# Singapore Mathematical Society

## Singapore Mathematical Olympiad (SMO) 2024

### Junior Section (Round 1)

**Tuesday, 28 May 2024** **0930-1200 hrs**

**Instructions to contestants**

1. *Answer ALL 25 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*

**PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.**

**Multiple Choice Questions**

1. If $x^2 + 4x + 16 = 0$, what is the value of $x^3$?

(A) 4 (B) 8 (C) 16 (D) 64 (E) 128

2. Let $a$ be a real number that satisfies $-1 < a < 0$. Which of the following is true?

(A) $\pi^a < \frac{1}{\pi} < \left(\frac{1}{\pi}\right)^a$

(B) $\pi^a < \left(\frac{1}{\pi}\right)^a < \frac{1}{\pi}$

(C) $\frac{1}{\pi} < \left(\frac{1}{\pi}\right)^a < \pi^a$

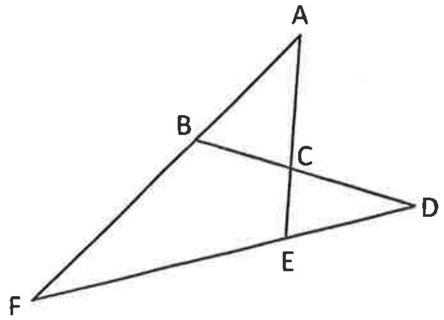
(D) $\frac{1}{\pi} < \pi^a < \left(\frac{1}{\pi}\right)^a$

(E) $\left(\frac{1}{\pi}\right)^a < \pi^a < \frac{1}{\pi}$

3. How many non-congruent triangles are there whose sides have integer lengths and the longest side has length 10 units?

(A) 25 (B) 30 (C) 35 (D) 40 (E) 45

4. In the diagram below, the points B and E lie on AF and DF respectively, and AE and BD intersect at C. If AB = AC, BD = BF and EA = EF, find ∠BAC.




(A) 30° (B) 33° (C) 36° (D) 38° (E) 40°

5. Let $x, y$ and $z$ be real numbers such that $x \neq 0$, $y - z \neq 0$ and $z + x \neq 0$. If $\frac{2}{x} = \frac{4}{y-z} = \frac{5}{z+x}$, find the value of $\frac{7x-y}{y+2z}$.

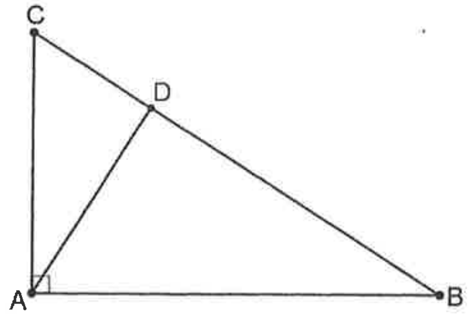
(A) $\frac{11}{17}$ (B) $-\frac{11}{17}$ (C) $\frac{7}{13}$ (D) $-\frac{7}{13}$ (E) 1

**Short Questions**

6. Let $N$ be a 2-digit whole number. When 2692 is divided by $N$, the remainder is 13, and when 2978 is divided by $N$, the remainder is 14. Find the sum of all the possible values of $N$.

7. If $x$ is a positive integer such that $2^x + 2^{89} = 512^{10}$, find the value of $x$.

8. The diagram below shows a right-angled triangle ABC. The sides AC and AB are in the ratio 3: 5. The point D lies on BC such that AD is perpendicular to BC. Furthermore, DB is 8 cm longer than CD. What is the length of BC in cm?




9. Let $n$ be a positive integer. Suppose that $a_1, a_2, a_3, \ldots$ is a sequence of numbers defined by

$$a_1 = \sqrt{(n+3)(n-1)+4}, \ a_k = \sqrt{(n+2k+1)a_{k-1}+4} \text{ for } k \geq 2.$$

If $a_{100} = 2024$, find the value of $n$.

10. Let $N$ be the smallest positive integer such that the sum of its digits is 2024. What is the sum of the digits of the number $N + 2$?

11. If $a$ and $b$ are nonzero real numbers such that $\frac{1}{b}-\frac{1}{a}=4$, find the value of $\frac{3a+7ab-3b}{a-3ab-b}$.

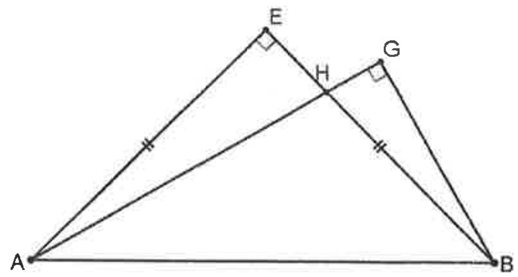
12. If the 5-digit whole number $\overline{11ab6}$ is a perfect square, find the value of $a+b$.

13. Let $N = 1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + 30 \times 31$. Find the value of $N$.

14. Let $x$ and $y$ be nonzero real numbers where $x \neq y$. If $x^2 + \sqrt{3}y = 4$ and $y^2 + \sqrt{3}x = 4$, find the value of $\left(\frac{y}{x}\right)^2 + \left(\frac{x}{y}\right)^2$.

15. Find the sum of all 2-digit even numbers $N$ with the following property: $N$ is a multiple of the product of its two digits.

16. In the diagram below, AEB is an isosceles right-angled triangle and ABG is a 30°–60°–90° right-angled triangle with ∠GAB = 30°. The sides AG and BE intersect at H. If the area of triangle AHE is 50 cm$^2$, find the area of triangle BGH in cm$^2$.




17. Find the smallest positive integer $n$ such that $\sqrt{n}-\sqrt{n-1}<\frac{1}{99}$.

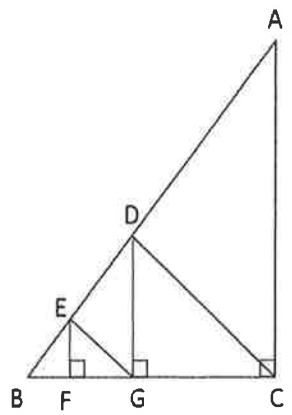
18. Let $a$, $b$ and $c$ be real numbers such that $a+b+c=8$ and $ab+bc+ca=0$. Find the maximum value of $3(a+b)$.

19. In the table below, every row and every column is an infinite arithmetic progression.

| | | | | | |
|---|---|---|---|---|---|
| 1 | 3 | 5 | 7 | 9 | ⋯ |
| 3 | 6 | 9 | 12 | 15 | ⋯ |
| 5 | 9 | 13 | 17 | 21 | ⋯ |
| 7 | 12 | 17 | 22 | 27 | ⋯ |
| 9 | 15 | 21 | 27 | 33 | ⋯ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋯ |

How many times does the number 2025 appear in the table?

20. In the diagram below, ABC is a right-angled triangle. Points D and E lie on AB while points F and G lie on BC such that ΔEFG and ΔDGC are right-angled isosceles triangles. It is given that DC = 3EG and the area of ΔDGC = 1 cm$^2$. What is the area of ΔADC in cm$^2$?

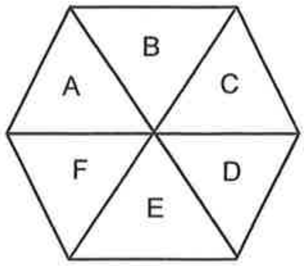



21. How many different 4-tuples $(a, b, c, d)$ are there, where $a$, $b$, $c$ and $d$ are positive integers, such that

$$a > b > c > d,\ a + b + c + d = 2024 \ \text{ and } \ a^2 - b^2 + c^2 - d^2 = 2024?$$

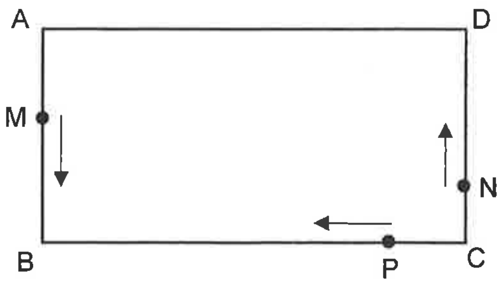
22. Points A and B lie on the graph of $y = x^2 + 5x - 8$ such that A, B and the origin O are collinear and $|OB| = 2|OA|$. It is given that A lies in the first quadrant. Find $|AB|^2$.

23. The diagram below shows a regular hexagon that is divided into six congruent triangular regions A, B, C, D, E and F. Two triangular regions are adjacent if they share a common side. For example, A and B are adjacent but A and C are not adjacent. In how many ways can we colour these regions A, B, C, D, E and F using six different colours such that adjacent regions do not receive the same colour? (Note that not all six colours need to be used in colouring the six regions and non-adjacent regions can receive the same colour.)

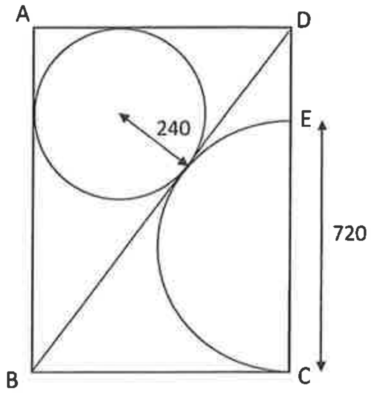



24. The diagram below shows three toy cars moving in a rectangular circuit ABCD where AB = 10 m and BC = 20 m. Toy cars M and N start from the vertices A and C respectively and move in an anti-clockwise direction with constant speeds 10 m/min and 4 m/min respectively. Toy car P starts from C and moves in a clockwise direction with a constant speed of 8 m/min.

The three toy cars start their motion at the same time. Assume that when any two toy cars meet, there is no collision and the cars will continue with their motion. Find the total time elapsed in minutes when all the three toy cars meet simultaneously for the fifth time.



25. In the diagram below, ABCD is a rectangle. A circle of radius 240 mm is inscribed in ΔABD and BD is a common tangent to both the circle and a semicircle whose diameter CE lies on CD. It is given that CE = 720 mm. Find the perimeter of rectangle ABCD in mm.



# Singapore Mathematical Society

## Singapore Mathematical Olympiad (SMO) 2024

### Senior Section (Round 1)

**Tuesday, 28 May 2024** **0930 – 1200 hrs**

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## Multiple Choice Questions

1. Find the largest positive integer $A$ such that $2^x + \dfrac{2025}{2^x} - A > 0$ for all real number $x$.

(A) 59 (B) 69 (C) 79 (D) 89 (E) 99

2. If $x = \dfrac{1}{\log_{\frac{2024}{2023}} 7} + \dfrac{1}{\log_{\frac{2023}{2022}} 7} + \dfrac{1}{\log_{\frac{2022}{2021}} 7}$, find $7^x$.

(A) $\dfrac{2021}{2024}$ (B) $\dfrac{2024}{2021}$ (C) $\dfrac{2022}{2024}$ (D) $\dfrac{2024}{2022}$ (E) 2024

3. Simplify $\dfrac{2024}{\sqrt{4+\sqrt{12}}} + \dfrac{2024}{\sqrt{4-\sqrt{12}}}$.

(A) 1012 (B) $1012\sqrt{3}$ (C) 2024 (D) $2024\sqrt{3}$ (E) $1012 + 1012\sqrt{3}$

4. Suppose $x^{\frac{1}{3}} + 12 = y^{\frac{1}{3}}$ for some real numbers $x$ and $y$. Find the minimum possible value of $y - x$.

(A) 432 (B) 532 (C) 632 (D) 732 (E) None of the above

5. Find the largest possible value of $\dfrac{\sqrt{2}\cos(2x)}{\sin(x)+\cos(x)}$.

(A) $\dfrac{1}{2}$ (B) 1 (C) $\dfrac{3}{2}$ (D) 2 (E) $\dfrac{5}{2}$

## Short Questions

6. If $\sqrt{x+\sqrt{x}} + \sqrt{x-\sqrt{x}} = 4$, find the value of $15x$.

7. Find the smallest positive integer $K$ such that

$$x^2 - 200x + y^2 = 0 \quad \text{and} \quad x + y \leq K.$$

8. Given that $\dfrac{\cos(x)}{\sin(3x)} - \dfrac{\sin(x)}{\cos(3x)} - 2 \cdot \dfrac{\sin(4x)}{\cos(6x)} = 2024$. Find the value of $\dfrac{\cos(10x)}{\sin(12x)}$.

9. Find the smallest positive integer $k$ such that the coefficient of $x^k$ in the expansion of $\left(5x^3 + \frac{1}{\sqrt{x}}\right)^{2024}$ is **not** zero.

10. Let

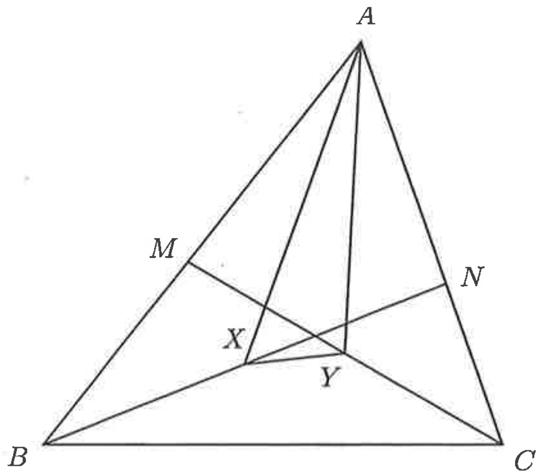
$$P = \left(2024^2+1\right)\left(2024^{2^2}+1\right)\left(2024^{2^3}+1\right)\cdots\left(2024^{2^{10}}+1\right) \times 2025 + \frac{1}{2023}.$$

Find the smallest positive integer $N$ such that $N > \log_{2024} P$.

11. Let $\triangle ABC$ be a triangle with area 1000. Let $M$ and $N$ be points on $AB$ and $AC$ respectively such that

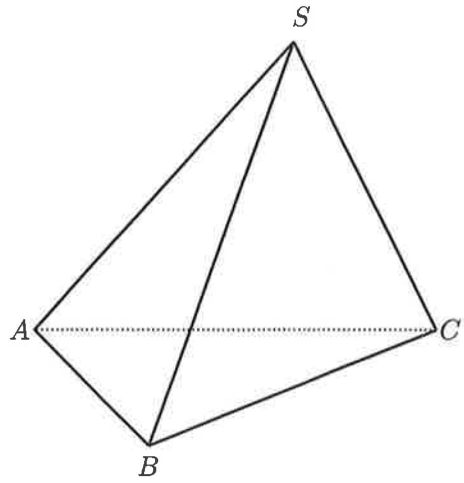
$$AM : MB = 3 : 2 \quad \text{and} \quad AN : NC = 7 : 3.$$

Let $X$ and $Y$ be the midpoints of $BN$ and $CM$ respectively. Find the area of $\triangle AXY$.




12. Find the largest positive integer $n \leq 10000$ such that $1 + 2024n^2$ is a perfect square.

13. In a tetrahedron $SABC$, the faces $SBC$ and $ABC$ are perpendicular to each other. The angles $\angle ASB$, $\angle BSC$, $\angle ASC$ are all $60°$, and $SB = SC = 4$. Find the square of the volume of the tetrahedron.




14. Let $a, b, c$ be the three real roots of the cubic equation

$$2x^3 - 4x^2 - 21x - 8 = 0$$

Given that

$$S = \frac{1}{ab + c - 1} + \frac{1}{bc + a - 1} + \frac{1}{ca + b - 1}$$

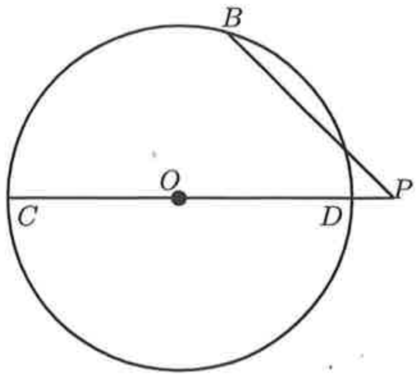
is a rational number that can be expressed as a fraction in the lowest form $\frac{m}{n}$, find the value of $m^2 + n^2$.

15. Consider the equation

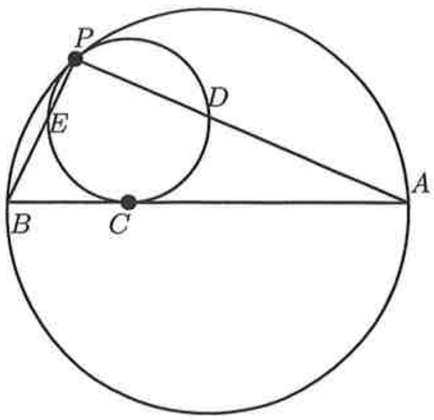
$$\frac{\sqrt{3}-1}{\sin x} + \frac{\sqrt{3}+1}{\cos x} = 4\sqrt{2}.$$

For the range $0 < x < \pi/2$, the sum of the solutions of the equation can be expressed in the form $\frac{m\pi}{n}$, where $\frac{m}{n}$ is a fraction in the lowest form. Find $m + n$.

16. An engineer constructs a circle with centre $O$ and diamater $CD$ on level ground, and builds a vertical tower of height 20 at the centre. $B$ is another point on the circumference and $P$ is on $CD$ produced such that $PB$ is a secant line of the circle. Given that $PB = 33$, $PC = 77$, and $CD = 74$, find the minimum possible distance of any point on $PB$ to the top of the tower.




17. $P$ is a common point of tangency of two circles. $BA$ is a chord of the larger circle which is tangent to the smaller circle at a point $C$. $PB$ and $PA$ intersect the smaller circle at points $E$ and $D$ respectively. If $BA = 15$, $PE = 2$, and $PD = 3$, find the length $CA$.




18. On each face of a cube, an integer greater than 2 is written. Each vertex of the cube is the intersection of three unique faces, and each edge is the intersection of two unique faces. Assign to each vertex the product of the numbers written on the faces intersecting the vertex, and assign to

each edge the product of the numbers written on the faces intersecting the edge. The sum of the numbers assigned to the eight vertices is equal to 2024. Find the maximum possible value of an edge.

19. Find the sum of the squares of each of the roots of the equation

$$x^2 - 4\lfloor x \rfloor - 12 = 0,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$.

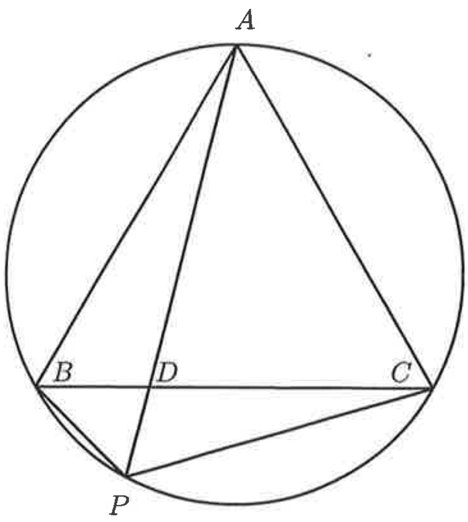
20. Calculate the remainder when $1901^{2024}$ is divided by 1216.

21. Let $P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ be a polynomial with non-negative integer coefficients satisfying $0 \le a_i \le 17$ for all $i$. If $P(18) = 367616$, find the value of $P(3)$.

22. Evaluate the sum

$$\frac{2}{1+\tan\left(\frac{\pi}{260}\right)} + \frac{2}{1+\tan\left(\frac{2\pi}{260}\right)} + \frac{2}{1+\tan\left(\frac{3\pi}{260}\right)} + \cdots + \frac{2}{1+\tan\left(\frac{129\pi}{260}\right)}.$$

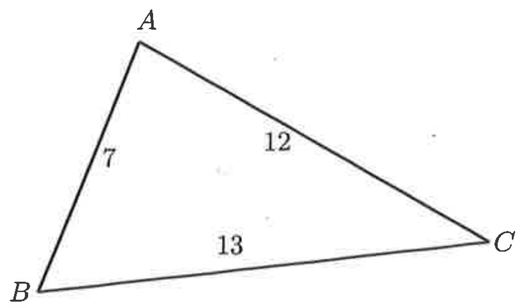
23. An equilateral triangle $ABC$ is inscribed in a circle and $P$ is a point on the minor arc $BC$. Point $D$ is the intersection of $AP$ and $BC$.




Suppose that $BP = 5$, $CP = 20$. Find the length of $AD$.

24. Find the number of positive integers $x < 9000$ such that $x^3 + 95$ is divisible by 96.

25. A scalene triangle $\triangle ABC$ has sides $AB = 7$, $AC = 12$, and $BC = 13$. Write $\tan\frac{A-B}{2}\tan\frac{C}{2}$ as a fraction $\frac{m}{n}$ in its lowest form and find $m + n$.


A
7
12
13
B
C

## Singapore Mathematical Society
## Singapore Mathematical Olympiad (SMO) 2024
## (Open Section, Round 1)

**Wednesday, 30 May 2024** **0930-1200 hrs**

**Instructions to contestants**

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

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In this paper, let $\lfloor x \rfloor$ denote the greatest integer not exceeding $x$, and $\lceil x \rceil$ denote the smallest integer not less than $x$. For examples, $\lfloor 5 \rfloor = 5$, $\lfloor 2.8 \rfloor = 2$, $\lfloor -2.3 \rfloor = -3$; $\lceil 5 \rceil = 5$, $\lceil 2.8 \rceil = 3$, $\lceil -2.3 \rceil = -2$

1. Let $S_k = 1 + 2 + 3 + \cdots + k$ for any positive integer $k$. Find $S_1 + S_2 + S_3 + \cdots + S_{20}$.

2. Let $S = \sum_{r=1}^{64} r\binom{64}{r}$, where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ and $0! = 1$. Find $\log_2 S$

3. Let $x$ be the largest number in the interval $[0, 2\pi]$ such that $(\sin x)^{2024} - (\cos x)^{2024} = 1$. Find $\lfloor x \rfloor$.

   (Note: If you think that such a number $x$ does not exist, enter your answer "99999".)

4. Find the number of real numbers $x$ that satisfies the equation $|x-2| + |x-3| = |2x-5|$.

   (Note: If you think that there are no such numbers, enter "0"; if you think that there are infinitely many such numbers, enter "99999".)

5. Among all the real numbers that satisfies the inequality $e^x \geq 1 + 2e^{-x}$, find the minimum value of $\lceil e^x + e^{-x} \rceil$.

6. Find the smallest positive integer $C$ greater than 2024 such that the sets $A = \{2x^2+2x+C : x \in \mathbb{Z}\}$ and $B = \{x^2 + 2024x + 2 : x \in \mathbb{Z}\}$ are disjoint.

7. Let $ABCD$ be a convex quadrilateral inscribed in a circle $\omega$. The bisector of $\angle BAC$ meets $\omega$ at $E$ $(\neq A)$, the bisector of $\angle ABD$ meets $\omega$ at $F$ $(\neq B)$, $AE$ intersects $BF$ at $P$ and $CF$ intersects $DE$ at $Q$. Suppose $EF = 20$, $PQ = 11$. Find the area of the quadrilateral $PEQF$.

8. Let $f(x) = \sqrt{x^2+1} + \sqrt{(4-x)^2+4}$. Find the minimum value of $f(x)$.

9. It is known that $a \geq 0$ satisfies $\sqrt{4+\sqrt{4+\sqrt{4+\sqrt{4+a}}}} = a$. find the value of $(2a-1)^2$.

10. A rectangle with sides parallel to the horizontal and vertical axes is inscribed in the region bounded by the graph of $y = 60 - x^2$ and the $x$-axis. If the area of the largest such rectangle has area $k\sqrt{5}$, find the value of $k$.

11. Let $x$ be a real number satisfying the equation $x^{x^5} = 100$. Find the value of $\lfloor x^5 \rfloor$.

12. Let $a, b, c, d, e$ be distinct integers with $a + b + c + d + e = 9$. If $m$ is an integer such that

$$(m-a)(m-b)(m-c)(m-d)(m-e) = 2009,$$

determine the value of $m$.

13. Let $\{x\}$ be the fractional part of the number $x$, i.e., $\{x\} = x - \lfloor x \rfloor$. If $S = \int_0^9 \{x\}^2 dx$, find $\lfloor S \rfloor$.

14. The solution of the inequality $|(x+1)(x-6)| > |(x+4)(x-2)|$ can be expressed as $x < a$ or $b < x < c$. If $S = |a| + |b| + |c|$, find $\lfloor 14S \rfloor$.

15. Given that $x, y > 0$ and $x\sqrt{2-y^2} + y\sqrt{2-x^2} = 2$, find the value of $x^2 + y^2$.

16. A convex polygon has $n$ sides such that no three diagonals are concurrent. It is known that all its diagonals divide the polygon into 2500 regions. Determine $n$.

17. Find the number of integers $n$ between $-2029$ and 2029 inclusive such that $(n+2)^2 + n^2$ is divisible by 2029.

18. Let $f$ be a function such that for any real number $x$,we have$f(x) + 2f(2-x) = x + x^2$. Find the value of $f(1) + f(2) + f(3) + \cdots + f(34)$.

19. Find the largest positive prime integer $p$ such that $p$ divides

$$S(p) = 1^{p-2} + 2^{p-2} + 3^{p-2} + 4^{p-2} + 5^{p-2} + 6^{p-2} + 7^{p-2} + 8^{p-2}.$$

20. Let $f$ be a function such that $f(x) + f\left(\frac{1}{1-x}\right) = 1 + \frac{1}{x}$ for all $x \notin \{0, 1\}$. Find the value of $\lfloor 180 \cdot f(10) \rfloor$.

21. Let $C$ be the circle with equation $(x-a)^2 + (y-b)^2 = r^2$, where at least one of the $a$ and $b$ are irrational numbers. Find the maximum possible number of points $(p, q)$ on $C$ where both $p$ and $q$ are rational numbers.

22. On the plane there are 2024 points coloured either red or blue such that each red point is the centre of a circle passing through 3 blue points. Determine the least number of blue points.

23. It is given that the positive real numbers $x_1, \ldots, x_{2026}$ satisfy $\frac{x_1^2}{x_1^2+1} + \cdots + \frac{x_{2026}^2}{x_{2026}^2+1} = 2025$. Find the maximum value of $\frac{x_1}{x_1^2+1} + \cdots + \frac{x_{2026}}{x_{2026}^2+1}$.

24. Let $n$ denote the numbers of ways of arranging all the letters of the word MATHEMATICS in one row such that

(1) both M's precede both T's; and

(2) neither the two M's nor the two T's are next to each other.

Determine the value of $\frac{n}{6!}$.

25. The incircle of the triangle $ABC$ centered at $I$ touches the sides $BC$, $CA$, $AB$ at $D$, $E$, $F$, respectively. Let $D'$ be the intersection of the extension of $ID$ with the circle through $B, I, C$; $E'$ the intersection of the extension of $IE$ with the circle through $A, I, C$; and $F'$ the intersection of the extension of $IF$ with the circle through $A, I, B$. Suppose $AB = 52, BC = 56, CA = 60$. Find $DD' + EE' + FF'$.