Exploring University Mathematics in Depth and Rigour

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Preface

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1 Introduction

1.1 Symbols and Notation

This section gives some of the standard mathematical symbols which you are likely to meet in your course.

1.1.1 Greek letters

There are certain conventions which symbols are used for which purpose in mathematics, for example x and y usually denote real variables; a, b and c usually denote real constants; i, j, k, m and n usually denote integers. Consequently, we soon exhaust our usual alphabet and we have to turn to Greek. The following lists those which are used in mathematics (two lower case letters, omicron and upsilon, and several upper case letters, are rarely used because of their similarity to other symbols).

Lower case

α	alpha	ι	iota	ρ	rho
β	beta	κ	kappa	σ	sigma
γ	gamma	λ	lambda	τ	tau
δ	delta	μ	mu	ϕ	phi
ϵ	epsilon	ν	nu	χ	chi
ζ	zeta	ξ	xi	ψ	psi
η	eta	π	pi	ω	omega
θ	theta				

Upper case

Γ	Gamma	Ξ	Xi	Φ	Phi
Δ	Delta	Π	Pi	Ψ	Psi
Θ	Theta	Σ	Sigma	Ω	Omega
Λ	Lambda				

1.1.2 Set Notation

- $\in$ an element of
- $\notin$ not an element of
- $\subseteq$ a subset of
- $\subset$ a proper subset of
- $\cup$ union of sets, defined as

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

- $\cap$ intersection of sets, defined as

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

- $\varnothing$ empty set

The following are standard ways of denoting some special sets of numbers.

$\mathbb{C}$	Set of complex numbers
$\mathbb{N}$	Set of positive integers
$\mathbb{Q}$	Set of rational numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{Z}$	Set of integers
(a, b)	Set of all real numbers x where $a < x \leq b$
$[a, b]$	Set of all real numbers x where $a \leq x \leq b$

1.1.3 Logic Notation

Zero-order logic

- $\wedge$ and

- $\vee$ or
- $\neg$ not

These three symbols are related via de Morgan:

$$\neg(A \vee B) \text{ iff } \neg A \wedge \neg B$$

We have a very common logical symbol implies " $\implies$ ", which is defined by

$$A \implies B \text{ iff } -A \vee B$$

First-order logic

- $\forall$ for all
- $\exists$ there exists

forall x P(x) iff exists x -P(x) exists x P(x) iff forall x -P(x)

1.2 Proofs

Apart from understanding the statements of abstract theorems, you should become used to proving them rigorously. The most important theorems will be proved in books and lectures, but you may be asked to reproduce the proofs in examinations. Moreover, you will frequently be set problems which require you to prove abstract statements which you have not seen before. In all cases, the aim will be to write out accurate and efficient proofs. There are three basic skills concerning proofs:

(i) identifying the essential points of a standard proof,

(ii) constructing proofs of facts which were previously unknown to you,

(iii) setting out proofs accurately, clearly, and efficiently.

Part I
Analysis

2
Logic and Sets

2.1 Supremum and Infimum

Set A is **bounded from above** if it has an **upper bound** M:

$$\exists M \in \mathbb{R} \text{ s.t. } \forall x \in A, x \leq M$$

If M is an upper bound for S and $M \in S$, then M is the **maximum** of S, denoted as $M = \max S$.

Set A is **bounded from below** if it has a **lower bound** m:

$$\exists m \in \mathbb{R} \text{ s.t. } \forall x \in A, x \leq m$$

Set A is **bounded** in the real numbers if it is bounded above and below.

The **supremum** of a set A, denoted $\sup A$, is defined as the smallest real number M such that $\forall x \in A, x \leq M$.

The **infimum** of a set A, denoted $\inf A$, is defined as the largest real number m such that $\forall x \in A, x \geq m$.

If the supremum or infimum belongs to the set, then we call them the maximum or minimum value of the set.

If we are dealing with rational numbers, the sup/inf of a set may not exist. For example, a set of numbers in $\mathbb{Q}$, defined by $\{[\pi \cdot 10^n]/10^n\}$. 3,3.1,3.14,3.141,3.1415,3.14159,... But this set does not have an infimum in $\mathbb{Q}$

This will not happen in the reals, in fact we have the following theorem:

Theorem 2.1.1

Any set in the reals bounded from above/below must have a supremum/infimum.

Proof. We prove this using Dedekind cuts.

Let S be a real number set. We consider the rational number set $A = \{x \in \mathbb{Q} \mid \exists y \in S\}$. Set B is defined to be the complement of A in $\mathbb{Q}$.

We go through the definitions to check that $(A|B)$ is a Dedekind cut.

1. Since $S \neq \varnothing$, pick $y \in S$, then $[y] - 1$ is a real number smaller than some element in S, hence $[y] - 1 \in A$ and thus $A \neq \varnothing$.

   Since we're given that S is bounded, $\exists M > 0$ as the upper bound for S, thus $B \neq \varnothing$.

   (Note that an upper bound is simply a number that is bigger than anything from the set, and is not the supremum

2. We defined B to be the complement of A in $\mathbb{Q}$, so this condition is trivial.

3. For any $x, y \in A$, if $x < y$ and $y \in A$, then $\exists z \in S$ such that $y < z \implies x < z \implies x \in A$.

4. Suppose otherwise that $x \in A$ is the largest element in A, then $\exists y \in S$ such that $x < y$ We then pick a rational number z between x and y. Since we still have $z < y$, we have $z \in A$ but $z > x$, contradictory to z being the largest.

   Now there's actually an issue with the proof for property 4 here How exactly are we finding z?

   First $x \in \mathbb{Q}$. Then $y \in \mathbb{R}$ so we rewrite it as $y = (C|D)$ via definition.

   $x < y$ translates to the fact that $x \in C$.

   Since y is real, by definition we know that C must not have a largest element.

   In particular, x is not largest and we can pick $z \in C$ such that $z > x$. This is in fact the z that we need

Now that all the properties of a real number are validated, we may finally conclude that $\alpha = (A|B)$ is indeed a real number.

Now we need to show that $\alpha = \sup S$.

Let $x \in S$. If x is not the maximum value of S, i.e. $\exists y \in S, x < y$, then $x \in A$ and thus $x < \alpha$.

If x is the maximum value of S, then for any rational number $y < x$ we have $y \in A$, and for any rational number $y \geq x$ we have $y \in B$. Thus $x = (A|B) = \alpha$.

In conclusion, $x \leq \alpha$ for all $x \in S$.

For any upper bound x of S, since $\forall y \in S, x \geq y$ we have $x \in B$ and thus $x \geq \alpha$.

$\therefore$ α is the smallest upper bound of S and thus $\sup S = \alpha$ exists. □

Problem 1. Find, with proof, the supremum and/or infimum of 1/n.

Proof. 1. sup1/n = max1/n = 1 inf1/n=0 as for all positive a, we can pick n=[1/a]+1, then a¿1/n □

Problem 2. Find, with proof, the supremum and/or infimum of sin n.

Proof. The answer is easy to guess: ±1

For the supremum, we need to show that 1 is the smallest we can pick, so for any $a = 1 - \epsilon < 1$ we want to find an integer n close enough to $2k\pi + \pi/2$ so that $\sin n > a$.

Whenever we want to show the approximations between rational and irrational numbers we should think of the pigeonhole principle.

$$2k\pi + \frac{\pi}{2} = 6k + (2\pi - 6)k + \frac{\pi}{2}$$

Consider the set of fractional parts $\{(2\pi - 6)k\}$. Since this an infinite set, for any small number δ there is always two elements $\{(2\pi - 6)a\} < \{(2\pi - 6)b\}$ such that

$$|\{(2\pi - 6)b\} - \{(2\pi - 6)a\}| < \epsilon$$

Then $\{(2\pi - 6)(b - a)\} < \delta$

We then multiply by some number m (basically adding one by one) so that

$$0 \leq \{(2\pi - 6) \cdot m(b - a)\} - \left(2 - \frac{\pi}{2}\right) < \delta$$

Picking $k = m(b - a)$ thus gives

$$\begin{aligned} 2k\pi + \pi/2 &= 6k + (2\pi - 6)k + \pi/2 \\ &= 6k + [(2\pi - 6)k] + 2 + (2\pi - 6)k - (2 - \pi/2) \end{aligned}$$

Thus $n = 6k + [(2\pi - 6)k] + 2$ satisfies $|2k\pi + \pi/2 - n| < \delta$

Now we're not exactly done here because we still need to talk about how well $\sin n$ approximates to 1.

We need one trigonometric fact: $\sin x < x$ for $x > 0$. This simply states that the area of a sector in the unit circle is larger than the triangle determined by its endpoints.

$$\begin{aligned}
\sin n &= sin\left(n - \left(2k\pi + \frac{\pi}{2}\right) + \left(2k\pi + \frac{\pi}{2}\right)\right) \\
&= \cos\left(n - \left(2k\pi + \frac{\pi}{2}\right)\right) \\
&= \cos\theta
\end{aligned}$$

$$1 - \sin n = 2\sin^2\frac{\theta}{2} = 2\sin^2\left|\frac{\theta}{2}\right| \leq \frac{\theta^2}{2} < \delta$$

Hence we simply pick $\delta = \epsilon$ to ensure that $1 - \sin n < \epsilon$, and we're done. □

Problem 3. Find, with proof, the supremum and/or infimum of

$$\left\{\frac{[n\alpha]}{n}\right\}$$

where $[x]$ is the Gauss function.

Proof. □

2.2 Injectivity, Surjectivity, Bijectivity

Let $f : X \to Y$ be a function.

Definition 2.2.1: Injectivity

The function f is injective if each element of the codomain Y has at most one element of the domain X that maps to it.

$$\forall x, x' \in X(f(x) = f(x') \implies x = x') \tag{2.1}$$

Lemma 2.2.1. If $f : X \to Y$ is injective and $g : Y \to Z$ is injective, then $g \circ f : X \to Z$ is injective.

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be arbitrary injective functions. We want prove that the function $g \circ f : X \to Z$ is also injective.

To do so, we will prove $\forall x, x' \in X$ that

$$(g \circ f)(x) = (g \circ f)(x') \implies x = x'$$

Suppose that $(g \circ f)(x) = (g \circ f)(x')$. Expanding out the definition of $g \circ f$, this means that $g(f(x)) = g(f(x'))$.

Since g is injective and $g(f(x)) = g(f(x'))$, we know $f(x) = f(x')$.

Similarly, since f is injective and $f(x) = f(x')$, we know that $x = x'$, as required. □

Definition 2.2.2: Surjectivity

The function f is surjective if *every* element of the codomain Y is mapped to at least one element of the domain X.

$$\forall y \in Y \exists x \in X \text{ s.t. } f(x) = y \tag{2.2}$$

Lemma 2.2.2. If $f : X \to Y$ is surjective and $g : Y \to Z$ is surjective, then $g \circ f : X \to Z$ is surjective.

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be arbitrary surjective functions. We want to prove that the function $g \circ f : X \to Z$ is subjective.

To do so, we want to prove that for any $z \in Z$, there is some $x \in X$ such that $(g \circ f)(x) = z$. Equivalently, we want to prove that for any $z \in Z$, there is some $x \in X$ such that $g(f(x)) = z$.

Consider any $z \in Z$. Since $g : Y \to Z$ is surjective, there is some $y \in Y$ such that $g(y) = z$. Similarly, since $f : X \to Y$ is surjective, there is some $x \in X$ such that $f(x) = y$. This means that there is some $x \in X$ such that $g(f(x)) = g(y) = z$, as required. □

Definition 2.2.3: Bijectivity

The function f is bijective if it is both injective and surjective: each element of the codomain Y is mapped to a unique element of the domain X.

A function $f : \mathbb{R} \to \mathbb{R}$ on real line is a special function. f is

- injective iff any horizontal line intersects at at most one point,
- surjective iff any horizontal line intersects at at least one point, and
- bijective iff any horizontal line intersects at exactly one point.

2.3 Inverses

Let $f : X \to Y$ be a function. $f^{-1} : Y \to X$ is the **inverse** of f if

$$\forall x \in X, y \in Y (f(x) = y \iff f^{-1}(y) = x) \tag{2.3}$$

In other words, if f maps x to y, then f^{-1} maps y back to x.

Lemma 2.3.1. Let $f : X \to Y$ be invertible and let $f^{-1} : Y \to X$ be its inverse. Then for any $x \in X$ and for any $y \in Y$, we have

$$f^{-1}(f(x)) = x \text{ and } f(f^{-1}(y)) = y.$$

Proof. First, consider any $x \in X$, we want to prove that $f^{-1}(f(x)) = x$. To see this, let $y = f(x)$. Since $f(x) = y$ and f^{-1} is the inverse, we have

$$f^{-1}(y) = x. \tag{1}$$

Substituting $y = f(x)$ into equation (1) tells us $f^{-1}(f(x)) = x$, as required.

Next, consider any $y \in Y$. We will prove that $f(f^{-1}(y)) = y$. To see this, let $x = f^{-1}(y)$. Since f^{-1} is the inverse of f, this means that

$$f(x) = y. \tag{2}$$

Plugging $x = f^{-1}(y)$ into equation (2) tells us $f(f^{-1}(y)) = y$, as required. □

Lemma 2.3.2. Let $f : X \to Y$. Then f is invertible if and only if f is a bijection.

To prove this result, we need to prove that:

- if $f : X \to Y$ is invertible, then f is a bijection, and
- if $f : X \to Y$ is a bijection, then f is invertible.

Proof. □

2.4 Cartesian Product

Definition 2.4.1: C

rtesian product of two sets A and B is defined as

$$A \times B = \{(a, b) \mid a \in A \land b \in B\} \tag{2.4}$$

For example,

$$\left\{0,\quad 1,\quad 2\right\} \times \left\{a,\quad b,\quad c\right\} = \begin{Bmatrix} (0,a), & (0,b), & (0,c) \\ (1,a), & (1,b), & (1,c) \\ (2,a), & (2,b), & (2,c) \end{Bmatrix}$$

We denote $A^2 = A \times A$. For example,

$$\left\{0,\quad 1,\quad 2\right\}^2 = \begin{Bmatrix} (0,0), & (0,1), & (0,2) \\ (1,0), & (1,1), & (1,2) \\ (2,0), & (2,1), & (2,2) \end{Bmatrix}$$

3 ZFC, Ordinals, and Cardinals

3.1 ZFC Axioms

Definition 3.1.1: Set

A **set** is a collection of objects satisfying a certain set of axioms. (These axioms are stated below.) Each object in the set is called an **element** of the set.

Remark. The **membership property** is the most basic set-theoretic property. We denote it by $\in$. Thus we read $X \in Y$ as "X is an element of Y".

Since the axioms form our definition of a set, we need an axiom to postulate that sets indeed do exist. More specifically, that at least one set exists.

Theorem 3.1.1: Axiom of Existence

There exists a set which has no elements.

Now that we have established that at least one set exists, we need a way to show uniqueness of sets. Intuitively, there should only be one set that has no elements, but we need the next axiom to prove this.

Theorem 3.1.2: Axiom of Extensionality

If every element of X is an element of Y and every element of Y is an element of X, then $X = Y$.

From the Axiom of Extensionality, we see that $X = Y$ is a property based on the elements contained in X and Y. To generalize, if two sets have the same elements, then they are identical. We can now set out to prove the uniqueness of the set with no elements.

Lemma 3.1.1. There exists only one set with no elements.

Proof. Suppose there exists two sets A and B which both have no elements.

If $x \in A$ then $x \in B$.

If $y \in B$ then $y \in A$.

Therefore by the Axiom of Extensionality, $A = B$.

($x \in A$ is a false antecedent and so "$x \in A \implies x \in B$" is automatically true. The same is also true for $y \in B$.) □

Definition 3.1.2: T

e unique set with no elements is called the **empty set** and is denoted by $\varnothing$.

Now that we have established that a unique set exists, we are naturally interested in the existence and uniqueness of other sets.

Theorem 3.1.3: Axiom Schema of Comprehension

Let $P(x)$ be a property of x. For any set A, there exists a set B such that $x \in B$ if and only if $x \in A$ and $P(x)$ holds.

Lemma 3.1.2. For every set A, there is a unique set B such that $x \in B$ if and only if $x \in A$ and $P(x)$.

Proof. Suppose B' is another set such that $x \in B'$ if and only if $x \in A$ and $P(x)$.

If $x \in B$ implies $x \in A$ and $P(x)$, then $x \in B'$.

If $x \in B'$ implies $x \in A$ and $P(x)$, then $x \in B$.

Thus we have $x \in B$ if and only if $x \in B'$.

Therefore $B = B'$. □

Theorem 3.1.4: Axiom of Pair

For any sets A and B, there exists a set C such that $x \in C$ if and only if $x = A$ or $x = B$.

Definition 3.1.3: W

define the **unordered pair** of A and B as the set having exactly A and B as its elements and denote it by $\{A, B\}$.

Theorem 3.1.5: Axiom of Union

For any set S, there exists a set U such that $x \in U$ if and only if $x \in A$ for some $A \in S$.

Definition 3.1.4: W

call the set U the **union** of S and denote it by $\bigcup S$.

Definition 3.1.5: W

call A a **subset** of B if every element of A belongs to B. We denote this by $A \subseteq B$.

Theorem 3.1.6: Axiom of Power Set

For any S, there exists P such that $X \in P$ if and only if $X \subseteq S$.

Definition 3.1.6: W

call P the **power set** of S and denote it by $\mathscr{P}(S)$.

Theorem 3.1.7: Axiom of Infinity

An inductive set exists.

We will revisit the Axiom of Infinity in more depth. Inductive sets will be defined later in the book. They are crucial in defining the set of natural numbers.

Theorem 3.1.8: Axiom Schema of Replacement

Let $P(x, y)$ be a property such that for every x there is a unique y for which $P(x, y)$ holds. For every A there exists B such that for every $x \in A$, there is $y \in B$ for which $P(x, y)$ holds.

Definition 3.1.7: T

e **union** of A and B is the set of all x which belong in either A, B, or both, denoted by $A \cup B$.

Remark. $A \cup B$ exists by our system of Axioms.

By Axiom of Pair, we have $\{A, B\}$. Apply Axiom of Union on $\{A, B\}$ to arrive at $A \cup B$.

Definition 3.1.8: T

e **intersection** of A and B is the set of all x which belong to both A and B, denoted by $A \cap B$.

Remark. $A \cap B$ also exists by our system of Axioms.

We can apply Axiom Schema of Comprehension to set A and the property $P(x) : x \in B$. It is easy to show that $A \cap B = \{x \in A \,|\, x \in B\}$.

Definition 3.1.9: T

e **difference** of A and B is the set of all $x \in A$ such that $x \notin B$, denoted by $A - B$.

Remark. We can apply the Axiom Schema of Comprehension to set A and the property $P(x) : x \notin B$ to arrive at $A - B = \{x \in A \,|\, x \notin B\}$.

3.2 Relations and Functions

4

Equivalence Classes and Quotient Sets

Product space

1. Suppose that A and B are two arbitrary sets, then a **relation** R between A and B is a subset of the product space $A \times B$, i.e.

$$R \subset A \times B.$$

   Visually speaking, it is uniquely determined by a simple bipartite graph over A and B.

   $a \in A$ and $b \in B$ are said to be **related** if $(a, b) \in R$, which is also denoted as $a \sim b$.

   On the bipartite graph, this is represented by an edge between a and b.

2. A **binary relation** in a set A is a relation between A and itself, i.e.

$$R \subset A \times A.$$

   Properties:

   - reflexive: $(a, a) \in R \; \forall a \in A$
   - transitive: $(a, b) \in R \wedge (b, c) \in R \implies (a, c) \in R$
   - symmetric: $(a, b) \in R \implies (b, a) \in R$
   - anti-symmetric: $(a, b) \in R \implies (b, a) \notin R \; \forall a \neq b$

3. An **equivalence relation** is a binary relation that satisfies the following properties:

- reflexive: $a \sim a$
- symmetric: if $a \sim b$, then $b \sim a$
- transitive: if $a \sim b$ and $b \sim c$, then $a \sim c$

4. Let A be a set and $a \in A$, then the **equivalence class** of a is defined as by the subset of A whose elements are equivalent to a, i.e.

$$[a] = \{x \in A \mid x \sim a\}$$

Properties of equivalence classes:

- Every two equivalence classes are disjoint
- The union of equivalence classes form the entire set

You can translate these properties into the point of view from the elements: Every element belongs to one and only one equivalence class as

- No element belongs to two distinct classes
- All elements belong to an equivalence class

5. The **set of equivalence classes** (quotient sets) are the set of all equivalence classes, denoted by $A/\sim$.

One very important kind of question is whether these constructions are well-defined.

So, what's the purpose of this question? Well it turns out that definitions over equivalence classes often leaves gaps.

We often define such concepts by considering what's commonly referred to as representatives of the equivalence classes

Problem 4. (Modular Arithmetic) Define the ring of integers modulo n:

$$\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/\sim \text{ where } x \sim y \iff x - y \in n\mathbb{Z}.$$

The equivalence classes are called congruence classes modulo n.

(a) Define the sum of two congruence classes modulo n, $[x], [y] \in \mathbb{Z}/n\mathbb{Z}$, by
$$[x] + [y] = [x + y]$$
Show that the above definition is well-defined.

(b) Define the product of two congruence classes modulo n and show that such a definition is well-defined.

Solution.
(a) We often define such concepts by considering what's commonly referred to as representatives of the equivalence classes

For example, here we define [x]+[y]=[x+y] for two elements [x] and [y] in Z/nZ

So what we know here are the classes [x] and [y]

But what exactly is x and y? It turns out that they are just some element in the equivalence classes that was arbitrarily picked out

We then perform the sum x+y, and consequently, we used this to point towards the class [x+y]

However, x and y are arbitrarily picked.

We want to show that, regardless of which representatives are chosen from the equivalence classes $[x]$ and $[y]$, we will always obtain the same result.

In the definition itself, we have defined that, for the two representatives x and y we define [x]+[y]=[x+y]

So now, let's say that we take two other arbitrary representatives, $x' \in [x]$ and $y' \in [y]$

Then by definition, we have

$$[x] + [y] = [x' + y']$$

Thus, our goal is to show that

$$[x' + y'] = [x + y]$$

Now what is the precise meaning of this expression? It is to say that the two sides of the equation are referring to the same equivalence class.

Therefore, the expression above is completely equivalent to the condition

$$x' + y' \sim x + y$$

We then check that this final expression is indeed true: Since $x' \in [x]$ and $y' \in [y]$, we have

$$\begin{aligned}
&x' \sim x \text{ and } y' \sim y \\
&\implies x' - x, y' - y \in n\mathbb{Z} \\
&\implies (x' + y') - (x + y) = (x' - x) + (y' - y) \in n\mathbb{Z}
\end{aligned}$$

(b) The product of two congruence classes is defined by

$$[x][y] = [xy]$$

For any other representatives x', y' we have

$$\begin{aligned}
&x'y' - xy \\
&= x'y' - xy' + xy' - xy \\
&= (x' - x)y' + x(y' - y) \in n\mathbb{Z}
\end{aligned}$$

Thus $[x'y'] = [xy]$ and the product is well-defined $\square$

Problem 5. Let $A = \mathbb{R}$ and for any $x, y \in A$, $x \sim y$ iff $x - y \in \mathbb{Z}$. For any two equivalence classes $[x], [y] \in A/\sim$, define

$$[x] + [y] = [x + y] \text{ and } -[x] = [-x]$$

(a) Show that the above definitions are well-defined.

(b) Find a one-to-one correspondence $\phi : X \to Y$ between $X = A/\sim$ and $Y : |z| = 1$, i.e. the unit circle in $\mathbb{C}$, such that for any $[x_1], [x_2] \in X$ we have

$$\phi([x_1])\phi([x_2]) = \phi([x_1 + x_2])$$

(c) Show that for any $[x] \in X$,

$$\phi(-[x]) = \phi([x])^{-1}$$

Solution.
(a)

$$(x' + y') - (x + y) = (x' - x) + (y' - y) \in \mathbb{Z}$$

Thus $[x' + y'] = [x + y]$

$$(-x') - (-x) = -(x' - x) \in \mathbb{Z}$$

Thus $[-x'] = [-x]$.

(b) Complex numbers in the polar form: $z = re^{i\theta}$

Then the correspondence is given by $\phi([x]) = e^{2\pi i x}$

$$[x] = [y] \iff x - y \in \mathbb{Z} \iff e^{2\pi i(x-y)} = 1 \iff e^{2\pi i x} = e^{2\pi i y}$$

Hence this is a bijection.

Before that, we also need to show that ϕ is well-defined, which is almost the same as the above.

If we choose another representative x' then

$$\phi([x]) = e^{2\pi i x'} = e^{2\pi i x} \cdot e^{2\pi i(x'-x)} = e^{2\pi i x}$$

(c) You can either refer to the specific correspondence $\phi([x]) = e^{2\pi i x}$ or use its properties.

$$\phi(-[x])\phi([x]) = \phi([-x])\phi([x]) = \phi([-x + x]) = \phi([0]) = 1$$

□

Problem 6. (Set of Rational Numbers) Let $\mathbb{Z}$ be the set of integers, and let $\mathbb{Z}^*$ be the set of nonzero integers. We define

$$\mathbb{Q} = \{(a,b) \mid a \in \mathbb{Z}, b \in \mathbb{Z}^*\}/\sim$$

where

$$(a,b) \sim (c,d) \iff ad = bc.$$

Let $\frac{a}{b}$ denote the equivalence class for (a,b). Such an equivalence class is called a rational number.

(a) For any two rational numbers $\frac{a}{b}$, $\frac{c}{d}$, their sum is determined by

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

Show that the above definition is well-defined.

(b) Define the product of two rational numbers and show that such a definition is well-defined.

(c) Prove that for every equivalence class $\frac{a}{b} \in \mathbb{Q}$, there exists a unique integer pair (p,q) satisfying the following properties:

$$q > 0, (p,q) = 1 \text{ and } (p,q) \in \frac{a}{b}.$$

(d) Using the partial order of $\mathbb{Z}$, define the partial order of $\mathbb{Q}$.

Solution.
(a) For this problem, we are dealing with a "hidden" equivalence class.

The expressions a/b and c/d themselves are derived from their representatives (a,b) and (c,d)

So suppose that we choose other representatives (a',b') and (c',d'), then the sum would be a'/b'+c'/d'=(a'd'+b'c')/b'd'

We now have to show that (ad+bc)/bd = (a'd'+b'c')/b'd' iff (ad+bc,bd) $\sim (a'd'+b'c', b'd') iff (ad+bc)b'd' = (a'd'+b'c')bd$

a/b=a'/b' (a,b)$\sim (a',b') ab' = a'b$

(ad+bc)b'd' =ab'dd'+bb'cd' =a'bdd'+bb'c'd =(a'd'+b'c')bd

(b) The definition would be a/b $\cdot$ c/d = ac/bd

This is actually a lot simpler to check

a'c'bd=(a'b)(c'd)=(ab')(cd')=acb'd' Hence a'c'/b'd'=ac/bd

(c) We basically try to do this step by step as we would in simplifying fractions

First you can pick b to be positive, otherwise we swap a and b with -a and -b

Then simplify the common factors

For this one we let $(a, b) = d$, and $a = dp, b = dq$

Then (p,q) is the pair that we need

Alright, for the final one we'll first need to talk about partial orders

Given a set A equipped with an equivalence relation =, a **partial order** in A is a binary relation denoted by $\leq$ (with the same notation conventions as of ~) satisfying the following properties:

1. Reflexivity: $a \leq a$
2. Antisymmetry: $a \leq b \wedge b \leq a \implies a = b$
3. Transitivity: $a \leq b \wedge b \leq c \implies a \leq c$

In order to define the partial order we need to account for whether the denominators are negative

a/b $\leq c/d, andifb, d > 0thenwecansafelydrawaconnectiontotheexpressionad \leq bc$

In order to show that this does in fact give a partial order we check that 1) $ab \leq ab$ and hence $a/b \leq a/b$

2) If a/b $\leq c/dandc/d \leq a/b, thenad \leq bcandbc \leq ad, hencead = bcandthusa/b = c/d$

3) is trickier due to complications arising from inequalities and multiplication

If $\frac{a}{b} \leq \frac{c}{d}$ and $\frac{c}{d} \leq \frac{e}{f}$, note that $b, d, f > 0$ and so $ad \leq bc$ and $cf \leq de$. i) $e < 0$, then $c < 0$ and $a < 0$, thus $-ad \geq -bc$, $-cf \geq -de$ and we have $acdf \geq bcde$ $af \leq be(c < 0, d > 0)$ Thus $a/b \leq e/f$

ii) $e \geq 0$ but $a < 0$, then $af < 0 \leq be$ and thus $a/b < e/f$

iii) $a \geq 0$, then $c \geq 0$ and $e \geq 0$, and we have the ordinary case. □

5 The Real and Complex Number Systems

5.1 Rational Numbers

This book assumes familiarity with the **rational numbers** (i.e., the numbers of the form $\frac{m}{n}$, where m, n are integers and $n \neq 0$).

5.2 Dedekind cuts

The reader should already be aware that the set of rational numbers $\mathbb{Q}$ contains *gaps* at irrational numbers such as $\sqrt{2}$ and π. This section serves to construct $\mathbb{R}$ from $\mathbb{Q}$.

In 1872, German mathematician Richard Dedekind pointed out that a real number x can be determined by its lower set A and upper set B:

$$A \coloneqq \{a : \mathbb{Q} \mid a < x\}$$

$$B \coloneqq \{b : \mathbb{Q} \mid x < b\}$$

He defined a “real number” as a pair of sets of rational numbers, the lower and upper sets shown above. Such a pair of sets of rational numbers are known as a **Dedekind cut**.

- A is a **lower set**: $\forall a, b \in \mathbb{R}$, if $a < b$ where $b \in A$, then $a \in A$.
- B is an **upper set**: $\forall a, b \in \mathbb{R}$, if $a < b$ where $a \in B$, then $b \in B$.

Definition 5.2.1: Dedekind cut

A non-empty subset $(A, B) \subset \mathbb{Q}$ is a Dedekind cut if:

1. A, B must both be non-empty:

$$A \neq 0, B \neq 0$$

2. A, B are disjoint:

$$A \cup B = \mathbb{Q}, A \cap B = \varnothing$$

3. (Downwards closed) A is closed downwards:

$$\forall x, y \in \mathbb{Q} \text{ with } x < y, y \in A \implies x \in A$$

4. (No maximal element) A does not contain a greatest element:

$$\forall x \in A, \exists y \in A \text{ s.t. } x < y$$

Definition 5.2.2: Real numbers

The set of real numbers $\mathbb{R}$ is the set of all Dedekind cuts.

5.2.1 Order relations

α and β are real numbers. Let $\alpha = (A, B)$, $\beta = (C, D)$. Then

$$\alpha < \beta \iff A \subset C$$

Note that since B is the complement of A, α is completely determined by A itself.

This ordering on the real numbers satisfies the following properties:

- $x < y$ and $y < z \implies x < z$
- Exactly one of $x < y$, $x = y$ or $x > y$ holds
- $x < y \implies x + z < y + z$

Property 5.2.1. For any two real numbers α and β, one of the following must hold:

$$\alpha < \beta,\ \alpha = \beta,\ \alpha > \beta$$

Proof. The proof is by contradiction.

Now note that $\alpha \leq \beta \iff A \subseteq C$ ($A = C$ is possible).

Suppose otherwise, that all three of the above are false, then neither of the sets A and C can be a subset of the other.

We pick two rational numbers from each set: Pick p where $p \in A$, $p \notin C$, pick q where $q \in C$, $q \notin A$

- Obviously we cannot have $p = q$.
- If $p < q$, then since $q \in C$, according to property 3, we have $p \in C$, a contradiction.
- Similarly for $p > q$, we would find that $q \in A$, a contradiction.

Hence our assumption is false.

$\therefore$ One of the three cases $\alpha < \beta$, $\alpha = \beta$, $\alpha > \beta$ must hold. $\square$

5.2.2 Addition of reals

Definition 5.2.3: L

t $\alpha = (A, B)$, $\beta = (C, D)$, then $\alpha + \beta = (X, Y)$ where

$$X = \{a + c \,|\, a \in A, c \in C\}$$

Proof. To show that (X, Y) is a Dedekind cut, we check the conditions for Dedekind cuts.

- Property 1 is trivial.
- Property 2 is by definition.
- Property 3:
  Let $x, y \in X$ satisfy $x < y$, $y \in X$. Let $y = a + c$, $a \in A$, $c \in C$. Let $\epsilon = y - x$. Let $a' = a - \frac{\epsilon}{2}$, $c' = c - \frac{\epsilon}{2}$. Then

  $$a' + c' = a + c - \epsilon = x$$

  $a' < a, a \in A \implies a' \in A$. Similarly, $c' \in C$.
  $\therefore x = a' + c' \in X$.
- Property 4:
  $\forall a + c \in X, a \in A, c \in C$,
  $\exists a' \in A, c' \in C$ s.t $a < a', c < c'$.
  $\therefore a' + c' \in X$ satisfies $a + c < a' + c'$.

$\square$

Property 5.2.2. Addition is **commutative**:

$$\alpha + \beta = \beta + \alpha$$

Proof. The proof is trivial. $\square$

Property 5.2.3. Addition is **associative**:

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$$

Proof. Let $\alpha = (A, A')$, $\beta = (B, B')$, $\gamma = (C, C')$

$$\beta + \gamma = (B + C, (B + C)')$$

In this notation we only need to show that $A + (B + C) = (A + B) + C$.

$$\begin{aligned}
&x \in A + (B + C) \\
&\text{iff } \exists a \in A, p \in B + C, \text{s.t. } x = a + p \\
&\text{iff } \exists a \in A, b \in B, c \in C, \text{s.t. } x = a + b + c \\
&\text{iff } x \in (A + B) + C
\end{aligned}$$

Hence proven. □

Property 5.2.4. Prove that

$$\alpha + 0 = \alpha = 0 + \alpha$$

Proof. Let $0 = (O, O')$ where $O = \{x < 0\}, O' = \{x \geq 0\}$.
Let $\alpha = (A, B)$, then $\alpha + 0 = (C, D)$ where

$$\begin{aligned}
C &= \{a + \epsilon \,|\, a \in A, \epsilon < 0\} \\
&= \{a - \epsilon \,|\, a \in A, \epsilon > 0\}
\end{aligned}$$

$$a - \epsilon < a, a \in A \implies a - \epsilon \in A \implies C \subseteq A$$

According to Property 4, $\forall a \in A, \exists a' \in A$ s.t. $a < a'$.
Let $\epsilon = a' - a > 0$, then

$$a = a' - \epsilon, a' \in A, \epsilon > 0 \implies a \in C$$

So $A = C$
$\therefore \alpha + 0 = \alpha$ □

Property 5.2.5. Express $-\alpha$ in terms of α; show

$$\alpha + (-\alpha) = 0 = (-\alpha) + \alpha$$

Proof. We split this into two cases.

Case 1: α is a rational number, then $\alpha = (A, B)$ where we simply have $A = \{x < \alpha\}$, $B = \{x \geq \alpha\}$.

Let $-\alpha = (A', B')$, where $A' = \{x < -\alpha\}$, $B' = \{x \geq -\alpha\}$. We see that $\alpha + (-\alpha) \leq 0$ is obvious.

On the other hand, since $0 = (O, O')$, for any $\epsilon < 0$ we have

$$\epsilon = \left(\alpha + \frac{\epsilon}{2}\right) + \left(-\alpha + \frac{\epsilon}{2}\right) \in A + A'$$

Hence $\alpha + (-\alpha) = 0$.

Case 2: α is irrational, let $\alpha = (A, B)$ where B does not have a lowest value. Then $-B = \{-x \mid x \in B\}$ does not have a highest value.

We wish to define $-\alpha = (-B, -A)$, but first we need to show that this is well-defined.

- Property 1 is trivial.
- Property 2: Prove that $-A$ and B are disjoint.

  Note that $\forall x \in \mathbb{R}$, if $x = -y$, then exactly one out of $y \in A$ and $y \in B$ is true $\implies$ exactly one out of $x \in -B$ and $x \in -A$ is true.
- Property 3: Prove $-B$ is closed downwards.

  Suppose otherwise, that $x < y, y \in -B$ but $x \notin -B$. Then $-y \in B$, $-x \notin B$. Since A is the complement of B, $-y \notin A$, $-x \in A$. But $-y < -x$, a contradiction.
- Property 4 is already guaranteed by the irrationality of α.

All of these properties imply that the real numbers form a commutative group by addition. □

5.2.3 Multiplication of reals

define multiplication of real numbers; you will need to define them for positive real numbers first

We can skip the distributive law though

Positive Multiplication: Let $\alpha = (A, B), \beta = (C, D)$, where α, β are both non-negative, then we define $\alpha \times \beta$ to be the pair (X, Y) where

X is the set of all products ac where $a \in A, c \in C$ and at least one of the two numbers is non-negative. Y is the set of all products bd where $b \in B, d \in D$.

5.3 Intermediate Value Theorem

5.4 Bolzano-Weiersstrass Theorem

5.5 Connectedness of $\mathbb{R}$

Part II

Number Theory

6 Number Theory

Number Theory is the study of integers.

6.1 Modular Arithmetic

6.2 Primes

Part III

Algebraic Number Theory

7 Groups, Rings and Fields

Readings:

- "Topics in Algebra" by Herstein (Chapter 2)
- Video lectures by Benedict Gross

7.1 What groups are

Definition 7.1.1: Group

A **group** is a pair $G = (G, \star)$ consisting of a set of elements G, and a binary operation $\star$ on G, such that:

- G has an **identity element**, usually denoted 1_G or just 1, with the property that

$$\forall g \in G(1_G \star g = g \star 1_G = g)$$

- The operation is **associative**, meaning

$$\forall a, b, c \in G(a \star (b \star c) = (a \star b) \star c)$$

Consequently we generally omit the parentheses.

- Each element $g \in G$ has an **inverse**, that is, an element $h \in G$ such that

$$g \star h = h \star g = 1_G$$

It is not required that $\star$ is commutative ($a \star b = b \star a$). We say that a group is **abelian** if the operation is commutative and **non-abelian** otherwise.

8 Galois Theory

Readings

- Notes by Tom Leinster

8.1

Part IV

Calculus

9 Single Variable Calculus

applications of calculus involving parametric, polar and vector functions polynomial approximations and convergence of series. Intermediate Value Theorem, Mean Value Theorem, limits of functions, asymptotic and unbounded behavior

Book: "Understanding Analysis" by Stephen Abbott. Videos: Lectures by Francis Su

9.1 Limits

Indeterminate forms:

- $\frac{0}{0}$
- $\frac{\infty}{\infty}$
- $0 \times \infty$
- $\infty - \infty$
- 0^0
- 1^∞
- ∞^0

9.1.1 Evaluating Limits

Methods:

- Plug in value.
- Cancel common factors.
- Multiply by the conjugate of the numerator or denominator.

9.1.2 Epsilon-Delta Definition

Definition 9.1.1: L

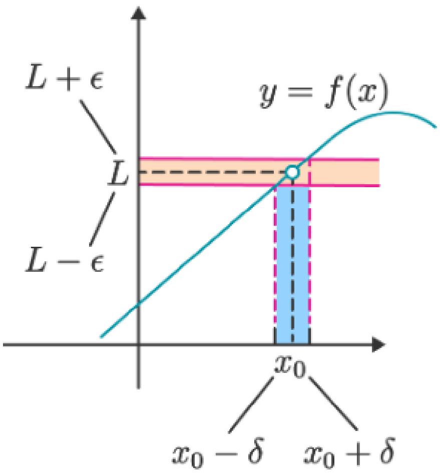
t $f(x)$ be a function defined on an open interval around x_0. We say that the limit of $f(x)$ as x approaches x_0 is L, i.e.

$$\lim_{x \to x_0} f(x) = L$$

if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in \mathbb{R}$,

$$|x - x_0| < \delta \implies \left| f(x) - L \right| < \epsilon$$

Visualising this graphically,



As ϵ becomes smaller and smaller, there always exists a δ that satisfies the property that for any x in the open interval $(x_0 - \delta, x_0 + \delta)$, the value of $f(x)$ lies in the interval $(L - \epsilon, L + \epsilon)$.

Example 1. Prove that

$$\lim_{x \to 3} 2x + 4 = 10.$$

Before the proof, we work backwards to find the value of δ in terms of ϵ and x_0, which we then declare in our proof.

$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall x \in \mathbb{R}$

$$|x-3| < \delta \implies \left|f(x) - 10\right| < \epsilon$$

Let $\epsilon > 0$ be given.

$$\begin{aligned} \left|f(x) - 10\right| &= |2x + 4 - 10| \\ &= |2x - 6| \\ &= 2\,|x - 3| < \epsilon \end{aligned}$$

Notice

$$|x-3| < \frac{\epsilon}{2}.$$

We can thus define

$$\delta \coloneqq \frac{\epsilon}{2}.$$

We now write our proof.

Proof. Let $\epsilon > 0$ be given. Choose $\delta = \frac{\epsilon}{2}$.
(Note that this value of δ comes from our preliminary working above.)
Then $\forall x \in \mathbb{R}$,

$$\begin{aligned} |x-3| &< \delta = \frac{\epsilon}{2} \\ 2|x-3| &< \epsilon \\ |2x-6| &< \epsilon \\ |2x+4-10| &< \epsilon \\ |f(x)-10| &< \epsilon \end{aligned}$$

□

Example 2. Use the formal definition of the limit to verify that

$$\lim_{x \to 3} \sqrt{2x+3} = 3.$$

We must prove that $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\sqrt{2x+3} - 3 < \epsilon$ whenever

$|x-3|<\delta$.

$$\begin{aligned}\sqrt{2x+3}-3 &= \left|\frac{(2x+3)-3^2}{\sqrt{2x+3}+3}\right| = \left|\frac{2x-6}{\sqrt{2x+3}+3}\right| \\ &\leq \left|\frac{2(x-3)}{3}\right| \\ &= \frac{2}{3}|x-3| < \frac{2}{3}\delta\end{aligned}$$

Hence, we can define

$$\epsilon \coloneqq \frac{2}{3}\delta$$

which we can use in our proof.

9.1.3 Continuity

We say $f(x)$ is continuous at x_0 when

$$\lim_{x\to x_0} f(x) = f(x_0)$$

9.1.4 Important limits

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \tag{9.1}$$

$$\lim_{x\to 0}\frac{1-\cos x}{x} = 0 \tag{9.2}$$

sum rule

product rule

quotient rule

9.1.5 L'Hôpital's Rule

Theorem 9.1.1: L'Hôpital's Rule

9.2 Derivative

9.2.1 Definition

Definition 9.2.1: Derivative

The **derivative** of $f(x)$ with respect to x is the function $f'(x)$, which is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{9.3}$$

Definition 9.2.2: Differentiability

A function $f(x)$ is differentiable at $x = x_0$ if $f'(x_0)$ exists. A function $f(x)$ is differentiable on an interval if the derivative exists for each and every point in the interval.

Definition 9.2.3: Continuity

If $f(x)$ is differentiable at $x = x_0$, then $f(x)$ is continuous at $x = x_0$.

Proof.

$$\begin{aligned}
\lim_{x \to x_0} (f(x) - f(x_0)) &= \lim_{x \to x_0} \frac{(f(x) - f(x_0))(x - x_0)}{x - x_0} \\
&= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) \\
&= f'(x_0) \cdot 0 \\
&= 0
\end{aligned}$$

□

9.2.2 Differentiation rules

Scalar multiplication

Addition rule

Theorem 9.2.1: Addition rule

$$(f + g)' = f' + g' \tag{9.4}$$

Proof.

$$\begin{aligned}
(f+g)'(x) &= \lim_{h\to 0}\frac{(f+g)(x+h)-(f+g)(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}\\
&= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right]\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= f'(x)+g'(x)
\end{aligned}$$

□

Power rule

Theorem 9.2.2: Power rule

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1} \tag{9.5}$$

Proof. Using implicit differentiation,

$$\begin{aligned}
y &= x^n\\
\ln y &= \ln x^n\\
\ln y &= n\ln x\\
\frac{y'}{y} &= n\frac{1}{x}
\end{aligned}$$

$$y' = y\frac{n}{x} = x^n\left(\frac{n}{x}\right) = nx^{n-1}$$

□

Product rule

Theorem 9.2.3: Product rule

$$(fg)' = f'g + fg' \tag{9.6}$$

Proof.

$$
\begin{aligned}
(fg)'(x) &= \lim_{h\to 0} \frac{(fg)(x+h)-(fg)(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x)-f(x)g(x)+f(x+h)g(x+h)-f(x+h)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x)-f(x)g(x)}{h} + \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x+h)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} g(x) + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} f(x+h) \\
&= \left[\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\right] g(x) + \left[\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}\right] f(x) \\
&= f'(x)g(x) + f(x)g'(x)
\end{aligned}
$$

□

Quotient rule

Theorem 9.2.4: Quotient rule

$$
\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \tag{9.7}
$$

Proof.

$$
\begin{aligned}
\left[\frac{f(x)}{g(x)}\right]' &= \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\frac{f(x+h)g(x)-f(x)g(x+h)}{g(x+h)g(x)}\\
&= \lim_{h\to 0}\frac{1}{h}\frac{f(x+h)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(x+h)}{g(x+h)g(x)}\\
&= \lim_{h\to 0}\frac{1}{g(x+h)g(x)}\left[\frac{f(x+h)g(x)-f(x)g(x)}{h}+\frac{f(x)g(x)-f(x)g(x+h)}{h}\right]\\
&= \lim_{h\to 0}\frac{1}{g(x+h)g(x)}\left[g(x)\frac{f(x+h)-f(x)}{h}-f(x)\frac{g(x)+g(x+h)}{h}\right]\\
&= \frac{1}{g^2(x)}[g(x)f'(x)-f(x)g'(x)]\\
&= \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}
\end{aligned}
$$

□

Chain rule

Theorem 9.2.5: Chain rule

If f and g are both differentiable functions and we define $F(x) = (f \circ g)(x)$, then the derivative of $F(x)$ is

$$F'(x) = f'(g(x))g'(x) \tag{9.8}$$

Sine

Cosine

9.2.3 Implicit differentiation

9.3 Integral

9.3.1 Definition

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \tag{9.9}$$

9.3.2 Integration rules

9.3.3 Integration techniques

Trigonometry

U-Substitution

$$\int f(g(x))g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u \tag{9.10}$$

where $u = g(x)$.

Integration by parts (and DI method)

From the product rule used for differentiation, we obtain

$$\int fg'\,\mathrm{d}x = fg - \int f'g\,\mathrm{d}x \tag{9.11}$$

Alternatively, we can rewrite this as

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u \tag{9.12}$$

9.3.4 Line and double integrals

to compute arc lengths, areas of curves

applications of integrals to find area and volume

9.4 Riemann Sums

A Riemann sum is an approximation of an integral by a finite sum.

Definition 9.4.1: Riemann Sum

Let f be defined on the closed interval $[a, b]$ and let Δx be a partition of $[a, b]$, with

$$a = x_1 < x_2 < \cdots < x_n < x_{n+1} = b.$$

Let Δx_i denote the length of the ith subinterval $[x_i, x_{i+1}]$ and let c_i denote any value in the ith subinterval.
The sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i$$

is a Riemann sum of f on $[a, b]$.
As the subinterval becomes infinitesimally small,

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i \tag{9.13}$$

Given a graph $y = f(x)$, and we want to find the integral on the x-interval $[0, 1]$.
Split the interval $[0, 1]$ into n equal subintervals

$$\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \cdots, \left[\frac{n-1}{n}, 1\right].$$

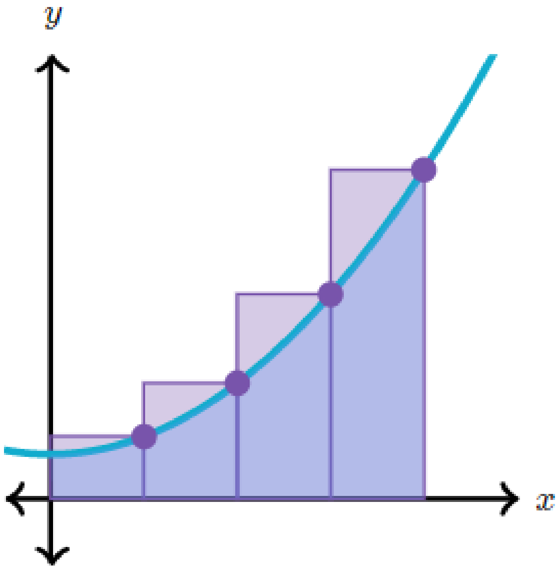
Consider the height of the rectangles. We take the right value. Hence for the kth subinterval $\left[\dfrac{k-1}{n}, \dfrac{k}{n}\right]$ where $k = 1, \ldots, n$, height of rectangle is $f\left(\dfrac{k}{n}\right)$.
Area of kth rectangle is

$$\frac{1}{n} \cdot f\left(\frac{k}{n}\right).$$

Therefore, the integral is obtained by summing up the area of n rectangles, which gives us

$$\int_0^1 f(x)\,\mathrm{d}x = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) \tag{9.14}$$

where there are infinitely many rectangles, i.e. $n \to \infty$.



Example 3. (SMO/2020) Find the value of

$$S = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n(n+k)}}.$$

Solution.

$$\begin{aligned} S &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \sqrt{\frac{1}{1+\frac{k}{n}}} \\ &= \int_0^1 \frac{1}{\sqrt{1+x}}\,\mathrm{d}x \\ &= 2\sqrt{2} - 2 \end{aligned}$$

□

9.5 Fundamental Theorem of Calculus

Using the definition of the derivative, we differentiate the following integral:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(s)\,\mathrm{d}s &= \lim_{h\to 0}\frac{\int_a^{x+h} f(s)\,\mathrm{d}s - \int_a^x f(s)\,\mathrm{d}s}{h}\\ &= \lim_{h\to 0}\frac{\int_x^{x+h} f(s)\,\mathrm{d}s}{h}\\ &= \lim_{h\to 0}\frac{hf(x)}{h}\\ &= f(x)\end{aligned}$$

9.6 Differential Equations

https://www.math.hkust.edu.hk/ machas/differential-equations.pdf

9.6.1 Ordinary differential equations

Definition 9.6.1: Ordinary differential equation

An **ordinary differential equation** involves

- an independent variable, x
- a function of x, y, which is a dependent variable
- derivatives of y with respect to x.

A first-order differential equation involves the first derivative of y with respect to x, whereas a second-order differential equation involves the second derivative of y with respect to x, and so on.

First-order differential equations

Second-order differential equations

9.7 Fourier Series

https://math.mit.edu/ gs/cse/websections/cse41.pdf

Basically creating a wave by combining infinitely many sine waves: square wave = sin x + sin 3x / 3 + sin 5x / 5 + ...

$$f(x) = a_0 \tag{9.15}$$

9.8 Laplace transform

10 Multivariable Calculus

10.1 Partial derivatives

Definition 10.1.1: Partial derivative

For a continuous function $z = f(x, y)$, the **partial derivative** of $f(x, y)$ with respect to x is defined as

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}. \tag{10.1}$$

Similarly, the partial derivative of $f(x, y)$ with respect to y is defined as

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

One of the most common notations used to represent partial derivatives is

$$\frac{\partial f}{\partial x}$$

which denotes the partial derivative of f with respect to x. We call the symbol ∂ "del".

Notations for second partial derivatives:

- Second partial derivative of $f(x, y)$ with respect to x then x:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

- Second partial derivative of $f(x, y)$ with respect to y then y:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

- Second partial derivative of $f(x,y)$ with respect to x then y:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = f_{xy}$$

10.1.1 How To Do Partial Derivatives?

Keep in mind that we only need to find the derivative of functions with respect to one variable by keeping the rest of the variables constant.

- Identify the variable we're differentiating. For example, when working with $\frac{\partial f}{\partial x}$, we differentiate f with respect to x.
- Treat the rest of the variables as constants.
- Apply fundamental derivative rules to differentiate f with respect to the variable.

$$\begin{aligned}\frac{\partial f}{\partial x} &= \underbrace{\frac{\partial}{\partial x}(2x^2 - 4xy + y^2)}_{\substack{\text{Take the derivative of } x\\ \text{Treat } y \text{ as a constant}}}\\ &= 2(2x) - 4(1)y + 0\\ &= 4x - 4y\end{aligned}$$

Similarly, taking second order partial derivatives,

$$\begin{aligned}\frac{\partial^2 f}{\partial x\,\partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\\ &= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}(12x^2y - 3xy^2)\right]\\ &= \frac{\partial}{\partial x}[12x^2(1) - 3x(2y)]\\ &= \frac{\partial}{\partial x}(12x^2 - 6xy)\\ &= 12(2x) - 6y(1)\\ &= 24x - 6y\end{aligned}$$

10.2 Partial differential equations

10.2.1 Definitions and Terminology

A **partial differential equation** is an equation involving a function and/or its partial derivatives. For example,

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$$

where $f(x,t)$ is a function of multiple variables.

We can classify PDEs based on:

- **Order.**

  The order is the number corresponding to the order of the highest partial derivative in the equation.

  For instance, the order of the following PDE is 2.

  $$\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t}$$

  This also applies to mixed partial derivatives. For instance, the order of the following PDE is 3.

  $$\frac{\partial^3 f}{\partial x^2\, \partial y} = \frac{\partial f}{\partial t}$$

- **Number of independent variables.**

  An independent variable is what we differentiate with respect to.

- **Linearity.**

  A linear PDE is one in which the *dependent* variable (the one being differentiated) appears only in a linear fashion.

  For instance, the two PDEs above are linear as the partial derivatives are not being raised to a power or multiplied with each other.

  The following PDE is non-linear.

  $$f\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t}$$

- **Homogeneity.**

  A homogenous PDE is one in which every term only involves the dependent variable and/or its derivatives.

  The first two PDEs above are homogenous as every term contains f or its derivatives.

  The following PDE is non-homogenous as there are two terms that do not contain f.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t} + x^2 + \tan t$$

- **Coefficient type.**

  The coefficient here refers to the coefficient of the term involving the dependent variable and its derivatives. It can be either constant or variable.

  For instance, the coefficients of the terms in the first two examples are 1. We say that these two PDEs have constant coefficients.

  The following PDE has variable coefficients.

$$\tan x \frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t}$$

- **Parabolic, Hyperbolic, or Elliptic.**

  We can do this classification for linear 2nd order PDEs which take the form of

$$A\frac{\partial^2 f}{\partial x^2} + B\frac{\partial^2 f}{\partial x\,\partial y} + C\frac{\partial^2 f}{\partial y^2} + D\frac{\partial f}{\partial x} + E\frac{\partial f}{\partial y} + Ff = G$$

  where the coefficients are generally functions of x or y.

  For a **hyperbolic** PDE, $B^2 - 4AC > 0$. Using variable substitutions to change x and y to η and ε respectively, we can reduce the PDE to

$$\frac{\partial^2 f}{\partial \eta^2} - \frac{\partial^2 f}{\partial \varepsilon^2} + g = 0$$

  where g denotes the first and lower order terms. This is similar to the equation of a hyperbola: $x^2 - y^2 = 1$.

  For a **parabolic** PDE, $B^2 - 4AC = 0$. Using variable substitutions, we can reduce the PDE to

$$\frac{\partial^2 f}{\partial \eta^2} + g = 0.$$

This is similar to the equation of a parabola: $x^2 + y = 0$.

For an **elliptic** PDE, $B^2 - 4AC < 0$. Using variable substitutions, we can reduce the PDE to

$$\frac{\partial^2 f}{\partial \eta^2} + \frac{\partial^2 f}{\partial \varepsilon^2} + g = 0.$$

This is similar to the equation of an ellipse: $x^2 + y^2 = 1$.

Note that if the coefficients are constants, the PDE can be hyperbolic, parabolic or elliptic. However, if the coefficients are variables, then it is possible for the PDE to be hyperbolic in some regions, and elliptic or parabolic in some regions.

10.2.2 Solutions and Auxiliary Conditions

There are a lot of solutions to a given PDE, hence it is important for us to know the auxiliary conditions, i.e. boundary and initial conditions, which dictate which technique we use to solve the PDE.

- A boundary condition expresses the behavior of a function on the boundary (border) of its area of definition. An initial condition is like a boundary condition, but then for the time-direction.

Part V

Linear Algebra

11 Basics

Book: "Linear Algebra Done Right" by Sheldon Axler Videos: Sheldon Axler's Playlist

Major topics: Systems of linear equations, matrices, determinants, Euclidean spaces, linear combinations and linear span, subspaces, linear independence, bases and dimension, rank of a matrix, inner products, eigenvalues and eigenvectors, diagonalization, linear transformations between Euclidean spaces

11.1 Vectors

We consider an xy-plane, where a 2D vector is rooted at the origin. For notation-wise, a 2D vector consists of a pair of numbers a and b, written as

$$\begin{bmatrix} a \\ b \end{bmatrix}$$

where the vector is a units long parallel to the x-axis, and b units parallel to the y-axis.

In 3 dimensions, we add the z-axis, perpendicular to the xy-plane. Notation-wise, each vector is associated with a triplet of numbers, denoted as

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

11.1.1 Vector algebra

Vector addition

To add two vectors in the list-of-numbers conception of vectors, match up their terms and add them each together.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$$

Scalar multiplication

Scaling is to multiply a vector $\vec{\mathbf{v}}$ by a certain number n.

$$n\vec{\mathbf{v}} = n \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} nx \\ ny \end{bmatrix}$$

11.2 Linear combinations, span and bases

In the xy-coordinate system, there are two special vectors. The one pointing to the right with length 1 is $\hat{\imath}$, the unit vector in the x-direction. The other one is pointing straight up with length 1 is $\hat{\jmath}$, the unit vector in the y-direction. Collectivevly, $\hat{\imath}$ and $\hat{\jmath}$ are known as the **basis vectors** of the xy-coordinate system.

For a vector $\vec{\mathbf{v}}$,

$$\vec{\mathbf{v}} = \begin{bmatrix} x \\ y \end{bmatrix} = x\hat{\imath} + y\hat{\jmath}$$

We can also choose other pairs of vectors as the basis vectors to describe vectors.

11.3 Matrices

11.4 Determinants

11.5 Vector Spaces

11.6 Eigenvalues and Eigenvectors

11.7 Orthogonality and Least Squares

11.8 Abstract vector spaces

Part VI

Topology

12

Euclidean n-space

12.1 n-dimensional Euclidean space

$\mathbb{R}^n$, as a set, is defined as the set of vertical vectors with n coordinates in the real numbers.

Algebraically, $\mathbb{R}^n$ is an n-dimensional vector space over $\mathbb{R}$. Vectors in $\mathbb{R}^n$ are expressed as vertical vectors

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

We usually express the above vector compactly as follows:

$$x = (x_1, \ldots, x_n)^T$$

12.1.1 Properties

Algebraically, $\mathbb{R}^n$ is a vector space over $\mathbb{R}$. This means that $\mathbb{R}^n$ has the following extra properties.

These properties make up the algebraic structure of $\mathbb{R}^n$, which may then be further expanded in linear algebra However, for now we will continue on with the analytical/topological aspects of Euclidean space.

Addition

For any two vectors x, y, we may perform addition

$$x + y = (x_1 + y_1, \ldots, x_n + y_n)^T$$

Properties of addition:

1. $x + y = y + x$
2. $(x + y) + z = x + (y + z)$
3. The zero vector $0 = (0, \ldots, 0)$ satisfies $x + 0 = 0 + x = x$
4. For any vector x, its negative $-x$ satisfies $x + (-x) = (-x) + x = 0$

Scalar multiplication

For any vector x and real number (scalar) k, we may perform scalar multiplication

$$kx = (kx_1, \ldots, kx_n)^T$$

Properties of scalar multiplication:

1. $0 \cdot x = 0$, $1 \cdot x = x$
2. $(kl)x = k(lx) = l(kx)$
3. $k(x + y) = kx + ky$
4. $(k + l)x = kx + lx$

13 Point Set Topology

Online Notes with Problems: MAT327 Course Notes http://www.math.toronto.edu/iva

Broadly, geometry is the study of measuring quantities. Mathematicians then use these measurements to make conclusions about properties of the spaces being studied. Topology, on the other hand, studies spaces by asking questions from a qualitative perspective.

13.0.1 Metric Spaces

The Euclidean space is built upon the vector space $\mathbb{R}^n$. Specifically speaking, it is $\mathbb{R}^n$ endowed with two extra notions: the norm and the metric.

Definition 13.0.1: Norm of a space

The norm of the Euclidean space $\|\cdot\|$ is a real-valued function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$. Given a vector $x = (x_1, \ldots, x_n)^T$ in $\mathbb{R}^n$, the **norm** of x is defined as

$$\|x\| = \sqrt{\sum_{i=1}^{n} {x_i}^2} = \sqrt{{x_1}^2 + \cdot + {x_n}^2}.$$

Norms are required to satisfy the following fundamental properties:

1. Positive Definiteness: For any vector x, $\|x\| \geq 0$, and $\|x\| = 0 \iff x = 0$.

2. Absolute Homogeneity: For any vector x and scalar a, $\|ax\| = |a| \cdot \|x\|$

3. Subadditivity (Triangle Inequality): For any two vectors x and y, $\|x + y\| \leq \|x\| + \|y\|$

Definition 13.0.2: Metric space of a set

A metric space is a pair (X, d), where X is a set and $d : X \times X \to \mathbb{R}_{\geq 0}$ is the **distance function**. d should satisfy

1. Positive Definiteness: For any two elements x and y, $d(x, y) \geq 0$, and $d(x, y) = 0 \iff x = y$
2. Symmetry: For any two elements x and y, $d(x, y) = d(y, x)$
3. Triangle Inequality: For any three elements x, y, z, $d(x, z) \leq d(x, y) + d(y, z)$

The **metric** d of the Euclidean space is a real-valued function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Given two vectors $x = (x_1, \ldots, x_n)^T$, $y = (y_1, \ldots, y_n)^T$ in $\mathbb{R}^n$, the distance between x and y is defined as

$$d(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = \sqrt{(x_1 - y_1)^2 + \cdot + (x_n - y_n)^2}.$$

Once we have a notion of distance, we can discuss open sets. The idea of topological spaces will be to bypass the notion of distance and simply consider these open sets.

Definition 13.0.3: Ball

Given a metric space (X, d) and a point $p \in X$, the open ball of radius $r \in \mathbb{R}_{>0}$ around p is

$$B_r(p) = q \in X \mid d(p, q) < r$$

A **ball** in $\mathbb{R}^n$ is determined by its center $x \in \mathbb{R}^n$ and its radius $r > 0$, and is denoted by $B(x, r)$.

Open balls are instances of open sets.

Definition 13.0.4: Punctured ball

A **punctured ball** in $\mathbb{R}^n$ is a ball excluding its center, and is denoted by $B_0(x, r)$.

13.0.2 Some Concepts in Euclidean Space

Definition 13.0.5: Bounded set

A set E in $\mathbb{R}^n$ is a **bounded set** if there exists $M > 0$ such that $\|x\| \le M \forall x \in E$.

Problem 7. Exercise 1: Given E, F in $\mathbb{R}^n$ and real number k, define

$$kE = \{kx \mid x \in E\}$$

$$E + F = \{x + y \mid x \in E, y \in F\}$$

(a) Show that if E is bounded, then kE is bounded;

(b) Show that if E and F are bounded, then $E + F$ is bounded

Definition 13.0.6: Diameter of set

iven a set $E \subset \mathbb{R}^n$, the **diameter** of E is defined as

$$diam E = \sup_{x,y \in E} d(x,y).$$

Problem 8. Find the diameter of the open unit ball in $\mathbb{R}^n$ given by

$$B = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$$

Solution. First note that

$$d(x,y) = \|x - y\| \le \|x\| + \|-y\| = \|x\| + \|y\| < 1 + 1 = 2$$

On the other hand, for any $\epsilon > 0$, we pick

$$x = (1 - \frac{\epsilon}{4}, 0, \ldots, 0), y = (-\left(1 - \frac{\epsilon}{4}\right), 0, \ldots, 0)$$

Then

$$d(x,y) = 2 - \frac{\epsilon}{2} > 2 - \epsilon$$

Therefore $diam B = 2$. □

Problem 9. Given a set E in $\mathbb{R}^n$, show that E is bounded iff $diam E < +\infty$.

Solution.
Forward direction:

If E is bounded, then there exists $M > 0$ such that $\forall x \in E$, $\|x\| \leq M$.

Thus $\forall x, y \in E$,

$$d(x,y) = \|x - y\| \leq \|x\| + \|y\| \leq 2M.$$

Thus $diamE = \sup d(x,y) \leq 2M < +\infty$

Backward direction:

Suppose that $diamE = r$.

Pick a random point $x \in E$, suppose that $||x|| = R$

Then for any other $y \in E$,

$$\|y\| = \left\|x + (y - x)\right\| \leq \|x\| + \|y - x\| \leq R + r$$

Thus, by picking $M = R + r$, we obtain $\|y\| \leq M \forall y \in E$, and we're done.

Basically you use x to confine E within a ball, which is then confined within an even bigger ball centered at the origin. □

Definition 13.0.7: Distance between sets

Given two sets $E, F \subset \mathbb{R}^n$, the **distance between sets** E and F is defined as

$$d(E,F) = \inf_{x \in E, y \in F} \|x - y\|.$$

Obviously $d(E,F) > 0$ implies that E and F are disjoint, but E and F may still be disjoint even if $d(E,F) = 0$, e.g. the closed intervals $E = (-1,0), F = (0,1)$.

Problem 10. Suppose that E and F are sets in $\mathbb{R}^n$ where F is finite, then E and F are disjoint iff $d(E,F) > 0$.

Topology in Euclidean Space

Before we move on, we need to talk about how we think about topology. The concept first begins with an attempt to say that two points are close to one another.

Of course, we did define the metric earlier But as it turns out, this particular notion can be made extremely abstract

Specifically speaking, we could theoretically define closeness simply with set theory

Imagine that in some random set X, there is a predetermined family of subsets scrA in P(X) (scr: script; cursive)

Now for some element x in X, suppose that we can pick a set in scr A containing x

We may denote this set as U(x) Then, from the perspective of U(x), a point y in X would seem to be close to x if y also lies in U(x)

: Ah actually the family of subsets is usually denoted as scrN : The family scrN is called the neighbourhood system : There is also the notion of the neighbourhood system of a particular point, scrN(x)=U $\in scrN|x \in U :$
$Now, here'stheeasiestparttoconfuse : Theword'system'intheaboveterminologyisactu$
$That'sbecausetheterminologyof'neighbourhood'isusedasfollows : Wesaythatasubset$
$scrNsuchthatx \in UandUisasubsetofN : AhI'mverysorrybutImessedupthetermino$

According to wikipedia, the neighbourhood system actually refers to all neighbourhoods What I was talking about earlier should've been called a neighbourhood basis

: The neighbourhood basis is denoted by scrB

Okay let's redo the entire thing

1. Neighbourhood Basis Given a set X, we define a family of subsets in X, denoted by $scrB$, to describe points close to each other; points that belong to the same set U in $scrB$ are considered to be close to each other with respect to U.

2. Neighbourhood Given a point x in X, we use the term **neighbourhood** to describe a particular construction for x; N is said to be a neighbourhood of x, if there exists U in $scrB$ containing x such that $U \subset N$.

2'. Neighbourhood System Given a point x in X, the **neighbourhood system** of x, denoted $scrN(x)$, is the set of all neighbourhoods of x.

There is still a lot of problems regarding the above definition 1. If M and N are neighbourhoods of x, is M $\cap Naneighbourhoodofx?Supposedlytheanswershouldbey$

People realized that the above requirements can be formalized simply with the neighbourhood systems themselves

These are the axioms for the neighbourhood systems

1. $scrN(x)$ is nonempty, and $\forall U \in scrN(x), x \in U$
2. If $U, V \in scrN(x)$, then $\exists W \in scrN(x)$ s.t. $W \subset U \cap V$
3. If $U \in scrN(x)$ and $y \in U$, then $\exists V \in scrN(y)$ s.t. $V \subset U$

As for the Euclidean plane, we have a natural way of defining the neighbourhood systems First we pick the neighbourhood basis to be scrB = B(x,ϵ)$|x \in \mathbb{R}^n, \epsilon > 0$

Then we say that N is a neighbourhood of x if there exists $\epsilon > 0 such that B(x, \epsilon) is in N.B$

Once we have neighbourhood systems, we can then define the two most important kinds of sets in topology, open and closed sets.

13.1 Terminology

13.1.1 Neighbourhood, interior and open sets

Definition 13.1.1: Neighbourhood

A set $A \subset \mathbb{R}^n$ containing x is known as a **neighbourhood** of x if $B(x, \epsilon) \subset A$ for some $\epsilon > 0$.

Definition 13.1.2: Interior point

An element $x \in A$ is known as an **interior point** if A is a neighbourhood of x.

Definition 13.1.3: Interior of set

The **interior** of a set A, denoted by A°, is the set of all interior points in A.

Definition 13.1.4: Open set

A set $A \subset \mathbb{R}^n$ is known as an **open set** if $A^\circ = A$, i.e. all points in A are interior points.

13.1.2 Limit points, closure and closed sets

Definition 13.1.5: Limit point

An element $x \in A$ is known as a **limit point** of A if $B_0(x, \epsilon) \cap A \subset \varnothing$ for all $\epsilon > 0$.

Definition 13.1.6: Induced set

The **induced set** of set A, denoted by A', is the set of all limit points of A.

Definition 13.1.7: Closure of set

The **closure** of a set A, denoted by $\bar{A}$, is the union set $A \cup A'$.

Definition 13.1.8: Closed set

A set $A \subset \mathbb{R}^n$ is known as a **closed set** if $\bar{A} = A$, i.e. all limit points of A are contained in A.

13.1.3 Further topological constructions of points

Definition 13.1.9: Isolated point

An element $x \in A$ is known as an **isolated point** of A if it is not a limit point of A.

Definition 13.1.10: Boundary of set

The **boundary** of a set A, denoted by ∂A, is the set difference $\bar{A} \backslash A^\circ$.

Definition 13.1.11: Boundary point

An element $x \in \mathbb{R}^n$ is known as a **boundary point** of A if it is in ∂A.

Definition 13.1.12: Exterior point

An element $x \in \mathbb{R}^n$ is known as an **exterior point** of A if it is an interior point of A^c.

13.1.4 Further topological constructions of sets

Definition 13.1.13: Compact set

A set $A \subset \mathbb{R}^n$ is **compact** if it is a bounded closed set.

Definition 13.1.14: Dense subset

A subset $B \subset A$ is known as a **dense subset** of A if $\bar{B} = A$.

Definition 13.1.15: Nowhere dense set

A set $A \subset \mathbb{R}^n$ is **nowhere dense** if its closure has no interior, i.e. $(\bar{A})^\circ = \varnothing$.

In general topology, these are the axioms used to define open and closed sets: A is open iff A$^c isclosedAnarbitraryunionofopensetsisopen; afiniteintersectiono$

At the moment we only consider them to be certain properties regarding open and closed sets in R^n

Property P1 : Forwards direction: Let A be open, we consider the punctured balls of x not in A : (if x is not in A, we consider the punctured balls centered at x) : Our goal is to show that B$_0(x, r) always intersects with A^c :$
$Sosupposeotherwisethat B_0(x,) isasubsetof A for some > 0 : Ahnosorry, weconsiderxn$
$Thethingiswewanttoshowthat A^c isclosed, i.e. alllimitpointsof A^c arein A^c :$
$Sosupposeotherwisethatxisalimitpointof A^c thatisnotin A^c : xisalimitpointof A^c, henc$
$0, B_0(x,) alwaysintersectswith A^c : Thisisequivalenttosayingthat B_0(x,) isneverasubs$
$A : However, xisnotin A^c, soxisin A : Butif Aisopen, thenthereexists > 0 suchthat B(x,$
$Backwardsdirection : Let A^c beclosed, supposeotherwisethat Aisnotopen, i.e. thereisapo$
$Thatistosay, B(x,) alwaysintersectswith A^c : Me : Backwardsdirection :$
$Let A^c beclosed, supposeotherwisethat Aisnotopen, i.e. thereisapointxsuchthat B(x,) isn$
$--------------Iforgottomentionthatthepointxisin A : Soifxisin A, then B(x,) A^c$
$B_0(x,) A^c : Butthismeansthat B_0(x,) A^c isneverempty, hencexisalimitpointof A^c :$
$However, xisin A, contradictoryto A^c beingclosedandthusshouldcontainallofitslimitpo$
$P2 : Let Abeanarbitraryunionofopensets U_i wherei I : Thenforanyx A, supposethatx U_i$
$Ontheotherhand, let U and V beopensetsandletx U V : Since U and V areopen, wecanpick_1$
$Thenwesimplypick = min_{1,2} sothat B(x,) isin U V : P3followsfromde Morgan's Law, F$
$Ex2 : Comparethefollowingsets : 1.(A B)ř, AřBř2.(A B)ř, AřBř3.bar(A B), bar Abar B$
$Comparetheirsizes, i.e. determineiftheyareequal, orifonesetmaybeasubsetoftheothe$
$Ex3 : Provethatthesetofexteriorpoints, (A^c)řisthesameas(bar A)^c : Ex4 :$
$Regardingalternativedescriptions : 1. Aisaneighbourhoodofxiffthereexistsanopense$
$0, B(x,) containsinfinitelymanyelementsof A(youdon'tneedtomentionthepuncturedb$
$0, B(x,) intersectswithboth Aand A^c : Ex5 : Regardingclosures(Thefollowingproperti$
$set'counterparts) : 1. A'' = A'2. bar Aisclosed, i.e. bar(bar A) = bar A : Bytheway, it'simp$
$Thisisactuallystandardincalculus, but Iamusingtheterminologyasyouwouldseethemi$
$settopology : (Though B(x, r)/B_r(x), U(x, r)/U_r(x) arealsoconsideredtobestandardnot$
$(Evenworse, differentauthorsmayusethemtorefertodifferenttypesof"basisneighb$
$y : |y - x| < r(balls) N(x, r) = y : |y_i - x_i| < r(cubes) : Intermsoftopologyhowever, using$

$2{-}1 : (AB)$ř$maybebiggerInRweconsidertheintervalsA = (-1, 0], B = [0, 1), thenA$ř$B$ř
$(-1, 0)(0, 1), (AB)$ř $= (-1, 1) : ForxinA$řBř$, wehaveeitherxinA$ř$orxinB$ř$, sothereissor$
$2{-}2 : EqualIfxin(AB)$ř$, thenthereexistsaballUcenteredatxsuchthatUisinbothAandB$
$2{-}3 : Equal2{-}4 : barAbarBmaybebiggerWellyoucoulddothesefirstbutthey'reactuallyc$
$1, 2{-}2soI'llmoveon : Ex3 : xin(A^c)$ř$iffthereexists > 0suchthatB(x,)isinA^ciffB(x,)$
$\emptyset iffxnotinAandB_0(x,)A = \emptyset iffxnotinAA' = barAiffxin(barA^c) : 4{-}1 :$
$WeshowthatB(x,)isopen : ForallyB(x,), then|y - x| < ForallzB(y, -|y -$
$x|), |z-x||z-y|+|y-x| < -|y-x|+|y-x| = ThereforeB(y, -|y-x|)isasubsetofB(x,)$

4-2: We construct a sequence x$_n recursivelyasfollows : Pickx_1B_0(x,)APickx_{n+1}B_0(x,$
$x|)AItiseasytoseethattheballsabovearegettingsmallersoallx_narebothmutuallydistinc$

4-3: x is a boundary point iff xbarA A° Forwards: 1. x is in A, then all B(x,) intersects with A at x, but since x is not in A° they must always intersect with A$^c aswell2.xnotinA, thenallB(x,)intersectswithA^catx, butsincexisinbarA, x$
$Backwards : 1.xisinA, thensinceB(x,)alwaysintersectswithA^c, xcannotbeinA$ř$2.xisn$
$Infactwecandescribetheclosurewithoutreferingtopuncturedballsandinducedsets :$
$xinbarAiffB(x,)alwaysintersectswithA : Alsoasasidenote, A$ř$dA(A^c)$ř $=$
$R^n : Ahsorry5{-}1isincorrect : IthinktheactualpropertyshouldbethatA'isclosed :$
$Acounterexamplewouldbetheset1, 1/2, 1/3, 1/4, ...A' = 0, A'' = \emptyset$

In order to show that A' is closed, we need to show that if x is a limit point of A', then x is in A', i.e. x is a limit point of A So we need to show that limit points of A' are always limit points of A Let x be a limit point of A', then for all ¿0, B$_0(x, /2)intersectswithA'andwemaypickyB_0(x, /2)A'$

Now here's the tricky part Since yA', y is a limit point of A, hence B$_0(y, |y-$
$x|)intersectswithAandthuswemaypickzB_0(y, |y - x|)A$

We show that zB$_0(x,) : |z - x||z - y| + |y - x| < 2|y - x| <, hencezB(x,)|z - y| <$
$|x - y|, hencezxTherefore, zB_0(x,)$

As for 5-2, it is just 5-1 and 2-3

For homework, you'll work out some properties regarding dense sets

1. A is a dense set in X iff A intersects with all open sets in X 2. If A is dense in X and B is dense in A, then B is dense in X 3. If A and B are dense in X where A is open, then AB is dense in X

14 Algebraic Topology

Book: Algebraic Topology by Allen Hatcher https://pi.math.cornell.edu/ hatcher/AT/ page.html Videos: Lectures by Pierre Albin

Part VII

Complex Analysis

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Complex Analysis

“Visual Complex Functions: an Introduction with Phase Portraits” by Elias Wegert

“Complex Analysis” by Serge Lang

Videos: Wesleyan University Playlist

Part VIII

Differential Geometry

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Differential Geometry

Introduction to Differentiable Manifolds and Riemannian Geometry

Part IX

Others

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Probability

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Game Theory

Game Theory is the study of strategically interdependent behaviour.

18.1 Strict Dominance

18.1.1 Prisoner's Dilemma

To start off, we will take a look at the **Prisoner's Dilemma**, which goes as follows:

> Two thieves plan to rob a store, but the police arrest them for trespassing. The police suspect that they planned to break in but lack the evidence to support such an accusation. They require a confession to charge the suspects. The police offer them the following deal:
>
> - If no one confesses, both are charged a *one month* jail sentence each for trespassing.
> - If a rat confesses and the other does not, the rat is not charged but the other is charged a *twelve month* jail sentence for robbery.
> - If both confess, both are charged an *eight month* jail sentence each.
>
> If both criminals are self-interested and only care about minimising their jail time, should they take the interrogator's deal?

We condense the above information into a **payoff matrix** as shown below, where we have two players, A and B. The horizontal rows represent A's choices, while the vertical columns represent B's choices, and each cell contains a combination of their payoffs.

	quiet	confess
quiet	$-1, -1$	$-12, 0$
confess	$0, -12$	$-8, -8$

18.1.2 Split or Steal

The game goes as follows:

> Each of two players, Sarah and Steve, has to pick one of two balls: inside one ball appears the word '**split**' and inside the other the word '**steal**' (each player is first asked to secretly check which of the two balls in front of him/her is the split ball and which is the steal ball). They make their decisions simultaneously.

The possible outcomes are shown in the figure below, where each row is labelled with a possible choice for Sarah and each column with a possible choice for Steven. Each cell in the table thus corresponds to a possible pair of choices and the resulting outcome is written inside the cell.

		Steven	
		Split	Steal
Sarah	Split	Sarah gets \$50,000 Steven gets \$50,000	Sarah gets nothing Steven gets \$100,000
	Steal	Sarah gets \$100,000 Steven gets nothing	Sarah gets nothing Steven gets nothing

18.2 Nash Equilibrium

Nash Equilibrium is a set of optimal strategies that work against *all* counter-steategies. This means that if any given player were told the strategies of all their opponents, they still would choose to retain their original strategy.

18.2.1 Matrix games

18.3 Fair Division

18.3.1 Rental harmony problem

Sperner's lemma

https://www.cs.cmu.edu/ arielpro/15896/docs/paper19b.pdf