Notes

Singapore Physics Olympiad 2023

Topic 3: Kinematics of Rotational Motion

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Notes

Unit of angles

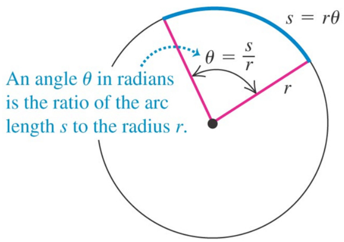
- The **radian** is a unit of angular measurement and it can be expressed as the arc length s along a circle divided by the radius r

$$\boxed{\theta \equiv \frac{s}{r}}$$

- Conversion: $360^\circ = 2\pi\,\text{rad}$

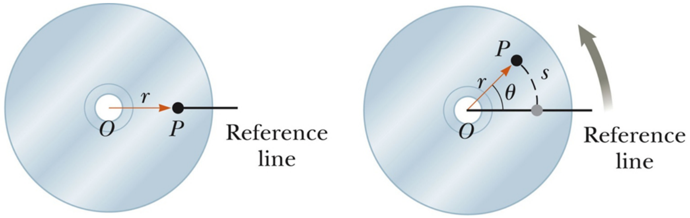
$$\boxed{\theta\,[\text{rad}] = \frac{\pi}{180^\circ} \times \theta\,[\text{degrees}]}$$

$$1\,\text{rad} = \frac{360^\circ}{2\pi} \approx 57.30^\circ$$



Angular position

- **Rigid object**: Every part of the object is fixed in position *relative* to all other parts of the object
- When a rigid object rotates through a given angle, all parts of the object rotate through the same angle
- **Angular position**: Angle θ, measured in radians, in which the point P has moved through within a time interval with respect to the *fixed* reference line



Notes

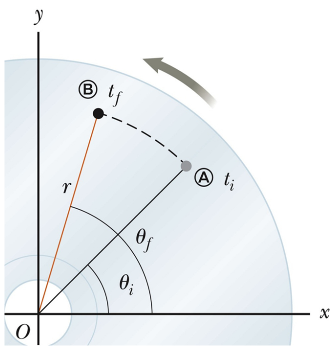
Angular displacement

- **Angular displacement** $\Delta\theta$ is defined as the angle the rigid object rotates through during some time interval

$$\boxed{\Delta\theta = \theta_f - \theta_i}$$

SI unit: radian (rad)

- Every point on the rigid object undergoes the same angular displacement in any given time interval



Notes

Angular velocity

- **Average angular velocity** of the rotating rigid object is the ratio of the angular displacement to the time interval:

$$\boxed{\omega_{\mathrm{av},z} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}}$$

SI unit: radian per second $(\mathrm{rad/s})$

- **Instantaneous angular velocity** is defined as the limit of the average angular speed as the time interval approaches zero:

$$\boxed{\omega_z(t) = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{\mathrm{d}\theta(t)}{\mathrm{d}t}}$$

SI unit: radian per second $(\mathrm{rad/s})$

- Every part of a rotating rigid object has the same (instantaneous) angular velocity at any instant of time

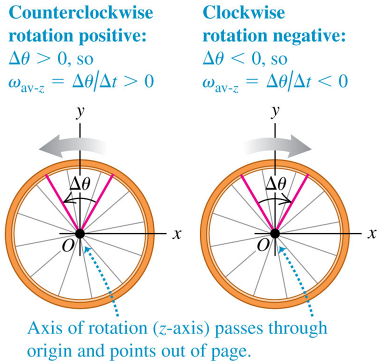
Notes

'Sense' of rotation

- A rigid object's average angular velocity and instantaneous angular velocity can be positive or negative

- The angular velocity is positive if the object is rotating in the direction of increasing θ (counter-clockwise direction)

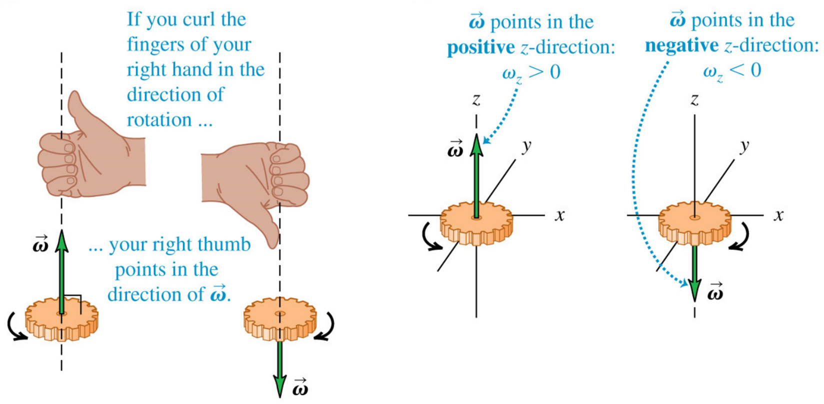
- The angular velocity is negative if the object is rotating in the direction of decreasing θ (clockwise direction)



Notes

Angular velocity is a vector

- **Right hand rule**: The direction of the angular velocity vector can be visualized by sweeping out the angle with the fingers of right hand. The thumb points in the direction of the angular velocity vector.



Notes

Angular acceleration

- **Average angular acceleration**: The ratio of the change in angular velocity to the time interval it takes for the rigid object to undergo the change

$$\boxed{\alpha_{\mathsf{av},z} = \frac{\omega_{f,z} - \omega_{i,z}}{t_f - t_i} = \frac{\Delta\omega_z}{\Delta t}} \qquad \text{SI unit: } (\mathrm{rad/s^2})$$

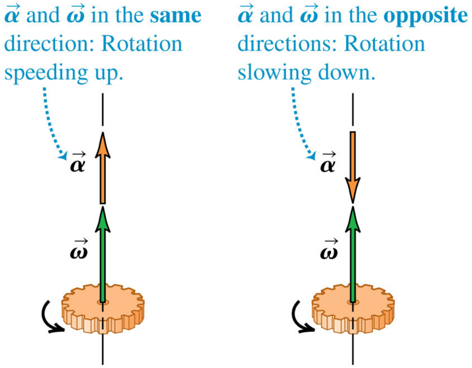
- **Instantaneous angular acceleration**: The limit of the average angular acceleration as the time interval approaches zero

$$\boxed{\alpha_z(t) = \lim_{\Delta t \to 0} \frac{\Delta\omega_z}{\Delta t} = \frac{\mathrm{d}\omega_z(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2}} \qquad \text{SI unit: } (\mathrm{rad/s^2})$$

Notes

Angular acceleration is a vector

- When the rotational axis is fixed, the angular acceleration and angular velocity vectors both lie along that axis



Notes

Rotation with constant angular acceleration α_z

- Angular velocity as a function of time:

$$\alpha_z(t) = \frac{\mathrm{d}\omega_z(t)}{\mathrm{d}t} = \alpha_z = \text{constant} \quad \Rightarrow \quad \boxed{\omega_z(t) = \omega_{0z} + \alpha_z t}$$

- Angular position as a function of time:

$$\omega_z(t) = \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} \quad \Rightarrow \quad \boxed{\theta(t) = \theta_0 + \omega_{0z} t + \frac{1}{2}\alpha_z t^2}$$

- Eliminating time dependence:

$$\boxed{\omega_z^2(t) = \omega_{0z}^2 + 2\alpha_z\left[\theta(t) - \theta_0\right]}$$

Notes

Comparison: Motion with constant accelerations

Constant linear acceleration

$$a_x(t) = a_x = \text{constant}$$
$$v_x(t) = v_{0x} + a_x t$$
$$x(t) = x_0 + v_{0x} t + \frac{1}{2} a_x t^2$$
$$v_x^2(t) = v_{0x}^2 + 2a_x [x(t) - x_0]$$

Constant angular acceleration

$$\alpha_z(t) = \alpha_z = \text{constant}$$
$$\omega_z(t) = \omega_{0z} + \alpha_z t$$
$$\theta(t) = \theta_0 + \omega_{0z} t + \frac{1}{2} \alpha_z t^2$$
$$\omega_z^2(t) = \omega_{0z}^2 + 2\alpha_z [\theta(t) - \theta_0]$$

There is a one-to-one correspondence between them:

$$\theta \leftrightarrow x \,, \omega \leftrightarrow v \,, \alpha \leftrightarrow a$$

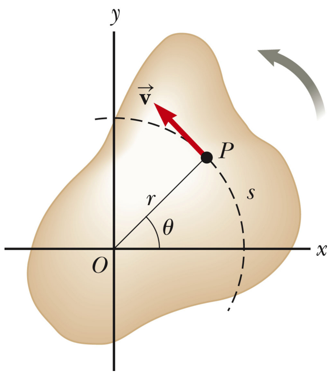
For constant a and α, we can write analogous equations for rotational motion as in linear motion.

Notes

Linear speed in rigid object rotation

- When a rigid object rotates about a fixed axis, every particle of the object moves in a circle whose center is on the axis of rotation

$$s = r\theta$$
$$\Rightarrow \quad v = \frac{\mathrm{d}s}{\mathrm{d}t} = r\,\frac{\mathrm{d}\theta}{\mathrm{d}t}$$
$$\Rightarrow \quad \boxed{v = r\omega}$$



- **Tangential speed** of a point on a rotating rigid object equals to the perpendicular distance of that point from the axis of rotation multiplied by the angular speed

Notes

Tangential acceleration

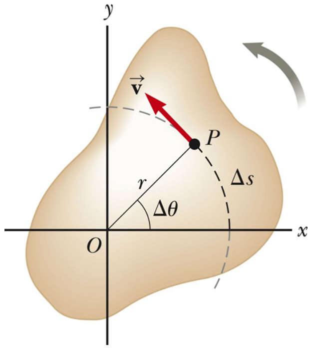
Notes

- **Tangential acceleration**, the component parallel to the instantaneous velocity, acts to change the *magnitude* of the particle's velocity

$$\Delta v = r\,\Delta\omega$$

$$\Rightarrow \quad \lim_{\Delta t\to 0}\frac{\Delta v}{\Delta t} = r\lim_{\Delta t\to 0}\frac{\Delta \omega}{\Delta t}$$

$$\Rightarrow \quad \boxed{a_t = r\alpha}$$



- Tangential acceleration of a point on the rotating rigid object equals to the point's perpendicular distance from the axis of rotation multiplied by the angular acceleration

Centripetal acceleration

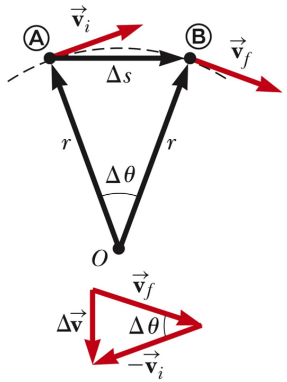
Notes

- **Centripetal acceleration**, the component of the particle's acceleration directed toward the rotational axis, is associated with the change of *direction* of the particle's velocity

$$\frac{\Delta v}{v} = \frac{\Delta s}{r} \quad \Rightarrow \quad \Delta v = \frac{v}{r}\,\Delta s$$

$$\Rightarrow \quad \lim_{\Delta t\to 0}\frac{\Delta v}{\Delta t} = \frac{v}{r}\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t}$$

$$\Rightarrow \quad \boxed{a_r = \frac{v^2}{r} = r\omega^2}$$

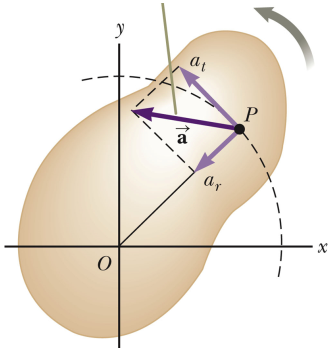


Linear acceleration in rigid object rotation

- Tangential component of the acceleration is due to changing speed
- Centripetal component of the acceleration is due to changing direction
- Total linear acceleration:

$$\vec{\mathbf{a}} = \vec{\mathbf{a}}_t + \vec{\mathbf{a}}_r$$

$$\Rightarrow \quad a = \sqrt{a_t^2 + a_r^2} = r\sqrt{\alpha^2 + \omega^4}$$



Notes

Rotational kinetic energy

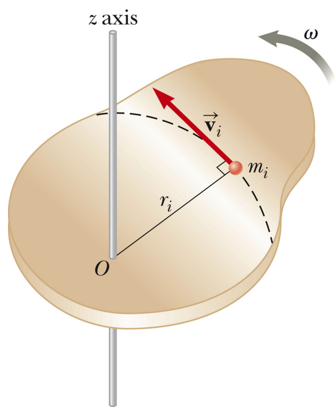
- Treating a rigid object as a *collection* of particles rotating about a fixed z axis with an angular speed ω
- Kinetic energy of the i-th particle:

$$K_i = \frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i\left(r_i\omega\right)^2 = \frac{1}{2}m_i r_i^2\omega^2$$

- **Rotational kinetic energy**:

$$K = \sum_i K_i = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$

$$\Rightarrow \quad \boxed{K = \frac{1}{2}I\omega^2}$$



Notes

Defining moment of inertia

Notes

- **Moment of inertia** of a system of particles:

$$\boxed{I = \sum_i m_i r_i^2}$$

SI unit: $\mathrm{kg} \cdot \mathrm{m}^2$

- Moment of inertia of a continuous rigid object can be calculated by imaging the object to be divided into many small elements each of mass Δm_i

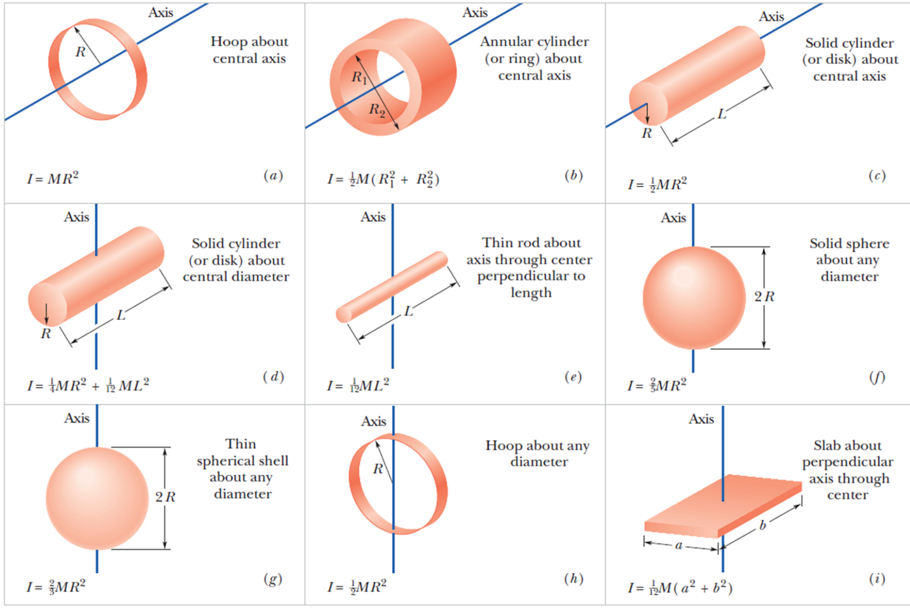
$$\boxed{I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \, \Delta m_i = \int r^2 \, \mathrm{d}m}$$

Moment of inertia

Notes

- Moment of inertia is a measure of the *resistance* of an object to *changes in its rotational motion*, similar to mass being a measure of an object's resistance to changes in its translational motion
- Mass is an intrinsic property of an object but moment of inertia depends on the choice of rotational axis
- Moment of inertia depends on the mass and how the mass is distributed around the rotational axis

Moments of inertia of homogeneous rigid objects



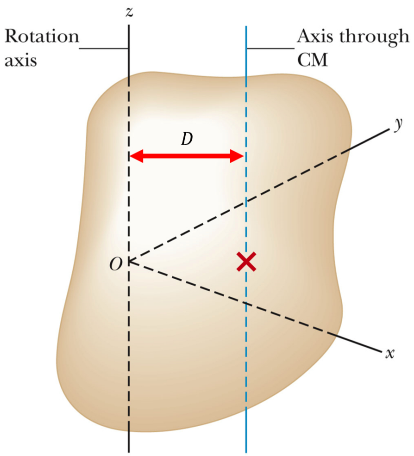
Notes

Parallel axis theorem

- **Parallel axis theorem**: relationship between the moments of inertia of a rigid object of mass M about two parallel axes

$$\boxed{I = I_{\text{CM}} + MD^2}$$

- I_{CM} is the moment of inertia about an axis through the center of mass and I is the moment of inertia about a parallel axis at a distance D from the first axis

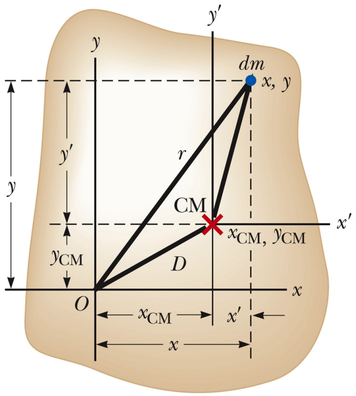


Notes

Proof of the parallel axis theorem

Notes

$$
\begin{aligned}
I &= \int r^2 \, \mathrm{d}m = \int \left(x^2 + y^2\right) \, \mathrm{d}m \\
&= \int \left[\left(x' + x_{\mathsf{CM}}\right)^2 + \left(y' + y_{\mathsf{CM}}\right)^2\right] \, \mathrm{d}m \\
&= \int \left(x'^2 + y'^2\right) \, \mathrm{d}m + \left(x_{\mathsf{CM}}^2 + y_{\mathsf{CM}}^2\right) \int \mathrm{d}m \\
&\quad + 2x_{\mathsf{CM}} \int x' \, \mathrm{d}m + 2y_{\mathsf{CM}} \int y' \, \mathrm{d}m \\
&= I_{\mathsf{CM}} + MD^2
\end{aligned}
$$

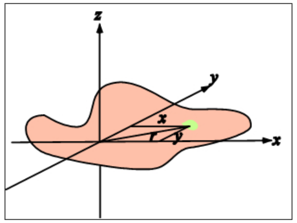


Perpendicular axis theorem

Notes

- **Perpendicular axis theorem**: sum of moments of inertia about any two perpendicular axes in the plane of the body is equal to the moment of inertia about an axis through the point of intersection perpendicular to the plane of the object

$$
\begin{aligned}
I_z &= \int r^2 \, \mathrm{d}m = \int \left(x^2 + y^2\right) \, \mathrm{d}m \\
&= \int x^2 \, \mathrm{d}m + \int y^2 \, \mathrm{d}m \\
\Rightarrow &\quad \boxed{I_z = I_x + I_y}
\end{aligned}
$$



This theorem works only for planar figures (2D bodies), i.e., bodies of negligible thickness – not a sphere, cone, etc.

Problem 1

Notes

A small block of mass m is placed on the top of a smooth sphere of radius R. Then the sphere is imparted a constant acceleration a_0 in the horizontal direction and the block begins sliding down. Find

(a) the velocity of the block relative to the sphere at the moment of break-off.

(b) the angle θ_0 between the vertical and the radius vector drawn from the center of the sphere to the break off point.

Problem 2

Notes

(a) Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis perpendicular to the rod and passing O at an arbitrary distance h from one end.

(b) Initially the rod is at rest. It is given a constant angular acceleration of magnitude α around the axis through O. Find how much work is done on the rod in a time t?

(c) At time t, what is the linear acceleration of the point on the rod farthest from the axis?

Problem 3

Notes

A rod of length L and M has a linear density proportional to the distance from one end.

(a) Calculate the moment of inertial of the rod about an axis through the light end of the rod perpendicular to the rod.

(b) What about the moment of inertia about the axis through the heavier end?

(c) Calculate the moment of inertia about an axis through the center of mass.

Problem 4

Notes

Calculate the moment of inertia of a cylinder of mass M, radius R and length L about an axis at right angles to its cylindrical axis and passing through the center of mass.