Notes

Singapore Physics Olympiad 2023

Topic 4: Dynamics of Rotational Motion

Email: phyhcmk@gmail.com

Latest update: July 5, 2023 10:32am

Vector (cross) products of two vectors

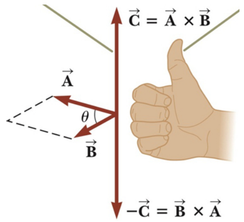
Notes

- **Vector product** of two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is another vector $\vec{\mathbf{C}}$ defined as:

$$\boxed{\vec{\mathbf{C}} \equiv \vec{\mathbf{A}} \times \vec{\mathbf{B}}}$$

$\vec{\mathbf{C}}$ is read as "$\vec{\mathbf{A}}$ cross $\vec{\mathbf{B}}$".

- Magnitude of vector $\vec{\mathbf{C}}$ is $AB\sin\theta$ where θ is the angle between vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$
- Its magnitude is equal to the area of parallelogram formed by the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$
- Direction of $\vec{\mathbf{C}}$ is perpendicular to the plane formed by $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ and it given by the **right-hand rule**



Some properties of vector product

Notes

- Vector product is *not* commutative: $\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\vec{\mathbf{B}} \times \vec{\mathbf{A}}$
- If $\vec{\mathbf{A}}$ is parallel to $\vec{\mathbf{B}}$, then $\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \vec{\mathbf{0}}$
- If $\vec{\mathbf{A}}$ is perpendicular to $\vec{\mathbf{B}}$, then $\left|\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right| = AB$
- Vector product obeys the distributive law:

$$\vec{\mathbf{A}} \times \left(\vec{\mathbf{B}} + \vec{\mathbf{C}}\right) = \vec{\mathbf{A}} \times \vec{\mathbf{B}} + \vec{\mathbf{A}} \times \vec{\mathbf{C}}$$

- Derivative of the cross product with respect to some variable such as t:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right) = \vec{\mathbf{A}} \times \frac{\mathrm{d}\vec{\mathbf{B}}}{\mathrm{d}t} + \frac{\mathrm{d}\vec{\mathbf{A}}}{\mathrm{d}t} \times \vec{\mathbf{B}}$$

Calculating cross product using components

Notes

- Cross products between Cartesian unit vectors:

$$\begin{aligned}
\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}} &= \hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}} = \mathbf{0} \\
\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}} &= -\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{k}} \\
\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{k}} &= -\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{\imath}} \\
\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\imath}} &= -\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{k}} = \hat{\boldsymbol{\jmath}}
\end{aligned}$$

- Cross products between two vectors:

$$\vec{\mathbf{A}} = A_x\,\hat{\boldsymbol{\imath}} + A_y\,\hat{\boldsymbol{\jmath}} + A_z\,\hat{\boldsymbol{k}} \qquad \vec{\mathbf{B}} = B_x\,\hat{\boldsymbol{\imath}} + B_y\,\hat{\boldsymbol{\jmath}} + B_z\,\hat{\boldsymbol{k}}$$

$$\boxed{\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \left(A_y B_z - A_z B_y\right)\hat{\boldsymbol{\imath}} + \left(A_z B_x - A_x B_z\right)\hat{\boldsymbol{\jmath}} + \left(A_x B_y - A_y B_x\right)\hat{\boldsymbol{k}}}$$

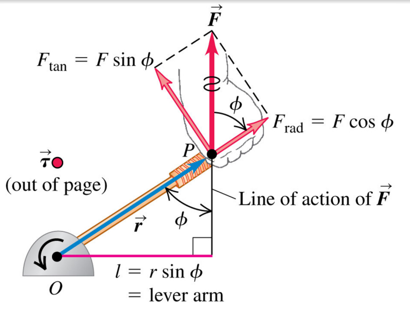
Definition of torque

- **Torque** (or moment) is a measure of the tendency of a force to rotate an object about some axis

$$\boxed{\tau = F\ell = rF\sin\phi = F_{\text{tan}}r}$$

SI unit: Newton-meter ($\mathrm{N \cdot m}$)

The value of torque depends on the chosen axis of rotation.



- $\ell = r\sin\phi$ is called the **lever arm** which is the perpendicular distance from the axis of rotation to the *line of action of the force* (an imaginary line extending out both ends of the force vector)

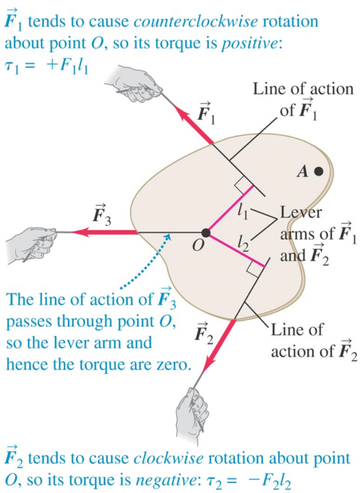
Notes

Sign convention of torque

- If the turning tendency of the force is counterclockwise (clockwise), the torque will be positive (negative)

- Net torque with respect to point O:

$$\sum \tau = \tau_1 + \tau_2 + \tau_3$$
$$= +F_1\ell_1 - F_2\ell_2 + 0$$

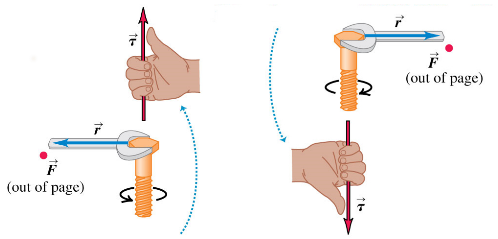


Notes

Torque as a vector

- When a force $\vec{\mathbf{F}}$ acts at a point having a positive vector $\vec{\mathbf{r}}$ with respect to an origin O, the torque $\vec{\tau}$ of the force with respect to O is the vector quantity

$$\boxed{\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}} \quad \Rightarrow \quad \tau = |\vec{\tau}| = rF\sin\phi$$



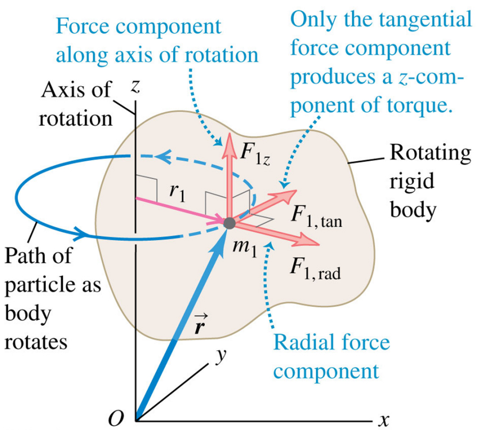
Direction of $\vec{\tau}$ is perpendicular to both $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$ with a sense given by the right-hand rule.

Notes

Newton's second law for rotational motion

- Net torque on a rigid object equals the object's moment of inertia about the rotation axis times its angular acceleration:

$$F_{1,\text{tan}} = m_1 a_{1,\text{tan}}$$
$$\tau_{1z} = F_{1,\text{tan}} r_1 = m_1 r_1^2 \alpha_z$$
$$\sum \tau_{iz} = \left(\sum m_i r_i^2\right) \alpha_z$$
$$\boxed{\sum \tau_z = I\alpha_z}$$



Notes

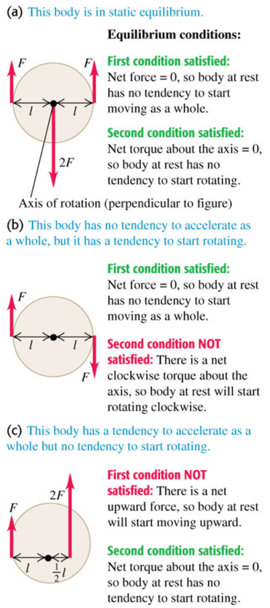
Conditions for equilibrium

- First condition for equilibrium (translational equilibrium): the vector sum of all external forces acting on the object is zero

$$\sum \vec{\mathbf{F}} = \vec{\mathbf{0}}$$

- Second condition for equilibrium (rotational equilibrium): the sum of torques due to all external forces acting on the object about *any* point is zero

$$\sum \vec{\tau} = \mathbf{0} \quad \text{(about any point)}$$



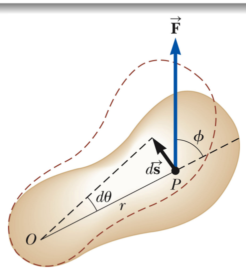
Notes

Work done in rotational motion

- Work done on the rigid object as it rotates through an infinitesimal distance $\mathrm{d}s = r\,\mathrm{d}\theta$:

$$\begin{aligned} \mathrm{d}W &= \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{s}} = F\,\mathrm{d}s\cos\left(90^\circ - \phi\right) \\ &= \left(F\sin\phi\right) r\,\mathrm{d}\theta = \left(rF\sin\phi\right)\,\mathrm{d}\theta \\ &= \tau\,\mathrm{d}\theta \end{aligned}$$

The radial component of the force does no work as it is perpendicular to the displacement.



- Work done in rotational motion:

$$\boxed{W = \int_{\theta_1}^{\theta_2} \tau\,\mathrm{d}\theta} \quad \longleftrightarrow \quad \boxed{W = \int_{x_1}^{x_2} F\,\mathrm{d}x}$$

Notes

Power in rotational motion

Notes

- Work done for the infinitesimal rotation:

$$\mathrm{d}W = \tau\,\mathrm{d}\theta$$

- Rate at which work is being done as the rigid object rotates about the fixed axis through the angle $\mathrm{d}\theta$ in a time interval $\mathrm{d}t$:

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \tau\,\frac{\mathrm{d}\theta}{\mathrm{d}t} = \tau\,\omega$$

- Power in rotational motion:

$$\boxed{P = \tau\,\omega} \quad \longleftrightarrow \quad \boxed{P = F\,v}$$

Work-energy theorem for rotational motion

Notes

- Newton's second law for rotational motion:

$$\tau = I\,\alpha = I\,\frac{\mathrm{d}\omega}{\mathrm{d}t} = I\,\frac{\mathrm{d}\omega}{\mathrm{d}\theta}\,\frac{\mathrm{d}\theta}{\mathrm{d}t} = I\omega\,\frac{\mathrm{d}\omega}{\mathrm{d}\theta}$$

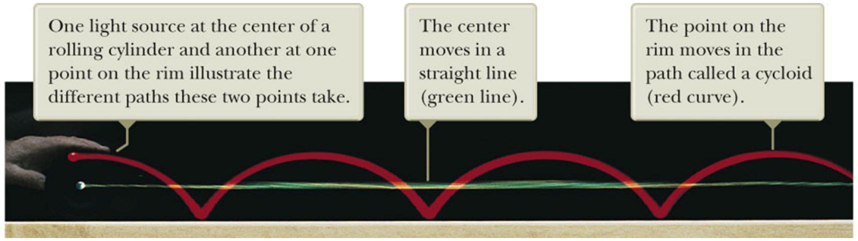
- Work done for the infinitesimal rotation:

$$\mathrm{d}W = \tau\,\mathrm{d}\theta = I\omega\,\mathrm{d}\omega$$

- Net work done by external forces in rotating a rigid object about a fixed axis equals to the change in the object's rotational kinetic energy:

$$W = \int_{\omega_i}^{\omega_f} I\omega\,\mathrm{d}\omega \quad \Rightarrow \quad \boxed{W = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2}$$

Rolling object



- The red curve shows the path moved by a point on the rim of the object – this path is called a *cycloid*
- The green line shows the path of the center of mass of the object
- An object rolls without slipping in pure rolling motion

Notes

Condition for pure rolling motion

- When the rigid object rotates through an angle θ, its center of mass moves a linear distance s

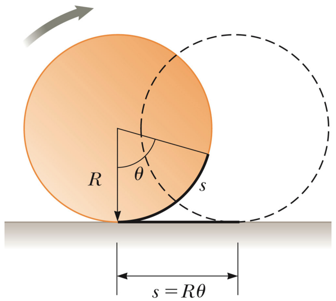
$$s = R\theta$$

- Linear speed of the center of mass:

$$v_{\mathsf{CM}} = \frac{\mathrm{d}s}{\mathrm{d}t} = R\frac{\mathrm{d}\theta}{\mathrm{d}t} = R\omega$$

- Linear acceleration of the center of mass:

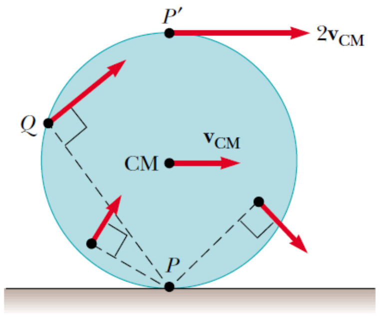
$$a_{\mathsf{CM}} = \frac{\mathrm{d}v_{\mathsf{CM}}}{\mathrm{d}t} = R\frac{\mathrm{d}\omega}{\mathrm{d}t} = R\alpha$$



Notes

Linear velocities of pure rolling object

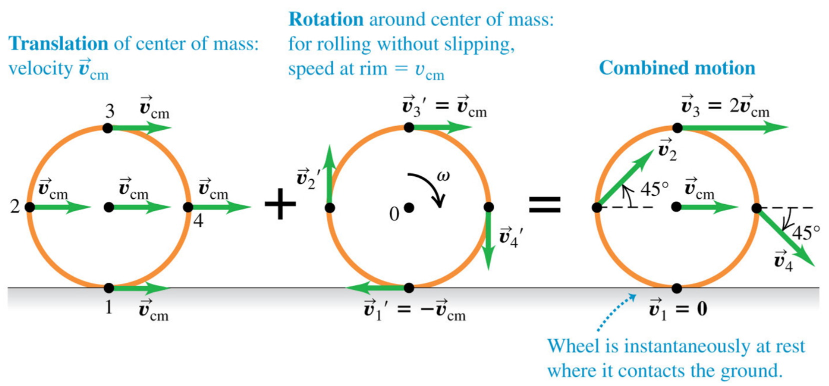
- All points on the rigid object have the same angular speed but different linear velocities
- Linear velocity of any point is in a direction perpendicular to the line from that point to the contact point P
- At any instant, the part of the rigid object that is at point P is at rest relative to the surface because slipping does not occur



Notes

Decomposition of pure rolling motion

- Pure rolling motion can be modelled as a *combination* of pure translational motion of the center of mass and the pure rotational motion of the rigid object around the center of mass



Notes

Kinetic energy of pure rolling object

- Imagine the rigid object rotating about an "instantaneous axis" of rotation that passes through the point of contact P
- Moment of inertia about the "instantaneous axis" (parallel-axis theorem):

$$I_P = I_{\mathsf{CM}} + MR^2$$

- Rotational kinetic energy about the contact point P:

$$K = \frac{1}{2}I_P\omega^2 = \frac{1}{2}I_{\mathsf{CM}}\omega^2 + \frac{1}{2}MR^2\omega^2 = \frac{1}{2}I_{\mathsf{CM}}\omega^2 + \frac{1}{2}Mv_{\mathsf{CM}}^2$$

- Total kinetic energy of a rigid body undergoing pure rolling motion is the sum of the rotational kinetic energy about its center of mass and the translational kinetic energy of the its center of mass

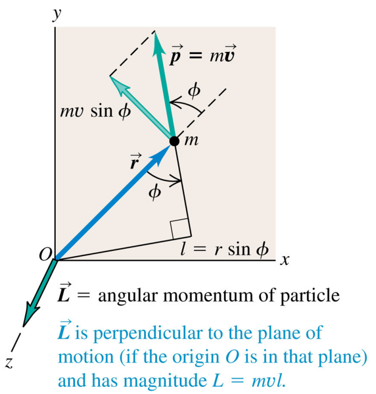
Notes

Angular momentum of a particle

- **Angular momentum** of a particle with respect to the origin O of an inertial frame is the vector product of the particle's position vector $\vec{\mathbf{r}}$ relative to O and its momentum $\vec{\mathbf{p}}$:

$$\boxed{\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}} \qquad \text{SI unit: } \mathrm{kg \cdot m^2/s}$$

Both magnitude and direction of the angular momentum $\vec{\mathbf{L}}$ depends on the choice of the origin.



Notes

Torque and angular momentum

Notes

- When a net force $\vec{\mathbf{F}}$ acts on a particle, its velocity and momentum change, so its angular momentum may also change

$$\begin{aligned}\frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{d}t} &= \frac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}t} \times \vec{\mathbf{p}} + \vec{\mathbf{r}} \times \frac{\vec{\mathbf{p}}}{\mathrm{d}t} \\ &= \vec{\mathbf{v}} \times \vec{\mathbf{p}} + \vec{\mathbf{r}} \times \vec{\mathbf{F}} \\ &= \vec{\boldsymbol{\tau}}\end{aligned}$$

- Newton's second law for rotational motion: torque acting on a particle is equal to the time rate of change of the particle's angular momentum

$$\boxed{\vec{\boldsymbol{\tau}} = \frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{d}t}}$$

Angular momentum of a system of particles

Notes

- Total angular momentum of a system of particles is defined as the vector sum of the angular momenta of the individual particles:

$$\vec{\mathbf{L}}_{\text{tot}} = \vec{\mathbf{L}}_1 + \vec{\mathbf{L}}_2 + \cdots + \vec{\mathbf{L}}_N = \sum_i \vec{\mathbf{L}}_i$$

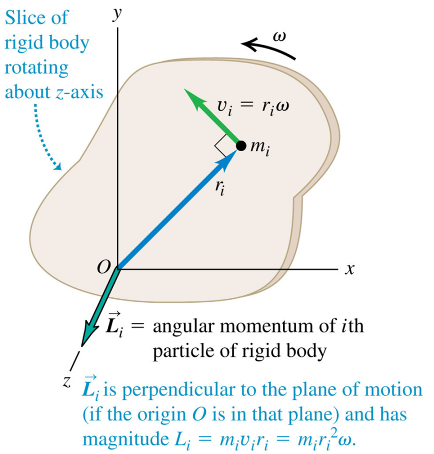
- Net external torque acting on a system about some axis passing through an origin in an inertial frame equals the time rate of change of the total angular momentum of the system about that origin:

$$\sum_i \vec{\boldsymbol{\tau}}_i = \sum_i \frac{\mathrm{d}\vec{\mathbf{L}}_i}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\sum_i \vec{\mathbf{L}}_i = \frac{\mathrm{d}\vec{\mathbf{L}}_{\text{tot}}}{\mathrm{d}t}$$

Angular momentum of a rotating rigid object

- Total angular momentum:

$$\left|\vec{\mathbf{L}}_{\text{tot}}\right| = \sum_i \left|\vec{\mathbf{L}}_i\right| = \sum_i m_i v_i r_i$$
$$= \sum_i m_i (r_i\omega) r_i = \sum_i \left(m_i r_i^2\right)\omega$$
$$= \left(\sum_i m_i r_i^2\right)\omega = I\omega$$



- Net external torque:

$$\sum_i \tau_i = \frac{\mathrm{d}\left|\vec{\mathbf{L}}_{\text{tot}}\right|}{\mathrm{d}t} = I\,\frac{\mathrm{d}\omega}{\mathrm{d}t} = I\alpha$$

Notes

Conservation of angular momentum

- Total angular momentum of a system is constant is the resultant external torque acting on the system is zero

$$\sum_i \vec{\boldsymbol{\tau}}_i = \frac{\mathrm{d}\vec{\mathbf{L}}_{\text{tot}}}{\mathrm{d}t} = 0 \quad \Rightarrow \quad \vec{\mathbf{L}}_{\text{tot}} = \text{constant}$$

- For a system of particles:

$$\sum_i \vec{\mathbf{L}}_i = \text{constant}$$

- For a rigid object rotation about a symmetry axis:

$$I_i\omega_i = I_f\omega_f$$

Notes

Problem 1

Notes

A uniform disc of radius R is spinning about the vertical axis and placed on a horizontal surface. If the initial angular speed is ω and the coefficient of friction is μ, determine the time before which the disc comes to rest.

Problem 2

Notes

A uniform rod of mass M and length L is placed vertically with one end pinned to a frictionless horizontal floor. It starts to fall when it is given a small displacement. When the rod makes an angle θ with the vertical, find

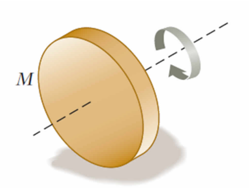
(a) the radial acceleration of the top of the pole;
(b) the tangential acceleration of the top of the pole;
(c) the reaction force at the end of the rod contacting the floor if the pin is removed when it starts to fall.

Problem 3

Notes

A uniform solid disk of radius R and mass M is set into rotation with an angular speed ω_0 about an axis through the center. While still rotating at this speed, the disk is placed into contact with a horizontal surface and immediately released. Assume that the coefficient of friction between the disk and surface is μ.

(a) What is the time interval after setting the disk down before pure rolling motion begins?
(b) How far does the disk travel before pure rolling begins?
(c) What is the angular speed of the disk once pure rolling takes place? What about the speed of the center of mass?
(d) How much energy was dissipated in heat?



Problem 4

Notes

A bead of mass m and initial speed v_0 strikes a uniform thin rod of mass m and length L perpendicularly at one end. The rod is initially rests on a frictionless horizontal plane.

(a) If the other end of the rod is fixed on a hinge which allows the rod to rotate freely in the horizontal plane, and the bead stays on the rod after collision, find the mechanical energy loss due to the collision.
(b) If the rod is free to move on the plane and the bead stays on the rod after collision, find the mechanical energy loss due to collision.
(c) If the rod is free to move on the plane and the collision is elastic, find the angular speed of the rod, and the speeds of the bead and the center of mass of the rod. Assuming that the velocity of the bead is perpendicular to the rod right after the collision.