H2 Mathematics

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Abstract

This set of notes follows the Singapore GCE A-Level H2 Mathematics syllabus.

Graphing Calculator

The use of a graphing calculator is crucial, not only because of its ability to solve equations and plot graphs that are rather challenging to do by hand, but also to save precious time (e.g. use GC to solve system of linear equations instead of by hand).

Here are some combinations of keys that you might find helpful.

General

- Reset: `2` `8` `ON`
- Switch on: `ON`
- Switch off: `OFF`
- Return to main page: `QUIT`
- Display MATH menu: `MATH`
- Change calculator mode: `MODE`

Basic input methods

- Input fractions: `MATH` `FRAC`
- Change decimal to fraction: `MATH` `1:▸Frac`
- Change fraction to decimal: `MATH` `2:▸Dec`
- Cube root: `MATH` `4:` $\sqrt[3]{}$
- Equality and inequality symbols: `TEST` `CONDITIONS`

Graphing

- Input equation of graph: `Y=`
- View graph of equation: `GRAPH`
- Restrict domain/range of graph: `WINDOW`
- Zoom in or out: `ZOOM`
- Find y value at a specific x value: `CALC` `1:value`
- Find x-intercept: `CALC` `2:zero`
- Find point of intersection between graphs: `CALC` `5:intersect`
- Find minimum or maximum point of graph: `CALC` `3:minimum` or `4:maximum`
- Parametric functions: `MODE` `FUNCTION` `PARAMETRIC` then proceed to graph
- Conic sections: `APPS` `2:Conics`
- Piecewise function: `MATH` `B:piecewise`

Algebra

- Solve quadratic equation: `APPS` `4:Plysmlt2` `1:POLYNOMIAL ROOT FINDER`
- Solve system of linear equations: `APPS` `4:Plysmlt2` `2:SIMULTANEOUS EQN SOLVER`
- Find solution(s) to a complicated equation: graph the functions, then find point(s) of intersection

Sequences and Series

- Use graph to determine behaviour of sequence/series: `TABLE`
- Evaluate summation: `MATH` `0:summation`

Calculus

- Evaluate derivative/gradient at a point (graph): `CALC` `6:dy/dx`
- Evaluate integral given the lower and upper limits (graph): `CALC` `7:∫f(x)dx`
- Evaluate derivative at an x value: `MATH` `8:nDeriv`
- Evaluate definite integral: `MATH` `9:fnInt`

Probability

- Permutation: `MATH` `PROB` `2:nPr`
- Combination: `MATH` `PROB` `3:nCr`
- Factorial: `MATH` `PROB` `4:!`

Statistics

- Calculate expectation and variance of discrete random variable:

Correlation and Regression

- Scatter diagram: input values into lists `L1` and `L2`, `STAT PLOTS` `1:Plot1` `Xlist` `Ylist` `9:ZoomStat`
- Value of r: input values, `STAT` `CALC` `8:LinReg(a+bx)`
- Regression line on scatter diagram: `F4` `1:Y1` to store equation into Y1
- Linearise: bring cursor to `L3`, key in `1/L1` to generate values of $\frac{1}{x}$

Contents

Part I

Pure Mathematics

§1 Functions

- **Domain**: set of possible inputs

  Range: set of possible outputs

  Vertical line test is used to check for functions: For a function, every vertical line $x = k$, $k \in D_f$ cuts the graph *at most once*.

- A **one-one function** is a function where no two distinct elements in the given domain have the same image under f.

  $$\forall x_1, x_2 \in D_f, x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$

  This means that every output corresponds to exactly one input.

  Horizontal line test is used to check for one-one functions: Every horizontal line $y = k$, $k \in R_f$ cuts the graph *at most once*. Conversely, for a function that is not one-one, provide a *specific counter-example* of a horizontal line that cuts the graph at more than one point.

- Let f be a one-one function. Then f has an **inverse function** f^{-1} defined as

  $$f^{-1}(y) = x \iff f(x) = y, \forall x \in D_f$$

  $D_{f^{-1}} = R_f$, $R_{f^{-1}} = D_f$

  For the inverse function f^{-1} to exist, f is a one-one function (check using horizontal line test).

  To find the inverse of a given function, make x the subject.

  Graphically, the graphs of f and f^{-1} are **reflections** of each other in the line $y = x$.[1] Hence $y = f(x)$, $y = f^{-1}(x)$ and $y = x$ **intersect at one same point**.[2]

- A **composite function** gf is where *all* elements in the domain of function f are directly mapped to the elements in the range of function g.

  The composite function gf exists iff $R_f \subseteq D_g$.[3]

  $D_{gf} = D_f$, $R_{gf} = R_g$

  To determine the range of a composite function, use **two-stage mapping**.

  An **identity function** is a function which returns the same value, which was used as its input.

  $$f^{\breve{}1} f(x) = f f^{\breve{}1}(x) = x$$

> **Remark.** Even though the composite functions $f^{\breve{}1} f$ and $f f^{\breve{}1}$ have the same rule, they may have different domains. $D_{f^{\breve{}1} f} = D_f$ whereas $D_{f f^{\breve{}1}} = D_{f^{-1}}$.

[1] Proof: Let (a, b) be a point on the curve $y = f(x)$. Then (b, a) is a point on the curve $y = f^{-1}(x)$ since $f(a) = b \iff a = f^{-1}(b)$.

[2] However, this is not the case for functions where $y = f(x)$ and $y = f^{-1}(x)$ do not intersect in the given domain, or for functions where $f(x) = f^{-1}(x)$.

[3] Proof: The domain of function g must include values of the range of f, so that the function g is well-defined as every element in its domain, as well as in R_f, is mapped to something.

Problems

§2 Graphs

- Features to include in graph sketch:
  1. **Stationary points**
     - Maximum point
     - Minimum point
     - Point of inflexion
  2. **Intercepts**
  3. **Asymptotes**
     - Horizontal asymptote: line $y = a$ where $x \to \pm\infty$, $y \to a$
     - Vertical asymptote: line $x = a$ where $x \to a$, $y \to \pm\infty$
     - Oblique asymptote: line $y = ax + b$ where $x \to \pm\infty$, $y - (ax + b) \to 0$

  To determine the restriction on possible values of x or y, use *discriminant*.

- Conic Sections
  - **Circle** with centre (h, k) and radius r:
  $$(x-h)^2 + (y-k)^2 = r^2 \tag{1}$$
  - **Ellipse** with centre (h, k) and semi-major axis of a units, semi-minor axis of b units:
  $$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \tag{2}$$
  - **Horizontal hyperbola** with centre (h, k):
  $$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \tag{3}$$
    Distance from turning points to centre: a units (in x-direction)
    Equations of oblique asymptotes: $y - k = \pm\frac{b}{a}(x-h)$
  - **Vertical hyperbola** with centre (h, k):
  $$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1 \tag{4}$$
    Distance from turning points to centre: b units (in y-direction)
    Equations of oblique asymptotes: $y - k = \pm\frac{b}{a}(x-h)$
  - **Regular parabola** with vertex (h, k):
  $$y = a(x-h)^2 + k \tag{5}$$
  - **Sideways parabola** with vertex (h, k):
  $$x = a(y-k)^2 + h \tag{6}$$
  - Rectangular hyperbola
    In the case of horizontal and vertical asymptotes,
  $$y = \frac{ax+b}{cx+d} \tag{7}$$

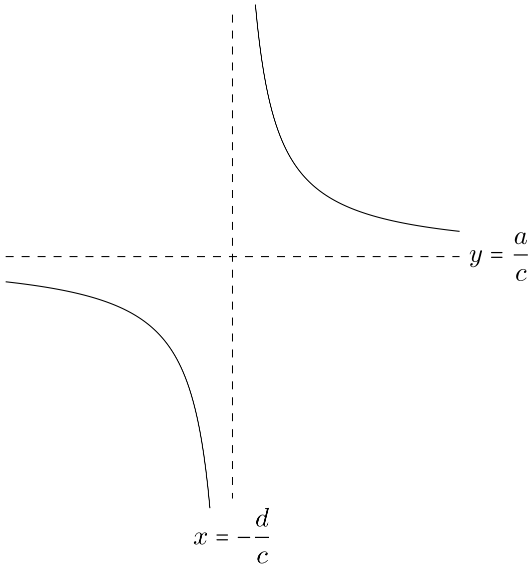
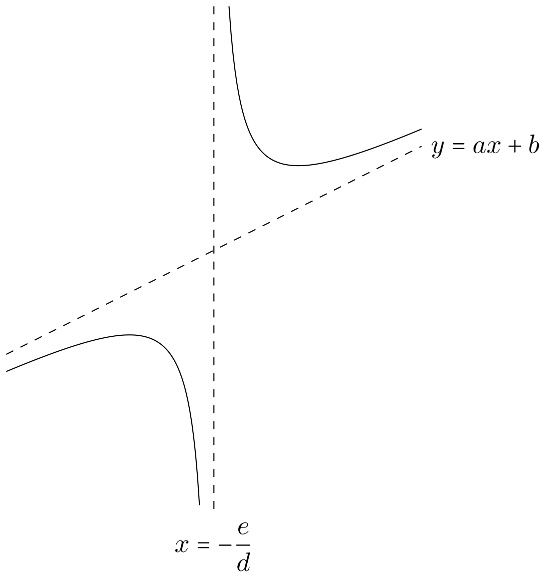


Figure 1: Rectangular hyperbola

In the case of oblique asymptotes,

$$y = ax + b + \frac{c}{dx + e} \tag{8}$$



- Parametric equations
  - To sketch the Cartesian graph of parametric equations, using `MODE` to change the mode from `FUNCTION` to `PARAMETRIC` on GC. Remember to take note of the domain of the parameter t.
  - To find the Cartesian equation, eliminate the parameter t by solving the equations simultaneously. A Cartesian equation should only contain the variables x and y.
  - To convert a Cartesian equation to its parametric form, if involving trigonometric expressions, the Pythagorean trigonometric identity $\sin^2 x + \cos^2 x = 1$ may be handy.
- **Transformations**

  To get a function $cf(bx + a) + d$ from $f(x)$,

  1. translate by a units in the negative x direction
  2. scale by factor of $\frac{1}{b}$ parallel to x-axis
  3. scale by factor of c parallel to the y-axis
  4. translate by d units in the positive y direction.

Translation

Equation	Replace	Graph	Point
$y = f(x) + a$	$y \mapsto y - a$	Translate a units in positive y-direction	$(x, y) \to (x, y + a)$
$y = f(x) - a$	$y \mapsto y + a$	Translate a units in negative y-direction	$(x, y) \to (x, y - a)$
$y = f(x - a)$	$x \mapsto x - a$	Translate a units in positive x-direction	$(x, y) \to (x + a, y)$
$y = f(x + a)$	$x \mapsto x + a$	Translate a units in negative x-direction	$(x, y) \to (x - a, y)$

Reflection

Equation	Replace	Graph	Point
$y = -f(x)$	$y \mapsto -y$	Reflected in x-axis	$(x, y) \to (x, -y)$
$y = f(-x)$	$x \mapsto -x$	Reflected in y-axis	$(x, y) \to (-x, y)$

Scaling

Equation	Replace	Graph	Point
$y = af(x)$	$y \mapsto \dfrac{y}{a}$	Scale by factor of a parallel to y-axis	$(x, y) \to (x, ay)$
$y = f\left(\dfrac{x}{a}\right)$	$x \mapsto \dfrac{x}{a}$	Scale by factor of a parallel to x-axis	$(x, y) \to (ax, y)$

Modulus

Equation	Graph	Point
$y = \lvert f(x) \rvert$	Retain the portion of graph above x-axis, reflect the portion below x-axis in the x-axis	$(x, y) \to (x, \lvert y \rvert)$
$y = f(\lvert x \rvert)$	For positive x, retain the portion of graph. For negative x, reflect the positive portion of graph in the y-axis	$(x, y) \to (\lvert x \rvert, y)$

Reciprocal

$y = \dfrac{1}{f(x)}$

Derivative

$y = f'(x)$

Problems

§3 Equations and Inequalities

- **Systems of linear equations** can be solved efficiently using `PlySmlt2` on GC.

  Types of solutions:

  1. Unique solution
  2. Infinitely many solutions
  3. No solutions

  Questions often involve practical problems, from which systems of linear equations are set up.

  Important points to take note when solving inequalities:

  1. Do not cross multiply without knowing whether terms are positive or not.
  2. Know the difference between "and" and "or", i.e. intersection and union of sets.
  3. Solutions should not be equal to roots of denominator.

  Some manipulations before using the methods below:

  - Directly deduce after moving all terms to one side.
  - Multiply the square of a term in the denominator.
  - Either the numerator or denominator is always positive (working is required to show this).

- Test-value method

  To use the test-value method:

  1. Indicate the critical value(s) on a number line.
  2. Choose an x-value within each interval as the *test-value*.
  3. Plug in the test-value to evaluate whether the polynomial is positive or negative within that interval.

- Graphical method

  For polynomial inequalities in the form of $P(x) > 0$, sketching the graph of $y = P(x)$ and its x-intercepts gives us the solution.

  Alternatively, for inequalities in the form of $P(x) > Q(x)$, sketch the graph of $y = P(x)$ and $y = Q(x)$, then identify the region of the graph where the inequality holds.

- Solutions of related inequalities

  To deduce solutions of related inequalities, replace x with some expression of x using the solutions of inequalities solved in earlier parts of the question.

Problems

§4 Sequences and Series

- Sequences

  A **sequence** is a set of numbers $u_1, u_2, \ldots, u_n$ where $n \in \mathbb{Z}^+$. A sequence can be generated by giving a formula u_n for the nth term.

  When describing the behaviour of a sequence, describe its

  1. **Trend**: (strictly) increasing/decreasing, constant, alternating
     For a strictly increasing sequence, show that $u_{n+1} > u_n$.
     For a strictly decreasing sequence, show that $u_{n+1} < u_n$.
  2. **Convergence**: converges, diverges
     For a convergent sequence, show that as $n \to \infty^+$, u_n approaches a *unique value*. For a divergent sequence, show that as $n \to \infty^+$, u_n approaches ∞^+ or ∞^-.

- Series

  A **series** is the <u>sum of terms</u> of a sequence. The sum to n terms is denoted by S_n.

  To find the term for a series when given the sequence,

  $$u_n = S_n - S_{n-1} \tag{9}$$

  For the sum to infinity S_∞ of a series to exist, the series converges; conversely, the sum to infinity does not exist if the series diverges.

- Arithmetic Progression

  An **arithmetic progression** is a sequence in which successive terms differ by a common difference.

  The formula for the nth term is

  $$u_n = a + (n-1)d \tag{10}$$

  The formula for the sum is

  $$S_n = \frac{n}{2}[2a + (n-1)d] \tag{11}$$

  To show that u_n is an AP, show that $u_n - u_{n-1}$ is a constant.

- Geometric Progression

  An **geometric progression** is a sequence in which successive terms differ by a common ratio.

  The formula for the nth term is

  $$u_n = ar^{n-1} \tag{12}$$

  The formula for the sum is

  $$\begin{aligned} S_n &= \frac{a(1-r^n)}{1-r} \qquad |r| < 1 \\ S_n &= \frac{a(r^n-1)}{r-1} \qquad |r| > 1 \end{aligned} \tag{13}$$

  To show that u_n is a GP, $\dfrac{u_n}{u_{n-1}}$ is a constant.

  The formula for sum to infinity is

  $$S_\infty = \frac{a}{1-r} \qquad |r| < 1 \tag{14}$$

  To show that the sum to infinity of a GP exists, the GP converges, so show that common ratio $|r| < 1$.

- Summation Series

  - Standard series

    $$\sum_{r=1}^{n} r = \frac{n(n+1)}{2} \tag{15}$$

    $$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6} \tag{16}$$

    $$\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4} \tag{17}$$

    > **Remark.** The lower limit must be 1.

  - **Method of differences**

    The general term $u_r = f(r) - f(r-1)$, then

    $$\begin{aligned} \sum_{r=1}^{n} u_r &= \sum_{r=1}^{n} (f(r) - f(r-1)) \\ &= \begin{cases} \cancel{f(1)} - f(0) + \\ \cancel{f(2)} - \cancel{f(1)} + \\ \vdots \\ \cancel{f(n-1)} - \cancel{f(n-2)} + \\ f(n) - \cancel{f(n-1)} \end{cases} \\ &= f(n) - f(0) \end{aligned}$$

    > **Remark.** Must show **diagonal** cancellation of intermediate terms in the working.

Problems

§5 Differentiation

- Differentiation rules
  - Scalar multiplication
$$\frac{\mathrm{d}}{\mathrm{d}x}kf(x) = k\frac{\mathrm{d}}{\mathrm{d}x}f(x) \tag{18}$$
  - Sum/Difference rule
$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x) \pm g(x)] = \frac{\mathrm{d}}{\mathrm{d}x}f(x) + \frac{\mathrm{d}}{\mathrm{d}x}g(x) \tag{19}$$
  - Product rule
$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x) \cdot g(x)] = f(x)\frac{\mathrm{d}}{\mathrm{d}x}g(x) + g(x)\frac{\mathrm{d}}{\mathrm{d}x}f(x) \tag{20}$$
  - Quotient rule
$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)} \tag{21}$$
  - Chain rule
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} \tag{22}$$
- New functions
  Apply Chain Rule wherever necessary.
  - Exponential functions
$$\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x \ln a$$
  - Logarithmic functions
$$\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\log_a x = \frac{1}{x \ln a}$$
  - Trigonometric functions
$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x \tan x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\operatorname{cosec}^2 x$$
  - Inverse trigonometric functions
$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1} x = \frac{1}{1+x^2}$$
- Applications
  - Increasing and decreasing function
    As long as $f(x)$ is continuous,
    * $f'(x) > 0 \implies$ curve is increasing
    * $f'(x) < 0 \implies$ curve is decreasing
    * $f'(x) = 0 \implies$ curve is stationary
  - Concavity
    * $f''(x) > 0 \implies$ curve is concave upwards
    * $f''(x) < 0 \implies$ curve is concave downwards
  - Nature of stationary points
$$f'(a) = 0$$
    First derivative test:
    Second derivative test:
    * $f''(a) > 0 \implies$ minimum point
    * $f''(a) < 0 \implies$ maximum point
  - Tangents and normals
    Equation of **tangent** to curve $y = f(x)$ at point (a, b):
$$y - b = f'(a)(x - a)$$
    where gradient of tangent $= f'(a)$.
    Equation of **normal** to curve $y = f(x)$ at point (a, b):
$$y - b = -\frac{1}{f'(a)}(x - a)$$
    where gradient of normal $= -\frac{1}{f'(a)}$.
  - Maximisation and minimisation problems
  - Connected rates of change
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$
  - Implicit differentiation
    To differentiate an implicit function, make use of the Chain Rule.
  - Parametric differentiation
    If x and y are functions of a parameter t, by applying Chain Rule, we have
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$

Problems

§6 Maclaurin Series

- **Binomial theorem**: $\forall n \in \mathbb{Z}^+$
$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \cdots + b^n \tag{23}$$
Binomial series: $\forall n \in \mathbb{R}, n \neq 0$, which includes **negative** and **fractional** n
$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r + \cdots \tag{24}$$
which has validity range $|x| < 1$.

- The **Taylor series** is a power series centered at $x = a$:
$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(0)}{3!}(x-a)^3 + \frac{f^{(4)}(0)}{4!}(x-a)^4 + \cdots \tag{25}$$
The **Maclaurin series** is a special case of the Taylor series, centered at $x = 0$:
$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots \tag{26}$$

- Expansion of **standard series** and their validity range:
$$\begin{aligned}
(1+x)^n &= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!} + \cdots && |x| < 1 \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots && \text{all } x \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots && \text{all } x \text{ in radians} \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots && \text{all } x \text{ in radians} \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots && -1 < x \leq 1
\end{aligned}$$

- **Small angle approximation**
Approximation of trigonometric functions for a sufficiently small x such that higher powers of x can be neglected.
$$\begin{aligned}
\sin x &\approx x \\
\cos x &\approx 1 - \frac{x^2}{2} \\
\tan x &\approx x
\end{aligned} \tag{27}$$

Problems

§7 Integration Techniques

- Standard functions

$$\int (ax+b)^n \,\mathrm{d}x = \frac{(ax+b)^{n+1}}{a(n+1)}$$

$$\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln|ax+b|$$

$$\int e^{ax+b}\,\mathrm{d}x = \frac{1}{a}e^{ax+b}$$

- Systematic integration

$$\int f'(x)[f(x)]^n \,\mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1}$$

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)|$$

$$\int f'(x)e^{f(x)}\,\mathrm{d}x = e^{f(x)}$$

- Trigonometric functions

$$\int \sin x \,\mathrm{d}x = -\cos x$$

$$\int \cos x \,\mathrm{d}x = \sin x$$

$$\int \sec^2 x \,\mathrm{d}x = \tan x$$

$$\int \operatorname{cosec}^2 x \,\mathrm{d}x = -\cot x$$

$$\int \sec x \tan x \,\mathrm{d}x = \sec x$$

$$\int \operatorname{cosec} x \cot x \,\mathrm{d}x = -\operatorname{cosec} x$$

$$\int \tan x \,\mathrm{d}x = \ln|\sec x|$$

$$\int \sec x \,\mathrm{d}x = \ln|\tan x + \sec x|$$

$$\int \operatorname{cosec} x \,\mathrm{d}x = \ln|\operatorname{cosec} x - \cot x| = \ln\left|\tan\frac{x}{2}\right|$$

$$\int \cot x \,\mathrm{d}x = \ln|\sin x|$$

Some transformations:

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

More importantly,

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

Sum to product:

$$\sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$$

$$\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$$

$$\cos P + \cos Q = 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$$

$$\cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$$

- Algebraic fractions

$$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \sin^{-1}\frac{x}{a}$$

$$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

- Partial fractions

$$\int \frac{1}{x^2-a^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$$

$$\int \frac{1}{a^2-x^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|$$

- Substitution

- Integration by parts

$$\int uv' = uv - \int u'v$$

Guideline on choosing "u":

Abbreviation	**Function Type**
L	logarithmic
I	inverse trigonometric
A	algebraic
T	trigonometric
E	exponential

Using **DI method** (also known as tabular method),

Example 7.1. Evaluate

$$\int x^2 e^x \,\mathrm{d}x\,.$$

Solution. We choose x^2 as "D", and e^x as "I".

sign	**D**	**I**
$+$	x^2	e^x
$-$	$2x$	e^x
$+$	2	e^x
$-$	0	e^x

Multiplying terms diagonally and summing them up gives us

$$\int x^2 e^x \,\mathrm{d}x = x^2 e^x - 2xe^x + 2e^x + c.$$

□

Problems

§8 Applications of Integration

- Integral is a *limit of sum.*

  In general, $\int_a^b f(x)\,\mathrm{d}x$ may be thought of as the value of the area bounded by the curve $y = f(x)$, x-axis, $x = a$, $x = b$ where $a \le b$.

  Since this area can be divided into vertical strips of equal width Δx, where area ΔA of one strip is $f(x)\Delta x$, the area A is given by

$$\lim_{\Delta x \to 0} \sum_{x=a}^{b} f(x)\Delta x = \int_a^b f(x)\,\mathrm{d}x \tag{28}$$

- Area under curve

  When calculating area of the region required, it should always be **positive**. This means **adding a negative sign** to an integral that is negative.

  For area between curve and x-axis,

  1. Sketch the curve
  2. Observe where the curve cuts the x-axis.
  3. Split the area into one above x-axis (positive), one below x-axis (negative).

  For area between curve and y-axis,

  1. Sketch the curve
  2. Observe where the curve cuts the y-axis.
  3. Split the area into one right of y-axis (positive), one left of y-axis (negative).

  When 2 or more curves are involved,

  1. Evaluate point(s) of intersection.
  2. Split the area into different parts.

  For all $x \in [a, b]$, for $f(x) \ge g(x)$, area between curves is

$$\int_a^b [f(x) - g(x)]\,\mathrm{d}x \tag{29}$$

  For all $y \in [a, b]$, for $f(y) \ge g(y)$, area between curves is

$$\int_a^b [f(y) - g(y)]\,\mathrm{d}y \tag{30}$$

  Remember this as: “upper" minus “lower" curve

  Area under curve parametrically: For $x = h(t)$ and $y = g(t)$, area under curve

$$\int_a^b y\,\mathrm{d}x = \int_{t_1}^{t_2} g(t)\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t \tag{31}$$

  where t_1 and t_2 are values of t when $x = a$ and $x = b$ respectively.

  Similarly,

$$\int_c^d x\,\mathrm{d}y = \int_{t_3}^{t_4} h(t)\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t \tag{32}$$

  where t_3 and t_4 are values of t when $y = c$ and $y = d$ respectively.

  > **Remark.** We do not find the Cartesian equation of the curve.

- Solid of revolution (Volume)

  Rotate region bounded by $y = f(x)$, x-axis, $x = a$, $x = b$ 2π around x-axis, volume of revolution is

$$\pi\int_a^b y^2\,\mathrm{d}x = \pi\int_a^b \left[f(x)\right]^2\mathrm{d}x \tag{33}$$

  Rotate region bounded by $x = f(y)$, y-axis, $y = a$, $y = b$ 2π around y-axis, volume of revolution is

$$\pi\int_a^b x^2\,\mathrm{d}y = \pi\int_a^b \left[f(y)\right]^2\mathrm{d}y \tag{34}$$

  Volume of standard shapes

  – Cone: $\frac{1}{3}\pi r^2 h$
  – Cylinder: $\pi r^2 h$

  Rotate region bounded by two curves $y = f(x)$ and $y = g(x)$ 2π around x-axis, volume of revolution is

$$\pi\int_a^b \left[f(x)\right]^2 - \left[g(x)\right]^2\mathrm{d}x \tag{35}$$

  Rotate region bounded by two curves $x = f(y)$ and $x = g(y)$ 2π around y-axis, volume of revolution is

$$\pi\int_a^b \left[f(y)\right]^2 - \left[g(y)\right]^2\mathrm{d}y \tag{36}$$

Problems

§9 Differential Equations

- **1st order differential equation**: has derivatives up to the 1st derivative

  – Separation of variables

> **Example 9.1.** Find the general solution of
> $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{2x(x+1)} \qquad \text{for } x > 0$$
> and sketch some of the family of solution curves.

Solution.

$$\begin{aligned}
&\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{2x(x+1)} \\
\implies &\int \frac{2}{y}\,\mathrm{d}y = \int \frac{1}{x(x+1)}\,\mathrm{d}x = \int \left(\frac{1}{x} - \frac{1}{x+1}\right)\mathrm{d}x \\
\implies &2\ln y = \ln x - \ln(x+1) + \ln A \\
\implies &y^2 = \frac{Ax}{x+1}
\end{aligned}$$

□

  – Using substitutions

> **Example 9.2.** Use the substitution $z = x + y$ to solve the differential equation
> $$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos(x+y).$$

Solution. From $z = x + y$,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 1 + \frac{\mathrm{d}y}{\mathrm{d}x} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}x} - 1$$

Substituting this and solving by separable variables,

$$\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}x} &= 1 + \cos z \\
\int \frac{1}{1+\cos z}\,\mathrm{d}z &= \int \mathrm{d}x \\
\frac{1}{2}\sec^2\frac{z}{2}\,\mathrm{d}z &= x + c \quad \text{[double angle formula]} \\
\tan\frac{z}{2} &= x + c
\end{aligned}$$

Substituting back gives us

$$\tan\frac{x+y}{2} = x + c$$

□

- **2nd order differential equation**: has derivatives up to the 2nd derivative

  Direct integration

Problems

§10 Vectors

- **Magnitude** of a vector $\mathbf{a}$ is denoted by $|\mathbf{a}|$. **Unit vector** of vector $\mathbf{a}$ is denoted by $\hat{\mathbf{a}}$, where $|\hat{\mathbf{a}}| = 1$.

$$\frac{\mathbf{a}}{|\mathbf{a}|} = \hat{\mathbf{a}} \tag{37}$$

  Convention for unit vectors: $\mathbf{i}$ is unit vector along x-axis, $\mathbf{j}$ is unit vector along y-axis, $\mathbf{k}$ is unit vector along z-axis.

  Ratio theorem: if point P divides AB in the ratio of $\lambda : \mu$, then

$$\overrightarrow{OP} = \frac{\mu \cdot \overrightarrow{OA} + \lambda \cdot \overrightarrow{OB}}{\mu + \lambda} \tag{38}$$

  Midpoint theorem is a special case where P is the midpoint:

$$\overrightarrow{OX} = \frac{\overrightarrow{OA} + \overrightarrow{OB}}{2}$$

  Equal vectors

$$\mathbf{a} = \mathbf{b} \iff \text{same magnitude and direction}$$

  Parallel vectors

$$\mathbf{a} \parallel \mathbf{b} \iff \exists \lambda \in \mathbb{R}, \lambda \neq 0 \text{ s.t. } \mathbf{a} = \lambda \mathbf{b}$$

  That is, one vector is a *scalar multiple* of the other.

  Collinear points

$$A, B, C \text{ collinear} \iff \overrightarrow{AB} \parallel \overrightarrow{AC}$$

  (and the two vectors share a common point A)

  Coplanar vectors

$$\mathbf{a}, \mathbf{b}, \mathbf{c} \text{ coplanar} \iff \exists \lambda, \mu \in \mathbb{R} \text{ s.t. } \mathbf{a} = \lambda \mathbf{b} + \mu \mathbf{c}$$

  where $\mathbf{b}$ and $\mathbf{c}$ are non-parallel, non-zero vectors. That is, one of the vectors can be expressed as a *unique linear combination* of the other two vectors.

  Parallelogram

$$OACB \text{ is a parallogram} \iff \overrightarrow{BC} = \overrightarrow{OA}$$

  That is, two opposite sides are equal.

- **Dot product** (or scalar product) is defined as

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta \tag{39}$$

  where θ is the angle in the interval $[0, \pi]$ between vectors $\mathbf{a}$ and $\mathbf{b}$.

  > **Remark.** The angle between two vectors which either *both* point outwards or inwards.

  For 3D vectors,

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{40}$$

  > **Remark.** Geometrically, dot product measures the **alignment** between the two vectors.

  Properties:

  – Commutative law applies: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

  – Distributive law applies: $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{c} + \mathbf{d}) = \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{d}$
    Distributive property of scalar multiplication: $(\lambda \mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\lambda \mathbf{b}) = \lambda(\mathbf{a} \cdot \mathbf{b})$

  – Associative law applies: $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$

  Applications

  – Find length of vector

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$$

  – Show perpendicular vectors

$$\mathbf{a} \perp \mathbf{b} \iff \mathbf{a} \cdot \mathbf{b} = 0$$

    where $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors.

  – Show parallel vectors

$$\mathbf{a} \cdot \mathbf{b} = \pm |\mathbf{a}||\mathbf{b}|$$

    where positive sign implies same direction, negative sign implies opposite directions.

  – Angle between two vectors

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

    If $\mathbf{a} \cdot \mathbf{b} > 0$, then angle between $\mathbf{a}$ and $\mathbf{b}$ is acute.
    If $\mathbf{a} \cdot \mathbf{b} < 0$, then angle between $\mathbf{a}$ and $\mathbf{b}$ is obtuse.

  – Length of projection of $\mathbf{a}$ onto $\mathbf{b}$

$$|\mathbf{a} \cdot \hat{\mathbf{b}}|$$

    > **Remark.** Modulus sign, since length must be positive.

  – Vector projection of $\mathbf{a}$ onto $\mathbf{b}$

$$(\mathbf{a} \cdot \hat{\mathbf{b}}) \hat{\mathbf{b}}$$

    > **Remark.** No modulus sign, since vectors can take on both positive and negative values for direction.

- **Cross product** is defined as

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \, \hat{\mathbf{n}} \tag{41}$$

  where θ is the angle in the interval $[0, \pi]$ between vectors $\mathbf{a}$ and $\mathbf{b}$, $\hat{\mathbf{n}}$ denotes the unit normal vector (perpendicular to both vectors).

  For 3D vectors,

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \tag{42}$$

> **Remark.** Geometrically, cross product produces a **new vector** perpendicular to the two vectors.

Properties:

- Commutative law does not apply, in fact $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$. Note that $\mathbf{a}\times\mathbf{b}| = |\mathbf{b}\times\mathbf{a}|$.
- Distributive law applies: $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = (\mathbf{a}\times\mathbf{b}) + (\mathbf{a}\times\mathbf{c})$
  Distributive property of scalar multiplication: $(\lambda\mathbf{a})\times\mathbf{b} = \mathbf{a}\times(\lambda\mathbf{b}) = \lambda(\mathbf{a}\times\mathbf{b})$
- Associative law does not apply: $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} \neq \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$

Applications

- Find normal vector[4]

$$\mathbf{c} = \mathbf{a}\times\mathbf{b}$$

> **Remark.** Normal vectors are not unique; any vector parallel to $\mathbf{a}\times\mathbf{b}$ is a normal to the plane containing $\mathbf{a}$ and $\mathbf{b}$.

- Show parallel vectors

$$\mathbf{a}\parallel\mathbf{b} \iff \mathbf{a}\times\mathbf{b} = 0$$

- Show perpendicular vectors

$$\mathbf{a}\perp\mathbf{b} \iff |\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|$$

- Shortest distance from point to vector or line

$$|\mathbf{a}\times\hat{\mathbf{b}}|$$

- Area of parallelogram

$$|\mathbf{a}\times\mathbf{b}|$$

where $\mathbf{a}$ and $\mathbf{b}$ are two adjacent sides of the parallelogram.

> **Remark.** Modulus sign, since area must be positive.

- Area of triangle

$$\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$$

where $\mathbf{a}$ and $\mathbf{b}$ are two adjacent sides of the triangle.

> **Remark.** Modulus sign, since area must be positive.

- **Lines**

  Equation of a line

  1. **Vector form**

  $$\ell : \mathbf{r} = \mathbf{a} + \lambda\mathbf{m}, \lambda\in\mathbb{R} \tag{43}$$

  where $\mathbf{r}$ is general position vector of any point on the line; $\mathbf{a}$ is position vector of fixed point; $\mathbf{m}$ is *direction vector*.

  2. **Cartesian form**

  $$\frac{x-a_1}{m_1} = \frac{y-a_2}{m_2} = \frac{z-a_3}{m_3} = \lambda \tag{44}$$

  where $\mathbf{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, $\mathbf{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\mathbf{m} = m_1\hat{\imath} + m_2\hat{\jmath} + m_3\hat{k}$, $m_1, m_2, m_3 \neq 0$

  3. **Parametric form**

  $$x = a_1+\lambda m_1, \quad y = a_2+\lambda m_2, \quad z = a_3+\lambda m_3 \tag{45}$$

Relationships between two lines ($\ell_1 : \mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{m}_1$ and $\ell_2 : \mathbf{r} = \mathbf{a}_2 + \lambda\mathbf{m}_2$)

- **Parallel** (coplanar)
  Direction vectors are parallel to each other: $\mathbf{m}_1 \parallel \mathbf{m}_2$.
- **Intersecting** (coplanar)
  Direction vectors are not parallel to each other: $\mathbf{m}_1 \nparallel \mathbf{m}_2$.
  Solving simultaneously (by equating the two lines) gives one unique solution (λ, μ).
- **Skew**[5] (non-coplanar)
  The lines do not satisfy the conditions in both cases above, i.e. no unique solution (λ, μ).

Applications

- Determine whether a point lies on a line
  To check if point P lies on line ℓ, substitute $\mathbf{p}$ into equation for ℓ. Compare each of the x, y, z components, form three equations and solve for λ. If solution is consistent i.e. λ obtained from each equation is the same, then P lies on the line.
- Acute angle between two lines
  For two lines with direction vectors $\mathbf{m}_1$ and $\mathbf{m}_2$, using dot product,

$$\cos\theta = \frac{|\mathbf{m}_1\cdot\mathbf{m}_2|}{|\mathbf{m}_1||\mathbf{m}_2|}.$$

- Foot of perpendicular from point to line
  Let F be foot of perpendicular from P to ℓ. Since F lies on ℓ, $\overrightarrow{OF} = \mathbf{a} + \lambda\mathbf{m}$ for some λ (which we want to determine). Find $\overrightarrow{PF}$, in terms of λ. Since $\overrightarrow{PF}\perp\mathbf{m}$, using dot product,

$$\overrightarrow{PF}\cdot\mathbf{m} = 0,$$

  solve for λ. Then substitute the value of λ into the equation of $\overrightarrow{OF}$.
- Perpendicular distance from point to line
  Let h be perpendicular distance from P to ℓ. Using cross product,

$$h = |\overrightarrow{AP}\times\hat{\mathbf{m}}|.$$

[4] vector perpendicular to two given vectors $\mathbf{a}$ and $\mathbf{b}$
[5] Do not intersect and are not parallel

– Length of projection of vector onto line

For line with direction vector $\mathbf{m}$ and vector $\mathbf{v}$,

$$|\mathbf{v} \cdot \hat{\mathbf{m}}|.$$

– Point of reflection of point in line

Use ratio theorem

- **Planes**

Equation of a plane

1. **Vector equation (parametric form)**

$$\pi : \mathbf{r} = \mathbf{a} + \lambda \mathbf{m}_1 + \mu \mathbf{m}_2, \lambda, \mu \in \mathbb{R} \tag{46}$$

where $\mathbf{a}$ is position vector of fixed point, $\mathbf{m}_1$ and $\mathbf{m}_2$ are non-zero, non-parallel vectors that are parallel to the plane.

2. **Vector equation (scalar product form)**

$$\pi : \mathbf{r} \cdot \mathbf{n} = D \text{ where } D = \mathbf{a} \cdot \mathbf{n} \tag{47}$$

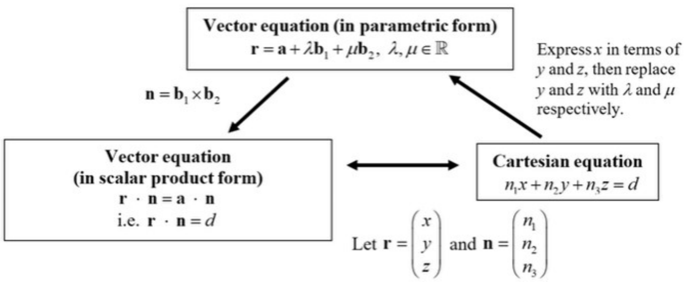
where $\mathbf{r}$ is position vector of any point, $\mathbf{a}$ is position vector of fixed point, $\mathbf{n}$ is normal vector of plane.[6]

$D = 0$ if plane passes through origin (since dot product of $\mathbf{0}$ with any vector is 0).

3. **Cartesian equation**

$$\pi : n_1 x + n_2 y + n_3 z = D \tag{48}$$

where $\mathbf{n} = n_1 \hat{i} + n_2 \hat{j} + n_3 \hat{k}$, $\mathbf{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$.[7]



Relationships between a line and a plane ($\ell : \mathbf{r} = \mathbf{a} + \lambda \mathbf{m}$ and $\pi : \mathbf{r} \cdot \mathbf{n} = D$)

– ℓ and π **do not intersect**

$\mathbf{m} \cdot \mathbf{n} = 0$ and $\mathbf{a} \cdot \mathbf{n} \neq D$; that is, ℓ and π are parallel, and line and plane have no common point.

When solving line and plane simultaneously, no solution.

– ℓ **lies on** π

$\mathbf{m} \cdot \mathbf{n} = 0$ and $\mathbf{a} \cdot \mathbf{n} = D$; that is, ℓ and π are parallel, and line and plane have infinitely many common points.

When solving line and plane simultaneously, infinitely many solutions.

– ℓ and π **intersect**

$\mathbf{m} \cdot \mathbf{n} = 0$; that is, ℓ and π are not parallel, and line and plane have one common point.

When solving line and plane simultaneously, one solution.

Applications

– Form vector equation in parametric form

Find position vector of a fixed point, and two direction vectors.

– Convert from vector equation in parametric form to scalar product form

To find $\mathbf{n}$, given two vectors parallel to the plane, take cross product.

– Convert from Cartesian equation to vector equation (parametric)

Express x in terms of y and z (to reduce number of variables to 2). Then replace y and z with λ and μ.

– Point lying on a plane

A point lies on a plane if its position vector satisfies equation of plane.

For example, unique (λ, μ) satisfies position vector of a point.

– Perpendicular distance from point to plane

Let F denote foot of perpendicular from Q to $\pi : \mathbf{r} \cdot \mathbf{n} = D$. Using length of projection,

$$QF = |\overrightarrow{QF}| = |\overrightarrow{QA} \cdot \hat{\mathbf{n}}|$$

given position vector $\mathbf{q}$, and position vector $\mathbf{a}$ or D.

Alternative: Find foot of perpendicular, find vector, take magnitude

– Foot of perpendicular from point to plane

Consider line ℓ_{QF} passing through Q and F. $\mathbf{n}$ can be used as its direction vector. Hence

$$\begin{cases} \pi : \mathbf{r} \cdot \mathbf{n} = D \\ \ell_{QF} : \mathbf{r} = \overrightarrow{OQ} + \lambda \mathbf{n} \end{cases}$$

Since line and plane intersect at P, solve simultaneously to find λ, substitute back in to find $\overrightarrow{OF}$.

– Acute angle between line and plane

– Acute angle between two planes

– Foot of perpendicular

– Relationship among three planes

Reduce to relationship between a line and a plane.

[6] If you rearrange it, it is $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$. Since $\mathbf{r} - \mathbf{a}$ is the vector joining $\mathbf{a}$ to $\mathbf{r}$, this equation says that $\mathbf{n}$ is orthogonal (perpendicular) to the vector joining $\mathbf{a}$ to $\mathbf{r}$ for any $\mathbf{r}$ on the plane. If you draw a diagram, you should be able to intuitively see that this is the case.

[7] This can be derived by expressing vectors in column form, then taking dot product.

Problems

Problem 1. Given three points P_1, P_2, P_3 in 3-dimensional space (with position vectors $\mathbf{p}_i$ for point P_i). Given two points C_1 and C_2 in the same space (with position vectors $\mathbf{c}_i$ for point C_i), determine if C_1 and C_2 are on the different sides of the plane formed by points P_1, P_2, P_3.

Solution. Firstly find the plane passing through the three points P_1, P_2, P_3. It can be done by taking the normal to the plane to be $(\mathbf{p}_1 - \mathbf{p}_2) \times (\mathbf{p}_1 - \mathbf{p}_3) = \mathbf{n}$.

Notice that the normal $\mathbf{n}$ lies on one of the two sides.

Then find the two vectors connecting the plane to C_1 and C_2, i.e. $\mathbf{p}_1 - \mathbf{c}_1$ and $\mathbf{p}_1 - \mathbf{c}_2$.

Taking the scalar product with the normal indicates how much angle is in between the normal and the two directions. Hence if $\mathbf{n} \cdot (\mathbf{p}_1 - \mathbf{c}_1)$ and $\mathbf{n} \cdot (\mathbf{p}_1 - \mathbf{c}_2)$ have different signs, they lie on different sides of the plane. $\square$

§11 Complex Numbers

- The **imaginary number** is denoted by $i = \sqrt{-1}$. A **complex number** is of the form $a + bi$ where a and b are real numbers. The set of complex numbers is denoted by $\mathbb{C} = \{z \mid z = a + bi, a, b \in \mathbb{R}\}$.

  a is the **real part** of z, denoted by $\text{Re}(z)$; b is the **imaginary part** of z, denoted by $\text{Im}(z)$.

- **Cartesian form**

  $a + bi$ is the **Cartesian form** of complex number z.

  - **Compare real and imaginary parts**
  - Equality: equal if and only if corresponding real and imaginary parts are equal

  $$a + bi = c + di \iff a = c \text{ and } b = d$$

  - Addition and subtraction: add or subtract real and imaginary parts
  - Multiplication: expand brackets in the usual fashion and remember that $i^2 = -1$
  - The **conjugate** is $\bar{z} = a - bi$. Multiplying a complex number with its conjugate eliminates the imaginary part:

  $$z\bar{z} = a^2 + b^2$$

  A very useful property is

  $$z\bar{z} = |z|^2$$

  - Division: multiply numerator and denominator by the conjugate of denominator
  - **Argand diagram**: consists of **real axis** and **imaginary axis**. The point (a, b) represents complex number $a + bi$.

    Geometric representation of complex numbers:

    * **Addition and subtraction**: similar to that of vectors.
    * **Conjugate**: reflection about the real axis.
    * **Multiplication by** i: rotate 90° anti-clockwise about origin.
    * **Scalar multiplication**: scaling

- Complex roots of polynomial equations

  - **Fundamental Theorem of Algebra**

Theorem 11.1 (Fundamental Theorem of Algebra)
Let $P(z) = a_n z^n + \cdots + a_1 z + a_0$ be a polynomial of degree $n \geq 1$ with (complex) coefficients a_k.
Then the roots of $P(z) = 0$ are complex; there are n (not necessarily distinct) complex numbers $\gamma_1, \ldots, \gamma_n$ such that

$$P(z) = a_n(z - \gamma_1)(z - \gamma_2)\cdots(z - \gamma_n).$$

The theorem shows that a degree n polynomial has n roots in $\mathbb{C}$ (including repetitions).

  - **Conjugate Root Theorem**

Theorem 11.2 (Conjugate Root Theorem)
Complex roots of a polynomial equation with <u>real</u> coefficients occur in conjugate pairs.

- **Polar form** expresses a point in space by distance from origin r and angle measured anticlockwise from real axis θ.

  - Trigonometric form

  $$z = r(\cos\theta + i\sin\theta) \tag{49}$$

  where r is the **modulus** $|z|$:

  $$|z| = r = \sqrt{a^2 + b^2}$$

  and θ is the **argument** $\arg(z)$:

  $$\arg(z) = \theta = \tan^{-1}\frac{b}{a}$$

  where $\theta \in (-\pi, \pi]$.

  - Exponential form

Theorem 11.3 (Euler's Formula)
For a complex number z with modulus r and argument θ, z can be expressed as

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}. \tag{50}$$

Theorem 11.4 (Euler's Identity)
A special case of Euler's Formula, in which $r = 1$ and $\theta = \pi$:

$$e^{i\pi} + 1 = 0. \tag{51}$$

NOT IN SYLLABUS

- **Triangle inequality**

  For complex numbers z_1 and z_2,

  $$|z_1| + |z_2| \geq |z_1 + z_2| \tag{52}$$

  with equality only if one of them is 0 or $\arg(z_1) = \arg(z_2)$ i.e. z_1 and z_2 are on the same ray from the origin.

- Proof of **Conjugate Root Theorem**

  Proof. Consider the equation

  $$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$$

  where for $a_0, \ldots, a_n \in \mathbb{R}$, $a_n \neq 0$, $n \in \mathbb{Z}^+$.

  Suppose β is a non-real root of the equation, then

  $$a_n \beta^n + a_{n-1} \beta^{n-1} + \cdots + a_1 \beta + a_0 = 0.$$

  Taking conjugates on both sides of the equation,

  $$a_n \overline{\beta^n} + a_{n-1} \overline{\beta^{n-1}} + \cdots + a_1 \overline{\beta} + a_0 = 0$$

  Note that $\overline{\beta^k} = \overline{\beta}^k$. Thus we have

  $$a_n \bar{\beta}^n + a_{n-1} \bar{\beta}^{n-1} + \cdots + a_1 \bar{\beta} + a_0 = 0$$

  which means that the complex conjugate $\bar{\beta}$ is also a non-real root of the given equation. □

- Proof of **Euler's Formula**

  Proof. Recall that the Taylor Series for e^x is given by

  $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

  Substituting ix for x gives us

  $$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \cdots$$

  Separating real and imaginary parts,

  $$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

  Observe that the real part is the Taylor series of $\cos x$, while the imaginary part is the Taylor series of $\sin x$. Hence proven. □

- **de Moivre's Theorem**

  Theorem 11.5 (de Moivre's Theorem)

  For a real number θ and integer n we have that

  $$\cos n\theta + i \sin n\theta = (\cos\theta + i \sin\theta)^n.$$

  Proof. The proof easily follows from mathematical induction. □

  to derive trigonometric identities

- **Roots of unity**

  The n-th roots of unity, that is the solutions of the equation $z^n = 1$, are

  $$z = e^{i\frac{2k\pi}{n}}, \quad k = 0, 1, 2, \ldots, n-1.$$

  When plotted these n-th roots of unity form a regular n-gon inscribed within the unit circle with a vertex at 1. More generally, for $c \neq 0$, the n solutions of $z^n = c$ make a regular n-gon inscribed in the circle $|z| = |c|^{\frac{1}{n}}$.

- Loci of simple equations and inequalities

  - Circle: $|z - c| \leq r$, where $r \in \mathbb{R}^+, c \in \mathbb{C}$

- Perpendicular bisector of line segment AB: $|z-a| = |z-b|$, where $a, b \in \mathbb{C}$
- Half-line with end point A (excluding A), inclined at angle α to positive real axis: $\arg(z-a) = \alpha$
- Apollonius circle: $|z-a| = k|z-b|$, where $k > 0, k \neq 1$[8]
- Part of a circle with end points A and B: $\arg(z-a) - \arg(z-b) = \alpha$

- **Extended complex plane**
  Let $A, C \in \mathbb{R}$, $B \in \mathbb{C}$, with A, B not both zero. Then

$$Az\bar{z} + B\bar{z} + \bar{B}z + C = 0 \tag{1}$$

represents

(a) a line in direction iB when $A = 0$;

(b) a circle, if $A \neq 0$ and $|B|^2 \geq AC$, with centre $-\dfrac{B}{A}$ and radius $|A|^{-1}\sqrt{|B|^2 - AC}$;

and otherwise has no solutions. Moreover every circle and line can be represented in the form of (1).

Proof. If $A \neq 0$ then we can rearrange (1) as

$$\begin{aligned} z\bar{z} + \frac{B}{A}\bar{z} + \frac{\bar{B}}{A}z + \frac{C}{A} &= 0 \\ \left(z + \frac{B}{A}\right)\overline{\left(z + \frac{B}{A}\right)} &= \frac{B\bar{B}}{A^2} - \frac{C}{A} \\ \left|z + \frac{B}{A}\right|^2 &= \frac{|B|^2 - AC}{A^2}. \end{aligned}$$

If $|B|^2 \geq AC$ then this is a circle with centre $-\dfrac{B}{A}$ and radius $|A|^{-1}\sqrt{|B|^2 - AC}$ and otherwise there are no solutions to (1). Conversely, note that the equation of a general circle is $|z-a| = r$ where r where $a \in \mathbb{C}, r \geq 0$. This is can be rearranged as

$$z\bar{z} - a\bar{z} - \bar{a}z + \left(|a|^2 - r^2\right) = 0$$

which is of the form of (1) with $A = 1$, $B = -a$, and $C = |a|^2 - r^2$.

If $A = 0$ then we have the equation $B\bar{z} + \bar{B}z + C = 0$. If we write $B = u + iv$ and $z = x + yi$ then

$$(u + iv)(x - yi) + (u - vi)(x + yi) + c = 0$$

which rearranges to

$$2ux + 2vy + C = 0 \tag{2}$$

which is the equation of a line. Moreover we see that every line appears in this form by choosing u, v, C appropriately. The line is paralell to the vector $(v, -u)$ or equivalently $v - ui = i(u + iv) = iB$. □

[8] You can prove this by putting $z = x + iy$ and squaring both sides.

Problems

Problem 2 (STEP 2 2020 Q7)**.** In this question, $w = \dfrac{2}{z-2}$.

(i) Let z be the complex number $3+ti$, where $t \in \mathbb{R}$. Show that $|w-1|$ is independent of t. Hence show that, if z is a complex number on the line $\mathrm{Re}(z) = 3$ in the Argand diagram, then w lies on a circle in the Argand diagram with centre 1.

Let V be the line $\mathrm{Re}(z) = p$, where p is a real constant not equal to 2. Show that, if z lies on V, then w lies on a circle whose centre and radius you should give in terms of p. For which z on V is $\mathrm{Im}(w) > 0$?

(ii) Let H be the line $\mathrm{Im}(z) = q$, where q is a non-zero real constant. Show that, if z lies on H, then w lies on a circle whose centre and radius you should give in terms of q. For which z on H is $\mathrm{Re}(w) > 0$?

Solution.

(i)

$$|w-1|^2 = \left|\frac{1-ti}{1+ti}\right|^2 = \frac{(1-ti)(1+ti)}{(1+ti)(1-ti)} = 1,$$

which is independent of t.

Points on the line $\mathrm{Re}(z) = 3$ have the form $z = 3+ti$ and the points satisfying $|w-1| = 1$ lie on a circle with centre 1.

If $z = p + ti$, then

$$|w-c|^2 = \left|\frac{2-(p-2)c-cti}{(p-2)+ti}\right|^2 = \frac{\big(2-(p-2)c\big)^2 + c^2t^2}{(p-2)^2+t^2}$$

which is independent of t when $\big(2-(p-2)c\big)^2 = c^2(p-2)^2$, i.e., $c = \dfrac{1}{p-2}$. Thus the circle has centre at $\dfrac{1}{p-2}$ and radius $\dfrac{1}{|p-2|}$.

$$w = \frac{2}{(p-2)+ti} = \frac{2(p-2)-2ti}{(p-2)^2+t^2},$$

so $\mathrm{Im}(w) > 0$ when $t < 0$; that is, for those z on V with negative imaginary part.

(ii) If $z = t + qi$ then

$$|w-ci|^2 = \left|\frac{2+cq-(t-2)ci}{(t-2)+qi}\right|^2 = \frac{c^2(t-2)^2+(cq+2)^2}{(t-2)^2+q^2}$$

which is independent of t when $(cq+2)^2 = c^2q^2$, i.e., $c = -\dfrac{1}{q}$, so the circle has centre $-\dfrac{1}{q}i$ and radius $\sqrt{c^2} = \dfrac{1}{|q|}$.

$$w = \frac{2}{(t-2)+qi} = \frac{2(t-2)-2qi}{(t-2)^2+q^2},$$

so $\mathrm{Re}(w) > 0$ when $t > 2$; that is, for those z on H with real part greater than 2.

$\square$

Part II

Statistics

§12 Permutation and Combination

- Counting principles
  - **Addition principle**: If there are r choices for performing a particular task, and the number of ways to carry out the k-th choice is k_n for $k = 1, 2, 3, \ldots, r$, then the total number of ways of performing the particular task is equal to the sum of the number of ways for all the r different choices, i.e.,

    $$n_1 + n_2 + n_3 + \cdots + n_r.$$

  - **Multiplication principle**: If one task can be performed in m ways, and following this, a second task can be performed in n ways (regardless of which way the first task was performed), then the number of ways of performing the 2 tasks in succession is $m \times n$.
- **Permutation**: ordered arrangement of objects
  - Permutation of n distinct objects (in a row)
    Given n distinct objects, total number of ways of arranging all these n objects in a row is

    $$n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 = n!$$

  - Permutation of r objects taken from n distinct objects (in a row)
    Given n distinct objects, the number of ways arranging r of these objects $(0 \le r \le n)$ is

    $$n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!} =^n \mathrm{P}_r$$

  - Permutations involving non-distinct/identical objects
    Suppose there are n objects, of which there are r different groups of indistinguishable objects. Total number of permutations of all n objects:

    $$\frac{n!}{n_1! \cdots n_r!}$$

    where n_i represents the number of identical objects in the i-th group for $i = 1, 2, 3 \ldots, r$.

    > **Remark.** We are essentially dividing away the order created by the identical objects had they been considered to be distinct.

  - Permutations of n distinct objects in a circle
    Positions are indistinguishable:

    $$\frac{n!}{n} = (n-1)!$$

    Positions are distinguishable:

    $$n \times (n-1)! = n!$$

    where there are n ways to number the seats.

    > **Remark.** Seats become distinguishable if seats are numbered, different colour and different shapes/sizes.

- **Combination**: unordered selection of a number of objects
  - No. of ways to choose r objects from n distinct objects:

    $$\frac{n!}{r!(n-r)!} = \binom{n}{r} =^n \mathrm{C}_r$$

  - Select at least 1 object from n distinct objects

    $$2^n - 1$$

> **Remark.** When restrictions are given, they must be satisfied first before the number ofcombinations is selected. This can be done by breaking down into different cases, calculating separately and adding/subtracting them.

Some useful techniques:

1. Complementary technique
2. Grouping technique (order must be taken into account)
3. Insertion technique (order must be taken to account – use for 3 or more objects)

> **Remark.** When subdividing into groups of equal number, remember to divide by the number of groups.

Problems

Problem 3 (N2017/II/6)**.**

Problem 4 (N2016/II/7)**.**

Problem 5 (N2015/II/11)**.**

§13 Probability

- **Trials**

  Probability is concerned with the outcome of trials. Trials are also called *experiments* or *observations* (multiple trials). Trials refers to an event whose outcome is unknown.

- **Sample space** S

  Set of all possible elementary outcomes of a trial. The probability of the sample space is always 1.

- **Event** E

  An event $E \subseteq S$ is the specification of the outcome of a trial. An event can consist of a single outcome or a set of outcomes. The **complement** of an event is everything in the sample space that is not that event. The probability of an event is always between 0 and 1.

- Operations on events

  - **Union** of A and B (denoted by $A \cup B$): the event that either A or B occurs (or both).
  - **Intersection** of A and B (denoted by $A \cap B$): the event that both A and B occurs.
  - **Complement** of A (denoted by A'): the event that A does not occur.

  > **Remark.** $A \cap A' = \varnothing$ and $A \cup A' = S$.

  A and B are **mutually exclusive** (or disjoint) if they cannot occur simultaneously, i.e. they do not share any common outcomes: $A \cap B = \varnothing$.

- **Probability** of an event:

$$P(A) = \frac{n(A)}{n(S)}$$

  where $n(A)$ is number of outcomes in A, $n(S)$ is total number of outcomes in sample space S.

  - $0 \leq P(A) \leq 1$
  - Combined events: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  - Mutually exclusive events: $P(A \cap B) = 0 \implies P(A \cup B) = P(A) + P(B)$
  - Complement: $P(A') = 1 - P(A)$

- **Conditional probability** of A given B, denoted by $P(A \mid B)$: probability of A occurring given that B has occurred.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \tag{53}$$

  A and B are **independent** if probability that A occurs is not affected by occurrence of B:

$$P(A \mid B) = P(A)$$

  or

$$P(A \cap B) = P(A) \times P(B).$$

- Useful approaches

  - Few stages with few outcomes: Tree diagram
  - Diff. probabilities for diff. events: Venn diagram
  - Small sample space: Table of outcomes
  - P&C method: Only WITHOUT replacement
  - Sequences and Series: For turn-by-turn situations

Problems

Problem 6 (N2023/II/5).

Problem 7 (N2023/II/6).

Problem 8 (N2022/II/6). Interesting problem involving geometric progression.

Problem 9 (N2022/II/7c).

Problem 10 (N2022/II/9).

Problem 11 (N2020/II/8).

Problem 12 (N2018/II/7).

Problem 13 (N2015/II/9).

§14 Discrete Random Variables

- A **random variable** X for an experiment is one where the event that X takes on a certain value x, corresponds exactly to a possible event E of the experiment. We express the event E as $\{X = x\}$ in this case.

  > **Remark.** The convention is to use capital letters, e.g. X and Y, to denote random variables, while corresponding lower case letters, e.g. x and y, are used to represent one of the values it can take.

  > **Remark.** A random variable can be either discrete or continuous.

  A **discrete random variable** can only take certain numerical values in an interval.

- Let X be a discrete random variable taking values $x_1, x_2, \dots, x_n$. Then the **probability distribution function** of X is the function f that maps each value x_k to the *probability* that $X = x_k$, i.e.

$$f(x) = P(X = x) \quad \text{for } x = x_1, x_2, \dots, x_n$$

- If X is a discrete random variable with probability distribution function $P(X = x)$ for $x = x_1, x_2, \dots$, then the **cumulative distribution function** of X is given by

$$P(X \leq x) = \sum_{r \leq x} P(X = r)$$

- The **expectation** (mean) of a discrete random variable X indicates its weighted average and is given by

$$\mu = \mathrm{E}(X) = \sum x P(X = x) \tag{54}$$

  For constants a and b,

  - $\mathrm{E}(a) = a$ (average of a constant value is itself)
  - $\mathrm{E}(aX) = a\,\mathrm{E}(X)$
  - $\mathrm{E}(aX \pm b) = a\,\mathrm{E}(X) \pm b$
  - $\mathrm{E}(aX \pm bY) = a\,\mathrm{E}(X) \pm b\,\mathrm{E}(Y)$

- **Variance**, $\mathrm{Var}(X)$ or σ^2: measures how widely spread the values of the discrete random variable are; the larger the variance, the more scattered the observations.

$$\mathrm{Var}(X) = \sum (x_i - \mu)^2 p_i \tag{55}$$

  However this definition is not useful; we introduce a more useful equation.

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 \tag{56}$$

Problems

Problem 14 (N2020/II/5)**.**

Problem 15 (N2018/II/8)**.**

§15 Binomial Distribution

- $X \sim B(n, p)$, where n is number of trials, p is probability of success.

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \tag{MF26}$$

  To find mode (value of x with highest probability), use the TABLE function in G.C. to get highest value.

- Conditions for binomial distribution:
  1. Finite number of trials
  2. Outcome of each trial is termed a "success" or "failure"
  3. *Probability of success is same for each trial
  4. *Trials are independent of each other
- Expectation: $\mathrm{E}(X) = np$
- Variance: $\mathrm{Var}(X) = np(1-p)$

NOT IN SYLLABUS: Poisson distribution

- Conditions for poisson distribution:
  1. The events occur at random and are independent of each other in a given interval of time or space*,
  2. The average number of events per interval is constant throughout the interval*
  3. The average number of events per interval is proportional to the size of the interval.
- $X \sim Po(\lambda)$, where λ is the mean number of occurences.

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, 3, \ldots$$

- Expectation: $\mathrm{E}(X) = \lambda$
- Variance: $\mathrm{Var}(X) = \lambda$

Problems

Problem 16 (N2023/II/10)**.**

Problem 17 (N2021/II/11)**.**

Problem 18 (N2020/II/9)**.**

Problem 19 (N2019/II/7)**.**

Problem 20 (N2018/II/6)**.**

Problem 21 (N1982/I/13)**.** The random variable X is the number of successes in n independent trials of an experiment in which the probability of success in any one trial is p.

Show that
$$\frac{P(X = k+1)}{P(X = k)} = \frac{(n-k)p}{(k+1)(1-p)}, \quad k = 0, 1, 2, \ldots, n-1.$$

Find the most probable number of success when $n = 10$ and $p = \frac{1}{4}$.

Solution. Given that $X \sim B(n,p)$, thus

$$\begin{aligned}
\frac{P(X=k+1)}{P(X=k)} &= \frac{(n-k)p}{(k+1)(1-p)} \\
&= \frac{\binom{n}{k+1}p^{k+1}(1-p)^{n-(k+1)}}{\binom{n}{k}p^k(1-p)^{n-k}} \\
&= \frac{\frac{n!}{(k+1)(n-k-1)!}}{\frac{n!}{k!(n-k)!}}p^{k+1-k}(1-p)^{n-k-1-(n-k)} \\
&= \frac{k!(n-k)!}{(k+1)!(n-k-1)!}p^1(1-p)^{-1} \\
&= \frac{(n-k)p}{(k+1)(1-p)}
\end{aligned}$$

Given $n = 10$, $p = \frac{1}{4}$. If k is the mode,

$$\begin{aligned}
&P(X=k+1) < P(X=K) \\
&\frac{P(X=k+1)}{P(X=k)} < 1 \\
&\frac{(10-k)\frac{1}{4}}{(k+1)\frac{3}{4}} < 1 \\
&10-k < 3(k+1) \\
&7 < 4k \\
&\frac{7}{4} < k
\end{aligned}
\qquad\qquad
\begin{aligned}
&P(X=k-1) < P(X=k) \\
&1 < \frac{P(X=k)}{P(X=k-1)} \\
&1 < \frac{(11-k)\frac{1}{4}}{(k)\frac{3}{4}} \\
&3k < 11-k \\
&4k < 11 \\
&k < \frac{11}{4}
\end{aligned}$$

Thus

$$\frac{7}{4} < k < \frac{11}{4}$$

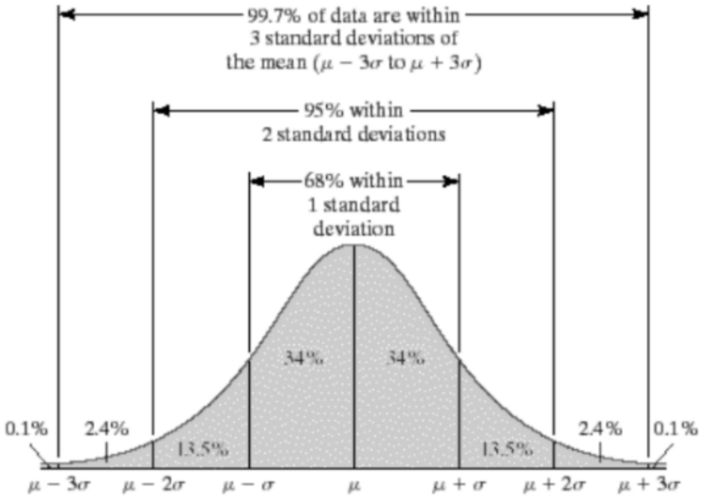
so mode is $k = 2$. $\square$

§16 Normal Distribution

- Normal distribution: $X \sim N(\mu, \sigma^2)$, where μ is mean, σ^2 is variance.
- Standard normal distribution: $Z \sim N(0, 1)$, where $\mu = 0$, $\sigma = 1$.
  To standardise (used to solve for unknown μ or σ),

$$Z = \frac{X - \mu}{\sigma}$$

- Normal curve is symmetrical



- Use GC to calculate • finding the value of $P(X < x_1)$ or a related probability, given the values of x_1, μ, σ
  use invNorm • finding a relationship between x_1, μ, σ given the value of $P(X < x_1)$, or a related probability
- Properties of expectation and variance
  • solving problems involving the use of E(aX + b) and Var (aX + b) • solving problems involving the use of E(aX + bY) and Var (aX + bY), where X and Y are independent

§17 Sampling

- **Random sampling**: every member of population has equal chance of being selected, and the selections are independent of each other.

  One type of sampling is **simple random sampling**: each possible sample of size n has the same chance of being chosen from population of size N.

  1. Create a list of the population (sampling frame), number members of population from 1 to N.
  2. Make a selection of n distinct members by using random number generator.

  Advantage: Free from bias

  Disadvantage: Not able to get access to some members chosen from the sample

- If $X \sim N(\mu, \sigma^2)$, then

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

- If $X_1, \dots, X_n$ is random sample of size n taken from non-normal or unknown distribution with mean μ, variance σ^2, then for sufficiently large n (at least 50), by **Central Limit Theorem** (CLT),

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \text{ approximately.}$$

- Estimation

  $\bar{x}$ is unbiased estimate of μ:

$$\bar{x} = \frac{\sum x}{n} \tag{57}$$

  s^2 is unbiased estimate of σ^2:

$$\begin{aligned} s^2 &= \frac{n}{n-1} \times \text{sample variance} \\ &= \frac{n}{n-1}\left(\frac{\sum(x-\bar{x})^2}{n}\right) \\ &= \frac{1}{n-1}\left(\sum x^2 - \frac{(\sum x)^2}{n}\right) \end{aligned} \tag{58}$$

  Unbiased estimate of common population variance from two samples:

$$s^2 = \frac{\sum(x_1-\bar{x}_1)^2 + \sum(x_2-\bar{x}_2)^2}{n_1+n_2-2} \tag{MF26}$$

  If data is represented in the form $x - c$,

$$\bar{x} = \frac{\sum(x-c)}{n} + c$$

  and

$$s^2 = \frac{1}{n-1}\left(\sum(x-c)^2 - \frac{\left(\sum(x-c)\right)^2}{n}\right)$$

§18 Hypothesis Testing

- **Null hypothesis** H_0: particular claim for a *value* for the population mean (status quo claim).

  Alternative hypothesis H_1: range of values that excludes the value specified by null hypothesis (suspicion).

  Test statistic: random variable whose value is calculated from sample data, usually sample mean $\bar{X}$. (Thus we consider the distribution of the test statistic.)

  Test value: value of the test statistic, usually observed sample mean $\bar{x}$. (what you actually get) in this case, observed sample mean.

  Probability value (p-value): probability of getting a test statistic as extreme or more extreme than the observed value.

  p-value is also the lowest significance level at which H_0 is rejected. (try to graph this out!)

  $$\begin{aligned} p\text{-value} \le \alpha &\implies \text{reject } H_0 \\ p\text{-value} > \alpha &\implies \text{do not reject } H_0 \end{aligned}$$

  Critical region: range of values of test statistic that leads to the rejection of H_0 (depends on type of test). The value of c which determines the critical region is known as the **critical value**.

  $$\begin{aligned} \bar{x} \text{ lies in critical region} &\implies \text{reject } H_0 \\ \bar{x} \text{ does not lie in critical region} &\implies \text{do not reject } H_0 \end{aligned}$$

  Significance level α: probability of rejecting H_0 when it is actually true (i.e. wrongly rejecting null hypothesis). (Decided before any sample values are taken, to avoid being influenced by sample readings.)

- • formulation of hypotheses and testing for a population mean based on: – a sample from a normal population of known variance – a large sample from any population

- **1-tail test**: alternative hypothesis H_1 looks for increase/decrease in μ.

  - For an increase, $H_1 : \mu > \mu_0$, critical region and p-value are in **right tail**.

    $$p\text{-value} = P(\bar{X} \ge \bar{x}), \quad \alpha = P(\bar{X} \ge c)$$

  - For a decrease, $H_1 : \mu < \mu_0$, critical region and p-value are in **left tail**.

    $$p\text{-value} = P(\bar{X} \le \bar{x}), \quad \alpha = P(\bar{X} \le c)$$

  2-tail test: alternative hypothesis $H_1 : \mu \ne \mu_0$ looks for a *change* in μ, without specifying whether it is an increase or decrease.

  $$\alpha = P(\bar{X} \le c_1) + P(\bar{X} \ge c_2) = 2P(\bar{X} \le c_1) = 2P(\bar{X} \ge c_2),$$

  $$p\text{-value} = \begin{cases} 2P(\bar{X} \le \bar{x}) & \text{if } \bar{x} < \mu_0, \\ 2P(\bar{X} \ge \bar{x}) & \text{if } \bar{x} > \mu_0. \end{cases}$$

- **Answering format for hypothesis test**:

  1. State null hypothesis $H_0 : \mu = \mu_0$ and alternative hypothesis H_1 (depends if left-tail, right-tail, or 2-tail)
  2. State significance level α (provided in question).
  3. Consider distribution of test statistic (if we are testing sample mean, then test statistic is $\bar{X}$).

  Critical value approach:

  4. Calculate critical value based on significance level, and test value based on sample data.
  5. If test value falls in critical region, reject H_0; otherwise, do not reject H_0.
     Write down conclusion in the context of question.

  p-value approach:

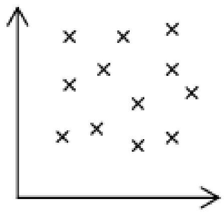
  6. Calculate p-value based on sample data.
  7. If $p \le \alpha$, reject H_0; otherwise, do not reject H_0.
     Write down conclusion in the context of question.

Remark. When concluding a hypothesis test, it is either "reject H_0" or "do not reject H_0".

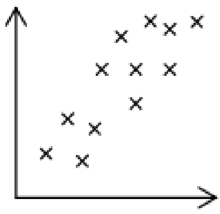
Note that "do not reject H_0" is not equivalent to "accept H_0". This is because under the framework of hypothesis testing, there is no way to prove H_0 is true; we can only assess whether there is sufficient evidence *against* it.

§19 Correlation and Regression

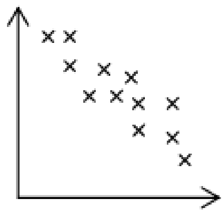
- **Scatter diagram**: a sketch where each axis represents a variable, each point represents an observation.
  - Need not start from $(0, 0)$.
  - Label axes according to context.
  - Indicate range of data values (minimum and maximum values).
  - Relative position of points should be accurate.



no correlation



positive correlation



negative correlation

Interpreting scatter diagram

1. Direction: positive / negative direction
2. Form: points lie on straight line (linear) / curve

> **Example 19.1.** Positive linear relationship, negative linear relationship, curvilinear relationship, no clear relationship.

- **Product moment correlation coefficient** r: measures strength and direction of a linear correlation between two variables.

$$r = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sqrt{\left(\sum(x-\bar{x})\right)\left(\sum(y-\bar{y})\right)}} = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sqrt{\left(\sum x^2 - \frac{(\sum x)^2}{n}\right)\left(\sum y^2 - \frac{(\sum y)^2}{n}\right)}} \qquad \text{(MF26)}$$

It can be shown that $-1 \le r \le 1$.

- $r \approx 1$: strong positive linear correlation
- $r \approx -1$: strong negative linear correlation
- $r = 0$: no linear correlation

> **Remark.** r is independent of units of measurement.

Identify **outlier**(s) from scatter diagram, remove them to calculate more accurate value of r.

Correlation does not imply causation.

- Regression line

  Least squares method

  - Regression line of y on x: line which minimises sum of squares of errors in prediction of y from points to line

$$y = a + bx$$

$$y - \bar{y} = b(x - \bar{x}), \quad b = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2} \qquad \text{(MF26)}$$

  - Regression line of x on y: line which minimises sum of squares of errors in prediction of x from points to line

$$x = c + dy$$

$$x - \bar{x} = d(y - \bar{y}), \quad d = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(y-\bar{y})^2}$$

    replace x with y and vice versa.

Remark. Regression lines pass through mean point $(\bar{x}, \bar{y})$.

Remark. The stronger the linear correlation, the closer the two regression lines are to each other.

Choosing a suitable regression line

- Dependent variable y, independent variable x: y on x
- Estimate x given value of y: x on y (least error for x)

Interpolation: estimate within given range of values

Extrapolation: estimate outside given range of values

Reliability of estimate*

1. Appropriateness of regression line used:
   Correct regression line should be used for estimate to be reliable, depends whether there is dependent & independent variable.
2. Strength of linear correlation, i.e. value of $|r|$:
   $|r|$ should be close to 1 for the estimate to be reliable.
3. Interpolation or extrapolation:
   Interpolation is likely to give a more reliable estimate than extrapolation.

   Example 19.2. Since $x = 10$ lies outside the given data range ($31 \leq x \leq 98$), it is an extrapolation, so the linear model might not hold out of this range. Hence estimated value is unreliable.
   Since $y = 75$ lies within the given data range ($54 \leq y \leq 96$), it is an interpolation. Hence estimated value is reliable.

- **Transformation to linearity**
  - square transformation: $y = a + bx^2$
  - reciprocal transformation: $y = ab^x \implies \ln y = \ln a + x \ln b$
  - logarithmic transformation: $y = ax^b \implies \ln y = \ln a + b \ln x$