The Blue Book

for

Mathematics Olympiad

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About the Author

List of prestigious math olympiad competitions:

- International Mathematics Olympiad (IMO)
- Asian Pacific Mathematics Olympiad (APMO)

Here are some useful references:

- Unofficial syllabus for math olympiads, by Evan Chen
- Notes on proof-writing style
- AoPS Contest Collections
- Yufei Zhao's olympiad handouts
- Evan Chen's olympiad handouts
- Olympiad Problems and Solutions, by Evan Chen
- Canadian Mathematical Society Resources
- Mathematical Reflections
- 3Blue1Brown

Contents

Part I

Number Theory

1 Divisibility

1.1 Division Algorithm

Theorem 1.1.1: Division Algorithm

For every integer pair a and b, there exists distinct integer quotient and remainders, q and r, that satisfy

$$a = bq + r, \quad 0 \leq r < b$$

Proof. We will prove that this is true for positive a and b. The other cases when one or both of a and b are negative follow very similarly. There are two parts to this proof:

- Proving that for every pair (a, b) we can find a corresponding quotient and remainder.
- Proving that this quotient and remainder pair are unique.

For proving the existence of the quotient and remainder, given two integers a and b with varying integer q, consider the set

$$\{a - bq, \quad a - bq \geq 0\}.$$

By the **well-ordering principle**, we know that this set must have a minimum, say when $q = q_1$. Clearly from the condition on the set, we must have $a - bq_1 = r \geq 0$. It now serves to prove that $a - bq_1 = r < b$. For the sake of contradiction, assume that $a - bq_1 \geq b$. However, then

$$a - b(q_1 + 1) \geq 0,$$

therefore it also should be a member of the above set. Furthermore,

$$a - b(q_1 + 1) < a - bq_1,$$

contradicting the minimality of q_1. Therefore, it is impossible for $a - bq_1 \geq b$, and we have $0 \leq a - bq_1 < b$. The second part of this proof is to show that the

quotient and remainder are unique. Assume for the sake of contradiction that a can be represented in two ways:

$$a = bq_1 + r_1 = bq_2 + r_2 \implies b(q_1 - q_2) = r_2 - r_1 \implies b \mid r_2 - r_1$$

However,

$$b > r_2 - r_1 > -b$$

since $0 \leq r_1, r_2 < b$. Since $r2 - r1$ is a multiple of b, we must have $r_2 - r_1 = 0 \implies r_2 = r_1$ and $q_2 = q_1$. □

1.2 Greatest Common Divisor and Lowest Common Multiple

Of special significance is the case in which the remainder in the Division Algorithm turns out to be zero.

Definition 1.2.1

An integer b is said to be **divisible** by an integer $a \neq 0$, denoted by $a \mid b$, if there exists some integer c such that $b = ac$. We write $a \nmid b$ to indicate that b is not divisible by a.

Remark. Note that if a is a divisor of b, then b is also divisible by $-a$ (why?), so the divisors of an integer always occur in pairs. To find all the divisors of a given integer, it is sufficient to obtain the positive divisors and then adjoin them to the corresponding negative integers. For this reason, we shall usually limit ourselves to the consideration of the positive divisors.

Using the definition above, for integers a, b, c, the following hold: (a) a — 0 , 1— a , a — a . (b) a —1 if and only if a = ±1. (c) If a — b and c — d , then ac — bd . (d) If a — b and b — c , then a — c (transitivity). (e) a — b and b — a if and only if a = ±b . (f) If a — b and b 0, then a ≤

$b.(g)Ifa|banda|c, thena|(bx+cy)forarbitraryintegersxandy.Itisalsoworthpointingou$
$1, 2, ..., n, thena|(b1x1 + b2x2 + ... + bnxn)$

for all integers xi.

1.3 Euclidean Algorithm

The **greatest common divisor** of integers a and b, denoted by $d = \gcd(a, b)$, is the largest integer d such that $d \mid a$ and $d \mid b$.

The Euclidean Algorithm makes use of the following properties:

- $\gcd(a, 0) = a$ for all positive integers a
- $\gcd(a, b) = \gcd(b, a \mod b)$ for all positive integers a and b, which is equivalent to $\gcd(a, b) = \gcd(a - b, b)$

Example 1.

Problems

Problem 1 (IMO/2023/1)**.** Determine all composite integers $n > 1$ that satisfy the following property: if $d_1, d_2, \ldots, d_k$ are all the positive divisors of n with $1 = d_1 < d_2 < \cdots < d_k = n$, then d_i divides $d_{i+1} + d_{i+2}$ for every $1 \leq i \leq k - 2$.

2

Modular Arithmetic

Modular arithmetic is a way of systematically ignoring differences involving a multiple of an integer.

For integer n, two integers are equal *modulo* n if they differ by multiples of n.

Definition 2.0.1: L

t n, x, y be integers. x is congruent to y modulo n if $n \mid x - y$.

$$x \equiv y \pmod{n}$$

$n \mid x - y$ is equivalent to the following statements:

- $n \mid y - x$.
- $x = y + jn$ for some $j \in \mathbb{Z}$.
- $y = x + kn$ for some $k \in \mathbb{Z}$.

2.1 Modular operations

Congruence modulo n is an equivalence relation.

1. (Reflexivity) $a \equiv a \pmod{n}$ for all $a \in \mathbb{Z}$.
2. (Symmetry) For $a, b \in \mathbb{Z}$, if $a \equiv b \pmod{n}$, then $b \equiv a \pmod{n}$.
3. (Transitivity) For $a, b, c \in \mathbb{Z}$, if $a \equiv b \pmod{n} \wedge b \equiv c \pmod{n} \implies a \equiv c \pmod{n}$.

Let $n \in \mathbb{Z}$.

1. If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then

$$a + c \equiv b + d \pmod{n}$$

2. If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then

$$ac \equiv bd \pmod{n}$$

3. If $a \equiv b \pmod{n}$, then

$$ac \equiv bc \pmod{n}$$

An **equivalence class** is the set of integers congruent to each other modulo n. $\mathbb{Z}_n$ denotes the set of equivalence classes under congruence modulo n.

For example, congruence classes modulo 3 is $\mathbb{Z}_3 = 0, 1, 2$.

2.2 Modular inverse

For coprime a and n, there exists an inverse of $a \pmod{n}$. In other words, there exists b such that

$$ab \equiv 1 \pmod{n}$$

There is a much faster method for finding the inverse of $a \pmod{n}$ that we will discuss in the next articles on the Extended Euclidean Algorithm.

2.3 Orders Modulo A Prime

https://web.evanchen.cc/handouts/ORPR/ORPR.pdf

2.4 Theorems

For $n = {p_1}^{a_1}{p_2}^{a_2}\dots{p_k}^{a_k}$,

- Number of factors:
$$\tau(n) = \prod_{i=1}^{k}(a_i + 1)$$
- Sum of factors:
$$d(n) = \prod_{i=1}^{k}(1 + p_i + \cdots + {p_i}^{a_i})$$
- Number of positive integers less than n which are coprime to n:
$$\phi(n) = n\prod_{i=1}^{k}\left(1 - \frac{1}{p_i}\right)$$

Theorem 2.4.1: Fermat's Little Theorem

For prime p and $1 \le a < p$,

$$a^p \equiv a \pmod p \tag{2.1}$$

Theorem 2.4.2: Euler's Totient Theorem

For coprime a and n,

$$a^{\phi(n)} \equiv 1 \pmod n \tag{2.2}$$

where $\phi(n)$ denotes the totient function.

Theorem 2.4.3: Wilson's Theorem

For odd prime p,

$$(p-1)! \equiv -1 \pmod p \tag{2.3}$$

Theorem 2.4.4: Chinese Remainder Theorem

Given pairwise coprime positive integers n_i and arbitrary integers a_i, the system of simultaneous congruences

$$\begin{aligned} x &\equiv a_1 \pmod n_1 \\ x &\equiv a_2 \pmod n_2 \\ &\vdots \\ x &\equiv a_k \pmod n_k \end{aligned}$$

has a unique solution modulo $n_1 n_2 \cdots n_k$.

2.5 Quadratic Residues

An integer a is a *quadratic residue* modulo p if $a \equiv x^2 \pmod{p}$, or $x^2 \equiv a \pmod{p}$ for some integer x. If a is not a quadratic residue modulo p, it is said to be a *quadratic non-residue* modulo p.

Quadratic residues

$$\begin{aligned} n^2 &\equiv 0/1 \pmod{3} \\ n^2 &\equiv 0/1 \pmod{4} \\ n^2 &\equiv 0/1/4 \pmod{5} \\ n^2 &\equiv 0/1/4 \pmod{8} \end{aligned}$$

Cubic residues

$$\begin{aligned} n^3 &\equiv 0/1/6 \pmod{7} \\ n^3 &\equiv 0/1/8 \pmod{9} \end{aligned}$$

Legendre symbol and quadratic reciprocity for odd primes:

Definition 2.5.1: Legendre symbol

Let p be an odd prime and let a be an integer. The Legendre symbol of a with respect to p is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } a \not\equiv 0 \pmod{p} \\ -1 & \text{if } a \text{ is a quadratic non-residue modulo } p \\ 0 & \text{if } a \equiv 0 \pmod{p} \end{cases} \tag{2.4}$$

Theorem 2.5.1: Euler's Criterion

If p is an odd prime, then for any residue class a, it is true that

$$\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p} \tag{2.5}$$

Proof. This can be proved using Fermat's Little Theorem. □

Euler's Criterion implies that

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right) \tag{2.6}$$

Theorem 2.5.2: Law of Quadratic Reciprocity

For distinct odd primes p, q,

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}} \tag{2.7}$$

Lemma 2.5.1 (Gauss's Lemma)**.** Suppose p is an odd prime and $p \nmid a$. Then consider the $\frac{p-1}{2}$ distinct integers

$$a \pmod p, 2a \pmod p, \ldots, \left(\frac{p-1}{2}\right)a \pmod p$$

Let n be the number of these integers that are larger than $\frac{p}{2}$, then

$$\left(\frac{a}{p}\right) = (-1)^n.$$

Proof. Evaluate the product $a \cdot 2a \cdots \frac{p-1}{2}a \pmod p$ in two different ways. By rearranging terms, we get

$$a^{\frac{p-1}{2}}\left(\frac{p-1}{2}\right)!$$

But the product can also be evaluated by noticing that each of the distinct integers in Gauss's lemma is either x or $p-x$ for $1 \le x \le \frac{p-1}{2}$, and showing that each of the x's is distinct. Multiplying them together modulo p gives $(\frac{p-1}{2})!$ multiplied by n minus signs due to the number of $p-x$ terms of which there are n, hence the sign is $(-1)^n$. The result follows by Euler's criterion and cancelling the $(\frac{p-1}{2})!$. □

Special cases

- $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$
- $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$
- $\left(\frac{-3}{p}\right) = 1 if p = 1 \pmod 6; -1 if p = 5 \pmod 6)$
- $(5/p) = 1 if p = 1, 9 (mod 10); -1 if p = 3, 7 (mod 10)$

3 Diophantine Equations

3.1 Linear combination

A Diophantine equation in the form $ax + by = c$ is known as a linear combination. There will always be an infinite number of solutions when $\gcd(a, b) = 1$ and $\gcd(a, b) \mid c$.

Theorem 3.1.1: Bezout's Lemma

For non-zero integers a and b, let $d = \gcd(a, b)$, there exists integers x and y that satisfy

$$ax + by = d.$$

An important case:

$$a, b \text{ coprime} \iff \exists x, y \text{ such that } ax + by = 1$$

Example 2. Find integers x and y that satisfy

$$102x + 38y = 2.$$

Solution. Apply the Euclidean algorithm on a and b to calculate $\gcd(a, b)$:

$$\begin{aligned}
102 &= 2 \times 38 + 26 \\
38 &= 1 \times 26 + 12 \\
26 &= 2 \times 12 + 2 \\
12 &= 6 \times 2 + 0 \\
6 &= 3 \times 2 + 0
\end{aligned}$$

Work backwards and substitute the numbers from above:

$$\begin{aligned} 2 &= 26 - 2 \times 12 \\ &= 3 \times 26 - 2 \times 38 \\ &= 3 \times 102 - 8 \times 38 \end{aligned}$$

$\therefore x = 3$, $y = -8$. □

3.2 Pythagorean Triples

All solutions of

$$a^2 + b^2 = c^2$$

are of the form

$$\begin{aligned} a &= k(m^2 - n^2) \\ b &= k(2mn) \\ c &= k(m^2 + n^2) \end{aligned}$$

3.3 Pell's Equation

Theorem 3.3.1: Pell's Equation

If $n > 0$ is not a perfect square, then the equation

$$x^2 - ny^2 = 1$$

has infinitely many solutions.

Note that $(x, y) = (1, 0)$ is a trivial solution.

Steps:

1. Find one non-trivial solution (x, y).
2. Let $\alpha^n = (x+y\sqrt{d})^n$ where $n = 2, 3, \ldots$. The coefficients of the integer and square root give us the values of x and y respectively.

Example 3. Find positive integers x and y that satisfy

$$x^2 - 2y^2 = 1.$$

Solution. We first observe that $(x, y) = (3, 2)$ is a solution.

$$\alpha = (3 + 2\sqrt{2})$$
$$\alpha^n = (3 + 2\sqrt{2})^n$$

For $n = 2$,

$$\begin{aligned} \alpha^2 &= (3 + 2\sqrt{2})^2 \\ &= 17 + 12\sqrt{2} \end{aligned}$$

From this, we deduce that another solution is $(x, y) = (17, 12)$.
Simply repeat the above method to find further solutions. □

Theorem 3.3.2: Frobenius coin problem

If a, b are relatively prime positive integers, the equation

$$ax + by = n$$

cannot be solved with non-negative x, y if $n = ab - a - b$, but can be solved for all $n > ab - a - b$.

Theorem 3.3.3: Fermat's Last Theorem

For $n > 2$, there are no non-zero solutions to

$$a^n + b^n = c^n.$$

4

Problems

Problem 2 (IMO/1988)**.** Let $a, b \in \mathbb{N}$. Prove that if $ab+1 \mid a^2+b^2$, then

$$\frac{a^2+b^2}{ab+1}$$

is a perfect square.

Proof. We use a method known as *Vieta Jumping.*
Let $\dfrac{a^2+b^2}{ab+1} = k \in \mathbb{N}$ and assume that k is not a perfect square. We are going to prove by contradiction.
Let $(a,b) = (A,B)$ be a solution to the above equation, such that $A+B$ is minimal. Without loss of generality, $A \geq B$. The equation becomes

$$A^2+B^2 = k(AB+1)$$
$$A^2-(kB)A+B^2-k = 0$$

We see that A is a root to the equation $x^2-(kB)x+B^2-k=0$. Let A, A_1 be the roots to this equation. By Vieta's Formula, we have

$$A_1 + A = kB \tag{1}$$

$$A_1 \cdot A = B^2 - k \tag{2}$$

From (1),

$$A_1 = kB - A \in \mathbb{Z}$$

From (2),

$$A_1 = \frac{B^2-k}{A} \neq 0$$

Note that

$$\frac{{A_1}^2+B^2}{A_1B+1} = k > 0$$

Since ${A_1}^2 + B^2 > 0$ and $k > 0$, we must have $A_1 B + 1 > 0$, so $A_1 > 0 \implies A_1 \in \mathbb{N}$.

However,

$$\begin{aligned}
\because B^2 - k &\leq A^2 - k \\
&< A^2 \\
\therefore \frac{B^2 - k}{A} &\leq A \\
A_1 &< A
\end{aligned}$$

(A_1, B) is another solution but $A_1 + B < A + B$, which contradicts its minimality.

$\therefore \dfrac{a^2 + b^2}{ab + 1}$ must be a perfect square. $\square$

Problem 3. (ITALY/2011) Given that p is a prime number, find integer solutions to

$$n^3 = p^2 - p - 1.$$

Solution. It is easy to see that $n < p$.

$$\begin{aligned} p^2 - p &= n^3 + 1 \\ p(p-1) &= (n+1)(n^2 - n + 1) \end{aligned}$$

Since p is prime, $p \mid n+1$ or $p \mid n^2 - n + 1$.

Case 1: $p \mid n+1$

Since $n < p$, thus $n + 1 \le p$. Hence, $n + 1 = p$. Substituting this into the original equation gives us

$$n^3 = n^2 + n - 1$$

$$(n-1)^2(n+1) = 0$$
$$n = 1$$

$\therefore (n, p) = (1, 2)$.

Case 2: $p \mid n^2 - n + 1$

Let $n^2 - n + 1 = kp$ where k is a positive integer. Then

$$\begin{aligned} p(p-1) &= (n+1)(n^2 - n + 1) \\ &= kp(n+1) \\ p - 1 &= k(n+1) \\ p &= kn + k + 1 \\ n^2 - n + 1 &= k(kn + k + 1) \\ n^2 - n(1 + k^2) - (k^2 + k - 1) &= 0 \end{aligned}$$

Taking discriminant,

$$\begin{aligned} \Delta &= (1 + k^2)^2 + 4(k^2 + k - 1) \\ &= k^4 + 6k^2 + 4k - 3 \end{aligned}$$

which is a perfect square.

Let $f(k) = k^4 + 6k^2 + 4k - 3$.

We find that $k = 3$ via trial and error, then $n = 11$, $p = 37$.

For $k \geq 4$, we can prove that $f(k)$ is not a perfect square; in fact, $f(k)$ lies between two consecutive perfect squares, as shown below:

$$(k^2 + 3)^2 < f(k) < (k^2 + 4)^2$$

which can be easily shown by expanding the terms.

$\therefore (n, p) = (11, 37)$ $\square$

Problem 4. (KOREA/2012) Find positive integer solutions to

$$2^m p^2 + 1 = q^5$$

where p, q are primes.

Solution. Rewriting the equation gives us

$$\begin{aligned} 2^m p^2 &= q^5 - 1 \\ &= (q-1)(q^4 + q^3 + q^2 + q + 1) \end{aligned}$$

We can observe that since $q \neq 2$, thus $q - 1$ is even and $q^4 + q^3 + q^2 + q + 1$ is odd.

Let $f(q) = q^4 + q^3 + q^2 + q + 1$. We want to prove that $q - 1$ and $f(q)$ are coprime.

$\square$

Problem 5. (USAMO/2003) Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.

Proof. This is immediate by induction on n. For $n = 1$ we take 5; moving forward if M is a working n-digit number then exactly one of

$$\begin{aligned}
N_1 &= 10^n + M \\
N_3 &= 3 \cdot 10^n + M \\
N_5 &= 5 \cdot 10^n + M \\
N_7 &= 7 \cdot 10^n + M \\
N_9 &= 9 \cdot 10^n + M
\end{aligned}$$

is divisible by 5^{n+1}; as they are all divisible by 5^n and $\frac{N_k}{5^n}$ are all distinct. □

Problem 6. (CANADA/1969) Show that there are no integers a, b, c for which $a^2 + b^2 - 8c = 6$.

Proof. Note that all perfect squares are equivalent to $0, 1, 4 \pmod 8$. Hence, the problem statement is equivalent to $a^2 + b^2 \equiv 6 \pmod 8$. It is impossible to obtain a sum of 6 with two of $0, 1, 4$, so our proof is complete. $\square$

Problem 7 (ALBANIA/2009)**.** Find all the natural numbers m, n such that $1 + 5 \cdot 2^m = n^2$.

Solution. We have $5 \cdot 2^m = (n-1)(n+1) \implies n-1 = 2^k$, or $n+1 = 2^k$

If $n-1 = 2^k \Rightarrow n+1 = 2^k + 2$

But $5|2^k + 2$, $2^k + 2 = 2^t \cdot 5 \implies t = 1, k = 3 \implies n = 9, m = 4$

If $n+1 = 2^k \implies n-1 = 2^k - 2$. But $5|2^k - 2$, $2^k - 2 = 2^t \cdot 5 \implies t = 1, 2^k = 12$

$\therefore (m, n) = (4, 9)$ is a unique solution. $\square$

Part II

Algebra

5 Polynomials

5.1 Elementary Manipulation

5.1.1 Factorisation

The reader should be familiar with the following factorisations:

- Difference of squares:

$$a^2 - b^2 = (a + b)(a - b)$$

- Sum of squares:

$$a^2 + b^2 = (a + b)^2 - 2ab$$

- Sum of cubes:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

- Difference of cubes:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

5.1.2 Manipulations

This requires constant practice.

5.2 Polynomials

Theorem 5.2.1: Quadratic formula

For $a, b, c \in \mathbb{R}, a \neq 0$, the quadratic equation $ax^2 + bx + c = 0$ has solutions

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{5.1}$$

Let Δ denote the discriminant, then $\Delta = b^2 - 4ac$.

- For $\Delta < 0$, the 2 roots are complex and conjugates to each other.
- For $\Delta = 0$, the 2 roots are real and repeated.
- For $\Delta > 0$, the 2 roots are real and distinct.

Theorem 5.2.2: Vieta's relations

For polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with complex coefficients with roots $r_1, r_2, \ldots, r_n$,

$$\begin{aligned}
&r_1 + r_2 + \cdots + r_n = -\frac{a_{n-1}}{a_n} \\
&r_1 r_2 + r_2 r_3 + \cdots + r_{n-1} r_n = \frac{a_{n-2}}{a_n} \\
&\vdots \\
&r_1 r_2 \cdots r_n = (-1)^n \frac{a_0}{a_n}
\end{aligned} \tag{5.2}$$

Theorem 5.2.3: Division Algorithm

Let $\deg(P(x))$ denote the degree of $P(x)$. If $f(x)$ and $d(x)$ are polynomials where $d(x) \neq 0$ and $\deg(d(x)) < \deg(f(x))$, then

$$f(x) = d(x)q(x) + r(x) \tag{5.3}$$

Theorem 5.2.4: Remainder Theorem

If the polynomial $f(x)$ is divided by $(x - c)$, then the remainder is $f(c)$.

Theorem 5.2.5: Factor Theorem

Let $f(x)$ be a polynomial.

$$f(c) = 0 \iff (x - c) \text{ is a factor of } f(x) \tag{5.4}$$

Theorem 5.2.6: Linear Factorisation Theorem

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $n \geq 1, a_n \neq 0$, then

$$f(x) = a_n(x - c_1)(x - c_2)\ldots(x - c_n)$$

where $c_1, c_2, \ldots c_n$ are complex numbers.

A polynomial equation of degree n has n roots, counting multiple roots (multiplicities) separately.

Imaginary roots occur in conjugate pairs; if $a + bi$ is a root ($b \neq 0$), then the imaginary number $a - bi$ is also a root.

Theorem 5.2.7: Fundamental Theorem of Algebra

A polynomial of degree n, where $n > 0$, with real or complex coefficients, has at least one real or complex root.

5.3 Inequalities

Theorem 5.3.1: AM-GM Inequality

For non-negative real $a_1, a_2, \ldots, a_n$,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \tag{5.5}$$

Equality holds iff $a_1 = a_2 = \cdots = a_n$.

Theorem 5.3.2: Cauchy-Schwarz Inquality

For real $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$,

$$({a_1}^2 + {a_2}^2 + \cdots + {a_n}^2)({b_1}^2 + {b_2}^2 + \cdots + {b_n}^2) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \tag{5.6}$$

Equality holds iff $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$.

Theorem 5.3.3: Titu's lemma

For positive real $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$,

$$\frac{{a_1}^2}{b_1} + \frac{{a_2}^2}{b_2} + \cdots + \frac{{a_n}^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n} \tag{5.7}$$

Equality holds iff $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$.

Theorem 5.3.4: Hölder's Inequality

For real $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, and positive real p, q that satisfy $\frac{1}{p} + \frac{1}{q} = 1$,

$$\left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |b_k|^q\right)^{\frac{1}{q}} \geq \sum_{k=1}^{n} |a_k b_k| \tag{5.8}$$

Theorem 5.3.5: Triangle inequality

For real $a_1, a_2, \ldots, a_n$,

$$|a_1| + |a_2| + \cdots + |a_n| \geq |a_1 + a_2 + \cdots + a_n| \tag{5.9}$$

Equality holds iff $a_1, a_2, \cdots, a_n$ are all non-negative.

Theorem 5.3.6: Schur's inequality

For non-negative real a, b, c and $n > 0$,

$$a^n(a-b)(a-c) + b^n(b-c)(b-a) + c^n(c-a)(c-b) \geq 0 \tag{5.10}$$

Equality holds iff either $a = b = c$ or when two of a, b, c are equal and the third is 0.

Proof. WLOG, let $a \geq b \geq c$. Note that

$$\begin{aligned} & a^n(a-b)(a-c) + b^n(b-a)(b-c) \\ & = a^n(a-b)(a-c) - b^n(a-b)(b-c) \\ & = (a-b)(a^n(a-c) - b^n(b-c)) \end{aligned}$$

Clearly, $a^n \geq b^n \geq 0$, and $a - c \geq b - c \geq 0$. Thus,

$$(a-b)(a^n(a-c) - b^n(b-c)) \geq 0 \implies a^n(a-b)(a-c) + b^n(b-a)(b-c) \geq 0$$

However, $c^n(c-a)(c-b) \geq 0$, and thus the proof is complete. □

When $n = 1$, we have the well-known inequality:

$$a^3 + b^3 + c^3 + 3abc \geq a^2b + a^2c + b^2a + b^2c + c^2a + c^2b$$

When $n = 2$, an equivalent form is:

$$a^4 + b^4 + c^4 + abc(a+b+c) \geq a^3b + a^3c + b^3a + b^3c + c^3a + c^3b$$

Theorem 5.3.7: Jensen's inequality

Let a real-valued function f be convex on the interval I. Let $x_1, \ldots, x_n \in I$ and $\omega_1, \ldots, \omega_n \geq 0$. Then we have

$$\frac{\omega_1 f(x_1) + \omega_2 f(x_2) + \cdots + \omega_n f(x_n)}{\omega_1 + \omega_2 + \cdots + \omega_n} \geq f\left(\frac{\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_n x_n}{\omega_1 + \omega_2 + \cdots + \omega_n}\right) \tag{5.11}$$

If f is concave, the direction of the inequality is flipped.
In particular, if we take the weights $\omega_1 = \omega_2 = \cdots = \omega_n = 1$, we get the inequality

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \geq f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right).$$

AM-GM inequality is one of the special cases of Jensen's inequality.

5.4 Absolute Value Equations

Common problem-solving techniques:

- Squaring both sides
- Casework: solve each case for x
- Sketching graph

Example 4. Find all real values of x such that

$$|x+2|+|2x+6|+|3x-3|=12.$$

Solution. The turning points of the three terms are $x=-2$, $x=-3$, $x=1$ respectively. Hence, we just need to check the cases:

- $x \leq 3$
- $-3 < x \leq -2$
- $-2 < x \leq 1$
- $x > 1$

$\therefore x=-\frac{5}{2}, x=\frac{7}{6}$ □

Problems

Problem 8. (CHINA/1979) Given that $x^2 - x + 1 = 0$, find the value of $x^{2015} - x^{2014}$.

Solution. Rewriting the equation gives us

$$\begin{aligned}
&(x^2 - x + 1)(x + 1) = 0 \\
&x^3 + 1 = 0 \\
&x^3 = -1
\end{aligned}$$

which we can substitute into the given expression.

$$\begin{aligned}
x^{2015} - x^{2014} &= x^{2014}(x - 1) \\
&= x^{2014} \cdot x^2 \\
&= x^{2016} \\
&= (x^3)^{672} \\
&= (-1)^6 = 1
\end{aligned}$$

$\square$

Problem 9. (MACEDONIA/2016) For $n \geq 3$, $a_1, a_2, \ldots, a_n \in \mathbb{R}^+$ satisfy

$$\frac{1}{1+{a_1}^4} + \frac{1}{1+{a_2}^4} + \cdots + \frac{1}{1+{a_n}^4} = 1.$$

Prove that

$$a_1 a_2 \cdots a_n \geq (n-1)^{\frac{n}{4}}.$$

Proof. Let $b_i = {a_i}^4$ where $b_i \geq 0$. The given condition becomes

$$\frac{1}{1+b_1} + \frac{1}{1+b_2} + \cdots + \frac{1}{1+b_n} = 1.$$

and we want to prove

$$b_1 b_2 \cdots b_n \geq (n-1)^n.$$

Let $t_i = \dfrac{1}{1+b_i}$. Rewriting the given condition gives us

$$t_1 + t_2 + \cdots + t_n = 1.$$

and we want to prove

$$\frac{1-t_1}{t_1} \cdot \frac{1-t_2}{t_2} \cdots \frac{1-t_n}{t_n} \geq (n-1)^n.$$

Using AM-GM, we have

$$\begin{aligned}
&\frac{1-t_1}{t_1} \cdot \frac{1-t_2}{t_2} \cdots \frac{1-t_n}{t_n} \\
&= \frac{t_2+t_3+\cdots+t_n}{t_1} \cdot \frac{t_1+t_3+\cdots+t_n}{t_2} \cdots \frac{t_1+t_2+\cdots+t_{n-1}}{t_n} \\
&\geq \frac{(n-1)(t_2 t_3 \cdots t_n)^{\frac{1}{n-1}}}{t_1} \cdot \frac{(n-1)(t_1 t_3 \cdots t_n)^{\frac{1}{n-1}}}{t_2} \cdots \frac{(n-1)(t_1 t_2 \cdots t_{n-1})^{\frac{1}{n-1}}}{t_n} \\
&= (n-1)^n
\end{aligned}$$

□

Problem 10. (USAMO/2012) Find all integers $n \geq 3$ such that among any n positive real numbers $a_1, a_2, \ldots, a_n$ with

$$\max(a_1, a_2, \ldots, a_n) \leq n \cdot \min(a_1, a_2, \ldots, a_n),$$

there exist three that are the side lengths of an acute triangle.

Solution. We claim that $n \geq 13$ are all the satisfying positive integers.

WLOG, let $a_1 \leq a_2 \leq \cdots \leq a_n$. Three positive real numbers $a \leq b \leq c$ are the side lengths of an acute triangle iff $a^2 + b^2 > c^2$.

Thus, if our n positive real numbers contain no such triple, we must have $a_i^2 + a_j^2 \leq a_k^2$ for all $i < j < k$.

We have the following claim:

Lemma 5.4.1. Let $S = \{a_1, a_2, \cdots, a_n\}$ be a set of $n \geq 3$ positive real numbers, where $a_1 \leq a_2 \leq \cdots \leq a_n$. If S contains no three numbers that are side lengths of an acute triangle, we have $a_i \geq F_i \cdot a_1^2$ for all $1 \leq i \leq n$, where F_i is the i-th Fibonacci number.

Proof. If $n = 3$, we must have $a_1^2 + a_2^2 \leq a_3^2$. And since $a_1^2 \leq a_2^2$ and $a_3^2 \geq a_1^2 + a_2^2 \geq 2a_2^2$, the claim holds for $n = 3$.

Assume that the claim holds for all $t \leq n$. Consider a set S of $n + 1$ real numbers such that $a_1 \leq a_2 \leq \cdots \leq a_{n+1}$ and S contains no three numbers that are side lengths of an acute triangle. Then, we must have

$$\begin{gathered}
a_1^2 + a_2^2 \leq a_3^2 \\
a_2^2 + a_3^2 \leq a_4^2 \\
\vdots \\
a_{n-1}^2 + a_n^2 \leq a_{n+1}^2.
\end{gathered}$$

Since the statement holds for all $t \leq n$, we have $a_i \geq F_i \cdot a_1^2$ for all $1 \leq i \leq n$. Thus, $a_{n+1}^2 \geq a_{n-1}^2 + a_n^2 \geq F_{n-1} \cdot a_1^2 + F_n \cdot a_1^2 = F_{n+1} \cdot a_1^2$. QED □

Now, if $n \geq 13$, we have $a_n \geq F_n \cdot a_1^2$. However, since $\max(a_1, a_2, \ldots, a_n) \leq n \cdot \min(a_1, a_2, \ldots, a_n)$, we have $n \cdot a_1 \geq a_n$, or $a_n^2 \geq n^2 \cdot a_1^2$. But for all $n \geq 13$, we have $n^2 < F_n$, hence $a_n \geq F_n \cdot a_1^2 > n^2 a_1^2 \geq a_n$, which is absurd. Thus for all $n \geq 13$, we will always have three numbers that are side lengths of an acute triangle.

For $n \leq 12$, the set $S = \{\sqrt{F_i} t \mid 1 \leq i \leq n,\ t \in \mathbb{R}^+,\ F_i \text{ is the } i\text{-th Fibonacci number}\}$ satisfies that it contains no three numbers that are side lengths of an acute triangle. □

Problem 11. (IMO/2000) Let a, b, c be positive real numbers so that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \le 1.$$

Proof. Let $a = \dfrac{x}{y}, b = \dfrac{y}{z}, c = \dfrac{z}{x}$, then

$$\prod_{cyc}(x - y + z) \le xyz \iff (x^3 + y^3 + z^3) + 3xyz \ge \sum_{cyc} x^2 y + \sum_{cyc} x^2 z$$

This holds by Schur's inequality. $\square$

Problem 12. Show that

$$\sum_{k=1}^{n} a_k^2 \geq a_1 a_2 + a_2 a_3 + \cdots + a_{n-1} a_n + a_n a_1$$

Proof. Multiply both sides by 2,

$$2 \sum_{k=1}^{n} a_k^2 \geq 2(a_1 a_2 + a_2 a_3 + \cdots + a_{n-1} a_n + a_n a_1)$$

Subtracting each side by the RHS,

$$(a_1 - a_n)^2 + (a_2 - a_1)^2 + (a_3 - a_2)^2 + \cdots + (a_n - a_{n-1})^2 \geq 0$$

which is always true. □

Problem 13. (CANADA/1969) Show that if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$ and p_1, p_2, p_3 are not all zero, then

$$\left(\frac{a_1}{b_1}\right)^n = \frac{p_1 {a_1}^n + p_2 {a_2}^n + p_3 {a_3}^n}{p_1 {b_1}^n + p_2 {b_2}^n + p_3 {b_3}^n}$$

for every positive integer n.

Proof. Instead of proving the two expressions equal, we prove that their difference equals zero.
Subtracting the LHS from the RHS,

$$\frac{p_1 {a_1}^n + p_2 {a_2}^n + p_3 {a_3}^n}{p_1 {b_1}^n + p_2 {b_2}^n + p_3 {b_3}^n} - \frac{{a_1}^n}{{b_1}^n} = 0$$

Finding a common denominator, the numerator becomes

$$\begin{aligned} &{b_1}^n (p_1 {a_1}^n + p_2 {a_2}^n + p_3 {a_3}^n) - {a_1}^n (p_1 {b_1}^n + p_2 {b_2}^n + p_3 {b_3}^n) \\ &= p_2 ({a_2}^n {b_1}^n - {a_1}^n {b_2}^n) + p_3 ({a_3}^n {b_1}^n - {a_1}^n {b_3}^n) = 0 \end{aligned}$$

(The denominator is irrelevant since it never equals zero)
From $\frac{a_1}{b_1} = \frac{a_2}{b_2}$, we have

$${a_1}^n {b_2}^n = {a_2}^n {b_1}^n$$

Similarly, from $\frac{a_1}{b_1} = \frac{a_3}{b_3}$, we have

$${a_1}^n {b_3}^n = {a_3}^n {b_1}^n$$

Hence, ${a_2}^n {b_1}^n - {a_1}^n {b_2}^n = {a_3}^n {b_1}^n - {a_1}^n {b_3}^n = 0$ and our proof is complete. □

6

Sequences and Series

6.1 Summation

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \tag{6.1}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \tag{6.2}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2 \tag{6.3}$$

Proof. These can be proven using mathematical induction. □

6.2 Telescoping sums

A sum in which subsequent terms cancel each other, leaving only initial and final terms. For example,

$$\begin{aligned} S &= \sum_{i=1}^{n-1} (a_i - a_{i+1}) \\ &= (a_1 - a_2) + (a_2 - a_3) + \ldots + (a_{n-2} - a_{n-1}) + (a_{n-1} - a_n) \\ &= a_1 - a_n \end{aligned}$$

Example 5. Evaluate the following sum:

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}}.$$

Solution.

$$\frac{1}{\sqrt{n+1}+\sqrt{n}} = \frac{\sqrt{n+1}-\sqrt{n}}{(\sqrt{n+1}+\sqrt{n})(\sqrt{n+1}-\sqrt{n})}$$
$$= \sqrt{n+1}-\sqrt{n}$$

Doing this for each fraction gives us

$$(\sqrt{2}-\sqrt{1})+(\sqrt{3}-\sqrt{2})+\cdots+(\sqrt{100}-\sqrt{99}) = \sqrt{100}-\sqrt{1} = 9$$

□

Example 6. Evaluate the following sum:

$$\sum_{n=1}^{2015} \frac{1}{n^2+3n+2}.$$

Solution. A common method is to use partial fractions which will cancel each other out.

$$\frac{1}{n^2+3n+2} = \frac{1}{n+1} - \frac{1}{n+2}$$

$$\sum_{n=1}^{2015}\left(\frac{1}{n+1}-\frac{1}{n+2}\right) = \left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{2016}-\frac{1}{2017}\right)$$
$$= \frac{1}{2}-\frac{1}{2017} = \frac{2015}{4034}$$

□

6.3 Power series

Theorem 6.3.1: Taylor Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{6.4}$$

Theorem 6.3.2: Maclaurin Series

This is a special case of Taylor Series, where $a = 0$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \tag{6.5}$$

The power series below can be easily computed:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{6.6}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{6.7}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{6.8}$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{6.9}$$

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots \tag{6.10}$$

$$\frac{1}{(1 - x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots \tag{6.11}$$

6.4 Behaviour of sequences

A sequence *converges* to a number L when

$$\lim_{n\to\infty} a_n = L$$

Conversely, a sequence that *diverges* is simply one that does not converge.

Example 7. Consider the sequence

$$a_n = \frac{3n^2 - 3n + 1}{2n^2 + 4n - 1}$$

Solution. Divide the numerator and denominator by the highest power of n.

$$\lim_{n\to\infty} \frac{3n^2 - 3n + 1}{2n^2 + 4n - 1} = \lim_{n\to\infty} \frac{3 - \frac{3}{n} + \frac{1}{n^2}}{2 + \frac{4}{n} - \frac{1}{n^2}}$$

The $\frac{3}{n}, \frac{1}{n^2}, \frac{4}{n}, \frac{1}{n^2}$ terms approach 0 as n tends to infinity, so we get

$$\lim_{n\to\infty} \frac{3 - \frac{3}{n} + \frac{1}{n^2}}{2 + \frac{4}{n} - \frac{1}{n^2}} = \frac{3 - 0 - 0}{2 + 0 + 0} = \frac{3}{2}$$

as the value of the limit. □

Theorem 6.4.1: Sandwich Theorem

If we have three sequences a_n, b_n, c_n related by inequalities $a_n \leq b_n \leq c_n$ then if the two "outer" sequences a_n and c_n both converge to the same number L, so does the sequence in the middle:

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n \implies \lim_{n\to\infty} b_n = L$$

Example 8. Consider the sequence

$$a_n = \frac{2+\sin n}{n}$$

Solution. Since $-1 \leq \sin n \leq 1$, we have the inequalities:

$$\frac{2+(-1)}{n} \leq \frac{2+\sin n}{n} \leq \frac{2+1}{n}$$

or in other words

$$\frac{1}{n} \leq \frac{2+\sin n}{n} \leq \frac{3}{n}$$

Since the sequence on the left and right converge to 0, the sandwiched sequence also converges to 0.

Hence

$$\lim_{n\to\infty} \frac{2+\sin n}{n} = 0$$

□

Note that if $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ each exist individually, then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

But take note that we would not be able to "split up" a sum in this way if either limit of a_n or b_n does not exist on its own.

6.5 Generating Functions

A generating function is just a different way of writing a sequence of numbers. Here we will be dealing mainly with sequences of numbers (a_n).

Definition 6.5.1

Let $(a_n)_{n\geq 0}$ be a sequence of numbers. The generating function associated with this sequence is the series

$$A(x) = \sum_{n\geq 0} a_n x^n.$$

Problems

7

Recurrence Relations

7.1 First Order Recurrence Relations

The homogeneous case can be written in the following way:

$$x_n = rx^{n-1}, x_0 = A$$

Its general solution is

$$x_n = Ar^n$$

which is a geometric sequence with ratio r.

7.2 Second Order Recurrence Relations

$$c_0x_n + c_1x_{n-1} + c_2x_{n-2} = 0$$

We first look for solutions in the form of $x_n = cr^n$. Plugging in the equation, we get

$$c_0cr^n + c_1cr^{n-1} + c_2cr^{n-2} = 0.$$

This simplifies to

$$c_0r^2 + c_1r + c_2 = 0$$

which is known as the *characteristic equation* of the recurrence.
The roots r_1, r_2 of the above equation are known as the *characteristic roots*.

In the case of distinct real roots, the general solution is

$$x_n = c_1{r_1}^n + c_2{r_2}^n$$

where c_1 and c_2 are constants to be found.

Problems

8 Functional Equations

8.1 Function

” Definition of a function, and concepts such as injectivity and surjectivity

8.2 Functional Equation

” Cauchy’s functional equation

An equation containing an unknown function is called a functional equation. A typical functional equation problem will ask you to find all functions satisfying a certain property. For such problems, you must prove *both* directions.

Common problem-solving techniques:

- Guess the function.
- Substitute values such as 1, 0, -1, x, or $-x$.
- Spotting recurrence relations.
- Spotting cyclic functions.
- Cauchy’s method.

Example 9. Let $a \neq 1$. Solve the equation

$$af(x) + f\left(\frac{1}{x}\right) = ax$$

where the domain of f is the set of all non-zero real numbers.

Solution. Replacing x by x^{-1}, we get

$$af\left(\frac{1}{x}\right) + f(x) = \frac{a}{x}$$

We therefore have

$$(a^2 - 1)f(x) = a^2x - \frac{a}{x}$$

and hence

$$f(x) = \frac{a^2x - \frac{a}{x}}{a^2 - 1}.$$

□

Example 10. (Fibonacci Equations) Let

$$f(n + 2) = f(n + 1) + f(n)$$

where $f(0) = 0, f(1) = 1$. Find a general formula for the sequence.

Solution. Consider the solution of the form

$$f(n) = \alpha^n$$

for some real number α. Then we have

$$\alpha^{n+2} = \alpha^{n+1} + \alpha^n$$

from which we conclude that $\alpha^2 - \alpha - 1$. Solving quadratically, we have

$$\alpha_1 = \frac{1 + \sqrt{5}}{2}, \alpha_2 = \frac{1 - \sqrt{5}}{2}.$$

Hence, a general solution of the sequence can be written as

$$f(n) = c_1\left(\frac{1 + \sqrt{5}}{2}\right)^n + c_2\left(\frac{1 - \sqrt{5}}{2}\right)^n$$

where c_1 and c_2 are coefficients to be determined using the initial values. By the initial conditions, we have

$$\begin{aligned} &c_1 + c_2 = 0 \\ &c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 1 \end{aligned}$$

Thus we have

$$c_1 = \frac{1}{\sqrt{5}},\ c_2 = -\frac{1}{\sqrt{5}}.$$

Hence this gives us

$$f(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

□

Problems

Problem 14. (IMO/2019) Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b,

$$f(2a) + 2f(b) = f(f(a+b)).$$

Solution. First, we substitute $a = 0$ to get

$$f(0) + 2f(b) = f(f(b)).$$

It follows that

$$f(f(a+b)) = 2f(a+b) + f(0),$$

so we have

$$2f(a+b) + f(0) = f(2a) + 2f(b).$$

Substituting $a = 1$ (the motivation is that, since $f(x)$ takes the integers to the integers, it might be useful to relate $f(x+1)$ with $f(x)$) yields

$$2f(b+1) + f(0) = f(2) + 2f(b).$$

Rearranging this a little bit, we get

$$f(b+1) - f(b) = \frac{f(2) - f(0)}{2}.$$

Clearly, $\frac{f(2)-f(0)}{2}$ is constant, so it follows that $f(x)$ is linear.

Now, we let $f(x) = gx + h$. Substituting this back, we find that either $g = h = 0$ or $g = 2$.

Hence, we have $\boxed{f(x) = 2x + h \text{ for some constant h}}$, or $\boxed{f \equiv 0}$. □

Problem 15 (IMO/2015)**.** Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the equation

$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$$

for all real x and y.

Solution. Let $P(x, y)$ denote the assertion. Then, $P(0, y)$ gives $f(f(y)) + f(0) = f(y) + yf(0)$. Therefore, $y = 0$ gives $f(f(0)) = 0$ and $y = f(0)$ gives $2f(0) = f(0)^2$. This implies $f(0) = 0$ or $f(0) = 2$.

Case 1: $f(0) = 2$ Then, $f(2) = 0$ and $f(f(y)) = f(y) + 2y - 2$. This implies f is injective and $f(y) = y$ if and only if $y = 1$. Now, $P(x, 1)$ gives $f(x + f(x + 1)) = x + f(x + 1)$, so $f(x + 1) = 1 - x$. Therefore, $f(x) = 2 - x$. This works because both sides are equal to $y + 2 - xy$.

Case 2: $f(0) = 0$ Then, $f(f(y)) = f(y)$. Now, $P(f(k), k - f(k))$ gives

$$f(2f(k)) + f(f(k)(k - f(k))) = 2f(k) + (k - f(k))f(k)$$

and $P(f(k), 0)$ gives

$$f(2f(k)) = 2f(k).$$

This means that $f(f(k)(k - f(k))) = (k - f(k))f(k)$. Therefore, $P(k - f(k), f(k))$ gives

$$f(k) + f(f(k)(k - f(k))) = f(k) + (k - f(k))f(k) = k - f(k)f(k - f(k)),$$

so

$$(k - f(k))(f(k) - 1) = -f(k - f(k)).$$

Therefore, if $f(a) - a = f(b) - b \neq 0$, then $f(a) = f(b)$, so $a = b$. Since $P(1, -1)$ gives $f(1) + f(-1) = 1 - f(1)$ and $P(-1, 1)$ gives $f(-1) + f(-1) = -1 + f(-1)$, we get $f(-1) = -1$ and $f(1) = 1$. Now, $P(1, y)$ gives $f(1 + f(1 + y)) - (1 + f(1 + y)) + f(y) - y = 0$, so if $g(x) = f(x) - x$, then $g(y) = -g(1 + f(1 + y))$. If $g(y) \neq 0$, then $g(y) = -g(1 + f(1 + y)) = g(1 + f(1 + 1 + f(1 + y)))$, so $y - 1 = f(f(y + 1) + 2)$. Therefore, $f(y - 1) = y - 1$. If $f(y+1) \neq y+1$, then $f(y) = y$, contradiction. Therefore, $f(y+1) = y+1$, so $f(y + 3) = y - 1$, which implies $f(y + 2) = y + 2$. However, $P(1, y + 2)$ gives $f(y) - y + f(y + 2) - (y + 2) = 0$, contradiction since $f(y + 2) = y + 2$ but $f(y) \neq y$. Therefore, we must have $f(y) = y$ for all y, which works since both sides are equal to $2x + y + xy$.

Therefore, the only solutions are $\boxed{f(x) = x}$ and $\boxed{f(x) = 2 - x}$. □

Problem 16. (CHINA/2016) Find all functions $f : \mathbb{Z} \to \mathbb{Z}$, such that for $\forall m, n \in \mathbb{Z}$,

$$f(f(m+n)) = f(m) + f(n).$$

Solution. Let $a = f(0)$ and $c = f(1) - f(0)$ $f(m) + f(1) = f(f(m+1)) = f(m+1) + f(0)$ and so $f(m+1) = f(m) + c$ and so $f(x) = cx + a$

Plugging this back into the original equation, we get $\boxed{\text{S1 : } f(x) = 0 \quad \forall x \in \mathbb{Z}}$, which indeed fits

$\boxed{\text{S2 : } f(x) = x + a \quad \forall x \in \mathbb{Z}}$, which indeed fits, whatever is $a \in \mathbb{Z}$ □

Part III
Geometry

9 Synthetic Geometry

9.1 Angles

The reader should be familiar with terms such as segment, endpoint, midpoint, ray, origin, line, collinear.

9.1.1 Phantom Points

9.2 Triangle

9.2.1 Triangle Centers

Definitions

Let ABC be a triangle.

Triangle centers:

1. Centroid G: Intersection of medians.
2. Circumcenter O: Center of the circle which passes through vertices of the triangle.
3. Incenter I: Intersection of the internal angle bisectors, which is also the center of the unique circle inside the triangle tangent to all three sides.
4. Excenter J: Center of the circle which is tangent to sides of the triangle but lies outside the triangle.

5. Orthocenter H: Intersection of altitudes.

Associated inscribed triangles:

1. Intouch triangle: The triangle whose vertices are the contact points of the incircle with the sides of ABC.
2. Medial triangle: The triangle whose vertices are the midpoints of the sides of ABC.
3. Orthic triangle: The triangle whose vertices are the feet of the altitudes.

Properties

Centroid medians cut triangle into 6 parts of equal area G divides medians into segments of ratio 2:1

Circumcenter R = a/2sinA = b/2sinB = c/2sinC

Incenter r=area/s (s is semiperimeter)

Orthocenter cyclic quadrilaterals H,A,B,C are incenter/excenters of orthic triangle

Point where angle bisector of A meets perp bisector of BC lies on circumcircle of ABC. This point is also the center of circle passing through B, C, I, J_A

Properties of the orthocenter

9.2.2 Congruency and Similarity

Triangle similarity

1. Two angles match.
2. Two sides have lengths in the same ratio, and the angle between those two sides matches.
3. Three sides have lengths in the same ratio.
4. The triangles are right-angled and the ratio between one side and hypotenuse is the same.

Triangle congruence Same conditions but check that one corresponding side matches.

9.2.3 Viviani's Theorem

Theorem 9.2.1: Viviani's Theorem

The sum of the distances from any interior point to the sides of an equilateral triangle equals the length of the triangle's altitude.

Proof. □

Theorem 9.2.2: T

e sum of the distances from any point P inside a parallelogram is independent of the location of P.
The converse also holds: If the sum of the distances from a point in the interior of a quadrilateral to the sides is independent of the location of the point, then the quadrilateral is a parallelogram.

Theorem 9.2.3: G

neralisation: In a 2k-sided polygon with opposite sides parallel, the sum of distances from an arbitrary point inside the polygon to its sides is constant and is independent of the point's position.

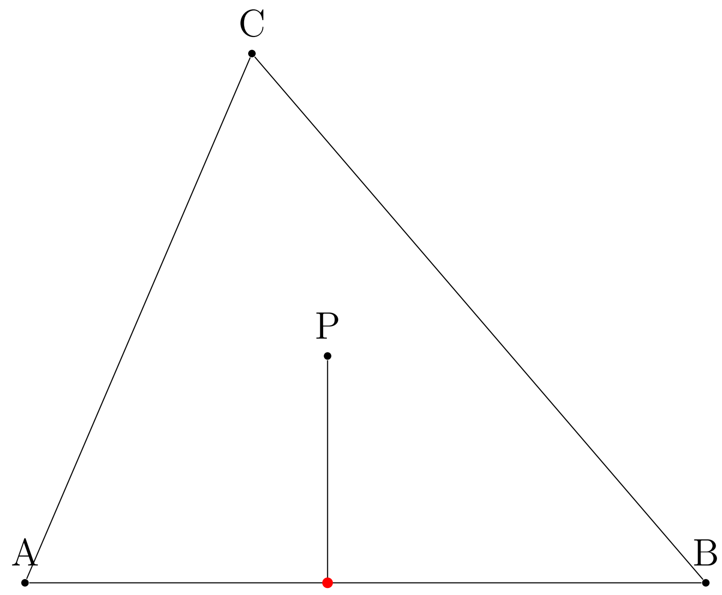
Regular polygons: Theorem 4. In every regular polygon, the sum of distances from an arbitrary point inside the polygon to its sides is constant and is independent of the point's position.

Theorem 5. The sum of distances from a point to the side lines of an equiangular polygon does not depend on the point and is that polygon's invariant.

Theorem 9.2.4: Carnot's Theorem

For a triangle ABC, P lies in the interior of the triangle. D, E, F denote the foots of perpendicular from P to sides BC, CA, AB respectively. The following equation holds:

$$AE^2 + BD^2 + CF^2 = AF^2 + BE^2 + CD^2 \tag{9.1}$$



Proof. By Pythagoras' Theorem:

$$PC^2 - CF^2 = PF^2$$
$$PA^2 - AF^2 = PF^2$$

This gives us

$$PC^2 - PA^2 = CF^2 - AF^2 \tag{1}$$

Similarly, we have

$$PA^2 - PB^2 = AE^2 - BE^2 \tag{2}$$
$$PB^2 - PC^2 = BD^2 - CD^2 \tag{3}$$

Adding (1) + (2) + (3) gives us our desired outcome. □

Theorem 9.2.5: Pythagorean Inequality

A generalisation of the Pythagorean theorem, which states that in a right triangle with sides $a \le b \le c$ we have $a^2 + b^2 = c^2$. This inequality extends this to obtuse and acute triangles. The inequality states:
For an acute triangle with sides $a \le b \le c$,

$$a^2 + b^2 > c^2$$

For an obtuse triangle with sides $a \le b \le c$,

$$a^2 + b^2 < c^2$$

This inequality is a direct result of the Cosine Law.

9.2.4 Area of triangle

$$[ABC] = \frac{ah_a}{2} = \frac{1}{2}ab\sin C = sr \tag{9.2}$$

where r denotes inradius, s denotes semiperimeter.

$$[ABC] = \frac{abc}{4R} \tag{9.3}$$

This formula is derived from Sine Rule where $\sin C = \dfrac{c}{2R}$.

Theorem 9.2.6: Heron's formula

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)} \tag{9.4}$$

where s denotes semiperimeter.

Theorem 9.2.7: Stewart's Theorem

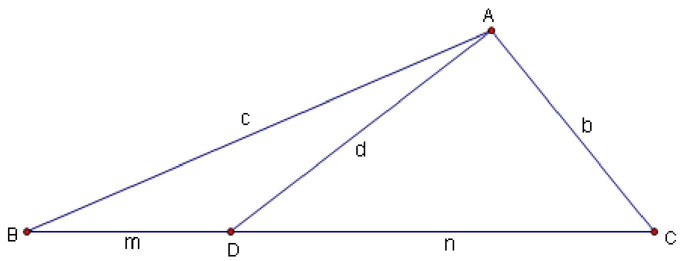
Given a triangle ΔABC, if cevian AD is drawn so that $BD = m$, $CD = n$, $AD = d$, we have

$$b^2m + c^2n = amn + d^2a.$$

Rewriting this,

$$man + dad = bmb + cnc. \tag{9.5}$$

This Theorem can be derived using Cosine Rule. You will usually use this Theorem to find the lengths of angle bisectors and medians.

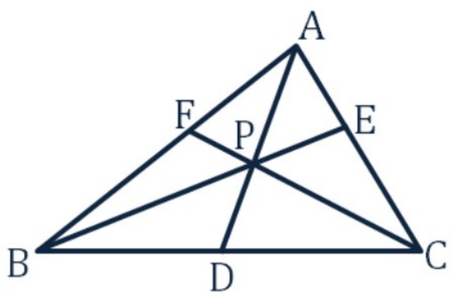


9.2.5 Ceva and Menelaus

Theorem 9.2.8: Ceva's Theorem

Given a triangle ΔABC with a point P inside the triangle, continue lines AP, BP, CP to intersect BC, CA, AB at D, E, F respectively.

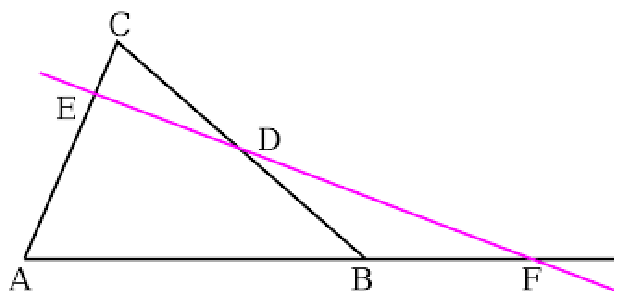
$$\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = 1 \tag{9.6}$$



Theorem 9.2.9: Menelaus' Theorem

Given a triangle ΔABC, and a transversal that intersects BC, AC, AB at points D, E, F respectively.

$$\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = 1 \tag{9.7}$$



9.3 Quadrilaterals

Theorem 9.3.1: British Flag Theorem

If $ABCD$ is a rectangle and P is a point inside of it, then we have

$$PA^2 + PC^2 = PB^2 + PD^2$$

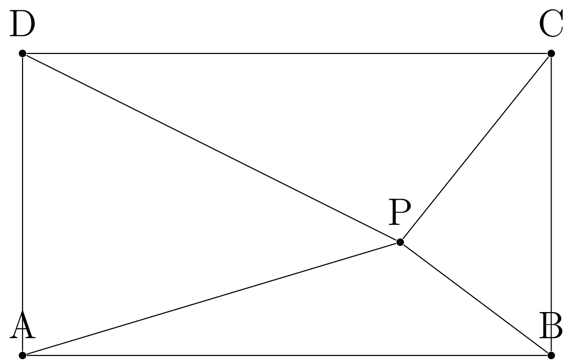


Figure 9.1: British flag theorem

Proof. This can be easily proven using Pythagoras' Theorem. □

9.4 Circle

The reader should be familiar with basic circle terminology, such as **center**, **radius**, **chord**, **diameter**, **tangent**, **secant**, **arc**, **sector**, **segment**, and **circumference**.

9.4.1 Angles

Angle subtended at the centre of a circle by a chord is twice the angle subtended on the circumference.

All angles subtended by a fixed chord in the same segment of a circle are equal.

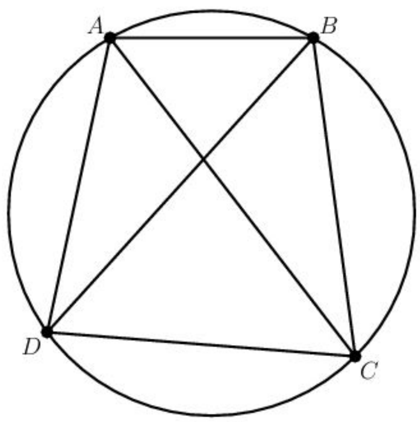
9.4.2 Cyclic Quadrilaterals

Angle chasing and cyclic quadrilaterals

Theorem 9.4.1: Ptolemy's Theorem

Given a cyclic quadrilateral $ABCD$, the product of lengths of diagonals is equal to the sum of products of lengths of the pairs of the opposite sides:

$$AC \cdot BD = AB \cdot CD + AD \cdot BC \tag{9.8}$$



Theorem 9.4.2: Ptolemy's Inequality

For four points A, B, C, D in the plane,

$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD \tag{9.9}$$

where equality holds if and only if $ABCD$ is a cyclic quadrilateral with diagonals AC and BD, or if A, B, C, D are collinear.

Proof. We construct a point P such that the triangles APB and DCB are similar and have the same orientation. This means that

$$BD = \frac{BA \cdot DC}{AP} \tag{1}$$

But since this is a spiral similarity, we also know that the triangles ABD and PBC are also similar, which implies that

$$BD = \frac{BC \cdot AD}{PC} \tag{2}$$

By the triangle inequality, we have $AP + PC \geq AC$. Multiplying both sides of the inequality by BD and using equations (1) and (2) gives us

$$BA \cdot DC + BC \cdot AD \geq AC \cdot BD$$

which is the desired inequality. Equality holds iff A, P, C are collinear. But since the triangles BAP and BDC are similar, this would imply that the angles BAC and BDC are congruent, i.e. that $ABCD$ is a cyclic quadrilateral. □

Theorem 9.4.3: Brahmagupta's Formula

Given a cyclic quadrilateral $ABCD$,

$$[ABCD] = \sqrt{(s-a)(s-b)(s-c)(s-d)} \tag{9.10}$$

where s denotes the semiperimeter.

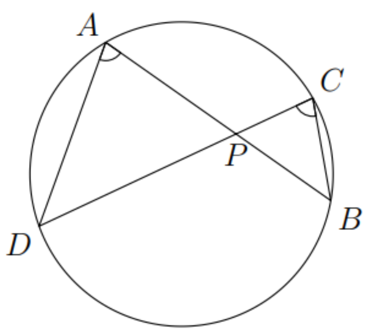
9.4.3 Power of a Point

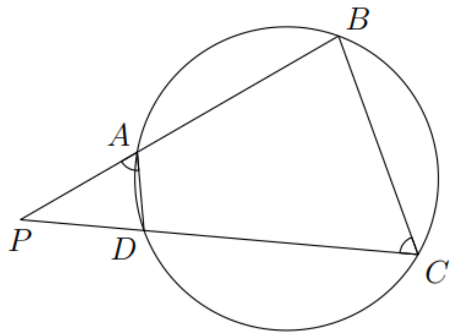
Power of a point is a frequently used tool in Olympiad geometry.

Theorem 9.4.4: Power of a point

Let Γ be a circle, and P be a point. Let a line through P meet Γ at points A and B, and another line through P meet Γ at points C and D. Then

$$PA \cdot PB = PC \cdot PD. \tag{9.11}$$





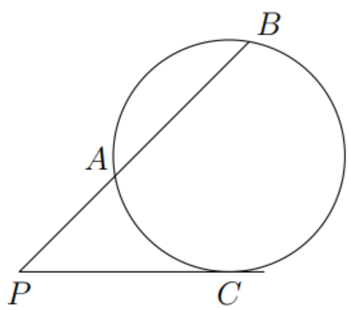
Proof. There are two configurations to consider, depending on whether P lies inside the circle or outside the circle.

When P lies inside the circle, we have $\angle PAD = \angle PCB$ and $\angle APD = \angle CPB$, so triangles PAD and PCB are similar. Hence $\dfrac{PA}{PD} = \dfrac{PC}{PB}$. Rearranging, we get $PA \cdot PB = PC \cdot PD$.

When P lies outside the circle, we have $\angle PAD = \angle PCB$ and $\angle APD = \angle CPB$, so again triangles PAD and PCB are similar. We get the same result in this case. $\square$

As a special case, when P lies outside the circle and $C = D(PC$ is a tangent), we have

$$PA \cdot PB = PC^2 \tag{9.12}$$



Theorem 9.4.5: Converse to Power of a Point

Let A, B, C, D be four distinct points. Let lines AB and CD intersect at P. Assume that either (1) P lies on both line segments AB and CD, or (2) P lies on neither line segments. Then A, B, C, D are concyclic if and only if $PA \cdot PB = PC \cdot PD$.

Proof. The expression $PA \cdot PB = PC \cdot PD$ can be rearranged as $\dfrac{PA}{PD} = \dfrac{PC}{PB}$. In both configurations described in the statement of the theorem, we have $\angle APD = \angle CPB$. It follows by angles and ratios that triangles APD and CPB are similar.

Thus $\angle PAD = \angle PCB$. In both cases this implies that A, B, C, D are concyclic. □

Suppose that Γ has center O and radius r. We say that the **power** of point P with respect to Γ is

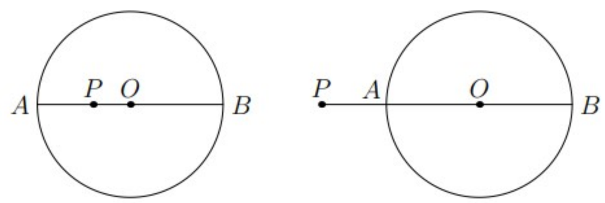
$$PO^2 - r^2$$

Let line PO meet Γ at points A and B, so that AB is a diameter. We will use *directed lengths*, meaning that for collinear points P, A, B, an expression such as $PA \cdot PB$ is assigned a positive value if PA and PB point in the same direction, and a negative value if they point in opposite directions. Then

$$PA \cdot PB = (PO + OA)(PO + OB) = (PO - r)(PO + r) = PO^2 - r^2,$$

which is the power of P. So the power of a point theorem says that this quantity equals to $PC \cdot PD$, where C and D are the intersections with Γ of any line through P.

By convention, the power of P is negative when P is inside the circle, and positive when P is outside the circle. When P is outside the circle, the power equals to the square of the length of the tangent from P to the circle.



Let Γ_1 and Γ_2 be two circles with different centers O_1 and O_2, and radii r_1 and r_2 respectively.

The radical axis of Γ_1 and Γ_2 is the set of points with equal powers with respect to both circles.

$$PO_1{}^2 - r_1{}^2 = PO_2{}^2 - r_2{}^2$$

which can be represented as

$$pow(P, \Gamma_1) = pow(P, \Gamma_2).$$

The radical axis is a line perpendicular to the line connecting the circles' centers (line ℓ).

Proof.

Lemma 9.4.1. Let P be a point in the plane, and let P' be the foot of the perpendicular from P to O_1O_2. Then

$$pow(P, \Gamma_1) - pow(P, \Gamma_2) = pow(P', \Gamma_1) - pow(P', \Gamma_2).$$

The proof of the lemma is an easy application of the Pythagorean Theorem.

Lemma 9.4.2. There is a unique point P on line O_1O_2 such that $pow(P, O_1) = pow(P, O_2)$.

Proof: First show that P lies between O_1 and O_2 via proof by contradiction, by using a bit of inequality theory and the fact that $O_1O_2 > r_1 + r_2$. Then, use the fact that $O_1P + PO_2 = O_1O_2$ (a constant) to prove the lemma.

The first lemma shows that every point on the plane can be equivalently mapped to a line on O_1O_2. The second lemma shows that only one point in this mapping satisfies the given condition. Combining these two lemmas shows that the radical axis is a line perpendicular to ℓ, hence proved. □

When Γ_1 and Γ_2 intersect, the intersection points A and B both have a power of 0 with respect to either circle, so A and B must lie on the radical axis. This shows that the radical axis *coincides with the common chord* when the circles intersect.

To show that some point lies on the radical axis or the common chord, we can show that the point has *equal powers with respect to the two circles.*

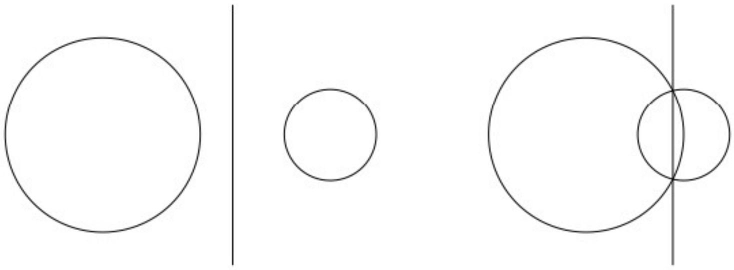


Figure 9.2: Radical axis

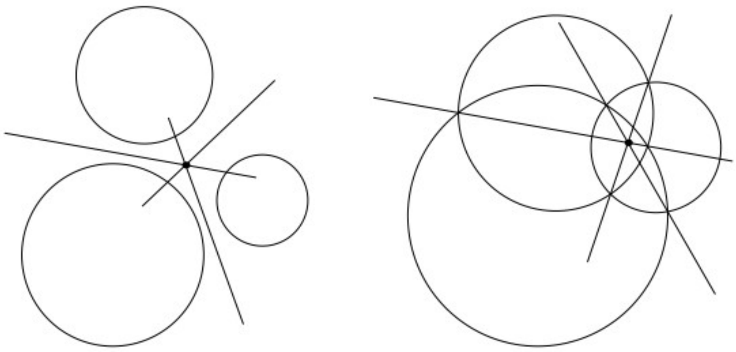


Figure 9.3: Radical center

Theorem 9.4.6: Radical Axis Theorem

Given three circles, no two concentric, the three pairwise radical axes (which are non-parallel) are concurrent, at a point known as the radical center.

Proof. Denote the three circles by $\Gamma_1, \Gamma_2, \Gamma_3$, and denote the radical axes of Γ_i and Γ_j by ℓ_{ij}.

Suppose that the radical axes are not all parallel. Let ℓ_{12} and ℓ_{13} meet at X. Since X lies on ℓ_{12}, it has equal powers with respect to Γ_1 and Γ_2. Since X lies on ℓ_{13}, it has equal powers with respect to Γ_1 and Γ_3. Therefore, X has equal powers with respect to all three circles, and hence it must lie on ℓ_{23} as well. □

9.4.4 Euler's Line and Nine-point Circle

Euler's line: O,G,H are collinear, with GH=2 OG $\mathrm{OI}^2 = R^2 - 2Rr$

Theorem 9.4.7: Nine Point Circle Theorem

The three midpoints of the three sides of any triangle, the feet of the three altitudes of that triangle, and the midpoints of the segments from the three vertices of the triangle to its orthocenter, all lie on a single circle.

The center of this circle lies on the Euler line, at the midpoint between the orthocenter and circumcenter.

The radius of this circle is half the circumradius of the triangle.

9.4.5 Simson line

9.4.6 Miquel's theorem

Theorem 9.4.8: Miquel's theorem

Let ABC be a triangle, and let X, Y, Z be points on lines BC, CA, AB respectively. Assume that the six points A, B, C, X, Y, Z are all distinct. Then the circumcircles of triangles AYZ, BZX, CXY pass through a common point.

Proof. The proof involves angle chasing. □

9.5 Trigonometry

Pythagorean identities

$$\sin^2 A + \cos^2 A = 1 \tag{9.13}$$

$$tan^2 A + 1 = \sec^2 A \tag{9.14}$$

$$1 + \cot^2 A = \operatorname{cosec}^2 A \tag{9.15}$$

Addition formulae

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \tag{9.16}$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \tag{9.17}$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \tag{9.18}$$

Double angle formulae:

$$\sin 2A = 2 \sin A \cos A \tag{9.19}$$

$$\begin{aligned} \cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 2 \\ &= 2 - 2\sin^2 A \end{aligned} \tag{9.20}$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \tag{9.21}$$

Half angle formulae

$$\sin \frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{2}} \tag{9.22}$$

$$\cos \frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{2}} \tag{9.23}$$

Sum to product

$$\sin A + \sin B = 2 \sin \frac{A + B}{2} \cos \frac{A - B}{2} \tag{9.24}$$

$$\sin A - \sin B = 2 \cos \frac{A + B}{2} \sin \frac{A - B}{2} \tag{9.25}$$

$$\cos A + \cos B = 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2} \tag{9.26}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2} \tag{9.27}$$

Product to sum

$$\sin A\cos B = \frac{1}{2}\sin(A+B)\sin(A-B) \tag{9.28}$$

R-formula

$$a\sin\theta \pm b\cos\theta = \sin(\theta \pm \alpha)$$

$$a\cos\theta \mp b\cos\theta = \cos(\theta \pm \alpha)$$

where $R = \sqrt{a^2+b^2}$, $\alpha = \tan^{-1}\frac{b}{a}$ where $0 < \alpha < \frac{\pi}{4}$.

Theorem 9.5.1: Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \tag{9.29}$$

where R denotes radius of circumcircle of triangle ABC.

Theorem 9.5.2: Cosine rule

$$a^2 = b^2 + c^2 - 2bc\cos A \tag{9.30}$$

9.6 Hyperbolic Functions

Hyperbolic sine function:

$$\sinh x = \frac{e^x - e^{-x}}{2} \tag{9.31}$$

Hyperbolic cosine function:

$$\cosh x = \frac{e^x + e^{-x}}{2} \tag{9.32}$$

$$\sinh x + \cosh x = e^x \tag{9.33}$$

Theorem 9.6.1: Osborn's rule

cos should be converted into cosh and sin into sinh, except when there is a product of two sines, when a sign change must be effected.

$$\cosh 2x = 1 + 2\sinh^2 x \tag{9.34}$$

$$\sinh 2A = 2\sinh A \cosh A \tag{9.35}$$

Problems

Problem 17. (OXFORD MAT) Evaluate

$$\sin^2 1° + \sin^2 2° + \cdots + \sin^2 89° + \sin^2 90°.$$

Solution. Recall the Pythagorean Identity $\sin^2 x + \cos^2 x = 1$.
Rewriting terms gives us

$$\begin{aligned}
&\sin^2 1° + \sin^2 2° + \cdots + \cos^2 2° + \cos^2 1° + 1 \\
&= (\sin^2 1° + \cos^2 1°) + \cdots + (\sin^2 44° + \cos^2 44°) + \sin^2 45° + 1 \\
&= 44(1) + \frac{1}{2} + 1 = 45\frac{1}{2}
\end{aligned}$$

□

10 Coordinate Systems

10.1 Cartesian Coordinates

10.1.1 Conic Sections

Definitions and basic properties of conic sections

10.2 Barycentric Coordinates

10.3 Complex Numbers

11 Advanced Techniques

11.1 Inversion in the plane

Definition and first properties Generalised lines and circles Inversion distance formula O Inversive geometry

11.2 Projective geometry

Cross ratios Projective transformations O Projective geometry, e.g. cross ratios, harmonic bundles, poles and polars, Pascal's theorem, and so on " Homothety

11.3 Complete quadrilaterals

Spiral similarity Miquel point of a cyclic quadrilateral

Part IV

Combinatorics

12
Combinatorics

” Basic counting arguments, e.g. writing expressions ” Principle of mathematical induction ” Recursion and recurrence relations ” The pigeonhole principle ” Definition of sets and functions O Elementary probability O Expected value and linearity of expectation O Basic properties and definitions from graph theory, e.g. connectedness and degree of a vertex O Definition and existence of the convex hull of a finite set of points !! Nontrivial results from graph theory, such as Hall’s marriage lemma or Turan’s theore

12.1 Permutations and Combinations

A **permutation** is an arrangement of objects in a specific order. Number of ways to arrange k distinguishable objects from n distinguishable objects:

$$ {}^{n}\mathrm{P}_{k} = \frac{n!}{(n-k)!} \tag{12.1} $$

A **combination** is a selection of objects without regard to the order

$$ {}^{n}\mathrm{C}_{k} = \binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{12.2} $$

Number of subsets of a set with n elements is 2^m.

Number of ways to choose k objects from n objects if repetition is allowed $= \binom{n+k-1}{k}$.

Number of paths from $(0,0)$ to (m,n) going 1 unit rightwards or upwards $= \binom{m+n}{n}$.

Number of k-tuples of positive integers which sum equals n is $\binom{n-1}{k-1}$.

Number of k-tuples of non-negative integers which sum equals n is $\binom{n+k-1}{k-1}$.

12.2 Binomial coefficients

12.2.1 Pascal's Triangle

We can observe that by means of expansion,

$$\binom{n}{k} = \binom{n}{n-k} \tag{12.3}$$

Each number in the Pascal's triangle is a binomial coefficient. Pascal's and hockey-stick identities:

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} \tag{12.4}$$

$$\sum_{r=k}^{n} \binom{r}{k} = \binom{n+1}{k+1} \tag{12.5}$$

$$\sum_{r=0}^{n} \binom{k+r}{r} = \binom{n+k+1}{n} \tag{12.6}$$

We also have

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m} \tag{12.7}$$

which can be easily proven via expansion.

Theorem 12.2.1: Vandermonde's Identity

$$\sum_{r=0}^{k} \binom{m}{r}\binom{n}{k-r} = \binom{m+n}{k} \tag{12.8}$$

12.2.2 Binomial Theorem

Theorem 12.2.2: Binomial Theorem

For $n \in \mathbb{Z}^+$, we have

$$\begin{aligned}(x+y)^n &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n}y^n \\ &= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k\end{aligned} \tag{12.9}$$

Proof. This can be proven using mathematical induction. □

This gives us a very useful identity:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n \tag{12.10}$$

Alternatively, we can think of this using Pascal's triangle. From one row to the row below, we are essentially duplicating each number, hence the total doubles on each iteration.

Sums:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \tag{12.11}$$

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1} \tag{12.12}$$

$$\sum_{k=0}^{n} k^2\binom{n}{k} = n(n+1)2^{n-2} \tag{12.13}$$

12.3 Catalan Numbers

Theorem 12.3.1: Catalan numbers

The Catalan numbers are given by the formula

$$C_n = \frac{1}{n+1}\binom{2n}{n} \tag{12.14}$$

12.3.1 Dyck Paths and Acceptable Sequences

The number of valid parenthesis expressions that consist of n right parentheses and n left parentheses is equal to the n-th Catalan number.

For example, $C_3 = 5$ and there are 5 ways to create valid expressions with 3 sets of parenthesis:

- ( ) ( ) ( )
- (( )) ( )
- ( ) (( ))
- ((( )))
- (( ) ( ))

Considering right parenthesis to be +1s, and left −1s, we can write this more formally as follows:

The number of sequences $a_1, \ldots, a_n$ of $2n$ terms that can be formed using n copies of +1s and n copies of −1s whose partial sums satisfy

12.3.2 Recurrence Relation; Generating Function

12.4 Principle of Inclusion-Exclusion

The **principle of inclusion and exclusion** is a counting technique that computes the number of elements that satisfy at least one of several properties while guaranteeing that elements satisfying more than one property are not counted twice.

The idea behind this principle is that summing the number of elements that satisfy at least one of two categories and subtracting the overlap prevents double counting.

For two sets,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

where $|S|$ denotes the cardinality (i.e. number of elements) of set S.

For three sets,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

More generally, if A_i are finite sets, then

$$\begin{aligned} |\bigcup_{i=1}^{n} A_i| = &\sum_{i=1}^{n} |A_i| - \sum_{1 \le i \le j \le n} |A_i \cap A_j| + \sum_{1 \le i \le j \le k \le n} |A_i \cap A_j \cap A_k| \\ &- \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|. \end{aligned} \tag{12.15}$$

12.5 Counting

Counting number of rectangles:

For a $m \times n$ grid, to form a rectangle, choose 2 points from the $m+1$ points along the column, and choose 2 points from the $n+1$ points along the row. Hence the number of rectangles we can form is

$$\binom{m+1}{2}\binom{n+1}{2}$$

Circle division: chords divide a circle number of chords = n choose 2 where there are n points number of intersection points = n choose 4 (any 4 points forms 2 chords, thus gives a unique intersection point)

12.6 Derangements

A derangement is a permutation with no fixed points. That is, a derangement of a set leaves no element in its original place. For example, the derangements of $\{1,2,3\}$ are $\{2,3,1\}$ and $\{3,1,2\}$, but $\{3,2,1\}$ is not a derangement of $\{1,2,3\}$ because 2 is a fixed point.
The number of derangements of an n-element set is denoted D_n. This number satisfies the recurrences

$$D_n = n \cdot D_{n-1} + (-1)^n$$

and

$$D_n = (n-1) \cdot (D_{n-1} + D_{n-2})$$

and is given by the formula

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

12.7 Probability

12.8 Pigeonhole Principle

Theorem 12.8.1: Pigeonhole Principle

If $k+1$ objects are placed into k boxes, then at least one box contains two or more objects.

Proof. We use a proof by contraposition.

Suppose none of the k boxes has more than one object. Then the total number of objects would be at most k. This contradicts the statement that we have $k+1$ objects. □

Theorem 12.8.2: Generalised Pigeonhole Principle

If n objects are placed into k boxes, then there is at least one box containing at least $\left\lceil \frac{n}{k} \right\rceil$ objects.

Proof. We use a proof by contradiction.

Suppose that none of the boxes contains more than $\left\lceil \frac{n}{k} \right\rceil - 1$ objects.

Then the total number of objects is

$$k\left(\left\lceil \frac{n}{k} \right\rceil - 1\right)$$

but

$$k\left(\left\lceil \frac{n}{k} \right\rceil - 1\right) < k\left[\left(\frac{n}{k} + 1\right) - 1\right] = n$$

where the inequality $\left\lceil \frac{n}{k} \right\rceil < \frac{n}{k} + 1$ was used.

This is a contradiction, because there are a total of n objects. □

Problems

Problem 18. Use a combinatorial proof to show that

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}.$$

Proof. For combinatorial proofs, we begin with a story. Consider a group of $2n$ animals, where n are dogs and n are cats.

RHS: Number of ways to pick n animals from a group of $2n$ animals.

For LHS, we try to understand what's going on in the summation:

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots$$

We see that each term looks like a case. For example, for the first term, pick 0 items from the first group, and pick n items from the second group. This shows that if we want to pick n animals, we can pick k dogs and $n-k$ cats.

LHS: Consider all cases where we pick k dogs and $n-k$ cats.

$\therefore$ LHS is the same as RHS as they both count the same number of things. Hence proven. $\square$

Problem 19. Evaluate

$$S = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}.$$

Solution. Writing the sum backwards yields

$$\begin{aligned} S &= n\binom{n}{n} + (n-1)\binom{n}{n-1} + \cdots + \binom{n}{1} \\ &= n\binom{n}{0} + (n-1)\binom{n}{1} + \cdots + \binom{n}{n-1} \end{aligned}$$

Add this to the original series to get

$$\begin{aligned} &2S = n\left[\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}\right] \\ &2S = n2^n \\ &\boxed{S = n2^{n-1}} \end{aligned}$$

□

This is the proof of the sum

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$$

Problem 20. (USAMO 2005) Legs L_1, L_2, L_3, L_4 of a square table each have length n, where n is a positive integer. For how many ordered 4-tuples (k_1, k_2, k_3, k_4) of non-negative integers can we cut a piece of length k_i from the end of leg L_i $(i = 1, 2, 3, 4)$ and still have a stable table?

(The table is stable if it can be placed so that all four of the leg ends touch the floor. Note that a cut leg of length 0 is permitted.)

Solution. The table is stable if $k_1 + k_3 = k_2 + k_4$. Let this common value be k such that that $k_1 + k_3 = k_2 + k_4 = k$. Let c_k be the number of ways to make the table stable for each value of k. We want to find $\sum_{k=0}^{2n} c_k$.

Note that each table leg is at least 0 and at most n, hence we'll break this into two sums so that it's easier to handle:

$$\sum_{k=0}^{n} c_k + \sum_{k=n+1}^{2n} c_k$$

Case 1: If $0 \le k \le n$, there are $k+1$ ways to partition k_1 and k_3, and another $k+1$ ways to partition k_2 and k_4. There are $(k+1)^2$ ways to partition k_i in this interval. Hence

$$\sum_{k=0}^{n} (k+1)^2$$

Case 2: If $n+1 \le k \le 2n$, each of the k_i is at most n and at least 0. There are $(2n-k+1)^2$ ways to partition the k_i in this interval. Hence

$$\sum_{k=n+1}^{2n} (2n-k+1)^2$$

Evaluating the sum gives us $\boxed{\dfrac{(n+1)(2n^2+4n+3)}{3}}$. □

13 Graph Theory

Readings:

- Graph Theory Vol. 3

13.1 Definitions and Terminology

1. A **graph** is a pair of sets $G = (V, E)$ where V is a set of vertices and E is a collection of edges whose endpoints are in V. It is possible that a graph can have infinitely many vertices and edges. Unless stated otherwise, we assume that all graphs are simple.[1]

2. Two vertices v, w are said to be **adjacent** if there is an edge joining v and w. An edge and a vertex are said to be **incident** if the vertex is an endpoint of the edge.

3. Given a vertex v, the **degree** of v is defined to be the number of edges containing v as an endpoint. If the degree of each vertex is the same, we can call that the degree of the graph.

4. A **path** in a graph G is defined to be a finite sequence of distinct vertices $v_0, v_1, \ldots, v_t$ such that v_i is adjacent to v_{i+1}. (A graph itself can also be called a path.) The **length** of a path is defined to be the number of edges in the path.

5. A **cycle** in a graph G is defined to be a finite sequence of distinct vertices $v_0, v_1, \ldots, v_t$ such that v_i is adjacent to v_{i+1} where the indices

[1] An edge whose endpoints are the same is called a **loop**. A graph where there is more than one edge joining a pair of vertices is called a **multigraph**. A graph without loops and is not a multigraph is said to be **simple**.

are taken modulo $t+1$. (A graph itself can also be called a cycle.) The **length** of a cycle is defined to be the number of vertices (or edges) in the path.

6. A graph is said to be **connected** if for any pair of vertices, there exists a path joining the two vertices. Otherwise, a graph is said to be **disconnected**.

7. The **distance** between two vertices u, v in a graph is defined to be the length of the shortest path joining u, v. (In the case the graph is disconnected, this may not be well-defined.)

8. Let $G = (V, E)$ be a graph. The **complement** $\overline{G}$ of G is a graph with the same vertex set as G and $E(\overline{G}) = e \notin E(G)$, i.e. $\overline{G}$ has edges exactly where there are no edges in G.

9. Let $G = (V, E)$ be a finite graph. A graph G is said to be **complete** if every pair of vertices in G is joined by an edge. A complete graph on n vertices is denoted by K_n.

10. A graph G is said to be **bipartite** if $V(G)$ can be partitioned into two non-empty disjoint sets A, B such that no edge has both endpoints in the same set. A graph is said to be **complete bipartite** if G is bipartite and all possible edges between the two sets A, B are drawn. In the case where $|A| = m, |B| = n$, such a graph is denoted by $K_{m,n}$.

11. Let $k \geq 2$. A graph G is said to be k**-partite** if $V(G)$ can be partitioned into k pairwise disjoint sets $A_1, \ldots, A_k$ such that no edge has both endpoints in the same set. A **complete** k**-partite** graph is defined similarly as a complete bipartite. In the case where $|A_i| = n_i$, such a graph is denoted by $K_{n_1,n_2,\ldots,n_k}$. (Note that a 2-partite graph is simply a bipartite graph.)

12. A graph is said to be **planar** if it can be drawn such that a pair of edges can only cross at a vertex.

Trivial graph: one vertex

Theorem 13.1.1: Euler's Characteristic Formula

For any connected planar graph, the number of vertices V minus the number of edges E plus the number of regions R equals 2.

$$V - E + R = 2 \tag{13.1}$$

(or V-E+F=2 for 3 dimensional polyhedra) To prove this, for trivial graph, V=1, F=1, E=0 Adding one edge, we either introduce a new vertex or face (if edge is connected to preexisting vertex)

vertex/edge colouring and Ramsey Theory

nature and properties

Connected graph Regular graph Complete graph Bipartite graph Directed graph

13.2 Trees and Balancing

A **tree** is defined to be a connected graph that does not contain any cycles.

A **cycle** in a graph means there is a path from an object back to itself.

Characterisation of trees: Let G be a connected graph with n vertices. The following statements are equivalent.

1. G does not contain any cycles
2. G contains exactly $n-1$ edges
3. For any two vertices, there exists exactly one path joining the two vertices
4. The removal of any edge disconnects the graph

Problems

Problems can include tournament, matching, and scheduling problems.

Problem 21. (Moser's circle problem) Determine the number of regions into which a circle is divided if n points on its circumference are joined by chords with no three internally concurrent.

Solution. Consider the graph which has points on the circumference and intersection points between chords as its vertices.

Let V, E, F denote the number of vertices, edges, regions respectively.

To count the number of intersection points, note that 4 points on the circumference give one unique intersection point between the two non-parallel chords formed by connecting two pairs of points which intersect inside the circle. Hence, number of intersection points is $\binom{n}{4}$.

$$V = n + \binom{n}{4}$$

Total number of edges includes n circular arcs, number of original chords formed from connecting pairs of points on the circumference E = no. of original lines + 2 x no. of intersection points E = n choose 2 + 2 x n choose 4 + n since there are n circular arcs

Using Euler's Characteristic Formula, we have

$$F = E - V - 1$$

$$\boxed{F = 1 + \binom{n}{2} + \binom{n}{4}}$$

□

Part V
Other Topics

14 Proofs

Before talking about proofs, we first discuss several elements of logic:

1. **Statements.**

   A statement, or a proposition, is a sentence which is either true or false, but not both.

   Given a statement P, $\sim P$ (read as "not P") is the negation of the statement P.

2. **Quantifiers.**

   We introduce the logical quantifiers: $\forall$ means "for all", $\exists$ means "there exists

   Negating statements with quantifiers can be tricky. We first look at the example statement below:

   P: The height of everyone in the room is at least 150cm.

   For this statement to be true, the height of every person must be at least 150cm. For it to be false, it suffices for there to be at least one person whose height is less than 150cm.

   $$P : \forall i \in \{\text{Alan}, \text{Beth}, \ldots, \text{Zach}\}, h_i \geq 150$$

   Then the negation is

   $$\sim P : \exists i \in \{\text{Alan}, \text{Beth}, \ldots, \text{Zach}\}, h_i < 150$$

3. **Conditional statements.**

14.1 Induction

Definition 14.1.1: Mathematical induction

Let $P(n)$ be a family of statements indexed by $n \in \mathbb{Z}_0^+$.
By **mathematical induction**, If $P(0)$ is true and $P(k)$ is true $\forall k \in \mathbb{N}$, then $P(n)$ is true $\forall n \in \mathbb{N}$.

Steps to write the proof:

1. **Base case.**

   Prove that $P(0)$ is true.

2. **Inductive step.**.

   $P(k) \implies P(k+1)$

Example 11. Prove that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

for all positive integers n.

Proof.
The statement is true for $n = 1$ (base case) because with $n = 1$, LHS = RHS = 1.

Assume that the statement is true for some $n = k$, where $k \in \mathbb{Z}^+$. By our induction hypothesis, we have $1 + 2 + 3 + \cdots + k = \dfrac{k(k+1)}{2}$.

To show that the statement is true for $k + 1$,

$$\begin{aligned} 1 + 2 + 3 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{(k+1)(k+2)}{2} \\ &= \frac{(k+1)[(k+1)+1]}{2} \end{aligned}$$

$\because$ The statement holds true for $n = k$ and $n = k + 1$,

$\therefore$ The statement is true for all positive integers n. $\square$

Definition 14.1.2: Cauchy induction

Cauchy induction, also known as forward-backward induction, is a variant of mathematical induction.

To write the proof, there are a few steps:

1. **Base case.**

   Prove that $P(0)$ is true.

2. **Inductive step**.

   $P(k) \implies P(2k)$

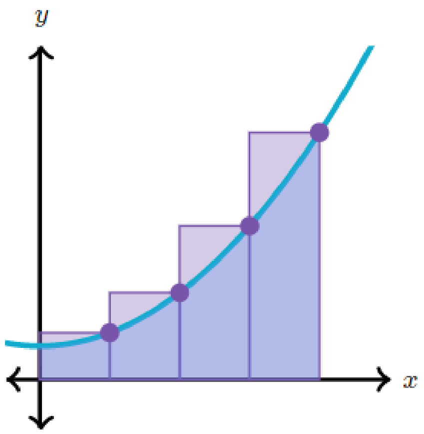
   $P(k) \implies P(k-1)$

15 Calculus

15.1 Riemann Sums

Given $y = f(x)$, we want to find the integral on the x-interval $[0, 1]$. Split the interval $[0, 1]$ into n equal subintervals

$$\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \cdots, \left[\frac{n-1}{n}, 1\right].$$



Consider the height of the rectangles. We take the right value. Hence for the k-th subinterval $\left[\frac{k-1}{n}, \frac{k}{n}\right]$ where $k = 1, \ldots, n$, the height of rectangle is $f\left(\frac{k}{n}\right)$.

Area of kth rectangle is

$$\frac{1}{n} \cdot f\left(\frac{k}{n}\right).$$

Therefore, the integral is obtained by summing up the area of n rectangles, which gives us

$$\int_0^1 f(x)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}f\left(\frac{k}{n}\right) \tag{15.1}$$

where there are infinitely many rectangles, i.e. $n \to \infty$.

Problems

Problem 22. (SMO/2020) Find the value of

$$S = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n(n+k)}}.$$

Solution.

$$\begin{aligned} S &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \sqrt{\frac{1}{1+\frac{k}{n}}} \\ &= \int_0^1 \frac{1}{\sqrt{1+x}}\,\mathrm{d}x \\ &= 2\sqrt{2} - 2 \end{aligned}$$

□