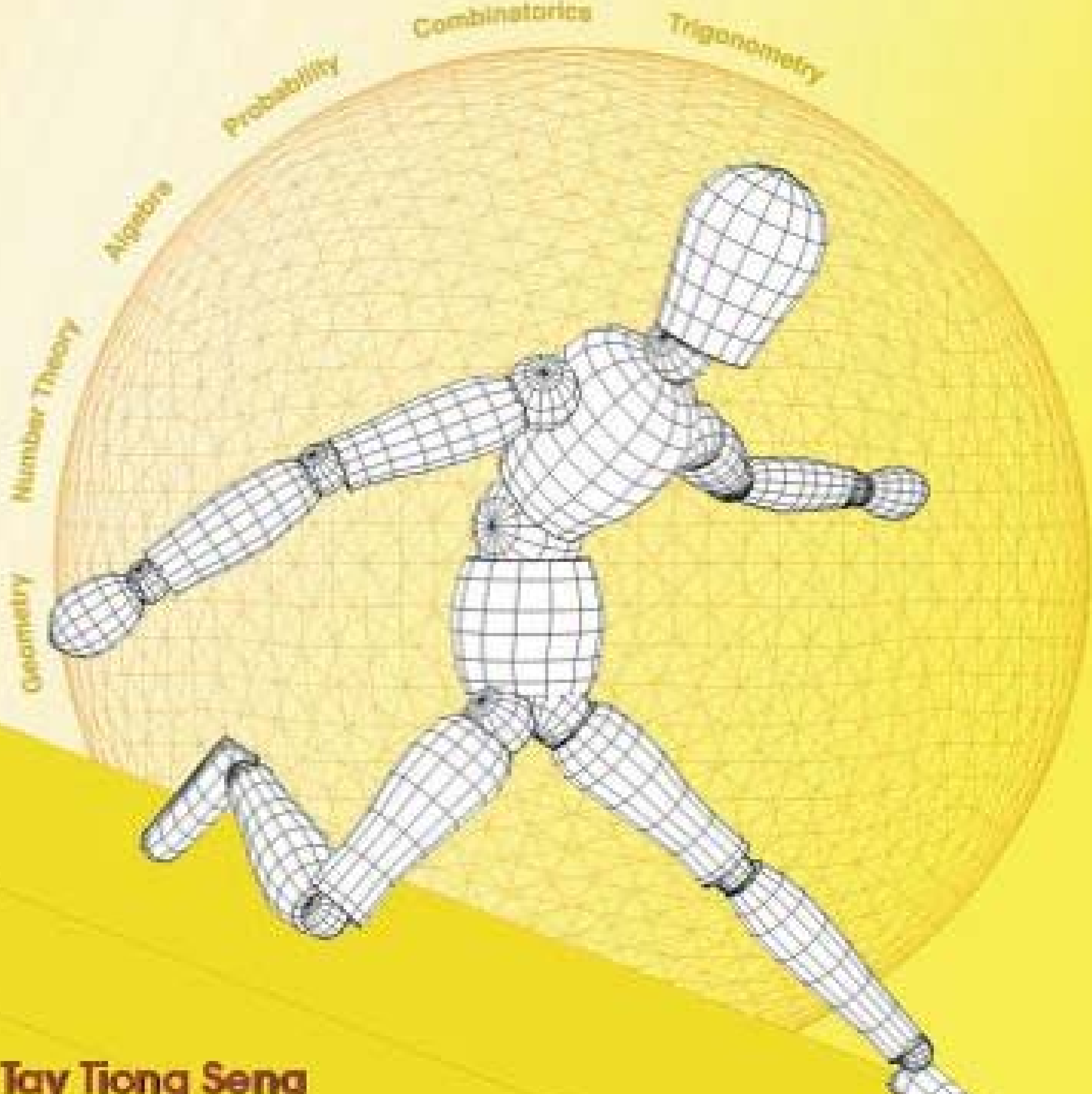
SINGAPORE MATHEMATICAL OLYMPIADS 2009



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Published by Singapore Mathematical Society

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Junior Section)

Tuesday, 2 June 2009 — 0930 – 1200 hrs

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answers on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.

For the other short questions, write your answer in the answer sheet and shade the appropriate bubble below your answer.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

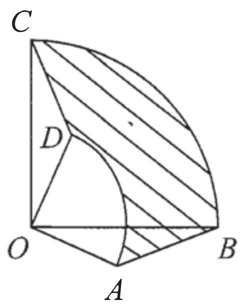
PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1 Let C_1 and C_2 be distinct circles of radius 7 cm that are in the same plane and tangent to each other. Find the number of circles of radius 26 cm in this plane that are tangent to both C_1 and C_2.

(A) 2
(B) 4
(C) 6
(D) 8
(E) none of the above

2 In the diagram below, the radius of quadrant OAD is 4 and the radius of quadrant OBC is 8. Given that $\angle COD = 30°$, find the area of the shaded region $ABCD$.



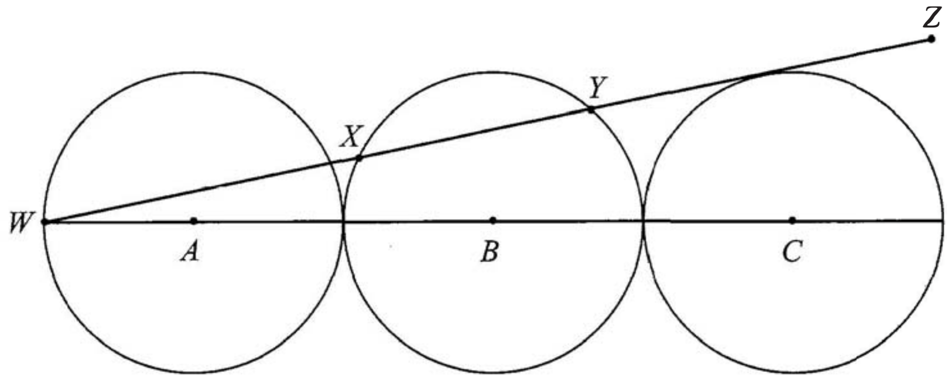
(A) 12π
(B) 13π
(C) 15π
(D) 16π
(E) none of the above

3 Let k be a real number. Find the maximum value of k such that the following inequality holds:

$$\sqrt{x-2} + \sqrt{7-x} \geq k.$$

(A) $\sqrt{5}$
(B) 3
(C) $\sqrt{2} + \sqrt{3}$
(D) $\sqrt{10}$
(E) $2\sqrt{3}$

4 Three circles of radius 20 are arranged with their respective centres A, B and C in a row. If the line WZ is tangent to the third circle, find the length of XY.



(A) 30
(B) 32
(C) 34
(D) 36
(E) 38

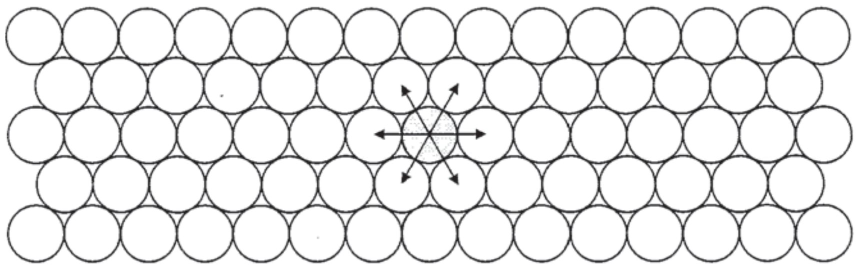
5 Given that x and y are both negative integers satisfying the equation $y = \dfrac{10x}{10-x}$, find the maximum value of y.

(A) -10
(B) -9
(C) -6
(D) -5
(E) None of the above

6 The sequence a_n satisfy $a_n = a_{n-1} + n^2$ and $a_0 = 2009$. Find a_{50}.

(A) 42434
(B) 42925
(C) 44934
(D) 45029
(E) 45359

7 Coins of the same size are arranged on a very large table (the infinite plane) such that each coin touches six other coins. Find the percentage of the plane that is covered by the coins.



(A) $\dfrac{20}{\sqrt{3}}\pi$ %

(B) $\dfrac{50}{\sqrt{3}}\pi$ %

(C) $16\sqrt{3}\,\pi$ %

(D) $17\sqrt{3}\,\pi$ %

(E) $18\sqrt{3}\,\pi$ %

8 Given that x and y are real numbers satisfying the following equations:

$$x + xy + y = 2 + 3\sqrt{2} \quad \text{and} \quad x^2 + y^2 = 6,$$

find the value of $|x + y + 1|$.

(A) $1 + \sqrt{3}$

(B) $2 - \sqrt{3}$

(C) $2 + \sqrt{3}$

(D) $3 - \sqrt{2}$

(E) $3 + \sqrt{2}$

9 Given that $y = (x - 16)(x - 14)(x + 14)(x + 16)$, find the minimum value of y.

(A) -896

(B) -897

(C) -898

(D) -899

(E) -900

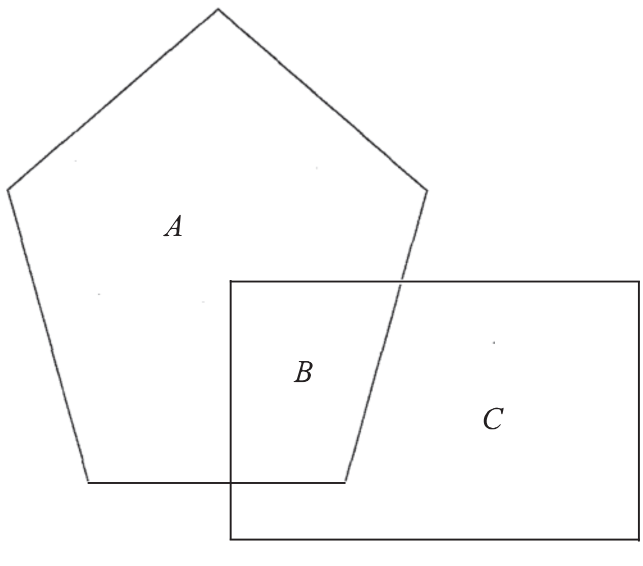
10 The number of positive integral solutions (a, b, c, d) satisfying $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=1$ with the condition that $a < b < c < d$ is

(A) 6
(B) 7
(C) 8
(D) 9
(E) 10

Short Questions

11 There are two models of LCD television on sale. One is a '20 inch' standard model while the other is a '20 inch' widescreen model. The ratio of the length to the height of the standard model is 4 : 3, while that of the widescreen model is 16 : 9. Television screens are measured by the length of their diagonals, so both models have the same diagonal length of 20 inches. If the ratio of the area of the standard model to that of the widescreen model is A : 300, find the value of A.

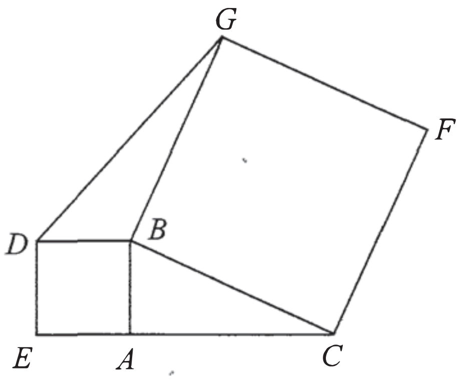
12 The diagram below shows a pentagon (made up of region A and region B) and a rectangle (made up of region B and region C) that overlaps. The overlapped region B is $\frac{3}{16}$ of the pentagon and $\frac{2}{9}$ of the rectangle. If the ratio of region A of the pentagon to region C of the rectangle is $\frac{m}{n}$ in its lowest term, find the value of $m+n$.



13 2009 students are taking a test which comprises ten true or false questions. Find the minimum number of answer scripts required to guarantee two scripts with at least nine identical answers.

14 The number of ways to arrange 5 boys and 6 girls in a row such that girls can be adjacent to other girls but boys cannot be adjacent to other boys is $6! \times k$. Find the value of k.

15 ABC is a right-angled triangle with $\angle BAC = 90°$. A square is constructed on the side AB and BC as shown. The area of the square $ABDE$ is 8 cm^2 and the area of the square $BCFG$ is 26 cm^2. Find the area of triangle DBG in cm^2.



16 The sum of $\frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + \frac{1}{4 \times 5 \times 6} + \dots + \frac{1}{13 \times 14 \times 15} + \frac{1}{14 \times 15 \times 16}$ is $\frac{m}{n}$ in its lowest terms. Find the value of $m + n$.

17 Given that $a + \frac{1}{a+1} = b + \frac{1}{b-1} - 2$ and $a - b + 2 \neq 0$, find the value of $ab - a + b$.

18 If $|x| + x + 5y = 2$ and $|y| - y + x = 7$, find the value of $x + y + 2009$.

19 Let p and q represent two consecutive prime numbers. For some fixed integer n, the set $\{n - 1, 3n - 19, 38 - 5n, 7n - 45\}$ represents $\{p, 2p, q, 2q\}$, but not necessarily in that order. Find the value of n.

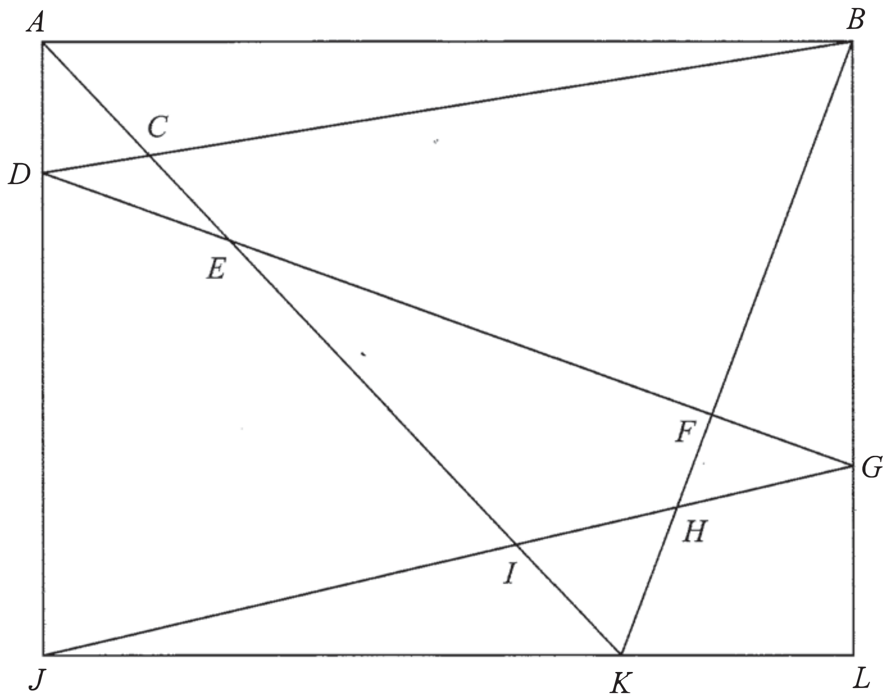
20 Find the number of ordered pairs of positive integers (x, y) that satisfy the equation

$$x\sqrt{y} + y\sqrt{x} + \sqrt{2009xy} - \sqrt{2009x} - \sqrt{2009y} - 2009 = 0.$$

21 Find the integer part of

$$\frac{1}{\frac{1}{2003}+\frac{1}{2004}+\frac{1}{2005}+\frac{1}{2006}+\frac{1}{2007}+\frac{1}{2008}+\frac{1}{2009}}.$$

22 The diagram below shows a rectangle *ABLJ*, where the area of *ACD*, *BCEF*, *DEIJ* and *FGH* are 22 cm^2, 500 cm^2, 482 cm^2 and 22 cm^2 respectively. Find the area of *HIK* in cm^2.



23 Evaluate $\sqrt[3]{77-20\sqrt{13}} + \sqrt[3]{77+20\sqrt{13}}$.

24 Find the number of integers in the set $\{1, 2, 3, \ldots, 2009\}$ whose sum of the digits is 11.

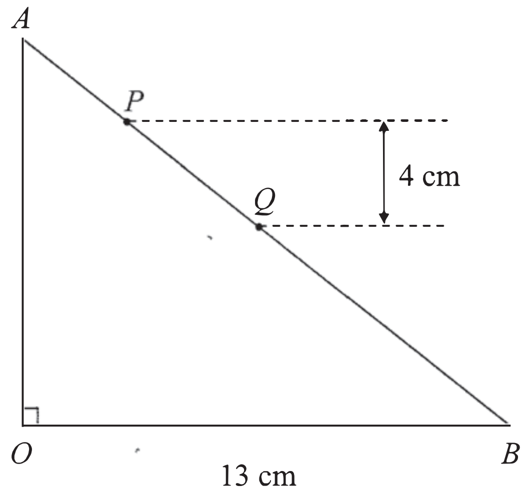
25 Given that

$$x + (1+x)^2 + (1+x)^3 + \ldots + (1+x)^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n,$$

where each a_r is an integer, $r = 0, 1, 2, \ldots, n$.

Find the value of n such that $a_0 + a_2 + a_3 + a_4 + \ldots + a_{n-2} + a_{n-1} = 60 - \dfrac{n(n+1)}{2}$.

26 In the diagram, OAB is a triangle with $\angle AOB = 90°$ and $OB = 13$ cm. P & Q are 2 points on AB such that $26AP = 22PQ = 11QB$. If the vertical height of $PQ = 4$ cm, find the area of the triangle OPQ in cm^2.



27 Let x_1, x_2, x_3, x_4 denote the four roots of the equation

$$x^4 + kx^2 + 90x - 2009 = 0.$$

If $x_1 x_2 = 49$, find the value of k.

28 Three sides OAB, OAC and OBC of a tetrahedron $OABC$ are right-angled triangles, i.e. $\angle AOB = \angle AOC = \angle BOC = 90°$. Given that $OA = 7$, $OB = 2$ and $OC = 6$, find the value of

$$(\text{Area of } \Delta OAB)^2 + (\text{Area of } \Delta OAC)^2 + (\text{Area of } \Delta OBC)^2 + (\text{Area of } \Delta ABC)^2.$$

29 Find the least positive integer n for which $\dfrac{n-10}{9n+11}$ is a non-zero reducible fraction.

30 Find the value of the smallest positive integer m such that the equation

$$x^2 + 2(m+5)x + (100m+9) = 0$$

has only integer solutions.

31 In a triangle ABC, the length of the altitudes AD and BE are 4 and 12 respectively. Find the largest possible integer value for the length of the third altitude CF.

32 A four digit number consists of two distinct pairs of repeated digits (for example 2211, 2626 and 7007). Find the total number of such possible numbers that are divisible by 7 or 101 but not both.

33 m and n are two positive integers satisfying $1 \le m \le n \le 40$. Find the number of pairs of (m, n) such that their product mn is divisible by 33.

34 Using the digits 0, 1, 2, 3 and 4, find the number of 13-digit sequences that can be written so that the difference between any two consecutive digits is 1.

Examples of such 13-digit sequences are 0123432123432, 2323432321234 and 3210101234323.

35 m and n are two positive integers of reverse order (for example 123 and 321) such that $mn = 1446921630$. Find the value of $m + n$.

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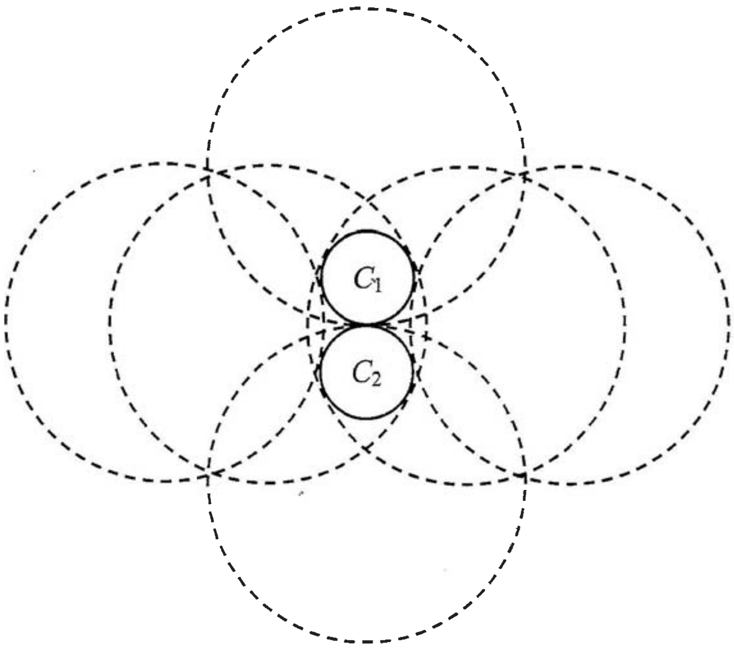
(Junior Section Solutions)

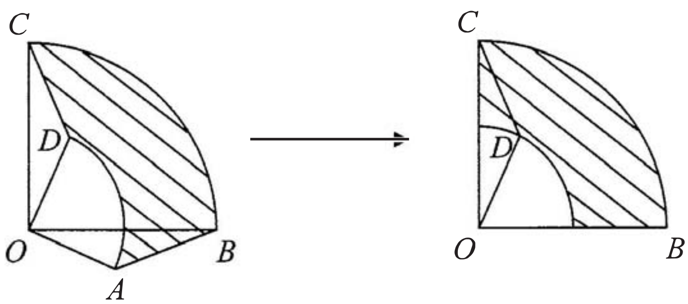
1 Answer: (C)

As seen from the diagram below, there are 6 circles that are tangent to both C_1 and C_2.



2 Answer: (A)

Rotate ΔOAB 90° anticlockwise about the point O to overlap with ΔODC.



$\therefore$ Shaded area $= \frac{1}{4}(64\pi - 16\pi) = 12\pi$.

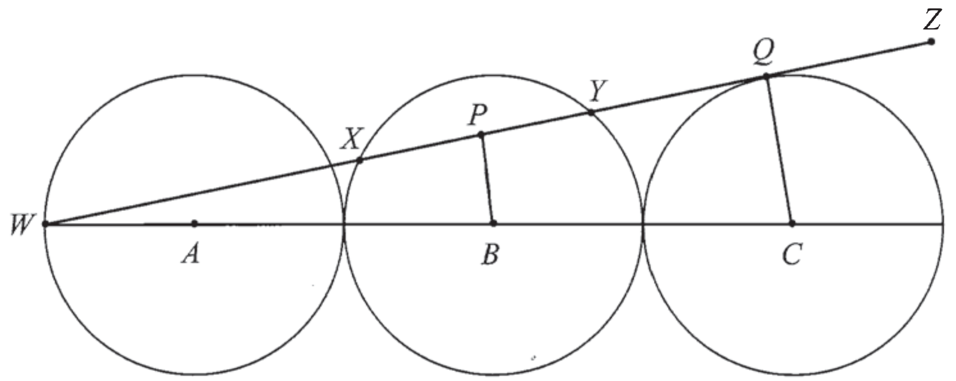
3 Answer: (D)

$$\left(\sqrt{x-2}+\sqrt{7-x}\right)^2 = 5+2\sqrt{(x-2)(7-x)} = 5+2\sqrt{6.25-(x-4.5)^2}.$$

Hence maximum value of $k=\sqrt{5+2(2.5)}=\sqrt{10}$.

4 Answer: (B)

Let P be the midpoint of XY and Q be the point where the line WZ meets the third circle.



Then by similar Δs, $\dfrac{PB}{60}=\dfrac{20}{100}\Rightarrow PB=12$. Hence $XY=2XP=2\sqrt{20^2-12^2}=32$.

5 Answer: (D)

$y=\dfrac{10x}{10-x}\Rightarrow x=10-\dfrac{100}{y+10}<0$, so $\dfrac{100}{y+10}>10\Rightarrow -10<y<0$.

Since both x and y are integers, $y+10$ is a factor of 100, i.e. $y=-9, -8, -6, -5$.

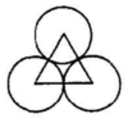
The maximum value of y is -5.

6 Answer: (C)

Let $b_n=a_n-a_{n-1}=n^2$. $\sum\limits_{n=1}^{50} b_n=\sum\limits_{n=1}^{50} n^2=\dfrac{20}{100}(50)(51)(101)=42925$.

On the other hand, $\sum\limits_{n=1}^{50} b_n=a_{50}-a_0\Rightarrow a_{50}=42925+a_0=44934$.

7 Answer: (B)



Consider the equilateral triangle formed by joining the centres of 3 adjacent coins. It is easy to see that the required percentage is given by the percentage of the triangle covered by these three coins.

We have $\dfrac{\frac{1}{2}\pi r^2}{\sqrt{3}\,r^2} \times 100\% = \dfrac{50}{\sqrt{3}}\,\pi\,\%$.

8 Answer: (E)

$x^2 + y^2 = 6 \Rightarrow (x + y)^2 = 2xy + 6$. So $2x + \{\,(x + y)^2 - 6\,\} + 2y = 2(2 + 3\sqrt{2}\,)$
$\Rightarrow\ (x + y)^2 + 2(x + y) + 1 = 11 + 6\sqrt{2} \Rightarrow (x + y + 1)^2 = (3 + \sqrt{2}\,)^2$.

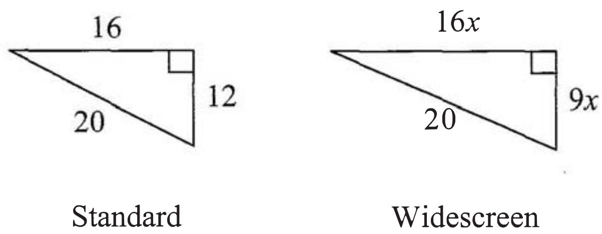
9 Answer: (E)

$y = (x^2 - 16^2)\,(x^2 - 14^2) = x^4 - 452x^2 + 50176 = (x^2 - 226)^2 - 900$.
So minimum value of $y = -\,900$.

10 Answer: (A)

$(a, b, c, d) = (2,3,7,42), (2,3,8,24), (2,3,9,18), (2,3,10,15), (2,4,5,20), (2,4,6,12)$.

11 Answer: (337)



Standard Widescreen

$(16x)^2 + (9x)^2 = 20^2 \Rightarrow 337x^2 = 400.\ \therefore\ \dfrac{\text{Area of Standard}}{\text{Area of Widescreen}} = \dfrac{(16)(12)}{(16x)(9x)} = \dfrac{337}{300}$.

12 Answer: (47)

Let the area of the pentagon and the rectangle be P and R respectively.
We have $\frac{3}{16}P = \frac{2}{9}R$. So $\frac{m}{n} = \frac{13}{16}P \div \frac{7}{9}R = \frac{26}{21} \Rightarrow m + n = 47$.

13 Answer: (513)

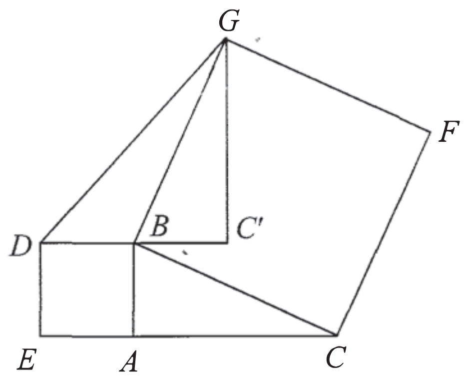
Minimum number of answer scripts is $2^9 + 1 = 513$.

14 Answer: (2520)

First we arrange the 6 girls in 6! ways. Next, there are 7 spaces between the 6 girls to insert the 5 boys. Hence $k = {}^7C_5 \times 5! = 2520$.

15 Answer: (6)

We rotate ΔBAC 90° anticlockwise about the point B to get $\Delta BC'G$.
$C'G = BC = \sqrt{26-8} = \sqrt{18}$.



$\therefore$ The area of $\Delta DBG = \frac{1}{2} \times DB \times C'G = \frac{1}{2}\sqrt{8}\sqrt{18} = 6 \text{ cm}^2$.

16 Answer: (173)

Note that $\frac{1}{n \times (n+1) \times (n+2)} = \frac{1}{2}\left(\frac{1}{n \times (n+1)} - \frac{1}{(n+1) \times (n+2)}\right)$.

Hence $\frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + \frac{1}{4 \times 5 \times 6} + \dots + \frac{1}{13 \times 14 \times 15} + \frac{1}{14 \times 15 \times 16}$

$= \frac{1}{2}\left(\frac{1}{2 \times 3} - \frac{1}{15 \times 16}\right) = \frac{13}{160}$.

17 Answer: (2)

Let $x = a + 1$, $y = b - 1$ ($x - y \neq 0$), then $x - 1 + \frac{1}{x} = y + 1 + \frac{1}{y} - 2 \Rightarrow x + \frac{1}{x} = y + \frac{1}{y}$

$\Rightarrow (x - y)\left(1 - \frac{1}{xy}\right) = 0 \Rightarrow xy = 1 \Rightarrow ab - a + b = 2.$

18 Answer: (2012)

Suppose $y \geq 0$. From $|y| - y + x = 7$ we have $x = 7$ and from $|x| + x + 5y = 2$ we have $y = -\frac{12}{5} < 0$ ($\rightarrow\leftarrow$). So $y < 0$.

Suppose $x \leq 0$. From $|x| + x + 5y = 2$ we have $y = \frac{2}{5} > 0$ ($\rightarrow\leftarrow$) . So $x > 0$.

Hence the two equations become $-2y + x = 7$ and $2x + 5y = 2 \Rightarrow x = \frac{13}{3}$, $y = -\frac{4}{3}$.

19 Answer: (7)

Since $3p + 3q = 6n - 27$, $p + q = 2n - 9$ which is odd. So $p = 2$, $q = 3$ and $n = 7$.

20 Answer: (6)

Note that $x\sqrt{y} + y\sqrt{x} + \sqrt{2009xy} - \sqrt{2009x} - \sqrt{2009y} - 2009$
$= (\sqrt{x} + \sqrt{y} - \sqrt{2009})(\sqrt{xy} - \sqrt{2009})$, so $xy = 2009$.
$\therefore$ $(x, y) = (1, 2009), (7, 287), (41, 49), (49, 41), (287, 7), (2009, 1)$.

21 Answer: (286)

Let $L = \frac{1}{2003} + \frac{1}{2004} + \frac{1}{2005} + \frac{1}{2006} + \frac{1}{2007} + \frac{1}{2008} + \frac{1}{2009}$.

Clearly $\frac{7}{2009} < L < \frac{7}{2003} \Rightarrow 286\frac{1}{7} < \frac{1}{L} < 287.$

22 Answer: (26)

Looking at half of the area of rectangle $ABLJ$, we have $(ABK) = (ABD) + (DGJ)$
$\Rightarrow (ABC) + (BCEF) + (EFHI) + (HIK) = (ABC) + (ACD) + (DEIJ) + (EFHI) + (FGH)$
$\Rightarrow (BCEF) + (HIK) = (ACD) + (DEIJ) + (FGH)$
$\Rightarrow 500 + (HIK) = 22 + 482 + 22 \Rightarrow (HIK) = 26.$

23 Answer: (7)

Let $X = \sqrt[3]{77 - 20\sqrt{13}}$, $Y = \sqrt[3]{77 + 20\sqrt{13}}$ and $A = X + Y$.
Since $X^3 + Y^3 = 154, XY = \sqrt[3]{77^2 - 20^2 \times 13} = 9, A^3 = (X + Y)^3 = X^3 + Y^3 + 3XY(X + Y)$
$\Rightarrow A^3 = 154 + 27A \Rightarrow A^3 - 27A - 154 = 0 \Rightarrow (A - 7)(A^2 + 7A + 22) = 0 \Rightarrow A = 7.$

24 Answer: (133)

We consider 3 sets of integers:

(a) 0001 ~ 0999. We count the number of nonnegative solutions of $a + b + c = 11$. It is ${}^{11+3-1}C_{3-1} = {}^{13}C_2 = 78$. Note that 11 should not be split as 0+0+11 nor 0+1+10. So the number of solutions is $78 - 3 - 6 = 69$.

(b) 1001 ~ 1999. We count the number of nonnegative solutions of $a + b + c = 10$. It is ${}^{10+3-1}C_{3-1} = {}^{12}C_2 = 66$. Note that 10 should not be split as 0+0+10. So the number of solutions is $66 - 3 = 63$.

(c) 2001 ~ 2009. We see that only 2009 satisfies the property.

$\therefore$ The total number of integers satisfying the property is $69 + 63 + 1 = 133$.

25 Answer: (5)

Let $x = 1$, we have $a_0 + a_1 + a_2 + \ldots + a_n = 1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 3$.
Also, $a_1 = 1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$, $a_n = 1$,
so $60 - \dfrac{n(n+1)}{2} + \dfrac{n(n+1)}{2} + 1 = 2^{n+1} - 3 \Rightarrow n = 5$.

26 Answer: (26)

Note that area of $\Delta OPQ = \dfrac{1}{2} \times 4\text{ cm} \times 13\text{ cm} = 26\text{ cm}^2$.

27 Answer: (7)

From given equation, $x_1 + x_2 + x_3 + x_4 = 0$, $= x_1x_2(x_3 + x_4) + x_3x_4(x_1 + x_2) = -90$ and $x_1x_2x_3x_4 = -2009$. Now $x_1x_2 = 49 \Rightarrow x_3x_4 = -41$. So $x_1 + x_2 = 1, x_3 + x_4 = -1$. Hence $k = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = 49 + (1) \times (-1) - 41 = 7$.

28 Answer: (1052)

Note that (Area of $\Delta OAB)^2$ = (Area of $\Delta OAC)^2$ + (Area of $\Delta OBC)^2$ + (Area of $\Delta ABC)^2$.
So (Area of $\Delta OAB)^2$ + (Area of $\Delta OAC)^2$ + (Area of $\Delta OBC)^2$ + (Area of $\Delta ABC)^2$
$= 2 \times$ {(Area of $\Delta OAC)^2$ + (Area of $\Delta OBC)^2$ + (Area of $\Delta ABC)^2$}
$= 2 \times \{ (1/2 \times 7 \times 6)^2 + (1/2 \times 7 \times 2)^2 + (1/2 \times 2 \times 6)^2 \} = 1052.$

29 Answer: (111)

Consider $\frac{9n+11}{n-10} = 9 + \frac{101}{n-10}$. If $\frac{n-10}{9n+11}$ is a non-zero reducible fraction, then $\frac{101}{n-10}$ is also a non-zero reducible fraction $\Rightarrow$ Least positive integer $n = 111$.

30 Answer: (90)

$x^2 + 2(m+5)x + (100m+9) = 0 \Rightarrow x = -(m+5) \pm \sqrt{(m-45)^2 - 2009}$. This yields integer solutions if and only if $(m-45)^2 - 2009$ is a perfect square, say n^2.

Hence $(m-45)^2 - n^2 = 2009 = 7^2 \times 41 \Rightarrow$

$$|m-45| + n = 2009 \text{ and } |m-45| - n = 1, \text{ or}$$
$$|m-45| + n = 287 \text{ and } |m-45| - n = 7, \text{ or}$$
$$|m-45| + n = 49 \text{ and } |m-45| - n = 41.$$

Solving, $n = 45 \pm 1005,\ 45 \pm 147,\ 45 \pm 45 \Rightarrow$ The smallest positive n is 90.

31 Answer: (5)

Area of the triangle is $\frac{1}{2} \times AD \times BC = \frac{1}{2} \times AC \times BE = \frac{1}{2} \times AB \times CF$.

Since $AD = 4$ and $BE = 12$, $BC : AC = 3 : 1$. Let $AC = x$, then $BC = 3x$.

Using Triangle Inequality, $AB < AC + BC$ and $BC < AB + AC \Rightarrow 2x < AB < 4x$.

From $CF = \frac{12x}{AB}$, $3 < CF < 6 \Rightarrow$ The largest integer value for CF is 5.

32 Answer: (97)

We let the 2 distinct digits be A and B with $1 \le A \le 9$, $0 \le B \le 9$ and $A \ne B$.

Case 1: $ABAB = 101 \times AB$. There are $9 \times 9 = 81$ possibilities.
However AB must not be a multiple of 7 (12 possibilities excluding 07 and 77) $\Rightarrow$
There are $81 - 12 = 69$.

Case 2: $AABB = 11 \times (100A + B)$. There are 11 possibilities that are factors of 7.
$(A,B) = (1,5)$, (2,3), (3,1), (3,8), (4,6), (5,4), (6,2), (6,9), (7,0), (8,5), (9,3).

Case 3: $ABBA = 11 \times (91A + 10B)$. B must be 0 or 7. There are 17 possibilities.

$\therefore$ Total such possible number $= 69 + 11 + 17 = 97$.

33 Answer: (64)

If $n = 33$, $m = 1, 2, 3, \ldots, 32$, so there are 32 pairs.
If $m = 33$, $n = 33, 34, \ldots, 40$, so there are 8 pairs.

If $n, m \ne 33$, we have 2 cases:

Case 1: $m = 3a$, $n = 11b$ ($a \ne 11$ and $b \ne 3$), thus $1 \le 3a \le 11b \le 40$.
So if $b = 1$, $a = 1, 2, 3$ and if $b = 2$, $a = 1, 2, 3, \ldots, 7$. There are 10 pairs.

Case 2: $m = 11a$, $n = 3b$ ($a \ne 3$ and $b \ne 11$), thus $1 \le 11a \le 3b \le 40$.
So if $a = 1$, $b = 4, 5, 6, 7, 8, 9, 10, 12, 13$ and if $a = 2$, $b = 8, 9, 10, 12, 13$.
There are $9 + 5 = 14$ pairs.

Hence we have altogether $32 + 8 + 10 + 14 = 64$ pairs.

34 Answer: (3402)

We call a sequence that satisfy the condition a "good" sequence.

Let A_n denote the number of "good" sequence that end in either 0 or 4,
B_n denote the number of "good" sequence that end in either 1 or 3,
C_n denote the number of "good" sequence that end in 2.

We have

(1) $A_{n+1} = B_n$ because each sequence in A_{n+1} can be converted to a sequence in B_n by deleting its last digit.

(2) $B_{n+1} = A_n + 2C_n$ because each sequence in A_n can be converted into a sequence in B_{n+1} by adding a 1 (if it ends with a 0) or a 3 (if it ends with a 4) to its end, and each sequence in C_n can be converted into a sequence in B_n by adding a 1 or a 3 to it.

(3) $C_{n+1} = B_n$ because each sequence in C_{n+1} can be converted to a sequence in B_n by deleting the 2 at its end.

Hence we can show that $B_{n+1} = 3B_{n-1}$ for $n \geq 2$.

Check that $B_1 = 2$ and $B_2 = 4$, so $B_{2n+1} = 2 \times 3^n$ and $B_{2n} = 4 \times 3^{n-1}$.

So $A_{13} + B_{13} + C_{13} = 2B_{12} + B_{13} = 2 \times 4 \times 3^5 + 2 \times 3^6 = 3402$.

35 Answer: (79497)

Clearly, m and n are both 5-digit numbers.

Next, it would be helpful that we know $mn = 2 \times 3^5 \times 5 \times 7 \times 11^2 \times 19 \times 37$.

Now since the last digit of mn is 0, we may assume $5 \mid m$ and $2 \mid n$. But the first digit of mn is 1 $\Rightarrow$ Last digit of m is 5 (not 0) and last digit of n is 2 (not 4, 6 or 8).

Also, $3^5 \mid mn$, so 9 divides at least one of m and n. On the other hand, $9 \mid m \Rightarrow 9 \mid n$. Similarly $11 \mid m \Rightarrow 11 \mid n$.

Set $n = 198k$. Then the last digit of k is 4 or 9.

Since the remaining factors 3, 7, 19, 37 are odd, the last digit of k must be 9.
We have only the following combinations: $k = 7 \times 37$ or $3 \times 7 \times 19$ or $3 \times 19 \times 37$.

Recall that the first digit of n is 5, so $50000 \leq 198k < 60000 \Rightarrow k = 7 \times 37$.

Hence $n = 198k = 51282$ and $m = 28215 \Rightarrow m + n = 79497$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Junior Section, Round 2)

1. In $\triangle ABC$, $\angle A = 2\angle B$. Let a, b, c be the lengths of its sides BC, CA, AB, respectively. Prove that

$$a^2 = b(b + c).$$

2. The set of 2000-digit integers are divided into two sets: the set M consisting all integers each of which can be represented as the product of two 1000-digit integers, and the set N which contains the other integers. Which of the sets M and N contains more elements?

3. Suppose $\overline{a_1a_2\dots a_{2009}}$ is a 2009-digit integer such that for each $i = 1, 2, \dots, 2007$, the 2-digit integer $\overline{a_ia_{i+1}}$ contains 3 distinct prime factors. Find a_{2008}. (Note: $\overline{xyz\dots}$ denotes an integer whose digits are $x, y, z, \dots$.)

4. Let S be the set of integers that can be written in the form $50m + 3n$ where m and n are non-negative integers. For example 3, 50, 53 are all in S. Find the sum of all positive integers not in S.

5. Let a, b be positive real numbers satisfying $a + b = 1$. Show that if $x_1, x_2, \dots, x_5$ are positive real numbers such that $x_1x_2\dots x_5 = 1$, then

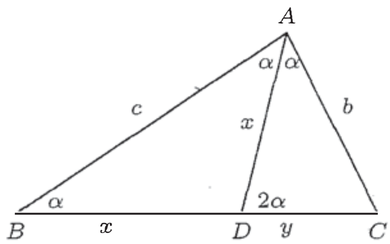
$$(ax_1 + b)(ax_2 + b)\cdots(ax_5 + b) \geq 1.$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Junior Section, Round 2 solutions)

1. Let AD be the angle bisector of $\angle A$. Then $\triangle ABC \simeq \triangle DAC$. Thus $AB/DA = AC/DC = BC/AC$. Let $BD = x$ and $DC = y$. Then $c/x = b/y = a/b$. Thus $cb = ax, b^2 = ay$. Thus $b^2 + cb = ax + ay$ and hence $b(b+c) = a^2$.



2. We solve the general case of $2n$-digit integers where $n \geq 2$. There are $10^{2n} - 10^{2n-1}$ $2n$-digit integers. There are $10^n - 10^{n-1}$ n-digit integers. Consider all the products of pairs of n-digit integers. The total number P of such products satisfies

$$\begin{aligned} P &\leq 10^n - 10^{n-1} + \frac{(10^n - 10^{n-1})(10^n - 10^{n-1} - 1)}{2} \\ &= \frac{10^{2n} - 10^{2n-1} - (10^{2n-1} - 10^{2n-2} - 10^n + 10^{n-1})}{2} \\ &< \frac{10^{2n} - 10^{2n-1}}{2}. \end{aligned}$$

These products include all the numbers in M. Thus $|M| < |N|$.

3. Two-digit numbers which contain three distinct prime factors are:

$30 = 2\cdot 3\cdot 5, 42 = 2\cdot 3\cdot 7, 60 = 4\cdot 3\cdot 5, 66 = 2\cdot 3\cdot 11, 70 = 2\cdot 5\cdot 7, 78 = 2\cdot 3\cdot 13, 84 = 4\cdot 3\cdot 7$

From here, we conclude that $a_i = 6$ for $i = 1, 2, \ldots, 2007$ and a_{2008} is either 6 or 0.

4. If x is the smallest integer in S such that $x \equiv i \pmod 3$, then $x + 3k \in S$ and $x - 3(k+1) \notin S$ for all $k \geq 0$. We have 3 is the smallest multiple of 3 that is in S; 50 is smallest number in S that is $\equiv 2 \pmod 3$ and 100 is the smallest number in S that is $\equiv 1 \pmod 3$. Thus the positive numbers not in S are $1, 4, \ldots, 97$ and $2, 5, \ldots, 47$. Their sum is

$$\frac{33(97+1)}{2} + \frac{16(2+47)}{2} = 2009.$$

5. The left hand side is

$$\begin{aligned}
a^5 x_1 x_2 \ldots x_5 &+ a^4 b(x_1x_2x_3x_4 + x_1x_2x_3x_5 + \cdots + x_2x_3x_4x_5) \\
&+ a^3b^2(x_1x_2x_3 + x_1x_2x_4 + \cdots + x_3x_4x_5) \\
&+ a^2b^3(x_1x_2 + x_1x_3 + \cdots + x_4x_5) + ab^4(x_1 + x_2 + \cdots + x_5) + b^5 \\
&\geq a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 = (a+b)^5 = 1.
\end{aligned}$$

The last is true since by AM-GM inequality,

$$\begin{aligned}
&x_1x_2x_3x_4 + x_1x_2x_3x_5 + \cdots + x_2x_3x_4x_5 \geq 5(x_1x_2x_3x_4x_5)^{4/5} = 5 \\
&x_1x_2x_3 + x_1x_2x_4 + \cdots + x_3x_4x_5 \geq 10(x_1x_2x_3x_4x_5)^{6/10} = 10 \\
&x_1x_2 + x_1x_3 + \cdots + x_4x_5 \geq 10(x_1x_2x_3x_4x_5)^{4/10} = 10 \\
&x_1 + x_2 + \cdots + x_5 \geq 5(x_1x_2x_3x_4x_5)^{1/5} = 5
\end{aligned}$$

(Note. For a proof of the general case, see Senior Q4)

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Senior Section)

Tuesday, 2 June 2009 0930 – 1200 hrs

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D *or* E) *corresponding to the correct answer.*

For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Multiple Choice Questions

1. Suppose that π is a plane and A and B are two points on the plane π. If the distance between A and B is 33 cm, how many lines are there in the plane such that the distance between each line and A is 7 cm and the distance between each line and B is 26 cm respectively?

   (A) 1
   (B) 2
   (C) 3
   (D) 4
   (E) Infinitely many

2. Let $y = (17 - x)(19 - x)(19 + x)(17 + x)$, where x is a real number. Find the smallest possible value of y.

   (A) −1296
   (B) −1295
   (C) −1294
   (D) −1293
   (E) −1292

3. If two real numbers a and b are randomly chosen from the interval (0, 1), find the probability that the equation $x^2 - \sqrt{a}\,x + b = 0$ has real roots.

   (A) $\frac{1}{8}$
   (B) $\frac{5}{16}$
   (C) $\frac{3}{16}$
   (D) $\frac{1}{4}$
   (E) $\frac{1}{3}$

4. If x and y are real numbers for which $|x| + x + 5y = 2$ and $|y| - y + x = 7$, find the value of $x + y$.

   (A) −3
   (B) −1
   (C) 1
   (D) 3
   (E) 5

5. In a triangle ABC, $\sin A = \frac{3}{5}$ and $\cos B = \frac{5}{13}$. Find the value of $\cos C$.

(A) $\frac{56}{65}$ or $\frac{16}{65}$

(B) $\frac{56}{65}$

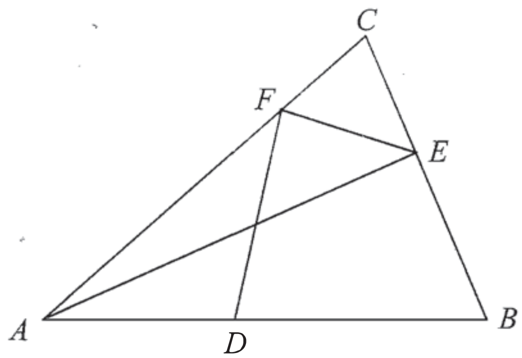
(C) $\frac{16}{65}$

(D) $-\frac{56}{65}$

(E) $\frac{56}{65}$ or $-\frac{16}{65}$

6. The area of a triangle ABC is 40 cm^2. Points D, E and F are on sides AB, BC and CA respectively, as shown in the figure below. If $AD = 3$ cm, $DB = 5$ cm, and the area of triangle ABE is equal to the area of quadrilateral $DBEF$, find the area of triangle AEC in cm^2.

(A) 11
(B) 12
(C) 13
(D) 14
(E) 15



7. Find the value of $\frac{3}{1!+2!+3!}+\frac{4}{2!+3!+4!}+\cdots+\frac{22}{20!+21!+22!}$.

(A) $1-\frac{1}{24!}$

(B) $\frac{1}{2}-\frac{1}{23!}$

(C) $\frac{1}{2}-\frac{1}{22!}$

(D) $1-\frac{1}{22!}$

(E) $\frac{1}{2}-\frac{1}{24!}$

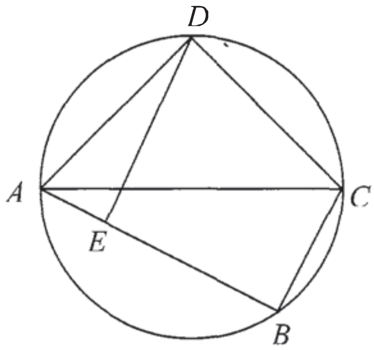
8. There are eight envelopes numbered 1 to 8. Find the number of ways in which 4 identical red buttons and 4 identical blue buttons can be put in the envelopes such that each envelope contains exactly one button, and the sum of the numbers on the envelopes containing the red buttons is more than the sum of the numbers on the envelopes containing the blue buttons.

(A) 35
(B) 34
(C) 32
(D) 31
(E) 62

9. Determine the number of acute-angled triangles (i.e., all angles are less than 90°) in which all angles (in degrees) are positive integers and the largest angle is three times the smallest angle.

(A) 3
(B) 4
(C) 5
(D) 6
(E) 7

10. Let $ABCD$ be a quadrilateral inscribed in a circle with diameter AC, and let E be the foot of perpendicular from D onto AB, as shown in the figure below. If $AD = DC$ and the area of quadrilateral $ABCD$ is 24 cm^2, find the length of DE in cm.



(A) $3\sqrt{2}$
(B) $2\sqrt{6}$
(C) $2\sqrt{7}$
(D) $4\sqrt{2}$
(E) 6

Short Questions

11. Find the number of positive divisors of $(2008^3 + (3 \times 2008 \times 2009) + 1)^2$.

12. Suppose that a, b and c are real numbers greater than 1. Find the value of

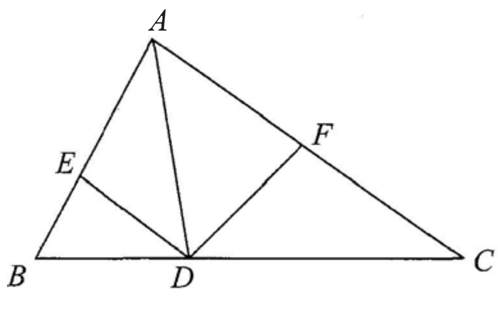
$$\frac{1}{1+\log_{a^2b}\left(\frac{c}{a}\right)}+\frac{1}{1+\log_{b^2c}\left(\frac{a}{b}\right)}+\frac{1}{1+\log_{c^2a}\left(\frac{b}{c}\right)}.$$

13. Find the remainder when $(1! \times 1) + (2! \times 2) + (3! \times 3) + \cdots + (286! \times 286)$ is divided by 2009.

14. Find the value of $\left(25+10\sqrt{5}\right)^{1/3}+\left(25-10\sqrt{5}\right)^{1/3}$.

15. Let $a=\dfrac{1+\sqrt{2009}}{2}$. Find the value of $(a^3-503a-500)^{10}$.

16. In the figure below, ABC is a triangle and D is a point on side BC. Point E is on side AB such that DE is the angle bisector of $\angle ADB$, and point F is on side AC such that DF is the angle bisector of $\angle ADC$. Find the value of $\dfrac{AE}{EB}\cdot\dfrac{BD}{DC}\cdot\dfrac{CF}{FA}$.



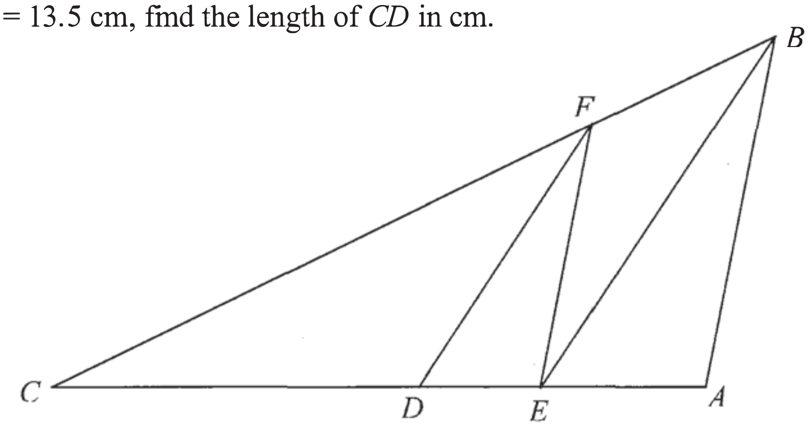
17. Find the value of

$$(\cot 25^\circ-1)(\cot 24^\circ-1)(\cot 23^\circ-1)(\cot 22^\circ-1)(\cot 21^\circ-1)(\cot 20^\circ-1).$$

18. Find the number of 2-element subsets $\{a, b\}$ of $\{1, 2, 3, \ldots, 99, 100\}$ such that $ab + a + b$ is a multiple of 7.

19. Let x be a real number such that $x^2 - 15x + 1 = 0$. Find the value of $x^4 + \frac{1}{x^4}$.

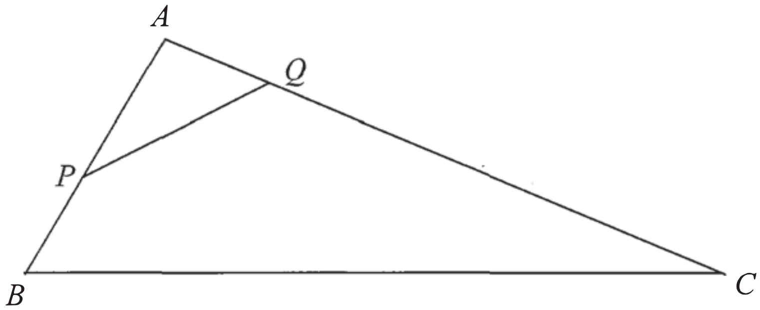
20. In the figure below, ABC is a triangle with AB = 10 cm and BC = 40 cm. Points D and E lie on side AC and point F on side BC such that EF is parallel to AB and DF is parallel to EB. Given that BE is an angle bisector of $\angle ABC$ and that AD = 13.5 cm, find the length of CD in cm.



21. Let $S = \{1, 2, 3, \dots, 64, 65\}$. Determine the number of ordered triples (x, y, z) such that $x, y, z \in S$, $x < z$ and $y < z$.

22. Given that $a_{n+1} = \frac{a_{n-1}}{1 + na_{n-1}a_n}$, where $n = 1, 2, 3, \dots$, and $a_0 = a_1 = 1$, find the value of $\frac{1}{a_{199}\, a_{200}}$.

23. In the figure below, ABC is a triangle with AB = 5 cm, BC = 13 cm and AC = 10 cm. Points P and Q lie on sides AB and AC respectively such that $\frac{\text{area of } \Delta APQ}{\text{area of } \Delta ABC} = \frac{1}{4}$. Given that the least possible length of PQ is k cm, find the value of k.



24. If x, y and z are real numbers such that $x + y + z = 9$ and $xy + yz + zx = 24$, find the largest possible value of z.

25. Find the number of 0 – 1 binary sequences formed by six 0's and six 1's such that no three 0's are together. For example, 110010100101 is such a sequence but 101011000101 and 110101100001 are not.

26. If $\dfrac{\cos 100^\circ}{1-4\sin 25^\circ \cos 25^\circ \cos 50^\circ} = \tan x^\circ$, find x.

27. Find the number of positive integers x, where $x \neq 9$, such that

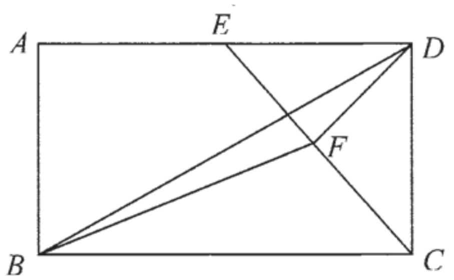
$$\log_{\frac{x}{9}}\left(\frac{x^2}{3}\right) < 6 + \log_3\left(\frac{9}{x}\right).$$

28. Let n be the positive integer such that

$$\frac{1}{9\sqrt{11}+11\sqrt{9}}+\frac{1}{11\sqrt{13}+13\sqrt{11}}+\frac{1}{13\sqrt{15}+15\sqrt{13}}+\cdots+\frac{1}{n\sqrt{n+2}+(n+2)\sqrt{n}}=\frac{1}{9}.$$

Find the value of n.

29. In the figure below, $ABCD$ is a rectangle, E is the midpoint of AD and F is the midpoint of CE. If the area of triangle BDF is 12 cm^2, find the area of rectangle $ABCD$ in cm^2.



30. In each of the following 6-digit positive integers: 555555, 555333, 818811, 300388, every digit in the number appears at least twice. Find the number of such 6-digit positive integers.

31. Let x and y be positive integers such that $27x + 35y \leq 945$. Find the largest possible value of xy.

32. Determine the coefficient of x^{29} in the expansion of $(1+x^5+x^7+x^9)^{16}$.

33. For $n = 1, 2, 3, \ldots$, let $a_n = n^2 + 100$, and let d_n denote the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges over all positive integers.

34. Using the digits 1, 2, 3, 4, 5, 6, 7, 8, we can form 8! (= 40320) 8-digit numbers in which the eight digits are all distinct. For $1 \le k \le 40320$, let a_k denote the kth number if these numbers are arranged in increasing order:

$$12345678,\ 12345687,\ 12345768,\ \ldots,\ 87654321;$$

that is, $a_1 = 12345678$, $a_2 = 12345687$, $\ldots$, $a_{40320} = 87654321$. Find $a_{2009} - a_{2008}$.

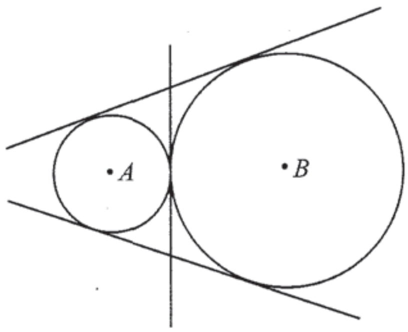
35. Let x be a positive integer, and write $a = \lfloor \log_{10} x \rfloor$ and $b = \left\lfloor \log_{10} \frac{100}{x} \right\rfloor$. Here $\lfloor c \rfloor$ denotes the greatest integer less than or equal to c. Find the largest possible value of $2a^2 - 3b^2$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Senior Section Solutions)

1. Answer: (C)
In the plane π, draw a circle of radius 7 cm centred at A and a circle of radius 26 cm centred at B.



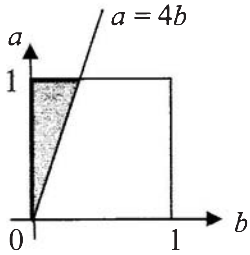
If ℓ is a line on the plane π, and the distance between ℓ and A is 7 cm and the distance between ℓ and B is 26 cm, then ℓ must be tangential to both circles. Clearly, there are 3 lines in the plane that are tangential to both circles, as shown in the figure above.

2. Answer: (A)
We have

$$\begin{aligned} y &= (17^2 - x^2)(19^2 - x^2) \\ &= x^4 - (17^2 + 19^2)x^2 + 17^2 \cdot 19^2 \\ &= x^4 - 650x^2 + 323^2 \\ &= (x^2 - 325)^2 + 323^2 - 325^2 \end{aligned}$$

Hence the smallest possible value of y is $323^2 - 325^2 = (-2)(648) = -1296$.

3. Answer: (A)
The discriminant of the equation is $a - 4b$.
Thus the equation has real roots if and only if $a \geq 4b$.
The shaded part in the figure on the right are all the points with coordinates (a, b) such that $0 < a, b < 1$ and $a \geq 4b$. As the area of the shaded part is $\frac{1}{8}$, it follows that the required probability is $\frac{1}{8}$.



4. Answer: (D)

If $x \le 0$, then $|x| = -x$, and we obtain from $|x| + x + 5y = 2$ that $y = \frac{2}{5}$. Thus y is positive, so $|y| - y + x = 7$ gives $x = 7$, which is a contradiction since $x \le 0$. Therefore we must have $x > 0$. Consequently, $|x| + x + 5y = 2$ gives the equation

$$2x + 5y = 2. \qquad (1)$$

If $y \ge 0$, then $|y| - y + x = 7$ gives $x = 7$. Substituting $x = 7$ into $|x| + x + 5y = 2$, we get $y = -\frac{12}{5}$, which contradicts $y \ge 0$. Hence we must have $y < 0$, and it follows from the equation $|y| - y + x = 7$ that

$$x - 2y = 7. \qquad (2)$$

Solving equations (1) and (2) gives $x = \frac{13}{3}$, $y = -\frac{4}{3}$. Therefore $x + y = 3$.

5. Answer: (C)

$\sin A = \frac{3}{5}$ implies that $\cos A = \frac{4}{5}$ or $-\frac{4}{5}$, and $\cos B = \frac{5}{13}$ implies that $\sin B = \frac{12}{13}$, since $0 < B < 180^\circ$.

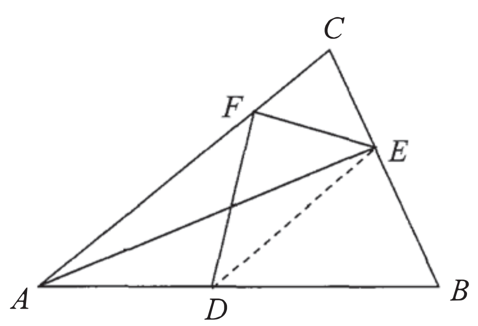
If $\cos A = -\frac{4}{5}$, then $\sin(A+B) = \sin A \cos B + \cos A \sin B = \frac{3}{5}\cdot\frac{5}{13} - \frac{4}{5}\cdot\frac{12}{13} < 0$, which is not possible since $0 < A + B < 180^\circ$ in a triangle. Thus we must have $\cos A = \frac{4}{5}$. Consequently, since $C = 180^\circ - (A + B)$, we have

$$\cos C = -\cos(A+B) = -\cos A\cos B + \sin A \sin B$$
$$= -\frac{4}{5}\cdot\frac{5}{13} + \frac{3}{5}\cdot\frac{12}{13} = \frac{16}{65}.$$

6. Answer: (E)

Since area of triangle ABE is equal to area of quadrilateral $DBEF$, we see that area of ΔDEA = area of ΔDEF. This implies that DE is parallel to AF.

Thus $\frac{CE}{CB} = \frac{AD}{AB} = \frac{3}{8}$. Since $\frac{\text{area of } \Delta AEC}{\text{area of } \Delta ABC} = \frac{CE}{CB}$, it follows that area of $\Delta AEC = \frac{3}{8} \times 40 = 15\ \text{cm}^2$.



7. Answer: (C)

First note that $(n-2)!+(n-1)!+n!=(n-2)![1+(n-1)+n(n-1)]=n^2(n-2)!$.
Therefore the given series can be written as

$$\sum_{n=3}^{22}\frac{n}{n^2(n-2)!}=\sum_{n=3}^{22}\frac{1}{n(n-2)!}=\sum_{n=3}^{22}\frac{n-1}{n(n-1)(n-2)!}$$

$$=\sum_{n=3}^{22}\frac{n-1}{n!}=\sum_{n=3}^{22}\left(\frac{1}{(n-1)!}-\frac{1}{n!}\right).$$

Summing the telescoping series, we obtain $\frac{1}{2!}-\frac{1}{22!}$.

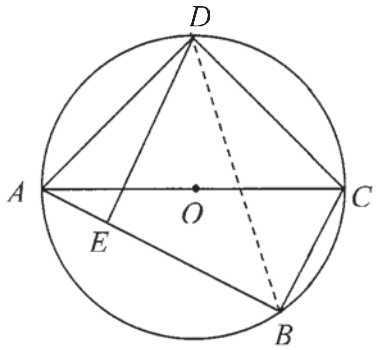
8. Answer: (D)

There are $\binom{8}{4}=70$ ways of putting 4 identical red buttons and 4 identical blue buttons in the envelopes. Since $1+2+3+\cdots+8=36$, there are 8 cases where the sum of the numbers on the envelopes containing the red buttons is equal to 18 (which is also equal to the sum of the numbers on the envelopes containing the blue buttons), namely, (8, 7, 2, 1), (8, 6, 3, 1), (8, 5, 4, 1), (8, 5, 3, 2), (7, 6, 4, 1), (7, 6, 3, 2), (7, 5, 4, 2) and (6, 5, 4, 3). Hence it follows that the required number of ways is $\frac{70-8}{2}=31$.

9. Answer: (B)

Let the angles of the acute-angled triangle be x°, y°, $3x^\circ$, where the smallest angle is x°. Then we have $x+y+3x=180$ and $0<x\le y\le 3x<90$. From the inequalities $x\le y\le 3x$, we obtain $5x\le x+y+3x\le 7x$, and hence it follows from the first equation that $\frac{180}{7}\le x\le 36$. Since x is an integer and $3x<90$, we deduce that $x=26, 27, 28, 29$. Hence there are 4 acute-angled triangles whose angles are respectively $(26^\circ, 76^\circ, 78^\circ)$, $(27^\circ, 72^\circ, 81^\circ)$, $(28^\circ, 68^\circ, 84^\circ)$ and $(29^\circ, 64^\circ, 87^\circ)$.

10. Answer: (B)



Let r be the radius of the circle with centre O.
Since $AD = DC$ and $\angle ADC = 90^\circ$, $\angle ACD = 45^\circ$. Thus $\angle ABD = 45^\circ$. As $\angle DEB = 90^\circ$, this implies that $DE = BE$. Let $x = DE = BE$. Since $BC /\!/ ED$ and

area of quadrilateral $ABCD$ = area of ΔAED + area of ΔEBD + area of ΔBCD,

we have

$$24 = \frac{1}{2}\cdot(AB - x)x + \frac{1}{2}x^2 + \frac{1}{2}\cdot BC\cdot x = \frac{1}{2}(AB + BC)x. \quad (1)$$

On the other hand, as OD is perpendicular to AC, and

area of quadrilateral $ABCD$ = area of ΔABC + area of ΔACD,

we have

$$24 = \frac{1}{2}\cdot AB\cdot BC + \frac{1}{2}AC\cdot OD = \frac{1}{2}\cdot AB\cdot BC + r^2. \quad (2)$$

Now equation (2) and $AB^2 + BC^2 = AC^2 = 4r^2$ imply that

$$(AB + BC)^2 = 4r^2 + 2\cdot AB\cdot BC = 4r^2 + 4(24 - r^2) = 96.$$

Therefore $AB + BC = 4\sqrt{6}$. Hence from equation (1), we obtain $x = 2\sqrt{6}$.

11. Answer: 91

Let $a = 2008$. Then

$$\begin{aligned}(2008^3 + (3 \times 2008 \times 2009) + 1)^2 &= (a^3 + 3a(a + 1) + 1)^2\\ &= (a^3 + 3a^2 + 3a + 1)^2\\ &= (a + 1)^6 = 2009^6 = 7^{12}\cdot 41^6.\end{aligned}$$

Hence the number of positive divisors is $(12 + 1)(6 + 1) = 91$.

12. Answer: 3

$$\frac{1}{1+\log_{a^2b}\left(\frac{c}{a}\right)}+\frac{1}{1+\log_{b^2c}\left(\frac{a}{b}\right)}+\frac{1}{1+\log_{c^2a}\left(\frac{b}{c}\right)}$$

$$=\frac{1}{\log_{a^2b}(a^2b)+\log_{a^2b}\left(\frac{c}{a}\right)}+\frac{1}{\log_{b^2c}(b^2c)+\log_{b^2c}\left(\frac{a}{b}\right)}+\frac{1}{\log_{c^2a}(c^2a)+\log_{c^2a}\left(\frac{b}{c}\right)}$$

$$=\frac{1}{\log_{a^2b}(abc)}+\frac{1}{\log_{b^2c}(abc)}+\frac{1}{\log_{c^2a}(abc)}$$

$$=\log_{abc}(a^2b)+\log_{abc}(b^2c)+\log_{abc}(c^2a)$$

$$=\log_{abc}(abc)^3=3.$$

13. Answer: 2008

Observe that $n! \times n = n! \times (n+1-1) = (n+1)! - n!$. Therefore

$$(1! \times 1) + (2! \times 2) + (3! \times 3) + \cdots + (286! \times 286)$$
$$= (2! - 1!) + (3! - 2!) + (4! - 3!) + \cdots + (287! - 286!)$$
$$= 287! - 1.$$

Since $2009 = 287 \times 7$, $287! - 1 \equiv -1 \equiv 2008 \pmod{2009}$. It follows that the remainder is 2008.

14. Answer: 5

Let $x=\left(25+10\sqrt{5}\right)^{1/3}+\left(25-10\sqrt{5}\right)^{1/3}$. Then

$$x^3=\left(25+10\sqrt{5}+25-10\sqrt{5}\right)+3\left(25^2-100(5)\right)^{1/3}\left[\left(25+10\sqrt{5}\right)^{1/3}+\left(25-10\sqrt{5}\right)^{1/3}\right],$$

which gives $x^3=50+15x$, or $(x-5)(x^2+5x+10)=0$. This equation admits only one real root $x = 5$.

15. Answer: 1024

$a=\dfrac{1+\sqrt{2009}}{2}$ gives $(2a-1)^2=2009$, which simplified to $a^2-a=502$. Now

$$\left(a^3-503a-500\right)^{10}=\left(a(a^2-a-502)+a^2-a-500\right)^{10}$$
$$=\left(a(a^2-a-502)+(a^2-a-502)+2\right)^{10}$$
$$=(0+0+2)^{10}=1024.$$

16. Answer: 1

Since DE is the angle bisector of $\angle ADB$, we have $\frac{AE}{EB} = \frac{AD}{BD}$. Similarly, since DF is the angle bisector of $\angle ADC$, $\frac{AF}{CF} = \frac{AD}{DC}$. Hence $\frac{AE}{EB} \cdot \frac{BD}{DC} \cdot \frac{CF}{FA} = 1$.

17. Answer: 8

First note that if $A + B = 45^\circ$, then $1 = \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$, and so $1 - \tan A - \tan B = \tan A \tan B$. Consequently,

$$(\cot A - 1)(\cot B - 1) = \frac{1 - \tan A - \tan B + \tan A \tan B}{\tan A \tan B} = \frac{2 \tan A \tan B}{\tan A \tan B} = 2.$$

Hence

$$\begin{aligned} &(\cot 25^\circ - 1)(\cot 24^\circ - 1)(\cot 23^\circ - 1)(\cot 22^\circ - 1)(\cot 21^\circ - 1)(\cot 20^\circ - 1) \\ &= (\cot 25^\circ - 1)(\cot 20^\circ - 1)(\cot 24^\circ - 1)(\cot 21^\circ - 1)(\cot 23^\circ - 1)(\cot 22^\circ - 1) \\ &= 8. \end{aligned}$$

18. Answer: 602

Note that $ab + a + b = (a + 1)(b + 1) - 1$. Thus $ab + a + b$ is a multiple of 7 if and only if $(a+1)(b+1) \equiv 1 \pmod 7$.

Let $A = \{1, 2, 3, \ldots, 99, 100\}$, and let $A_i = \{x \in A : x \equiv i \pmod 7\}$ for $i = 0, 1, 2, \ldots, 6$. It is easy to verify that for any $x \in A_i$ and $y \in A_j$, where $0 \le i \le j \le 6$, $xy \equiv 1 \pmod 7$ if and only if $i = j \in \{1, 6\}$, or $i = 2$ and $j = 4$, or $i = 3$ and $j = 5$.

Thus we consider three cases.

Case 1: $a+1, b+1 \in A_i$ for $i \in \{1, 6\}$.

Then $a, b \in A_i$ for $i \in \{0, 5\}$. As $|A_0| = |A_5| = 14$, the number of such subsets $\{a, b\}$ is $2\binom{14}{2} = 182$.

Case 2: $a + 1$ and $b + 1$ are contained in A_2 and A_4 respectively, but not in the same set.

Then a and b are contained in A_1 and A_3 respectively, but not in the same set.

Since $|A_1| = 15$ and $|A_3| = 14$, the number of such subsets $\{a, b\}$ is $14 \times 15 = 210$.

Case 3: $a + 1$ and $b + 1$ are contained in A_3 and A_5 respectively, but not in the same set.

Then a and b are contained in A_2 and A_4 respectively, but not in the same set.

Note that $|A_2| = 15$ and $|A_4| = 14$. Thus the number of such subsets $\{a, b\}$ is $14 \times 15 = 210$.

Hence the answer is $182 + 2 \times 210 = 602$.

19. Answer: 49727

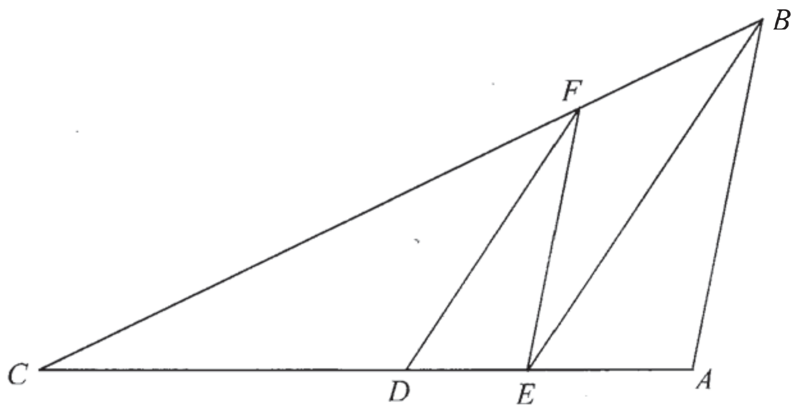
Since $x^2 - 15x + 1 = 0$, $x + \frac{1}{x} = 15$. Therefore

$$x^4 + \frac{1}{x^4} = \left(x + \frac{1}{x}\right)^4 - 4\left(x^2 + \frac{1}{x^2}\right) - 6$$

$$= \left(x + \frac{1}{x}\right)^4 - 4\left(x + \frac{1}{x}\right)^2 + 8 - 6$$

$$= 15^4 - 4 \times 15^2 + 2 = 49727.$$

20. Answer: 24



Since BE bisects $\angle ABC$, we have $AE : EC = AB : BC = 1 : 4$. Furthermore, since $EF \mathbin{/\!/} AB$ and $DF \mathbin{/\!/} EB$, we see that DF bisects $\angle EFC$. Hence $DE : DC = 1 : 4$. Let $AE = x$ and $DE = y$. Then we have $x + y = 13.5$ and $4x = 5y$. Solving the equations yields $x = 7.5$ and $y = 6$. It follows that $CD = 4y = 24$.

21. Answer: 89440

The number of such ordered triples (x, y, z) with $x = y$ is

$$\binom{65}{2} = 2080.$$

The number of such ordered triples (x, y, z) with $x \neq y$ is

$$2 \times \binom{65}{3} = 87360.$$

Hence the answer is $2080 + 87360 = 89440$.

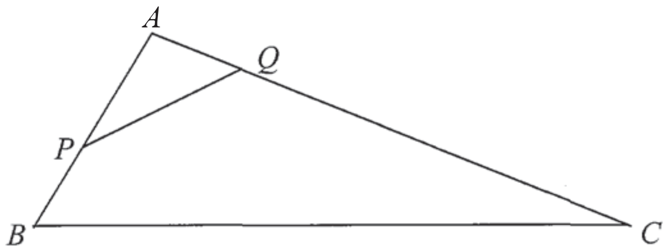
22. Answer: 19901

First note that $\dfrac{1}{a_{n+1}a_n}-\dfrac{1}{a_na_{n-1}}=\dfrac{1+na_{n-1}a_n}{a_{n-1}a_n}-\dfrac{1}{a_na_{n-1}}=n$. Therefore

$$\sum_{n=1}^{199}\left(\frac{1}{a_{n+1}a_n}-\frac{1}{a_na_{n-1}}\right)=\sum_{n=1}^{199}n=\frac{199\times 200}{2}=19900.$$

Hence $\dfrac{1}{a_{200}a_{199}}=1+19900=19901$.

23. Answer: 6



We have $\cos A=\dfrac{AB^2+AC^2-BC^2}{2(AB)(AC)}=\dfrac{5^2+10^2-13^2}{2(5)(10)}=-\dfrac{11}{25}$.

Let $AP=x$ cm and $AQ=y$ cm. Since area of $\Delta APQ=\dfrac{1}{2}xy\sin A$ and area of $\Delta ABC=\dfrac{1}{2}(AB)(AC)\sin A=\dfrac{1}{2}(5)(10)\sin A$, we obtain $\dfrac{xy}{50}=\dfrac{1}{4}$, that is, $xy=\dfrac{25}{2}$.

Hence

$$PQ^2=x^2+y^2-2xy\cos A=x^2+\left(\frac{25}{2x}\right)^2-25\left(-\frac{11}{25}\right)$$

$$=x^2+\frac{625}{4x^2}+11\geq 2\sqrt{x^2\cdot\frac{625}{4x^2}}+11=25+11=36.$$

Consequently, $PQ\geq 6$, with the equality attained when $x=y=\dfrac{5}{\sqrt{2}}$.

24. Answer: 5

Since $x+y=9-z$, $xy=24-z(x+y)=24-z(9-z)=z^2-9z+24$. Now note that x and y are roots of the quadratic equation $t^2+(z-9)t+(z^2-9z+24)=0$. As x and y are real, we have $(z-9)^2-4(z^2-9z+24)\geq 0$, which simplified to $z^2-6z+5\leq 0$. Solving the inequality yields $1\leq z\leq 5$. When $x=y=2$, $z=5$. Hence the largest possible value of z is 5.

25. Answer: 357

First put the six 1's in one sequence. Then there are 7 gaps before the first 1, between two adjacent 1's and after the last 1. For each such gap, we can put a single 0 or double 0's (that is, 00).

If there are exactly i double 0's, then there are exactly $6-2i$ single 0's, where $i =$ 0, 1, 2, 3. Therefore the number of such binary sequences with exactly i double 0's is $\binom{7}{i}\binom{7-i}{6-2i}$. Hence the answers is $\sum_{i=0}^{3}\binom{7}{i}\binom{7-i}{6-2i} = 357$.

26. Answer: 95

$$\frac{\cos 100^\circ}{1-4\sin 25^\circ \cos 25^\circ \cos 50^\circ} = \frac{\cos 100^\circ}{1-2\sin 50^\circ \cos 50^\circ} = \frac{\cos^2 50^\circ - \sin^2 50^\circ}{(\cos 50^\circ - \sin 50^\circ)^2}$$

$$= \frac{\cos 50^\circ + \sin 50^\circ}{\cos 50^\circ - \sin 50^\circ} = \frac{1+\tan 50^\circ}{1-\tan 50^\circ}$$

$$= \frac{\tan 45^\circ + \tan 50^\circ}{1-\tan 45^\circ \tan 50^\circ} = \tan 95^\circ.$$

Hence $x = 95$.

27. Answer: 223

$$\log_{\frac{x}{9}}\left(\frac{x^2}{3}\right) < 6 + \log_3\left(\frac{9}{x}\right)$$

$$\Leftrightarrow \frac{\log_3\left(\frac{x^2}{3}\right)}{\log_3\left(\frac{x}{9}\right)} < 6 + \log_3 9 - \log_3 x$$

$$\Leftrightarrow \frac{\log_3 x^2 - \log_3 3}{\log_3 x - \log_3 9} < 6 + \log_3 9 - \log_3 x.$$

Let $u = \log_3 x$. Then the inequality becomes $\frac{2u-1}{u-2} < 8 - u$, which is equivalent to $\frac{u^2 - 8u + 15}{u-2} < 0$. Solving the inequality gives $u < 2$ or $3 < u < 5$, that is, $\log_3 x < 2$ or $3 < \log_3 x < 5$. It follows that $0 < x < 9$ or $27 < x < 243$. Hence there are 223 such integers.

28. Answer: 79

First observe that

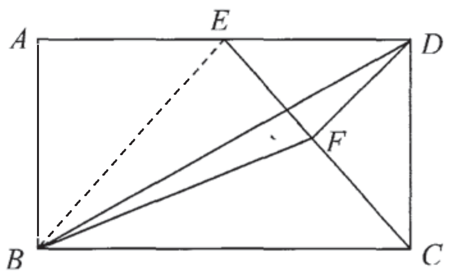
$$\frac{1}{x\sqrt{x+2}+(x+2)\sqrt{x}}=\frac{1}{\sqrt{x}\cdot\sqrt{x+2}}\left(\frac{1}{\sqrt{x}+\sqrt{x+2}}\right)$$
$$=\frac{1}{\sqrt{x}\cdot\sqrt{x+2}}\cdot\frac{\sqrt{x+2}-\sqrt{x}}{(x+2)-x}$$
$$=\frac{1}{2}\left(\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x+2}}\right).$$

Therefore

$$\frac{1}{9\sqrt{11}+11\sqrt{9}}+\frac{1}{11\sqrt{13}+13\sqrt{11}}+\cdots+\frac{1}{n\sqrt{n+2}+(n+2)\sqrt{n}}$$
$$=\frac{1}{2}\left(\frac{1}{\sqrt{9}}-\frac{1}{\sqrt{11}}\right)+\frac{1}{2}\left(\frac{1}{\sqrt{11}}-\frac{1}{\sqrt{13}}\right)+\cdots+\frac{1}{2}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+2}}\right)=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{\sqrt{n+2}}\right).$$

Now $\frac{1}{2}\left(\frac{1}{3}-\frac{1}{\sqrt{n+2}}\right)=\frac{1}{9}$ yields $n=79$.

29. Answer: 96



Let S be the area of rectangle $ABCD$. Then we have

$$\text{area of } \Delta CDF=\frac{1}{2}\times\text{area of } \Delta CDE=\frac{1}{2}\times\frac{1}{4}S=\frac{1}{8}S.$$

Next we have area of $\Delta BCF=\frac{1}{2}\times$ area of $\Delta BCE=\frac{1}{2}\times\frac{1}{2}S=\frac{1}{4}S.$

Now

$$12=\text{area of } \Delta BDF=\text{area of } \Delta BCD-\text{area of } \Delta BCF-\text{area of } \Delta CDF$$
$$=\frac{1}{2}S-\frac{1}{4}S-\frac{1}{8}S=\frac{1}{8}S.$$

Hence the area of rectangle $ABCD = 96$ cm^2.

30. Answer: 11754

First note that if every digit in the 6-digit number appears at least twice, then there cannot be four distinct digits in the number. In other words, the number can only be formed by using one digit, two distinct digits or three distinct digits respectively. Therefore we consider three cases.

Case 1: The 6-digit number is formed by only one digit.

Then the number of such 6-digit numbers is clearly 9.

Case 2: The 6-digit number is formed by two distinct digits.

First, the number of such 6-digit numbers formed by two given digits i and j, where $1 \le i < j \le 9$, is

$$\binom{6}{2}+\binom{6}{3}+\binom{6}{4}=50.$$

Next, the number of such 6-digit numbers formed by 0 and a given digit i, where $1 \le i \le 9$, is

$$\binom{5}{2}+\binom{5}{3}+\binom{5}{4}=25.$$

Therefore the total number of such 6-digit numbers formed by two distinct digits is

$$\binom{9}{2}\times 50+9\times 25=2025.$$

Case 3: The 6-digit number is formed by three distinct digits.

First, the number of such 6-digit numbers formed by three given digits i, j and k, where $1 \le i < j < k \le 9$, is

$$\binom{6}{2}\cdot\binom{4}{2}=90.$$

Next, the number of such 6-digit numbers formed by 0 and two given digits i and j, where $1 \le i < j \le 9$, is

$$\binom{5}{2}\cdot\binom{4}{2}=60.$$

Therefore the total number of such 6-digit numbers formed by three distinct digits is

$$\binom{9}{3}\times 90+\binom{9}{2}\times 60=9720.$$

Hence the answer is 9 + 2025 + 9720 = 11754.

31. Answer: 234

Since $27x + 35y \le 945$, we have $y \le \dfrac{945-27x}{35}$. It follows that

$$xy \le \frac{945x - 27x^2}{35} = \frac{27}{35}(35x - x^2) = \frac{27}{35}\left(\left(\frac{35}{2}\right)^2 - \left(x - \frac{35}{2}\right)^2\right).$$

Therefore, if $\left|x - \dfrac{35}{2}\right| \ge \dfrac{5}{2}$, that is, if $x \ge 20$ or $x \le 15$, then

$$xy \le \frac{27}{35}\left(\left(\frac{35}{2}\right)^2 - \left(\frac{5}{2}\right)^2\right) < 231.4.$$

If $x = 16$, then $y \le \dfrac{945-27(16)}{35} \le 14.7$. Thus $y \le 14$, and $xy \le 224$.
Similarly, if $x = 17$, then $y \le 13$, and $xy \le 221$.
If $x = 18$, then $y \le 13$, and $xy \le 234$.
If $x = 19$, then $y \le 12$, and $xy \le 228$.
In conclusion, the maximum value of xy is 234, which is attained at $x = 18$ and $y = 13$.

32. Answer: 65520

Note that

$$(1 + x^5 + x^7 + x^9)^{16} = \sum_{i=0}^{16} \binom{16}{i} x^{5i}(1 + x^2 + x^4)^i.$$

It is clear that if $i > 5$ or $i < 4$, then the coefficient of x^{29} in the expansion of $x^{5i}(1 + x^2 + x^4)^i$ is 0. Note also that if i is even, then the coefficient of x^{29} in the expansion of $x^{5i}(1 + x^2 + x^4)^i$ is also 0. Thus we only need to determine the coefficient of x^{29} in the expansion of $\dbinom{16}{i} x^{5i}(1 + x^2 + x^4)^i$ for $i = 5$.

When $i = 5$, we have

$$\begin{aligned}
\binom{16}{i} x^{5i}(1 + x^2 + x^4)^i &= \binom{16}{5} x^{25}(1 + x^2 + x^4)^5 \\
&= \binom{16}{5} x^{25} \sum_{j=0}^{5} \binom{5}{j} (x^2 + x^4)^j \\
&= \binom{16}{5} x^{25} \sum_{j=0}^{5} \binom{5}{j} x^{2j}(1 + x^2)^j.
\end{aligned}$$

It is clear that the coefficient of x^4 in the expansion of $\sum_{j=0}^{5}\binom{5}{j}x^{2j}(1+x^2)^j$ is $\binom{5}{1}+\binom{5}{2}=15$. Hence the answer is $\binom{16}{5}\times 15=65520$.

33. Answer: 401

For each $n = 1, 2, 3, \ldots$, since $d_n=\gcd(a_n,a_{n+1})$, we have $d_n \mid a_n$ and $d_n \mid a_{n+1}$. Thus $d_n \mid a_{n+1}-a_n$, that is, $d_n \mid (n+1)^2+100-(n^2+100)$, which gives $d_n \mid 2n+1$. Hence $d_n \mid 2(n^2+100)-n(2n+1)$, and we obtain $d_n \mid 200-n$. It follows that $d_n \mid 2(200-n)+2n+1$, that is, $d_n \mid 401$. Consequently, $1 \le d_n \le 401$ for all positive integers n.

Now when $n = 200$, we have $a_n = a_{200} = 200^2+100 = 401\times 100$ and $a_{n+1}=a_{201}=201^2+100=401\times 101$. Therefore $d_{200}=\gcd(a_{200},a_{201})=401$. Hence it follows that the maximum value of d_n when n ranges over all positive integers is 401, which is attained at $n = 200$.

34. Answer: 441

First we determine a_{2008} and a_{2009}. Suppose that $a_{2008}=\overline{x_1x_2x_3x_4x_5x_6x_7x_8}$, where the x_i's are distinct digits in $\{1,2,3,4,5,6,7,8\}$.

Let $A=\{a_k : k=1,2,\ldots,40320\}$.

Since 7! = 5040 > 2008, we deduce that $x_1=1$, as there are more than 2008 numbers in A such that the first digit is 1.

As $2 \times 6! < 2008 < 3 \times 6!$, we have $x_2=4$, as there are less than 2008 numbers in A such that the first digit is 1 and the second digit is 2 or 3, but there are more than 2008 numbers in A such that the first digit is 1 and the second digit is 2, 3 or 4. Similarly, since $2 \times 6! + 4 \times 5! < 2008 < 2 \times 6! + 5 \times 5!$, we see that the third digit x_3 is 7. By repeating the argument and using the inequalities

$2 \times 6! + 4 \times 5! + 3 \times 4! < 2008 < 2 \times 6! + 4 \times 5! + 4 \times 4!$ and

$2004 = 2 \times 6! + 4 \times 5! + 3 \times 4! + 2 \times 3! < 2008 < 2 \times 6! + 4 \times 5! + 3 \times 4! + 3 \times 3!$,

we obtain $x_4=6$, $x_5=5$. Note also that among the numbers in A of the form 1476****, the digit 5 first appears as the fifth digit in a_{2005} if the numbers are arranged in increasing order. Consequently, as the last three digits are 2, 3 and 8, we must have $a_{2005}=14765238$. It follows that $a_{2006}=14765283$, $a_{2007}=14765328$, $a_{2008}=14765382$, and $a_{2009}=14765823$. Hence $a_{2009}-a_{2008}=14765823-14765382=441$.

35. Answer: 24

Write $u = \log_{10} x$. Then $\log_{10} \dfrac{100}{x} = 2 - u$. Since $a = \lfloor \log_{10} x \rfloor$, we have

$$u = a + \gamma \quad \text{for some } 0 \leq \gamma < 1. \qquad (1)$$

Similarly, since $b = \lfloor 2 - u \rfloor$, we have

$$2 - u = b + \delta \quad \text{for some } 0 \leq \delta < 1. \qquad (2)$$

Then $0 \leq \gamma + \delta < 2$. Since $\gamma + \delta = u - a + (2 - u - b) = 2 - a - b$ is an integer, it follows that $\gamma + \delta = 0$ or $\gamma + \delta = 1$.

Case 1: $\gamma + \delta = 0$.

Then $\gamma = 0$ and $\delta = 0$, since $\gamma \geq 0$ and $\delta \geq 0$. Therefore

$$\begin{aligned} 2a^2 - 3b^2 &= 2u^2 - 3(2-u)^2 \\ &= -u^2 + 12u - 12 \\ &= 24 - (u-6)^2 \leq 24, \end{aligned}$$

and the maximum value is attained when $u = 6$.

Case 2: $\gamma + \delta = 1$.

Then we must have $0 < \gamma, \delta < 1$ by (1) and (2). Also, by (1) and (2), we have $b = \lfloor 2 - u \rfloor = \lfloor 2 - a - \gamma \rfloor = 1 - a$. Thus

$$\begin{aligned} 2a^2 - 3b^2 &= 2a^2 - 3(1-a)^2 \\ &= -a^2 + 6a - 3 \\ &= 6 - (a-3)^2 \leq 6. \end{aligned}$$

Hence the largest possible value of $2a^2 - 3b^2$ is 24, when $x = 10^6$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Senior Section, Round 2)

Saturday, 27 June 2009 **0930-1230**

INSTRUCTIONS TO CONTESTANTS

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

1. Let M and N be points on sides AB and AC of triangle ABC respectively. If

$$\frac{BM}{MA} + \frac{CN}{NA} = 1,$$

show that MN passes through the centroid of ABC.

2. Find all pairs of positive integers n, m that satisfy the equation $3 \cdot 2^m + 1 = n^2$.

3. L et A be an n-element subset of $\{1, 2, \ldots, 2009\}$ with the property that the difference between any two numbers in A is not a prime number. Find the largest possible value of n. Find a set with this number of elements. (Note: 1 is not a prime number.)

4. Let $a, b, c > 0$ such that $a + b + c = 1$. Show that if if $x_1, x_2, \ldots, x_5$ are positive real numbers such that $x_1 x_2 \ldots x_5 = 1$, then

$$(ax_1^2 + bx_1 + c)(ax_2^2 + bx_2 + c) \cdots (ax_5^2 + bx_5 + c) \geq 1.$$

5. In an archery competition, there are 30 contestants. The target is divided in two zones. A hit at zone 1 is awarded 10 points while a hit at zone 2 is awarded 5 points. No point is awarded for a miss. Each contestant shoots 16 arrows. At the end of the competition statistics show that more than 50% of the arrows hit zone 2. The number of arrows that hit zone 1 and miss the target are equal. Prove that there are two contestants with the same score.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Senior Section, Round 2 solutions)

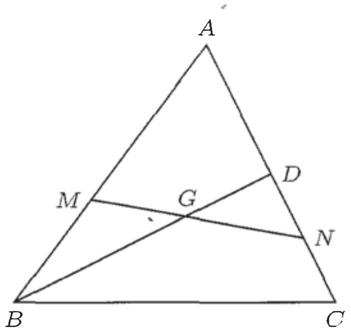
1. Let D be the midpoint of AC. Since $\frac{CN}{NA} < 1$, N lies in the segment CD. Let G be the intersection of BD and MN. By Menelaus' Theorem applied to the line MN and triangle ABD,

$$\frac{DG}{GB} \cdot \frac{BM}{MA} \cdot \frac{AN}{ND} = 1.$$

Thus

$$\begin{aligned}\frac{BG}{GD} &= \frac{BM}{MA} \cdot \frac{AN}{ND} = \left(1 - \frac{CN}{NA}\right) \cdot \frac{AN}{ND} \\ &= \frac{NA - CN}{ND} = \frac{(2CD - CN) - CN}{ND} \\ &= \frac{2ND}{ND} = 2.\end{aligned}$$

Therefore, G is the centroid of ABC.



2. We have $n^2 \equiv 1 \pmod 3$. Thus $n = 3k+1$ or $3k+2$ for some nonnegative integer k.

(i) $n = 3k+1$. After simplifying, we have $2^m = 3k^2 + 2k = k(3k+2)$. Thus k and $3k+2$ are both powers of 2. It is clear that $k = 2$ is a solution and $k = 1$ is not. If $k = 2^p$, where $p \geq 2$, then $3k + 2 = 2(3 \cdot 2^{p-1} + 1)$ is not a power of 2 as $3 \cdot 2^{p-1} + 1$ is odd. We have one solution: $n = 7, m = 4$.

(ii) $n = 3k+2$: Again we have $2^m = 3k^2 + 4k + 1 = (3k+1)(k+1)$ and both $k+1$ and $3k+1$ must be powers of 2. Both $k = 0, 1$ are solutions. When $k = 0$, $m = 0$, which is not admissible. For $k > 1$, we have $3k + 1 = 2k + (k+1) > 2k + 2$ and therefore $4(k+1) > 3k+1 > 2(k+1)$. Hence if $k + 1 = 2^p$ for some positive integer p, then

$2^{p+2} > 3k+1 > 2^{p+1}$ and we conclude that $3k+1$ cannot be a power of 2. Thus there is one solution in this case: $(n,m) = (5,3)$.

et A be an n-element subset of $\{1,2,\dots,2009\}$ with the property that the difference between any two numbers in A is not a prime number. Find the largest possible value of n. Find a set with this number of elements. (Note: 1 is not a prime number.)

3. If $n \in A$, then $n+i \notin A$, $i = 2,3,5,7$. Among $n+1, n+4, n+6$ at most one can be in A. Thus among any 8 consecutive integers, at most 2 can be in S. Hence $|A| \le 2\lceil 2009/8 \rceil = 504$. Such a set is $\{4k+1 : k = 0,1,\dots,502\}$.

4. We give a proof of the general case. Consider the expansion of

$$(ax_1^2 + bx_1 + c)(ax_2^2 + bx_2 + c)\cdots(ax_n^2 + bx_n + c).$$

The term in $a^i b^j c^k$, where $i+j+k = n$ is

$$a^i b^j c^k[(x_1x_2\dots x_i)^2(x_{i+1}x_{i+2}\dots x_{i+j}) + \cdots].$$

There are altogether $\binom{n}{i}\binom{n-i}{j}$ terms in the summation. (We choose i factors from which we take ax_t^2. From the remaining $n-i$ factors, we choose j to take the terms bx_s.) By symmetry, the number of terms containing x_i^2 is a constant, as is the number of terms containing the term x_i. Thus, when the terms in the summation are multiplied together, we get $(x_1x_2\dots x_n)^p = 1$ for some p. (For our purpose, it is not necessary to compute p. In fact $p = 2\binom{n-1}{i-1}\binom{n-i}{j} + \binom{n-1}{j-1}\binom{n-j}{i} = \frac{2i+j}{n}\binom{n}{i}\binom{n-i}{j}$.) By the AM-GM inequality, we have

$$a^i b^j c^k[(x_1x_2\dots x_i)^2(x_{i+1}x_{i+2}\dots x_{i+j}) + \cdots] \ge a^i b^j c^k \binom{n}{i}\binom{n-i}{j}.$$

Hence

$$(ax_1^2 + bx_1 + c)\cdots(ax_n^2 + bx_n + c) \ge \sum_{i+j+k=n} a^i b^j c^k \binom{n}{i}\binom{n-i}{j} = (a+b+c)^n = 1.$$

5. The number of arrows that that hit zone 1 is $< 30 \cdot 16/4 = 120$. If contestant i hits zone 1 a_i times, zone 2 b_i times and miss the target c_i times, then the total score is $10a_i + 5b_i = 5a_i + 5(a_i + b_i) = 5a_i + 5(16 - c_i) = 80 + 5(a_i - c_i)$. Suppose the scores are all distinct, then the 30 numbers $a_i - c_i$ must all be distinct. By the pigeonhole principle, half of these 30 numbers are either positive or negative. We consider the “positive” case. Without loss of generality, let $a_i - c_i > 0$ for $i = 1,\dots,15$. Then $a_i - c_i \ge i$. Therefore $a_i \ge i$. Hence $a_1 + \cdots + a_{15} \ge 120$. But $a_1 + \cdots + a_{30} < 120$, and we have a contradiction. The “negative” case is similar.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Open Section, Round 1)

Wednesday, 3 June 2009 **0930 – 1200**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

1. The expression $1000 \sin 10^0 \cos 20^0 \cos 30^0 \cos 40^0$ can be simplified as $a \sin b^0$, where a and b are positive integers with $0 < b < 90$. Find the value of $100a + b$.

2. Let A_1, A_2, A_3, A_4, A_5 and A_6 be six points on a circle in this order such that $\widehat{A_1A_2} = \widehat{A_2A_3}$, $\widehat{A_3A_4} = \widehat{A_4A_5}$ and $\widehat{A_5A_6} = \widehat{A_6A_1}$, where $\widehat{A_1A_2}$ denotes the arc length of the arc A_1A_2 etc. It is also known that $\angle A_1A_3A_5 = 72^\circ$. Find the size of $\angle A_4A_6A_2$ in degrees.

3. Let $P_1, P_2, \ldots, P_{41}$ be 41 distinct points on the segment BC of a triangle ABC, where AB = AC = 7. Evaluate the sum $\sum_{i=1}^{41}(\mathrm{AP}_i^2 + \mathrm{P}_i\mathrm{B} \cdot \mathrm{P}_i\mathrm{C})$.

4. Determine the largest value of x for which

$$\left|x^2 - 11x + 24\right| + \left|2x^2 + 6x - 56\right| = \left|x^2 + 17x - 80\right|.$$

5. Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$ be a polynomial in x where a_0, a_1, a_2, a_3, a_4 are constants and $a_5 = 7$. When divided by $x - 2004$, $x - 2005$, $x - 2006$, $x - 2007$ and $x - 2008$, $f(x)$ leaves a remainder of 72, -30, 32, -24 and 24 respectively. Find the value of f(2009).

6. Find the value of $\dfrac{\sin 80^o}{\sin 20^o} - \dfrac{\sqrt{3}}{2\sin 80^o}$.

7. Determine the number of 8-digit positive integers such that after deleting any one digit, the remaining 7-digit number is divisible by 7.

8. It is given that $\sqrt{a} - \sqrt{b} = 20$, where a and b are real numbers. Find the maximum possible value of $a - 5b$.

9. Let ABC be a triangle with sides AB = 7, BC = 8 and AC = 9. A unique circle can be drawn touching the side AC and the lines BA produced and BC produced. Let D be the centre of this circle. Find the value of BD^2.

10. If $x = \frac{1}{2}\left(\sqrt[3]{2009} - \frac{1}{\sqrt[3]{2009}}\right)$, find the value of $\left(x + \sqrt{1+x^2}\right)^3$.

11. Let $S = \{1, 2, 3, \ldots, 30\}$. Determine the number of vectors (x, y, z, w) with $x, y, z, w \in S$ such that $x < w$ and $y < z < w$.

12. Let f(n) be the number of 0's in the decimal representation of the positive integer n. For example, f(10001123) = 3 and f(1234567) = 0. Find the value of

$$f(1) + f(2) + f(3) + \ldots + f(99999).$$

13. It is given that k is a positive integer not exceeding 99. There are no natural numbers x and y such that $x^2 - ky^2 = 8$. Find the difference between the maximum and minimum possible values of k.

14. Let $S = \{1, 2, 3, 4, \ldots, 16\}$. In each of the following subsets of S,

$$\{6\}, \{1, 2, 3\}, \{5, 7, 9, 10, 11, 12\}, \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

the sum of all the elements is a multiple of 3. Find the total number of non-empty subsets A of S such that the sum of all elements in A is a multiple of 3.

15. A function $f : \mathbf{R} \to \mathbf{R}$ satisfies the relation $f(x)f(y) = f(2xy + 3) + 3f(x + y) - 3f(x) + 6x$, where

$x, y \in \mathbf{R}$. Find the value of f(2009).

16. Let $\{a_n\}$ be a sequence of positive integers such that $a_1 = 1$, $a_2 = 2009$ and for $n \geq 1$,

$a_{n+2}a_n - a_{n+1}^2 - a_{n+1}a_n = 0$. Determine the value of $\frac{a_{993}}{100a_{991}}$.

17. Determine the number of ways of tiling a 4x9 rectangle by tiles of size 1x2.

18. Find the number of 7-digit positive integers such that the digits from left to right are non-increasing. (Examples of 7-digit non-increasing numbers are 9998766 and 5555555; An example of a number that is NOT non-increasing is 7776556)

19. Determine the largest prime number less than 5000 of the form $a^n - 1$, where a and n are positive integers, and n is greater than 1.

20. Determine the least constant M such that

$$\frac{x_1}{x_1+x_2}+\frac{x_2}{x_2+x_3}+\frac{x_3}{x_3+x_4}+\cdots+\frac{x_{2009}}{x_{2009}+x_1}<M,$$

for any positive real numbers $x_1, x_2, x_3, \ldots, x_{2009}$.

21. Six numbers are randomly selected from the integers 1 to 45 inclusive. Let p be the probability that at least three of the numbers are consecutive. Find the value of $\lfloor 1000p \rfloor$. (Note: $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x).

22. Evaluate $\displaystyle\sum_{k=0}^{\infty} \frac{2}{\pi} \tan^{-1}\left(\frac{2}{(2k+1)^2}\right)$.

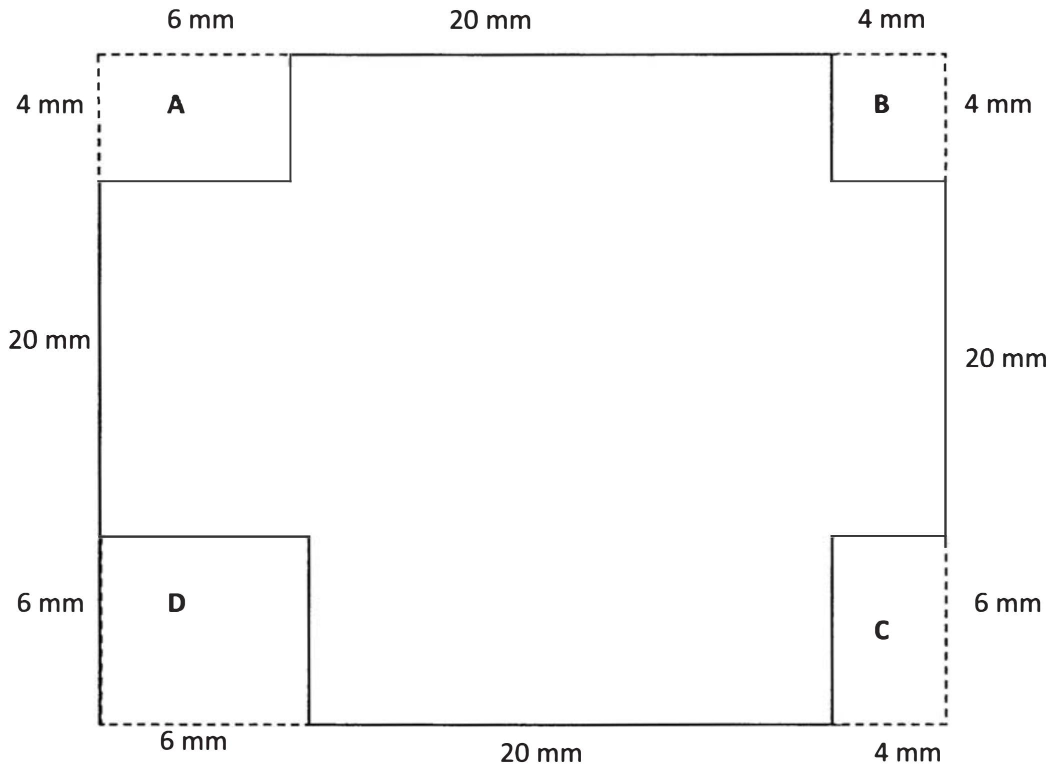
23. Determine the largest prime factor of the sum $\displaystyle\sum_{k=1}^{11} k^5$.

24. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $x_1 = 3, x_2 = 24$ and

$$x_{n+2} = \frac{1}{4}x_{n+1} + \frac{3}{4}x_n$$

for every positive integers n. Determine the value of $\displaystyle\lim_{n\to\infty} x_n$.

25. A square piece of graph paper of side length 30 mm contains 900 smallest squares each of side length 1mm each. Its four rectangular corners, denoted by A, B, C, D in clockwise order, are cut away from the square piece of graph paper. The resultant graph paper, which has the shape of a cross, is shown in the figure below. Let N denote the total number of rectangles, **excluding** all the squares which are contained in the resultant graph paper. Find the value of $\frac{1}{10}N$.



Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Open Section, Round 1 Solution)

1. Answer: 12560

$$\sin 10^0 \cos 20^0 \cos 30^0 \cos 40^0 = \frac{2\sin 10^0 \cos 10^0 \cos 20^0 \cos 30^0 \cos 40^0}{2\cos 10^0}$$

$$= \frac{\sin 20^0 \cos 20^0 \cos 30^0 \cos 40^0}{2\cos 10^0}$$

$$= \frac{\sin 40^0 \cos 40^0 \cos 30^0}{4\cos 10^0}$$

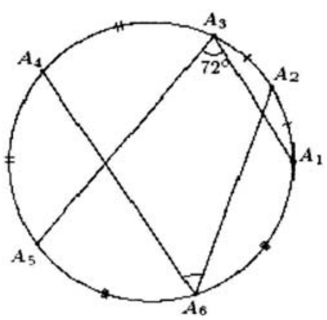
$$= \frac{\sin 80^0 \cos 30^0}{8\cos 10^0}$$

$$= \frac{\cos 10^0 \sin 60^0}{8\cos 10^0}$$

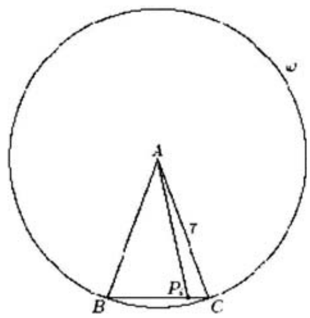
Hence $1000\sin 10^0 \cos 20^0 \cos 30^0 \cos 40^0 = 125\sin 60^0$, showing that $a = 125$ and $b = 60$. So, $100a + b = 12560$.

2. Answer: 54.

First, observe that $\angle A_1A_6A_5 = 180^o - 72^o = 108^o$. Hence $\angle A_4A_6A_2 = \angle A_1A_6A_5 - \angle A_1A_6A_2 - \angle A_4A_6A_5 = 108^o - \angle A_2A_6A_3 - \angle A_3A_6A_4 = 108^o - \angle A_4A_6A_2$. Thus, $2\angle A_4A_6A_2 = 108^o$, resulting in $\angle A_4A_6A_2 = 54^o$.



3. Answer: 2009



Construct a circle ω centred at A with radius AB = AC = 7. The power of P_i with respect to ω is $P_iA^2 - 7^2$, which is also equal to $-BP_i \cdot P_iC$. Thus, $P_iA^2 + BP_i \cdot P_iC = 7^2 = 49$. Therefore, the sum equals 49 x 41 = 2009.

4. Answer: 8

Since $2x^2 + 6x - 56 - (x^2 - 11x + 24) = x^2 + 17x - 80$, the given equation holds if and only if

$$\left(x^2 - 11x + 24\right)\left(2x^2 + 6x - 56\right) \le 0,$$

Since $|a - b| = |a| + |b|$ if and only if $ab \le 0$. The above inequality reduces to

$$(x-3)(x-8)(x-4)(x+7) \le 0.$$

Since

$$\{x : (x-3)(x-8)(x-4)(x+7) \le 0\} = [-7,3] \cup [4,8],$$

we conclude that the largest value of x is 8.

5. Answer: 1742.

We have

$$f(x) = \frac{(x-2005)(x-2006)(x-2007)(x-2008)}{(-1)(-2)(-3)(-4)}(72) + \frac{(x-2004)(x-2006)(x-2007)(x-2008)}{(1)(-1)(-2)(-3)}(-30) + \frac{(x-2004)(x-2005)(x-2007)(x-2008)}{(2)(1)(-1)(-2)}(32) + \frac{(x-2004)(x-2005)(x-2006)(x-2008)}{(3)(2)(1)(-1)}(-24) + \frac{(x-2004)(x-2005)(x-2006)(x-2007)}{(4)(3)(2)(1)}(24) + 7(x-2004)(x-2005)(x-2006)(x-2007)(x-2008),$$

So that f(2009) = 1742.

6. Answer: 2

$$\frac{\sin 80^o}{\sin 20^o} - \frac{\sqrt{3}}{2\sin 80^o} = \frac{\sin 80^o}{\sin 20^o} - \frac{\sin 60^o}{\sin 80^o}$$

$$= \frac{\sin^2 80^o - \sin 20^o \sin 60^o}{\sin 20^o \sin 80^o} = \frac{1 - \cos 160^o + \cos 80^o - \cos 40^o}{2\sin 20^o \sin 80^o}$$

$$= \frac{1 - \cos 40^o + \cos 80^o - \cos 160^o}{2\sin 20^o \sin 80^o}$$

$$= \frac{2\sin^2 20^o + 2\sin 120^o \sin 40^o}{2\sin 20^o \sin 80^o} = \frac{2\sin^2 20^o + 2\sqrt{3}\sin 20^o \cos 20^o}{2\sin 20^o \sin 80^o}$$

$$= \frac{\sin 20^o + \sqrt{3}\cos 20^o}{\sin 80^o} = \frac{2\sin(20^o + 60^o)}{\sin 80^o} = 2$$

7. Answer: 64

Let $N = \overline{abcdefgh}$ be such a number. By deleting a and b, we get $\overline{bcdefgh}$ and $\overline{acdefgh}$ respectively. Both of them are divisible by 7, hence their difference 1000000 $(b - a)$ is also divisible by 7, therefore $b - a$ is divisible by 7. By the similar argument, $c - b$, $d - c$, $e - d$, $f - e$, $g - f$, $h - g$ are divisible by 7. In other word, all the digits of N are congruent modulo 7. If N contains digits that are greater than 7, then one can subtract 7 from each digit to get a new number N'. Then N satisfies the requirements in the question if and only if N' does. Since all the digits are congruent modulo 7, it remains to consider numbers of the form $\overline{pppppppp}$, where $p = 0, 1, 2, 3, 4, 5, 6$. By deleting a digit in this number, we get the number $\overline{ppppppp} = 1111111p$, which is divisible by 7 if and only if p is the digit 0 or 7. However, the first two digits of N must be 7 since the number N has 8 digits and that any number we get by deleting a digit in N has 7 digits. On the other hand, the remaining 6 digits can be independently 0 or 7. Consequently, there are $2^6 = 64$ choices of such numbers.

8. Answer: 500

$$\sqrt{a} = \sqrt{b} + 20$$
$$a = b + 400 + 40\sqrt{b}$$
$$a - 5b = 400 + 40\sqrt{b} - 4b$$
$$a - 5b = 400 - 4(\sqrt{b} - 5)^2 + 100$$
$$a - 5b \le 500.$$

9. Answer: 224

Let the circle with centre D and radius r touch the tangent lines AC, BA produced and BC produced at the points E, F and G respectively. Then r = DE = DF = DF. Hence, triangles BDF and BDG are congruent, and hence ∠ABD = ∠CBD = ½ ∠ABC. We have

$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{8^2+7^2-9^2}{2(8)(7)} = \frac{2}{7}$, and hence $\sin\frac{B}{2} = \sqrt{\frac{1-\cos B}{2}} = \sqrt{\frac{5}{14}}$.

To find r, we have

$$(ABD) + (BCD) - (ACD) = (ABC),$$

where (ABD) denotes the area of triangle ABD, etc.

Hence $\frac{1}{2}cr + \frac{1}{2}ar - \frac{1}{2}br = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{8+9+7}{2} = 12$.

Solving, we get $r = 4\sqrt{5}$. Considering, triangle BDF, we have $BD = \frac{r}{\sin\frac{B}{2}} = 4\sqrt{14}$. Thus, we have $BD^2 = 224$.

10. Answer: 2009

Since $x = \frac{1}{2}\left(\sqrt[3]{2009} - \frac{1}{\sqrt[3]{2009}}\right)$, we have $(\sqrt[3]{2009})^2 - 2x\sqrt[3]{2009} - 1 = 0$ .We see that $\sqrt[3]{2009}$ is a root of the equation $t^2 - 2xt - 1 = 0$. Thus

$\sqrt[3]{2009} = \frac{2x + \sqrt{4x^2+4}}{2} = x + \sqrt{x^2+1}$ or $\sqrt[3]{2009} = \frac{2x - \sqrt{4x^2+4}}{2} = x - \sqrt{x^2+1} < 0$,

which is not possible. Thus $\left(x + \sqrt{1+x^2}\right)^3 = 2009$.

11. Answer: 90335

There are two cases to consider: Case (1) $x \in \{y, z\}$ and Case (2): $x \notin \{y, z\}$. For Case (1), there are $2\binom{30}{3}$ ways and for Case (2), there are $3\binom{30}{4}$ ways. Hence, total number of ways = 90335.

12. Answer: 38889

Let $S = \{1, 2, 3,\ldots, 99999\}$, and $S_i = \{n \in S\text{: } f(n) = i\}$ for $i \geq 0$. Thus,

$$S = \cup_{0\leq i\leq 4} S_i.$$

For $0 \leq i \leq 4$, if $n \in S_i$, and n has exactly k digits in the decimal representation, then exactly we have $k - i$ digits are non-zero. Thus,

$$|S_i| = \sum_{k=i+1}^{5} \binom{k-1}{i} 9^{k-i}.$$

Then, it is clear that

$$M = |S_1| + 2|S_2| + 3|S_3| + 4|S_4| = 38889.$$

13. Answer: 96

Note that $3^2 - 1\times1^2 = 8$ and $4^2 - 2\times2^2 = 8$. Suppose $k \equiv 0 \pmod 3$. Note that $a^2 \equiv 0,1 \pmod 3$ for all natural numbers a. Thus, $x^2 - ky^2 \equiv 0,1 - 0 \pmod 3 \equiv 0,1 \pmod 3$ but $8 \equiv 2 \pmod 3$. Hence, there are no natural numbers x and y such that $x^2 - ky^2 = 8$ if k is a multiple of 3. Therefore, $\max\{k\} - \min\{k\} = 99 - 3 = 96$.

14. Answer: 21855.

Let $S_i = \{x \in S\text{: } x \equiv i \pmod 3\}$ for $i = 0,1, 2$. Note that $|S_0| = 5$, $|S_1| = 6$ and $|S_2| = 5$. Let ∂ be the set of all subsets A of S such that $\sum_{x\in\partial} x$ is a multiple of 3. Note that for any $A \subseteq S$,

$$\sum_{x\in A} x = \sum_{i=0}^{2} \sum_{x\in A\cap S_i} x \equiv |A \cap S_1| + 2|A \cap S_2| \pmod 3.$$

Thus, $A \in \partial$ if and only if $|A \cap S_1| \equiv |A\cap S_2| \pmod 3$. Thus, it is clear that

$|\partial| = 2^{|S_0|} m$, where

$$m = \left\{\binom{6}{0} + \binom{6}{3} + \binom{6}{6}\right\}\left\{\binom{5}{0} + \binom{5}{3}\right\} + \left\{\binom{6}{1} + \binom{6}{4}\right\}\left\{\binom{5}{1} + \binom{5}{4}\right\}$$
$$+ \left\{\binom{6}{2} + \binom{6}{5}\right\}\left\{\binom{5}{2} + \binom{5}{5}\right\} = 683.$$

Hence $|\partial| = 2^5 \times 683 = 21856$. Since we want only non-empty subsets, we have 21855.

15. Answer: 4021

Given $f(x)f(y) = f(2xy + 3) + 3f(x + y) - 3f(x) + 6x$, so if interchanging x and y we have

$f(y)f(x) = f(2xy + 3) + 3f(x + y) - 3f(y) + 6y$.

Subtracting, we have $-3f(x) + 6x = -3f(y) + 6y$ for all $x, y \in \mathbf{R}$, showing that $f(x) - 2x$ is a constant, let it be k. So, $f(x) = 2x + k$.

Substitute back to the given functional equation, we have

$$(2x + k)(2y + k) = 2(2xy + 3) + k + 3[2(x + y) + k] - 3(2x + k) + 6x$$

$$4xy + 2kx + 2ky + k^2 = 4xy + 6 + k + 6x + 6y + 3k - 6x - 3k + 6x$$
$$2k(x + y) - 6(x + y) = k - k^2 + 6$$
$$(k - 3)(k + 2) = 2(x + y)(3 - k)$$

$(k - 3)(k + 2 + 2x + 2y) = 0$ for all $x, y \in \mathbf{R}$.

Thus $k = 3$. Hence $f(2009) = 2(2009) + 3 = 4021$.

16. Answer: 89970

$$a_{n+2}a_n - a_{n+1}^2 - a_{n+1}a_n = 0 \quad \Rightarrow \quad \frac{a_{n+2}a_n - a_{n+1}^2 - a_{n+1}a_n}{a_{n+1}a_n} = 0$$

$\dfrac{a_{n+2}}{a_{n+1}} - \dfrac{a_{n+1}}{a_n} = 1$. From here, we see that $\left\{\dfrac{a_{n+1}}{a_n}\right\}$ is an arithmetic sequence with first term 2009 and common difference 1. Thus $\dfrac{a_{n+1}}{a_n} = n + 2008$, and that

$\dfrac{a_{993}}{a_{992}} = 992 + 2008 = 3000$ and $\dfrac{a_{992}}{a_{991}} = 991 + 2008 = 2999$. We therefore have

$$\frac{a_{993}}{100a_{991}} = 30(2999) = 89970.$$

17. Answer: 6336

Let f_n be the number of ways of tiling a 4 x n rectangle. Also, let g_n be the number of ways of tiling a 4 x n rectangle with the top or bottom two squares in the last column missing, and let h_n be the number of ways of tiling a 4 x n rectangle with the top and bottom squares in the last column missing. Set up a system of recurrence relations involving f_n , g_n and h_n by considering the ways to cover the nth column of a 4 x n rectangle. If two vertical tiles are used, then there are f_{n-1} ways. If one vertical tile and two adjacent horizontal tiles are used, then there are $2g_{n-1}$ ways. If one vertical tile and two non-adjacent horizontal tiles are used, then there are h_{n-1} ways. If four horizontal tiles are used, then there are f_{n-2} ways. Similarly, one can establish the recurrence relations for g_n and h_n. In conclusion, we obtain for $n \geq 2$,

$$\begin{aligned} f_n &= f_{n-1} + f_{n-2} + 2g_{n-1} + h_{n-1} \\ g_n &= g_{n-1} + f_{n-1} \\ h_n &= h_{n-2} + f_{n-1}, \end{aligned}$$

With initial conditions $f_0 = f_1 = g_1 = h_1 = 1$ and $h_0 = 0$. Solving f_n recursively, we obtain $f_9 =$ 6336.

18. Answer: 11439

Each 16-digit binary sequence containing exactly nine '0's and seven '1's can be matched uniquely to such a 7-digit integer or 0000000 as follows: Each '1' will be replaced by a digit from 0 to 9 in this way: the number of '0's to the right of a particular '1' indicates the value of the digit. For example, 0110000010101101 ~ 8832110 and 1111000000000111 ~ 9999000.

Thus, required number = $\binom{16}{7} - 1$.

19. Answer: 127

Since $a^n - 1 = (a-1)(a^{n-1} + a^{n-2} + \cdots + a + 1)$ is a prime, then $a = 2$. Suppose $n = rs$ is a composite, then $2^{rs} - 1 = (2^r - 1)(2^{r(s-1)} + 2^{r(s-2)} + \cdots + 2^r + 1)$, where each factor on the right is greater than 1, contradicting the fact that $a^n - 1$ is a prime. Therefore n must be a prime. The largest prime n such that $2^n - 1 < 5000$ is 11. However, $2^{11} - 1 = 2047$ $= 23 \times 89$, which is not a prime. Since $2^7 - 1 = 127$ is a prime number, the answer to this question is 127.

20. Answer: 2008

Let $S = \frac{x_1}{x_1+x_2} + \frac{x_2}{x_2+x_3} + \frac{x_3}{x_3+x_4} + \cdots + \frac{x_{2009}}{x_{2009}+x_1}$. Then it is clear that $S > \frac{x_1}{x_1+x_{2+\cdots}+x_{2009}} + \frac{x_2}{x_1+x_2+\cdots+x_{2009}} + \cdots + \frac{x_{2009}}{x_1+x_{2+\cdots}+x_{2009}} = 1$. Next, set

$S' = \frac{x_{2009}}{x_{2009}+x_{2008}} + \frac{x_{2008}}{x_{2008}+x_{2007}} + \frac{x_{2007}}{x_{2007}+x_{2006}} + \cdots + \frac{x_1}{x_1+x_{2009}}$. By the same reasoning, $S' > 1$.

Note that $S + S' = 2009$. We claim that $S < 2008$. Suppose that $S \geq 2008$, then we must have $2009 = S + S' > 2008 + 1 = 2009$, which is absurd. Hence our claim that $S < 2008$ is true.

We shall next show that M is the least possible bound for S. Consider the numbers $x_i = a^i$, where $I = 1, 2, 3, \ldots, 2009$. Direct computation yields $S = \frac{2008}{a+1} + \frac{a^{2008}}{a^{2008}+1}$. When a is chosen to be arbitrarily close to 0, this expression gets arbitrarily close to 2008.

21. Answer: 56

The number of selections such that no two of the numbers are consecutive = The number of binary sequences containing 6 '1's and 34 '0's = $\binom{40}{6}$.

The number of selections such that exactly two of the numbers are consecutive = The number of binary sequences containing a '11', 4 '1's and 35 '0's = $\binom{5}{1}\binom{40}{5}$.

The number of selections with exactly two sets of two consecutive numbers but no three numbers are consecutive = The number of binary sequences containing 2 '11's, 2 '1's and 36 '0's = $\binom{4}{2}\binom{40}{4}$.

The number of selections with exactly three sets of two consecutive numbers but no three numbers are consecutive = The number of binary sequences containing 3 '11's and 37 '0's = $\binom{40}{3}$.

Thus, $1000p = 1000\left[\binom{45}{6} - \binom{40}{6} - \binom{5}{1}\binom{40}{5} - \binom{4}{2}\binom{40}{4} - \binom{40}{3}\right] \div \binom{45}{6} = 56.28..$

22. Answer: 1

$$\sum_{k=0}^{\infty}\frac{2}{\pi}\tan^{-1}\left(\frac{2}{(2k+1)^2}\right)$$

$$=\lim_{n\to\infty}\sum_{k=0}^{n}\frac{2}{\pi}\tan^{-1}\left(\frac{2}{(2k+1)^2}\right)$$

$$=\lim_{n\to\infty}\sum_{k=0}^{n}\frac{2}{\pi}\tan^{-1}\left(\frac{2k+2-2k}{1+(2k)(2k+2)}\right)$$

$$=\lim_{n\to\infty}\sum_{k=0}^{n}\frac{2}{\pi}\left(\tan^{-1}(2k+2)-\tan^{-1}(2k)\right)$$

$$=\lim_{n\to\infty}\frac{2}{\pi}\tan^{-1}(2n+2)=1.$$

23. Answer: 263

We consider the sum $\sum_{k=1}^{11}\left(6k^5+2k^3\right)$ and observe that

$$\sum_{k=1}^{11}\left(6k^5+2k^3\right)$$

$$=\sum_{k=1}^{11}\left\{k^3(k+1)^3-k^3(k-1)^3\right\}$$

$$=11^3\times 12^3.$$

Hence

$$\sum_{k=1}^{11}k^5$$

$$=\frac{11^3\times 12^3}{6}-\frac{2}{6}\sum_{k=1}^{11}k^3$$

$$=2\times 11^3\times 12^2-\frac{1}{3}\left(\frac{11^2\times 12^2}{4}\right)$$

$$=11^2\times 12\times\left(2\times 11\times 12-1\right)$$

$$=11^2\times 12\times 263.$$

Finally, since $\sqrt{256}=16$ and the numbers 2, 3, 5, 7, 11 and 13 do not divide 263, we conclude that 263 is the largest prime factor of $\sum_{k=1}^{11}k^5$.

24. Answer: 15

For each positive integer n, we have

$$x_{n+2} - x_{n+1} = -\frac{3}{4}\left(x_{n+1} - x_n\right)$$

and so we have $x_{n+1} - x_n = \left(-\frac{3}{4}\right)^{n-1}\left(x_2 - x_1\right) = 21\left(-\frac{3}{4}\right)^{n-1}$.

Therefore, $x_n = x_1 + 21\sum_{k=1}^{n-1}\left(-\frac{3}{4}\right)^{k-1}$ for all positive integers $n \geq 3$. --- (1)

Now, we let $n \to \infty$ in (1) to conclude that $\lim_{n\to\infty} x_n = x_1 + 21\sum_{k=1}^{\infty}\left(-\frac{3}{4}\right)^{k-1} = 3 + \frac{21}{1+0.75} = 15.$

25. Answer: 14413

The number of rectangles (including squares)

= (20 + 19 + 18 +…+2 + 1)(30 + 29 + 28 +…..+1) x 2 – (20 + 19 + 18 +….. + 1)^2

= 151200.

The number of squares

= (20 x 30 + 19 x 29 + 18 x 28 + ….. + 1 x 11) x 2 – (20^2 + 19^2 + 18^2 + …. + 1^2)

= 7070.

N = the number of rectangles less all squares = 151200 – 7070 = 144130.

Hence N / 10 = 14413.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Open Section, Round 2)

Saturday, 4 July 2009 **0900-1330**

INSTRUCTIONS TO CONTESTANTS

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

1. Let O be the center of the circle inscribed in a rhombus $ABCD$. Points E, F, G, H are chosen on sides AB, BC, CD and DA respectively so that EF and GH are tangent to the inscribed circle. Show that EH and FG are parallel.

2. A palindromic number is a number which is unchanged when the order of its digits is reversed. Prove that the arithmetic progression 18, 37, ... contains infinitely many palindromic numbers.

3. For k a positive integer, define A_n for $n = 1, 2, \ldots,$ by

$$A_{n+1} = \frac{nA_n + 2(n+1)^{2k}}{n+2}, \quad A_1 = 1.$$

Prove that A_n is an integer for all $n \geq 1$, and A_n is odd if and only if $n \equiv 1$ or $2 \pmod 4$.

4. Find the largest constant C such that

$$\sum_{i=1}^{4}(x_i + \frac{1}{x_i})^3 \geq C$$

for all positive real numbers $x_1, \cdots, x_4$ such that

$$x_1^3 + x_3^3 + 3x_1x_3 = x_2 + x_4 = 1.$$

5. Find all integers x, y and z with $2 \leq x \leq y \leq z$ such that

$$xy \equiv 1 \pmod z, \quad xz \equiv 1 \pmod y, \quad yz \equiv 1 \pmod x.$$

Singapore Mathematical Society

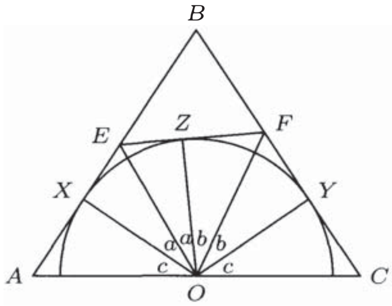
Singapore Mathematical Olympiad (SMO) 2009

(Open Section, Round 2 solutions)

1. The figure shows half of the rhombus (which is an isosceles triangle), where X, Y, Z are points of tangency of the circle to the sides AB, CB and EF respectively. Note that

$$\angle XOE = \angle EOZ, \quad \angle ZOF = \angle FOY, \quad \angle AOX = \angle COY.$$

In particular, $a + b + c = 90°$.



Thus

$$\angle AEO = 90° - a = b + c = \angle COF.$$

Hence the triangles AOE and CFO are similar. It follows that $AE \cdot CF = AO^2$. Similarly, on the lower half of the rhombus, $AO^2 = AH \cdot CG$. Then $AE/AH = CG/CF$ and hence the triangles AEH and CGF are similar. Thus $\angle AEH = \angle CGF$. Since AB is parallel to CD, it follows that EH is parallel to FG.

2. Let $a_i = 18 + 19i$. We'll show that there are infinitely many i such that a_i consists of only the digit 1, i.e.

$$a_i = 18 + 19i = \frac{10^k - 1}{9}.$$

This yields $10^k \equiv 11 \pmod{19}$. Thus any positive integer of the form $\frac{10^k-1}{9}$, where $10^k \equiv 11 \pmod{19}$ is in the AP. Since $10^6 \equiv 11$ and $10^{18} \equiv 1 \pmod{19}$, we have $10^{18t+6} \equiv 11 \pmod{19}$ for any t. Thus there are infinitely many palindromic numbers.

3. We have

$$(n + 2)A_{n+1} - nA_n = 2(n + 1)^{2k}$$
$$(n + 1)A_n - (n - 1)A_{n-1} = 2(n)^{2k}$$

From these we get

$$(n+1)(n+2)A_{n+1} - n(n+1)A_n = 2(n+1)^{2k+1}$$
$$n(n+1)A_n - (n-1)nA_{n-1} = 2(n)^{2k+1}$$
$$(n+1)(n+2)A_{n+1} - (n-1)nA_{n-1} = 2(n+1)^{2k+1} + 2(n)^{2k+1}$$

Using this recurrence, we obtain

$$A_n = \frac{2S(n)}{n(n+1)} \quad \text{where} \quad S(n) = 1^t + 2^t + \cdots + n^t,\ t = 2k+1.$$

Since

$$2S(n) = \sum_{i=0}^{n} \left((n-i)^t + i^t\right) = \sum_{i=1}^{n} \left((n+1-i)^t + i^t\right)$$

we see that $n(n+1) \mid 2S(n)$. Thus A_n is an integer for all n.

(i) $n \equiv 1$ or $2 \pmod 4$. Then $S(n)$ is odd since it has an odd number of odd terms. Thus A_n is odd.

(ii) $n \equiv 0 \pmod 4$. Then $(n/2)^t \equiv 0 \pmod n$. Thus

$$S(n) = \sum_{i=0}^{n/2} \left((n-i)^t + i^t\right) - \left(\frac{n}{2}\right)^t \equiv 0 \pmod n.$$

Thus A_n is even.

(iii) $n \equiv 3 \pmod 4$. Then $((n+1)/2)^t \equiv 0 \pmod{n+1}$. Thus

$$S(n) = \sum_{i=1}^{(n+1)/2} \left((n+1-i)^t + i^t\right) - \left(\frac{n+1}{2}\right)^t \equiv 0 \pmod{n+1}.$$

Thus A_n is even.

4. First note that

$$\begin{aligned} x_1^3 + x_3^3 + 3x_1x_3 - 1 &= x_1^3 + x_3^3 - (1)^3 - 3x_1x_3(-1) \\ &= (x_1 + x_3 - 1)\left((x_1+x_3)^2 + (x_1+x_3) + 1\right) - 3x_1x_3(x_1+x_3-1) \\ &= (x_1+x_3-1)[(x_1-x_3)^2 + (x_1+1)(x_3+1)]. \end{aligned}$$

It is equal to zero only when either $x_1 + x_3 = 1$ or $x_1 = x_3 = -1$. Thus we must have $x_1 + x_3 = 1$ as they are positive. It now suffixes to show that the following is sharp:

$$\sum_{i=1}^{2} \left(y_i + \frac{1}{y_i}\right)^3 \geq 125/4 \ \text{ when } y_1 + y_2 = 1 \ \text{ and } y_1, y_2 > 0.$$

To this end, it is clear that the function $f(x) = (x + 1/x)^3$ is convex. Thus,

$$f(x) + f(1 - x) \geq 2f(1/2) = 125/4.$$

5. The only solution is $(x, y, z) = (2, 3, 5)$.

First of all, observe that $(x, y) = (x, z) = (y, z) = 1$. Then $2 \leq x < y < z$, and combining the three given congruences we can express it as

$$xy + xz + yz - 1 \equiv 0 \pmod{x,\ y,\ \ z}.$$

Since x, y and z are pairwise coprime, we have

$$xy + xz + yz - 1 \equiv 0 \pmod{xyz}.$$

It follows that $xy + xz + yz - 1 = k(xyz)$ for some integer $k \geq 1$. Dividing by xyz, we obtain that

$$\frac{1}{z} + \frac{1}{y} + \frac{1}{x} = \frac{1}{xyz} + k > 1.$$

Since $x < y < z$, it follows that

$$1 < \frac{1}{x} + \frac{1}{y} + \frac{1}{z} < \frac{3}{x}$$

and this gives $x = 2$ as the only value. In this case, the inequalities give

$$\frac{1}{2} < \frac{1}{y} + \frac{1}{z} < \frac{2}{y},$$

which implies that $y = 3$. It follows that the only possible values of z are 4 and 5. Hence, for $2 \leq x < y < z$, the solutions are $(x, y, z) = (2, 3, 4)$ and $(2, 3, 5)$. Since 2 and 4 are not relatively prime, the only solution is $(x, y, z) = (2, 3, 5)$.