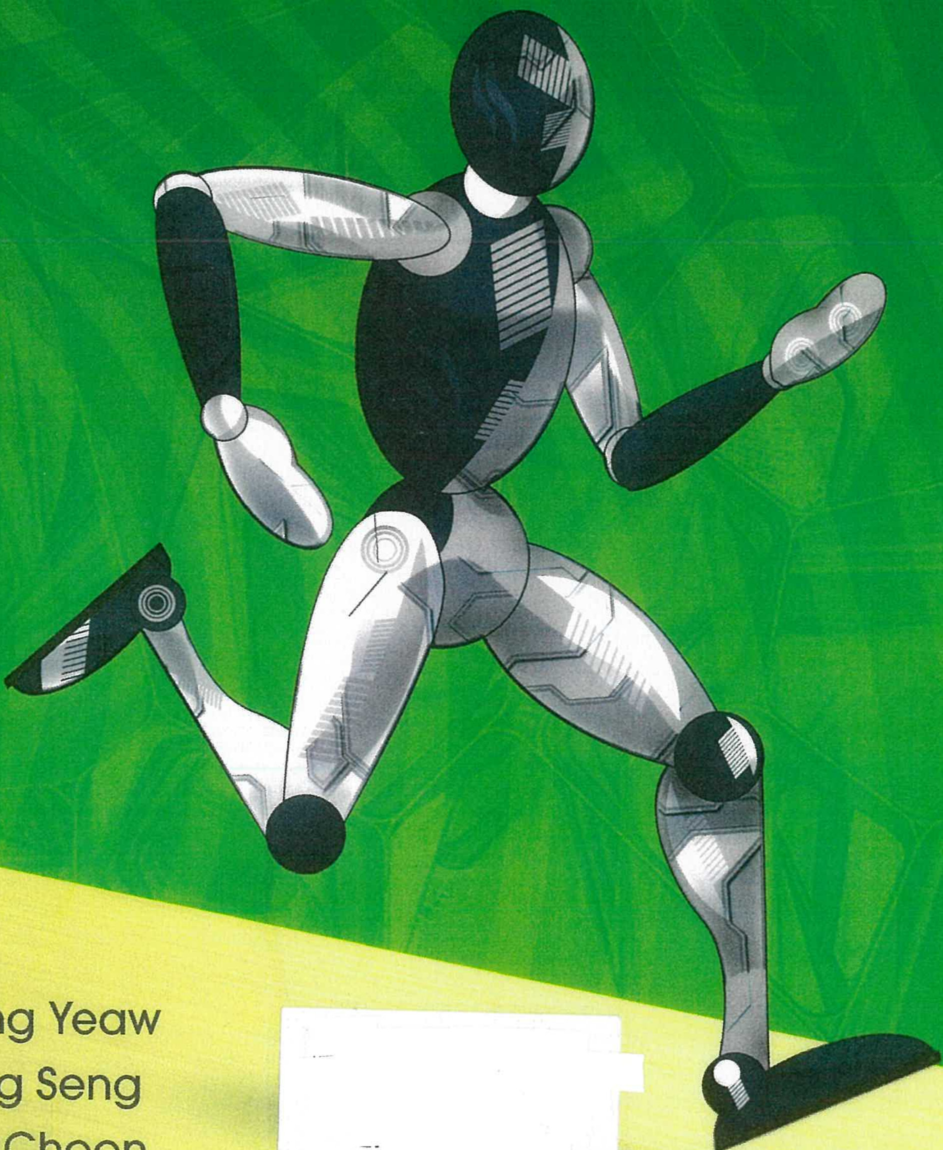
Singapore Mathematical Olympiads 2019



Ku Cheng Yeaw
Tay Tiong Seng
Toh Pee Choon
Toh Tin Lam

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Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2019

Junior Section (Round 1)

Tuesday, 4 June 2019 0930-1200 hrs

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1. Which of the five numbers

$$2^{30}, 3^{19}, 4^{14}, 6^{12} \text{ and } 9^{10}$$

has the largest value?

(A) 2^{30} (B) 3^{19} (C) 4^{14} (D) 6^{12} (E) 9^{10}

2. In a strange island, there are only two types of inhabitants: truth-tellers who only tell the truth and liars who only tell lies. One day, you meet two such inhabitants A and B. A said: "Exactly one of us is a truth-teller." B kept silent. Which of the following must be true?

(A) Both A and B are truth-tellers (B) Both A and B are liars
(C) A is a truth-teller and B is a liar (D) A is a liar and B is a truth-teller
(E) Not enough information to decide

3. If $x - \frac{1}{x} = 3$, what is the value of $\frac{21}{2} - \frac{3}{6}x^2 + \frac{3}{2}x$?

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

4. Let x, y and z be positive integers satisfying

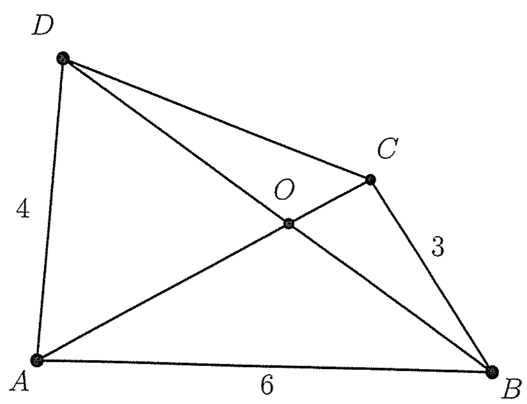
$$x^2 + y^2 + z^2 = 2(xy + 1) \qquad \text{and} \qquad x + y + z = 2022.$$

If x_1 and x_2 are two distinct solutions for x, what is the value of $x_1 + x_2$?

(A) 2019 (B) 2020 (C) 2021 (D) 2022 (E) 2023

5. In a quadrilateral $ABCD$, the diagonals AC and BD intersect at the point O. Suppose that $\angle BAD + \angle ACB = 180°$, $|BC| = 3$, $|AD| = 4$, $|AC| = 5$ and $|AB| = 6$. What is the value of $\frac{|OD|}{|OB|}$?

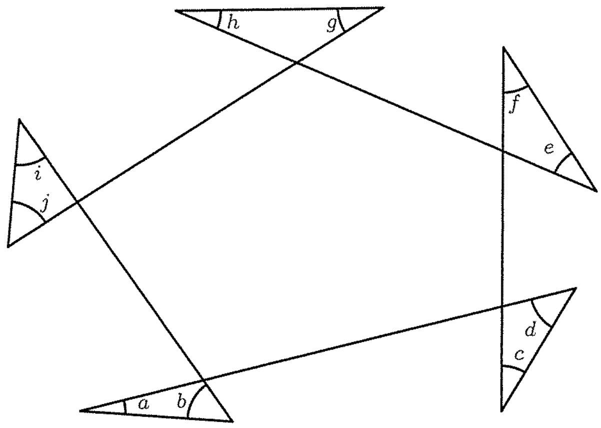
(A) $\frac{2}{3}$ (B) $\frac{8}{9}$ (C) $\frac{9}{10}$ (D) $\frac{10}{9}$ (E) $\frac{9}{8}$



Short Questions

6. In the following diagram, all lines are straight. What is the value (in °) of

$$\angle a + \angle b + \angle c + \angle d + \angle e + \angle f + \angle g + \angle h + \angle i + \angle j?$$



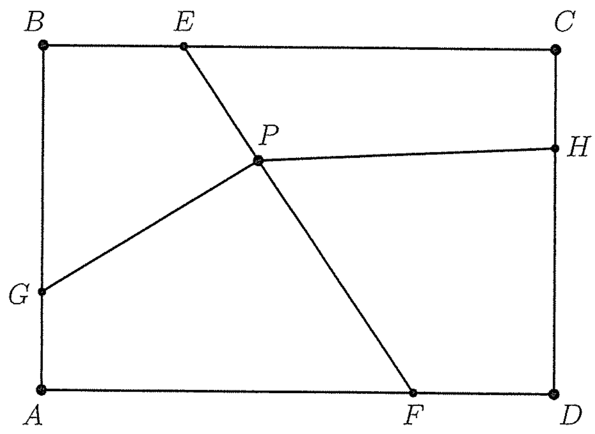
7. Let x be a positive integer. Suppose that the highest common factor of x and 80 is 40 and the highest common factor of x and 252 is 12. What is the minimum value of x?

8. Suppose that m and n are positive integers where $\dfrac{100m}{n}$ is a perfect cube greater than 1. What is the minimum value of $m + n$?

9. What is the largest possible two-digit positive integer that is 18 more than the product of its two digits?

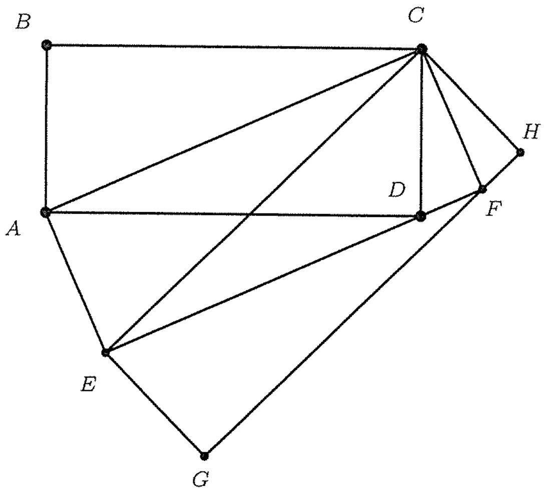
10. A teacher spent \$77 to purchase 25 gifts for her class. There were three different types with respective unit prices \$2, \$4, and \$5. If she bought more than three gifts of each type, how many of the \$2 gifts did she buy?

11. In the following diagram, $ABCD$ is a rectangle with $|AB| = 4$ and $|BC| = 6$. Points E and F lie on the sides BC and AD respectively such that $|BE| = |FD| = 2$. Points G and H lie on the sides AB and CD respectively such that $|AG| = |CH| = 1$. Suppose P lies on EF, such that the quadrilateral $BGPE$ has area 5. What is the area of the quadrilateral $FDHP$?



12. Two boxes A and B contain an equal number of toys. If one of the toys, with value \$5, is transferred from A to B, the average value of the toys in box A will decrease by \$1, while the average value of the toys in box B will increase by \$1. Suppose every toy has a positive value, what is the total value (in \$) of the toys in the two boxes?

13. In the following diagram, $ABCD$, $ACFE$ and $ECHG$ are all rectangles. If $|AB| = 6$ and $|BC| = 8$, what is area of the rectangle $ECHG$?



14. What is the value of $\sqrt{199^2 + 800}$?

15. A student wrote the following 99 numbers on the board:

$$1, -2, 3, -4, 5, -6, \ldots, -98, 99.$$

She arbitrarily picked two of these numbers, say a and b, and replaced them by the sum $a + b$. For example, if she had picked 37 and -24, she would erase those two numbers and add the number 13 to the board. Suppose that she repeated the process, and after 98 replacements, there was one number left on the board. What is the largest possible value of this number?

16. If the equation $\dfrac{x-1}{x-5} = \dfrac{m}{10-2x}$ has no solutions in x, what is the value of $|m|$?

17. If

$$S = \frac{1 + 3 + 5 + \cdots + 2019}{2 + 4 + 6 + \cdots + 2020},$$

what is the value of $1011S$?

18. A five-digit positive integer x has the following properties:
(i) x has distinct digits which are from $\{1, 2, 3, 4, 5\}$;
(ii) $x > 23456$.
How many possible values can x take?

19. In the figure below, each distinct letter represents a unique digit such that the arithmetic holds. What digit does the letter L represent?

$$\begin{array}{cccc}
 & \mathrm{M} & \mathrm{A} & \mathrm{P} \\
\times & & \mathrm{H} & \mathrm{I} \\
\hline
 & \mathrm{S} & \mathrm{I} & \mathrm{P} \\
\mathrm{H} & \mathrm{I} & \mathrm{P} & \\
\hline
\mathrm{C} & \mathrm{L} & \mathrm{A} & \mathrm{P}
\end{array}$$

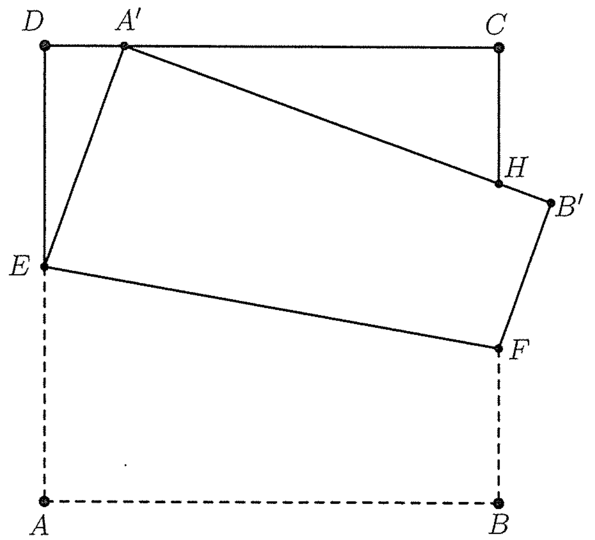
20. If x is a nonnegative real number, find the minimum value of

$$\sqrt{x^2+4}+\sqrt{x^2-24x+153}.$$

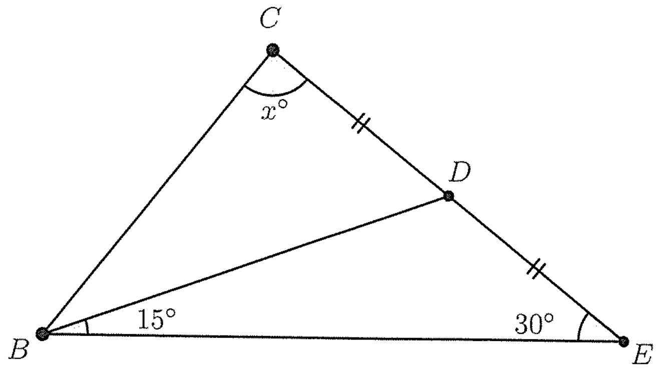
21. A positive integer is said to be "twelvish" if the sum of digits in its decimal representation is equal to 12. For example, the first four twelvish integers are $39, 48, 57$ and 66. What is the total number of twelvish integers between 1 and 999?

22. Two secondary one and m secondary two students took part in a round-robin chess tournament. In other words, each student played with every other student exactly once. For each match, the winner receives 3 points and the loser 0 points. If a match ends in a draw, both contestants receive 1 point each. If the total number of points received by all students was 130, and the number of matches that ended in a draw was less than half of the total number of matches played, what is the value of m?

23. $ABCD$ is a square sheet of paper with sides of length 6. The paper is folded along a crease line EF so that points A and B now lie on A' and B' respectively as indicated in the diagram. If H is the intersection of $A'B'$ and BC, what is the perimeter of the triangle $A'CH$?



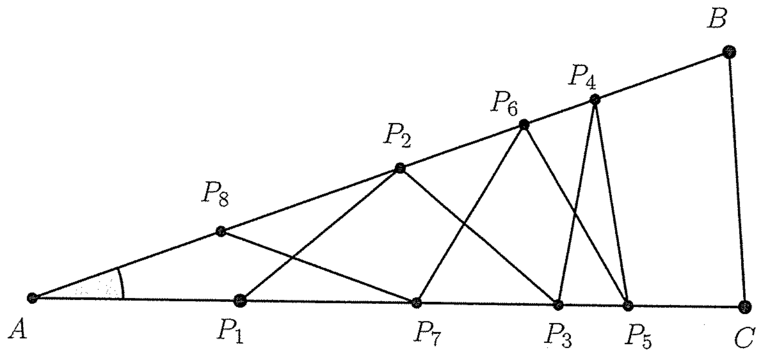
24. In the following diagram, $\angle BED = 30°$ and $\angle DBE = 15°$. If $|CD| = |DE|$, what is the value of x?



25. In the following diagram, $P_1, P_2, \cdots, P_8$ are points on $\triangle ABC$ such that

$$|AP_1| = |AP_8| = |P_iP_{i+1}|, \quad \text{for all } i = 1, 2, \cdots, 7.$$

What is the value (in °) of $\angle BAC$?



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Singapore Mathematical Olympiad (SMO) 2019

Junior Section (Round 1 Solutions)

1. Answer: (E)

   We have $9^{10} = 3^{20} > 3^{19}$ and $9^{10} > 8^{10} = 2^{30} > 2^{28} = 4^{14}$. It remains to compare 9^{10} with 6^{12}. Now
   $$6^{12} = 2^{12} \times 3^{12} = 8^4 \times 9^6 < 9^{10}.$$
   So 9^{10} has the largest value.

2. Answer: (E)

   Suppose A is a truth-teller, then his statement would be true and B would be a liar. However, if A is a liar, then his statement would be false which means B cannot be a truth-teller. So B is also a liar. Without more information, we cannot decide between these two cases.

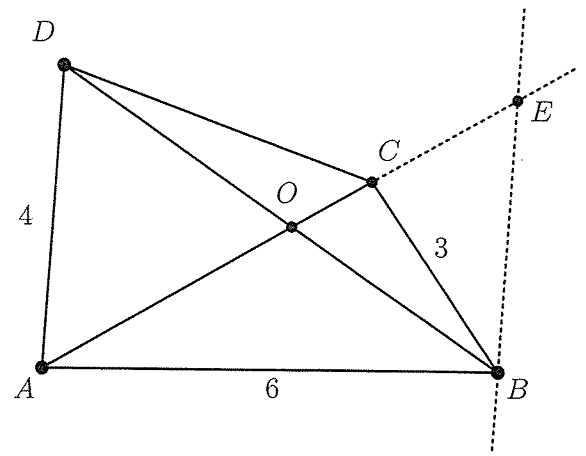
3. Answer: (E)

   Since $x - \frac{1}{x} = 3$, we have $x^2 - 3x = 1$. Thus
   $$\frac{21}{2} - \frac{1}{2}x^2 + \frac{3}{2}x = \frac{21}{2} - \frac{1}{2}(x^2 - 3x) = 10.$$

4. Answer: (C)

   The first equation is equivalent to
   $$(x-y)^2 = 2 - z^2.$$
   Since x, y and z are positive integers, we have $z = 1$, leaving us with $x - y = \pm 1$.
   If $x - y = 1$, $2022 = x + y + z = 2x - 1 + 1$ which means $x = 1011$.
   If $x - y = -1$, $2022 = x + y + z = 2x + 1 + 1$ which means $x = 1010$.
   Thus $x_1 + x_2 = 1011 + 1010 = 2021$.

5. Answer: (D)



Construct a line parallel to AD through B. Extend AC to intersect the constructed line at E as shown. Since AD is parallel to BE, $\angle ABE = 180° - \angle BAD = \angle ACB$. Thus $\triangle ABE$ is similar to $\triangle ACB$, which means

$$\frac{|BE|}{|AB|} = \frac{|CB|}{|AC|} \implies |BE| = \frac{6 \times 3}{5}.$$

We also have $\triangle AOD$ is similar to $\triangle EOB$, which means

$$\frac{|OD|}{|AD|} = \frac{|OB|}{|EB|} \implies \frac{|OD|}{|OB|} = \frac{4 \times 5}{18} = \frac{10}{9}.$$

6. Answer: 360

The values of the five interior angles of the pentagon are

$$180° - \angle a - \angle b, 180° - \angle c - \angle d, 180° - \angle e - \angle f, 180° - \angle g - \angle h, 180° - \angle i - \angle j.$$

Since the sum of these angles equals $3 \times 180°$, we have

$$\angle a + \angle b + \angle c + \angle d + \angle e + \angle f + \angle g + \angle h + \angle i + \angle j = 360°.$$

7. Answer: 120

Since 40 and 12 are common factors of x, the minimum value of x is the lowest common multiple of 40 and 12 which is 120.

8. Answer: 9

We require $\dfrac{2^2 \cdot 5^2 \cdot m}{n} = x^3$. If 10 is a factor of x, then 10 must also be a factor of m and the minimum value of $m + n \geq 10 + 1 = 11$. Next, if only 2 is a factor of x, then 2 must be a factor m while 25 must be a factor of n. In this case, $m + n \geq 2 + 25 = 27$. Likewise, if only 5 is factor of x, 5 must be a factor m while 4 must be a factor of n. We would then have $m + n \geq 5 + 4 = 9$. Finally, if neither 2 nor 5 is a factor of x, 100 must be a factor of n which means $n \geq 100$. Thus the minimum value of $m + n$ occurs when $m = 5$ and $n = 4$ giving $m + n = 9$.

9. Answer: 99

Let x and y be the tenths and ones digit respectively. Then $10x + y = 18 + xy$. This is equivalent to

$$8 = 18 - 10 = 10x + y - xy - 10 = (x - 1)(10 - y).$$

There are four possible pairs of (x, y) that satisfy the above, namely $(2, 2)$, $(3, 6)$, $(5, 8)$ and $(9, 9)$. The last corresponds to $99 = 18 + 9 \times 9$.

10. Answer: 14

Let x, y and z be the number of gifts that are worth \$2, \$4 and \$5 respectively. We have

$$2x + 4y + 5z = 77 \qquad \text{and} \qquad x + y + z = 25.$$

Eliminating x results in $2y + 3z = 27$. For integer solutions, z must be odd. Furthermore, since both $y > 3$ and $z > 3$, the only possibility is $z = 5$ and $y = 6$ which gives $x = 14$.

11. Answer: 8

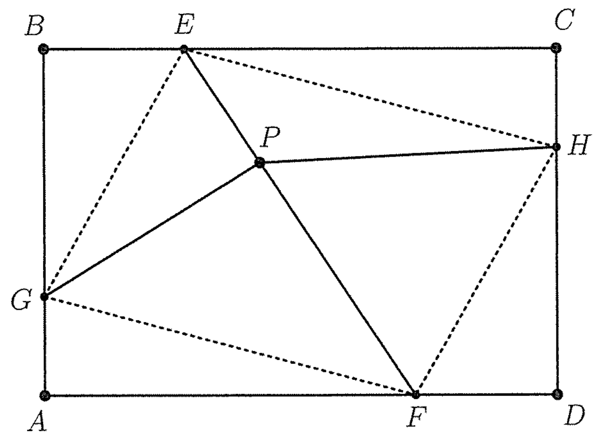
Using the notation $[A_1A_2\cdots A_k]$ to denote the area of the polygon enclosed by points $A_1, A_2, \ldots, A_k$, we have

$$[BGE] = [DHF] = 3 \qquad \text{and} \qquad [AFG] = [CEH] = 2.$$

Since $[AFEB] = 12$, we know $[EGF] = 12 - 3 - 2 = 7$. Similarly $[FHE] = 7$. So

$$[FPG] = 7 - [EGP] = 7 - (5 - 3) = 5.$$

The ratio $|EP| : |PF|$ equals $[EGP] : [FPG] = 2 : 5$. So using the same ratio in $\triangle FHE$, we can conclude that $[FHP] = 5$, thus $[FDHP] = 5 + 3 = 8$.



12. Answer: 12

Let n be the total number of toys in each box, a and b be the value of the toys in box A and B respectively. We have

$$\frac{a-5}{n-1} + 1 = \frac{a}{n} \implies a = n(6-n)$$
$$\frac{b+5}{n+1} - 1 = \frac{b}{n} \implies b = n(4-n).$$

We first note that $n \neq 1$ as the average value in A would have reduced from \$5 to \$0. Since every toy has a positive value, $b > 0$ which means $n < 4$. Thus either $n = 2$ or $n = 3$. In both cases, we obtain the same value of $a + b = 10n - 2n^2 = 12$.

13. Answer: 48

Using the notation $[A_1A_2\cdots A_k]$ to denote the area of the polygon enclosed by points $A_1, A_2, \ldots, A_k$, we have

$$[ACD] = \frac{6 \times 8}{2} = 24.$$

However, in rectangle $ACFE$,

$$[ACD] = [ECF] = \frac{1}{2}[ACFE].$$

Thus $[ECHG] = 2[ECF] = 2 \times 24 = 48$.

14. Answer: 201

Set $x = 199$, then $4x = 796$, thus

$$\sqrt{199^2 + 800} = \sqrt{x^2 + 4x + 4} = \sqrt{(x+2)^2} = 201.$$

15. Answer: 50

We can see that the value of the final number is independent of the order in which the numbers are picked, and equals the sum

$$\sum_{k=0}^{49}(2k+1) - \sum_{k=1}^{49} 2k = 50.$$

16. Answer: 8

We have

$$\frac{x-1}{x-5} = \frac{m}{10-2x} = \frac{m}{(-2)(x-5)}.$$

Assuming $x \neq 5$, the above reduces to

$$x = 1 - \frac{m}{2},$$

which means the equation always has a solution in x unless $1 - \frac{m}{2} = 5$. This gives $m = -8$ and thus $|m| = 8$.

17. Answer: 1010

Note that

$$\begin{aligned} S &= \frac{(1+2019) + (3+2017) + \cdots + (1009+1011)}{(2+2020) + (4+2018) + \cdots + (1010+1012)} \\ &= \frac{2020 \times 505}{2022 \times 505} \\ &= \frac{2020}{2022} = \frac{1010}{1011}. \end{aligned}$$

Thus $1011S = 1010$.

18. Answer: 86

We consider the following cases.

(a) $x > 30000$: The first digit is $3, 4$ or 5, with no other restrictions. Hence there are $3 \times 4! = 72$ possibilities.

(b) $24000 < x < 30000$: The first digit is 2 and the second digit is 4 or 5 with no other restrictions. Hence there are $2 \times 3! = 12$ possibilities.

(c) $23456 < x < 24000$: The first three digits are 235 with no other restrictions. Hence there are $1 \times 2! = 2$ possibilities.

Hence the total number of possibilities is $72 + 12 + 2 = 86$.

19. Answer: 6

Observe that I $\times$ AP and H $\times$ AP have the same last two digits IP. In other words,

$$\mathrm{I} \times \mathrm{AP} \equiv \mathrm{IP} \pmod{100} \qquad \text{and} \qquad \mathrm{H} \times \mathrm{AP} \equiv \mathrm{IP} \pmod{100}.$$

This means that (H$-$I)$\times$ AP is divisible by 100. If P is 0, I+P would not give A in the second column from the right, so P cannot be 0. Furthermore, as (H$-$I) consists of only a single digit, we conclude that AP is 25 or 75. The latter is impossible since we know the sum I+P ends with an A, which means I would be 2 and lead to a contradiction. So AP is 25 and I must be 7. Furthermore, (H$-$I) must be divisible by 4, so we can deduce H must be 3. The rest of the entries can then deduced accordingly to give

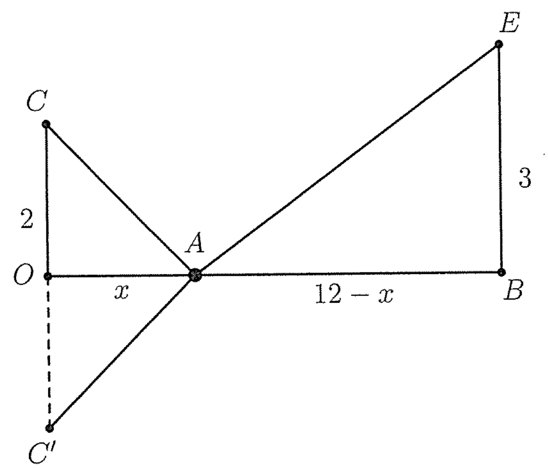
$$\begin{array}{cccc} & 1 & 2 & 5 \\ \times & & 3 & 7 \\ \hline & 8 & 7 & 5 \\ 3 & 7 & 5 & \\ \hline 4 & 6 & 2 & 5. \end{array}$$

20. Answer: 13

Consider the following diagram where $|OB| = 12$, and both $|OC| = 2$ and $|BE| = 3$ are perpendicular to OB. If $|OA| = x$ and C' is the reflection of C about the line OB, then the required expression written as

$$\sqrt{x^2 + 2^2} + \sqrt{(12 - x)^2 + 3^2}$$

represents the distance $|AC'| + |AE|$.



This distance is minimized when C', A and E are collinear giving us the minimum value of

$$\sqrt{12^2 + 5^2} = 13.$$

21. Answer: 73

There are 7 two digit twelvish integers, namely

$$\{39, 48, 57, 66, 75, 84, 93\}.$$

Each of the above correspond to another two twelvish integers that contain 0 as its second or third digit respectively. For example, 48 gives rise to 408 and 480. This brings the subtotal to $7 + 14 = 21$.

We now consider three digit twelvish integers that contain 1 as one of its digits. The remaining two digits can only be one of these four pairs $(2, 9)$, $(3, 8)$, $(4, 7)$ or $(5, 6)$. Note that all digits are distinct and there are 6 ways to arrange them, giving a total of $4 \times 6 = 24$ twelvish integers which contain 1.

Next we consider three digit twelvish integers with repeated digits. Other than 444, the other possibilities are $(2, 2, 8)$, $(3, 3, 6)$ and $(2, 5, 5)$. In each case, there are three ways to arrange them. So we have $1 + 9 = 10$ twelvish integers with repeated digits. (Note the $(0, 6, 6)$ case was already counted.)

Finally the remaining three digit twelvish integers must be permutations of $(2, 3, 7)$, $(2, 4, 6)$ and $(3, 4, 5)$. Each has six arrangements, giving a total 18.

The final tally is $21 + 24 + 10 + 18 = 73$.

22. Answer: 8

Let w denote that number of matches where one contestant won and d denote the number of matches that were drawn. We have $130 = 3w + 2d$. Furthermore,

$$w + d = \binom{m+2}{2} = \frac{(m+2)(m+1)}{2},$$

and thus combining the two equations gives $w = 130 - (m + 2)(m + 1)$. Now w must be more than half of the total matches played, and at same time be less than the total. So

$$\frac{(m+2)(m+1)}{4} < 130 - (m+2)(m+1) \leq \frac{(m+2)(m+1)}{2}.$$

We can reduce the above to

$$87 \leq (m+2)(m+1) < 104.$$

The only positive integer m that satisfies the above is $m = 8$.

23. Answer: 12

Since $\angle EA'H = 90°$, we have $\angle DEA' = \angle CA'H$ and so $\triangle DEA'$ and $\triangle CA'H$ are similar. Let $|DA'| = x$ and $|DE| = y$, then $|A'C| = 6 - x$ and $|EA'| = 6 - y$. We have

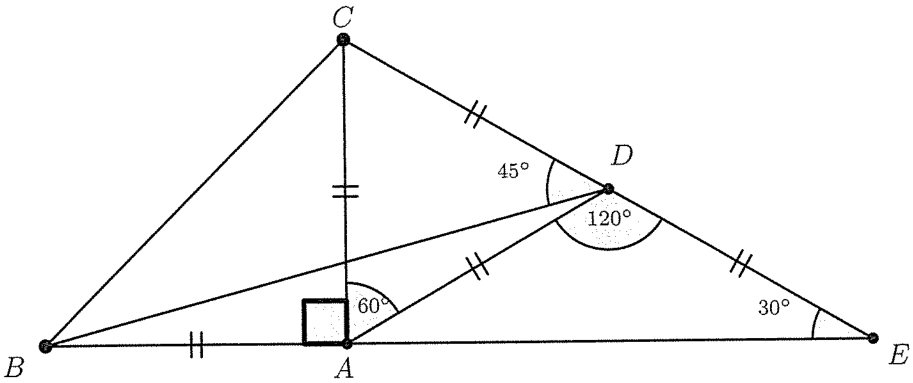
$$(6 - y)^2 = x^2 + y^2 \implies 36 - x^2 = 12y.$$

Since the perimeter of $\triangle DEA' = 6 + x$, by similarity, the perimeter of $\triangle A'CH$ is

$$(6 + x) \times \frac{6 - x}{y} = \frac{36 - x^2}{y} = \frac{12y}{y} = 12.$$

24. Answer: 105

Construct from D a line segment of length $|DE|$, intersecting BE at A to form an isosceles triangle $\triangle AED$. Since $\angle ADE = 120°$, then $\angle ADC = 60°$. Since $\angle BDC = 45°$, we have $\angle ADB = 15°$. Thus $\triangle ABD$ is also isosceles and $\angle BAD = 150°$. Construct CA to form an equilateral triangle because $\angle ADC = 60°$. Hence $\angle BAC = 90°$ and $\triangle BAC$ is an isosceles right-angled triangle. Thus $x° = 105°$.



25. Answer: 20

Let $\angle P_2AP_1 = x$. Since $|AP_1| = |P_1P_2|$, $\triangle AP_1P_2$ is isosceles and so $\angle AP_2P_1 = \angle P_2AP_1$. We can now deduce that the exterior angle

$$\angle P_2P_1P_7 = \angle AP_2P_1 + \angle P_2AP_1 = 2x.$$

Now $\triangle P_1P_2P_3$ is isosceles and so $\angle P_2P_3P_1 = 2x$. We can then deduce that the exterior angle

$$\angle P_4P_2P_3 = 2\angle P_2P_3P_1 - \angle AP_2P_1 = 3x.$$

Similarly, $\triangle P_2P_3P_4$ is isosceles and so $\angle P_2P_4P_3 = 3x$. We can then deduce that the exterior angle

$$\angle P_4P_3P_5 = 2\angle P_2P_4P_3 - \angle P_2P_3P_1 = 4x.$$

Finally, $\angle P_4P_5P_3 = \angle P_4P_3P_5 = 4x$.

An analogous argument establishes that $\angle P_5P_4P_6 = 4x$. So in $\triangle AP_4P_5$, we have the sum of interior angles equals to $9x$ which means $x = 20°$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2019

(Junior Section, Round 2)

Saturday, 29 June 2019 **0900-1200**

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1. In the triangle ABC, $AC = BC$, $\angle C = 90^\circ$, D is the midpoint of BC, E is the point on AB such that AD is perpendicular to CE. Prove that $AE = 2EB$.

2. There are 315 marbles divided into three piles of 81, 115 and 119. In each move Ah Meng can either merge several piles into a single pile or divide a pile with an even number of marbles into 2 equal piles. Can Ah Meng divide the marbles into 315 piles, each with a single marble?

3. Find all positive integers m, n such that $\frac{2m-1}{n}$ and $\frac{2n-1}{m}$ are both integers.

4. Let $a > b > 0$. Prove that

$$\sqrt{2}a^3 + \frac{3}{ab - b^2} \geq 10.$$

When does equality hold?

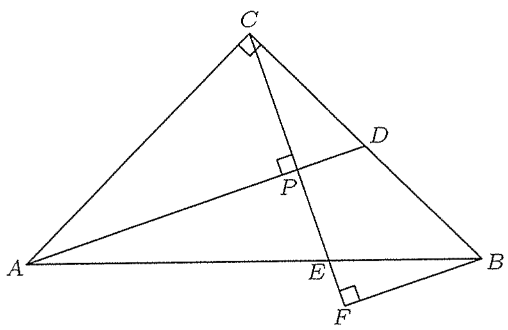
5. Let n be a positive integer and consider an arrangement of $2n$ blocks in a straight line, where n of them are red and the rest blue. A *swap* refers to choosing two consecutive blocks and then swapping their positions. Let A be the minimum number of swaps needed to make the first n blocks all red and B be the minimum number of swaps needed to make the first n blocks all blue. Show that $A + B$ is independent of the starting arrangement and determine its value.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2019

(Junior Section, Round 2 solutions)

1. Let AD intersect CE at P. Then $\angle APC = 90°$. It follows that $AP \cdot PD = CP^2$. Thus $AP/PD = (AP \cdot PD)/PD^2 = CP^2/PD^2 = AC^2/CD^2 = 4$. Let F be the foot of the perpendicular from B onto the line CE. Then PD is parallel to FB and $BF = 2DP$ since D is the midpoint of BC. Also the triangle APE and BFE are similar. Thus $AE/EB = AP/BF = AP/(2PD) = 2$.



2. The answer is no. First observe that when all the piles are multiples of an odd prime p, then after the next move, all the piles are still multiples of p. Since at the start, all piles are odd, Ah Meng can only merge two of the piles, ending with one of the following pairs: (81, 234), (115, 200), (119, 196), which are multiples of 3, 5 and 7 respectively. Therefore it is not possible to obtain a pile with a single marble.

3. Suppose m, n are such integers. Then there exist integers s, t such that $2m - 1 = sn$ and $2n - 1 = tm$. Then

$$(4 - st)m = s + 2 \quad \text{and} \quad (4 - st)n = t + 2.$$

Since $s+2 > 0$, $4-st > 0$. Thus $st < 4$. If $st = 1$, then $s = t = 1$ and we get $m = n = 1$. Since s, t are both odd, $st \neq 2$. If $st = 3$, then we get $s = 1, t = 3$ or $s = 3, t = 1$. From these we get $m = 3, n = 5$ or $m = 5, n = 3$.

4. We have

$$ab - b^2 = b(a-b) \le \left(\frac{b + (a-b)}{2}\right)^2 = \frac{a^2}{4}.$$

Thus

$$\begin{aligned} \text{LHS} &\ge \sqrt{2}a^3 + \frac{12}{a^4} \\ &= \frac{a^3}{\sqrt{2}} + \frac{a^3}{\sqrt{2}} + \frac{4}{a^2} + \frac{4}{a^2} + \frac{4}{a^2} \\ &\ge 5\left(\frac{a^6 4^3}{2a^6}\right)^{1/5} = 10. \end{aligned}$$

Equality holds when $b = a - b$ and $a^3/\sqrt{2} = 4/a^2$. Solving these we get $a = \sqrt{2}$, $b = \sqrt{2}/2$.

5. For each $i = 1, \ldots, n$ let r_i be the number of blue blocks to the left of the i^{th} red block (counting from the left) and b_i be the number of red blocks to the i^{th} blue block. Let $X = \sum r_i$ and $Y = \sum b_i$. It is eady to see that $A \le X$ as we can move the first red block to the left in r_i swaps, the 2nd red block to the left in r_2 swaps, etc. Also since the i^{th} red block has to swap positions with the r_i blue blocks to its left, we see that $A \ge X$. Therefore $A = X$. Likewise $B = Y$. Next we note X is simply the number of pairs (B_i, R_j) where the the blue block B_i is to the left of the red block R_j. Likewise Y is the number of pairs (R_i, B_j) where the the red block R_i is to the left of the blue block B_j. Thus $X + Y$ is the number of pairs $\{R_i, B_j\}$. Thus $X + Y = n^2$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2019

Senior Section (Round 1)

Tuesday, 4 June 2019 — 0930 – 1200 hrs

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1. The roots of the quadratic equation $x^2 - 7mx + 5n = 0$ are m and n, where $m \neq 0$ and $n \neq 0$. Find a quadratic equation whose roots are $\frac{m}{n}$ and $\frac{n}{m}$.

   (A) $6x^2 - 37x + 1 = 0$ (B) $6x^2 - 50x - 7 = 0$ (C) $6x^2 - 50x + 7 = 0$

   (D) $6x^2 - 37x + 6 = 0$ (E) $x^2 - 37x + 1 = 0$

2. Simplify
$$\left(\sqrt{10} - \sqrt{2}\right)^{\frac{1}{3}} \left(\sqrt{10} + \sqrt{2}\right)^{\frac{7}{3}}.$$

   (A) $24 + 4\sqrt{5}$ (B) $24 + 6\sqrt{5}$ (C) $24 + 8\sqrt{5}$ (D) $24 + 10\sqrt{5}$ (E) $24 + 12\sqrt{5}$

3. Let $a = 4^{3000}$, $b = 6^{2500}$ and $c = 7^{2000}$. Which of the following statement is true?

   (A) $a < b < c$ (B) $a < c < b$ (C) $b < a < c$

   (D) $c < a < b$ (E) $c < b < a$

4. If $\log_{21} 3 = x$, express $\log_7 9$ in terms of x.

   (A) $\dfrac{2x}{2-x}$ (B) $\dfrac{2x}{1-x}$ (C) $\dfrac{2x}{x-2}$ (D) $\dfrac{2x}{x-1}$ (E) $\dfrac{x}{1-x}$

5. Suppose that $\sin x = \dfrac{12}{13}$ and $\cos y = -\dfrac{4}{5}$, where $0° \leq x \leq 90°$ and $90° \leq y \leq 180°$. Find the value of $\cos(x+y)$.

   (A) $-\dfrac{56}{65}$ (B) $\dfrac{56}{65}$ (C) $-\dfrac{16}{65}$ (D) $\dfrac{16}{65}$ (E) None of the above

Short Questions

6. Find the largest positive integer n such that $n+8$ is a factor of $n^3 + 13n^2 + 40n + 40$.

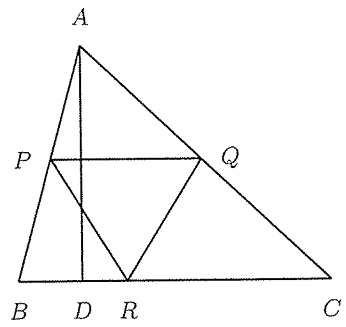
7. Suppose $\tan x = 5$. Find the value of $\dfrac{6 + \sin 2x}{1 + \cos 2x}$.

8. Suppose x and y are real numbers such that
$$\begin{aligned} |x-y| + 3x - y &= 70, \text{ and} \\ |y-x| + 3y + x &= 50. \end{aligned}$$

   Find the maximum possible value of $x + 2y$.

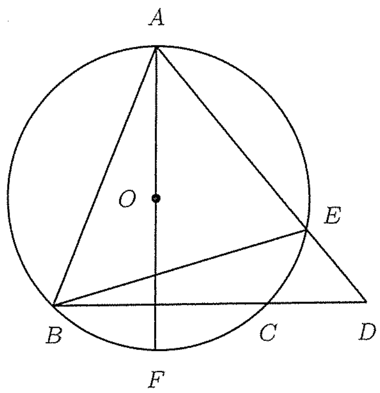
9. The coordinates of the vertices of a triangle $\triangle ABC$ are $A(6,0)$, $B(0,8)$ and $C(x,y)$ such that $x^2 - 16x + y^2 - 12y + 91 = 0$. Find the largest possible value of the area of the triangle $\triangle ABC$.

10. In the figure below, AD is perpendicular to the BC, PQ is parallel to BC, and the triangle $\triangle PQR$ is an equilateral triangle whose area (in meter^2) is equal to the length of AD (in meter). Find the smallest possible value of the length of BC.



11. Find the value of $448\left(\dfrac{\sin 12^\circ \sin 39^\circ \sin 51^\circ}{\sin 24^\circ}\right)$.

12. In the figure below, the chord AF passes through the origin O of the circle, and is perpendicular to the chord BC. It is given that $AB = 17$ cm, $CD = 5$ cm. Suppose $\dfrac{BE}{ED} = \dfrac{m}{n}$, where m and n are positive integers which are relatively prime. What is the value of $m+n$?



13. Let $P(x)$ be the polynomial that results from the expansion of the following expression:

$$(2x^3 + 3x^2 + x)^5 \left(\frac{x}{6} + \frac{1}{2}\right)^5.$$

Find the sum of the coefficients of x^{2k+1}, where $k = 0, 1, 2, 3, \ldots, 9$.

14. Find the value of the following expression:

$$\frac{2\left(1^2+2^2+3^2+\ldots+49^2+50^2\right)+(1\times 2)+(2\times 3)+(3\times 4)+\ldots+(48\times 49)+(49\times 50)}{100}.$$

15. Let M be the maximum possible value of $\dfrac{15x^2-x^3-39x-54}{x+1}$, where x is a positive integer. Find the value of $9M$.

16. Find the maximum possible value of $x+y+z$ where x, y, z are integers satisfying the following system of equations:

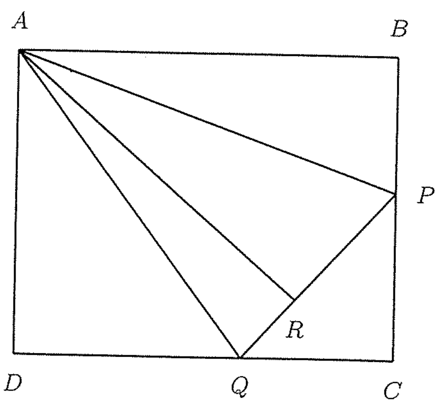
$$\begin{aligned} x^2z+y^2z+8xy &= 200, \\ 2x^2+2y^2+xyz &= 50. \end{aligned}$$

17. Find the remainder when 10^{43} is divided by 126.

18. Suppose $(\log_2 x)^2+4(\log_2(\log_3 y))^2 = 4(\log_2 x)(\log_2(\log_3 y))$. If $x=49$ and y is a positive integer, find y.

19. The figure below shows a rectangle $ABCD$ with $AB=16$ cm and $BC=15$ cm. Let P be a point on the side BC such that $BP=7$ cm, and let Q be a point on the side CD such that $CQ=6$ cm.

Find the length of AR (in cm), where R is the foot of the perpendicular from A to PQ.



20. A sequence $x_0, x_1, x_2, x_3, \ldots$ of integers satisfies the following conditions: $x_0=1$, and for any positive integer $n\geq 1$, $|x_n-1|=|x_{n-1}+2|$. Find the maximum possible value of $2019-(x_1+x_2+\cdots+x_{2018})$.

21. Consider a square $ABCD$ on the xy-plane where the coordinates of its vertices are given by $A(13,0)$, $B(23,13)$, $C(10,23)$ and $D(0,10)$. A lattice point is a point with integer coordinates. Find the number of lattice points in the interior of the square.

22. Eleven distinct chemicals $C_1, C_2, \ldots, C_{11}$ are to be stored in three different warehouses. Each warehouse stores at least one chemical. A pair $\{C_i, C_j\}$ of chemicals, where $i \neq j$, is either compatible or incompatible. Any two incompatible chemicals cannot be stored in the same warehouse. However, a pair of compatible chemicals may or may not be stored in the same warehouse. Find the maximum possible number of pairs of incompatible chemicals that can be found among the stored chemicals.

23. Let k be a positive integer and let the function f be defined as follows:

$$f(x) = \frac{\pi^x}{\pi^x + \pi^{2k-x}}.$$

Suppose the function $g(k)$ is defined as follows:

$$g(k) = f(0) + f\left(\frac{k}{2019}\right) + f\left(\frac{2k}{2019}\right) + f\left(\frac{3k}{2019}\right) + \ldots + f\left(\frac{4037k}{2019}\right) + f(2k).$$

Find the greatest positive integer n such that $g(k) \geq n$ for all $k \geq 1$.

24. Some students sat for a test. The first group of students scored an average of 91 marks and were given Grade A. The second group of students scored an average of 80 marks and were given Grade B. The last group of students scored an average of 70 marks and were given Grade C. The numbers of students in all three groups are prime numbers and the total score of all the students is 1785. Determine the total number of students.

25. Suppose a and b are positive integers satisfying

$$a^2 - 2b^2 = 1.$$

If $500 < a + b < 1000$, find $a + b$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2019

Senior Section (Round 1 Solutions)

1. **Answer.** (D)

   Solution. Note that $m + n = 7m$ (E1), $mn = 5n$ (E2). Since $n \neq 0$, it follows from (E2) that $m = 5$. So, from (E1), $n = 6m = 30$. Thus,

$$\begin{aligned} \frac{m}{n} + \frac{n}{m} &= \frac{5}{30} + \frac{30}{5} \\ &= \frac{1}{6} + 6 = \frac{37}{6}, \\ \frac{m}{n} \cdot \frac{n}{m} &= 1. \end{aligned}$$

   A quadratic equation whose roots are $\frac{m}{n}$ and $\frac{n}{m}$ is

$$\begin{aligned} x^2 - \left(\frac{m}{n} + \frac{n}{m}\right)x + \frac{m}{n} \cdot \frac{n}{m} &= 0 \\ x^2 - \frac{37}{6}x + 1 &= 0 \\ 6x^2 - 37x + 6 &= 0. \end{aligned}$$

2. **Answer.** (C)

   Solution.

$$\begin{aligned} \left(\sqrt{10} - \sqrt{2}\right)^{\frac{1}{3}} \left(\sqrt{10} + \sqrt{2}\right)^{\frac{7}{3}} &= \left(\sqrt{10} - \sqrt{2}\right)^{\frac{1}{3}} \left(\sqrt{10} + \sqrt{2}\right)^{\frac{1}{3}} \left(\sqrt{10} + \sqrt{2}\right)^{2} \\ &= \left((\sqrt{10} - \sqrt{2})(\sqrt{10} + \sqrt{2})\right)^{\frac{1}{3}} \left(\sqrt{10} + \sqrt{2}\right)^{2} \\ &= (10-2)^{\frac{1}{3}}(12 + 2\sqrt{20}) \\ &= 2(12 + 4\sqrt{5}) \\ &= 24 + 8\sqrt{5}. \end{aligned}$$

3. **Answer.** (D)

   Solution. Note that $a = 4^{3000} = (4^3)^{1000} = 64^{1000}$, $c = 7^{2000} = (7^2)^{1000} = 49^{1000} < 64^{1000} = a$. Thus, $c < a$. On the other hand, $b = 6^{2500} = (6^{5/2})^{1000}$. Since $6^5 = 7776 > 4^6 = 4096 \Longrightarrow 6^{5/2} > 4^3$, we deduce that $b > a$. Hence, $c < a < b$.

4. **Answer.** (B)

Solution. Note that $\frac{1}{x} = \log_3 21 = \log_3 7 + \log_3 3 = \log_3 7 + 1$. So $\log_3 7 = \frac{1}{x} - 1$.

$$\begin{aligned}
\log_7 9 &= \log_7 3^2 \\
&= 2 \cdot \frac{1}{\log_3 7} \\
&= \frac{2}{\frac{1}{x} - 1} \\
&= \frac{2x}{1-x}.
\end{aligned}$$

5. **Answer.** (A)

Solution.

$$\begin{aligned}
\cos(x+y) &= \cos x \cos y - \sin x \sin y \\
&= \frac{5}{13}\left(-\frac{4}{5}\right) - \frac{12}{13}\left(\frac{3}{5}\right) = -\frac{56}{65}
\end{aligned}$$

6. **Answer.** 32

Solution. By long division, we have

$$\frac{n^3 + 13n^2 + 40n + 40}{n+8} = n^2 + 5n + \frac{40}{n+8}.$$

Since $n+8$ is a factor, $\frac{40}{n+8}$ must be an integer. So the largest positive value of n is 32.

7. **Answer.** 83

Solution.

$$\begin{aligned}
\frac{6 + \sin 2x}{1 + \cos 2x} &= \frac{6 + 2\sin x \cos x}{1 + 2\cos^2 x - 1} \\
&= 3\sec^2 x + \tan x \\
&= 3(\tan^2 x + 1) + \tan x \\
&= 3(26) + 5 \\
&= 83.
\end{aligned}$$

8. **Answer.** 30

Solution. If $x \geq y$, then we have $4x - 2y = 70$ and $2x + 2y = 50 \Longrightarrow x = 20$, $2y = 10$ whence $x + 2y = 30$. If $x < y$, then we have $2x = 70$ and $4y = 50 \Longrightarrow x = 35$, $2y = 25$ $\Longrightarrow x > y$, contradicting the assumption that $x < y$. Hence, the maximum possible value is 30.

9. **Answer.** 41

Solution. By completing squares, we notice that point C lies on the circle ω of radius 3 centred at $P(8,6)$ whose equation is given by $(x-8)^2+(y-6)^2=3^2$. Let O denote the origin $(0,0)$. Then the line OP is perpendicular to the line AB. Let Q be the intersection point of AB and OP. By considering the area of the triangle $\triangle OAB$, we have

$$\frac{1}{2}\cdot OA\cdot OB=\frac{1}{2}\cdot OQ\cdot AB,$$

whence $OQ=4.8$.

Let R, S be the two intersection points of the circle ω and the line $y=\frac{3}{4}x$, where S lies above R.

Then $QR=OP-OQ-PR=10-4.8-3=2.2$. The maximum area of the triangle ABC occurs when $C=S$, and so the maximum area is

$$\frac{1}{2}\cdot AB\cdot QS=\frac{1}{2}\cdot 10\cdot(2.2+3+3)=41.$$

10. **Answer.** 8

Solution. Let S be the intersection point of AD and PQ. Let $a=PQ$, $b=BC$. Since $\triangle PQR$ is equilateral, we have $SD=\frac{\sqrt{3}}{2}PQ=\frac{\sqrt{3}a}{2}$. By the similarity of the triangles $\triangle APQ$ and $\triangle ABC$, we have

$$\frac{a}{b}=\frac{PQ}{BC}=\frac{AS}{AD}=\frac{AD-SD}{AD}=\frac{AD-\frac{\sqrt{3}a}{2}}{AD}=1-\frac{\sqrt{3}a}{2AD}.$$

It is given that the area of the triangle $\triangle PQR$ is

$$AD=\frac{1}{2}PQ\cdot SD=\frac{1}{2}(PQ)\frac{\sqrt{3}}{2}(PQ)=\frac{\sqrt{3}}{4}a^2.$$

Thus,

$$\begin{aligned}\frac{a}{b}&=1-\frac{\sqrt{3}a}{2\cdot\frac{\sqrt{3}}{4}a^2}\\&=1-\frac{2}{a}\\a^2&=ab-2b\\a^2-ba+2b&=0,\end{aligned}$$

which has real solutions (for a) provided $b^2-4(2b)\geq 0\implies b(b-8)\geq 0$. Since $b>0$, we deduce that $b\geq 8$. The smallest possible value for b is 8, since we can take $b=8$, $a=4$ and $AD=4\sqrt{3}$.

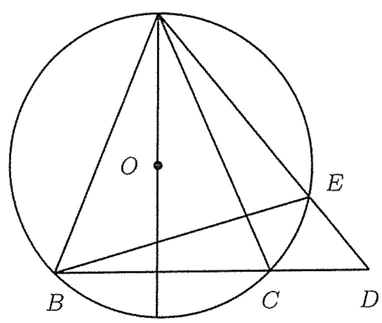
11. **Answer.** 112

Solution.

$$
\begin{aligned}
448\left(\frac{\sin 12^\circ \sin 39^\circ \sin 51^\circ}{\sin 24^\circ}\right) &= 224\left(\frac{2\sin 12^\circ \cos 12^\circ \sin 39^\circ \sin 51^\circ}{\sin 24^\circ \cos 12^\circ}\right) \\
&= 224\left(\frac{\sin 24^\circ \sin 39^\circ \sin 51^\circ}{\sin 24^\circ \cos 12^\circ}\right) \\
&= 112\left(\frac{2\sin 39^\circ \cos 39^\circ}{\cos 12^\circ}\right) \\
&= 112\left(\frac{\sin 78^\circ}{\cos 12^\circ}\right) \\
&= 112\left(\frac{\cos 12^\circ}{\cos 12^\circ}\right) = 112.
\end{aligned}
$$

12. **Answer.** 22

Solution. Adding the chord AC, we notice that $\angle CAD = \angle EBD$, and so the triangles $\triangle ACD$ and $\triangle BED$ are similar. This implies that $\frac{AC}{CD} = \frac{BE}{ED}$. Since $AC = AB$, we deduce that $\frac{BE}{ED} = \frac{AB}{CD} = \frac{17}{5}$, whence $m + n = 17 + 5 = 22$.



13. **Answer.** 512

Solution. Let $P(x) = \sum_{n=0}^{20} c_n x^n$. Then

$$
\begin{aligned}
4^5 = P(1) &= \sum_{n=0}^{20} c_n, && (1) \\
0 = P(-1) &= \sum_{n=0}^{20} c_n(-1)^n. && (2)
\end{aligned}
$$

Subtracting (2) from (1), we have

$$4^5 - 0 = 2(c_1 + c_3 + c_5 + \cdots + c_{19}),$$

whence

$$c_1 + c_3 + c_5 + \cdots + c_{19} = 512.$$

14. **Answer.** 1275

Solution. We first note that $a^3 - b^3 = (a-b)(a^2+ab+b^2)$. Hence, we have the following.

$$\begin{aligned}
50^2 &= 50^2 \\
50^2 + 50 \times 49 + 49^2 &= (50-49)(50^2 + 50 \times 49 + 49^2) = 50^3 - 49^3 \\
49^2 + 49 \times 48 + 48^2 &= (49-48)(49^2 + 49 \times 48 + 48^2) = 49^3 - 48^3 \\
48^2 + 48 \times 47 + 47^2 &= (48-47)(48^2 + 48 \times 47 + 47^2) = 48^3 - 47^3 \\
&\vdots \\
3^2 + 3 \times 2 + 2^2 &= (3-2)(3^2 + 3 \times 2 + 2^2) = 3^3 - 2^3 \\
2^2 + 2 \times 1 + 1^2 &= (2-1)(2^2 + 2 \times 1 + 1^2) = 2^3 - 1^3 \\
1^2 &= 1^2
\end{aligned}$$

Summing up all the above expressions, we obtain the following:

$$2\left(1^2 + 2^2 + 3^2 + \ldots + 49^2 + 50^2\right) + (1\times 2) + (2\times 3) + (3\times 4) + \ldots + (48\times 49) + (49\times 50)$$

$$= 50^2 + 50^3 - 1^3 + 1^2 = 125000 + 2500 = 127500.$$

Hence, dividing the above by 100, we obtain the answer 1275.

15. **Answer.** 82

Solution. By long division, we have

$$f(x) = g(x) + \frac{1}{x+1}, \text{ where } g(x) = -(x-5)(x-11).$$

The quadratic function $g(x)$ achieves its maximum at $x = 8$. Since $\frac{1}{x+1}$ decreases for all $x \geq 8$, we deduce that $f(x) \leq f(8)$ for all $x \geq 8$. On the other hand, $\frac{1}{x+1}$ achieves its maximum at $x = 1$. Since the quadratic function $g(x)$ is increasing on $[1,7]$, we deduce that $f(x) = g(x) + \frac{1}{x+1} \leq g(7) + \frac{1}{1+1} = 8 + \frac{1}{2} < f(8) = 9 + \frac{1}{9}$. We deduce that $f(x) \leq f(8)$ for all positive integers x.

Hence $M = 9 + \frac{1}{9} = \frac{82}{9}$. Thus, $9M = 82$.

16. **Answer.** 13

Solution Multiplying the equation $2x^2 + 2y^2 + xyz = 50$ by 2 and subtract it from the first equation, we have

$$x^2z + y^2z + 8xy - 4x^2 - 4y^2 - 2xyz = 100$$

$$(x-y)^2(z-4) = 2^2 5^2.$$

So $(x-y)^2$ is either 2^2 or 5^2 or 10^2.

If $(x-y)=2^2$, then $z-4=25 \Longrightarrow z=29$. It follows from the equation $2x^2+2y^2+xyz=50$ that $x^2+y^2+\frac{29xy}{2}=25 \Longrightarrow (x-y)^2+2xy+\frac{29xy}{2}=25 \Longrightarrow \frac{33xy}{2}=21$, contradicting the fact that x and y are integers.

If $(x-y)=5^2$, then $z-4=4 \Longrightarrow z=8$. It follows from the equation $2x^2+2y^2+xyz=50$ that $x^2+y^2+\frac{8xy}{2}=25 \Longrightarrow (x-y)^2+2xy+4xy=25 \Longrightarrow 6xy=0$. If $x=0$, then $y=\pm 5$; if $y=0$ then $x=\pm 5$. The solutions are $(x,y,z)=(0,\pm 5,8)$, $(x,y,z)=(\pm 5,0,8)$.

If $(x-y)=10^2$, then $z-4=1 \Longrightarrow z=5$. It follows from the equation $2x^2+2y^2+xyz=50$ that $x^2+y^2+\frac{5xy}{2}=25 \Longrightarrow (x-y)^2+2xy+\frac{5xy}{2}=25 \Longrightarrow \frac{9xy}{2}=-75$, contradicting the fact that x and y are integers.

Hence, the maximum possible value of $x+y+z$ is $5+0+8=13$.

17. **Answer.** 10

Solution

We first note that $10^{42}=2^{42}\times 5^{42}=2^{7\times 6}\times 5^{3\times 14}=128^6\times 125^{14}=(126+2)^6\times(126-1)^{14}$.

Removing multiples of 126, the remainder will be congruent to $2^6\times(-1)^{14}\equiv 64 \pmod{126}$.

Hence, $10^{43}\equiv 64\times 10\equiv 640\equiv 10 \pmod{126}$.

Therefore, 10^{43} divided by 126 gives a remainder of 10.

18. **Answer.** 2187

Solution. Rearranging the equation, we have $(\log_2 x-2\log_2(\log_3 y))^2=0$. Thus,

$$\log_2 x=2\log_2(\log_3 y) \implies x=(\log_3 y)^2 \Longrightarrow y=3^{\sqrt{x}}=3^7=81(27)=2187.$$

19. **Answer.** 17

Solution. Using elementary calculations, the area of the triangle $\triangle ABP$ is $\frac{1}{2}\times 16\times 7=56\text{cm}^2$, the area of the triangle $\triangle CPQ$ is $\frac{1}{2}\times 8\times 6=24\text{cm}^2$, and the area of the triangle $\triangle ADQ$ is $\frac{1}{2}\times 15\times 10=75\text{cm}^2$.

Since the area of rectangle $ABCD$ is $16\times 15=240$ cm^2, the area of triangle APQ is given by $240-75-24-56=85$ cm^2.

By Pythagoras Theorem, $PQ=\sqrt{6^2+8^2}=\sqrt{100}=10$ cm.

Hence, $AR=\frac{2\times\text{Area}}{\text{Base}}=\frac{2\times 85}{10}=17$ cm.

20. **Answer.** 3028

Solution. Squaring both sides of $|x_n - 1| = |x_{n-1} + 2|$, we have

$$\begin{aligned}(x_n - 1)^2 &= (x_{n-1} + 2)^2 \\ x_n^2 - x_{n-1}^2 &= 2x_n + 4x_{n-1} + 3.\end{aligned}$$

Thus,

$$\begin{aligned}x_{2019}^2 - x_{2018}^2 &= 2x_{2019} + 4x_{2018} + 3 \\ x_{2018}^2 - x_{2017}^2 &= 2x_{2018} + 4x_{2017} + 3 \\ &\vdots \\ x_2^2 - x_1^2 &= 2x_2 + 4x_1 + 3 \\ x_1^2 - x_0^2 &= 2x_1 + 4x_0 + 3.\end{aligned}$$

Summing all the equations, we have

$$x_{2019}^2 - x_0^2 = 6(x_1 + x_2 + \cdots + x_{2018}) + 4x_0 + 2x_{2019} + 3(2019).$$

Let $y = x_{2019}$, and $S = x_1 + x_2 + \cdots + x_{2018}$. Then

$$\begin{aligned}y^2 - 1 &= 6S + 4 + 2y + 3(2019) \\ (y-1)^2 &= 6S + 6 + 3(2019). \\ \frac{(y-1)^2}{3} &= 2S + 2021.\end{aligned}$$

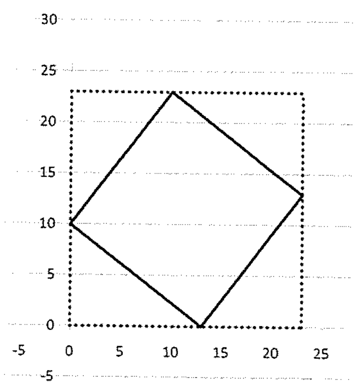
Therefore, $\frac{(y-1)^2}{3}$ must be an integer. Also, S is minimum if and only if $\frac{(y-1)^2}{3}$ is minimum. Note that $\frac{(y-1)^2}{3}$ cannot be 0; otherwise, $2S + 2021 = 0$ implies that S is not an integer. The smallest possible value of $\frac{(y-1)^2}{3}$ is 3, whence $(y-1)^2 = 9$. This is attainable by choosing $x_{2k} = 1$, $x_{2k+1} = -2$ for all $k = 0, 1, 2, \ldots$.

Hence, the maximum possible value M of $2019 - S$ is attained when S achieves its minimum possible value, i.e. $M = 2019 - \left(\frac{3-2021}{2}\right) = 2019 - (-1009) = 3028$.

21. **Answer.** 268

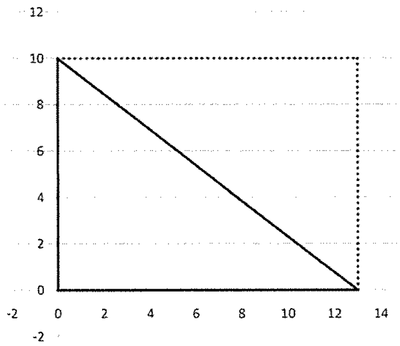
Solution.

Figure below shows the imbedding of square into a square with sides parallel to the axes, which is represented as a larger square with dotted sides.



There are $(13 + 10 - 1) \times (13 + 10 - 1) = 22 \times 22 = 484$ lattice points in the interior of the square with dotted lines.

We need to remove the number of lattice points in the interior of the four congruent triangles to calculate the number of lattice points in the interior of the square. Figure below shows one of the four congruent triangles imbedded into a rectangle with sides parallel to the axes.



There are $(13 - 1) \times (10 - 1) = 12 \times 9 = 108$ lattice points in the rectangle and so there are $\frac{108}{2} = 54$ lattice points in the triangle (note: no interior lattice points lie on the sides of the triangle).

Hence, there are $484 - 4 \times 54 = 268$ lattice points in the square with vertices at $(13, 0)$, $(23, 13)$, $(10, 23)$ and $(0, 10)$.

22. **Answer.** 40

Solution. Let W_1, W_2, W_3 be the three warehouses containing a, b, c chemicals respectively. Without loss of generality, assume that $1 \leq a \leq b \leq c$. Note that $a + b + c = 11$, $a \geq 1$, $b \geq 1$, $c \geq 1$.

There cannot be any pair of incompatible chemicals in the same warehouse. So any pair of incompatible chemicals must come from different warehouses. Let e be the total number of pairs of incompatible chemicals. This implies that

$$e \leq ab + bc + ac.$$

We also know that $a + b + c = 11$, and so $c = 11 - a - b$. Thus,

$$e \leq ab + (b + a)(11 - a - b) = ab + 11b + 11a - (a + b)^2 = 11a + 11b - a^2 - b^2 - ab.$$

If a is fixed, then quadratic expression $11a + 11b - a^2 - b^2 - ab$ is maximized when $11 - 2b - a = 0$, i.e. $2b = 11 - a$, $b = (11 - a)/2$. Since b is an integer, for our purpose, we can take $b = \lfloor (11 - a)/2 \rfloor$.

Also, $a \leq 3$ (otherwise, $a \geq 4$ implies that both b and c are also at least 4 and thus the number of chemicals would exceed 11).

So we have the following table for comparison, other possibilities will not give more edges:

a	b	$c = 11 - a - b$	$ab + bc + ac$
1	5	5	35
2	4	5	38
3	4	4	40

Thus, $e \leq 40$, and moreover this is attainable when the three warehouses have 3, 4, 4 chemicals respectively, such that no two chemicals from a warehouse are incompatible.

23. **Answer.** 2019

Solution. We first note that

$$\begin{aligned} f(x) + f(2k - x) &= \frac{\pi^x}{\pi^x + \pi^{2k-x}} + \frac{\pi^{2k-x}}{\pi^{2k-x} + \pi^{2k-(2k-x)}} \\ &= \frac{\pi^x}{\pi^x + \pi^{2k-x}} + \frac{\pi^{2k-x}}{\pi^{2k-x} + \pi^x} = 1 \end{aligned}$$

Thus we have the following.

$$\begin{aligned} f(0) + f(2k) &= 1 \\ f\left(\frac{k}{2019}\right) + f\left(\frac{4037k}{2019}\right) &= 1 \\ f\left(\frac{2k}{2019}\right) + f\left(\frac{4036k}{2019}\right) &= 1 \\ &\vdots \\ f\left(\frac{2018k}{2019}\right) + f\left(\frac{2020k}{2019}\right) &= 1 \\ f\left(\frac{2019k}{2019}\right) = f(k) &= \frac{\pi^k}{\pi^k + \pi^k} = \frac{1}{2} \end{aligned}$$

Summing all the above expressions, we have $g(k) = 2019\frac{1}{2}$ for all $k \geq 1$.

Hence, $g(k) \geq 2019$ for all $k \geq 1$, that is, $n = 2019$.

24. **Answer.** 23

Solution. Let p, q and r be prime numbers that denote the number of students in the first, second and third groups respectively. Since the total score of all the students is 1785, we have $91p + 80q + 70r = 1785$.

We first note that $1,785 = 3 \times 5 \times 7 \times 17$.

Since 80, 70 and 1785 are all divisible by 5 and $91p = 1785 - 80q - 70r$, p must be a prime that is divisible by 5, that is, $p = 5$.

Likewise, since 91, 70 and 1785 are all divisible by 7 and $80q = 1785 - 91p - 70r$, q must be a prime that is divisible by 7, that is, $q = 7$.

Hence, we have $91 \times 5 + 80 \times 7 + 70r = 1785$, that is, $r = 11$. So there is a total of $p + q + r = 5 + 7 + 11 = 23$ students.

25. **Answer.** 985

Solution. If (a, b) is a positive integer solution to $x^2 - 2y^2 = 1$, let $a' = 3a - 4b$ and $b' = 3b - 2a$, then $(a')^2 - 2(b')^2 = a^2 - 2b^2 = 1$. Note that there is no solution (a, b) with $b = 1$, and there is only one solution with $b = 0$, i.e. $(a, b) = (1, 0)$. If $b \geq 2$, then

$$(3a)^2 = 9(1 + 2b^2) > 16b^2 = (4b)^2 \quad \text{and} \quad (2a)^2 = 4(1 + 2b^2) \leq 9b^2 = (3b)^2.$$

Hence, (a', b') is a nonnegative integer solution to $x^2 - 2y^2 = 1$. Note that $a' + b' = a - b < a + b$, and $a = 3a' + 4b'$, $b = 2a' + 3b'$.

Repeating the process, we arrive at a nonnegative integer solution to $x^2 - 2y^2 = 1$, such that either $x = 0$ or $y = 0$. Since there is no solution with $y = 1$, this solution must be $(1, 0)$.

It follows that every nonnegative integer solution to $x^2 - 2y^2 = 1$ can be generated by the iteration $(a_0, b_0) = (1, 0)$ and $(a_{n+1}, b_{n+1}) = (3a_n + 4b_n, 2a_n + 3b_n)$.

One has $(a_n, b_n) = (1, 0), (3, 2), (17, 12), (99, 70), (577, 408), \ldots$.

Since $500 < a + b < 1000$, we have $(a, b) = (577, 408)$ and $a + b = 985$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2019

(Senior Section, Round 2)

Saturday, 29 June 2019 **0900-1300**

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1. In a parallelogram $ABCD$, the bisector of $\angle A$ intersects BC at M and the extension of DC at N. Let O be the circumcentre of the triangle MCN. Prove that $\angle OBC = \angle ODC$.

2. Let n, m be positive integers such that $n \geq 2m+1$. Given n distinct points and nm distinct line segments, each joining a pair of points, prove that there exist distinct points $A_0, A_1, \ldots, A_m$, so that for each $i = 0, 1, \ldots, m-1$, there is a line segment joining A_i to A_{i+1}.

3. Let $a_1, a_2, \ldots, a_{2000}$ be distinct positive integers such that $1 \leq a_1 < a_2 < \cdots < a_{2000} < 4000$ such that the LCM(least common multiple) of any two of them is ≥ 4000. Prove that $a_1 > 1334$.

4. Let m, n, k be positive integers so that $m \geq n$ and $1 + 2 + \cdots + n = mk$. Show that the numbers $1, 2, \ldots, n$ can be partitioned into k groups so that the sum of the numbers in each of the groups is m.

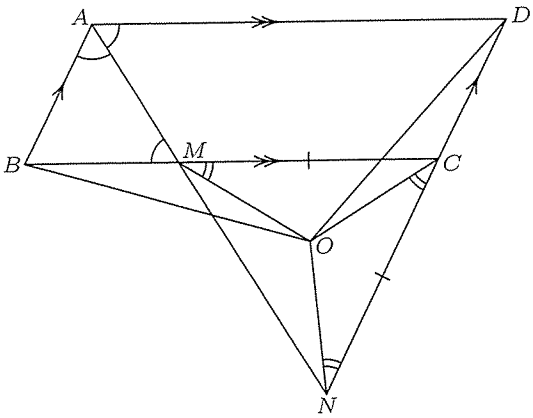
5. Determine all integer $n \geq 2$ such that it is possible to construct an $n \times n$ array where each entry is either 0, 1 or -1 so that the sums of elements in every row and every column are distinct.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2019

(Senior Section, Round 2 solutions)

1.



Since $AD \parallel BC$ and $AB \parallel DC$, we have $\angle CMN = \angle DAM = \angle BAN = \angle CNM$ so that $CM = CN$. Since $OM = ON = OC$, we have $\triangle OCM \equiv \triangle OCN$ so that $\angle OMC = \angle ONC = \angle OCN$. Consequently, $\angle OMB = \angle OCD$. Since $\angle BMA = \angle DAM = \angle BAM$, we have $BM = BA = CD$. As $OM = OC$, we thus have $\triangle OMB \equiv \triangle OCD$ so that $\angle OBM = \angle ODC$. That is $\angle OBC = \angle ODC$.

2. Fix m. We shall prove by induction on n. When $n = 2m+1$, there is a line segment joining every pair of points. Thus the conclusion and the base case holds.

Now suppose that for some $k \geq 2m+1$, the conclusion holds. Consider $k+1$ points and $(k+1)m$ line segments. If there is a point B which is joined to $\leq m$ line segments, then among the remaining k points, there are at least km line segments and the conclusion holds by induction hypothesis.

Now suppose that every point is joined to $\geq m+1$ line segments. Pick any point to be A_0. Pick any point that is joined to A_0 as A_1. Now A_1 is joined to $\geq m$ points besides A_0. Pick any of those points as A_2. This process can be continued until we pick A_m. Thus the inductive step is established and we are done.

3. Write $a_i = 2^{b_i} t_i$ where t_i is odd. Since the LCM is ≥ 4000, $t_1, \ldots, t_{2000}$ are distinct for if $t_i = t_j$ and $i < j$, then $a_i \mid a_j$ and $\operatorname{lcm}(a_i, a_j) = a_j$. Since there are exactly 2000 odd positive integers ≤ 4000, $t_1, \ldots, t_{2000}$ are all the odd integers ≤ 4000.

If $a_1 \leq 1333$, then $3a_1 = 2^{b_1} \times 3t_1 \leq 3999$. Therefore there exists $j > 1$ so that $3t_1 = t_j$. Hence $a_j = 2^{b_j} \times 3t_1$ and $a_1 = 2^{b_1}t_1$. Their LCM is either $3a_1$ or a_j but both are < 4000.

4. Clearly the assertion is true when $m = n$ and $m = n+1$. We shall prove by induction on n. The base case $n = 1$ is trivially true. Now suppose that the assertion is true for each $n < p$ and we shall prove that it is also true for $n = p$.

Case 1: $m \geq 2p$. Here

$$k = \frac{p(p+1)}{2m} \leq \frac{p+1}{4}.$$

Therefore $p \geq 4k - 1$ and so $p - 2k \geq 2k - 1 > 0$. Also $1 + 2 + \cdots + (p - 2k) = (1+2+\cdots+p) - [(p-2k+1) + \cdots + p] = km - k(2p - 2k + 1)$ is divisible by k. Since $p \geq 4k - 1$, we have

$$\frac{1 + 2 + \cdots + (p - 2k)}{p - 2k} = \frac{p - 2k + 1}{2} \geq k$$

or

$$m^* = \frac{1 + 2 + \cdots + (p - 2k)}{k} \geq p - 2k.$$

By the induction hypothesis, the numbers $1, 2, \ldots, p - 2k$ can be partitioned into k groups so that the sum of the elements in each group is m^*. The numbers $p - 2k + 1, p - 2k + 2, \ldots, p$ can be paired into k pairs with each pair summing to $2p - 2k + 1$. Add one pair to each group and we have the desired partition.

Case 2: m is even and $p + 1 < m < 2p$

Case 3: m is odd and $p + 1 < m < 2p$

These two cases are similar.

5. Let a_{ij} be the entry at (i, j), r_i be the sum of row i and c_i be the sum of column i. Since the sum ranges from $-n$ to n , only one of the numbers is not a sum. Therefore we can partitiion the sums into n nonnegative terms and n nonpositive terms. By permuting the rows and the columns, we can assume that $r_i \geq 0$, $i \leq k$, $r_i < 0$ for $i > k$, $c_i \geq 0$ for $i \leq n - k$ and $c_i < 0$ for $i > n - k$. We have

$$\sum_{i=1}^{n} |r_i| + \sum_{i=1}^{n} |c_i| \geq 2\Big(\sum_{i=1}^{n} i\Big) - n = n^2.$$

Next divide the array into 4 subarrays so that A is the lower left $k \times (n - k)$ subarray, B is the top right $(n - k) \times k$ subarray, C is the bottom right $k \times k$ subarray and D is the top left $(n-k)\times(n-k)$ subarray. The contribution of a_{ij} with $(i,j)\in A,B$ is $2a_{ij}$ and is 0 with $(i,j)\in C,D$. Thus we also have

$$\sum_{i=1}^{n}|r_i|+\sum_{i=1}^{n}|c_i|=2\sum_{i=1}^{k}\sum_{j=1}^{n-k}a_{ij}+2\sum_{i=k+1}^{n}\sum_{j=n-k+1}^{n}a_{ij}\le 4k(n-k).$$

Therefore

$$n^2\le 4k(n-k)\quad\Rightarrow\quad (n-2k)^2\le 0.$$

Therefore n must be even.

Finally, we construct, recursively such an array for all even n. When $n=2$, we the have the following:

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Suppose that for an even positive integer k, there exists such an $k\times k$ array A_k where the row sums and column sums are distinct and their values are $-k+1,-k+2,\ldots,k$. We construct the following array A_{k+2}:

$$A_{k+2}=\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ -1 & 0 & -1 & -1 & \cdots & -1 \\ -1 & 1 & & & & \\ -1 & 1 & & & & \\ \vdots & \vdots & & A_k & & \\ -1 & 1 & & & & \\ -1 & 1 & & & & \end{pmatrix}$$

The row sums and column sums are distinct and range in values from $-k-1$ to $k+2$.

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2019
(Open Section, Round 1)

Thursday, 6 June 2019 | 0930-1200 hrs

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

In this paper, let $\lfloor x \rfloor$ denote the greatest integer not exceeding x. For examples, $\lfloor 5 \rfloor = 5$, $\lfloor 2.8 \rfloor = 2$, and $\lfloor -2.3 \rfloor = -3$.

1. Find the largest positive integer k such that $\dfrac{x(k-x)-400}{x^2-x+k-38}$ is always negative for all real values of x.

2. Among all the nonzero real numbers x, y and z satisfying $x+y+z=0$, find the largest possible value of
$$\frac{y+z}{yz}(y^2+z^2-x^2)+\frac{z+x}{zx}(z^2+x^2-y^2)+\frac{x+y}{xy}(x^2+y^2-z^2).$$

3. Assume that $0<\alpha<\frac{\pi}{2}$ and $\cos(\alpha+\frac{\pi}{6})=\frac{4}{5}$. Determine the smallest possible integer R such that $100\sin(2\alpha+\frac{\pi}{12})<R$.

4. Determine the number of integer solutions of the following inequality:
$$x-2019<\log_{10}(x+2019).$$

5. Assume that $(-1,0)$ is a point on the curve $y=ax^2+bx+c$, where a,b,c are three fixed numbers not exceeding $\frac{1}{2}$. If the inequality $x \le ax^2+bx+c \le \frac{1}{2}(x^2+1)$ holds for all real numbers x, what is the value of $2020(a^2+b^2-c^2)$?

6. Determine the maximum value of $x+y$ where x and y are numbers satisfying
$$(x+\sqrt{1+x^2})(y+\sqrt{1+y^2})=1.$$

7. Find the maximum integer m with $m=y+x$ such that x and y are real numbers satisfying $x^2+4y^2+x+y=0$.

8. Among all the real numbers a, b and c satisfying the equation $a+b+c=0$ and $abc=10$, find the largest possible value of $a^3+b^3+c^3$.

9. Suppose that the solution set of the inequality $|x-1|+|x-6| \le 9$ is given by $\{x : a \le x \le b\}$ for some real numbers a and b. Find the value of $b-a$.

10. Let $\{a_n\}$ be an infinite sequence with $a_1=2$ and $a_{n+1}=a_n+4n+2$ for all $n \ge 1$. Determine $\lim\limits_{n\to\infty} 2000\left(\frac{1}{a_2-2}+\frac{1}{a_3-2}+\cdots+\frac{1}{a_n-2}\right)$.

11. Let $a > b > 0$. Let A be the minimum value of

$$2019a + \frac{1}{2019(a-b)b}.$$

Determine $\lfloor A \rfloor$.

12. Let $x_1 + x_2 + \cdots + x_{99} = 9702$ and $x_1^2 + x_2^2 + \cdots + x_{99}^2 = 960498$. Determine the largest possible value of x_1.

13. Find the largest positive integer n satisfying the following condition: there exist n positive integers $a_1, a_2, \ldots, a_n$ such that $|a_i - a_j|$ is a product of two distinct prime integers for all $i \neq j$.

14. Let a, b, c be integers satisfying $ab+bc+ca = 1$ and $(1+a^2)(1+b^2)(1+c^2) = 6923904100$. Determine the value of $|(a+b)(b+c)(c+a)|$.

15. In the triangle ABC, $AB = 209, BC = 171, CA = 190$, M is the midpoint of BC. Point L is on the extension of BA and point N is on the extension of MA such that C, L, N are collinear and $AL = NL$. Find the length of AN.

16. Let $\frac{a}{b}$ be the fraction with the smallest positive denominator such that

$$\frac{386}{2019} < \frac{a}{b} < \frac{35}{183}.$$

Determine the value of $a + b$.

17. Two circles are tangent to each other internally at a point T. Let the chord AB of the larger circle be tangent to the smaller circle at a point P. Suppose $AB = 59/\sqrt{3}, TA = 32$, and $TB = 27$. Find TP.

18. In triangle ABC, $BC = 10, CA = 9, \cos C = 5/14$. The bisector of $\angle C$ meets the circumcircle at $N(\neq C)$, the altitude from A meets the circumcircle at $L(\neq A)$ and NL intersects BC at M. Find the length of MC.

19. Let x_1 be the only real root of $\log_2 x = 2020 - x$ and x_2 be the only real root of $2^x = 2020 - x$. Determine the value of $x_1 + x_2$.

20. Determine the minimum integer x such that the following inequality holds:

$$\sqrt[3]{10x - 200} + \sqrt[3]{10x - 300} + 20x - 500 > 0.$$

21. Let $S = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{e^{k/n}}{n}$. Find $\lfloor S \rfloor$.

22. Two points A and B are randomly selected on a line segment OC. The probability that the segments OA, AB and BC form the sides of a triangle is p. Find the value of $\left\lfloor \frac{1}{p} \right\rfloor$.

(Note: If you think the answer of p is 0, enter your answer as 99999.)

23. Find the largest positive integer k such that 3^k is a factor of the value of the following expression:
$$2019!^2 + 2019!^4 + 2019!^6 + \cdots + 2019!^{2020}.$$

24. Let $S = \{2i - 1 : i = 1, 2, \cdots, 2019\}$. Determine the minimum value of k, where k is a positive integer, such that for any subset A of S with $|A| \geq k$, A contains two numbers a and b with the property that $b = 3^s a$ for some integer $s \geq 1$.

25. Ah Leong is going to fill in each cell of a 7×14 table with one of the four letters S, I, M, O so that in each 2×2 block, exactly one of the 4 letters is used. Determine the number of ways that this can be done.

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2019
(Open Section, Round 1)
(Solution)

1. **Answer.** 39.

   Solution. The expression is always negative only when the numerator is always negative and the denominator always positive. The numerator can be simplified as $-x^2+kx-400$, so that imposing the discriminant to be negative, we obtain $-40 < k < 40$. Imposing the discriminant of the denominator to be negative, we obtain $k > 38\frac{1}{4}$. Combining the two conditions, we have $38\frac{1}{4} < k < 40$, so that the only integer k has to be 39. $\square$

2. **Answer.** 0.

   Solution. By using $y+z=-x$, it can be shown that $\frac{y+z}{yz}(y^2+z^2-x^2)=2x$. Similarly, each of the other two terms in the expression can be expressed as $2y$ and $2z$ respectively. Hence the sum equals $2x+2y+2z=2(x+y+z)=0$. $\square$

3. **Answer.** 49.

   Solution. As $\cos(\alpha+\frac{\pi}{6})=\frac{4}{5}$, we have

$$\begin{aligned}
\sin(2(\alpha+\frac{\pi}{6})) &= 2\sin(\alpha+\frac{\pi}{6})\cos(\alpha+\frac{\pi}{6}) \\
&= 2\sqrt{1-\left(\frac{4}{5}\right)^2}\times\frac{4}{5} \\
&= 2\times\frac{3}{5}\times\frac{4}{5} \\
&= \frac{24}{25},
\end{aligned}$$

and

$$\cos(2(\alpha+\frac{\pi}{6})) = 2\cos^2(\alpha+\frac{\pi}{6})-1=\frac{7}{25}.$$

So

$$\begin{aligned}
\sin(2\alpha+\frac{\pi}{12}) &= \sin(2\alpha+\frac{\pi}{3}-\frac{\pi}{4}) \\
&= \sin(2\alpha+\frac{\pi}{3})\cos\frac{\pi}{4}-\cos(2\alpha+\frac{\pi}{3})\sin\frac{\pi}{4} \\
&= \frac{1}{\sqrt{2}}\left(\sin(2\alpha+\frac{\pi}{3})-\cos(2\alpha+\frac{\pi}{3})\right) \\
&= \frac{1}{\sqrt{2}}(\frac{24}{25}-\frac{7}{25}) \\
&= \frac{1}{\sqrt{2}}\frac{17}{25}.
\end{aligned}$$

Let $X = 100\sin(2\alpha+\frac{\pi}{12}) = 100\sin(2\alpha+\frac{\pi}{12}) = 4\times\frac{17}{\sqrt{2}}$, then $X^2 = 16\times\frac{17^2}{2} = 2312 < 49^2$, but $X^2 > 48^2$. Hence the answer is 49. $\square$

4. **Answer.** 4041.

Solution. Clearly, $x > -2019$ for any solution x of

$$x - 2019 < \log_{10}(x + 2019).$$

Thus, $-2018, -2017, \cdots, 2019$ are certainly solutions of the above inequality. Note the above inequality is equivalent to

$$10^{x-2019} < x + 2019.$$

It is easy to verify that $2020, 2021, 2022$ are solutions of the above inequality, but any integer x with $x \geq 2023$ is not a solution. Hence the answer is $1 + 2018 + 2022 = 4041$. $\square$

5. **Answer.** 505

Solution. As $(-1, 0)$ is a point on the curve $y = ax^2 + bx + c$, we know that $0 = a - b + c$. As the inequality $x \leq ax^2 + bx + c \leq \frac{1}{2}(x^2 + 1)$ holds for all real numbers x, taking $x = 1$ gives that $1 \leq a + b + c \leq \frac{1}{2}(1^2 + 1) = 1$, i.e.,

$$a + b + c = 1.$$

Hence

$$a + c = b = \frac{1}{2}.$$

As $b = \frac{1}{2}$ and $ax^2 + bx + c \leq \frac{1}{2}(x^2 + 1)$ holds for all real numbers x,

$$(\frac{1}{2} - a)x^2 - \frac{1}{2}x + (\frac{1}{2} - c) \geq 0$$

holds for all real numbers x, implying that

$$(\frac{1}{2})^2 - 4(\frac{1}{2} - a)(\frac{1}{2} - c) \leq 0,$$

i.e.,

$$(1 - 2a)(1 - 2c) \geq \frac{1}{4}$$

and

$$(1 - 2a)^{1/2}(1 - 2c)^{1/2} \geq \frac{1}{2}.$$

But

$$(1 - 2a)^{1/2}(1 - 2c)^{1/2} \leq \frac{1}{2}(1 - 2a + 1 - 2c) = (1 - a - c) = \frac{1}{2},$$

where the equality holds if and only if $1 - 2a = 1 - 2c$, i.e., $a = c$. Hence $a = c = \frac{1}{4}$, and $b = \frac{1}{2}$, and the value of

$$2020(a^2 + b^2 - c^2) = 2020(1/2)^2 = 505.$$

$\square$

6. **Answer.** 0.

Solution. Note that

$$\frac{1}{x+\sqrt{1+x^2}} = \frac{1}{x+\sqrt{1+x^2}} \cdot \frac{x-\sqrt{1+x^2}}{x-\sqrt{1+x^2}} = \frac{x-\sqrt{1+x^2}}{x^2-(1+x^2)} = -x+\sqrt{1+x^2}.$$

Thus $y+\sqrt{1+y^2} = -x+\sqrt{1+x^2}$. Similarly, $x+\sqrt{1+x^2} = -y+\sqrt{1+y^2}$. Adding the two equations gives $x+y=0$. Thus $x+y$ is constantly zero. □

7. **Answer.** 0.

As $m = y+3x$, $y = m-x$. By the given condition,

$$x^2 + 4(m-x)^2 + m = 0,$$

which implies that

$$4m^2 + 5m \leq 0.$$

Solving the above inequality gives the maximum integer solution $m = 0$. □

8. **Answer.** 30

Solution. Since $a+b+c=0$, we know that $a+b=-c$, $a+c=-b$ and $b+c=-a$. By considering the expansion of $(a+b+c)^3$, we obtain

$$\begin{aligned}
(a+b+c)^3 &= a^3+b^3+c^3+3a^2b+3a^2c+3a^2d+3b^2a+3b^2c+3b^2d \\
&\quad +3c^2a+3c^2b+3c^2d+3d^2a+3d^2b+3d^2c+6abc \\
&= a^3+b^3+c^3+3a^2(b+c)+3b^2(a+c)+3c^2(a+b)+6abc \\
&= a^3+b^3+c^3+3a^2(-a)+3b^2(-b)+3c^2(-c)+6abc \\
0 &= -2(a^3+b^3+c^3)+6abc
\end{aligned}$$

leading to that $a^3+b^3+c^3 = 30$. □

9. **Answer.** 9.

Solution. Consider the function $f(x) = |x-1| + |x-6|$. It can be expressed as

$$f(x) = \begin{cases} 7-2x, & x \leq 1; \\ 5 & 1 \leq x \leq 6; \\ 2x-7, & x > 6. \end{cases}$$

Solving it graphically, the solution set is $-1 \leq x \leq 8$, so that $b-a=9$. □

10. **Answer.** 750.

Solution. By using method of difference to find the sum $\sum_{k=1}^{n}(a_{k+1}-a_k)$, it is not difficult to obtain that $a_n = 2n^2$ is the expression for a_n. So

$$\begin{aligned}
&\frac{1}{a_2-2}+\frac{1}{a_3-2}+\cdots+\frac{1}{a_n-2}\\
=&\frac{1}{2\times 2^2-2}+\frac{1}{2\times 3^2-2}+\cdots+\frac{1}{2\times n^2-2}\\
=&\frac{1}{2}\left(\frac{1}{(2-1)(2+1)}+\frac{1}{(3-1)(3+1)}+\cdots+\frac{1}{(n-1)(n+1)}\right)\\
=&\frac{1}{4}\left(\frac{1}{(2-1)}-\frac{1}{(2+1)}+\frac{1}{(3-1)}-\frac{1}{(3+1)}+\cdots+\frac{1}{(n-1)}-\frac{1}{(n+1)}\right)\\
=&\frac{1}{4}\left(\frac{1}{1}+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
&\lim_{n\to\infty} 2000\left(\frac{1}{a_2-2}+\frac{1}{a_3-2}+\cdots+\frac{1}{a_n-2}\right)\\
=&\lim_{n\to\infty} 500\left(\frac{1}{1}+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}\right)=750.
\end{aligned}$$

□

11. **Answer.** 37.

Solution. Note that for any real numbers x, y, z,

$$x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-zx).$$

Thus, if x, y, z are positive, then $x^3+y^3+z^3 \geq 3xyz$ holds, where the equality holds whenever $x=y=z$ holds. Observe that

$$2019a+\frac{1}{2019(a-b)b}=2019(a-b)+2019b+\frac{1}{2019(a-b)b}$$

$$\geq 3\left(2019(a-b)\times b\times\frac{1}{(a-b)b}\right)^{1/3}=3(2019)^{1/3}.$$

Thus $A=3(2019)^{1/3}$. Observe also that $12^3<2019<13^3$. Thus $12<A/3<13$ and so $36<A<39$. But, $(37/3)^3<2019<(38/3)^3$, implying that $37<A<38$. □

12. **Answer.** 196.

Solution. By Cauchy-Schwarz inequality, we have

$$(9702-x_1)^2=(1\cdot x_2+\cdots+1\cdot x_{99})^2\leq 98(x_2^2+\cdots+x_{99}^2)=98(960498-x_1^2).$$

That is $9702^2-19404x_1+x_1^2\leq 98\times 960498-98x_1^2$. As $98\times 960498=9702^2$, this simplifies to $99x_1^2-19404x_1\leq 0$, or equivalently, $99x_1(x_1-196)\leq 0$. Thus $0\leq x_1\leq 196$.

Since equality holds when $x_2=x_3=\cdots=x_{99}$, which is equivalent to $x_1=0$ or $x_1=196$. Thus the largest value of x_1 is 196, and $x_2=x_3=\cdots=x_{99}=(9702-196)/98=97$. Note that the minimum values of x_1 is 0, with $x_2=x_3=\cdots=x_{99}=9702/98=99$. The same proof shows that $x_i\geq 0$ for all i. □

13. **Answer.** 4.

Solution. The 4 numbers $11, 33, 66, 88$ satisfy the condition of the problem. So $n \geq 4$. We shall show $n = 4$. Suppose $n \geq 5$. Then there are 2 numbers a_i and a_j such that $a_i \equiv a_j \pmod 4$. But $|a_i - a_j|$ is a not prime since it is divisible by 4. This contradiction shows that $n \leq 4$. □

14. **Answer.** 83210.

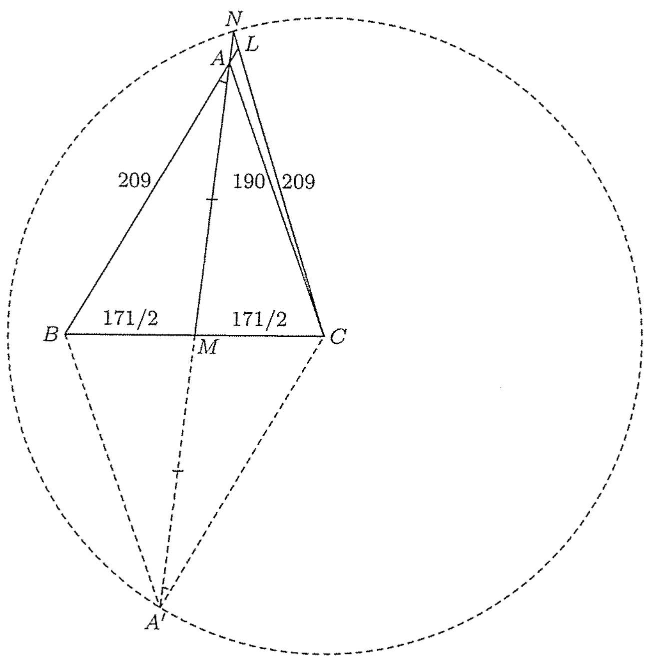
Solution. As $1 = ab + bc + ca$, we have $1 + a^2 = a^2 + ab + bc + ca = (a+b)(c+a)$. Similarly, $1 + b^2 = (a+b)(b+c)$ and $1 + c^2 = (b+c)(c+a)$. Therefore,

$$(1+a^2)(1+b^2)(1+c^2) = ((a+b)(b+c)(c+a))^2.$$

Thus $|(a+b)(b+c)(c+a)| = \sqrt{6923904100} = 83210$. For example, taking $a = -23, b = 28, c = 129$, we have $ab + bc + ca = 1$ and $(1+a^2)(1+b^2)(1+c^2) = 6923904100$. □

15. **Answer.** 21.

Solution.



First we prove that $CN = AB$. Let A' be the on the extension of AM such that $AM = MA'$. Then $ABA'C$ is a parallelogram. Then $\angle CNA' = \angle LNA = \angle LAN = \angle BAA' = \angle CA'N$. This means $CN = CA' = AB = 209$. By cosine rule, $\cos B = \frac{17}{33}$ and $AM = \frac{361}{2}$. Thus $\cos\angle BAM = \frac{191}{209}$. We have $A'N = 2CN\cos\angle MNC = 2AB\cos\angle BAM = 2 \times 209 \times \frac{191}{209} = 382$. Therefore, $AN = A'N - AA' = A'N - 2AM = 382 - 361 = 21$. □

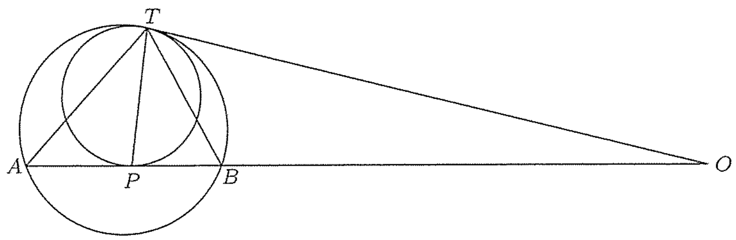
16. **Answer.** 299.

Solution. We have $\dfrac{386}{2019} < \dfrac{a}{b} < \dfrac{35}{183} \Leftrightarrow \dfrac{183}{35} < \dfrac{b}{a} < \dfrac{2019}{386} \Leftrightarrow 5 + \dfrac{8}{35} < \dfrac{b}{a} < 5 + \dfrac{89}{386}$. Thus $b > 5a$. Let $b = 5a + x$. The above inequality becomes $\dfrac{8}{35} < \dfrac{x}{a} < \dfrac{89}{386}$. That is $\dfrac{386x}{89} < a < \dfrac{35x}{8}$. The least positive integer x such that the interval $(\dfrac{386x}{89}, \dfrac{35x}{8})$ contains an integer is

$x = 11$, and $a = 48$ is the only integer inside $(\frac{386 \times 11}{89}, \frac{35 \times 11}{8}) = (47.708, 48.125)$. Thus $b = 5 \times 48 + 11 = 251$ and $a + b = 299$. If $x \geq 12$, then $a > \frac{386 \times 12}{89} > 52$ and hence $b > 5 \times 52 = 260 > 251$. □

17. **Answer.** 24.

Solution.



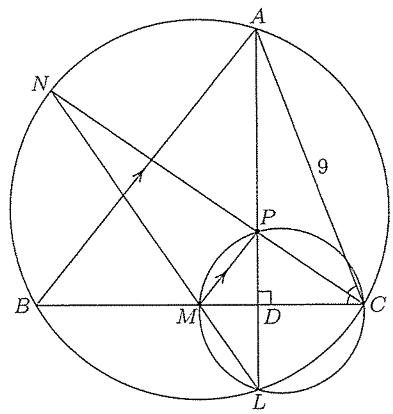
Let the tangent at T meet the extension of the chord AB at O. Then $\angle BTO = \angle TAB$. Thus $\triangle OAT$ is similar to $\triangle OTB$ so that $\frac{TA}{TB} = \frac{OT}{OB}$. Since $OT = OP$, we have $\frac{TA}{TB} = \frac{OP}{OB}$. On the other hand, $OP^2 = OA \cdot OB$.

Therefore $\frac{TA}{TB} = \frac{OP}{OB} = \frac{OA}{OP}$. Thus $\frac{TA}{TB} = \frac{OA-OP}{OP-OB} = \frac{AP}{BP}$.

Consider $\triangle TAB$. By Stewart's theorem, $TP = 24$. □

18. **Answer.** 5.

Solution.



Let CN intersect AL at P. The $\angle PCM = \angle PCA = \angle NCA = \angle NLA = \angle MLP$ so that P, C, L, M are concyclic. Thus $\angle PMC = \angle PLC = \angle PLC = \angle ABC$ so that MP is parallel to BA. It follows that $MD = \frac{PD \times BD}{AD}$. Below are the calculations that lead to the answer.

$\sin C = \frac{3\sqrt{19}}{14}$, $\tan\frac{C}{2} = \frac{3}{\sqrt{19}}$, $AD = 9 \times \frac{3\sqrt{19}}{14} = \frac{27\sqrt{19}}{14}$, $CD = 9 \times 5/14 = \frac{45}{14}$, $BD = 10 - \frac{45}{14} = \frac{95}{14}$, $PD = CD \tan\frac{C}{2} = \frac{135}{14\sqrt{19}}$. Thus $MD = \frac{PD \times BD}{AD} = \frac{\frac{135}{14\sqrt{19}} \times \frac{95}{14}}{\frac{27\sqrt{19}}{14}} = \frac{25}{14}$. Consequently, $MC = \frac{25}{14} + \frac{45}{14} = 5$, and M is the midpoint of BC. □

19. **Answer.** 2020.

Solution. By the given conditions,

$$\log_2 x_1 = 2020 - x_1, \quad i.e., x_1 = 2^{2020-x_1}$$

and

$$2^{x_2} = 2020 - x_2.$$

Thus

$$2^{x_2} - 2^{2020-x_1} = 2020 - x_2 - x_1,$$

i.e.,

$$2^{x_2} + x_2 = 2^{2020-x_1} + (2020 - x_1).$$

As the function $2^z + z$ is monotone increasing, the above equality implies that $x_2 = 2020 - x_1$. Thus $x_1 + x_2 = 2020$. □

20. **Answer.** 26.

Solution. The following inequalities are equivalent:

$$\sqrt[3]{10x-200} + \sqrt[3]{10x-300} + 20x - 500 > 0;$$

$$\sqrt[3]{10x-200} + 10x - 200 > 300 - 10x + \sqrt[3]{300-10x}.$$

As the function $\sqrt[3]{z} + z$ is monotone increasing, the above inequality holds if and only if the following inequality holds:

$$10x - 200 > 300 - 10x,$$

i.e., $20x > 500$, $x > 25$. Hence the answer is 26. □

21. **Answer.** 1

Solution. By direct interpretation, S can be interpreted as $\int_0^1 e^x dx$ which can be evaluated to be $e - 1$. Hence $\lfloor S \rfloor = 1$. □

22. **Answer.** 4

Solution. Without loss of generality, assume that the length $OC = 1$. Let the two points selected be denoted by x and y with $0 \le x < y \le 1$. When plotted on an x-y plane, all the possible pairs (x, y) lie in the region bounded by $y \ge x$, $x > 0$ and $y < 1$, which is the triangular region with the vertices $(0,0)$, $(1,1)$ and $(0,1)$. Area of this region $= \frac{1}{2}$. If the three lengths can form the side of a triangle, the sum of any two lengths must be greater than the third side by Triangle Inequality. Hence, it defines the region bounded by $x < \frac{1}{2}$, $y > \frac{1}{2}$ and $y - x < \frac{1}{2}$, which is the the triangular region defined by $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 1)$ and $(0, \frac{1}{2})$. Area of this region $= \frac{1}{8}$. Hence required probability $p = \frac{1}{4}$ so that $\left\lfloor \frac{1}{p} \right\rfloor = \frac{1}{p} = 4$.
□

23. **Answer.** 2010.

Solution. Note that

$$A = 2019!^2 + 2019!^4 + \cdots + 2019!^{2020} = 2019!^2(1 + 2019!^2 + \cdots + 2019!^{2018}).$$

Thus, for any integer $i \geq 1$, 3^i divides A if and only if 3^i divides $2019!^2$.

Observe that the largest power number m in the prime factorization of 2019! is equal to

$$m = \left\lfloor \frac{2019}{3} \right\rfloor + \left\lfloor \frac{2019}{3^2} \right\rfloor + \cdots + \left\lfloor \frac{2019}{3^r} \right\rfloor,$$

where r is the largest possible integer such that $3^r \leq 2019$. Thus $r = 6$ and is

$$m = \left\lfloor \frac{2019}{3} \right\rfloor + \left\lfloor \frac{2019}{3^2} \right\rfloor + \cdots + \left\lfloor \frac{2019}{3^6} \right\rfloor = 1005.$$

Hence the answer is $2m = 2010$. $\square$

24. **Answer.** 1347.

Solution. Note that the numbers in S are

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, \cdots, 4037,$$

which can be partitioned into exactly 673 subsets, as shown below:

$$\{1, 3, 5\}, \{7, 9, 11\}, \{13, 15, 17\}, \{19, 21, 23\}, \cdots, \{4033, 4035, 4037\}.$$

Notice that each subset above has exactly one multiple of 3. Thus, S contains exactly $2 \times 673 = 1346$ numbers which are not multiples of 3. Let $c_1, c_2, \cdots, c_{1346}$ denote these numbers. Clearly, the following set A does not contain any two numbers a, b in A and some integer $s \geq 1$ such that $b = 3^s a$:

$$A = \{c_1, c_2, \cdots, c_{1346}\}.$$

Now we partition S into 1346 subsets $S_1, S_2, \cdots, S_{1346}$, where

$$S_i = \{c_i \times 3^s \in S : s \geq 0\}.$$

For any subset A of S with $|A| \geq 1347$, we have $|A \cap S_i| \geq 2$ for some i, implying that $a = b \times 3^s$ for some numbers $a, b \in S_i$ and some integer $s \geq 1$. Hence the answer is 1347. $\square$

25. **Answer.** 99048.

Solution. Label the columns of the table by $1, 2, \ldots, 14$, the rows by $1, \ldots, 7$, and denote the cell in the ith column and jth row by (i, j). Consider a way $\mathcal{W}$ of filling in the table satisfying the condition that in each 2×2 block, exactly one of the 4 letters is used. We claim that either any row in $\mathcal{W}$ contains only two letters or any column in $\mathcal{W}$ contains only 2 letters. To prove this assertion, we consider 2 cases.

Case 1. $\mathcal{W}$ does not contain a rectangle of size 1×3 containing 3 distinct letters. Then in any row of $\mathcal{W}$ we have two alternating letters.

Case 2. $\mathcal{W}$ contains a rectangle of size 1×3 containing 3 distinct letters. Let the cells $(x, y), (x+1, y)$ and $(x+2, y)$ be filled with three distinct letters say S, I, M. Then the cell $(x+1, y+1)$ must be filled in with the letter O, $(x, y+1)$ with the letter M, $(x+2, y+1)$

with the letter S, $(x, y+2)$ with the letter S, $(x+1, y+2)$ with the letter I, $(x+2, y+2)$ with the letter M, and so on. Similarly, the cell $(x+1, y-1)$ is filled in with the letter O, $(x, y-1)$ with the letter M and $(x+2, y-1)$ with the letter S, etc. We conclude that column x contains only the letters S and M, column $x+1$ contains only the letters I and O. It follows that any column having the same parity as x contains only letters S and M, and any column having the same parity as $x+1$ contains only letters I and O.

This proves that either any row in $\mathcal{W}$ contains only two letters or any column in $\mathcal{W}$ contains only 2 letters.

The number of ways of filling in the table such that in any column of odd column index, two of the letters are filled in alternately and in any column of even column index the other two letters are filled in alternately is equal to $\binom{4}{2} \times 2^{14}$. The corresponding number of ways when the rows are filled in using two letters equal to $\binom{4}{2} \times 2^{7}$. The ways in which the letter of the cell (i, j) depend only on the parity of i and j are counted in both cases. There are $4! = 24$ such ways. Therefore, the number of ways equals to $6 \times (2^{14} + 2^{7}) - 24 = 99048$.

$\square$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2019

(Open Section, Round 2)

Saturday, 6 July 2019 **0900-1300**

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1. In the acute-angled triangle ABC with circumcircle ω and orthocenter H, points D and E are the feet of perpendiculars from A onto BC and from B onto AC respectively. Let P be a point on the minor arc BC of ω. Points M and N are the feet of perpendiculars from P onto lines BC and AC respectively. Let PH and MN intersect at R. Prove that $\angle DMR = \angle MDR$.

2. Find all functions $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $f(-f(x) - f(y)) = 1 - x - y$ for all $x, y \in \mathbb{Z}$.

3. A robot is placed at a point P on the x-axis but different from $(0,0)$ and $(1,0)$ and can move only along the axis either to the left or to the right. Two players play the following game. Player A gives a distance and B gives a direction and the robot will move the indicated distance along the indicated direction. Player A aims to move the robot to either $(0,0)$ or $(1,0)$. Player B's aim is to stop A from achieving his aim. For which P can A win?

4. Let $p \equiv 2 \pmod 3$ be a prime, k a positive integer and let $P(x) = 3x^{(2p-1)/3} + 3x^{(p+1)/3} + x + 1$. For any integer n, let $R(n)$ denote the remainder when n is divided by p and let $S = \{0, 1, \ldots, p-1\}$. At each step, you can either (a) replace every element i of S with $R(P(i))$ or (b) replace every element i of S with $R(i^k)$. Determine all k such that there exists a finite sequence of steps that reduces S to $\{0\}$.

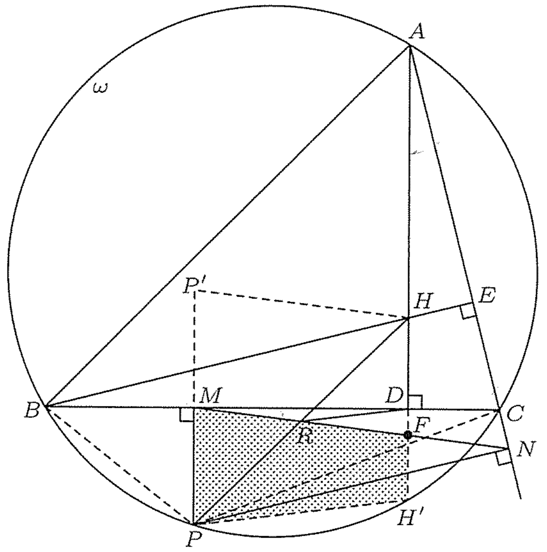
5. In an $m \times n$ chessboard $(m, n \geq 2)$, some dominoes are placed (without overlap) with each domino covering exactly two adjacent cells. Show that if no more dominoes can be added to the grid, then at least $2/3$ of the chessboard is covered by dominoes.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2019

(Open Section, Round 2 solutions)

1.



Let P' and H' be the reflections of P and H across BC respectively. It follows that $PP'HH'$ is an isosceles trapezium with PP' parallel to $H'H$ and $PH' = P'H$. Also it is a well-known result that H' lies on ω. Let MN intersect AH' at F. Note that P, M, C, N are concyclic. Hence $\angle MFH' = 90° + \angle CMN = 90° + \angle CPN = \angle PCA = \angle PH'F$. Thus $MPH'F$ is an isosceles trapezium with PM parallel to $H'F$ and $PH' = MF$. Therefore, $\angle PP'H = \angle P'PH' = \angle PMF$ so that $P'H$ is parallel to MF. Since M is the midpoint of PP' and MR is parallel to $P'H$, we have R is the midpoint of PH. Consequently, $MR = \frac{1}{2}P'H = \frac{1}{2}PH' = \frac{1}{2}MF$. Thus R is the midpoint of MF. It follows that $\angle DMR = \angle MDR$.

2. The answer is $f(x) = x - 1$. One can check easily that $f(x) = x - 1$ satisfies the functional equation. Suppose $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ satisfies

$$f(-f(x) - f(y)) = 1 - x - y.$$

Substituting $y = 1$, we have $f(-f(x) - f(1)) = -x$. Thus, $f(-2f(1)) = -1$.

Let $n \in \mathbb{Z}$. Then $f(-f(n)-f(1)) = -n$. Letting $x = -f(n)-f(1)$ and $y = -2f(1)$, we get $f(n+1) = 1+(f(n)+f(1))+2f(1)$. That is $f(n+1) = f(n)+3f(1)+1$. Using both forward and backward induction starting at $n = 1$, one can prove that

$$f(n) = (3n-2)f(1) + n - 1..$$

Letting $n = 0$, we have $f(0) = -2f(1) - 1$. Substituting $x = 0$ and $y = 1$, we get $f(-f(0) - f(1)) = 0$. That is $f(f(1) + 1) = 0$. If we let $n = f(1) + 1$, we get $(3(f(1) + 1) - 2)f(1) + (f(1) + 1) - 1 = 0$. Simplifying, we have $f(1)(3f(1) + 2) = 0$. Since $f(1)$ is an integer, we must have $f(1) = 0$. Consequently, we have $f(n) = n - 1$.

3. We ignore the y-axis and only consider the number line. If $P \notin [0, 1]$, then B wins as he only needs to choose a direction that moves the robot away from $[0, 1]$. If P is at 0 or 1, then A clearly wins. If $P = \frac{m}{2^n}$ with $m, n \in \mathbb{Z}^*$ and $m < 2^n$, A wins. Here we may assume that m is odd. A simply say $1/2^n$. Then the robot will move to either $\frac{m+1}{2^n}$ or $\frac{m-1}{2^n}$. If these are not either 1 or 0, then after canceling common factors, the robot is at $\frac{k}{2^l}$ with $k < 2^l$ and $l < n$. Now A just says $1/2^l$, and so on. Eventually, the denominator becomes 1 and P is at either 0 or 1.

Suppose $x \in (0, 1)$ is a number not of the form $\frac{m}{2^n}$, whatever distance d that A chooses, then at least one of $x + d$ and $x - d$ is also not of the form. Then B simply chooses the direction that goes to that number. So the robot will never reach 0 or 1.

4. The answer is all k such that $\gcd(k, p - 1) > 1$. Note that all congruences are mod p. Using Fermat's Little Theorem, we have, for every $n \in \mathbb{Z}$,

$$P(n^3) = 3n^{2p-1} + 3n^{p+1} + n^3 + 1 \equiv 3n + 3n^2 + n^3 + 1 \equiv (n + 1)^3.$$
$$\therefore \quad R(P(n^3)) \equiv (n + 1)^3. \qquad (1)$$

If k is an integer such that $\gcd(k, p - 1) = 1$, then there exist integers a, b such that $ak + b(p - 1) = 1$. Then if $x^k \equiv y^k$, since $x^{p-1} \equiv 1 \equiv y^{p-1}$, we have

$$x^{ak+b(p-1)} \equiv y^{ak+b(p-1)} \quad \Rightarrow \quad x \equiv y.$$

Therefore

$$\{0^k, 1^k, \ldots, (p - 1)^k\} = \{0, 1, \ldots, p - 1\} \qquad (2).$$

Now since $p \equiv 2 \pmod 3$, we have $\gcd(3, p - 1) = 1$. So, by (1) and (2),

$$\{R(P(0)), \ldots, R(P((p-1)))\} = \{R(P(0^3)), \ldots, R(P((p-1)^3))\} = \{0, 1, \ldots, p-1\}. \ (3)$$

Therefore , by (2) and (3), when $\gcd(k, p - 1) = 1$, we cannot reduce S to $\{0\}$.

Now let $\gcd(k, p - 1) = e > 1$. We shall show given any set $\{a, b\}$ with $0 \le a, b \le p - 1$, we can reduce it to $\{0\}$. Then by repeatedly applying it to pairs of elements of S, we can reduce it to just $\{0\}$. Let $a \equiv m^3$ and $b \equiv (m + d)^3$ for some integers m, d. Then, using(1), i.e., operation (a), repeatedly, we can replace the pair a, b by $R(c^3), R(c+d)^3)$ where c is any integer.

Next we shall show that by choosing an appropriate c, the pair can be replaced by a single integer using operation (b). Since $e \mid p - 1$ and $e > 1$, there exists $s \not\equiv 1$ such that

$s^e \equiv 1$. Since $e \mid k$, we also have $s^k \equiv 1$. There exists integer n such that $s \equiv n^3$. Now, we claim there exists c such that $sc^3 \equiv (c+d)^3$. This is equivalent to $nc \equiv c+d$ which gives us $c(n-1) \equiv d$. Since $s \not\equiv 1$, we have $n \not\equiv 1$ too and so the equation $c(n-1) \equiv d$ always has a solution in c. Letting c be a solution. We then have

$$R((c^3)^k)) \equiv c^{3k} \equiv (sc^3)^k \equiv (c+d)^{3k} \equiv R((c+d)^3)^k)$$

Thus we have reduced the pair $R(c^3), R(c+d)^3)$, using operation (b), to a single integer which we call x. If we let $x \equiv y^3$ where $y < p$, it then suffices to apply (a) again $p - y$ times to replace y^3 by 0 and we are done.

5. Let D, B, D', B', B'' denote the number of dominoes, blanks, dominoes at the boundary, blanks at the boundary, blanks at the corner respectively. No two blanks are adjacent. A pair is either a domino and a blank or a domino and the exterior or a blank and the exterior that share an edge. Let P be the number of such pairs. Each domino is involved in at most 4 pairs, each blank not at the corner is involved in 4 pairs while each corner blank is involved in 3 pairs. Thus we have

$$4D + B' \geq P \geq 4B - B'' + D' \Rightarrow 4D \geq 4B - B'' + D' - B' \Rightarrow D \geq B - 1$$

since $B" \leq 4$ and $D' \geq B'$ as no two blanks are adjacent. We will now demonstrate that equality in $D \geq B - 1$ is impossible. For equality to hold, the following conditions must simultaneously be met:

- All four corners are blanks ($B'' = 4$).
- Along the border, blanks and dominoes alternate ($B' = D'$)
- Each domino is adjacent to, on each side, a *blank* ~~or~~ the border of the grid.
- Each blank is adjacent to, on each side, a domino or the border of the grid.

Consider the longer side of a domino ("A"). It is either adjacent to the border, or a blank and a domino ("B"). However, domino B's shorter side must be adjacent to a blank (or the border), so that cannot be adjacent to A. Hence, along the longer side of any domino A, it must be adjacent to a domino with a similar orientation as itself. Now we note that all dominoes must have the same orientation. Otherwise, if there exists a horizontal domino H, we can get a chain of dominoes from the top edge of H to the top edge of the grid, and similarly a chain of dominoes from the bottom edge of H to the bottom edge of the grid. This prevents a chain of adjacent vertical dominoes from the left edge of the grid to the right edge of the grid, which can be constructed given a vertical domino V in a similar fashion. Without loss of generality, assume that there are only horizontal dominoes. Consider the bottom-most layer of cells of the grid. They must alternate between blanks and dominoes, with blanks on both ends. However, the next row above it must begin with domino on one end and then alternate with blank and domino to the other end. But the other end would contain 2 adjacent dominoes, a contradiction.

Singapore Mathematical Olympiad 2019 Results

School Award (Category 1)

Within each category of award, the schools are arranged according to alphabetical order.

Gold Award	
Hwa Chong Institution	NUS High School of Mathematics and Science
Raffles Girls' School (Secondary)	Raffles Institution
Silver Award	
Anglo-Chinese School (Independent)	Catholic High School
Maris Stella High School	Nanyang Girls' High School
River Valley High School	St Joseph's Institution
Victoria School	
Bronze Award	
Anderson Secondary School	Cedar Girls' Secondary School
CHIJ St Nicholas Girls' School	Chung Cheng High (Main)
Dunman High School	Nan Chiau High School
Nan Hua High School	National Junior College
Ngee Ann Secondary School	Singapore Chinese Girls' School
Temasek Junior College	
Commendation Award	
Anglo-Chinese International School	Bukit Panjang Govt High School
Clementi Town Secondary School	Commonwealth Secondary School
Crescent Girls' School	Dunman Secondary School
Fuhua Secondary School	Gan Eng Seng School
Kranji Secondary School	Methodist Girls' School
NPS International School	St Andrew's Secondary School
Swiss Cottage Secondary School	Tanjong Katong Girls' School
Tanjong Katong Secondary School	Temasek Secondary School
United World College of South East Asia	Xinmin Secondary School
Yishun Town Secondary School	Zhonghua Secondary School

Singapore Mathematical Olympiad 2019 Results

School Award (Category 2)

Within each category of award, the schools are arranged according to alphabetical order.

Gold Award
Hwa Chong Institution
National Junior College
NUS High School of Mathematics and Science
Raffles Institution
Silver Award
Anglo-Chinese School (Independent)
Nanyang Junior College
Bronze Award
Dunman High School
Temasek Junior College
Victoria Junior College
Commendation Award
Anderson Secondary School
Anderson Serangoon Junior College
Anglo-Chinese Junior College
Catholic Junior College
Crescent Girls' School
Eunoia Junior College
Hwa Chong International School
Jurong Pioneer Junior College
Nan Hua High School
Nanyang Girls' High School
Ngee Ann Secondary School
NPS International School
River Valley High School
St Joseph's Institution
St Joseph's Institution International
Tampines Meridian Junior College
Victoria School

Singapore Mathematical Olympiad 2019 Results

Individual Award (Junior Section)

RANK	NAME	SCHOOL
1	Yu Qinxi	Raffles Institution
2	James Lim Jia Sin (Lin Jiaxin)	NUS High School of Mathematics and Science
2	Mario Tanijaya	NUS High School of Mathematics and Science
4	Oh Zhi Yuan	NUS High School of Mathematics and Science
5	Liu Zihan	Raffles Institution
6	Su Zixuan	Raffles Institution
7	Edden Chew Keyn-Hantz	NUS High School of Mathematics and Science
7	Yu Hanzhang	Raffles Girls' School (Secondary)
9	Andrew Yar Kwok Wing	Raffles Institution
10	Ryan Suwandi	NUS High School of Mathematics and Science
11	Liang Yi Kuan	Raffles Institution
12	Michael Liew Wensheng	Anglo-Chinese School (Independent)
13	Zhang Chenxi	Raffles Girls' School (Secondary)
14	Hugo Maximus Lim	NUS High School of Mathematics and Science
15	Ma Xingjian	Raffles Institution
16	Chang Chia Wei	Raffles Institution
17	Drew Michael Terren Ramirez	NUS High School of Mathematics and Science
18	Lim Sue Han Justin	NUS High School of Mathematics and Science
19	Kong U Ham	Nanyang Girls' High School
20	Tay Eewan	NUS High School of Mathematics and Science
21	Tan Chun Ern	Hwa Chong Institution
22	Cheong Eason	NUS High School of Mathematics and Science
22	Ziv Ng	Raffles Institution
24	Ryan Wong Wern Jieh	Hwa Chong Institution
25	He Jingyang James	Raffles Institution
26	Daryle Ang Xuan Ye	NUS High School of Mathematics and Science
27	Davin Khor Wen Ning	Hwa Chong Institution
28	Lai Le Hao Jerome	Raffles Institution
29	Gabriel Goh Hao Xiang	Hwa Chong Institution
30	Xu Ziyu	Hwa Chong Institution
30	Lee I-Shiang	NUS High School of Mathematics and Science

Singapore Mathematical Olympiad 2019 Results

Individual Award (Senior Section)

RANK	NAME	SCHOOL
1	Boo Tse Yang Lucas	Raffles Institution
2	Low Choo Ray	Raffles Institution
3	Krishnan Akash	NUS High School of Mathematics and Science
4	Wu Xinyue	Nanyang Girls' High School
5	Lim Jin Tao	NUS High School of Mathematics and Science
6	Chieu Le Heng (Jiang Leheng)	NUS High School of Mathematics and Science
7	Shevonne Chia	NUS High School of Mathematics and Science
8	Wang Yuxuan	Raffles Institution
9	Wang Yunrui	Raffles Institution
10	Lee Wei Ming	NUS High School of Mathematics and Science
11	Tan Yun Hui	NUS High School of Mathematics and Science
12	Hu Xinghui	Raffles Girls' School (Secondary)
13	Lin Sixun	St Joseph's Institution
14	Liu Fenghua	Victoria School
15	Le Tran Cong Hung	Anglo-Chinese School (Independent)
16	Li Hantao	Raffles Institution
17	Ong Yan Sheng	NUS High School of Mathematics and Science
18	Pan Yifan	Raffles Institution
19	Shen Xing Yang	NUS High School of Mathematics and Science
20	Justin Adrian Halim	Raffles Institution
21	Wang Sili	Raffles Institution
22	Kuai En Kai Ethan	NUS High School of Mathematics and Science
23	Chen Yuxin	Temasek Secondary School
24	Toh Jing En Daniel	NUS High School of Mathematics and Science
25	Kiefer Ong Xian Yao	Hwa Chong Institution
26	Yuan Chenghao	St Joseph's Institution
27	Russell Ng Jung Hng	NUS High School of Mathematics and Science
28	Ong Ee Shua, Etienne	Raffles Institution
29	Kwee Tze Wei, Bernard	Hwa Chong Institution
30	Wang Chenyang	St Joseph's Institution

Singapore Mathematical Olympiad 2019 Results

Individual Award (Open Section)

RANK	NAME	SCHOOL
1	Boo Tse Yang Lucas	Raffles Institution
2	Ng Yu Peng	Hwa Chong Institution
3	Cheng Puhua	Raffles Institution
4	David Toh Hui Kai	Raffles Institution
5	Tan Xu Chen	Raffles Institution
6	Ng Yangyi, Aloysius	Raffles Institution
7	Tan Wee Kean	Raffles Institution
8	Khor Jun Wei	Raffles Institution
9	Low Choo Ray	Raffles Institution
10	Leong Eu-Shaun	Raffles Institution
11	Yang Yue	Hwa Chong Institution
12	Shi Cheng	Hwa Chong Institution
13	Ang Boon Han Nathaniel	Anglo-Chinese School (Independent)
13	Ding Tian Yao	St Joseph's Institution International
15	Jamie Lim Jia Sin (Lin Jiaxin)	NUS High School of Mathematics and Science
16	Ong Ee Shua, Etienne	Raffles Institution
17	Mario Tanijaya	NUS High School of Mathematics and Science
18	Lim An Jun	Hwa Chong Institution
19	Shen Xinyi	Raffles Institution
20	Nguyen Khoi Nguyen	St Joseph's Institution
21	Luke Tan	Raffles Institution
22	Teow Hua Jun	Hwa Chong Institution
23	Le Viet Hung	Anderson Secondary School
24	Tan Si Jie	NUS High School of Mathematics and Science
25	Dong Leyi	Hwa Chong Institution
26	Liu Siyong	Raffles Institution
27	Wang Yunrui	Raffles Institution
28	Sun Longxuan	Hwa Chong Institution
29	Daniel Choo Zhenghao	Raffles Institution
30	Zhang Yu Chi	Raffles Institution