 & -3 & 4 & -3 \end{array}\right] \xleftrightarrow{R_3 = R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -4 \\ 0 & 2 & -2 & 0 \\ 0 & -3 & 8 & 5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -4 \\ 0 & 2 & -2 & 0 \\ 0 & -3 & 8 & 5 \end{array}\right] \xleftrightarrow{R_2 = R_2 \times 1/2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -4 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 8 & 5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -4 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 8 & 5 \end{array}\right] \xleftrightarrow{R_3 = R_3 + 3R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 5 \end{array}\right] \xleftrightarrow{R_3 = R_3 \times 1/5} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Since, the row reduced augmented matrix is in triangular form, we can apply the backward substitution the linear system associated to the row reduced augmented matrix is

$$x_1 - 2x_3 = -4$$
$$x_2 - x_3 = 0$$
$$x_3 = 1$$

Thus, we have $(x_1, x_2, x_3) = (-2, 1, 1)$

Now NOTE!!! After Reduction, if you have the following cases; here is what to do

① $\left[\begin{array}{ccc|c} 1 & a & c & A \\ 0 & 1 & d & B \\ 0 & 0 & 1 & C \end{array}\right]$ You have a Unique Solution

We are considering the case in the last row. C can be any number.

② $\left[\begin{array}{ccc|c} 1 & a & c & A \\ 0 & 1 & d & B \\ 0 & 0 & 0 & 0 \end{array}\right]$ You have a Solution, here the Solution is Infinitely Many

③ $\left[\begin{array}{ccc|c} 1 & a & c & A \\ 0 & 1 & d & B \\ 0 & 0 & 0 & C \end{array}\right]$ No Solution

Here $C \neq 0$. Thus, we have no Solution.

<u>Example</u>: Using elementary row operations, Solve the System of equations

$$x_1 + 2x_3 = 1$$
$$x_2 + x_3 = 0$$
$$2x_1 + 4x_3 = 1$$

Solution: The augmented Matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{array}\right] \longleftarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

Clearly, we have the third case, where $0 = -1$. Thus, the system of equations has no solution.

Example: For what value of k will the linear system below be consistent.

$$x_1 + 3x_2 = 4$$
$$9x_1 + x_2 = k$$
$$-3x_1 + 7x_2 = 2.$$

Solution

Augmented Matrix

$$\left[\begin{array}{cc|c} 1 & 3 & 4 \\ 9 & 1 & k \\ -3 & 7 & 2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 3 & 4 \\ 9 & 1 & k \\ -3 & 7 & 2 \end{array}\right] \underset{R_3 = R_3 + 3R_1}{\overset{R_2 = R_2 - 9R_1}{\longleftrightarrow}} \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & -26 & k-36 \\ 0 & 16 & 14 \end{array}\right]$$

$$\overset{R_2 = R_2 \times -1/26}{\longleftrightarrow} \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 1 & k-36/26 \\ 0 & 16 & 14 \end{array}\right]$$

$$\overset{R_3 = R_3 - 16R_2}{\longleftrightarrow} \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 1 & \frac{k-36}{26} \\ 0 & 0 & \frac{470-8k}{13} \end{array}\right]$$

We complete this in class.

Example: Determine which values of k, if any, will give (a) Unique Solution (b) No Solution (c) Infinitely Many Solutions to the following system of equations.

$$x + 2y - z = 1$$
$$2x + 3y + z = 1$$
$$-4x - 5y + (k^2-9)z = k+1$$

Solution

$$\left[\begin{array}{ccc|c} 4 & 2 & -1 & 1 \\ 2 & 3 & 1 & 1 \\ -4 & -5 & k^2-9 & k+1 \end{array}\right]$$

$R_2 \Rightarrow R_2 - 2R_1$
$R_3 \Rightarrow R_3 + 4R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & -1 \\ 0 & 3 & k^2-13 & k+5 \end{array}\right]$$

$R_2 \Rightarrow R_2 \times (-1)$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 3 & k^2-13 & k+5 \end{array}\right]$$

$R_3 \Rightarrow R_3 - 3R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & k^2-4 & k+2 \end{array}\right]$$

(a) Unique Solution. $[A|B]$ $\quad A \neq 0,\ B \neq 0$

We know that $k^2 - 4 \neq 0$ and $k+2 \neq 0$.
$\Rightarrow k \neq 2,\ k \neq -2 \qquad k \neq -2$.

The solution is $k \in \mathbb{R}$ such that $k \neq -2$ & $k \neq 2$.

If $k = 1$, we have the last row to be
$0 \quad 0 \quad -3 \mid 3$

(b) No Solution. $[0|A]$ $\quad A \neq 0$

We know that $k^2 - 4 = 0$
$\Rightarrow k = 2,\ k = -2$, However, if

$k=2,\ k=-2$ $\qquad$ $k=-2$.

Note if $K=2$, we will have $K+2=4$, thus we will not have the desire solution. Thus

$$\boxed{K=-2}$$

Example: Find all values of k which the system has (a) Unique solution (b) No solution (c) Infinitely many solutions.

$$x+z=k^2$$
$$2x+y+3z=-3k$$
$$3x+y+4z=-2.$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & k^2 \\ 2 & 1 & 3 & -3k \\ 3 & 1 & 4 & -2 \end{array}\right]$$

$R_2 \Rightarrow R_2 - 2R_1$
$R_3 \Rightarrow R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & k^2 \\ 0 & 1 & 1 & -3k-2k^2 \\ 0 & 1 & 1 & -2-3k^2 \end{array}\right]$$

$R_3 \Rightarrow R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & k^2 \\ 0 & 1 & 1 & -3k-2k^2 \\ 0 & 0 & 0 & -2+3k-k^2 \end{array}\right]$$

NOTE the last row $[0 \ 0 \ 0 \mid -2+3k-k^2]$

(a) Unique Solution $[A \mid B]$, $A\neq 0, B\neq 0$:
there is no value of k for unique solution.

(b) No solution $[0 \mid A]$, $A\neq 0]$
thus, we have
$$-2+3k-k^2\neq 0$$
$$k^2-3k+2\neq 0$$
$$k\neq 1,\ k\neq 2.$$

$k \in \mathbb{R}$ such that $k \neq$, $k \neq 2$.

© Infinitely Many Solutions $[0 \mid 0]$
Thus, we must have

$$k^2 - 3k + 2 = 0$$
$$k = 1 \text{ and } k = 2.$$

Thus, for infinitely many solutions $k = 1$, $k = 2$.

* Find the Solution of the System of equations

$$2x + 2y + 6z = 14$$
$$2x - y + 3z = 5$$

Solution

$$\left[\begin{array}{ccc|c} 2 & 2 & 6 & 14 \\ 2 & -1 & 3 & 5 \end{array}\right] \xrightarrow{R_1 \Rightarrow R_1 \times 1/2} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 7 \\ 2 & -1 & 3 & 5 \end{array}\right]$$

$$\xrightarrow{R_2 = R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 7 \\ 0 & -3 & -3 & -9 \end{array}\right]$$

$$R_2 \Rightarrow R_2 \times -1/3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 7 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

Note that there is no space for a third pivot, so this is as far as the matrix can be reduced. The system now takes the form

$$x + y + 3z = 7$$
$$y + z = 3 \Rightarrow y = 3 - z.$$

Here, is what we introduce the notion of free variable. So, we say, let
$z = t \in \mathbb{R}$, so that

$$y = 3 - t$$
$$x = 7 - 3z - y = 7 - 3t - (3 - t)$$
$$= 7 - 3t - 3 + t$$
$$= 7 - 3 - 3t + t$$

$$x_1 + 3x_2 - 4x_3 + 4x_4 = 4$$
$$x_1 + 4x_2 - 7x_3 + 6x_4 = 3$$

Soln

$$\left[\begin{array}{cccc|c} 1 & 3 & -4 & 4 & 4 \\ 1 & 4 & -7 & 6 & 3 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 3 & -4 & 4 & 4 \\ 0 & 1 & -3 & 2 & -1 \end{array}\right]$$

Clearly, we have

$$x_1 + 3x_2 - 4x_3 + 4x_4 = 4$$
$$x_2 - 3x_3 + 2x_4 = -1$$
$$\Rightarrow x_2 = -1 + 3x_3 - 2x_4.$$

Thus, we have x_3 and x_4 to be free variables. Thus, we say Let $x_3 = s \in \mathbb{R}$ $x_4 = t \in \mathbb{R}$.

$$x_2 = -1 + 3s - 2t.$$

$$x_1 = 4 - 3x_2 - 4x_3 + 4x_4$$
$$= 4 - 3(-1 + 3s - 2t) - 4s + 4t$$
$$= 7 - 5s + 2t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 - 5s + 2t \\ -1 + 3s - 2t \\ s \\ t \end{bmatrix}$$

Example: Solve the system of equations

$$x + 9y - z = 27$$
$$x - 8y + 16z = 10$$
$$2x + y + 15z = 37$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 9 & -1 & 27 \\ 1 & -8 & 16 & 10 \\ 2 & 1 & 15 & 37 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 9 & -1 & 27 \\ 0 & -17 & 17 & -17 \\ 0 & -17 & 17 & -17 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 9 & -1 & 27 \\ 0 & 1 & -1 & 1 \\ 0 & -17 & 17 & -17 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 9 & -1 & 27 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We don't have to consider the last row. we start from the second row

$$x + 9y - z = 27$$

$$y - z = 1$$

$$\Rightarrow y = 1 + z \quad \text{let } z = t \in \mathbb{R}$$

$$y = 1 + t$$

$$x = 27 - 9y + z = 27 - 9(1+t) + t$$

$$= 18 - 8t$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 18 - 8t \\ 1 + t \\ t \end{bmatrix}$$

Example: Find the solution of the system of equations

$$-2x_1 + 4x_2 + 2x_3 - 8x_4 + 4x_5 = -8$$

$$3x_1 - 6x_2 - 2x_3 + 11x_4 - 7x_5 = 13$$

$$x_1 - 2x_2 - 5x_3 + 8x_4 + x_5 = -2$$

$$\left[\begin{array}{ccccc|c} 3 & -6 & -1 & 10 & -8 & 14 \\ 1 & -2 & -5 & 8 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & -5 & 8 & 1 & 3 \\ 3 & -6 & -1 & 10 & -8 & 14 \\ -2 & 4 & 2 & -8 & 4 & -8 \end{array}\right]$$

OR

$$\left[\begin{array}{ccccc|c} \underline{1} & -2 & -1 & 4 & -2 & 4 \\ 3 & -6 & -1 & 10 & -8 & 14 \\ 1 & -2 & -5 & 8 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & -1 & 4 & -2 & 4 \\ 0 & 0 & 2 & -2 & -2 & 2 \\ 0 & 0 & -4 & 4 & 3 & 7 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & -1 & 4 & -2 & 4 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & -4 & 4 & 3 & 7 \end{array}\right]$$

$$\left[\begin{array}{ccc} 1 & -2 & 0 \\ 0 & 0 & [illegible] \\ 0 & 0 & [illegible] \end{array}\right.$$

We will complete this in class.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ t \\ ? \end{bmatrix}$$

See you in class.