Undergraduate Physics

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Preface

The book is divided into the following sections.

Part I covers Classical Mechanics, closely following [GPS11].

?? covers Electromagnetism, which follows [Jac99].

Contents

I
Classical Mechanics

1 Elementary Principles

§1.1 Mechanics of a Particle

The position of a particle is specified by a vector $\mathbf{r}$ from some given origin. The trajectory of the particle with respect to time is described by

$$\mathbf{r} = \mathbf{r}(t).$$

Considering the vector as three-dimensional, we write

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

or, more concisely,

$$\mathbf{r}(t) = \left(x(t), y(t), z(t)\right).$$

Notation. We sometimes use $\mathbf{r}$ and $\mathbf{x}$ interchangeably to denote the position vector.

The **velocity** of the particle is

$$\mathbf{v} \coloneqq \dot{\mathbf{r}} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$$

and the **acceleration** is

$$\mathbf{a} \coloneqq \ddot{\mathbf{r}} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}.$$

The **linear momentum** of a particle, with (inertial) mass m, is defined as

$$\mathbf{p} \coloneqq m\mathbf{v}.$$

Newtonian mechanics is a framework which allows us to determine the trajectory $\mathbf{x}(t)$ of a particle in any given situation. This framework is usually presented as three axioms known as **Newton's laws of motion**:

> **Law** (Newton's 1st Law)
> In the absence of an external resultant force, a particle moves with constant velocity.

The tendency of an object to keep moving once it is set in motion is called **inertia**.

A frame of reference in which Newton's first law is valid is called an **inertial frame of reference**. Because Newton's first law is used to define what we mean by an inertial frame of reference, it is sometimes called the *law of inertia*.

> **Law** (Newton's 2nd Law)
> The rate of change of momentum of a particle is proportional to the external resultant force acting upon it:
>
> $$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}. \tag{1.1}$$

In cases where mass is constant with respect to time, Eq. (1.1) reduces to the more familiar form:

$$\mathbf{F} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = m\mathbf{a}.$$

Newton's equation is a second order differential equation; this means that we will have a unique solution only if we specify two initial conditions. These are usually taken to be the position $\mathbf{x}(t_0)$ and the velocity $\dot{\mathbf{x}}(t_0)$ at some initial time t_0.

Law (Newton's 3rd Law)

Every action force has an equal and opposite reaction force.

Many of the important conclusions of mechanics can be expressed in the form of conservation theorems, which indicate under what conditions various mechanical quantities are constant in time.

Theorem (Conservation of linear momentum)

If the resultant force $\mathbf{F}$ is zero, then $\dot{\mathbf{p}} = \mathbf{0}$; that is, linear momentum $\mathbf{p}$ is conserved.

Proof. This follows directly from Newton's 2nd Law:

$$\mathbf{F} = 0 \implies \dot{\mathbf{p}} = 0$$

so $\mathbf{p}$ is conserved. □

The **angular momentum** of a particle about point O, denoted by $\mathbf{L}$, is defined as

$$\mathbf{L} := \mathbf{r} \times \mathbf{p}$$

where $\mathbf{r}$ is the position vector of the particle from O.

We define the **torque** (or **moment of force**) about O as

$$\mathbf{N} := \mathbf{r} \times \mathbf{F}.$$

Theorem (Conservation of angular momentum)

If the total torque $\mathbf{N}$ is zero, then $\dot{\mathbf{L}} = \mathbf{0}$; that is, angular momentum $\mathbf{L}$ is conserved.

Proof. From the definition of torque,

$$\mathbf{N} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}).$$

Note that by using the vector identity,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times m\mathbf{v}) = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v})$$

where the first term on RHS obviously vanishes. Thus

$$\mathbf{N} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times m\mathbf{v}) = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \dot{\mathbf{L}}.$$

Hence

$$\mathbf{N} = 0 \implies \dot{\mathbf{L}} = 0$$

so $\mathbf{L}$ is conserved. □

Remark. This is analogous to Newton's 2nd Law.

The **work done** by a force $\mathbf{F}$ on a particle in going from point 1 to point 2 is defined as the line integral

$$W_{12} := \int_1^2 \mathbf{F} \cdot \mathrm{d}\mathbf{s}\,.$$

The **kinetic energy** of a particle is defined as

$$T := \frac{1}{2}mv^2.$$

Theorem (Work-energy theorem)

The work done by the net force on a particle is equal to the change in the particle's kinetic energy:

$$W_{12} = T_2 - T_1. \tag{1.2}$$

Proof. For constant mass (as will be assumed from now on unless otherwise specified), the integral for work done reduces to

$$\int \mathbf{F} \cdot \mathrm{d}\mathbf{s} = \int \left(m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right)(\mathbf{v}\,\mathrm{d}t) = m\int \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \cdot \mathbf{v}\,\mathrm{d}t = \frac{1}{2}m\int \frac{\mathrm{d}}{\mathrm{d}t}\left(v^2\right)\mathrm{d}t$$

and thus

$$W_{12} = \frac{1}{2}m\left({v_2}^2 - {v_1}^2\right) = T_2 - T_1.$$

□

If the force field is such that the work W_{12} is independent of the path from point 1 to 2, then the force (and the system) is said to be **conservative**. Hence the work done around such a closed circuit is zero:

$$\oint \mathbf{F} \cdot \mathrm{d}\mathbf{s} = 0. \tag{1.3}$$

Remark. Physically it is clear that a system cannot be conservative if friction or other dissipation forces are present, because $\mathbf{F} \cdot \mathrm{d}\mathbf{s}$ due to friction is always positive and the integral cannot vanish.

Thus we can write the force $\mathbf{F}$ as the gradient of some scalar function V of position, known as the **potential**:

$$\mathbf{F} = -\nabla V(\mathbf{r}). \tag{1.4}$$

Remark. Note that in Eq. (1.4) we can add to V any quantity constant in space, without affecting the results. Hence the zero level of V is arbitrary.

Theorem (Conservation of energy)

If the forces acting on a particle are conservative, then the total energy of the particle $T + V$ is conserved.

Proof. For a conservative system, the work done by the forces is

$$W_{12} = V_1 - V_2.$$

Combining this with Eq. (1.2) gives

$$T_1 + V_1 = T_2 + V_2. \tag{1.5}$$

□

§1.2 Mechanics of a System of Particles

In generalising the ideas of the previous section to systems of many particles, we must distinguish between the *external forces* acting on the particles due to sources outside the system, and *internal forces* on, say, some particle i due to all other particles in the system. Thus, the Newton's second law for the i-th particle is

$$\dot{\mathbf{p}}_i = \sum_j \mathbf{F}_{ji} + \mathbf{F}_i^{(e)}$$

where $\mathbf{F}_i^{(e)}$ denotes the external force exerted on the i-th particle, $\mathbf{F}_{ji}$ denotes the internal force exerted by the j-th particle on the i-th particle.

Summing over all particles gives

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\sum_i m_i\mathbf{r}_i = \sum_i \mathbf{F}_i^{(e)} + \sum_{i\neq j}\mathbf{F}_{ji}.$$

Note that the first sum on the RHS is simply the total external force $\mathbf{F}^{(e)}$, while the second term vanishes by Newton's 3rd Law as $\mathbf{F}_{ij} + \mathbf{F}_{ji} = \mathbf{0}$. To reduce the LHS, we define a vector $\mathbf{R}$, called the **centre of mass** of the system, defined as

$$\mathbf{R} = \frac{\sum m_i\mathbf{r}_i}{m_i} = \frac{1}{M}\sum m_i\mathbf{r}_i.$$

Thus the above equation reduces to

$$M\frac{\mathrm{d}^2\mathbf{R}}{\mathrm{d}t^2} = \mathbf{F}^{(e)} \tag{1.6}$$

which states that the centre of mass moves as if the total external force were acting on the entire mass of the system concentrated at the centre of mass. Purely internal forces, if the obey Newton's 3rd Law, therefore have no effect on the motion of the centre of mass.

An oft-quoted example is the motion of an exploding shell – the centre of mass of the fragments traveling as if the shell were still in a single piece.

The total linear momentum of the system is given by

$$\mathbf{P} = \sum m_i\frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} = M\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t}$$

which is the total mass of the system times the velocity of the centre of mass.

Theorem (Conservation of linear momentum of system of particles)
If the total external force is zero, total linear momentum of the system is conserved.

Proof. This directly follows from the above equation. □

Theorem (Conservation of angular momentum of system of particles)
If the applied (external) torque is zero, then $\dot{\mathbf{L}} = \mathbf{0}$; that is, angular momentum $\mathbf{L}$ is conserved.

Proof. We obtain the total angular momentum of the system by forming the cross product $\mathbf{r}_i \times \mathbf{p}_i$ and summing over i.

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i.$$

Taking the derivative with respect to time,

$$\dot{\mathbf{L}} = \sum_i \mathbf{r}_i \times \dot{\mathbf{p}}_i.$$

Applying Newton's 2nd Law on each of the i particles, where $\dot{\mathbf{p}}_i = \sum_j \mathbf{F}_{ji} + \mathbf{F}_i^{(e)}$, gives

$$\dot{\mathbf{L}} = \sum_i \left(\mathbf{r}_i \times \mathbf{F}_i^{(e)}\right) + \sum_{i \neq j} \mathbf{r}_i \times \mathbf{F}_{ji}.$$

The first term on the RHS is simply the external torque $\mathbf{N}^{(e)}$. Assuming Newton's 3rd Law holds, we can write the last term on the RHS as a sum of the pairs of the form

$$\mathbf{r}_i \times \mathbf{F}_{ji} + \mathbf{r}_j \times \mathbf{F}_{ij} = (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ji}.$$

Note that $\mathbf{r}_i - \mathbf{r}_j$ is identical with the vector $\mathbf{r}_{ij}$ from j to i, so the RHS of the above equation can be written as

$$\mathbf{r}_{ij} \times \mathbf{F}_{ji}.$$

If the internal force between i-th and j-th partiles acts along $\mathbf{r}_{ij}$, then all of these cross products equal to zero. The sum over pairs is zero under this assumption and thus

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{N}^{(e)}. \tag{1.7}$$

□

The total linear momentum of the system is the same as if the entire mass were concentrated at the centre of mass and moving with it. The analogous theorem for angular momentum is more complicated. Taking the origin O as reference point, the total angular momentum of the system is

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i.$$

Let $\mathbf{R}$ be the position vector from O to the centre of mass, and let $\mathbf{r}'_i$ be the position vector from the centre of mass to the i-th particle. We define

$$\mathbf{r}_i = \mathbf{r}'_i + \mathbf{R}.$$

and

$$\mathbf{v}_i = \mathbf{v}'_i + \mathbf{v}$$

where $\mathbf{v} = \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t}$ is the velocity of the centre of mass relative to O, and $\mathbf{v}'_i = \frac{\mathrm{d}\mathbf{r}'_i}{\mathrm{d}t}$ is the velocity of the i-th particle relative to the centre of mass of the system.

Hence the total angular momentum is

$$\begin{aligned}\mathbf{L} &= \sum_i \mathbf{r}_i \times \mathbf{p}_i = \sum_i (\mathbf{R} + \mathbf{r}_i) \times m_i \left(\mathbf{v} + \mathbf{v}'_i\right) \\ &= \sum_i \mathbf{R} \times m_i \mathbf{v} + \sum_i \mathbf{r}'_i \times m_i \mathbf{v}' + \left(\sum_i m_i \mathbf{r}'_i\right) \times \mathbf{v} + \mathbf{R} \times \frac{\mathrm{d}}{\mathrm{d}t} \sum_i m_i \mathbf{r}'_i.\end{aligned}$$

Note that since $\mathbf{r}'_i = \mathbf{r}_i - \mathbf{R}$,

$$\sum_i m_i \mathbf{r}'_i = \sum_i m_i (\mathbf{r}_i - \mathbf{R}) = \sum_i m_i \mathbf{r}_i - \sum_i m_i \mathbf{R} = M\mathbf{R} - M\mathbf{R} = 0.$$

Since $\sum m_i \mathbf{r}'_i = 0$, the terms $(\sum_i m_i \mathbf{r}'_i) \times \mathbf{v}$ and $\mathbf{R} \times \frac{\mathrm{d}}{\mathrm{d}t} \sum_i m_i \mathbf{r}'_i$ vanish. Rewriting the remaining terms, the total angular momentum about O is

$$\mathbf{L} = \mathbf{R} \times M\mathbf{v} + \sum_i \mathbf{r}'_i \times \mathbf{p}'_i. \tag{1.8}$$

In words, the total angular momentum about a point O is the angular momentum of motion concentrated at the centre of mass, plus the angular momentum of motion about the centre of mass.

Finally, consider the kinetic energy of a system.

Theorem (Work-energy theorem for system of particles)
The work done in moving the system is equal to the change in the kinetic energy of the system:

$$W_{12} = T_2 - T_1.$$

Proof. As in the case of a single particle, we calculate the work done by all forces in moving the system from an initial configuration 1, to a final configuration 2:

$$W_{12} = \sum_i \int_1^2 \mathbf{F}_i \cdot \mathrm{d}\mathbf{s}_i = \sum_i \int_1^2 \mathbf{F}_i^{(e)} \cdot \mathrm{d}\mathbf{s}_i + \sum_{i \neq j} \int_1^2 \mathbf{F}_{ji} \cdot \mathrm{d}\mathbf{s}_i \,. \tag{1.9}$$

Again, the equations of motion can be used to reduce the integrals to

$$\begin{aligned}\sum_i \int_1^2 \mathbf{F}_i \cdot \mathrm{d}\mathbf{s}_i &= \sum_i \int_1^2 m_i \dot{\mathbf{v}}_i \cdot \mathbf{v}_i \,\mathrm{d}t \\ &= \sum_i \int_1^2 \mathrm{d}\left(\frac{1}{2} m_i {v_i}^2\right)\end{aligned}$$

Hence, the work done can still be written as the difference of the final and initial kinetic energies:

$$W_{12} = T_2 - T_1,$$

where the total kinetic energy of the system T is

$$T = \frac{1}{2} \sum_i m_i {v_i}^2.$$

□

We now present the kinetic energy T in an alternative form. Making use of the transformation to center-of-mass coordinates $\mathbf{v}_i = \mathbf{v}'_i + \mathbf{v}$, we may also write T as

$$\begin{aligned}T &= \frac{1}{2} \sum_i m_i \mathbf{v}_i \cdot \mathbf{v}_i \\ &= \frac{1}{2} \sum_i m_i (\mathbf{v} + \mathbf{v}'_i) \cdot (\mathbf{v} + \mathbf{v}'_i) \\ &= \frac{1}{2} \sum_i m_i v^2 + \frac{1}{2} \sum_i m_i v_i'^2 + \mathbf{v} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_i m_i \mathbf{r}'_i \right)\end{aligned}$$

and by the reasoning already employed in calculating the angular momentum, the last term vanishes, leaving

$$T = \frac{1}{2} M v^2 + \frac{1}{2} \sum_i m_i v_i'^2. \tag{1.10}$$

We see that the kinetic energy also consists of two parts: the kinetic energy obtained if all the mass were concentrated at the center of mass, plus the kinetic energy of motion about the center of mass.

Now consider the RHS of Eq. (1.9). In the special case that the external forces are derivable in terms of the gradient of a potential, the first term can be written as

$$\sum_i \int_1^2 \mathbf{F}_i^{(e)} \cdot \mathrm{d}\mathbf{s}_i = -\sum_i \int_1^2 \nabla_i V_i \cdot \mathrm{d}\mathbf{s}_i = -\sum_i V_i \Big|_1^2$$

where the subscript i on the del operator indicates that the derivatives are with respect to the components of $\mathbf{r}_i$. If the internal forces are also conservative, then the mutual forces between the ith and jth particles, $\mathbf{F}_{ij}$ and $\mathbf{F}_{ji}$, can be obtained from a potential function V_{ij}. To satisfy the strong law of action and reaction, V_{ij} can be a function only of the distance between the particles:

$$V_{ij} = V_{ij}(|\mathbf{r}_i - \mathbf{r}_j|).$$

The two forces are then automatically equal and opposite,

$$\mathbf{F}_{ji} - \nabla_i V_{ij} = \nabla_j V_{ij} = -\mathbf{F}_{ij}$$

and lie along the line joining the two particles,

$$\nabla V_{ij}(|\mathbf{r}_i - \mathbf{r}_j|) = (\mathbf{r}_i - \mathbf{r}_j) f$$

where f is some scalar function. If V_{ij} were also a function of the difference of some other pair of vectors associated with the particles, such as their velocities or (to step into the domain of modern physics) their intrinsic "spin" angular momenta, then the forces would still be equal and opposite, but would not necessarily lie along the direction between the particles.

When the forces are all conservative, the second term in Eq. (1.9) can be rewritten as a sum over pairs of particles, the terms for each pair being of the form

$$-\int_1^2 \left(\nabla_i V_{ij} \cdot \mathrm{d}\mathbf{s}_i + \nabla_j V_{ij} \cdot \mathrm{d}\mathbf{s}_j\right).$$

If the difference vector $\mathbf{r}_i - \mathbf{r}_j$ is denoted by $\mathbf{r}_{ij}$, and if ∇_{ij} stands for the gradient with respect to $\mathbf{r}_{ij}$, then

$$\nabla_i V_{ij} = \nabla_{ij} V_{ij} = -\nabla_j V_{ij},$$

and

$$\mathrm{d}\mathbf{s}_i - \mathrm{d}\mathbf{s}_j = \mathrm{d}\mathbf{r}_i - \mathrm{d}\mathbf{r}_j = \mathrm{d}\mathbf{r}_{ij}\,,$$

so that the term for the ij pair has the form

$$-\int \nabla_{ij} V_{ij} \cdot \mathrm{d}\mathbf{r}_{ij}\,.$$

The total work arising from internal forces then reduces to

$$-\frac{1}{2}\sum_{i\neq j}\int_1^2 \nabla_{ij}\nabla_{ij} \cdot \mathrm{d}\mathbf{r}_{ij} = -\frac{1}{2}\sum_{i\neq j} V_{ij}\Big|_1^2.$$

The factor $\frac{1}{2}$ appears because in summing over both i and j each member of a given pair is included twice, first in the i summation and then in the j summation.

From these considerations, it is clear that if the external and internal forces are both derivable from potentials it is possible to define a total potential energy V of the system:

$$V = \sum_i V_i + \frac{1}{2}\sum_{i\neq j} V_{ij} \tag{1.11}$$

such that the total energy $T + V$ is conserved, the analog of the conservation theorem for a single particle. We call the second term on the RHS of Eq. (1.11) the internal potential energy of the system.

§1.3 Constraints and Generalised Coordinates

There are certain constraints in realistic systems. Some examples include: gas molecules within a container are constrained by the walls of the vessel to move only inside the container; a particle placed on the surface of a solid sphere is subject to the constraint that it can move only on the surface or in the region exterior to the sphere.

Constraints may be classified in various ways. If the conditions of constraint can be expressed as equations connecting the coordinates of the particles (and possibly the time) having the form

$$f(\mathbf{r}_1, \mathbf{r}_2, \dots, t) = 0,$$

then the constraints are said to be **holonomic**.

> **Example**
> In a rigid body, the constraints are expressed by equations of the form
>
> $$(\mathbf{r}_i - \mathbf{r}_j)^2 - {c_{ij}}^2 = 0,$$
>
> which states that the magnitude of the relative position vector between two points in a rigid body is always the same, since the rigid body is not deformed.

Constraints not expressible in this fashion are called **non-holonomic**.

> **Example**
> The constraint involved in the example of a particle placed on the surface of a sphere is nonholonomic, for it can be expressed as an inequality
>
> $$r^2 - a^2 \geq 0$$
>
> where a is the radius of the sphere, $\mathbf{r}$ is the position vector of the particle from the centre of sphere.

Constraints are further classified according to the time dependence of the constraints. A constraint is **rheonomous** if it is time dependent; otherwise, it is **scleronomous**.

Due to constraints, the coordinates $\mathbf{r}_i$ are no longer all independent, since they are connected by the equations of constraint; hence the equations of motion are not all independent.

In the case of holonomic constraints, we can introduce **generalised coordinates**. A system of N particles in $\mathbb{R}^3$, free from constraints, has $3N$ independent coordinates or **degrees of freedom**. If there exist k holonomic constraints, then we can eliminate k coordinates, and we are left with $3N-k$ independent coordinates $\{q_i\}$; the system is said to have $3N-k$ degrees of freedom. Thus we express each of the old coordinates $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N$ in terms of the new $3N-k$ independent variables $q_1, q_2, \dots, q_{3N-k}$:

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{r}_1(q_1, q_2, \dots, q_{3N-k}, t) \\ &\vdots \\ \mathbf{r}_N &= \mathbf{r}_N(q_1, q_2, \dots, q_{3N-k}, t) \end{aligned} \tag{1.12}$$

We call Eq. (1.12) **transformation equations**, which include the constraints.

Example (Double pendulum)

A **double pendulum** moving in a plane consists of two masses connected by an inextensible light rod and suspended by a similar rod fastened to one of the particles.

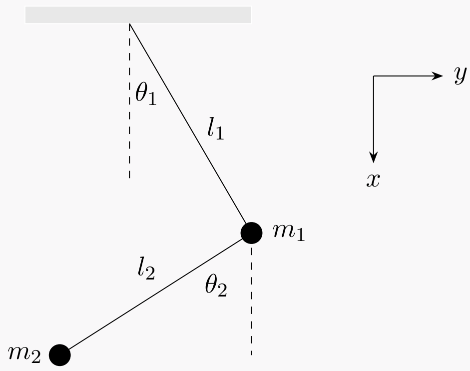

Figure 1.1: Double pendulum

Since $N = 2$, we would expect $3N = 6$ degrees of freedom. To introduce generalised coordinates for θ_1 and θ_2, we need to consider the constraints:

- Movement in xy-plane:
$$z_1 = 0, \quad z_2 = 0.$$
- Constant rod lengths:
$$\begin{aligned} {x_1}^2 + {y_1}^2 - {l_1}^2 &= 0 \\ (x_2 - x_1)^2 + (y_2 - y_1)^2 - {l_2}^2 &= 0 \end{aligned}$$

Hence there are $6 - 4 = 2$ independent degrees of freedom.

Consider the position of m_1. We have $x_1 = l_1 \cos\theta_1$ and $y_1 = l_1 \sin\theta_1$, so

$$\mathbf{r}_1 = (x_1, y_1) = (l_1 \cos\theta_1, l_1 \sin\theta_1) = f(\theta_1)$$

so $\mathbf{r}_1$ is a function of only θ_1.

§1.4 D'Alembert's Principle and Lagrange's Equations

Newton's equations are the fundamental laws of non-relativistic mechanics but their vector nature makes them simple to use only in Cartesian coordinates. The Lagrange equations represent a reformulation of Newton's laws to enable us to use them easily in a general coordinate system which is not Cartesian. Important examples are polar coordinates in the plane, spherical or cylindrical coordinates in three dimensions. The great power of the Lagrange method is that its basic equations take the same form in all coordinate systems.

A **virtual displacement**[1] of the system is defined as an arbitrary displacement $\delta\mathbf{r}_i$ of each particle but with the time frozen, in accordiance with the given constraints. In other words, it is not the physical displacement that would happen in a time δt, rather it is a mathematical displacement which we can carry out conceptually at a frozen instant of time.

For a system in equilibrium, the total force on each particle is zero so $\mathbf{F}_i = 0$. Then the virtual work of the force $\mathbf{F}_i$ in the displacement $\delta\mathbf{r}_i$ is $\mathbf{F}_i \cdot \delta\mathbf{r}_i = 0$. Summing over all particles,

$$\sum_i \mathbf{F}_i \cdot \delta\mathbf{r}_i = 0.$$

We now decompose $\mathbf{F}_i$ into two parts, namely the applied force $\mathbf{F}_i^{(a)}$ and the force of constraint $\mathbf{f}_i$:

$$\mathbf{F}_i = \mathbf{F}_i^{(a)} + \mathbf{f}_i.$$

Remark. The force of constraint is due to the constraints of the system. For example, for gas molecules in a container, the force of constraint is the force exerted on the molecules by the walls of the containe when gas molecules collide with the walls.

Then we can write

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta\mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta\mathbf{r}_i = 0.$$

We now restrict ourselves to systems for which the net virtual work of the forces of constraint is zero[2]; that is, $\sum_i \mathbf{f}_i \cdot \delta\mathbf{r}_i = 0$. Then

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta\mathbf{r}_i = 0, \tag{1.13}$$

which means that the virtual work of the applied forces vanishes. Eq. (1.13) is known as the **principle of virtual work**.

Consider now a system in motion. Then the equation of motion

$$\mathbf{F}_i = \dot{\mathbf{p}}_i$$

can be written as

$$\mathbf{F}_i - \dot{\mathbf{p}}_i = 0$$

[1] The displacement is called virtual to distinguish it from an actual displacement of the system $\mathrm{d}\mathbf{r}$ occurring in a time interval $\mathrm{d}t$, during which the forces and constraints may be changing.

[2] We have seen that this condition holds true for rigid bodies, such as a pendulum, and it is valid for a large number of other constraints. Thus, if a particle is constrained to move on a surface, the force of constraint is perpendicular to the surface, while the virtual displacement must be tangent to it, and hence the virtual work vanishes.

This is no longer true if sliding friction forces are present, and we must exclude such systems from our formulation.

Note that if a particle is constrained to a surface that is itself moving in time, the force of constraint is instantaneously perpendicular to the surface and the work during a virtual displacement is still zero even though the work during an actual displacement in the time $\mathrm{d}t$ does not necessarily vanish.

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