Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2023

Junior Section (Round 1)

Tuesday, 30 May 2023 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1. In a certain company, one-third of the employees have a university degree. Half of the employees who have a university degree are men and 40% of the employees who do not have a university degree are women. If 102 of the employees are men, how many of the employees are women?

   (A) 58 (B) 78 (C) 98 (D) 108 (E) 118

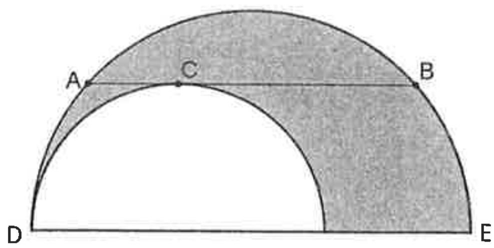
2. How many non-congruent triangles with integer side lengths have perimeter 7?

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

3. The median and mean of five distinct numbers, $4, 7, 10, 11, N$, are equal. Find the sum of all possible values of N.

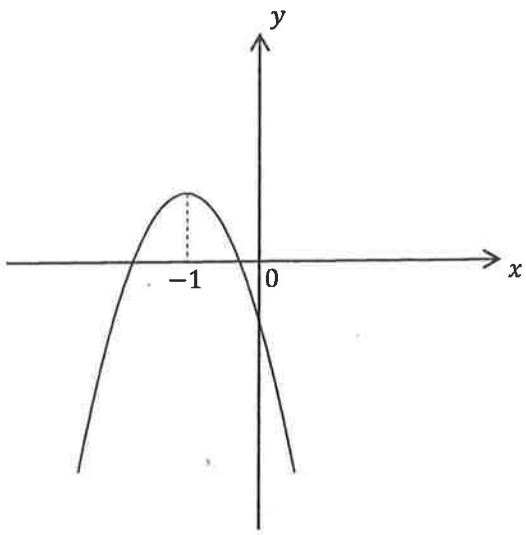
   (A) 18 (B) 21 (C) 26 (D) 29 (E) 35

4. The following diagram shows two semicircles whose diameters lie on the same line. AB is a chord of the larger semicircle that is tangent to the smaller semicircle at the point C and is parallel to the diameter DE of the larger semicircle. If $|AB| = 16$ cm, what is the area of the shaded region in cm^2?



   (A) 8π (B) 16π (C) 32π (D) 48π (E) 64π

5. The graph (not drawn to scale) of $y = ax^2 + bx + c$ with the maximum point at $x = -1$ is shown below. Which of the following must be true?



(I) $a - b + c > 0$ (II) $a + b - c > 0$ (III) $abc > 0$ (IV) $abc < 0$

(A) (I) and (IV)

(B) (II) and (IV)

(C) (I) and (III)

(D) (II) and (III)

(E) (I)

Short Questions

6. The product of the ages of three adults is 26390. Find the sum of their ages. (A person is an adult if he or she is at least 21 years old.)

7. Let n be a positive integer such that $n + 11$ is a factor of $n^2 + 121$. Find the largest possible value of n.

8. Find the largest integer less than or equal to $(3 + \sqrt{5})^3$.

9. The product of the two-digit number $\overline{x4}$ and the three-digit number $\overline{3yz}$ is 7656. Find the value of $x + y + z$.

10. If x and y are real numbers such that $x + y = 12$ and $xy = 10$, find the value of $x^4 + y^4$.

11. Let a and b be two positive integers such that $a^2 + 4b^2 \leq 1105$. Find the largest possible value of the product ab.

12. Find the value of the integer n such that the following equation holds:

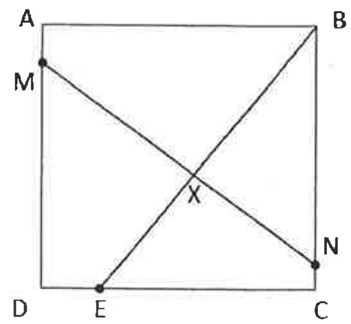
$$\frac{\sqrt{5} + n\sqrt{3} - 2\sqrt{2}}{(\sqrt{5} + \sqrt{3})(\sqrt{3} - \sqrt{2})} = \sqrt{5} + \sqrt{2}.$$

13. The lengths of the sides of a triangle are $\log_{10} 12$, $\log_{10} 75$ and $\log_{10} n$ where n is a positive integer. Find the number of possible values for n.

14. Let x be a real number. What is the minimum value of the following expression?

$$\frac{20x^2 + 10x + 3}{2x^2 + x + 1}$$

15. In the following diagram, ABCD is a square of side 16 cm. E lies on CD such that $|DE| = 4$ cm. M and N lie on AD and BC respectively such that MN is perpendicular to BE. X is the intersection of MN and BE. If $|MX| = 11$ cm and $|BN| = x$ cm, what is the value of x?



16. Find the smallest positive integer k such that every subset of $\{1, 2, \dots, 2023\}$ with exactly k integers contains three integers a, b, c with $a < b < c$ and $ab = c$.

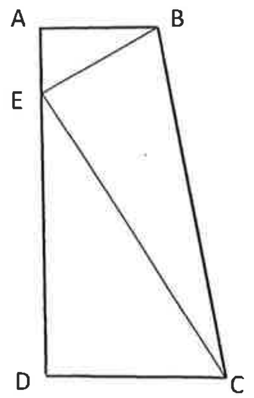
17. The difference between the least common multiple (LCM) and highest common factor (HCF) of a positive integer n and 18 is 627. Find the value of n.

18. The sum of all the interior angles except one of a convex polygon is 2023°. What is the number of sides of this polygon? (A polygon is convex if every interior angle is between 0° and 180°.)

19. If $\sqrt{19 - 8\sqrt{3}}$ is a root of the equation $x^2 - ax + b = 0$ where a and b are rational numbers, find the value of $a + b$.

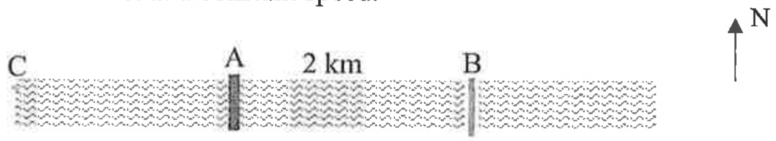
20. Find the smallest positive integer n such that the equation $x^2 + y^3 = n^4$ has a solution in positive integers x and y.

21. In the following diagram, AB is parallel to DC, |AB| = 6 cm, |AD| = 17 cm, |DC| = 10 cm and angle DAB = 90°. E lies on AD such that BE is perpendicular to EC. If the area of triangle BEC = k cm^2, what is the largest possible value of k?



22. The angle between the hour-hand and the minute-hand of a clock is denoted by $\theta°$ where $0 \leq \theta \leq 180$. If the clock now shows 3 : y (that is, 3 o'clock and y minutes) and $\theta = 175$, find the value of y, where y is an integer and $0 \leq y < 60$.

23. The diagram below shows a river with two bridges at A and B that are 2 km apart. The water flows from west to east at a constant speed.



A man rowed a boat upstream. He dropped his water bottle at A but he only noticed it at C after rowing for another 20 minutes upstream from A. He immediately turned back and rowed his boat downstream, and was able to pick up his water bottle at B. Assume that the man rowed his boat at a constant speed.

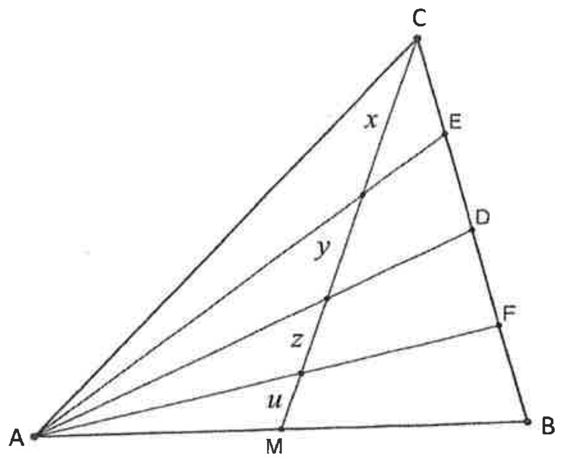
Find the speed of the water flowing in the river in km/h.

24. Four positive integers x, y, z and w satisfy the following equations:

$$xy + x + y = 104$$
$$yz + y + z = 146$$
$$zw + z + w = 524$$

If the product $xyzw = 2^7 \times 3^2 \times 5 \times 7$, find the value of $x + y + z + w$.

25. In the following diagram, ABC is a triangle. Points E, D and F lie on the side BC and divide the side into four equal parts. M is the midpoint of AB and CM intersects the line segments AE, AD and AF into the ratio $x : y : z : u$ where $x \geq y \geq z \geq u$ and x, y, z and u are integers. Find the least possible value of $x + y + z + u$.

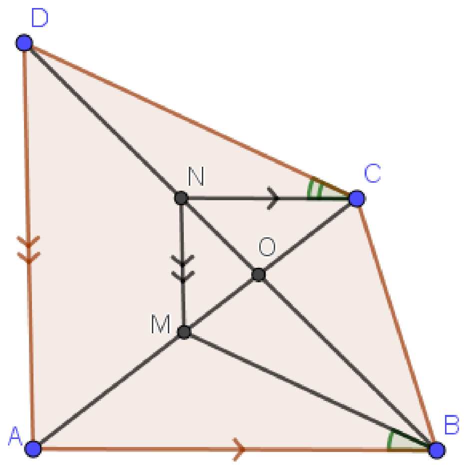


SMO 2023 (Junior Section) Answers

1. B
2. B
3. D
4. C
5. A
6. 90
7. 231
8. 143
9. 12
10. 15176
11. 276
12. 3
13. 893
14. 2
15. 15
16. 1981
17. 105
18. 14
19. 21
20. 6
21. 75
22. 50
23. 3
24. 64
25. 105

SMO Junior 2023 2nd Round (Solutions)

1. In a convex quadrilateral $ABCD$, the diagonals intersect at O, M and N are points on the segments OA and OD respectively. Suppose MN is parallel to AD and NC is parallel to AB. Prove that $\angle ABM = \angle NCD$.



Solution:

We claim $CD||BM$.

Since $CN||AB$, $\Delta CON \sim \Delta AOB$. Hence, $\frac{CO}{AO} = \frac{ON}{OB}$, giving $AO \cdot ON = CO \cdot OB$.

Since $MN||AD$, $\Delta MON \sim \Delta AOD$. Hence, $\frac{ON}{OD} = \frac{MO}{AO}$, giving $AO \cdot ON = MO \cdot OD$.

Equating, $CO \cdot OB = MO \cdot OD \Rightarrow \frac{CO}{OD} = \frac{MO}{OB}$. By SAS similarity, $\Delta COD \sim \Delta MOB$.

This implies $\angle DCO = \angle OMB$. Since we also have $\angle NCO = \angle OAB$ (from $CN||AB$), we conclude $\angle DCN = \angle DCO - \angle NCO = \angle OMB - \angle OAB = \angle ABM$ as desired.

2. What is the maximum number of integers that can be chosen from $1,2,\ldots,99$ so that the chosen integers can be arranged in a circle with the property that the product of every pair of neighbouring integers is a 3-digit integer.

Solution:

We claim there are at most 59 numbers.

Proof:

Clearly, 1 cannot be used in any arrangement.

Now call the 30 numbers $2,\ldots,31$ "small" and remaining numbers $32,33,\ldots,99$ "big". Note that since $32 \times 33 > 999$, no two big numbers can be adjacent in such an arrangement.

Since there are 30 small numbers, this implies there are at most 30 big numbers, otherwise two of them would be adjacent. Hence, there are at most 60 big numbers in total.

However, if there are exactly 60 numbers, then the small and big numbers must alternate. Yet among the big numbers, 31 can only be adjacent to 32 since $31 \times 33 = 1023 > 1000$. So 60 numbers is not achievable, and we have at most 59.

Construction:

Arrange the numbers 2 to 60 (in a circle) as follows:

$$2,60,3,59,4,58,\ldots,30,32,31.$$

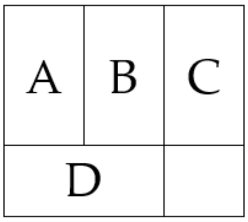
In this construction, each pair of adjacent numbers (other than $(31,2)$) clearly sums to either 62 or 63. Thus, their product ranges from $2 \times 60 = 120$ to $31 \times 32 = 992$, which are all 3-digit numbers.

However, $2 \times 31 = 62$ is not a 3-digit number, so we rectify this by simply swapping the numbers 2 and 4, giving us:

$$4,60,3,59,2,58,\ldots,30,32,31.$$

The new products formed are $4 \times 60 = 240, 3 \times 60 = 180, 3 \times 59 = 177$ and $4 \times 31 = 124$ which are all 3-digit numbers, so the construction is valid.

3. Define a domino to be a 1×2 rectangular block. A 2023×2023 square grid is filled with non-overlapping dominoes, leaving a single 1×1 gap. John then repeatedly slides dominoes into the gap; each domino is moved at most once. What is the maximum number of times that John could have moved a domino? (Example: In the 3×3 grid shown below, John could move 2 dominoes: D, followed by A.)



Solution:

We claim the maximum number of moves is $1012^2 - 1 = 1024143$.

Proof:

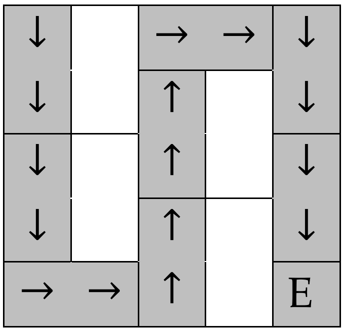
Label the squares in the grid (0,0) to (2022,2022).

Consider the position of the gap after a domino is moved. It is easy to see that the parity of the coordinates of the square will not change. Additionally, the same square cannot contain the gap twice, otherwise this implies that a domino was moved into that square, and was moved out of the square again later, which is disallowed.

Considering coordinates mod 2, in the 2023×2023 grid, there are 1012^2 squares labelled (0,0), $1012 \cdot 1011$ square labelled (0,1) or (1,0), and 1011^2 squares labelled (1,1). Hence, the number of moves is at most $1012^2 - 1$, if all the (0,0) squares are the 'gap square' at some point.

Construction:

This is attainable by connecting these squares in a 'snake' pattern, and sliding dominos accordingly; the remaining 1×2022 rectangles can be also tiled with dominos that do not move throughout. For example, here is the path for a 5×5 grid. Since each (0,0) mod 2 square is used exactly once, there must have been exactly $1012^2 - 1$ moves.



4. Two distinct 2-digit prime numbers p, q can be written one after the other in 2 different ways to form two 4-digit numbers. For example, 11 and 13 yield 1113 and 1311. If the two 4-digit numbers formed are both divisible by the average value of p and q, find all possible pairs (p, q).

Solution:

WLOG assume $p < q$.

The condition implies that $\frac{p+q}{2}$ is a factor of both $100p + q$ and $100q + p$.

Then $\frac{p+q}{2}$ is also a factor of $(100q + p) - 2 \cdot \frac{p+q}{2} = 99q$.

Since $\frac{p+q}{2} < \frac{q+q}{2} = q$ and q is prime, $\gcd\left(\frac{p+q}{2}, q\right) = 1$. Thus, $\frac{p+q}{2}$ divides 99.

Clearly, $12 = \frac{11+13}{2} \leq \frac{p+q}{2} \leq \frac{89+97}{2} = 93$. Hence, the only possible factor is 33, so $p + q = 66$ is the only possibility. Moreover, $100p + q = 99p + (p + q) = 99p + 66$ will clearly be divisible by $\frac{p+q}{2} = 33$. So any pair with $p + q = 66$ works.

Listing, the possible pairs are
(13,53), (19,47), (23,43), (29,37), (37,29), (43,23), (47,19) and (53,13).

5. Find all positive integers k such that there exists positive integers a, b such that
$$a^2 + 4 = (k^2 - 4)b^2.$$

Solution:

We claim that only $k = 3$ works. It is clear that for $k = 3$, $a = b = 1$ will satisfy the equation.

Since $a^2 < (k^2 - 4)b^2 < k^2b^2$, let $a = kb - c$ where c is a positive integer. Then substituting and simplifying, we get

$$k = \frac{4b^2 + 4 + c^2}{2bc}$$

Clearly, this implies c is even. So letting $c = 2x$, we get:

$$k = \frac{b^2 + x^2 + 1}{bx}.$$

(*Remark: It is a "well-known", but hard-to-prove fact that the only possible integer value of the RHS expression is 3. The proof uses Vieta jumping, and is reproduced below.*

Vieta Jumping has never been tested at the SMO Junior level; in fact, its only previous appearance in the SMO was in Senior 2nd Round 2022 Q4.)

Suppose there exists a $k \neq 3$ with such b, x. Let (b, x) be the pair with minimal sum such that $\frac{b^2+x^2+1}{bx} = k$. WLOG assume $x \geq b > 0$.

Then $x^2 - (kb)x + (b^2 + 1) = 0$.

Let x' be the 2nd root to this quadratic. By Vieta's,

$$x + x' = kb$$

$$xx' = b^2 + 1$$

Since $x' = kb - x, x'$ is an integer. Since $x' = \frac{b^2+1}{x}, x'$ is positive. Hence, (b, x') is another pair which satisfies the equation, by the minimality assumption, $x' \geq x$.

Then $b^2 + 1 = xx' \geq x^2 \geq b^2$.

Clearly, $b^2 + 1$ is not a square. Thus, $x^2 = b^2$, implying that $x = b$.

Substituting back, we get $k = \frac{b^2+b^2+1}{b^2} = 2 + \frac{1}{b^2}$, which implies that the only integer combination is $b = 1, k = 3$, contradicting $k \neq 3$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2023

Senior Section (Round 1)

Tuesday, 30 May 2023 **0930 – 1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1. Find the value of m such that $2x^2 + 3x + m$ has a minimum value of 9.

   (A) $\frac{9}{8}$ (B) $-\frac{9}{8}$ (C) $\frac{81}{8}$ (D) $-\frac{81}{8}$ (E) $\frac{63}{8}$

2. Which of the following is true?

   (A) $\sin(105°) - \cos(105°) = \frac{\sqrt{3}}{2}$ (B) $\sin(105°) - \cos(105°) = \frac{\sqrt{3}}{\sqrt{2}}$

   (C) $\sin(105°) + \cos(105°) = \frac{1}{2}$ (D) $\sin(105°) + \cos(105°) = \frac{1}{\sqrt{3}}$

   (E) None of the above.

3. If $\log_{\sqrt{2}} x = 10 - 3\log_{\sqrt{2}} 10$, find x.

   (A) 0.32 (B) 0.032 (C) 0.0032 (D) 0.64 (E) 0.064

4. If $(x-5)^2 + (y-5)^2 = 5^2$, find the smallest value of $(x+5)^2 + (y+5)^2$.

   (A) $225 - 100\sqrt{2}$ (B) $225 + 100\sqrt{2}$ (C) $225\sqrt{2}$ (D) $100\sqrt{2}$
   (E) None of the above

5. Suppose $\cos(180° + x) = \frac{4}{5}$, where $90° < x < 180°$. Find $\tan(2x)$.

   (A) $\frac{24}{7}$ (B) $\frac{7}{24}$ (C) $-\frac{24}{7}$ (D) $-\frac{7}{24}$ (E) $-\frac{24}{25}$

Short Questions

6. Suppose the roots of $x^2 + 11x + 3 = 0$ are p and q, and the roots of $x^2 + Bx - C = 0$ are $p+1$ and $q+1$. Find C.

7. If the smallest possible value of $(A-x)(23-x)(A+x)(23+x)$ is $-(48)^2$, find the value of A if $A > 0$.

8. Find the smallest positive odd integer greater than 1 that is a factor of

$$(2023)^{2023} + (2026)^{2026} + (2029)^{2029}.$$

9. Find the remainder of $7^{2023} + 9^{2023}$ when divided by 64.

10. Let $x, y, z > 1$, and let A be a positive number such that $\log_x A = 30$, $\log_y A = 50$ and $\log_{xy}(Az) = 150$. Find $(\log_A z)^2$.

11. Find the largest integer that is less than

$$\frac{3^{10} - 2^{10}}{10!}\left(\frac{1}{1!9!2} + \frac{1}{2!8!2^2} + \frac{1}{3!7!2^3} + \cdots + \frac{1}{9!1!2^9}\right)^{-1}.$$

Here, $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

12. Consider the following simultaneous equations:

$$\begin{aligned} xy^2 + xyz &= 91, \\ xyz - y^2z &= 72, \end{aligned}$$

where x, y, and z are positive integers. Find the maximum value of xz.

13. Let x be a real number such that

$$\frac{\sin^4 x + \cos^4 x}{\sin^2 x + \cos^4 x} = \frac{8}{11}.$$

Assuming $\sin^2 x > \frac{1}{2}$, find the value of $\sqrt{28}(\sin^4 x - \cos^4 x)$.

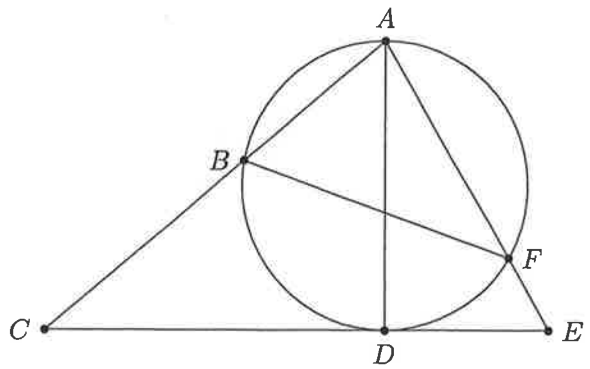
14. A sequence a_1, a_2, $\ldots$, is defined by

$$a_1 = 5,\ a_2 = 7,\ a_{n+1} = \frac{a_n + 1}{a_{n-1}} \quad \text{for } n \geq 2.$$

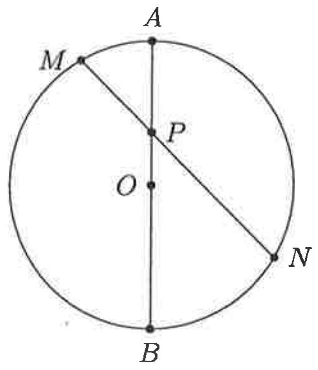
Find the value of $100 \times a_{2023}$.

15. Let C be a constant such that the equation $5\cos x + 6\sin x - C = 0$ have two distint roots a and b, where $0 < b < a < \pi$. Find the value of $61 \times \sin(a + b)$.

16. In the diagram below, CE is tangent to the circle at point D, AD is a diameter of the circle, and ABC, AFE are straight lines. It is given that $\frac{AB}{AC} = \frac{16}{41}$ and $\frac{AF}{AE} = \frac{49}{74}$. Let $\tan(\angle CAE) = \frac{m}{n}$, where m, n are positive integers and $\frac{m}{n}$ is a fraction in its lowest form. Find the sum $m + n$.



17. In the diagram below, AB is a diameter of the circle with centre O, MN is a chord of the circle that intersects AB at P, $\angle BON$ and $\angle MOA$ are acute angles, $\angle MPA = 45°$, $MP = \sqrt{56}$, and $NP = 12$. Find the radius of the circle.



18. Let $f(x) = \cos^2\left(\frac{\pi x}{2}\right)$. Find the value of

$$f\left(\frac{1}{2023}\right) + f\left(\frac{2}{2023}\right) + \cdots + f\left(\frac{2021}{2023}\right) + f\left(\frac{2022}{2023}\right).$$

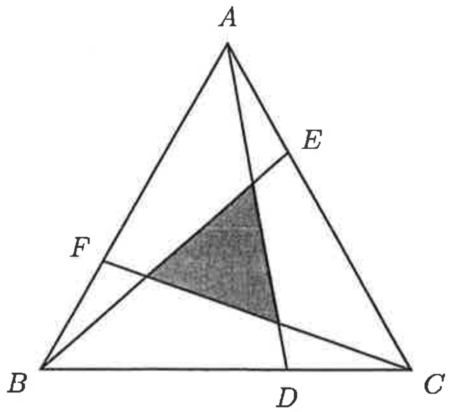
19. Find the remainder when 3^{2023} is divided by 215.

20. Find the sum of the prime divisors of 64000027.

21. Let $\triangle ABC$ be an equilateral triangle. D, E, F are points on the sides such that

$$BD : DC = CE : EA = AF : FB = 2 : 1.$$

Suppose the area of the triangle bounded by AD, BE and CF is 2023. Find the area of $\triangle ABC$.



22. Find the number of possible ways of arranging m ones and n zeros in a row such that

there are in total $2k+1$ strings of ones and zeros. For example,

$$1110001001110001$$

consists of 4 strings of ones and 3 strings of zeros.

23. Suppose that there exist numbers a, b, c and a function f such that for any real numbers x and y,

$$f(x+y)+f(x-y)=2f(x)+2f(y)+ax+by+c.$$

It is given that

$$f(2)=3, \quad f(3)=-5, \quad \text{and} \quad f(5)=7.$$

Find the value of $f(123)$.

24. Let f be a function such that for any nonzero number x,

$$6xf(x)+5x^2f(1/x)+10=0.$$

Find the value of $f(10)$.

25. Find the number of triangles such that all the sides are integers and the area equals the perimeter (in number).

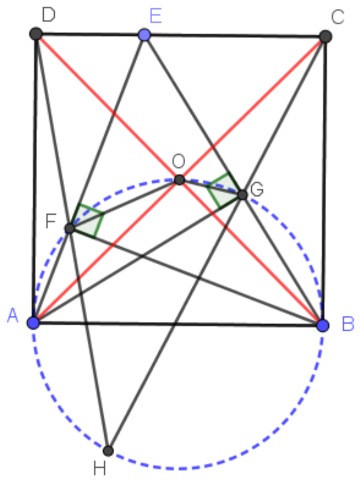
SMO 2023 (Senior Section) Answers

1. C
2. B
3. B
4. A
5. C
6. 7
7. 25
8. 3
9. 48
10. 49
11. 1024
12. 84
13. 2
14. 160
15. 60
16. 58
17. 10
18. 1011
19. 37
20. 514
21. 14161
22. 47628
23. 66795
24. 454
25. 5

SMO Senior 2023 Rd.2

1. Let $ABCD$ be a square, E be a point on the side DC, F and G be the feet of the altitudes from B to AE and from A to BE, respectively. Suppose DF and CG intersect at H. Prove that $\angle AHB = 90°$.

Solution:



Let AC, BD intersect at O. Since $ABCD$ is a square, $\angle AOB = 90°$. Hence A, F, O, G, B lie on the circle with diameter AB.

Note that $\angle EFO = \angle OBA = 45° = \angle EDO$, so D, E, O, F are concyclic. Similarly, $\angle EGO = \angle OAB = 45° = \angle ECO$, so E, C, O, G are concyclic.

Thus $\angle FOG = 360° - \angle EOF - \angle EOG = (180° - \angle EOF) + (180° - \angle EOG) = \angle EDF + \angle ECG = 180° - \angle FHG$, so F, O, G, H are concyclic. This implies H also lies on the circle with diameter AB, so $\angle AHB = 90°$.

Remarks:

1) One might note that O is a Miquel Point in ΔHCD after finding the two cyclic quadrilaterals $DEOF$ and $ECOG$.
2) Coordinate Geometry is also rather suitable for this problem, since $ABCD$ is a square.

2. Find all positive integers k such that there exist positive integers a, b such that
$$a^2 + 4 = (k^2 - 4)b^2.$$

Solution:

We claim that only $k = 3$ works. It is clear that for $k = 3$, $a = b = 1$ will satisfy the equation.

Since $a^2 < (k^2 - 4)b^2 < k^2b^2$, let $a = kb - c$ where c is a positive integer. Then substituting and simplifying, we get

$$k = \frac{4b^2 + 4 + c^2}{2bc}$$

Clearly, this implies c is even. So letting $c = 2x$, we get:

$$k = \frac{b^2 + x^2 + 1}{bx}.$$

(*Remark: It is a "well-known", but hard-to-prove fact that the only possible integer value of the RHS expression is 3. The proof uses Vieta jumping, and is reproduced below.)*

Suppose there exists a $k \neq 3$ with such b, x. Let (b, x) be the pair with minimal sum such that $\frac{b^2+x^2+1}{bx} = k$. WLOG assume $x \geq b > 0$.

Then $x^2 - (kb)x + (b^2 + 1) = 0$.

Let x' be the 2nd root to this quadratic. By Vieta's,

$$x + x' = kb$$
$$xx' = b^2 + 1$$

Since $x' = kb - x, x'$ is an integer. Since $x' = \frac{b^2+1}{x}$, x' is positive. Hence, (b, x') is another pair which satisfies the equation, by the minimality assumption, $x' \geq x$.

Then $b^2 + 1 = xx' \geq x^2 \geq b^2$.

Clearly, $b^2 + 1$ is not a square. Thus, $x^2 = b^2$, implying that $x = b$.

Substituting back, we get $k = \frac{b^2+b^2+1}{b^2} = 2 + \frac{1}{b^2}$, which implies that the only integer combination is $b = 1, k = 3$, contradicting $k \neq 3$.

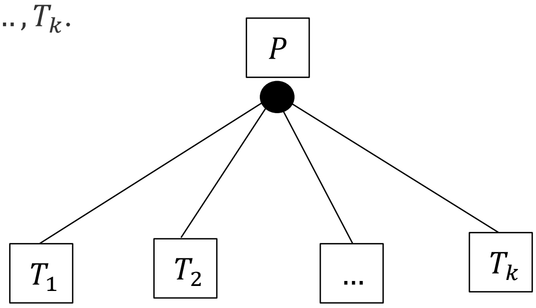
3. Let n be a positive integer. There are n islands with $n-1$ bridges connecting them such that one can travel from any island to another. One afternoon, a fire breaks out in one of the islands. Every morning, it spreads to all neighbouring islands. (Two islands are neighbours if they are connected by a bridge.) To control the spread, one bridge is destroyed every night until the fire has nowhere to spread the next day. Let X be the minimum possible number of bridges one has to destroy before the fire stops spreading. Find the maximum possible value of X over all possible configurations of bridges and islands where the fire starts at.

Solution:

We claim that the maximum possible value of X is $\lfloor\sqrt{n-1}\rfloor$.

We use a graph representation, where the islands are vertices and bridges are edges. Moreover, it is a connected graph with $n-1$ edges and n vertices, which implies it must be a tree (i.e. have no cycles).

We may represent it as a *rooted* tree where the root node P is the location where the fire starts at, and P is connected to the subtrees $T_1, T_2, \dots, T_k$.



<u>Construction:</u> For any positive integer k, consider the case with $n = k^2 + 1$ vertices, and each of $T_1, T_2, \dots, T_k$ is a line graph of k vertices. Clearly, the fire will spread along each T_i for up to k turns until a bridge is destroyed. After $k-1$ bridges are destroyed, some tree T_i would not have had any bridges destroyed. Thus, $X = k$ in this case. For any $k^2 + 1 < n \leq (k+1)^2$, we may simply add more vertices to T_k and the argument still holds. This shows that $X = \lfloor\sqrt{n-1}\rfloor$ is attainable.

<u>Optimality:</u>

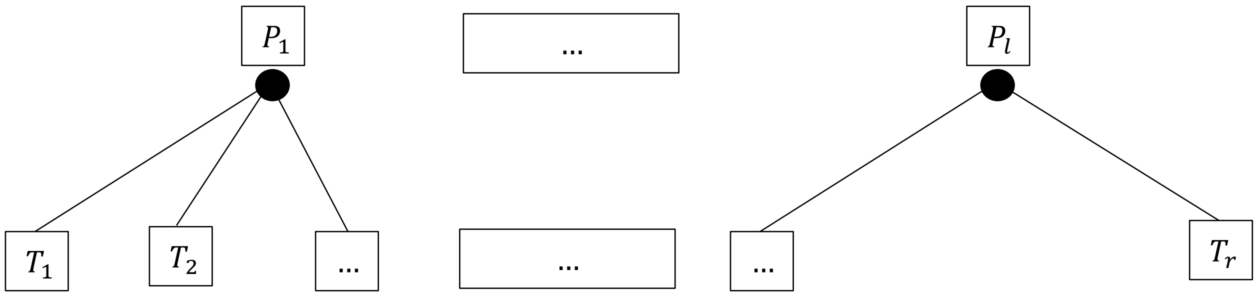
Without loss of generality, one may assume that an island on fire disappears after spreading to all its neighbouring islands. The configuration can then be viewed as a forest (i.e. a collection of trees) with their root nodes on fire; on each turn, each tree's root node is removed and its adjacent nodes become root nodes (on fire) of the respective subtrees.

At any point in time, call an island *fireable* if it is not yet on fire, but there exists a path from another island on fire to it.

Claim: If there are m *fireable* islands just before the time to destroy a bridge, then it suffices to destroy $\lfloor\sqrt{m}\rfloor$ bridges to stop the fire from spreading.

Proof (by strong induction on m): Consider a configuration with $k^2 \leq m < (k+1)^2$ *fireable* islands, and we aim to destroy at most k bridges.

Let the configuration contain l trees with root nodes $P_1, P_2, \ldots, P_l$ on fire, and suppose they are connected to subtrees $T_1, T_2, \ldots, T_r$ with $n_1, n_2, \ldots, n_r$ vertices respectively.



WLOG let $n_1 \leq n_2 \leq \cdots \leq n_r$. By definition, $n_1 + n_2 + \cdots + n_r = m$.

Moreover, if any $n_i \leq k$, we can 'ignore' the subtree T_i entirely, since the fire would definitely have finished spreading within k days. Thus, WLOG we may assume all trees have $\geq k+1$ vertices. Then since $m < (k+1)^2$, $r \leq k$. Thus, $n_r \geq \left\lceil \frac{m}{r} \right\rceil$.

We now destroy the bridge from P_l to the root node of T_r. The next morning, we 'remove' T_r, and all other root nodes are on fire. We will be left with $m - (r-1) - n_r \leq m + 1 - r - \frac{m}{r} \leq (m+1) - 2\sqrt{m}$ (AM − GM) $= \left(\sqrt{m} - 1\right)^2$ fireable nodes. Moreover, equality only can hold if m is a perfect square, i.e. $m = k^2$, and $r = \frac{m}{r}$, i.e. $k = r$. However, if $m = k^2, n_1 \geq k+1 \Rightarrow r \leq \left\lfloor \frac{k^2}{k+1} \right\rfloor = k - 1$, thus equality cannot hold. So there will be strictly less than $\left(\sqrt{m} - 1\right)^2$ fireable nodes. By the induction hypothesis, destroying a further $\lfloor\sqrt{m}\rfloor - 2$ bridges is sufficient for these, so we have stopped the fire by destroying $\lfloor\sqrt{m}\rfloor - 1$ bridges as desired.

Applying this claim to the general case gives the desired conclusion, since there are $n - 1$ *fireable* islands at first.

Remarks:

1) The construction is not too difficult here, the surprising thing is that it works! The main idea is the notion that we can let small subtrees "burn up quickly", so the worst case would be multiple medium-sized subtrees.
2) The proof is much clearer on a specific case (e.g. $n = 100$); the intuition is slightly obfuscated by the number of variables used.

4. Find all positive integers m, n satisfying $n! + 2^{n-1} = 2^m$.

Solution:

We claim $(m, n) = (1,1), (2,2), (5,4)$ are the only solutions. Suppose there are other solutions. Let $v_2(n)$ denote the largest k such that 2^k divides n. Observe that $m = v_2(RHS) = v_2(LHS)$.

Additionally, by Legendre's formula, $v_2(n!) = n - s_2(n)$, where $s_2(n)$ is the number of 1s in the binary representation of n.

We now consider two cases.

<u>Case 1:</u> n is not a power of 2.

Then $s_2(n) > 1$, so $v_2(n!) \leq n - 2$. Thus, $m = v_2(n! + 2^{n-1}) = v_2(n!)$. This clearly implies $2^m \leq n! < LHS$, contradiction.

<u>Case 2:</u> $n = 2^k$ for some $k \geq 3$.

Then $v_2(n!) = n - 1 = v_2(2^{n-1})$, so we may factor the LHS as $2^{n-1}\left(\frac{n!}{2^{n-1}} + 1\right)$. Removing all factors of 2 from $1,2,3, \ldots, n$ allows us to write $\frac{n!}{2^{n-1}}$ as $(1) \cdot (1 \cdot 3) \cdot (1 \cdot 3 \cdot 5 \cdot 7) \cdot \ldots \left(1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2^k - 1)\right)$.

Note that since $1 \cdot 3 \cdot 5 \cdot 7 \equiv 1 \pmod 8$, this product is congruent to $3 \pmod 8$ for $k \geq 3$. Hence, $\frac{n!}{2^{n-1}} + 1 \equiv 4 \pmod 8$, so $v_2\left(2^{n-1}\left(\frac{n!}{2^{n-1}} + 1\right)\right) = (n-1) + 2 = n + 1$.

Thus, $m = n + 1$, which gives $n! + 2^{n-1} = 2^{n+1}$.

But then $n! = 2^{n+1} - 2^{n-1} = 3 \cdot 2^{n-1}$, which clearly fails since $5 | n!$ for $n \geq 2^3$.

Therefore, there are indeed no other solutions.

Remark: Legendre's Formula exists in 2 forms; the other is the more familiar $v_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots$, which is commonly used to count zeroes at the end of $n!$.

5. Colour a 20000×20000 square grid using 2000 different colours with 1 colour in each square. Two squares are neighbours if they share a vertex. A path is a sequence of squares so that 2 successive squares are neighbours. Mark k of the squares. For each unmarked square x, there is exactly 1 marked square y of the same colour so that x and y are connected by a path of squares of the same colour. For any 2 marked squares of the same colour, any path connecting them must pass through squares of all the colours. Find the maximum value of k.

Solution:

We claim that the maximum value of k is simply 20000. This can be achieved by colouring entire rows with colours $1,2,3,\dots,2000$ and repeating this cycle 10 times, and then marking the first square of each row. Clearly, this satisfies the condition.

<u>Proof of Optimality:</u>

We may divide the square into connected *regions*, each consisting of squares of a single colour. The first condition implies that each region has exactly 1 marked square; moreover, the position of the marked square does not matter. Thus, we are actually aiming to maximise the number of regions.

To obtain $k > 20000$, obviously, some colour must appear in more than 1 region. WLOG suppose there are ≥ 2 regions of colour 1, and one of the paths between these regions passes through colours $2,3,4,\dots,2000$ in that order.

$$\boxed{1} \longrightarrow \boxed{2} \longrightarrow \boxed{\dots} \longrightarrow \boxed{2000} \longrightarrow \boxed{1}$$

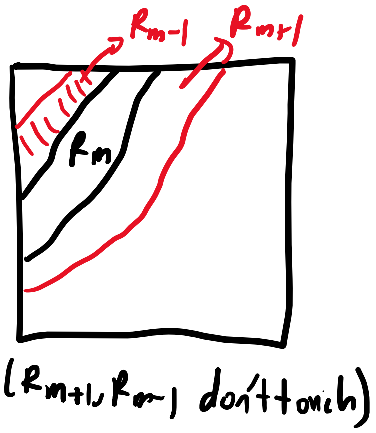
If any of the regions coloured $2,3,\dots,2000$ are connected to another additional region with colour k, then there clearly exists a path between the colour-k regions that does not pass through the colour 1. (If $k = 1$, then there is a path between colour-1 regions that does not pass through either colour 2 or 2000).

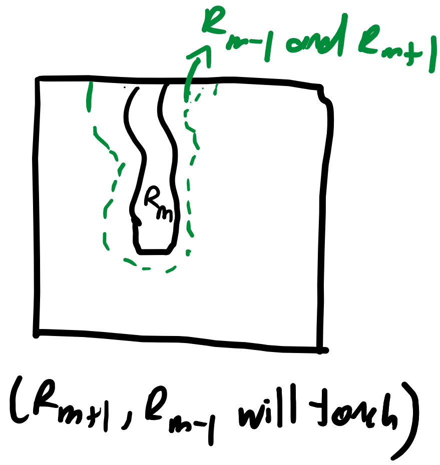
By this reasoning, it is easy to see that the regions can only touch each other in the order $1,2,3,\dots,2000,1,2,3,\dots,2000,\dots$. $(*)$

Now label these regions $R_1, R_2, \dots, R_n$ in order, and consider the shortest path between any square in R_1 and any square in R_n. Since it has to pass through $R_1, R_2, R_3, \dots, R_n$ in some order, this path has at least n squares. But clearly, any two squares have a path between them with at most 20000 squares. Hence $n \leq 20000$.

(∗) There is one potential case to consider – what if they touch each other in a cyclical fashion, ending in a colour-2000 region touching the first colour-1 region. (The shortest path argument would not quite work in that case.)

We claim, however, that this is not possible. Supposing by contradiction that it is possible, each region R_m must touch exactly 2 other regions R_{m+1} and R_{m-1}, which themselves do not touch. Observe that this is only possible if R_m 'disconnects' the square:





But then this implies that regions on the R_{m-1} "side" and R_{m+1} "side" will never have a path between them that doesn't pass through R_m, so a cycle is not possible.

Remark: Oddly enough, this is rather easy for Q5. However, the problem formulation is rather confusing, which makes it challenging to get started.

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2023
(Open Section, Round 1)

Wednesday, 31 May 2023 — 0930-1200 hrs

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

In this paper, let $\mathbb{R}$ denote the set of all real numbers, and $\lfloor x \rfloor$ denote the greatest integer not exceeding x. For examples, $\lfloor 5 \rfloor = 5$, $\lfloor 2.8 \rfloor = 2$, and $\lfloor -2.3 \rfloor = -3$.

1. The graph C with equation $y = \dfrac{ax^2 + bx + c}{x+2}$ has an oblique asymptote with equation $y = 4x - 6$ and one of the stationary points at $x = -4$. Find the value of $a + b + c$.

2. If $x = \dfrac{1}{1} + \dfrac{1}{1+2} + \dfrac{1}{1+2+3} + \dfrac{1}{1+2+3+4} + \cdots + \dfrac{1}{1+2+3+\cdots+100}$, find the value of $\lfloor 1010x \rfloor$.

3. The set of all possible values of x for which the sum of the infinite series

$$1 + \frac{1}{6}(x^2 - 5x) + \frac{1}{6^2}(x^2 - 5x)^2 + \frac{1}{6^3}(x^2 - 5x)^3 + \cdots$$

exists can be expressed as $(a, b) \bigcup (c, d)$, where $a < b < c < d$. Find $d - a$.

4. Find the value of $\lfloor y \rfloor$, where $y = \displaystyle\sum_{k=0}^{\infty} (2k+1)(0.5)^{2k}$.

(Hint: Consider the series expansion of $(1 - x)^{-2}$.)

5. The solution of the inequality $|x - 1| + |x + 1| \overset{<}{\cancel{\geq}} ax + b$ is $-1 < x < 2$. Find the value of $\lfloor a + b \rfloor$.

6. The equation $x^4 - 4x^2 + qx - r = 0$ has three equal roots. Find the value of $\left\lfloor \dfrac{3q^2}{r^2} \right\rfloor$.

7 The parabolas $y = x^2 - 16x + 50$ and $x = y^2$ intersect at 4 distinct points which lie on a circle centred at (a, b). Find $|a - b|$.

8. In the 3-dimensional Euclidean space with origin O and three mutually perpendicular x-, y- and z-axes, two planes $x + y + 3z = 4$ and $2x - z = 6$ intersect at the line $\mathbf{r} \times \begin{pmatrix} -1 \\ a \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ c \\ d \end{pmatrix}$. Find the value of $|a + b + c + d|$.

9. Let x, y, z be real numbers with $3x + 4y + 5z = 100$. Find the minimum value of $x^2 + y^2 + z^2$.

10. Find the area of the region represented by the equation $\lfloor x \rfloor + \lfloor y \rfloor = 1$ in the region $0 \leq x < 2$.

(Note: If you think that there is no area defined by the graph, enter "0"; if you think that the area is infinite, enter "9999".)

11. Let ABC be a triangle satisfying the following conditions that $\angle A + \angle C = 2\angle B$, and $\frac{1}{\cos A} + \frac{1}{\cos C} = \frac{-\sqrt{2}}{\cos B}$. Determine the value of $\dfrac{2022\cos(\frac{A-C}{2})}{\sqrt{2}}$.

12. Find x which satisfies the following equation

$$\frac{x-2019}{1} + \frac{x-2018}{2} + \frac{x-2017}{3} + \cdots + \frac{x+2}{2022} + \frac{x+3}{2023} = 2023.$$

13. Assume that x is a positive number such that $x - \frac{1}{x} = 3$ and

$$\frac{x^{10}+x^8+x^2+1}{x^{10}+x^6+x^4+1} = \frac{m}{n},$$

where m and n are positive integers without common factors larger than 1. Determine the value of $m+n$.

14. Consider the set of all possible pairs (x, y) of real numbers that satisfy $(x-4)^2 + (y-3)^2 = 9$. If S is the largest possible value of $\frac{y}{x}$, find the value of $\lfloor 7S \rfloor$.

15. Let x, y be positive integers with $16x^2 + y^2 + 7xy \leq 2023$. Find the maximum value of $4x + y$.

16. Let x be the largest real number such that

$$\sqrt{x - \frac{1}{x}} + \sqrt{1 - \frac{1}{x}} = x.$$

Determine the value of $(2x-1)^4$.

17. Two positive integers m and n differ by 10 and the digits in the decimal representation of mn are all equal to 9. Determine $m+n$.

18. Let $\{a_n\}$ be a sequence of positive numbers, and let $S_n = a_1 + a_2 + \cdots + a_n$. For any positive integer n, let $b_n = \frac{1}{2}\left(\frac{a_{n+1}}{a_n} + \frac{a_n}{a_{n+1}}\right)$. Given that $\frac{a_n+2}{2} = \sqrt{2S_n}$ holds for all positive integers n, determine the limit $\lim_{n\to+\infty}(b_1 + b_2 + \cdots + b_n - n)$.

19. Let ABC be a triangle with $AB = c, AC = b$ and $BC = a$, and satisfies the conditions $\tan C = \frac{\sin A + \sin B}{\cos A + \cos B}$, $\sin(B-A) = \cos C$ and that area of triangle $ABC = 3 + \sqrt{3}$. Determine the value of $a^2 + c^2$.

20. Let $g : \mathbb{R} \to \mathbb{R}$, $g(0) = 4$ and that

$$g(xy+1) = g(x)g(y) - g(y) - x + 2023.$$

Find the value of $g(2023)$.

21. In the triangle ABC, D is the midpoint of AC, E is the midpoint of BD, and the lines BA and CE are tangent to the circumcircle of the triangle ADE at A and E respectively. Suppose the circumradius of the triangle AED is $(\frac{64}{7})^{\frac{1}{4}}$. Find the area of the triangle ABC.

22. $ABCD$ is a parallelogram such that $\angle ABC < 90°$ and $\sin \angle ABC = \frac{4}{5}$. The point K is on the extension of BC such that $DC = DK$; the point L is on the extension of DC such that $BC = BL$. The bisector of $\angle CDK$ intersects the bisector of $\angle LBC$ at Q. Suppose the circumradius of the triangle ABD is 25. Find the length of KL.

23. A group of 200 monkeys is given the task of picking up all 3000 peanuts on the ground. Determine the maximum number k such that there must be k monkeys picking up the same number of peanuts.

[It is possible that some lazy monkeys may not pick up any peanuts at all.]

24. A chain of n identical circles $C_1, C_2, \ldots, C_n$ of equal radii and centres on the x-axis lie inside the ellipse E: $\frac{x^2}{2023} + \frac{y^2}{333} = 1$ such that C_1 is tangent to E internally at $(-\sqrt{2023}, 0)$, C_n is tangent to E internally at $(\sqrt{2023}, 0)$, and C_i is tangent to C_{i+1} externally for $i = 1, \ldots, n-1$. Determine the smallest possible value of n.

25. Let $p > 2023$ be a prime. Determine the number of positive integers n such that

$$(n-p)^2 + 2023(2023 - 2n - 2p)$$

is a perfect square.

SMO 2023 (Open Section) Answers

1. 10

2. 2000

3. 7

4. 2

5. 3

6. 32

7. 8

8. 33

9. 200

10. 2

11. 1011

12. 2020

13. 229

14. 24

15. 46

16. 25

17. 64

18. 1

19. 20

20. void

21. 7

22. 48

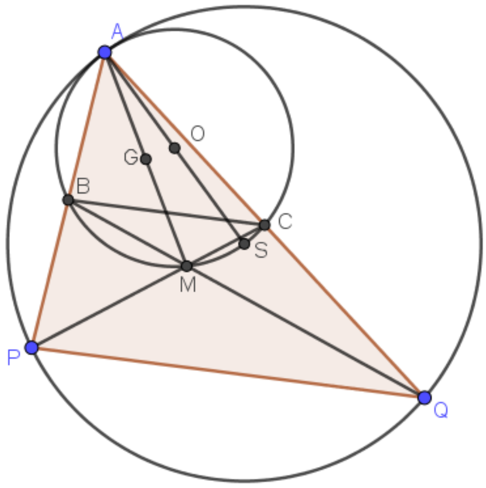
23. 7

24. 7

25. 6

SMO Open 2023 Rd.2

1. In a scalene triangle ABC with centroid G and circumcircle ω centred at O, the extension of AG meets ω at M, lines AB and CM intersect at P; and lines AC and BM intersect at Q. Suppose the circumcentre S of the triangle APQ lies on ω and A, O, S are collinear. Prove that $\angle AGO = 90°$.



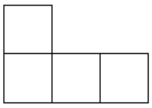
Solution:

Consider the homothety h centred at A which sends O to S and ω to the circumcentre of ΔAPQ. Since $AS = 2AO$, h has scale factor 2, so B, C are midpoints of AP, AQ respectively.

This implies that M is the centroid of ΔAPQ. Thus, $h(G) = M$, which implies $GO||MS$. Thus $\angle AGO = \angle AMS = 90°$.

Remark: A homothety is just a scaling with respect to a particular point (A, in this case). If one is unfamiliar with the concept, the statements can easily be replaced by a series of claims that use similar triangles (or the midpoint theorem, in this case.)

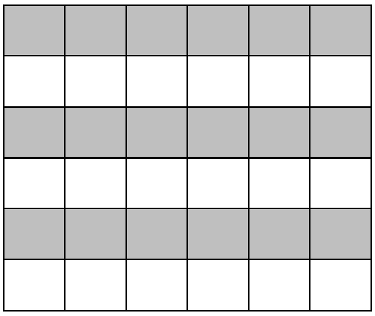
2. A grid of cells is tiled with dominoes such that every cell is covered by exactly one domino. A subset S of dominoes is chosen. Is it true that at least one of the following 2 statements is false?
   (1) There are 2022 more horizontal dominoes than vertical dominoes in S.
   (2) The cells covered by the dominoes in S can be tiled completely and exactly by the tetrominoes shown below (including rotations & reflections):



Solution:

Yes, at least one of the statements is false.

Suppose otherwise that both statements (1) and (2) are true, and let S contain n vertical and $n + 2022$ horizontal dominoes. Then there would be $n + 1011$ L-shapes.



Shade the grid such that the rows are alternately shaded and unshaded as shown above. Note that:

- A horizontal domino has 2 shaded or 2 unshaded squares.
- A vertical domino has 1 shaded and 1 unshaded square.
- An L-shape has 3 shaded and 1 unshaded square, or vice versa.

This means that the number of shaded squares is congruent to n (mod 2) since there are n vertical dominoes, but also $n + 1011$ (mod 2) since there are $n + 1011$ L-shapes, contradiction.

Remark: Colouring-based ideas would be the first things to consider when handling a 'tiling' problem, especially if we want to show something is impossible.

3. Let $n \geq 2$ be a positive integer. For any integer a, let $Q_a(x)$ denote the polynomial $x^n + ax$. Let p be a prime number and S_a be the set
$$S_a = \{b | 0 \leq b \leq p-1, \exists c \in \mathbb{Z}, Q_a(c) \equiv b \pmod{p}\}.$$
Show that the expression $\frac{1}{p-1}\sum_{a=1}^{p-1}|S_a|$ is an integer.

Solution:

We define (a, b) to be an *achievable* pair if $x^n + ax \equiv b$ (mod n) has a solution, $1 \leq a \leq p-1$ and $1 \leq b \leq p-1$. (All subsequent congruences and values are taken modulo n.)

Then $\sum_{a=1}^{p-1}|S_a|$ is simply the number of achievable pairs, plus the $p-1$ pairs $(1,0), (2,0), \ldots, (p-1,0)$ which are attained when $x = 0$. So it remains to show that the number of achievable pairs is a multiple of $p-1$. Moreover, we can assume WLOG that $x \not\equiv 0$.

Suppose that (a_0, b_0) is an achievable pair, and $x^n + a_0x \equiv b_0$.

Then for any $1 \leq k \leq p-1$,

$$k^n x^n + k^n a_0 x \equiv k^n b_0$$

$$\Leftrightarrow (kx)^n + (a_0k^{n-1})(kx) \equiv k^n b_0$$

This implies that (a_0k^{n-1}, b_0k^n) is also an achievable pair.

In general, for each set of the form $S_{a,b} = \{(ak^{n-1}, bk^n): 1 \leq k \leq p-1\}$ (call these type S), the pairs are either all achievable, or all not achievable.

Claim: The set of all pairs $T = \{(a,b): 1 \leq a, b \leq p-1\}$ is a disjoint union of type S sets. Moreover, each type S set has size $p-1$. (This will imply that the number of achievable pairs is a multiple of $p-1$ as desired.)

Proof: Clearly, each pair (a, b) lies in $S_{a,b}$, and $S_{a,b} \subset T$ for any a, b. We are left to show two claims:

a) Each set $S_{a,b}$ has size $p-1$.

Suppose that $(ak^{n-1}, bk^n) = (ak_0^{n-1}, bk_0^n)$ for some k, k_0. Then $\left(\frac{k}{k_0}\right)^{n-1} \equiv \left(\frac{k}{k_0}\right)^n \equiv 1$, so $\frac{k}{k_0} \equiv 1$, and $k = k_0$. Hence, each set has $p-1$ distinct elements.

b) The sets $S_{a,b}$ and $S_{a',b'}$ are either equal or disjoint.

Suppose $S_{a,b} \cap S_{a',b'} \neq \emptyset$. Then $(ak^{n-1}, bk^n) = (a'k_0^{n-1}, b'k_0^n)$ for some k, k_0.

Then $(a, b) = \left(a'\left(\frac{k_0}{k}\right)^{n-1}, b'\left(\frac{k_0}{k}\right)^n\right)$.

Let $(ak_1^{n-1}, bk_1^n) \in S_{a,b}$. Then $(ak_1^{n-1}, bk_1^n) = \left(a'\left(\frac{k_0 k_1}{k}\right)^{n-1}, b'\left(\frac{k_0 k_1}{k}\right)^n\right) \in S_{a',b'}$. So $S_{a,b} \subseteq S_{a',b'}$. Since both sets have size $p-1$, $S_{a,b} = S_{a',b'}$.

Hence, any sets are either disjoint or equal, showing that they are disjoint sets of size $p-1$ whose union is T.

Remark: This is also 2022 Open Q5, almost verbatim. It is odd that an SMO Open problem would repeat literally 1 year later. Although we shouldn't expect this to become a pattern, it would be a good idea to familiarize ourselves with recent SMO solutions for future contests, just in case.

4. Find all functions $f: \mathbb{Z} \to \mathbb{Z}$ such that

$$f(x+y)\left(\left(f(x)-f(y)\right)^2 + f(xy)\right) = f(x^3) + f(y^3).$$

Solution:

Let $P(x,y)$ be the assertion that $f(x+y)\left(\left(f(x)-f(y)\right)^2 + f(xy)\right) = f(x^3) + f(y^3)$.

$P(0,0)$: $f(0)^2 = 2f(0) \Rightarrow f(0) = 0$ or $f(0) = 2$.

$P(1,1)$: $f(2)f(1) = 2f(1) \Rightarrow f(1) = 0$ or $f(2) = 2$.

We now split into cases.

Case 1: $f(0) = 0$ and $f(1) = 0$.

$P(x,0)$: $f(x)^3 = f(x^3)$

$P(x,1)$: $f(x+1) \cdot \left(f(x)^2 + f(x)\right) = f(x^3) = f(x)^3$.

Suppose that $f(a) \neq 0$ for some a. Then letting $x = a$ and dividing by $f(a)$,

$$f(a+1) \cdot [f(a)+1] = f(a)^2.$$

Clearly, $f(a) \neq -1$. So $f(a+1) = \frac{f(a)^2}{f(a)+1} = f(a) - 1 + \frac{1}{f(a)+1}$.

This is only an integer if $f(a) = -2$. Thus, the only non-zero value in the range of f should be -2. But $f(a^3) = f(a)^3 = -8$, contradiction.

Hence, $\boldsymbol{f(x) = 0}$ **for all** $\boldsymbol{x}$. [1st Solution]

Case 2: $f(0) = 0$, $f(2) = 2$ (and $f(1) \neq 0$)**.**

As before, $P(x,0)$: $f(x)^3 = f(x^3)$. In particular, this means $f(1) = -1$ or 1, and $f(-1) = -1, 0$ or 1.

$P(2,-1)$: $f(1) \cdot \left(\left(f(2) - f(-1)\right)^2 + f(-2)\right) = f(8) + f(-1)$

$$\Rightarrow f(1) \cdot \left(\left(2 - f(-1)\right)^2 + f(-2)\right) = 8 + f(-1) \ (*)$$

$P(-1,-1)$: $f(-2) \cdot f(1) = 2f(-1)$.

If $f(-1) = 0$, then since $f(1) \neq 0$, $f(-2) = 0$. But substituting into $(*)$ gives $f(1) = 2$, contradiction. This implies $f(1) = \pm 1, f(-1) = \pm 1$ and $f(-2) = \pm 2$. Checking, the only combination which works is $f(1) = 1, f(-1) = -1$ and $f(-2) = -2$.

Now $P(x,1)$ gives $f(x+1) \cdot [f(x)^2 - f(x) + 1] = f(x)^3 + 1$, which implies $f(x+1) = f(x) + 1$ for all x. Clearly, this gives $\boldsymbol{f(x) = x}$ **for all** $\boldsymbol{x}$ [2nd solution].

Case 3: $f(0) = 2$, $f(1) = 0$

Then $P(1,0)$ gives $0 = 2$, contradiction.

Case 4: $f(0) = 2, f(2) = 2$ (and $f(1) \neq 0$).

$P(x, 0)$: $f(x)(f(x)^2 - 4f(x) + 6) = f(x^3) + 2$.
For $x = 1$ or -1, $x^3 = x$, so $f(x)(f(x)^2 - 4f(x) + 6) = f(x) + 2$.
Factorising, $f(x)^3 - 4f(x)^2 + 5f(x) - 2 = 0 \Rightarrow (f(x) - 2)(f(x) - 1)^2 = 0 \Rightarrow f(x) = 1$ or 2. Thus, $f(1)$ and $f(-1) = 1$ or 2.
$P(1, -1)$ gives $f(0) \cdot ((f(1) - f(-1)^2 + f(-1)) = f(1) + f(-1)$. Since $f(0) = 2$, $f(1) + f(-1)$ are even, i.e. $f(1) = f(-1) = 1$ or $f(1) = f(-1) = 2$.

Case 4a: $f(1) = f(-1) = 2$.
We show by induction on m that $f(x) = 2$ for $x \in [-m, m]$.
It is true for $m = 1$; suppose it is true for $m = k - 1$.
Note that from $P(x, 0)$, $f(x) = 2 \Rightarrow f(x^3) = 2$ as well.
Then $P(k-1, 1)$: $f(k)\left(\left(f(k-1) - f(1)\right)^2 + f(k-1)\right) = f((k-1)^3) + f(1)$

$$\Rightarrow f(k) \cdot 2 = 2 + 2$$
$$\Rightarrow f(k) = 2.$$

Likewise, $P(-(k-1), -1)$ gives $f(-k) = 2$.
This completes the induction, giving $\boldsymbol{f(x) = 2}$ **for all** $\boldsymbol{x}$ [3rd solution].

Case 4b: $f(1) = f(-1) = 1$.
Similarly, we may show by induction on m that $f(x) = 1$ for odd x and $f(x) = 2$ for even x where $x \in [-m, m]$.

So $\boldsymbol{f(x) = \begin{cases} 1, & x \text{ odd} \\ 2, & x \text{ even} \end{cases}}$ also works [4th solution].

Remark: This is a very tedious question – but none of the steps are particularly clever. It is just <u>very</u> long. We know that a much faster solution is unlikely to exist, since there are four different $f(x)$ that all work!

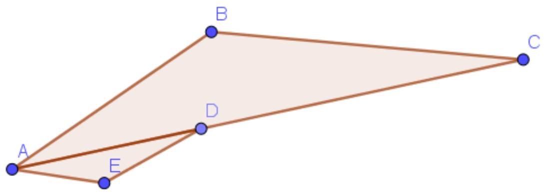
5. Determine all real numbers x between 0 and 180 such that it is possible to partition an equilateral triangle into finitely many triangles, each of which has an angle of $x°$.

Solutions:

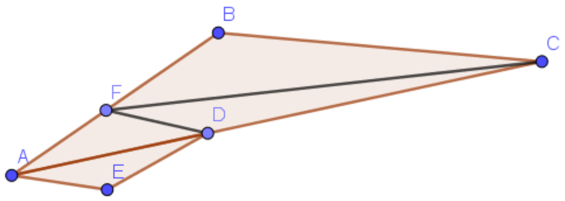
We claim that all $0 < x \leq 120$ work.

Proof of Impossibility:

Call a triangle with a $> 120°$ angle *large*. We simply show that we cannot partition the equilateral triangle into large triangles. For convenience, we claim that WLOG, we may assume that no triangles 'partially' share an edge. Suppose we have such a situation (e.g. AD lying on AC).



Then we further split ΔBCA into large triangles by letting F be on AB so that $FD||BC$:



Now let there be m triangle vertices on the sides of the original equilateral triangle (excluding the 3 original vertices), and n vertices in the interior which do not lie on the side of any triangle.

Then there are

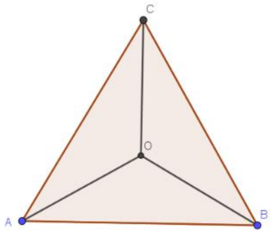
- No $x°$ angles on each of the 3 original vertices
- ≤ 1 $x°$ angle on each of the m "side" vertices
- ≤ 2 $x°$ angles on each of the n "interior" vertices.

So we have at most $m + 2n$ angles $> 120°$.

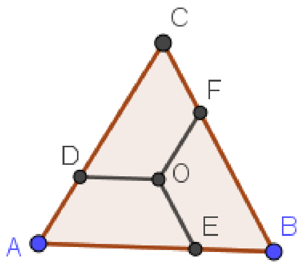
However, the sum of angles is $3 \times 60° + m \times 180° + n \times 360°$, giving $1 + m + 2n$ triangles, so some triangle does not have a $> 120°$ angle.

Constructions:

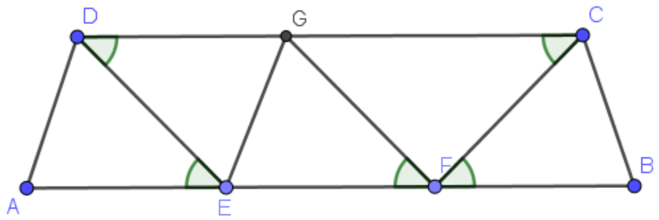
$x = 120°$ is simple; we just divide into three $120 - 30 - 30$ triangles.



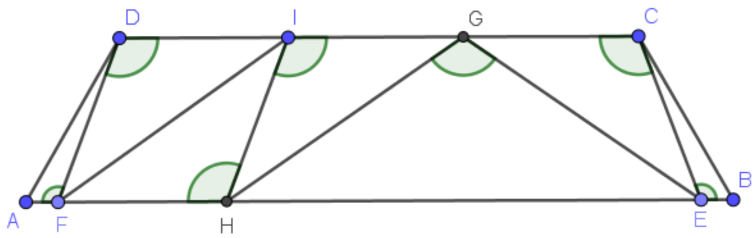
Otherwise, we split the equilateral triangle into 3 congruent trapeziums as shown, where O is the centroid of the equilateral triangle ABC.



It suffices to consider a particular trapezium, say, $ADOE$. Note that by drawing parallel lines to OD, we can divide it into smaller trapeziums with angles $60°, 60°, 120°, 120°$, and whose bases are at least k times of the height for arbitrarily large k. We show how to divide such a trapezium into triangles with angle $x°$.



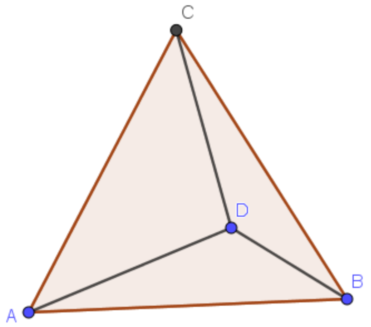
For $x < 90$, we construct on trapezium $ABCD$ (where $AB||CD$ and $AB > CD$) by letting E, F be on AB such that $\angle AED = \angle BFC = x°$, then letting G be on CD such that $DE||GF$, which ensures $\angle GDE = \angle GFE = \angle GCF = x°$.



For $x > 90$, we let E, F be on AB again so that $\angle AFD = \angle CEB = x°$. Then we let G be on CD so that $CE = CG$, reflect C over G onto I, and let H be on AB so that $HI||FD$. This ensures that all triangles have a $x°$ angle again.

Remarks: This is a nice, but difficult question. It is tempting to guess that something like $x = \frac{60}{n}, \frac{90}{n}, \frac{120}{n}$ *or* $\frac{360}{n}$ $(n \in \mathbb{Z}^+)$ *would be the answer.*

To motivate the answer ***not*** *being one of these, we may consider a simple configuration:*



It is very possible that for suitable D, $\angle BCD, \angle ADB$ *and* $\angle CAD$ *are multiples of, say,* 7° *or* 1.7° *- and more importantly, it seems impossible to show that this* ***never*** *happens! Since there's no reason to think that any particular angle is 'special enough', which suggests that an entire range of values should be achievable.*