H3 Mathematics

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Abstract

The syllabus can be found here. This document will mainly focus on exposing the reader to as wide a variety of questions as possible. In addition to detailed solutions, the thought processes and proving techniques involved in solving each problem will also be included.

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General Tips

- Practice makes perfect.

  While somewhat less applicable to H3 Mathematics due to the unpredictability of questions, practice can still help by exposing you to different problems, and the thought processes that you can glean from them can still be invaluable.

  Relatively routine and common questions can still take up a decent chunk of the paper, so do try to secure those marks whenever possible, especially in combinatorics

- Do not panic.

  As a rule of thumb, if you do not have any idea on how to start on a question within 3–5 minutes, move on. It is not uncommon for people to not know how to do questions, so don't waste your time on them, especially if there could be easier questions hidden behind, or if you have partial solutions to earlier problems that you feel you can complete.

- Build a strong H2 foundation.

  This tip is pretty self-explanatory. If your H2 sucks, it is unlikely that you can do well in H3. Plus, H2 counts towards your UAP and stuff, while H3 is like, an ego boost or something.

- Farm method marks.

  Unlike in Math Olympiad where the marking scheme is: "Oh you did not do the key part correctly, 0/7", H3 is extremely generous. For instance, in the 2019 A-level Paper, out of 7 marks in a part question, 4 were awarded for an attempted induction if the candidate wrote down

  - inductive statement,
  - base case, with brief justification, and
  - induction hypothesis.

  The above three steps don't take much effort but yield good returns.

Syllabus

1. H2 Mathematics content areas

   (a) Functions, e.g. graphs, symmetries, derivatives, integrals, differential equations, limiting behaviours, bounds.

   (b) Sequences and series, e.g. general terms, sum, limiting behaviours, bounds.

2. Additional content areas

   (a) Inequalities: AM–GM inequality, Cauchy–Schwarz inequality, triangle inequality.

   (b) Numbers: primes, coprimes, divisibility, greatest common divisor, division algorithm, congruences and modular arithmetic.

   (c) Counting: distribution problems, Stirling numbers of the second kind, recurrence equations, bijection principle, principle of inclusion and exclusion.

§1 Number Theory

1. Prove the following bi-conditional statement: For all integers a and b, $3 \mid ab$ if and only if $3 \mid a$ or $3 \mid b$.

   Proof. **'If' part:** If $3 \mid a$ or $3 \mid b$, then $3 \mid ab$.

   We are supposed to prove two cases: (i) If $3 \mid a$ then $3 \mid ab$; and (ii) If $3 \mid b$ then $3 \mid ab$.

   Without loss of generality, we just need to prove one case. (Notice by interchanging a and b, it will not change the statement.)

   So suppose $3 \mid a$. Then $a = 3k$ for some integer k. Then $ab = 3kb$ would imply $3 \mid ab$.

   This proved the 'If' part.

   'Only if' part: If $3 \mid ab$, then $3 \mid a$ or $3 \mid b$.

   We prove this by contrapositive.

   Suppose $3 \nmid a$ and $3 \nmid b$, then $3 \nmid ab$.

   From the hypothesis, we have the following four cases:

   Case (i) $a = 3k + 1$ and $b = 3h + 1$.

   Then $ab = (3k + 1)(3h + 1) = 9kh + 3(k + h) + 1 = 3(3kh + k + h) + 1$.

   The RHS has a remainder 1 when divided by 3. So $3 \nmid ab$.

   Case (ii) $a = 3k + 2$ and $b = 3h + 1$.

   Then $ab = (3k + 2)(3h + 1) = 9kh + 3k + 6h + 2 = 3(3kh + k + 2h) + 2$.

   The RHS has a remainder 2 when divided by 3. So $3 \nmid ab$.

   Case (iii) $a = 3k + 1$ and $b = 3h + 2$.

   Then $ab = (3k + 1)(3h + 2) = 9kh + 6k + 3h + 2 = 3(3kh + 2k + h) + 2$.

   The RHS has a remainder 2 when divided by 3. So $3 \nmid ab$.

   Case (iv) $a = 3k + 2$ and $b = 3h + 2$.

   Then $ab = (3k + 2)(3h + 2) = 9kh + 6(k + h) + 4 = 3(3kh + 2k + 2h + 1) + 1$.

   The RHS has a remainder 1 when divided by 3. So $3 \nmid ab$.

   In all cases, we have $3 \nmid ab$. □

2. Let a, b be integers, not both 0. Prove that $\gcd(a + b, a - b) \le \gcd(2a, 2b)$.

   Thought process: apply the definition of gcd

   Proving techniques: direct proof

   Proof. Let $e = \gcd(a + b, a - b)$. Then $e \mid (a + b)$ and $e \mid (a - b)$. So

   $$e \mid (a + b) + (a - b) \implies e \mid 2a$$

   and

   $$e \mid (a + b) - (a - b) \implies e \mid 2b$$

   This implies e is a common divisor of $2a$ and $2b$. So $e \le \gcd(2a, 2b)$. □

3. Let a and b be integers, not both 0. Show that $\gcd(a,b)$ is the smallest possible positive linear combination of a and b. (i.e. There is no positive integer $c < \gcd(a,b)$ such that $c = ax + by$ for some integers x and y.)

Proof. Prove by contradiction.

Suppose there is a positive integer $c < \gcd(a,b)$ such that $c = ax + by$ for some integers x and y.

Let $d = \gcd(a,b)$. Then $d \mid a$ and $d \mid b$, and hence $d \mid ax + by$. This means $d \mid c$.

Since c is positive, this implies $\gcd(a,b) = d \leq c$. This contradicts $c < \gcd(a,b)$.

Hence we conclude that there is no positive integer $c < \gcd(a,b)$ such that $c = ax+by$ for some integers x and y. $\square$

4. Use the Unique Factorisation Theorem to prove that, if a positive integer n is not a perfect square, then $\sqrt{n}$ is irrational.

[The Unique Factorisation Theorem states that every integer $n > 1$ has a unique standard factored form, i.e. there is exactly one way to express $n = n = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$ where $p_1 < p_2 < \cdots < p_t$ are distinct primes and $k_1, k_2, \ldots, k_t$ are some positive integers.]

Proof. Prove by contradiction.

Suppose n is not a perfect square and $\sqrt{n}$ is rational.

Then $\sqrt{n} = \frac{a}{b}$ for some integers a and b. Squaring both sides and clearing denominator gives

$$nb^2 = a^2. \tag{$*$}$$

Consider the standard factored forms of n, a and b:

$$n = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$$

$$\begin{aligned} a = q_1^{e_1} q_2^{e_2} \cdots q_u^{e_u} &\implies a^2 = q_1^{2e_1} q_2^{2e_2} \cdots q_u^{2e_u} \\ b = r_1^{f_1} r_2^{f_2} \cdots r_v^{f_v} &\implies b^2 = r_1^{2f_1} r_2^{2f_2} \cdots r_v^{2f_v} \end{aligned}$$

i.e. the powers of primes in the standard factored form of a^2 and b^2 are all even integers.

This means the powers k_i of primes p_i in the standard factored form of n are also even by Unique Factorisation Theorem (UFT):

Note that all p_i appear in the standard factored form of a^2 with even power $2c_i$, because of $(*)$. By UFT, p_i must also appear in the standard factored form of nb^2 with the same even power $2c_i$.

If $p_i \nmid b$, then $k_i = 2c_i$ which is even. If $p_i \mid b$, then p_i will appear in b^2 with even power $2d_i$. So $k_i + 2d_i = 2c_i$, and hence $k_i = 2(c_i - d_i)$, which is again even.

Hence $n = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t} = \left(p_1^{\frac{k_1}{2}} p_2^{\frac{k_2}{2}} \cdots p_t^{\frac{k_t}{2}} \right)^2$.

Since $\frac{k_i}{2}$ are all integers, $p_1^{\frac{k_1}{2}} p_2^{\frac{k_2}{2}} \cdots p_t^{\frac{k_t}{2}}$ is an integer and n is a perfect square. This contradicts the given hypothesis that n is not a perfect square.

So we conclude that when a positive integer n is not a perfect square, then $\sqrt{n}$ is irrational. □

5. (Euclid's proof) There are infinitely many primes.

   Thought process: it is difficult to directly prove such an existential statement

   Proving technique: prove by contradiction

   Proof. Prove by contradiction.

   Suppose otherwise, that the list of primes is finite. Let $p_1, \ldots, p_r$ be our finite list of primes. We want to show this is not the full list of the primes.

   Consider the number
   $$N = p_1 \cdots p_r + 1.$$
   Since $N > 1$, it has a prime factor p. The prime p cannot be any of $p_1, \ldots, p_r$ since N has remainder 1 when divided by each p_i. Therefore p is a prime not on our list, so the set of primes cannot be finite. □

6. If n is an integer, prove that 3 divides $n^3 - n$.

   Proof. Prove by cases. This is done by partitioning $\mathbb{Z}$ according to remainders when divided by d (i.e. equivalence classes).

   We prove the three cases: $n = 3k$, $n = 3k + 1$, and $n = 3k + 2$.

   Case 1: $n = 3k$ for some integer k

   Then
   $$n^3 - n = (3k)^3 - (3k) = 3(9k^3 - k).$$
   Since $9k^3 - k$ is an integer, $3 \mid n^3 - n$.

   Case 2: $n = 3k + 1$ for some integer k

   Then
   $$n^3 - n = (3k+1)^3 - (3k+1) = 3(9k^3 + 9k^2 + 2k).$$
   Since $9k^3 + 9k^2 + 2k$ is an integer, $3 \mid n^3 - n$.

   Case 3: $n = 3k + 2$ for some integer k

   The proof is similar and shall be left as an exercise. □

7. (2017 A-Level H3 Mathematics) Prove that there is no integer solution (x, y) with x being prime, such that
   $$1591x + 3913y = 9331.$$

Proof. First we find $\gcd(1591, 3913)$ using the Euclidean Algorithm.

$$\begin{aligned}
3913 &= 2 \times 1591 + 731 \\
1591 &= 2 \times 731 + 129 \\
731 &= 5 \times 129 + 86 \\
129 &= 1 \times 86 + 43 \\
86 &= 2 \times 43 + 0
\end{aligned}$$

Thus $\gcd(1591, 3913) = 43$. By Bezout's Lemma, there are integer solutions for $1591x + 3913y = 43$. Since $43 \mid 9331$, multiplying both sides by some constant, there are also integer solutions for $1591x + 3913y = 9331$.

To prove by contradiction, we assume that x is prime, and there exists some integer y such that $1591x + 3913y = 9331$. Dividing both sides by 43,

$$37x + 91y = 217. \tag{$\star$}$$

Observe that $7 \mid 91y$ and $7 \mid 217$, so $7 \mid 37x$.

Since $\gcd(7, 37) = 1$ so $7 \mid x$. By our assumption, x is a prime so $x = 7$.

Substituting $x = 7$ into $(\star)$, we get $y = -\dfrac{6}{13}$, which contradicts y being an integer.

Hence we conclude that x cannot be a prime. □

8. Prove that we can find 100 consecutive positive integers which are all composite numbers.

   Thought process: existential statement, either apply constructive or non-constructive proof to find such 100 integers

   Proving techniques: constructive proof

   Proof. We can prove this existential statement via constructive proof.

   Our goal is to find integers $n, n+1, n+2, \dots, n+99$, all of which are composite.

   Take $n = 101! + 2$. Then n has a factor of 2 and hence is composite. Similarly, $n + k = 101! + (k+2)$ has a factor $k + 2$ and hence is composite for $k = 1, 2, \dots, 99$.

   Hence the existential statement is proven. □

9. Prove that for every pair of irrational numbers p and q such that $p < q$, there is an irrational x such that $p < x < q$.

   Proof. Consider the average of p and q: $p < \dfrac{p+q}{2} < q$.

   If $\dfrac{p+q}{2}$ is irrational, take $x = \dfrac{p+q}{2}$ and we are done.

   If $\dfrac{p+q}{2}$ is rational, call it r, take the average of p and r: $p < \dfrac{p+r}{2} < r < q$. Since p is irrational and r is rational, $\dfrac{p+r}{2}$ is irrational. In this case, we take $x = \dfrac{3p+q}{4}$. □

10. Given n real numbers $a_1, a_2, \ldots, a_n$. Show that there exists an a_i $(1 \le i \le n)$ such that a_i is greater than or equal to the mean (average) value of the n numbers.

Proof. Prove by contradiction.

Let $\bar{a}$ denote the mean value of the n given numbers. Suppose $a_i < \bar{a}$ for all a_i. Then

$$\bar{a} = \frac{a_1 + a_2 + \cdots + a_n}{n} < \frac{\bar{a} + \bar{a} + \cdots + \bar{a}}{n} = \frac{n\bar{a}}{n} = a.$$

We derive $\bar{a} < \bar{a}$, which is a contradiction.

Hence there must be some a_i such that $a_i > \bar{a}$. □

11. Prove that the following statement is false: there is an irrational number a such that for all irrational number b, ab is rational.

Thought process: prove the negation of the statement: for every irrational number a, there is an irrational number b such that ab is irrational.

Proving technique: constructive proof (note that we can consider multiple cases and construct more than one b)

Proof. Given an irrational number a, let us consider $\dfrac{\sqrt{2}}{a}$.

Case (i): $\dfrac{\sqrt{2}}{a}$ is irrational.

Take $b = \dfrac{\sqrt{2}}{a}$. Then $ab = \sqrt{2}$ which is irrational.

Case (ii): $\dfrac{\sqrt{2}}{a}$ is rational.

Then the reciprocal $\dfrac{a}{\sqrt{2}}$. Since $\sqrt{6}$ is irrational, the product $\left(\dfrac{a}{\sqrt{2}}\right)\sqrt{6} = a\sqrt{3}$ is irrational. Take $b = \sqrt{3}$, which is irrational. Then $ab = a\sqrt{3}$ which is irrational. □

12. Prove that there are infinitely many prime numbers that are congruent to 3 modulo 4.

Proof. Prove by contradiction.

Suppose there are only finitely many primes that are congruent to 3 modulo 4. Let $p_1, p_2, \ldots, p_m$ be the list of all the primes that are congruent to 3 modulo 4.

We construct an integer M by $M = (p_1 p_2 \cdots p_m)^2 + 2$.

We have the following observation:

(i) $M \equiv 3 \mod 4$.

(ii) Every p_i divides $M - 2$.

(iii) None of the p_i divides M. [Otherwise, together with (ii), this will imply p_i divides 2, which is impossible.]

(iv) M is not a prime number. [Otherwise, by (i), M is a prime number congruent to 3 modulo 4. But $M \neq p_i$ for all $1 \leq i \leq m$. This contradicts the assumption that $p_1, p_2, \ldots, p_m$ are all the prime numbers congruent to 3 modulo 4.]

From the above discussion, we know that M is a composite number by (iv). So it has a prime factorization $M = q_1 q_2 \cdots q_k$.

Since M is odd, all these prime factors q_j must be odd, and hence q_j must be congruent to either 1 or 3 modulo 4.

By (iii), q_j cannot be any of the p_i. So all q_j must be congruent to 1 modulo 4. Then M, which is the product of q_j, must also be congruent to 1 modulo 4.

This contradicts (i) that M is congruent to 3 modulo 4.

Hence we conclude that there must be infinitely many primes that are congruent to 3 modulo 4. $\square$

13. Number and Proof Tutorial 2 (Additional Problems) - wait for solutions

14. (H3 Specimen paper) For any positive integer n, if one square is removed from a $2^n \times 2^n$ checkerboard, the remaining squares can be completely covered by triominoes (an L-shaped domino consisting of three squares).

Proof. **Base case**: $P(1)$ is clearly true.

Inductive step: $P(k) \implies P(k+1)$ is true for all k, i.e. if a $2^k \times 2^k$ checkerboard with a square removed can be completely covered by triominoes, then a $2^{k+1} \times 2^{k+1}$ checkerboard with a square removed can be completely covered by triominoes.

(i) Divide the $2^{k+1} \times 2^{k+1}$ checkerboard into four $2^k \times 2^k$ sub-boards.

(ii) One of the sub-boards include the removed square.

(iii) WLOG, assume the top left sub-board has the removed square.

(iv) By induction hypothesis, this sub-board can be covered by triominoes.

(v) For the top right sub-board, we cover it with trominoes with a remaining square at the bottom left corner.

(vi) For the bottom right sub-board, we cover it with trominoes with a remaining square at the top left corner.

(vii) For the bottom left sub-board, we cover it with trominoes with a remaining square at the top right corner.

(viii) The remaining three squares from (v) to (vii) are connected and can be covered by one triomino.

$\square$

Remark. Although it is easy to visualise this by drawing it out, always produce a *written* proof.

15. For every positive integer $n \geq 4$,

$$n! > 2^n.$$

Proof. Let $P(n): n! > 2^n$

Base case: $P(4)$

LHS: $4! = 4 \times 3 \times 2 \times 1 = 24$, RHS: $2^4 = 16 < 24$

So $P(4)$ is true.

Inductive step: $P(k) \implies P(k+1)$ for all $k \in \mathbb{Z}^+_{\geq 4}$

$$\begin{aligned} k! &> 2^k \\ (k+1)k! &> 2^k(k+1) \\ &> 2^k 2 \quad \text{since from } k \geq 4,\ k+1 \geq 5 > 2 \\ &= 2^{k+1} \end{aligned}$$

hence proven $P(k) \implies P(k+1)$ for integers $k \geq 4$.

By PMI, we have proven $P(n)$ for all integers $n \geq 4$. □

16. (2023 TJC Further Mathematics) Prove by mathematical induction, for $n \geq 2$,

$$\sqrt[n]{n} < 2 - \frac{1}{n}.$$

Proof. Let $P(n): \sqrt[n]{n} < 2 - \dfrac{1}{n}$ for $n \geq 2$.

Base case: $P(2)$

When $n = 2$, $\sqrt{2} = 1.41\cdots < 2 - \dfrac{1}{2} = 1.5$ which is true. Hence $P(2)$ is true.

Inductive step: $P(k) \implies P(k+1)$ for all $k \in \mathbb{Z}^+_{\geq 2}$

Assume $P(k)$ is true for $k \geq 2, k \in \mathbb{Z}^+$, i.e.

$$\sqrt[k]{k} < 2 - \frac{1}{k} \implies k < \left(2 - \frac{1}{k}\right)^k$$

We want to prove that $P(k+1)$ is true, i.e.

$$k + 1 < \left(2 - \frac{1}{k+1}\right)^{k+1}$$

Since $k > 2$, we have

$$\begin{aligned} \left(2 - \frac{1}{k+1}\right)^{k+1} &> \left(2 - \frac{1}{k}\right)^{k+1} \quad \because k > 2 \\ &= \left(2 - \frac{1}{k}\right)^k \left(2 - \frac{1}{k}\right) \\ &> k\left(2 - \frac{1}{k}\right) \quad \text{[by inductive hypothesis]} \\ &= 2k - 1 = k + k - 1 > k - 1 \because k > 2 \end{aligned}$$

Hence $P(k+1)$ is true.

Since $P(2)$ is true and $P(k) \implies P(k+1)$, by mathematical induction $P(n)$ is true. □

17. Prove that for all integers $n \geq 3$,

$$\left(1+\frac{1}{n}\right)^n < n$$

Proof. **Base case:** $P(3)$

On the LHS, $\left(1+\frac{1}{3}\right)^3 = \frac{64}{27} = 2\frac{10}{27} < 3$. Hence $P(3)$ is true.

Inductive step: $P(k) \implies P(k+1)$ for all $k \in \mathbb{Z}^+_{\geq 3}$

Our inductive hypothesis is

$$\left(1+\frac{1}{k}\right)^k < k$$

Multiplying both sides by $\left(1+\frac{1}{k}\right)$ (to get a $k+1$ in the power),

$$\left(1+\frac{1}{k}\right)^k \left(1+\frac{1}{k}\right) = \left(1+\frac{1}{k}\right)^{k+1} < k\left(1+\frac{1}{k}\right) = k+1$$

Since $k < k+1 \iff \frac{1}{k} > \frac{1}{k+1}$,

$$\left(1+\frac{1}{k}\right)^{k+1} > \left(1+\frac{1}{k+1}\right)^{k+1}$$

The rest of the proof follows easily. □

A sequence of integers F_i, where integer $1 \leq i \leq n$, is called the **Fibonacci sequence** if and only if it is defined recursively by $F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n > 2$.

18. Let F_i be the Fibonacci sequence. Prove that $3 \nmid n$ if and only if F_n is odd.

Proof. **Forward direction:** $3 \nmid n \implies F_n$ is odd

Backward direction: F_n is odd $\implies 3 \nmid n$ (We prove the contrapositive: $3 \mid n \implies F_n$ is even)

Hence we only need to prove the following via PMI:

- $(\forall n \in \mathbb{Z}^+ \text{ and } 3 \nmid n), F_n$ is odd
  Base case: $P(1), P(2)$
  Inductive step: $P(k) \implies P(k+3)$ for all $k \geq 1$
- $(\forall n \in \mathbb{Z}^+ \text{ and } 3 \mid n), F_n$ is even
  Base case: $P(3)$
  Inductive step: $P(k) \implies P(k+3)$ for all $k \geq 3$

[Note that we have partitioned the domain into two.]

Hence to show $\forall n \in \mathbb{Z}^+ \; P(n)$,

Base case: $P(1), P(2), P(3)$

Inductive step: $\forall k \in \mathbb{Z}^+ \; P(k) \implies P(k+3)$ □

19. Let a_i where integer $1 \le i \le n$ be a sequence of integers defined recursively by initial conditions $a_1 = 1$, $a_2 = 1$, $a_3 = 3$ and the recurrence relation $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n > 3$.

For all $n \in \mathbb{Z}^+$, prove that
$$a_n \le 2^{n-1}.$$

Proof. Let $P(n) : a_n \le 2^{n-1}$.

Given the recurrence relation, it could be possible to use $P(k), P(k+1), P(k+2)$ to prove $P(k+3)$ for all $k \in \mathbb{Z}^+$.

Base case: $P(1), P(2), P(3)$

$P(1) : a_1 = 1 \le 2^{1-1} = 1$ is true.

$P(2) : a_2 = 1 \le 2^{2-1} = 2$ is true.

$P(3) : a_3 = 3 \le 2^{3-1} = 4$ is true.

Inductive step: $P(k) \wedge P(k+1) \wedge P(k+2) \implies P(k+3)$ for all $k \in \mathbb{Z}^+$

By inductive hypothesis, for $k \in \mathbb{Z}^+$ we have $a_k \le 2^k, a_{k+1} \le 2^{k+1}, a_{k+2} \le 2^{k+2}$.

$$\begin{aligned}
a_{k+3} &= a_k + a_{k+1} + a_{k+2} && \text{[start from recurrence relation]} \\
&\le 2^k + 2^{k+1} + 2^{k+2} && \text{[use inductive hypothesis]} \\
&= 2^k(1 + 2 + 2^2) \\
&< 2^k(2^3) && \text{[approximation, since } 1 + 2 + 2^2 < 2^3\text{]} \\
&= 2^{k+3}
\end{aligned}$$

which is precisely $P(k+3) : a_{k+3} \le 2^{k+3}$. $\square$

§2 Sequences and Series

§3 Inequalities

§4 Distribution Problems

(including Bijection Principle)

§5 Recurrence Relations