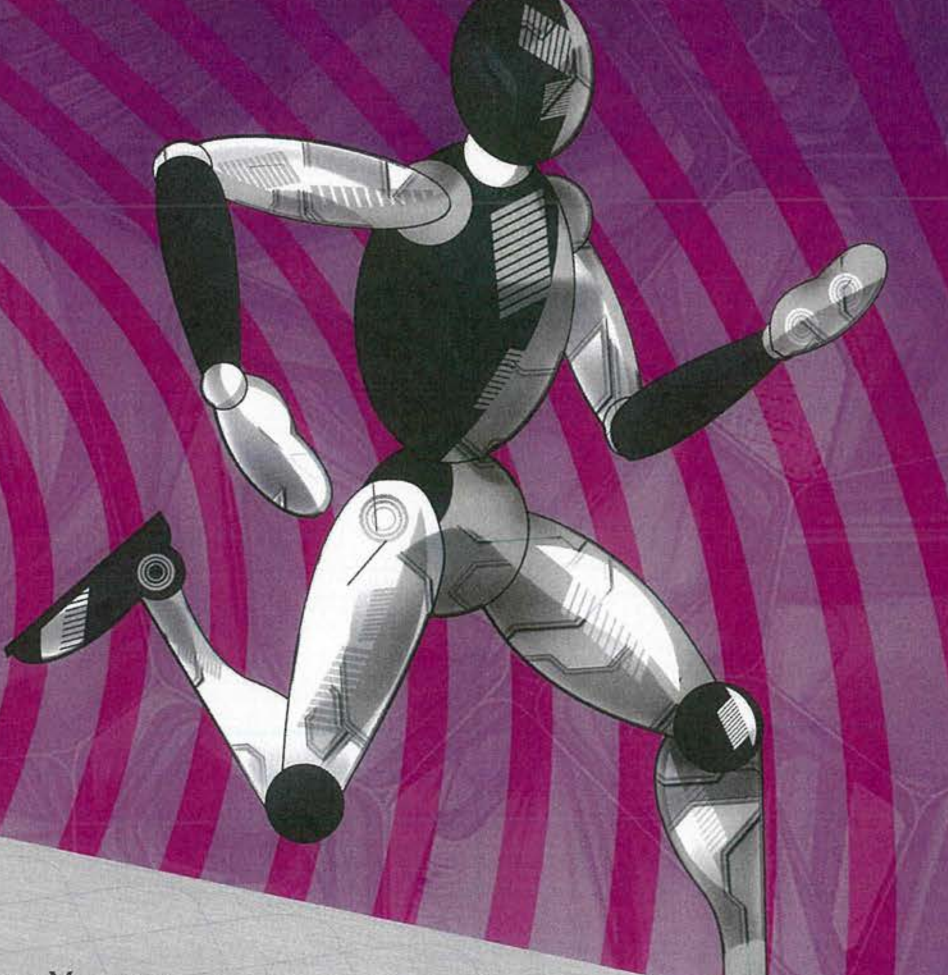
Singapore Mathematical Olympiads 2020



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Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2020

Junior Section (Round 1)

Monday, 7 September 2020 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*
8. *Throughout this paper, let* $[A_1A_2\dots A_r]$ *denote the area enclosed by the polygon* $A_1A_2\dots A_r$.
9. *Throughout this paper, let* $\overline{a_{n-1}a_{n-2}\dots a_0}$ *denote an* n*-digit number with the digits* a_i *in the corresponding position, i.e.* $\overline{a_{n-1}a_{n-2}\dots a_0} = a_{n-1}10^{n-1} + a_{n-2}10^{n-2} + \dots + a_0 10^0$.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1. Let $x = 2^{300}$, $y = 3^{200}$ and $z = 6^{100}$. Which of the following is true?

   (A) $x > y > z$ (B) $x > z > y$ (C) $y > z > x$ (D) $y > x > z$ (E) $z > x > y$

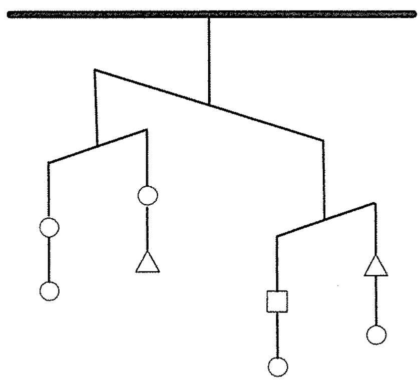
2. An expensive painting was stolen and the police rounded up five suspects Alfred, Boris, Chucky, Dan and Eddie. These were the statements that were recorded.

   Alfred: "Either Boris or Dan stole the painting."
   Boris: "I think Dan or Eddie is the guilty party."
   Chucky: "It must be Dan."
   Dan: "Boris or Eddie did it!"
   Eddie: "I am absolutely sure the thief is Alfred."

   The police knew that only one of the five suspects stole the painting and that all five were lying. Who stole the painting?

   (A) Alfred (B) Boris (C) Chucky (D) Dan (E) Eddie

3. The following diagram shows a system of balances hanging from the ceiling with three types of weights. The balances tip down to the heavier side. If we use $\square < \triangle$ to represent $\square$ is lighter than $\triangle$, which of the following is true?



   (A) $\square < \circ < \triangle$ (B) $\square < \triangle < \circ$ (C) $\triangle < \square < \circ$ (D) $\triangle < \circ < \square$
   (E) $\circ < \square < \triangle$

4. The integer 6 has exactly four positive factors, namely $1, 2, 3$ and 6. Likewise, the integer 8 has exactly four positive factors, $1, 2, 4$ and 8. How many integers from 9 to 50 (inclusive) have exactly four positive factors?

   (A) 10 (B) 11 (C) 12 (D) 13 (E) 14

5. A company sells a bottle of juice at \$1 per bottle and offers to give you a free bottle of juice if you recycle four empty bottles. Assuming there is no limit to the offer and you can recycle bottles that you receive from the offer, what is the largest number of bottles you can eventually consume if you have \$43?

   (A) 53 (B) 54 (C) 55 (D) 56 (E) 57

Short Questions

6. Let n be a positive integer. Suppose the lowest common multiple of 4, 5 and n is 2020. What is the sum of the smallest possible value of n and the largest possible value of n?

7. When the five-digit integer $\overline{2x6yx}$ is divided by the four-digit integer $\overline{5y27}$, the quotient is 4 and remainder is $\overline{x106}$, which is a four-digit integer. What is the value of the digit x?

8. How many multiples of 11 are there in the sequence $1100, 1101, 1102, \ldots, 2019, 2020$?

9. A quadruple (a, b, c, d) of positive integers is *skewed* if the median and mode of a, b, c, d are equal, but strictly greater than the mean of a, b, c, d. How many skewed (a, b, c, d) of positive integers are there that satisfy $a \leq b \leq c \leq d$ and $a + b + c + d = 40$?

10. Let T_n be a linear sequence given by

$$T_1 = 10,\ T_2 = 13,\ T_3 = 16, \ldots,$$

and S_n be a quadratic sequence given by

$$S_1 = 10,\ S_2 = 23,\ S_3 = 39, \ldots.$$

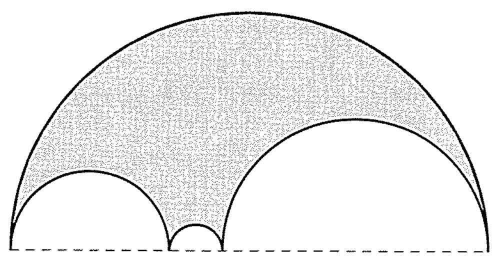
What is the value of $S_{101} - T_{101}$?

11. Let ABC be a triangle where D is the midpoint of BC and E lies on AC such that $AE : EC = 3 : 1$. Let F be the intersection of AD and BE. If the area of ABC is 280, what is the area of triangle BFD?

12. In the figure below, each distinct letter represents a unique distinct digit such that the arithmetic holds. If the letter K represents 6, what number does SHAKE represent?

$$\begin{array}{cccccc} & S & H & A & K & E \\ + & S & H & A & K & E \\ \hline R & A & T & T & L & E \end{array}$$

13. The diagram below shows a shape formed from four semi-circles whose radii are distinct integers. If the shape has perimeter of 18π and encloses an area of $k\pi$, where k is a prime, what is the value of k?



14. If the sum of the two diagonals of a kite is 100, what is the largest possible area of the kite?

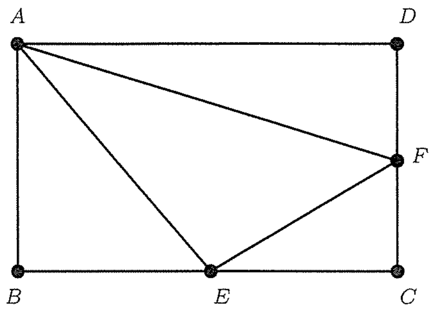
15. What is the smallest five-digit integer of the form $\overline{5x20y}$ that is divisible by 33?

16. What is the value of $2020^3 - 9(2020)(2017) - 2017^3$?

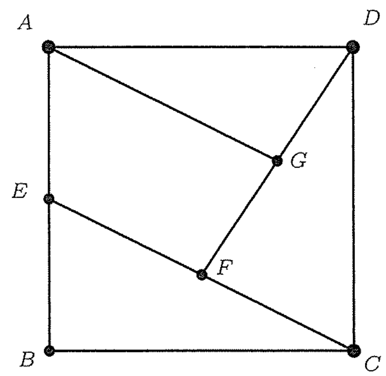
17. In the following diagram, $ABCD$ is a rectangle where E and F are points on BC and CD respectively. The area of triangle AEF, denoted $[AEF]$, is 2037. If

$$[AECF] = 2[ABE] = 3[ADF],$$

what is the area of the rectangle $ABCD$?



18. In the following diagram, $ABCD$ is a square of side length 64. E is the midpoint of AB, F is the midpoint of EC and G is the midpoint of FD. What is the area enclosed by the quadrilateral $AEFG$?



19. Let $X = 1234 \cdots 78798081$ be the integer that consists of all the integers from 1 to 81 written from left to right. What is the remainder of X when divided by 2020?

20. Let $A = \frac{1}{7} \times 3.14\dot{1}\dot{5}$, where $3.14\dot{1}\dot{5}$ is the rational number with recurring digits 15. In other words,

$$3.14\dot{1}\dot{5} = 3.14 + 0.0015 + 0.000015 + 0.00000015 + \cdots .$$

Suppose that $A = \frac{m}{n}$, where m and n are positive integers with no common factors larger than 1. What is the value of $m + n$?

21. Ali and Barry went running on a standard 400 metre track. They started simultaneously at the same location on the track but ran in opposite directions. Coincidentally, after 24 minutes, they ended at the same location where they started. Ali completed 12 rounds of the track in those 24 minutes while Barry completed 10 rounds. How many times did Ali and Barry pass each other during the run? (Exclude from your answer the times that they met at the start of the of run and when they completed the run after 24 minutes.)

22. If we have

$$(1-3x)+(1-3x)^2+\cdots+(1-3x)^{100}=a_0+a_1x+a_2x^2+\cdots+a_{100}x^{100},$$

for some integers $a_0, a_1, \ldots, a_{100}$, what is the value of

$$\left|\frac{a_1}{3}+\frac{a_2}{3^2}+\cdots+\frac{a_{100}}{3^{100}}\right|?$$

23. What is the value of $\sqrt{27-\sqrt{704}}+\sqrt{27+\sqrt{704}}$?

24. If $m>n$ are positive integers satisfying

$$(m^2-n^2)^2=1+80n,$$

what is the smallest possible value of mn?

25. What is the value of

$$\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{37}\right)+\left(\frac{2}{3}+\frac{2}{4}+\frac{2}{5}+\cdots+\frac{2}{37}\right)$$
$$+\left(\frac{3}{4}+\frac{3}{5}+\frac{3}{6}+\cdots+\frac{3}{37}\right)+\cdots+\left(\frac{35}{36}+\frac{35}{37}\right)+\frac{36}{37}\ ?$$

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Junior Section (Round 1 Solutions)

1. Answer: (D)

   Rewrite $x = 2^{300} = (2^3)^{100} = 8^{100}$ and $y = 3^{200} = (3^2)^{100} = 9^{100}$. Thus $z < x < y$.

2. Answer: (C)

   Since Alfred was lying, we can deduce that both Boris and Dan were not guilty. Similarly, the other statements reveal that everyone except Chucky was not guilty. Hence Chucky stole the painting.

3. Answer: (D)

   Comparing the first two vertical chains, we know that $\triangle < \circ$. Likewise, comparing the third and fourth vertical chains, we know that $\triangle < \square$. So $\triangle$ is the lightest. Finally, comparing the first pair and the last pair of chains we can conclude $\circ < \square$.

4. Answer: (D)

   Integers with exactly four positive factors must be either of the form p^3 or pq, where p and q are distinct primes. The only possibility of p^3 in the given range is 27. There are exactly twelve possibilities of pq, namely 2×5, 2×7, 2×11, 2×13, 2×17, 2×19, 2×23, 3×5, 3×7, 3×11, 3×13 and 5×7.

5. Answer: (E)

   First purchase 43 bottles and you can recycle 40 of these bottles for another 10 bottles. After consuming these 10, you would have $10 + 3$ empty bottles left. You can exchange 12 for another 3 bottles. After consuming these 3, you have $3 + 1$ empty bottles which can be exchanged for another bottle. So in total you can consume $43 + 10 + 3 + 1 = 57$ bottles.

6. Answer: 2121

   Since $2020 = 4 \times 5 \times 101$. Thus the smallest possible value of n is 101 while the largest possible value is 2020. Hence their sum is 2121.

7. Answer: 4

   First observe that since the remainder and quotient are both even, $\overline{2x6yx}$ must be even. Furthermore, as the remainder must be less than $\overline{5y27}$, x can only be 2 or 4. Now,

$$\begin{aligned}\overline{2x6yx} &= 4 \times \overline{5y27} + \overline{x106} \\ &= 4 \times (5027) + 4y \times 100 + x \times 1000 + 106 \\ &= 20214 + 4y \times 100 + x \times 1000.\end{aligned}$$

   Since the last two terms do not contribute to the tens and ones digits, we see that $x = 4$ and $y = 1$.

8. Answer: 84

   Since $11 \times 183 = 2013$ and $11 \times 184 = 2024$, we know that there are exactly 183 multiples of 11 in the sequence $1, 2, \ldots, 2020$. On the other hand, there are exactly 99 multiples of 11 in the sequence $1, 2, \ldots, 1099$. The required answer is thus the difference $183 - 99 = 84$.

9. Answer: 12

   Since the mode equals the median, we must have $b = c$ which must be strictly more than 10. The only possibilities are $b = c = 11$, $b = c = 12$ or $b = c = 13$, because in each case, we still require $c \leq d$. For $(a, 11, 11, 18 - a)$, a can range from 1 to 7. For $(a, 12, 12, 16 - a)$, a can range from 1 to 4. For $(a, 13, 13, 14 - a)$, a can only take the value of 1. So there are 12 possible skewed quadruples satisfying the given requirements.

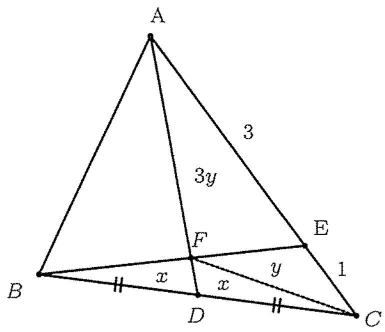
10. Answer: 15850

    Given that $T_n = an + b$, it is straightforward to determine that $T_n = 3n + 7$. We can also do likewise to deduce that $S_n = \frac{1}{2}(3n^2 + 17n)$. Thus

$$S_{101} - T_{101} = \frac{3(101)^2 + 17(101)}{2} - 310 = 15850.$$

    In fact, we have $S_n = T_1 + T_2 + \cdots + T_n$ which means $S_{101} - T_{101} = S_{100}$.

11. Answer: 20



    If we let the area of triangle BFD denoted as $[BFD]$ be x, then $[FCD] = x$. If we let $[FEC] = y$, then $[FAE] = 3y$, due to the ratio of the lengths of their respective bases. We know that

$$[ACD] = 3y + y + x = x + 4y = \frac{1}{2} \cdot 280 = 140.$$

    Similarly,

$$[BEC] = x + x + y = 2x + y = \frac{1}{4} \cdot 280 = 70.$$

    Solving the two equations for x gives $x = 20$.

12. Answer: 93860

    The first column from the right means E=0. The fifth and sixth columns from the right means R=1 and S is between 5 and 9, but not 6 which is already used.

$$\begin{array}{cccccc} & S & H & A & 6 & 0 \\ + & S & H & A & 6 & 0 \\ \hline 1 & A & T & T & 2 & 0. \end{array}$$

Since $6+6=12$, we can conclude T is odd. Furthermore, A and H must differ by 5 in order for 2A and 2H to have the same last digit. In particular, A=H+5. (A,H) cannot be the pairs (1,6) and (2,7). Neither can A be 9 which would result in T being 9. Thus A=8 and H=3 is the only possibility. T=7 and S=9 follow.

13. Answer: 23

Let r_1, r_2, r_3 and r_4 be the respective radii of the semi-circles ordered in a decreasing manner. Then, $2r_1 = 2r_2 + 2r_3 + 2r_4$ and the perimeter is

$$(r_1 + r_2 + r_3 + r_4)\pi = 2r_1\pi \implies r_1 = 9.$$

The enclosed area measures

$$k\pi = \frac{1}{2}\pi\Big(r_1^2 - (r_2^2 + r_3^2 + r_4^2)\Big).$$

Since the r_i are distinct integers, the following are the only possibilities

$$(r_2, r_3, r_4) = (6,2,1) \text{ or } (5,3,1) \text{ or } (4,3,2),$$

which results in areas of $20\pi, 23\pi$ and 26π respectively. Since k is prime, it must be 23.

14. Answer: 1250

The area of a kite is given by $\frac{1}{2}xy$, where x and y are the diagonals which satisfy $x+y=100$. We thus have to maximize

$$\frac{1}{2}xy = \frac{1}{2}x(100-x),$$

which is a quadratic with maximum occurring at $x=50$. Hence the largest possible area is $\frac{1}{2}\cdot 50^2 = 1250$.

Alternatively, by the AM-GM inequality

$$\frac{1}{2}xy \le \frac{1}{8}(x+y)^2 = \frac{1}{8}(100)^2 = 1250,$$

with equality when $x=y=50$.

15. Answer: 52206

From the divisibility tests for 3 for 11, we have

$$3 \mid 5 + x + 2 + 0 + y \implies 3 \mid x + y + 1$$

and

$$11 \mid y - 0 + 2 - x + 5 \implies 11 \mid 7 - x + y.$$

It is easy to check that only three pairs $(x,y) = (2,6), (5,9)$ or $(9,2)$ satisfy both conditions.

16. Answer: 27

Let $k = 2017$, then

$$(k+3)^3 - k^3 = 3k^2 \times 3 + 3k \times 3^2 + 3^3 = 9(k+3)(k) + 27.$$

Hence 27 is the required answer.

17. Answer: 5082

If we let $[ADF] = x$, then $[AECF] = 3x$ and $[ABE] = 1.5x$. Consequently, $[ABCD] = 5.5x$. We have

$$\frac{\frac{1}{2}|DF|}{|DC|} = \frac{x}{5.5x} \implies \frac{|DF|}{|DC|} = \frac{4}{11}$$

and

$$\frac{\frac{1}{2}|BE|}{|BC|} = \frac{1.5x}{5.5x} \implies \frac{|BE|}{|BC|} = \frac{6}{11}.$$

We can now set $|DF| = 4a$, $|FC| = 7a$, $|BE| = 6b$ and $|EC| = 5b$. This means that

$$[ADF] = x = 22ab, \qquad \text{and} \qquad [ECF] = \frac{1}{2}(5b)(7a).$$

Thus

$$[AEF] = 2037 = 3 \times 22ab - \frac{35}{2}ab = \frac{97}{2}ab.$$

This means $ab = 42$ and so

$$[ABCD] = 5.5x = \frac{11}{2} \times 22ab = 5082.$$

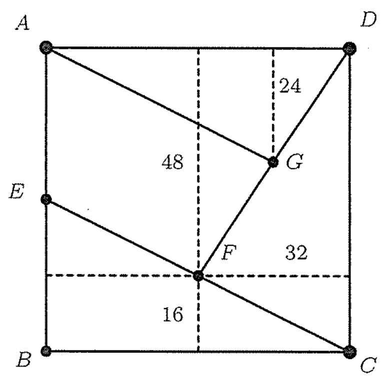
18. Answer: 1280

We have

$$[EBC] = \frac{1}{2} \times 32 \times 64 = 1024.$$

By drawing the line parallel to BC, passing through point F, and using congruent triangles, we observe that the altitude from F to CD is 32. Hence

$$[FCD] = \frac{1}{2} \times 32 \times 64 = 1024.$$



Using similar triangles, we can also observe that the altitude from F to BC is 16. This allows us to show that the altitude from G to AD is 24, thus

$$[ADG] = \frac{1}{2} \times 24 \times 64 = 768.$$

This means that the required area

$$[AEFG] = 64^2 - [EBC] - [FCD] - [ADG] = 1280.$$

19. Answer: 81

First note that $2020 = 20 \times 101$. Since $100^n \equiv (-1)^n \pmod{101}$, we have

$$X \equiv (81-80)+(79-78)+\cdots+(11-10)+(89-67)+(45-23)+1 \equiv 81 \pmod{101}.$$

On the other hand, $X \equiv 1 \equiv 81 \pmod{20}$. So $X - 81$ is divisible by both 20 and 101 and thus divisible by 2020. In other words, X has remainder 81 when divided by 2020.

20. Answer: 4781

Since

$$7A = 3.14\overline{15} \implies 700A = 314.15\overline{15}$$

and thus

$$693A = 311.01 \implies A = \frac{31101}{693 \times 100}.$$

It remains to check that $31101 = 3 \times 7 \times 1481$ and $693 = 3 \times 7 \times 33$. So

$$A = \frac{m}{n} = \frac{1481}{3300},$$

and $m + n = 4781$.

21. Answer: 21

Assuming that they both ran at constant speed throughout. Ali would pass Barry after completing every $\frac{12}{10+12}$ of 400 metres. Since he completed 12×400 metres, he would pass Barry

$$(12 \times 400) \div \left(\frac{12}{22} \times 400\right) = 22$$

times. That includes the very last time, so the required answer is 21. Note that the answer still holds even if they did not run at constant speed.

22. Answer: 100

First note that by expanding as a series in x, the constant term on the left side is 100, hence a_0 is 100. If we substitute $x = \frac{1}{3}$, then the left side equals 0, thus

$$a_0 + \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots + \frac{a_{100}}{3^{100}} = 0$$

which means

$$\left|\frac{a_1}{3} + \frac{a_2}{3^2} + \cdots + \frac{a_{100}}{3^{100}}\right| = |-100| = 100.$$

23. Answer: 8

Let $x = \sqrt{27-\sqrt{704}} + \sqrt{27+\sqrt{704}}$, then

$$\begin{aligned} x^2 &= 27 - \sqrt{704} + 27 + \sqrt{704} + 2\sqrt{(27-\sqrt{704})(27+\sqrt{704})} \\ &= 54 + 2\sqrt{27^2 - 704} \\ &= 64. \end{aligned}$$

Since x is positive, it must equal 8.

24. Answer: 380

We first observe that $1+80n$ must be a perfect square. For small values of n from 1 to 2, $1+80n$ is a perfect square only when $n=1$. In which case,

$$(m^2-1)^2 = 81 \implies m^2 = 1 \pm 9,$$

which has no integer solutions.

Suppose that $m \geq n+2$, then

$$1+80n = (m^2-n^2)^2 \geq (n^2+4n+4-n^2)^2 \implies 80n \geq 16(n+1)^2 - 1.$$

The above inequality holds only for $n \leq 2$ and we have seen that no positive integer solutions arise.

If $m = n+1$, then

$$1+80n = (2n+1)^2 \implies 20n = n^2+n \implies n = 19 \text{ or } 0.$$

Hence $n=19$ and $m=20$ are the required solutions giving $mn = 380$.

25. Answer: 333

The sum is equal to

$$\frac{1}{2} + \left(\frac{1}{3}+\frac{2}{3}\right) + \left(\frac{1}{4}+\frac{2}{4}+\frac{3}{4}\right) + \left(\frac{1}{5}+\frac{2}{5}+\frac{3}{5}+\frac{4}{5}\right) + \cdots + \left(\frac{1}{37}+\frac{2}{37}+\frac{3}{37}+\cdots+\frac{36}{37}\right),$$

which equals $\displaystyle\sum_{n=1}^{36} \frac{1+\cdots+n}{n+1} = \sum_{n=1}^{36} \frac{n(n+1)}{2(n+1)} = \frac{36 \times 37}{4} = 333.$

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Senior Section (Round 1)

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Multiple Choice Questions

1. Let b be a positive integer. If the minimum possible value of the quadratic function $5x^2 + bx + 506$ is 6, find the value of b.

   (A) 90 (B) 100 (C) 110 (D) 120 (E) 130

2. Which of the following is equal to

$$\sqrt{5+\sqrt{3}}+\sqrt{5-\sqrt{3}}\ ?$$

   (A) $\sqrt{10-\sqrt{22}}$ (B) $\sqrt{10+\sqrt{22}}$ (C) $\sqrt{10-2\sqrt{22}}$
   (D) $\sqrt{10+2\sqrt{22}}$ (E) None of the above

3. Simplify

$$\log_8 5 \cdot (\log_5 3 + \log_{25} 9 + \log_{125} 27).$$

   (A) $\log_2 3$ (B) $\log_3 2$ (C) $\log_2 9$ (D) $\log_3 16$ (E) $\log_2 27$

4. Let $a = 50^{\frac{1}{505}}$, $b = 10^{\frac{1}{303}}$ and $c = 6^{\frac{1}{202}}$. Which of the following is true?

   (A) $a < b < c$ (B) $a < c < b$ (C) $b < a < c$ (D) $b < c < a$ (E) $c < b < a$

5. Let $p = \log_{10}(\sin x)$, $q = (\sin x)^{10}$, $r = 10^{\sin x}$, where $0 < x < \dfrac{\pi}{2}$. Which of the following is true?

   (A) $p < q < r$ (B) $p < r < q$ (C) $q < r < p$ (D) $q < p < r$ (E) $r < p < q$

Short Questions

6. Find the minimum possible value of $|x-10| - |x-20| + |x-30|$, where x is any real number.

7. Parallelogram $ABCD$ has sides $AB = 39$ cm and $BC = 25$ cm. Find the length of diagonal AC (in cm) if diagonal $BD = 34$ cm.

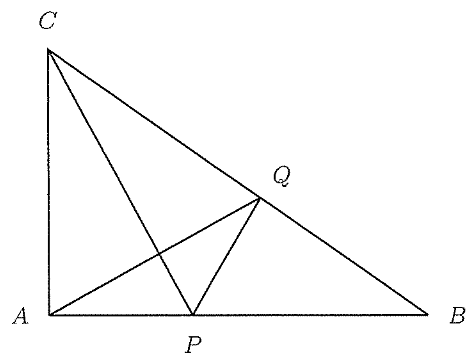
8. Suppose $\sin(45^\circ - x) = -\frac{1}{3}$, where $45^\circ < x < 90^\circ$. Find $(6\sin x - \sqrt{2})^2$.

9. If $8\cos x - 8\sin x = 3$, find the value of $55\tan x + \dfrac{55}{\tan x}$.

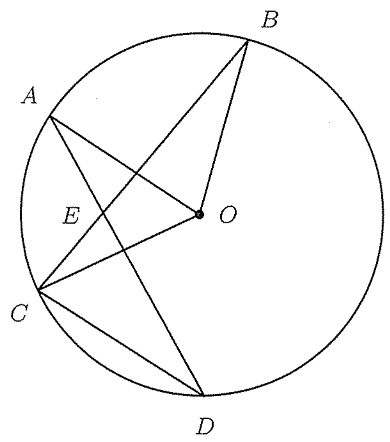
10. Find the number of ordered pairs (x, y), where x and y are integers, such that

$$x^2 + y^2 - 20x - 14y + 140 < 0.$$

11. The figure below shows a right-angled triangle ABC such that $\angle BAC = 90°$, $\angle ABC = 30°$ and $AB = 48$ cm. Let P be a point on side AB such that CP is the angle bisector of $\angle ACB$ and Q be a point on side BC such that line AQ is perpendicular to line CP. Determine the length of PQ.



12. In the figure below, the point O is the center of the circle, AD and BC intersect at E, and $\angle AEB = 70°$, $\angle AOB = 62°$. Find the angle $\angle OCD$ (in degree °).



13. Find the value of $\dfrac{4\cos 43°}{\sin 73°} - \dfrac{12\sin 43°}{\sqrt{3}\sin 253°}$.

14. If $\dfrac{x^2}{5} + \dfrac{y^2}{7} = 1$, find the largest possible value of $(x + y)^2$.

15. Find the coefficient of x^6 in the expansion of $(1 + x + 2x^2)^7$.

16. Suppose $(3x - y)^2 + \sqrt{x + 38 + 14\sqrt{x - 11}} + |z + x - y| = 7$. Find the value of $|x + y + z|$.

17. Suppose there are real numbers x, y, z satisfying the following equations:

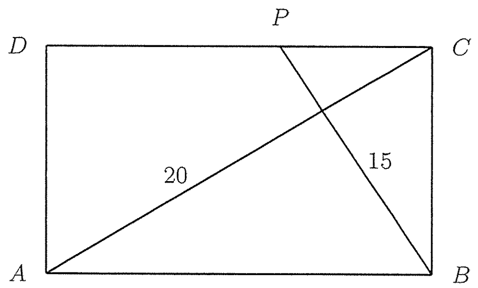
$$x + y + z = 60, \quad xy - z^2 = 900.$$

Find the maximum possible value of $|z|$.

18. Find the sum $\displaystyle\sum_{k=1}^{16} \log_2 \left(\sqrt{\sin^2 \frac{k\pi}{8} + 1} - \sin \frac{k\pi}{8} \right)$

19. Let a, b be positive real numbers, where $a > b$. Suppose there exists a real number x such that $(\log_2 ax)(\log_2 bx) + 25 = 0$. Find the minimum possible value of $\frac{a}{b}$.

20. The figure below shows a rectangle $ABCD$ such that the diagonal $AC = 20$ cm. Let P be a point on side CD such that BP is perpendicular to diagonal AC. Find the area of rectangle $ABCD$ (in cm^2) if $BP = 15$ cm.



21. Find the smallest positive integer that is greater than the following expression:

$$\left(\sqrt{7} + \sqrt{5}\right)^4.$$

22. Find the number of non-congruent right-angled triangles such that the length of all their sides are integers and that the hypotenuse has a length of 65 cm.

23. There are 6 couples, each comprising a husband and a wife. Find the number of ways to divide the 6 couples into 3 teams such that each team has exactly 4 members, and that the husband and the wife from the same couple are in different teams.

24. The **digit sum** of a number, say 987, is the sum of its digits, $9 + 8 + 7 = 24$. Let A be the digit sum of 2020^{2021}, and let B be the digit sum of A. Find the digit sum of B.

25. $40 = 2 \times 2 \times 2 \times 5$ is a positive divisor of 1440 that is a product of 4 prime numbers. $48 = 2 \times 2 \times 2 \times 2 \times 3$ is a positive divisor of 1440 that is a product of 5 prime numbers. Find the sum of all the positive divisors of 1440 that are products of an odd number of prime numbers.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2020

Senior Section (Round 1 Solutions)

1. **Answer.** (B)

Solution. By completing squares, we have

$$5x^2 + bx + 506 = 5\left(x + \frac{b}{10}\right)^2 - \frac{b^2}{20} + 506 \geq -\frac{b^2}{20} + 506,$$

where the lower bound is attained when $x = -\frac{b}{10}$. Thus, $-\frac{b^2}{20} + 506 = 6$ implies that $b = 100$. $\square$

2. **Answer.** (D)

Solution. Let $a = \sqrt{5+\sqrt{3}} + \sqrt{5-\sqrt{3}}$. Then

$$\begin{aligned} a^2 &= 5+\sqrt{3}+5-\sqrt{3}+2\sqrt{(5+\sqrt{3})(5-\sqrt{3})} \\ &= 10+2\sqrt{5^2-3} \\ &= 10+2\sqrt{22} \\ a &= \sqrt{10+2\sqrt{22}}. \quad \square \end{aligned}$$

3. **Answer.** (A)

Solution.

$$\begin{aligned} \log_8 5 \cdot (\log_5 3 + \log_{25} 9 + \log_{125} 27) &= \log_8 5 \cdot \left(\log_5 3 + \frac{\log_5 3^2}{\log_5 5^2} + \frac{\log_5 3^3}{\log_5 5^3}\right) \\ &= \log_8 5 \cdot (\log_5 3 + \log_5 3 + \log_5 3) \\ &= \log_8 5 \cdot \log_5 27 \\ &= \log_8 5 \cdot \frac{\log_8 27}{\log_8 5} \\ &= \log_8 27 \\ &= 3\log_8 3 = \frac{3}{\log_3 8} = \frac{3}{3\log_3 2} = \log_2 3. \quad \square \end{aligned}$$

4. **Answer.** (C)

Solution. We have $a^{3030} = (a^{505})^6 = 50^6$, $b^{3030} = (b^{303})^{10} = 10^{10}$, $c^{3030} = (c^{202})^{15} = 6^{15}$.

Notice that $5^2 > 2^4 \implies 5^6 > 5^4 2^4 = 10^4 \implies 50^6 = 5^6 10^6 > 10^4 10^6 = 10^{10}$. Thus, $a^{3030} > b^{3030}$.

Notice that $3^6 > 6^3 \implies 3^{15}2^9 = 3^6 6^9 > 6^3 6^9 = 6^{12} > 5^{12} \implies 6^{15} = 3^{15}2^9 2^6 > 5^{12}2^6 = (5^2 \times 2)^6 = 50^6$. Thus, $c^{3030} > a^{3030}$.

We deduce that $b^{3030} < a^{3030} < c^{3030}$, which implies that $b < a < c$. $\square$

5. **Answer.** A

Solution. $p = \log_{10}(\sin x) < 0$, since $\sin x < 1$. Note that $10p < 0 < 10^p \implies q = 10^{10p} < 10^{10^p} = r$. Hence, $p < 0 < q < r$. $\square$

6. **Answer.** 10

Solution. Let $f(x) = |x-10|-|x-20|+|x-30|$. For $x \leq 10$, $f(x) = (10-x)-(20-x)+(30-x) = 20-x \geq 10$. For $10 \leq x \leq 20$, $f(x) = (x-10)-(20-x)+(30-x) = x \geq 10$. For $20 \leq x \leq 30$, $f(x) = (x-10)-(x-20)+(30-x) = 40-x \geq 10$. For $x \geq 30$, $f(x) = (x-10)-(x-20)+(x-30) = x-20 \geq 10$. Hence, the minimum possible value of $f(x)$ is 10.

7. **Answer.** 56

Solution.

$$\begin{aligned}
\cos(\angle BAD) &= \frac{39^2+25^2-34^2}{2\cdot 39\cdot 25} = \frac{990}{2\cdot 39\cdot 25} \\
\cos(\angle ABC) &= -\cos(\angle BAD) = -\frac{990}{2\cdot 39\cdot 25} \\
AC^2 &= 39^2+25^2-2\cdot 39\cdot 25\cdot\left(-\frac{990}{2\cdot 39\cdot 25}\right) \\
&= 1521+625+990 = 3136 \\
AC &= \sqrt{3136} = 56.
\end{aligned}$$

$\square$

8. **Answer.** 16

Solution. Since $45° < x < 90°$, we have $-45° < 45° - x < 0°$. Thus, $\cos(45° - x) =$

$\sqrt{1-\sin^2(45^\circ - x)} = \sqrt{1-\frac{1}{9}} = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$.

$$\begin{aligned}
\sin x &= \sin(45^\circ - (45^\circ - x)) \\
&= \sin 45^\circ \cos(45^\circ - x) - \cos 45^\circ \sin(45^\circ - x) \\
&= \frac{\sqrt{2}}{2}\frac{2\sqrt{2}}{3} - \frac{\sqrt{2}}{2}\left(-\frac{1}{3}\right) \\
&= \frac{4+\sqrt{2}}{6}
\end{aligned}$$

Thus, $(6\sin x - \sqrt{2})^2 = 4^2 = 16$. □

9. **Answer.** 128

Solution.

$$\begin{aligned}
8\cos x - 8\sin x &= 3 \\
8 - 8\tan x &= 3\sec x \\
(8-8\tan x)^2 &= 9\sec^2 x \\
64 - 128\tan x + 64\tan^2 x &= 9\tan^2 x + 9 \\
55\tan^2 x + 55 &= 128\tan x \\
55\tan x + \frac{55}{\tan x} &= 128.
\end{aligned}$$

□

10. **Answer.** 25

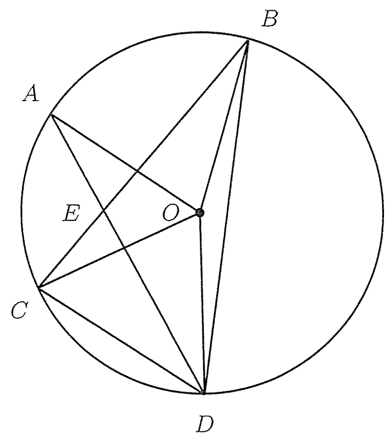
Solution. Notice that $x^2 + y^2 - 20x - 14y + 140 = 0 \iff (x-10)^2 + (y-7)^2 = 3^2$. So it is enough to count the number of coordinates (x, y), where x and y are integers, inside a circle centred at $(0, 0)$ with radius 3. There are 25 such coordinates. □

11. **Answer.** 16

Solution. Let M be the point of intersection of CP and AQ. Then $AM = MQ$. This implies that $AP = PQ$ and $\angle CQP = \angle CAP = 90^\circ$. Hence, $\frac{1}{2} = \sin 30^\circ = \sin\angle ABC = \frac{PQ}{PB}$. Let $x = AP$. Then $PB = 48 - x$ and $PQ = AP = x$. So $\frac{x}{48-x} = \frac{1}{2}$, $x = 16$.

12. **Answer.** 51

Solution. Draw the line OD and BD as shown below, and let $x^\circ = \angle COD$. Then $\angle CBD = \frac{x}{2}$. Also, notice that $\angle AOB = 2\angle ADB \Longrightarrow \angle ADB = \frac{1}{2}\angle AOB = 31^\circ$. Then $\angle ADB + \angle CBD = \angle AEB$ implies that $31 + \frac{x}{2} = 70 \Longrightarrow x = 78^\circ$. Hence, $\angle OCD = \frac{1}{2}(180 - x) = \frac{1}{2}(180 - 78) = 51^\circ$. □



$\square$

13. **Answer.** 8

Solution.

$$\begin{aligned}\frac{4\cos 43^\circ}{\sin 73^\circ} - \frac{12\sin 43^\circ}{\sqrt{3}\sin 253^\circ} &= \frac{4\cos 43^\circ}{\sin 73^\circ} + \frac{4\sqrt{3}\sin 43^\circ}{\sin 73^\circ} \\ &= \frac{8}{\sin 73^\circ}\left(\frac{1}{2}\cos 43^\circ + \frac{\sqrt{3}}{2}\sin 43^\circ\right) \\ &= \frac{8}{\sin 73^\circ}(\sin 30^\circ \cos 43^\circ + \cos 30^\circ \sin 43^\circ) \\ &= \frac{8}{\sin 73^\circ}\cdot \sin(30^\circ + 43^\circ) = 8.\end{aligned}$$

$\square$

14. **Answer.** 12

Solution. Let $z = x + y$. Then $y = z - x$, and

$$\begin{aligned}\frac{x^2}{5} + \frac{(z-x)^2}{7} &= 1 \\ 7x^2 + 5(z-x)^2 &= 35 \\ 7x^2 + 5(z^2 - 2xz + x^2) &= 35 \\ 12x^2 - 10zx + 5z^2 - 35 &= 0.\end{aligned}$$

Since x is a real number, the discriminant of the quadratic equation in x is non-negative:

$$(-10z)^2 - 4(12)(5z^2 - 35) \geq 0$$

$$100z^2 - 240z^2 + 1680 \geq 0$$

$$z^2 \leq 12.$$

Thus, the largest possible value of $(x+y)^2 = 12$, and this is attained when $x = \frac{5}{\sqrt{12}}$ and $y = \frac{7}{\sqrt{12}}$. $\square$

15. **Answer.** 1337

Solution.

$$\begin{aligned}(1+x+2x^2)^7 &= \sum_{i=0}^{7}\binom{7}{i}(x+2x^2)^i = \sum_{i=0}^{7}\binom{7}{i}x^i(1+2x)^i \\ &= \sum_{i=0}^{7}\binom{7}{i}x^i\sum_{j=0}^{i}\binom{i}{j}(2x)^j = \sum_{i=0}^{7}\sum_{j=0}^{i}\binom{7}{i}\binom{i}{j}2^jx^{i+j}. \qquad (1)\end{aligned}$$

For $i+j=6$, $j\le i$, we have the following possibilities:

i	j	$i+j$	$\binom{7}{i}\binom{i}{j}2^j$
3	3	6	$\binom{7}{3}\binom{3}{3}2^3 = 35\cdot 1\cdot 8 = 280$
4	2	6	$\binom{7}{4}\binom{4}{2}2^2 = 35\cdot 6\cdot 4 = 840$
5	1	6	$\binom{7}{5}\binom{5}{1}2^1 = 21\cdot 5\cdot 2 = 210$
6	0	6	$\binom{7}{6}\binom{6}{0}2^0 = 7\cdot 1\cdot 1 = 7$

The coefficient of x^6 in the expansion of $(1+x+2x^2)^7$ is

$$280+840+210+7=1337.$$

□

16. **Answer.** 66

Solution.

$$\begin{aligned}(3x-y)^2+\sqrt{x+38+14\sqrt{x-11}}+|z+x-y| &= 7\\ (3x-y)^2+\sqrt{x-11+14\sqrt{x-11}+49}+|z+x-y| &= 7\\ (3x-y)^2+\sqrt{(\sqrt{x-11}+7)^2}+|z+x-y| &= 7\\ (3x-y)^2+\sqrt{x-11}+7+|z+x-y| &= 7\\ (3x-y)^2+\sqrt{x-11}+|z+x-y| &= 0.\end{aligned}$$

This implies that $(3x-y)^2=0$, $\sqrt{x-11}=0$ and $|z+x-y|=0$, whence

$$x=11, y=33, z=22.$$

Thus, $|x+y+z| = 11+33+22=66.$ □

17. **Answer.** 40

Solution. Note that $x+y=60-z$ and $xy=900+z^2$. It follows that the quadratic equation $t^2-(x+y)t+xy = t^2-(60-z)t+(900+z^2)=0$ has real roots, i.e. $(60-z)^2-4(900+z^2)\ge 0 \Longrightarrow -40\le z\le 0 \Longrightarrow |z|\le 40$. Note that $x=y=50$, $z=-40$ is a solution to the given equations. Hence, the maximum possible value of $|z|$ is 40. □

18. **Answer.** 0

Solution Let $f(x) = \log_2(\sqrt{x^2+1} - x)$. The function f is defined for all real number x since $\sqrt{x^2+1} - x > 0$. Notice that $f(-x) = \log_2(\sqrt{x^2+1} + x) = \log_2\left(\frac{1}{\sqrt{x^2+1}-x}\right) = \log_2(\sqrt{x^2+1} - x)^{-1} = -\log_2(\sqrt{x^2+1} - x) = -f(x)$. So the function f is odd.

As k runs from 1 to 16, we have

$$\sin\frac{9\pi}{8} = -\sin\frac{\pi}{8}, \sin\frac{10\pi}{8} = -\sin\frac{2\pi}{8}, \ldots, \sin\frac{15\pi}{8} = -\sin\frac{7\pi}{8}, \sin\frac{16\pi}{8} = -\sin\frac{8\pi}{8}.$$

Let $x_k = \sin\frac{k\pi}{8}$. It follows that

$$\begin{aligned}
\sum_{k=1}^{16} \log_2\left(\sqrt{\sin^2\frac{k\pi}{8} + 1} - \sin\frac{k\pi}{8}\right) &= \sum_{k=1}^{16} f(x_k) \\
&= \sum_{k=1}^{8} f(x_k) + \sum_{k=9}^{16} f(x_k) \\
&= \sum_{k=1}^{8} f(x_k) + \sum_{k=1}^{8} f(-x_k) \\
&= \sum_{k=1}^{8}(f(x_k) + f(-x_k)) = \sum_{k=1}^{8}(f(x_k) - f(x_k)) = 0. \quad \square
\end{aligned}$$

19. **Answer.** 1024

Solution. We will drop the notation for the base and write $\log x$ in place of $\log_2 x$. Then $(\log a + \log x)(\log b + \log x) + 25 = 0$ implies that the following quadratic equation in $\log x$ has real roots: $(\log x)^2 + \log x(\log a + \log b) + (\log a)(\log b) + 25 = 0$. Thus,

$$\begin{aligned}
(\log a + \log b)^2 - 4(\log a \log b + 25) &\geq 0 \\
(\log a - \log b)^2 &\geq 10^2
\end{aligned}$$

which implies that either $\log a - \log b \geq 10$ or $\log a - \log b \leq -10$. The later is not possible since $a > b$. Hence, $\log\frac{a}{b} = \log a - \log b \geq 10 \implies \frac{a}{b} \geq 2^{10} = 1024$.

The lower bound is attained if $a = 2^{10}$, $b = 1$, $x = 2^{-5}$.

$\square$

20. **Answer.** 192

Solution. We first note that triangles ABC and BCP are similar. If $CP = x$ and $BC = kx$, then $AB = k^2x$. Applying Pythogoras theorem, we have the following relationships:

$$\begin{aligned}
x^2 + (kx)^2 &= x^2(1+k^2) = 15^2 = 225 && (2) \\
(kx)^2 + (k^2x)^2 &= x^2k^2(1+k^2) = 20^2 = 400. && (3)
\end{aligned}$$

Dividing (3) by (2) gives $k^2 = \left(\frac{20}{15}\right)^2 = \left(\frac{4}{3}\right)^2$. So $k = \frac{4}{3}$ and $x = \sqrt{\frac{15^2}{1+\left(\frac{4}{3}\right)^2}} = 9$.

Hence, $CP = x = 9$ cm, $BC = kx = \frac{4}{3} \times 9 = 12$ cm and $AB = k^2x = \left(\frac{4}{3}\right)^2 \times 9 = 16$ cm. Therefore, the area of rectangle $ABCD$ is $AB \times BC = 16 \times 12 = 192$ cm^2. $\square$

21. **Answer.** 568

Solution. We will use the following formula:

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

Note that

$$\begin{aligned}
\left(\sqrt{7}+\sqrt{5}\right)^4 + \left(\sqrt{7}-\sqrt{5}\right)^4 &= \sqrt{7}^4 + 4\sqrt{7}^3\sqrt{5} + 6\sqrt{7}^2\sqrt{5}^2 + 4\sqrt{7}\sqrt{5}^3 + \sqrt{5}^4 \\
&\quad + \sqrt{7}^4 - 4\sqrt{7}^3\sqrt{5} + 6\sqrt{7}^2\sqrt{5}^2 - 4\sqrt{7}\sqrt{5}^3 + \sqrt{5}^4 \\
&= 2\left(\sqrt{7}^4 + 6\sqrt{7}^2\sqrt{5}^2 + \sqrt{5}^4\right) \\
&= 2(49 + 210 + 25) \\
&= 568.
\end{aligned}$$

Since $2 < \sqrt{5} < \sqrt{7} < 3$, we deduce that $0 < \sqrt{7} - \sqrt{5} < 1$ and that $0 < \left(\sqrt{7}-\sqrt{5}\right)^4 < 1$. So we have the following inequality:

$$567 < \left(\sqrt{7}+\sqrt{5}\right)^4 < 568.$$

Hence, 568 is the smallest positive integer that is greater than $\left(\sqrt{7}+\sqrt{5}\right)^4$. $\square$

22. **Answer.** 4

Solution. Let x and y be the sides of the right-angled triangle. Then $x^2 + y^2 = 65^2 = 5^2 \cdot 13^2$. We consider the following cases.

- x and y have a factor of 5. Then $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{5}\right)^2 = 13^2 \Longrightarrow \frac{x}{5} = 5$, $\frac{y}{5} = 12$ is the only solution and $x = 25$, $y = 60$.
- x and y have a factor of 13. Then $\left(\frac{x}{13}\right)^2 + \left(\frac{y}{13}\right)^2 = 5^2 \Longrightarrow \frac{x}{13} = 3$, $\frac{y}{13} = 4$ is the only solution and $x = 39$ and $y = 52$.
- x and y are relatively coprime. Then

  $$y^2 = 65^2 - x^2 = (65+x)(65-x).$$

  Since there are no common factors that divide all x, y, 5 and 13, the terms $(65+x)$ and $(65-x)$ are perfect squares. Setting $65 - x = 1, 4, 9, 16, 25, 36, 49, 64$ and checking the corresponding values of $65 + x$ reveal two solutions:

  $$65 - x = 9,\ 65 + x = 121 \Longrightarrow x = 56, y = 33;$$

  $$65 - x = 49,\ 65 + x = 81 \Longrightarrow x = 16, y = 63.$$

  Hence, there are 4 non-congruent right-angled triangles whose hypotenuse is 65 cm long. $\square$

23. **Answer.** 960

Solution. We assign a child K_i to each couple comprising a husband H_i and wife W_i to form a family F_i, i.e. $F_i = \{H_i, W_i, K_i\}$, for $i = 1, 2, ..., 6$. We will count the number of ways to divide the 6 families into 3 teams such that each team has exactly 6 members, 2 of which are children, and that no two person in the same family belong to the same team.

We first count the number of ways to distribute the children such that there are exactly 2 children in a team. Since there are 6 children altogether, there are $\binom{6}{2}$ ways to assign 2 children to the first team, $\binom{4}{2}$ ways to assign another 2 children to the second team, and $\binom{2}{2}$ ways to assign the last 2 children to the third team. Since the order of the teams does not matter, there are

$$\frac{1}{3!} \times \binom{6}{2} \times \binom{4}{2} \times \binom{2}{2} = 15$$

ways to distribute the children such that there are exactly 2 children in a team.

Next, in each of these 15 instances, there are $2^6 = 64$ ways of assigning the husband and the wife (from the same couple) into the other 2 teams that do not contain their child.

Thus, there is a total of $15 \times 64 = 960$ ways to divide the 6 families into 3 teams such that each team has exactly 6 members, 2 of which are children, and that no two persons in the same family belong to the same team. Notice that in each of these 960 instances, removing all the children from the teams results in the 6 couples divided into 3 teams such that each team has 4 members, and that the husband and the wife are in different teams. Conversely, with any 3 teams satisfying the requirement of the question, there is a unique way of assigning the children resulting in 6 families divided into 3 teams satisfying the condition above.

Hence, there is a total of 960 ways to divide the 6 couples into 3 teams such that each team has exactly 4 members, , and that the husband and the wife are not in the same team. □

24. **Answer.** 7

Solution. 10000^{2021} has exactly $4 \times 2021 + 1 = 8085$ digits. Since each digit is at most 9, and $2020^{2021} < 10000^{2021}$, it follows that

$$A < 8085 \times 9 = 72765.$$

Since A has at most 5 digits, and the first digit is at most 7, we deduce that

$$B \leq 7 + 4 \times 9 = 43.$$

Let C be the digit sum of B. Then $C \leq 4 + 9 = 13$.

On the other hand, we have $2020 \equiv 4 \pmod 9 \implies 2020^{2021} \equiv 4^{2021} \equiv (4^3)^{673}4^2 \equiv 4^2 \equiv 7 \pmod 9$.

Since $C \equiv B \equiv A \equiv 2020^{2021} \equiv 7 \pmod 9$ and $C \leq 13$, we deduce that $C = 7$. $\square$

25. **Answer.** 2163

Solution. Let X and Y be the sum of all the positive divisors of 1440 that are products of an even number of prime numbers and an odd number of prime numbers respectively.

$$1440 = 2^5 \times 3^2 \times 5^1.$$

Thus

$$X + Y = (1 + 2 + 2^2 + 2^3 + 2^4 + 2^5) \times (1 + 3 + 3^2) \times (1 + 5) = 4914,$$

$$X - Y = (1 - 2 + 2^2 - 2^3 + 2^4 - 2^5) \times (1 - 3 + 3^2) \times (1 - 5) = 588.$$

Hence, $Y = \frac{1}{2}(4914 - 588) = 2163$, that is, the sum of all the positive divisors of 1440 that are product of an odd number of prime numbers is 2163. $\square$

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2020
(Open Section, Round 1)

Tuesday, 8 September 2020 | 0930-1200 hrs

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

In this paper, let $\mathbb{R}$ denote the set of all real numbers, $\lfloor x \rfloor$ denote the greatest integer not exceeding x and let $\lceil x \rceil$ denote the smallest integer not less than x. For examples, $\lfloor 5 \rfloor = 5$, $\lfloor 2.8 \rfloor = 2$, and $\lfloor -2.3 \rfloor = -3$; $\lceil 5 \rceil = 5$, $\lceil 2.8 \rceil = 3$, and $\lceil -2.3 \rceil = -2$.

1. If S is the sum of all the *real* roots of the equation $x^2 + \dfrac{1}{x^2} = 2020^2 + \dfrac{1}{2020^2}$, find $\lfloor S \rfloor$.

2. Find the largest positive integer x that satisfies the equation
$$(\lfloor x \rfloor - 2020)^2 + (\lceil x \rceil - 2030)^2 = (\lfloor x \rfloor - \lceil x \rceil + 10\rfloor)^2.$$
(Note: If you think that the above equation has no solution in positive integers, enter your answer as "0".)

3. Let $S_n = \dfrac{1}{1 \times 3} + \dfrac{1}{3 \times 5} + \dfrac{1}{5 \times 7} + \cdots + \dfrac{1}{(2n-1) \times (2n+1)}$. Find the value of n such that S_n takes the value of 0.48.

4. Given that the three planes in the Cartesian space with equations $2x + 4y + 6z = 5$, $3x + 5y + 2z = 6$ and $8x + 14y + az = b$ have a common line of intersection, find the value of $a + b$.

5. Let i be the complex number $\sqrt{-1}$, and n be the smallest positive integer such that $(\sqrt{3} + i)^n = a$, where a is a real number. Find the value of $\lfloor n - a \rfloor$.

6. In the three-dimensional Cartesian space, let $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ denote unit vectors along three mutually perpendicular directions x, y and z-axes respectively. Three straight lines l_1, l_2 and l_3 have equations defined by
$$l_1: \quad \mathbf{r} = (4+\lambda)\mathbf{i} + (5+\lambda)\mathbf{j} + (6+\lambda)\mathbf{k},$$
$$l_2: \quad \mathbf{r} = (4+3\mu)\mathbf{i} + (5-\mu)\mathbf{j} + (6-2\mu)\mathbf{k},$$
$$l_3: \quad \mathbf{r} = (1+6\nu)\mathbf{i} + (2+2\nu)\mathbf{j} + (3+\nu)\mathbf{k},$$
where μ, λ and ν are real numbers. If the area of the triangle enclosed by the three lines l_1, l_2 and l_3 is denoted by S, find the value of $10S^2$.

7. Given that $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(a^2 - b^2) = (a - b)\left(f(a) + f(b)\right)$$
for all real numbers a and b, and that $f(1) = \dfrac{1}{101}$. Find the value of $\displaystyle\sum_{k=1}^{100} f(k)$.

8. Find the sum of all the positive integers n such that $n^4 - 4n^3 + 22n^2 - 36n + 18$ is a perfect square.

   (Note: If you think that there are infinitely many positive integers n that satisfy the above conditions, enter your answer as "9999".)

9. Assume that

$$(x+2+m)^{2019} = a_0 + a_1(x+1) + a_2(x+1)^2 + \cdots + a_{2019}(x+1)^{2019}.$$

Find the largest possible integer m such that

$$(a_0 + a_2 + a_4 + \cdots + a_{2018})^2 - (a_1 + a_3 + a_5 + \cdots + a_{2019})^2 \leq 2020^{2019}.$$

10. Given that $S = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n(n+k)}}$. Find the value of $\lfloor (S+2)^2 \rfloor$.

11. Let $A = \{1, 2, \cdots, 10\}$. Count the number of ordered pairs (S_1, S_2), where S_1 and S_2 are non-intersecting and non-empty subsets of A such that the largest number in S_1 is smaller than the smallest number in S_2. For example, if $S_1 = \{1, 4\}$ and $S_2 = \{5, 6, 7\}$, then (S_1, S_2) is such an ordered pair.

12. Each cell of an 2020×2020 table is filled with a number which is either 1 or -1. For $i = 1, \ldots, 2020$, let R_i be the product of all the numbers in the ith row and let C_i be the product of all the numbers in the ith column. Suppose $R_i + C_i = 0$ for all $i = 1, \ldots, 2020$. What is the least number of -1's in the table?

13. Assume that the sequence $\{a_k\}_{k=1}^{\infty}$ follows an arithmetic progression with $a_2+a_4+a_9 = 24$. Find the maximum value of $S_8 \times S_{10}$, where S_k denotes the sum $a_1 + a_2 + \cdots + a_k$.

14. Consider all functions $g : \mathbb{R} \to \mathbb{R}$ satisfying the conditions that

 (i) $|g(a) - g(b)| \leq |a - b|$ for any $a, b \in \mathbb{R}$;

 (ii) $g(g(g(0))) = 0$.

 Find the *largest* possible value of $g(0)$.

15. A sequence $\{a_i\}_{i=1}^{\infty}$ is called a *good* sequence if $\frac{S_{2n}}{S_n}$ is a constant for all $n \geq 1$, where S_k denotes the sum $a_1 + a_2 + \cdots + a_k$. Suppose it is known that the sequence $\{a_i\}_{i=1}^{\infty}$ is a *good* sequence that follows an arithmetic progression. Determine a_{2020} if $a_1 = 1 \neq a_2$.

16. Determine the smallest positive integer p such that the system

$$\begin{cases} 6x + 4y + 3z = 0 \\ 4xy + 2yz + pxz = 0 \end{cases}$$

has more than one set of real solutions in x, y, z.

17. Let ABC be a triangle with $a = BC, b = AC$ and $c = AB$. It is given that $c = 100$ and

$$\frac{\cos A}{\cos B} = \frac{b}{a} = \frac{4}{3}.$$

Let P be a point on the inscribed circle of $\triangle ABC$. Find the maximum value of

$$PA^2 + PB^2 + PC^2.$$

18. Find the largest positive integer n less than 2020 such that $\binom{n-1}{k} - (-1)^k$ is divisible by n for $k = 0, 1, \ldots, n-1$.

19. Assume that $\{a_k\}_{k=1}^{\infty}$ is a sequence with the property that for any distinct positive integers m, n, p, q with $m + n = p + q$, the following equality always holds:

$$\frac{a_m + a_n}{(a_m+1)(a_n+1)} = \frac{a_p + a_q}{(a_p+1)(a_q+1)}.$$

Given $a_1 = 0$ and $a_2 = \frac{1}{2}$, determine $\frac{1}{1-a_5}$.

(Hint: Consider $c_k = \frac{1}{a_k+1} - \frac{1}{2}$ for all positive integer k.)

20. In the triangle ABC, the incircle touches the sides BC, CA, AB at D, E, F respectively. The line segments ED and AB are extended to intersect at P such that $AB = BP = PD$. Suppose $CA = 9$. Find the value of $(ABC)^2$, where (ABC) is the area of the triangle ABC.

21. In an acute-angled triangle ABC, $AB = 75, AC = 53$, the external bisector of $\angle A$ on CA produced meets the circumcircle of triangle ABC at E, and F is the foot of the perpendicular from E onto AB. Find the value of $AF \times FB$.

22. Let $\{a_k\}_{k=1}^{\infty}$ be an increasing sequence with $a_k < a_{k+1}$ for all $k = 1, 2, 3, \cdots$ formed by arranging all the terms in the set $\{2^r + 2^s + 2^t : 0 \leq r < s < t\}$ in increasing order. Find the largest value of the integer n such that $a_n \leq 2020$.

23. Let n be a positive integer and S be the set of all numbers that can be written in the form $\sum_{i=2}^{k} a_{i-1}a_i$ with $a_1, \ldots, a_k$ being positive integers that sum to n. Suppose the average value of all the numbers in S is 88199. Determine n.

24. Let x, y, z and w be real numbers such that $x + y + z + w = 5$. Find the minimum value of $(x+5)^2 + (y+10)^2 + (z+20)^2 + (w+40)^2$.

25. Let p and q be positive integers satisfying the equation $p^2 + q^2 = 3994(p - q)$. Determine the largest possible value of q.

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1. **Answer.** 0.

   Solution. By observation, four roots of the equation can be obtained as ± 2020, $\pm\dfrac{1}{2020}$. There are no other roots since a quartic equation can have only four roots. Hence the sum equals 0. $\square$

2. **Answer.** 2030.

   Solution. The given equation is equivalent to solving $a^2+b^2=(a-b)^2$, which is equivalent to $a=0$ *or* $b=0$. Hence we have $\lfloor x \rfloor = 2020$ or $\lceil x \rceil = 2030$; or equivalently, the solution is either $2020 \leq x < 2021$ or $2029 < x \leq 2030$. Hence the largest integer is 2030. $\square$

3. **Answer.** 12.

   Solution. By using the method of difference, it is easy to see that $S_n = \dfrac{n}{2n+1}$ and equating to 0.48 we obtain $n = 12$. $\square$

4. **Answer.** 27.

   Solution. To solve the system of equations algebraically by formulating them into a matrix and reducing them into a equivalent system of equations, we obtain

$$\begin{pmatrix} 2 & 4 & 6 & 5 \\ 3 & 5 & 2 & 6 \\ 8 & 14 & a & b \end{pmatrix} \to \cdots \to \begin{pmatrix} 1 & 2 & 3 & 2.5 \\ 0 & 1 & 7 & 1.5 \\ 0 & 0 & 10-a & 17-b \end{pmatrix}.$$

   Since the system of equation has infinitely many solutions, we must have $10 - a = 0$ so that $a = 10$ and $17 - b = 0$ so that $b = 17$, hence $a + b = 27$. $\square$

5. **Answer.** 70.

   Solution. Note that $(\sqrt{3}+i)^n = 2^n\left(\cos\dfrac{n\pi}{6} + i\sin\dfrac{n\pi}{6}\right)$. Hence the smallest positive integer $n = 6$, and in this case, $a = -2^6 = -64$, so that $n - a = 6 - (-64) = 70$. $\square$

6. **Answer.** 945.

   Solution. By direct computation, it is easy to check that l_1 and l_2 intersect at the point $A(4,5,6)$; the lines l_2 and l_3 intersect at $B(7,4,4)$ and the lines l_1 and l_3 intersect at the point $C(1,2,3)$. The area of such a triangle can be calculated by using cross product. Alternatively, it is easy to see that the vector $\mathbf{AC}$ is perpendicular to $\mathbf{AB}$. Hence the area of the triangle S is calculated as $\dfrac{1}{2}\sqrt{378}$, so that $10S = 945$. $\square$

7. **Answer.** 50.

Solution. It is clear that $f(0) = 0$. Let $a = -1, b = 0$ we obtain $f(-1) = -f(1)$. Let $a = x$ and $b = 1$, we obtain

$$f(x^2 - 1) = (x - 1)(f(x) + f(1)).$$

Let $a = x$ and $b = -1$, we obtain

$$f(x^2 - 1) = (x + 1)(f(x) - f(1)).$$

Equating the right hand sides of the above equations, we obtain $f(x) = f(1)x$ for all real numbers x.

$$\sum_{k=1}^{100} f(k) = f(1)\sum_{k=1}^{100} k = \frac{1}{101}\frac{100 \times 101}{2} = 50$$

which is the answer. □

8. **Answer.** 4

Solution. Let m be a positive integer such that

$$\begin{aligned} m^2 &= n^4 - 4n^3 + 22n^2 - 36n + 18 \\ &= (n^2 - 2n)^2 + 18(n^2 - 2n) + 18 \\ &= (n^2 - 2n + 9)^2 - 63 \end{aligned}$$

so that we have

$$\begin{aligned} \left(n^2 - 2n + 9\right)^2 - m^2 &= 63 \\ (n^2 - 2n + 9 - m)(n^2 - 2n + 9 + m) &= 63. \end{aligned}$$

Thus we only have three possible pairs: $n^2 - 2n + 9 + m = 63; n^2 - 2n + 9 - m = 1$ of which there is no integer solution for n; $n^2 - 2n + 9 + m = 21; n^2 - 2n + 9 - m = 3$ for which the positive integral solution is $n = 3$; and the next case is $n^2 - 2n + 9 + m = 9; n^2 - 2n + 9 - m = 7$ for which the positive integral solution is 1. Hence the sum of all the positive integral solutions is 4. □

9. **Answer.** 43.

Solution. As

$$(x + 2 + m)^{2019} = a_0 + a_1(x + 1) + a_2(x + 1)^2 + \cdots + a_{2019}(x + 1)^{2019},$$

letting $x = 0$ gives that

$$(m + 2)^{2019} = a_0 + a_1 + \cdots + a_{2019}$$

and letting $x = -2$ gives that

$$m^{2019} = a_0 - a_1 + a_2 - a_3 + \cdots - a_{2019}.$$

As

$$(a_0 + a_2 + a_4 + \cdots + a_{2018})^2 - (a_1 + a_3 + a_5 + \cdots + a_{2019})^2 \leq 2020^{2019},$$

we have

$$(a_0 + a_1 + \cdots + a_{2019})(a_0 - a_1 + a_2 - a_3 + \cdots - a_{2019}) \leq 2020^{2019}.$$

Thus

$$(m + 2)^{2019} m^{2019} \leq 2020^{2019},$$

i.e., $m(m + 2) \leq 2020$. The largest integer m satisfying this inequality is 43. Thus the answer is 43. □

10. **Answer.** 8.

Solution. It is easy to see that $S = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\sqrt{\frac{1}{1+\frac{k}{n}}} = \int_0^1 \frac{1}{\sqrt{1+x}}dx$, which can be evaluated to give the value of $2\sqrt{2}-2$. Hence $(S+2)^2 = 8$. □

11. **Answer.** 4097.

Solution. Let $\mathcal{S}$ be the set of ordered pairs (S_1, S_2), where S_1 and S_2 are non-empty subsets of A such that the largest number in S_1 is smaller than the smallest number in S_2. For any positive integer k such that $1 \le k \le 10$, let $\mathcal{S}_k$ be the subset of ordered pairs $(S_1, S_2) \in \mathcal{S}$ such that the largest number in S_1 is k. Obviously, $\mathcal{S}_{10} = \emptyset$ and for any $1 \le k \le 9$,

$$|\mathcal{S}_k| = 2^{k-1} \times (2^{10-k} - 1) = 2^9 - 2^{k-1}.$$

Thus,

$$|\mathcal{S}| = \sum_{k=1}^{9} 2^9 - \sum_{k=1}^{9} 2^{k-1} = 4097.$$

□

12. **Answer.** 1010.

Solution. First let's construct a table of 1 and -1 satisfying the condition of the problem. Denote the value of the cell in the ith row and the jth column by a_{ij}. Let

$$a_{1,2020} = a_{3,2018} = \cdots = a_{2k+1,2020-2k} = \cdots = a_{2019,2} = -1.$$

Let the values of the rest of a_{ij}'s all be 1. One can check that this table satisfies the condition of the problem. For this table, the number of -1's is 1010.

Now we prove that the number of -1's in such a table cannot be less than 1010. Suppose on the contrary that the number of -1's is less than or equal to 1009. Now we delete all the columns and rows containing -1's. Then not more than 2018 diagonal cells are deleted. (If $a_{p,q} = -1$, then the pth row and the qth column are deleted so that $a_{p,p}$ and $a_{q,q}$ are also deleted). So at least two diagonal cells remain. Consider either one of the undeleted diagonal cells, say its value is $a_{j,j}$. Neither the jth row nor the jth column contains -1's. Hence $R_j = C_j = 1$, a contradiction. □

13. **Answer.** 5120.

Solution. Let d denote the common difference of the sequence $\{a_k\}_{k=1}^{\infty}$. Then $a_n = a_1 + (n-1)d$. As $a_2 + a_4 + a_9 = 24$, we have

$$3a_1 + (d + 3d + 8d) = 24,$$

i.e.,

$$a_1 + 4d = 8.$$

Observe that

$$S_8 = 8a_1 + (1 + 2 + \cdots + 7)d = 8a_1 + 28d = 8(a_1 + 4d) - 4d = 64 - 4d$$

and

$$S_{10} = 10a_1 + (1+2+\cdots+9)d = 10a_1 + 45d = 10(a_1+4d) + 5d = 80 + 5d.$$

Thus,

$$\begin{aligned} S_8 \times S_{10} &= (64-4d)(80+5d) = 4(16-d)5(16+d) \\ &= 20(16^2 - d^2) \le 20 \times 16^2 = 5120. \end{aligned}$$

If $d = 0$, then $a_1 = 8$, implying that $S_8 = 8^2$ and $S_{10} = 8 \times 10$. Thus, in this case, $S_8 \times S_{10} = 8^2 \times 8 \times 10 = 5120$. □

14. **Answer.** 0.

Solution. We let g^k denote $g(g(g(\cdots g(x))\cdots)$, that is, the function g composing itself k times. By using (i) repeatedly,

$$|g(0)| = |g(0) - 0| \ge |g^2(0) - g(0)| \ge |g^3(0) - g^2(0)| = |g^2(0)|,$$

and also that

$$|g^2(0)| = |g^2(0) - 0| \ge |g^3(0) - g(0)| = |g(0)|.$$

Hence we must have $|g(0)| = |g^2(0)|$. If we have that $g(0) = g^2(0)$, then $g(0) = g^2(0) = g^3(0) = 0$. If we have that $g(0) = -g^2(0)$,

$$|g(0)| = |g(0) - 0| \ge |g^2(0) - g(0)| = 2|g(0)|,$$

we also must have $g(0) = 0$. □

15. **Answer.** 4039.

Solution. Let $a_2 = a_1 + d$. By the given condition, $d \ne 0$. Note also that

$$\begin{aligned} S_n &= a_1 + a_2 + \cdots + a_n = na_1 + d(1+2+\cdots+(n-1)) \\ &= na_1 + \frac{n(n-1)d}{2}. \end{aligned}$$

Thus

$$\frac{S_{2n}}{S_n} = \frac{2na_1 + \frac{2n(2n-1)d}{2}}{na_1 + \frac{n(n-1)d}{2}} = \frac{4a_1 + 2(2n-1)d}{2a_1 + (n-1)d} = c$$

for a constant c. This implies that

$$-cdn + 4dn - 2a_1c + cd + 4a_1 - 2d = 0.$$

As the above expression holds for all $n \ge 1$, we have

$$-cd + 4d = 0, \qquad -2a_1c + cd + 4a_1 - 2d = 0.$$

So $c = 4$ and $d = 2a_1$. As $a_1 = 1$, $d = 2$. Hence

$$a_{2020} = 1 + 2019 \times 2 = 4039.$$

□

16. **Answer.** 12.

Solution. Clearly the system has the solution $x = y = z = 0$ for every value of p. We need to see when it has more than one solution. From the first equation, we have $y = -\frac{3}{2}x - \frac{3}{4}z$. Substituting this into the second equation, we have $-4x(\frac{3}{2}x + \frac{3}{4}z) - 2(\frac{3}{2}x + \frac{3}{4}z)z + pxz = 0$. Simplifying, we have $12x^2 + 2xz(6-p) + 3z^2 = 0$. The discriminant of this quadratic equation in x is $D = 4z^2(6-p)^2 - 144z^2 = 4z^2((6-p)^2 - 36) = 4z^2p(p-12)$. If $z = 0$, then system has only the trivial solution. Thus $z \neq 0$. Then the quadratic equation in x has real solutions if and only if $p \leq 0$ or $p \geq 12$. Since p is a positive integer, consider $p = 12$. In this case, x, $z = 2x$ and $y = -3x$ is a family of solution for the pair of equations. Hence smallest $p = 12$. □

17. **Answer.** 8800.

Solution. By the given condition, $\dfrac{\cos A}{\cos B} = \dfrac{b}{a}$. By the Sine Rule, $\dfrac{\sin B}{\sin A} = \dfrac{b}{a}$. Thus,

$$\frac{\cos A}{\cos B} = \frac{\sin B}{\sin A},$$

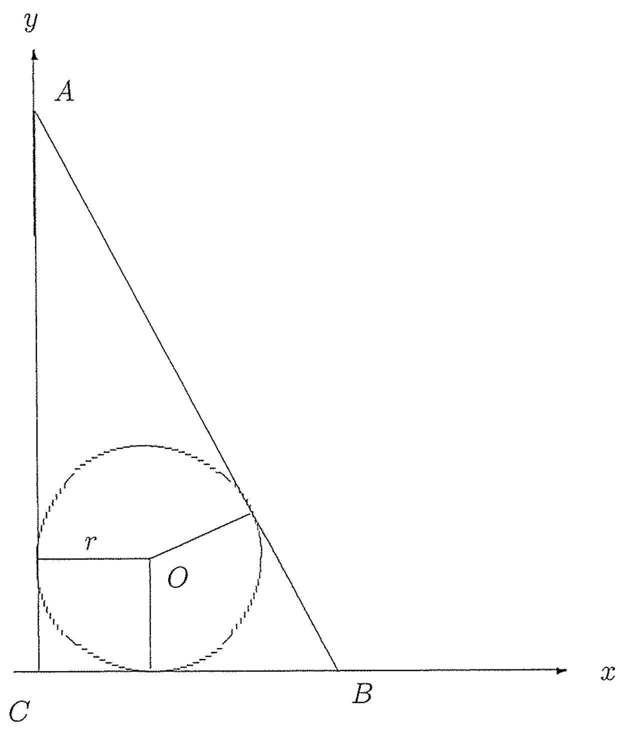
i.e., $\sin 2A = \sin 2B$. Thus, either $2A = 2B$ or $2A + 2B = 180°$, i.e., either $A = B$ or $A + B = 90°$. By the given condition, $\frac{b}{a} = \frac{4}{3}$, i.e., $3b = 4a$, implying that $A \neq B$. Hence, $A + B = 90°$. By the Pythygoras Theorem, we have

$$a^2 + b^2 = c^2 = 100^2.$$

It is known that $3b = 4a$. Solving the following equation system:

$$\begin{cases} a^2 + b^2 = 100^2; \\ 3b = 4a \end{cases}$$

gives that $a = 60$ and $b = 80$.



Assume that the radius of the inscribed circle is r. Then

$$\frac{1}{2}r(a+b+c) = \frac{1}{2}ab.$$

Thus $r = \dfrac{ab}{a+b+c} = \dfrac{60 \times 80}{60+80+100} = 20$. Assume that ABC is on the Cartesian axis system such that C is at the point $(0,0)$, B is at point $(60,0)$ and A is at point $(0,80)$. As the radius of the inscribed circle is 20, any point P on the inscribed circle has the position $(20+20\cos\theta, 20+20\sin\theta)$ for some θ with $0° \leq \theta \leq 360°$. Observe that

$$\begin{aligned} PA^2+PB^2+PC^2 &= (20+20\cos\theta)^2+(20+20\sin\theta-80)^2 \\ &\quad +(20+20\cos\theta-60)^2+(20+20\sin\theta)^2 \\ &\quad +(20+20\cos\theta)^2+(20+20\sin\theta)^2 \\ &= 800(10-\sin\theta). \end{aligned}$$

Its maximum value is obtained when $\sin\theta = -1$. Thus its maximum value is 8800. □

18. **Answer.** 2017.

Solution. We shall prove that n is a prime if and only if $\dbinom{n-1}{k} - (-1)^k$ is divisible by n for $k = 0, 1, \ldots, n-1$.

Suppose n is a prime, then $i^{-1} \pmod{n}$ exists for $i = 1, \ldots, n-1$. Thus for $k = 0, 1, \ldots, n-1$, we have $\dbinom{n-1}{k} = \displaystyle\prod_{i=1}^{k} \frac{n-i}{i} = \prod_{i=1}^{k}(n-i)i^{-1} \equiv \prod_{i=1}^{k}(-i)i^{-1} \equiv (-1)^k \pmod{n}$. Conversely, suppose $\dbinom{n-1}{k} \equiv (-1)^k \pmod{n}$ for $k = 0, 1, \ldots, n-1$. Assume on the contrary that n is a composite number. Let p be the smallest prime factor of n. Thus $1 < p < n$. Then $\dbinom{n-1}{p-1} \equiv (-1)^{p-1} \pmod{n}$ by the assumption. But $\dbinom{n-1}{p} = \dbinom{n-1}{p-1}(\frac{n}{p}-1) \equiv (-1)^{p-1}(\frac{n}{p}-1) \not\equiv (-1)^p \pmod{n}$, since $\frac{n}{p} \not\equiv 0 \pmod{n}$. The largest prime number less than 2020 is 2017. □

19. **Answer.** 41.

Solution. Let $c_k = \dfrac{1}{a_k+1} - \dfrac{1}{2}$ for all $k \geq 1$. Then,

$$a_k = -\frac{2c_k-1}{2c_k+1}.$$

Observe that

$$\frac{a_m+a_n}{(a_m+1)(a_n+1)} = -2c_mc_n + \frac{1}{2}.$$

Thus, the following equality

$$\frac{a_m+a_n}{(a_m+1)(a_n+1)} = \frac{a_p+a_q}{(a_p+1)(a_q+1)}$$

implies that $c_mc_n = c_pc_q$. By the given condition, for all integers $m, n, p, q \geq 1$ with $m+n = p+q$, $c_mc_n = c_pc_q$ always holds. As $a_1 = 0$ and $a_2 = \dfrac{1}{2}$, we have $c_1 = \dfrac{1}{2}$ and

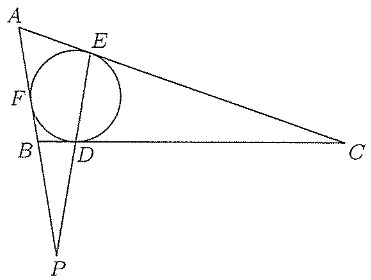
$c_2 = \dfrac{1}{6}$. We can show by induction that $c_n = \left(\dfrac{1}{2}\right)\left(\dfrac{1}{3}\right)^{n-1}$ holds for all $n \geq 1$ as follows: The result holds for $n \leq 2$. Suppose it holds for $n < r$, where $r \geq 3$. Then $c_r c_1 = c_{r-1}c_2$, which means that

$$c_r = c_{r-1}\frac{c_2}{c_1} = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^{r-2} \times \frac{1}{6} \times 2 = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^{r-1}.$$

Thus the claim holds and $c_5 = \dfrac{1}{2}\left(\dfrac{1}{3}\right)^4$ implying that $\dfrac{1}{1-a_5} = \dfrac{1}{1-\frac{1-2c_5}{1+2c_5}} = \dfrac{1+2c_5}{1+2c_5-(1-2c_5)} =$ 41. □

20. **Answer.** 140

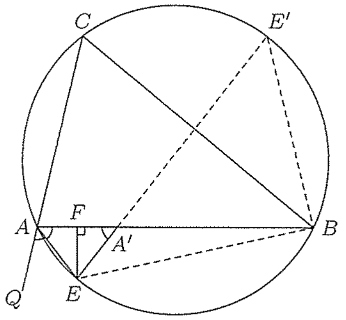
Solution.



Let $AB = c, BC = a, CA = b$, and $s = \dfrac{a+b+c}{2}$. By Menalaus' theorem applied to the transversal PDE, $\dfrac{AP}{PB}\dfrac{BD}{DC}\dfrac{CE}{EA} = -1$, that is, $-2 \times \dfrac{s-b}{s-c}\dfrac{s-c}{s-a} = -1$, or equivalently, $c = 3(b-a)$. Since the triangles PBD and CDE are similar, the points P, B, E, C are concyclic. Thus $AB \times AP = AE \times AC$. That is $2c^2 = (s-a)b$. Substituting $c = 3(b-a)$, we have $c = \dfrac{3a}{8}, b = \dfrac{9a}{8}$. Given $b = 9$, we have $a = 8, c = 3$. Thus $s = 10$. Using Heron's formula, $(ABC)^2 = s(s-a)(s-b)(s-c) = 10 \times 2 \times 1 \times 7 = 140$. □

21. **Answer.** 704.

Solution.



Let A' be the point on AB such that $A'F = FA$. Then $\triangle AEA'$ is isosceles. Extend EA' meeting the circumcircle of $\triangle ABC$ at E'. Join BE' and BE. Since $\angle ABC = \angle EBC - \angle ABE = \angle AA'E - \angle ABE = \angle E'EB$, we have $BE' = AC$. Also, $\triangle AEA'$ is similar to $\triangle E'BA'$ implies that $A'B = BE' = AC$. Hence, $2AF = AB - A'B = AB - AC$. Since $AB = 75, AC = 53$, we have $AF = 11$, $FB = 64$ and $AF \times FB = 704$. □

22. **Answer.** 165.

Solution. Note that for any integer $n > 0$,

$$1 + 2 + \cdots + 2^{n-1} = 2^n - 1 < 2^n.$$

Thus, for any $0 \le u < v < w$ and $0 \le r < s < t$, $2^u + 2^v + 2^w < 2^r + 2^s + 2^t$ if and only if one of the following holds:

(1) $w < t$;

(2) $w = t$ and $v < s$; or

(3) $w = t$, $v = s$ and $u < r$.

So all the numbers in the set $\{2^r + 2^s + 2^t : 0 \le r < s < t\}$ are pairwise distinct. For any $t \ge 2$, let

$$A_t = \{2^r + 2^s + 2^t : 0 \le r < s < t\}.$$

It is clear that for $t \ge 2$, $|A_t| = \sum_{i=1}^{t-1} i = (t-1)t/2 = \binom{t}{2}$. It is also clear that $\max A_t = 2^{t-2} + 2^{t-1} + 2^t = 7 \cdot 2^{t-2}$, and $\min A_{t+1} = 2^0 + 2^1 + 2^{t+1} = 2^{t+1} + 3 > \max A_t$. Observe that $1792 = \max A_{10} < 2020 < \min A_{11} = 2051$. Thus, if $a_n = 1792$, then $a_n < 2020$ and $2020 - a_n$ is as small as possible. Clearly, if $a_n = 1792 = \max A_{10}$, then $n = \sum_{t=2}^{10} \binom{t}{2} = \binom{11}{3} = 165$. □

23. **Answer.** 838.

Solution. Let $T = \{n-1, n, \ldots, \lfloor \frac{n^2}{4} \rfloor\}$. We shall show $S = T$. Given positive integers $a_1, \ldots, a_k$ with sum n, let $s = \sum_{i=2}^{k} a_{i-1}a_i$. Since $a_1, a_2 \ge 1$, we have $(a_1 - 1)(a_2 - 1) \ge 0$, so that $a_1a_2 \ge a_1 + a_2 - 1$. Thus $s = a_1a_2 + \sum_{i=3}^{k} a_{i-1}a_i \ge a_1 + a_2 - 1 + \sum_{i=3}^{k} a_i = n - 1$. Since $x(n-x)$ is maximized over the integers when $x = \lfloor \frac{n}{2} \rfloor$ with value $\lfloor \frac{n^2}{4} \rfloor$, we have $s \le (a_1 + a_3 + \cdots)(a_2 + a_4 + \cdots) \le \lfloor n^2/4 \rfloor$. Thus $S \subseteq T$.

Conversely, suppose $s \in T$. Note that $n - 1 = 1(n-1)$ and $\lfloor \frac{n^2}{4} \rfloor = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$. If s is the product of two numbers summing to n, then $s \in S$. Otherwise, $s = m(n-m) + j$ for some m and j with $1 \le m \le \lfloor \frac{n}{2} \rfloor$ and $1 \le j < n - 2m - 1$. In this case, s is realized using $(a_1, a_2, a_3, a_4) = (n - 2m - 1 - j, m, m + j, 1)$, so again $s \in S$.

If n is even, then the average value of all the numbers in T is $\frac{1}{2}(n - 1 + \frac{n^2}{4}) = \frac{(n+2)^2}{8} - 1$. Thus $\frac{(n+2)^2}{8} - 1 = 88199$ giving $n = 838$.

If $n = 2p + 1$ is odd, then the average value of all the numbers in T is $\frac{1}{2}(n - 1 + \lfloor \frac{n^2}{4} \rfloor) = \frac{1}{2}(2p + p^2 + p) = \frac{1}{2}(p^2 + 3p)$. Then $\frac{1}{2}(p^2 + 3p) = 88199$, which has no integer solution in p. Thus n cannot be odd. □

24. **Answer.** 1600.

Solution. Note that for any real numbers a, b, c, d,

$$(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2),$$

i.e.,

$$\frac{1}{4}(a+b+c+d)^2 \leq a^2+b^2+c^2+d^2.$$

Thus

$$\begin{aligned}
& (x+5)^2+(y+10)^2+(z+20)^2+(w+40)^2 \\
\geq\ & \frac{1}{4}\left((x+5)+(y+10)+(z+20)+(w+40)\right)^2 \\
=\ & \frac{1}{4}(x+y+z+w+75)^2 \\
=\ & \frac{1}{4}(80)^2 = 1600.
\end{aligned}$$

□

25. **Answer.** 290.

Solution. The given equation can be rewritten in the form $(p+q)^2+(3994-p+q)^2 = 3994^2$. Thus we need to solve the equation $x^2+y^2 = 3994^2$ in positive integers x, y, where $p = \frac{1}{2}(x-y+3994), q = \frac{1}{2}(x+y-3994)$.

Suppose x is odd. This implies y is odd. Taking mod 4 of the equation $x^2+y^2 = 3994^2$, we have $1+1 \equiv 0 \pmod 4$, a contradiction. Therefore x must be even. Then y is also even. Thus $x = 2u, y = 2v$, and the equation reduces to $u^2+v^2 = 1997^2$. Note that 1997 is a prime. Thus the solutions to the equation $u^2+v^2 = 1997^2$ are primitive and are given by $1997 = a^2+b^2, u = 2ab, v = a^2-b^2$, where $b < a$. Taking mod 5 of the equation $1997 = a^2+b^2$, we deduce that $a, b \equiv \pm 1 \pmod 5$. Similarly, by taking mod 3 of the equation $1997 = a^2+b^2$, we deduce that $a, b \equiv \pm 1 \pmod 3$. Therefore, $a, b \equiv 1, 4, 11, 14 \pmod{15}$. Since $b < a$, we have $1997/2 < a^2 < 1997$. That is $31 < a < 45$ so that $a = 34, 41, 44$. The only solution is $(a, b) = (34, 29)$ so that $(u, v) = (1972, 315)$. Thus $(x, y) = (3944, 630)$.

Consequently, the solutions to $x^2+y^2 = 3994^2$ are $(x, y) = (3944, 630), (630, 3944)$. Correspondingly, $(p, q) = (3654, 290), (340, 290)$. Thus $q = 290$. □

Singapore Mathematical Olympiad 2020 Results

School Award (Category 1)

Within each category of award, the schools are arranged according to alphabetical order.

Gold Award	
Hwa Chong Institution	Raffles Girls' School (Secondary)
Nanyang Girls' High School	Raffles Institution
NUS High School of Math and Science	
Silver Award	
Catholic High School	River Valley High School
Dunman High School	St Joseph's Institution
Maris Stella High School (Secondary)	Temasek Junior College
National Junior College	Victoria School
Bronze Award	
Anderson Secondary School	Methodist Girls' School
Anglo-Chinese School (Independent)	Nan Chiau High School
Cedar Girls' Secondary School	Ngee Ann Secondary School
CHIJ St Nicholas Girls' School	
Commendation Award	
Anglican High School	Presbyterian High School
Bowen Secondary School	Singapore Chinese Girls' School
Bukit Panjang Government High School	St Andrew's Secondary School
Chung Cheng High School (Main)	Swiss Cottage Secondary School
Clementi Town Secondary School	Tanjong Katong Secondary School
Commonwealth Secondary School	Temasek Secondary School
Dunman Secondary School	United World College SEA, Dover Campus
Hai Sing Catholic School	Xinmin Secondary School
Hwa Chong International School	Yio Chu Kang Secondary School
Nan Hua High School	Zhonghua Secondary School
Orchid Park Secondary School	

Singapore Mathematical Olympiad 2020 Results

School Award (Category 2)

Within each category of award, the schools are arranged according to alphabetical order.

Gold Award
Hwa Chong Institution
NUS High School of Math and Science
Raffles Institution
Temasek Junior College
Silver Award
Nanyang Junior College
National Junior College
Bronze Award
Dunman High School
St Joseph's Institution
Victoria Junior College
Commendation Award
Anderson Serangoon Junior College
Anglo-Chinese Junior College
Anglo-Chinese School (Independent)
Eunoia Junior College
Hwa Chong International School
Nanyang Girls' High School
River Valley High School
Tampines Meridian Junior College

Singapore Mathematical Olympiad 2020 Results

Individual Award (Junior Section)

RANK	NAME	SCHOOL
1	YU QINXI	Raffles Institution
2	Kong Jia Le	Hwa Chong Institution
2	Aidan Ong Ming Feng	Hwa Chong Institution
2	Wan Tse Peng Lucien	Hwa Chong Institution
2	Zhao Yaoqi	Hwa Chong Institution
2	Liu Yiming	Raffles Girls' School (Secondary)
2	KANG TAEYOUNG	Raffles Institution
2	ZHU JIE	Raffles Institution
Top 10 Award	DREW MICHAEL TERREN RAMIREZ	NUS High School of Math and Science
Top 10 Award	LIANG CHENXI	NUS High School of Math and Science
Top 10 Award	KIM JISOO	Raffles Girls' School (Secondary)
Top 10 Award	Peng Ruijia	Raffles Girls' School (Secondary)
Top 10 Award	CHEN YIXUAN	Raffles Institution
Top 10 Award	LIM ZHE XI, ZENNETH	Raffles Institution
Top 10 Award	PEH YAN XI	Raffles Institution
Top 10 Award	WOON QI RUI, ETHAN	Raffles Institution
Top 10 Award	CLEMENT YAP KUAN HONG	Raffles Institution
Top 10 Award	ZHANG LUMING	Raffles Institution

Singapore Mathematical Olympiad 2020 Results

Individual Award (Senior Section)

RANK	NAME	SCHOOL
1	LIU JIANQI	Victoria School
2	LIU ZIHAN	Raffles Institution
3	CAO YUEWEI	Dunman High School
3	SHEVONNE CHIA	NUS High School of Math and Science
3	MARIO TANIJAYA	NUS High School of Math and Science
3	ANDREW YAR KWOK WING	Raffles Institution
3	ZHOU HENGZHI	St Joseph's Institution
Top 10 Award	Zhou Kangyun	Hwa Chong Institution
Top 10 Award	WU XINYUE	Nanyang Girls' High School
Top 10 Award	LI YICHEN	National Junior College
Top 10 Award	Hu Xinghui	Raffles Girls' School (Secondary)
Top 10 Award	Yu Hanzhang	Raffles Girls' School (Secondary)
Top 10 Award	Zhang Chenxi	Raffles Girls' School (Secondary)

Singapore Mathematical Olympiad 2020 Results

Individual Award (Open Section)

RANK	NAME	SCHOOL
1	BOO TSE YANG LUCAS	Raffles Institution
2	Ng Yangyi Aloysius	Raffles Institution
3	Ang Boon Han Nathaniel	Anglo-Chinese School (Independent)
Top 10 Award	HUA JIANING	National Junior College
Top 10 Award	HO LI XIONG, TIMOTHY	NUS High School of Math and Science
Top 10 Award	TAN KIN HERN (CHEN JINGHENG)	NUS High School of Math and Science
Top 10 Award	YU ZHEYUAN	NUS High School of Math and Science
Top 10 Award	Hu Xinghui	Raffles Girls' School (Secondary)
Top 10 Award	LE VIET HUNG	Raffles Institution
Top 10 Award	LOW CHOO RAY	Raffles Institution
Top 10 Award	Tan Xu Chen	Raffles Institution
Top 10 Award	David Toh Hui Kai	Raffles Institution
Top 10 Award	WANG YUXUAN	Raffles Institution
Top 10 Award	Wang Yike	Victoria Junior College