H2 Mathematics

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Abstract

This set of notes follows the Singapore GCE A-Level H2 Mathematics syllabus.

Graphing Calculator

Here are some combinations of keys that you might find helpful.

General

- Reset: `2` `8` `on`
- Return to main page: `quit`
- Display MATH menu: `math`
- Change calculator mode: `mode`

Basic input methods

- Change decimal to fraction: `math` `1:▸Frac`
- Change fraction to decimal: `math` `2:▸Dec`
- Cube root: `math` `4:∛`
- Equality and inequality symbols: `test` `CONDITIONS`

Graphing

- Input equation of graph: `y=`
- View graph of equation: `graph`
- Restrict domain/range of graph: `window` `zoom` `ZoomFit`
- Zoom in or out: `zoom`
- Find y value at a specific x value: `calc` `1:value`
- Find x-intercept: `calc` `2:zero`
- Find point of intersection between graphs: `calc` `5:intersect`
- Find minimum or maximum point of graph: `calc` `3:minimum` or `4:maximum`
- Parametric functions: `mode` `FUNCTION` `PARAMETRIC` then proceed to graph
- Conic sections: `apps` `2:Conics`
- Piecewise function: `math` `B:piecewise`
- Composite function: input Y1 and Y2, `f4` to input Y2(Y1)

Equations and Inequalities

- Solve quadratic equation: `apps` `4:Plysmlt2` `1:POLYNOMIAL ROOT FINDER`
- Solve system of linear equations: `apps` `4:Plysmlt2` `2:SIMULTANEOUS EQN SOLVER`
- Find solution(s) to a complicated equation: graph the functions, then find point(s) of intersection

Sequences and Series

- Generate a sequence: `list` `OPS` `5:seq`
- Use graph to determine behaviour of sequence/series: `table`
- Evaluate summation: `math` `0:summation`

Calculus

- Evaluate derivative/gradient at a point (graph): `calc` `6:dy/dx`
- Evaluate integral given the lower and upper limits (graph): `calc` `7:∫f(x)dx`
- Evaluate derivative at an x value: `math` `8:nDeriv`
- Evaluate definite integral: `math` `9:fnInt`

Complex Numbers

- Changde mode settings: `mode` `REAL` `a+bi`
- Convert from rectangular to polar: `math` `CMPLX` `7:Polar`
- Simplifying complex expressions: just input
- Modulus: `math` `CMPLX` `5:abs`
- Argument: `math` `CMPLX` `4:angle`

Probability

- Permutation: `math` `PROB` `2:nPr`
- Combination: `math` `PROB` `3:nCr`
- Factorial: `math` `PROB` `4:!`

Distributions

- Binomial distribution (pdf): `distr` `A:binompdf`
- Binomial distribution (cdf): `distr` `B:binomcdf`
- Normal distribution (cdf): `distr` `2:normalcdf`
- Inverse normal: `distr` `3:invNorm`
- Get summary statistics: `stat` `1:edit` to input data, `stat` `1-Var Stats`
- Z-test: `stat` `TESTS` `1:Z-Test`

Correlation and Regression

- Scatter diagram: input values into lists `L1` and `L2`, `STAT PLOTS` `1:Plot1` `Xlist` `Ylist` `9:ZoomStat`
- Value of r: input values, `stat` `calc` `8:LinReg(a+bx)`
- Regression line on scatter diagram: `F4` `1:Y1` to store equation into Y1
- Linearise: bring cursor to `L3`, key in `1/L1` to generate values of $\frac{1}{x}$

Contents

Part I

Pure Mathematics

§1 Functions

- **Domain**: set of possible inputs

  Range: set of possible outputs

  Vertical line test: check for functions

$$f \text{ is a function} \iff \forall k \in D_f, x = k \text{ cuts the graph at most once.} \tag{1}$$

- **One-one function**: no two distinct elements in the given domain have the same image under f.

$$\forall x_1, x_2 \in D_f, x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$

  This means that every output corresponds to exactly one input.

  Horizontal line test: check for one-one functions

$$\begin{aligned} &f \text{ is one-one} \\ \iff &\forall k \in R_f, y = k \text{ cuts the graph at exactly one point.} \\ \iff &\forall k \in \mathbb{R}, y = k \text{ cuts the graph at most once.} \end{aligned} \tag{2}$$

  Conversely, for a function that is not one-one, provide a specific counter-example of a horizontal line that cuts the graph at more than one point.

- Let f be a one-one function. Then f has an **inverse function** f^{-1} defined as

$$f^{-1}(y) = x \iff f(x) = y, \forall x \in D_f$$

  Domain and range: $D_{f^{-1}} = R_f$, $R_{f^{-1}} = D_f$

  For the inverse function f^{-1} to exist, f is a one-one function (check using horizontal line test).

  To find the inverse of a given function, make x the subject.

  – If a function is a quadratic equation in terms of x, either complete the square, or apply quadratic formula, to make x the subject.

  – When applying square root, remember to add $\pm$ AND choose the sign by looking at the domain of f.

> **Example.** The function f is defined by $f : x \mapsto (x-3)^2+1$, $x \geq 3$. Show that the inverse function of f exists and find f^{-1} in similar form.

Solution. Since every horizontal line $y = k$, $k \geq 1$ cuts the graph $y = f(x)$ exactly once, f is one-one and f^{-1} exists.

Let $y = (x-3)^2 + 1$. Then

$$(x-3)^2 = y - 1 \implies x = 3 \pm \sqrt{y-1}.$$

Since $x \geq 3$, $x = 3 - \sqrt{y-1}$.

Hence $f^{-1} : x \mapsto 3 - \sqrt{x-1}$, $x \geq 1$. □

Graphically, f and f^{-1} are **reflections** of each other in the line $y = x$.[1] Hence $y = f(x)$, $y = f^{-1}(x)$ and $y = x$ **intersect at one same point**.[2]

- **Composite function** gf: *all* elements in the domain of function f are directly mapped to the elements in the range of function g.

$$gf \text{ exists} \iff R_f \subseteq D_g.^{3} \tag{3}$$

  Domain and range: $D_{gf} = D_f$, $R_{gf} = R_g$

> **Example.** The functions f and g are defined as follows:
>
> $$\begin{aligned} f &: x \mapsto (x+1)(x-12), \quad x \in \mathbb{R} \\ g &: x \mapsto x^2 + 2x, \quad x \in \mathbb{R}, x < -1 \end{aligned}$$
>
> Show that the composite function fg exists, and define fg in a similar form.

Solution.

$$\begin{aligned} g(x) &= x^2 + 2x, \quad x \in \mathbb{R}, x < -1 \\ &= (x+1)^2 - 1, \end{aligned}$$

thus $R_g = (-1, \infty)$.

Since $R_g = (-1, \infty) \subseteq \mathbb{R} = D_f$, fg exists.

$$\begin{aligned} fg : x &\mapsto f(x^2 + 2x), \quad x < -1 \\ &= (x^2 + 2x + 1)(x^2 + 2x - 12), \quad x < -1 \end{aligned}$$

□

To determine the range of a composite function gf,

– use **two-stage mapping**: find range of f, then on the graph of g, put range of f as the domain and find the corresponding range of g. OR

– sketch the graph of gf, then find the range of gf based on the domain of gf (i.e. domain of f)

[1] Proof: Let (a, b) be a point on the curve $y = f(x)$. Then (b, a) is a point on the curve $y = f^{-1}(x)$ since $f(a) = b \iff a = f^{-1}(b)$.

[2] However, this is not the case for functions where $y = f(x)$ and $y = f^{-1}(x)$ do not intersect in the given domain, or for functions where $f(x) = f^{-1}(x)$.

[3] Proof: The domain of function g must include values of the range of f, so that the function g is well-defined as every element in its domain, as well as in R_f, is mapped to something.

> **Example.** The functions f and g are defined by
>
> $$f : x \to \frac{2+x}{2-x}, \quad -2 \le x \le 1,$$
> $$g : x \to x^2 + 2x + 2, \quad x \in \mathbb{R}$$
>
> Show that the composite function gf exists and find the range of gf.

Solution. Using GC, $R_f = [0,3]$ and $R_g = [1,\infty)$.
Since $[0,3] = R_f \subseteq D_g = \mathbb{R}$, gf exists.
By two-stage mapping,

$$[-2,1] \xrightarrow{f} [0,3] \xrightarrow{g} [2,17]$$

where range of f is now the new domain of g, hence the corresponding new range is the range of gf.

Hence $R_{gf} = [2,17]$. □

Identity function: returns the same value, which was used as its input.

$$f^{-1}f(x) = ff^{-1}(x) = x$$

> **Remark.** Even though the composite functions $f^{-1}f$ and ff^{-1} have the same rule, they may have different domains. $D_{f^{-1}f} = D_f$ whereas $D_{ff^{-1}} = D_{f^{-1}}$.

Problem 1. The function f and g are defined by

$$f : x \mapsto \sqrt{2-x}, \quad x \le 2.$$
$$g : x \mapsto \frac{(x-1)^2}{x+5}, \quad x \in \mathbb{R}, a < x < -5.$$

(a) (i) State the smallest value of a for fg^{-1} to exist. [1]

(ii) With this smallest value of a, give a definition for the function fg^{-1} and find its exact range. [5]

(b) Given that $fh(x) = -x$, determinen $h(x)$ and hence state the largest domain for h. [2]

Solution.

(a) (i)

(ii)

(b)

□

Problem 2 (N2023/I/7).

Problem 3 (N2022/I/6).

Problem 4 (N2021/II/3).

Problem 5 (N2018/I/5).

Problem 6 (N2017/II/3).

Problem 7 (N2015/II/3).

§2 Graphs

- Features to include in graph sketch:
  1. **Stationary points**
     - Maximum point
     - Minimum point
     - Point of inflexion
  2. **Intercepts**
  3. **Asymptotes**
     - Horizontal asymptote: line $y = a$ where $x \to \pm\infty$, $y \to a$
     - Vertical asymptote: line $x = a$ where $x \to a$, $y \to \pm\infty$
     - Oblique asymptote: line $y = ax + b$ where $x \to \pm\infty$, $y - (ax + b) \to 0$

  To determine the restriction on possible values of x or y, use *discriminant*.

- Conic Sections
  - **Circle** with centre (h, k) and radius r:
  $$(x-h)^2 + (y-k)^2 = r^2 \tag{4}$$
  - **Ellipse** with centre (h, k) and semi-major axis of a units, semi-minor axis of b units:
  $$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \tag{5}$$
  - **Horizontal hyperbola** with centre (h, k):
  $$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \tag{6}$$
    Distance from turning points to centre: a units (in x-direction)

    Equations of oblique asymptotes: $y - k = \pm\frac{b}{a}(x - h)$
  - **Vertical hyperbola** with centre (h, k):
  $$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1 \tag{7}$$
    Distance from turning points to centre: b units (in y-direction)

    Equations of oblique asymptotes: $y - k = \pm\frac{b}{a}(x - h)$
  - **Regular parabola** with vertex (h, k):
  $$y = a(x-h)^2 + k \tag{8}$$
  - **Sideways parabola** with vertex (h, k):
  $$x = a(y-k)^2 + h \tag{9}$$
  - Rectangular hyperbola

    In the case of horizontal and vertical asymptotes,
  $$y = \frac{ax+b}{cx+d} \tag{10}$$

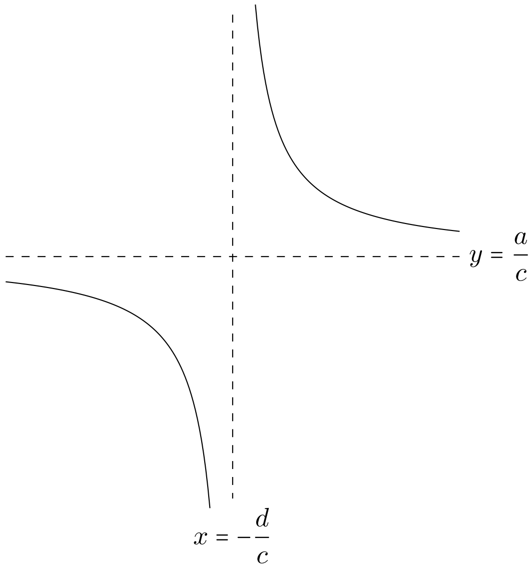
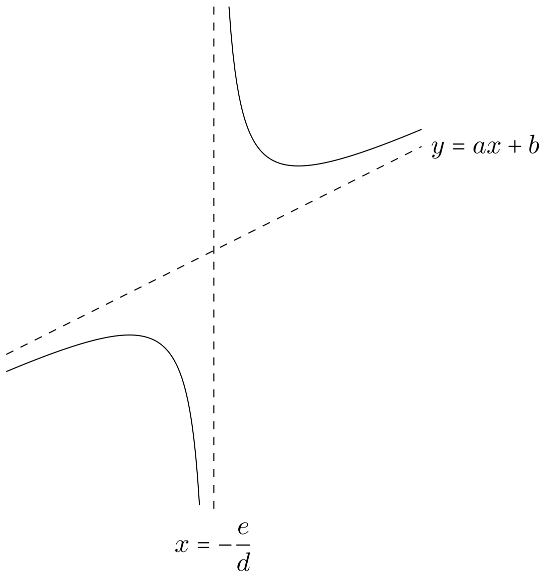


Figure 1: Rectangular hyperbola

In the case of oblique asymptotes,

$$y = ax + b + \frac{c}{dx + e} \tag{11}$$



- Parametric equations
  - To sketch the Cartesian graph of parametric equations, using `MODE` to change the mode from `FUNCTION` to `PARAMETRIC` on GC. Remember to take note of the domain of the parameter t.
  - To find the Cartesian equation, eliminate the parameter t by solving the equations simultaneously. A Cartesian equation should only contain the variables x and y.
  - To convert a Cartesian equation to its parametric form, if involving trigonometric expressions, the Pythagorean trigonometric identity $\sin^2 x + \cos^2 x = 1$ may be handy.
- **Transformations**

  To get a function $cf(bx + a) + d$ from $f(x)$,

$$f(x) \xrightarrow{1} f(x + a) \xrightarrow{2} f(bx + a) \xrightarrow{3} cf(bx + a) \xrightarrow{4} cf(bx + a)d$$

1. translate by a units in the negative x direction
2. scale by factor of $\frac{1}{b}$ parallel to x-axis

3. scale by factor of c parallel to the y-axis
4. translate by d units in the positive y direction.

Recommended order: translation, scaling, reflection

Type	Equation	Replacement	Graph
Translation	$y = f(x)+a$	$y \mapsto y-a$	Translate a units in positive y-direction
	$y = f(x)-a$	$y \mapsto y+a$	Translate a units in negative y-direction
	$y = f(x-a)$	$x \mapsto x-a$	Translate a units in positive x-direction
	$y = f(x+a)$	$x \mapsto x+a$	Translate a units in negative x-direction
Reflection	$y = -f(x)$	$y \mapsto -y$	Reflect in x-axis
	$y = f(-x)$	$x \mapsto -x$	Reflect in y-axis
Scaling	$y = af(x)$	$y \mapsto \dfrac{y}{a}$	Scale by factor of a parallel to y-axis
	$y = f\left(\dfrac{x}{a}\right)$	$x \mapsto \dfrac{x}{a}$	Scale by factor of a parallel to x-axis
Modulus	$y = \lvert f(x)\rvert$	$y \mapsto \lvert y\rvert$	Reflect $y < 0$ in the x-axis
	$y = f\left(\lvert x\rvert\right)$	$x \mapsto \lvert x\rvert$	Ignore $x < 0$, keep and reflect $x \geq 0$ in the y-axis
Reciprocal	$y = \dfrac{1}{f(x)}$	$y \mapsto \dfrac{1}{y}$	x-intercept becomes vertical asymptote (and vice versa), maximum point becomes minimum point (and vice versa), horizontal asymptote $y = a$ becomes horizontal asymptote $y = \frac{1}{a}$
Derivative	$y = f'(x)$	–	Vertical asymptote remains the same, horizontal aymptote $y = a$ becomes horizontal asymptote $y = 0$ (x-axis), oblique asymptote $y = mx + c$ becomes horizontal asymptote $y = m$, stationary point (a, b) becomes x-intercept $x = a$, point of inflexion (increasing/decreasing function) becomes turning point (max/min)

Problem 8. The plane p_2 is obtained by first translating $p_1 : \mathbf{r} \cdot \begin{pmatrix} 8 \\ -1 \\ 4 \end{pmatrix} = -5$ 2 units in the positive z-direction, and then reflecting in the xz-plane. Obtain an equation of p_2 in scalar product form.

Solution. Transformation: replacement of variables.

Converting p_1 to Cartesian form gives $8x - y + 4z = -5$.

$$8x - y + 4z \xrightarrow{\text{hello}}$$

□

§3 Equations and Inequalities

Equations

- **Systems of linear equations** can be solved efficiently using `PlySmlt2` on GC.
- Types of solutions:
  1. Unique solution
  2. Infinitely many solutions
  3. No solutions
- Questions often involve practical problems, from which systems of linear equations are set up.

Inequalities

- Important points to take note when solving inequalities:
  1. Do not cross multiply without knowing whether terms are positive or not.
  2. Know the difference between "and" and "or", i.e. intersection and union of sets.
  3. Solutions should not be equal to roots of denominator.

  Some manipulations before using the methods below:

  – Directly deduce after moving all terms to one side.
  – Multiply the square of a term in the denominator.
  – Either the numerator or denominator is always positive (working is required to show this).

- **Modulus function**:
$$\left|f(x)\right| = \begin{cases} f(x), & \text{if } f(x) \geq 0, \\ -f(x), & \text{if } f(x) < 0. \end{cases} \tag{12}$$
  From the definition of modulus, we have
$$\begin{aligned} |x| < a &\iff -a < x < a \\ |x| > a &\iff x > a \text{ or } x < -a \end{aligned}$$

- **Test-value method**
  1. Indicate the critical value(s) on a number line.
  2. Choose an x-value within each interval as the *test-value*.
  3. Plug in the test-value to evaluate whether the polynomial is positive or negative within that interval.

- **Graphical method**
  – For inequalities in the form of $P(x) > 0$, sketch graph of $y = P(x)$ and its x-intercepts, identify region of graph where $P(x) > 0$.
  – More generally, for inequalities in the form of $P(x) > Q(x)$, sketch the graph of $y = P(x)$ and $y = Q(x)$, then identify region of graph where the inequality holds.

- **Solutions of related inequalities**
  Replace x with some expression of x using the solutions of inequalities solved in earlier parts of the question.

Problem 9. Without using a graphic calculator, solve the inequality
$$\frac{2x^2+1}{2-x^2} \leq \frac{2+x^2}{x^2}.$$
[4]

Solution. Let $u = x^2$. Then the given inequality reduces to

$$\frac{2u+1}{2-u} \leq \frac{2+u}{u}.$$

Moving terms to one side and factorising,

$$\frac{(3u+4)(u-1)}{u(2-u)} \leq 0.$$

Idenfity critical values

$$u \leq -\frac{4}{3} \text{ (rej.)} \quad \text{or} \quad 0 < u \leq 1 \quad \text{or} \quad u > 2$$

thus

$$x^2 > 2 \quad \text{or} \quad x^2 le 1$$

and so

$$\boxed{x > \sqrt{2} \quad \text{or} \quad x < -\sqrt{2} \quad \text{or} \quad -1 \leq x \leq 1, x \neq 0}$$

□

Problem 10. The inequality $\dfrac{ax^2+bx+c}{x^2-3x+9} < 0$ is satisfied for all real values of x such that a, b and c are constants. State a relation involving a, b and c and also the range of values of a. [4]

Solution. Since

$$x^2 - 3x + 9 = \left(x - \frac{3}{2}\right)^2 + \frac{27}{4} \geq \frac{27}{4} > 0 \quad \forall x \in \mathbb{R}$$

thus $ax^2 + bx + c < 0$ for all $x \in \mathbb{R}$. By discriminant, $\boxed{b^2 - 4ac < 0}$. Obvious that $\boxed{a < 0}$. □

§4 Sequences and Series

For this syllabus we only deal with real sequences and series.

- **Sequence**: a set of numbers, denoted by (u_n), $n \in \mathbb{Z}^+$.

  A sequence can be generated by giving a formula for the n-th term, e.g. $u_n = f(n)$.

  Describing behaviour of a sequence

  1. **Trend**:
     - (strictly) increasing: $u_{n+1} > u_n$
     - (strictly) decreasing: $u_{n+1} < u_n$
     - alternating
  2. **Convergence**:
     - convergent: as $n \to +\infty$, $u_n \to L$
     - divergent: as $n \to +\infty$, $u_n \to +\infty$ or $u_n \to -\infty$.

  To find the limit of $\dfrac{P(n)}{Q(n)}$ where $P(n)$ and $Q(n)$ are polynomials, divide the numerator and denominator by the highest power available.

- **Series**: sum of terms of a sequence. Sum of n terms is denoted by S_n.

  To find the term for sequence when given the series,

  $$u_n = S_n - S_{n-1}. \tag{13}$$

  For the sum to infinity S_∞ of a series to exist, the series converges ($\exists L \in \mathbb{R}$ s.t. $\lim_{n\to\infty} S_n = L$); conversely, the sum to infinity does not exist if the series diverges.

- **Arithmetic progression**: a sequence in which successive terms differ by a common difference.

  The formula for n-th term is

  $$u_n = a + (n-1)d. \tag{14}$$

  The formula for the sum is

  $$S_n = \frac{n}{2}[2a + (n-1)d]. \tag{15}$$

  To show that u_n is an AP, show that $u_n - u_{n-1}$ is a constant.

- **Geometric progression**: a sequence in which successive terms differ by a common ratio.

  The formula for n-th term is

  $$u_n = ar^{n-1}. \tag{16}$$

  The formula for the sum is

  $$\begin{aligned} S_n &= \frac{a(1-r^n)}{1-r}, \quad |r|<1 \\ &= \frac{a(r^n-1)}{r-1}, \quad |r|>1 \end{aligned} \tag{17}$$

  To show that u_n is a GP, $\dfrac{u_n}{u_{n-1}}$ is a constant.

  The formula for sum to infinity is

  $$S_\infty = \frac{a}{1-r}, \quad |r|<1 \tag{18}$$

  To show that the sum to infinity of a GP exists, the GP converges, so show that common ratio $|r|<1$.

- Summation series

  - Standard algebraic series

    $$\sum_{r=1}^{n} r = \frac{n(n+1)}{2} \tag{19}$$

    $$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6} \tag{20}$$

    $$\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4} \tag{21}$$

    > **Remark.** The lower limit must be 1. Otherwise we have to use $\sum_{r=m}^{n} u_r = \sum_{r=1}^{n} u_r - \sum_{r=1}^{m-1} u_r$ to change lower limit.

  - **Method of differences**

    The general term $u_r = f(r) - f(r-1)$, then

    $$\begin{aligned} \sum_{r=1}^{n} u_r &= \sum_{r=1}^{n} (f(r) - f(r-1)) \\ &= \begin{cases} \cancel{f(1)} - f(0) + \\ \cancel{f(2)} - \cancel{f(1)} + \\ \vdots \\ \cancel{f(n-1)} - \cancel{f(n-2)} + \\ f(n) - \cancel{f(n-1)} \end{cases} \\ &= f(n) - f(0) \end{aligned}$$

    > **Remark.** Must show **diagonal** cancellation of intermediate terms in the working.

- When solving questions with practical scenarios, use a table to tabulate values e.g. n and u_n.

§5 Differentiation

- Differentiation rules
  - Scalar multiplication
$$(kf)' = kf' \tag{22}$$
  - Sum/Difference rule
$$(f \pm g)' = f' \pm g' \tag{23}$$
  - Product rule
$$(fg)' = f'g + fg' \tag{24}$$
  - Quotient rule
$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2} \tag{25}$$
  - Chain rule
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \tag{26}$$
- New functions
  - Exponential functions
$$\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x \ln a$$
  - Logarithmic functions
$$\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\log_a x = \frac{1}{x\ln a}$$
  - Trigonometric functions
$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x \tan x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\operatorname{cosec}^2 x$$
  - Inverse trigonometric functions
$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1} x = \frac{1}{1+x^2}$$

> **Remark.** Remember to apply chain rule whenever a function $f(x)$, instead of x, is involved.

- Applications
  - Increasing and decreasing function
    As long as f is continuous,

$f' > 0$	f increasing
$f' < 0$	f decreasing
$f' = 0$	f stationary

  - Concavity

$f'' > 0$	f concave
$f'' < 0$	f convex

  - Nature of stationary points
    First derivative test:
    * $f'(a^-) < 0, f'(a) = 0, f'(a^+) > 0 \implies$ minimum point
    * $f'(a^-) > 0, f'(a) = 0, f'(a^+) < 0 \implies$ maximum point
    * $+0+$ or $-0-$ inflexion point

    Second derivative test:

$f''(a) > 0$	a is minimum point
$f''(a) = 0$	unable to determine
$f''(a) < 0$	a is maximum point

  - Tangents and normals
    Equation of **tangent** to curve $y = f(x)$ at point (a, b):
$$y - b = f'(a)(x - a)$$
    where gradient of tangent $= f'(a)$.
    Equation of **normal** to curve $y = f(x)$ at point (a, b):
$$y - b = -\frac{1}{f'(a)}(x - a)$$
    where gradient of normal $= -\dfrac{1}{f'(a)}$.
  - Optimisation problems
    To maximise/minimise a quantity $A(x)$ as x varies,
    1. Solve $\frac{\mathrm{d}A}{\mathrm{d}x}$ to find the stationary values of A. (If there are two variables involved, reduce the equation to a function of 1 variable, through the use of another restriction equation obtained.)
    2. Use 1st or 2nd derivative tests to check for the nature of stationary points.
  - Connected rates of change
    Apply chain rule:
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

- **Implicit differentiation**: remember to apply chain rule.
- **Parametric differentiation**: if x and y are functions of a parameter t, applying chain rule gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

§6 Maclaurin Series

- **Binomial theorem**: $\forall n \in \mathbb{Z}^+$

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \cdots + b^n \tag{27}$$

Binomial series: $\forall n \in \mathbb{R}, n \neq 0$, which includes **negative** and **fractional** n

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r + \cdots \tag{28}$$

which has validity range $|x| < 1$.

> **Example.** Expanding in terms of ascending powers of x,
>
> $$(a+x)^n = a^n\left(1+\frac{x}{a}\right)^n = a^n\left[1 + n\left(\frac{x}{a}\right) + \frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{x}{a}\right)^3 + \cdots\right], \quad \left|\frac{x}{a}\right| < 1 \text{ i.e. } |x| < |a|.$$
>
> Expanding in terms of descending powers of x,
>
> $$(a+x)^n = x^n\left(1+\frac{a}{x}\right)^n = x^n\left[1 + n\left(\frac{a}{x}\right) + \frac{n(n-1)}{2!}\left(\frac{a}{x}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{a}{x}\right)^3 + \cdots\right], \quad \left|\frac{a}{x}\right| < 1 \text{ i.e. } |x| > |a|.$$

- The **Taylor series** is a power series centered at $x = a$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(0)}{3!}(x-a)^3 + \frac{f^{(4)}(0)}{4!}(x-a)^4 + \cdots \tag{29}$$

The **Maclaurin series** is a special case of the Taylor series, centered at $x = 0$:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots \tag{30}$$

To obtain the Maclaurin series of a given $f(x)$,

1. find $f'(x), f''(x), \dots$ until the desired order,
2. substitute $x = 0$ into $f'(x), f''(x), \dots$,
3. substitute $f'(0), f''(0), \dots$ into the Maclaurin expression above.

> **Remark.** It is often useful to apply implicit differentiation in obtaining Maclaurin series.

- Expansion of **standard series** and their validity range:

$$\begin{aligned}
(1+x)^n &= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!} + \cdots && |x| < 1 \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots && \text{all } x \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots && \text{all } x \text{ in radians} \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots && \text{all } x \text{ in radians} \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots && -1 < x \leq 1
\end{aligned} \tag{MF26}$$

- **Small angle approximation**: approximation of trigonometric functions for a sufficiently small x such that higher powers of x can be neglected.

$$\begin{aligned}
\sin x &\approx x \\
\cos x &\approx 1 - \frac{x^2}{2} \\
\tan x &\approx x
\end{aligned} \tag{31}$$

> **Remark.** When x is small, $\sin\left(x+\frac{\pi}{3}\right) \neq x + \frac{\pi}{3}$ as $x + \frac{\pi}{3}$ is not small; so need to use compound angle formula.

Remark. When there is reciprocal, binomial series is involved. For example,

$$\frac{\sin x}{\cos x + 1} \approx \frac{x}{2 - \frac{x^2}{2}} = x\left(2 - \frac{x^2}{2}\right)^{-1}.$$

Remark. Be familiar with sine and cosine rule.

§7 Integration Techniques

Note that the arbitrary constant C will be omitted throughout this text.

- Standard functions

$$\int (ax+b)^n \,\mathrm{d}x = \frac{(ax+b)^{n+1}}{a(n+1)}$$

$$\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln|ax+b|$$

$$\int e^{ax+b}\,\mathrm{d}x = \frac{1}{a}e^{ax+b}$$

- Systematic integration

$$\int f'(x)[f(x)]^n \,\mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1}$$

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)|$$

$$\int f'(x)e^{f(x)}\,\mathrm{d}x = e^{f(x)}$$

- Trigonometric functions

$$\int \sin x \,\mathrm{d}x = -\cos x$$

$$\int \cos x \,\mathrm{d}x = \sin x$$

$$\int \sec^2 x \,\mathrm{d}x = \tan x$$

$$\int \operatorname{cosec}^2 x \,\mathrm{d}x = -\cot x$$

$$\int \sec x \tan x \,\mathrm{d}x = \sec x$$

$$\int \operatorname{cosec} x \cot x \,\mathrm{d}x = -\operatorname{cosec} x$$

$$\int \tan x \,\mathrm{d}x = \ln|\sec x|$$

$$\int \sec x \,\mathrm{d}x = \ln|\tan x + \sec x|$$

$$\int \operatorname{cosec} x \,\mathrm{d}x = \ln|\operatorname{cosec} x - \cot x| = \ln\left|\tan\frac{x}{2}\right|$$

$$\int \cot x \,\mathrm{d}x = \ln|\sin x|$$

Some transformations:

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

More importantly,

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

Sum to product:

$$\sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$$

$$\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$$

$$\cos P + \cos Q = 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$$

$$\cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$$

- Algebraic fractions

$$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \sin^{-1}\frac{x}{a} \qquad \text{(MF26)}$$

$$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a} \qquad \text{(MF26)}$$

- Partial fractions

$$\int \frac{1}{x^2-a^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| \qquad \text{(MF26)}$$

$$\int \frac{1}{a^2-x^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| \qquad \text{(MF26)}$$

- Integration by substitution

Consider $\int f(x)\,\mathrm{d}x$ and if $x = \phi(t)$ is a function of t, then f is a function of t. Hence

$$\int f(x)\,\mathrm{d}x = \int f(t)\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t\,.$$

> **Remark.** For definite integrals, remember to change the limits after making the substitution.

- Integration by parts

$$\int uv' = uv - \int u'v \qquad (32)$$

Guideline on choosing "u":

Abbreviation	**Function type**
L	logarithmic
I	inverse trigonometric
A	algebraic
T	trigonometric
E	exponential

Using **DI method** (also known as tabular method),

Example. Evaluate

$$\int x^2 e^x \, \mathrm{d}x \, .$$

Solution. We choose x^2 as "D", and e^x as "I".

sign	**D**	**I**
$+$	x^2	e^x
$-$	$2x$	e^x
$+$	2	e^x
$-$	0	e^x

Multiplying terms diagonally and summing them up gives us

$$\int x^2 e^x \, \mathrm{d}x = x^2 e^x - 2xe^x + 2e^x + c.$$

$\square$

§8 Applications of Integration

- Integral is a *limit of sum*.

  $\int_a^b f(x)\,\mathrm{d}x$, the area bounded by the curve $y = f(x)$, x-axis, $x = a$, $x = b$ can be divided into n vertical strips of equal width $\frac{b-a}{n}$, where area of each strip is $\frac{b-a}{n} f(x)$, thus total area is given by, taking limits,

  $$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*) \frac{b-a}{n} \tag{33}$$

  where $f(x_k^*) \in \left[a + (k-1)\frac{b-a}{n}, a + k\frac{b-a}{n}\right]$.

- Area under curve

  When calculating area of the region required, it should always be **positive**. This means **adding a negative sign** to an integral that is negative.

  For area between curve and x-axis,

  1. Sketch the curve
  2. Observe where the curve cuts the x-axis.
  3. Split the area into one above x-axis (positive), one below x-axis (negative).

  For area between curve and y-axis,

  1. Sketch the curve
  2. Observe where the curve cuts the y-axis.
  3. Split the area into one right of y-axis (positive), one left of y-axis (negative).

  When 2 or more curves are involved,

  1. Evaluate point(s) of intersection.
  2. Split the area into different parts.

  For all $x \in [a, b]$, for $f(x) \geq g(x)$, area between curves is

  $$\int_a^b [f(x) - g(x)]\,\mathrm{d}x \tag{34}$$

  For all $y \in [a, b]$, for $f(y) \geq g(y)$, area between curves is

  $$\int_a^b [f(y) - g(y)]\,\mathrm{d}y \tag{35}$$

  Remember this as: "upper" minus "lower" curve

  Area under curve parametrically: For $x = h(t)$ and $y = g(t)$, area under curve

  $$\int_a^b y\,\mathrm{d}x = \int_{t_1}^{t_2} g(t) \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t \tag{36}$$

  where t_1 and t_2 are values of t when $x = a$ and $x = b$ respectively.

  Similarly,

  $$\int_c^d x\,\mathrm{d}y = \int_{t_3}^{t_4} h(t) \frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t \tag{37}$$

  where t_3 and t_4 are values of t when $y = c$ and $y = d$ respectively.

  > **Remark.** We do not find the Cartesian equation of the curve.

- Solid of revolution (Volume)

  Rotate region bounded by $y = f(x)$, x-axis, $x = a$, $x = b$ 2π around x-axis, volume of revolution is

  $$\pi \int_a^b y^2\,\mathrm{d}x = \pi \int_a^b \left[f(x)\right]^2 \mathrm{d}x \tag{38}$$

  Rotate region bounded by $x = f(y)$, y-axis, $y = a$, $y = b$ 2π around y-axis, volume of revolution is

  $$\pi \int_a^b x^2\,\mathrm{d}y = \pi \int_a^b \left[f(y)\right]^2 \mathrm{d}y \tag{39}$$

  Volume of standard shapes

  - Cone: $\frac{1}{3}\pi r^2 h$
  - Cylinder: $\pi r^2 h$

  Rotate region bounded by two curves $y = f(x)$ and $y = g(x)$ 2π around x-axis, volume of revolution is

  $$\pi \int_a^b \left[f(x)\right]^2 - \left[g(x)\right]^2 \mathrm{d}x \tag{40}$$

  Rotate region bounded by two curves $x = f(y)$ and $x = g(y)$ 2π around y-axis, volume of revolution is

  $$\pi \int_a^b \left[f(y)\right]^2 - \left[g(y)\right]^2 \mathrm{d}y \tag{41}$$

§9 Differential Equations

- **1st order differential equation**: has derivatives up to the 1st derivative
  - Direct integration
  - Separation of variables

> **Example.** Find the general solution of
> $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{2x(x+1)} \qquad \text{for } x > 0$$
> and sketch some of the family of solution curves.

Solution.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y}{2x(x+1)} \\
\int \frac{2}{y}\,\mathrm{d}y &= \int \frac{1}{x(x+1)}\,\mathrm{d}x = \int \left(\frac{1}{x} - \frac{1}{x+1}\right)\mathrm{d}x \\
2\ln y &= \ln x - \ln(x+1) + \ln A \\
&\boxed{y^2 = \frac{Ax}{x+1}}
\end{aligned}$$

□

  - Using substitutions

> **Example.** Use the substitution $z = x + y$ to solve the differential equation
> $$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos(x+y).$$

Solution. From $z = x + y$,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 1 + \frac{\mathrm{d}y}{\mathrm{d}x} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}x} - 1$$

Substituting this and solving by separable variables,

$$\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}x} &= 1 + \cos z \\
\int \frac{1}{1+\cos z}\,\mathrm{d}z &= \int \mathrm{d}x \\
\frac{1}{2}\sec^2\frac{z}{2}\,\mathrm{d}z &= x + c \qquad \text{[double angle formula]} \\
\tan\frac{z}{2} &= x + c
\end{aligned}$$

Substituting back gives us

$$\boxed{\tan\frac{x+y}{2} = x + c}$$

□

- **2nd order differential equation**: has derivatives up to the 2nd derivative
  Direct integration, two arbitrary constants expected.

§10 Vectors

- **Magnitude** of a vector $\mathbf{a}$ is denoted by $|\mathbf{a}|$. **Unit vector** of vector $\mathbf{a}$ is denoted by $\hat{\mathbf{a}}$, where $|\hat{\mathbf{a}}| = 1$.

$$\frac{\mathbf{a}}{|\mathbf{a}|} = \hat{\mathbf{a}} \tag{42}$$

Convention for unit vectors: $\mathbf{i}$ is unit vector along x-axis, $\mathbf{j}$ is unit vector along y-axis, $\mathbf{k}$ is unit vector along z-axis.

Ratio theorem: if point P divides AB in the ratio of $\lambda : \mu$, then

$$\overrightarrow{OP} = \frac{\mu \cdot \overrightarrow{OA} + \lambda \cdot \overrightarrow{OB}}{\mu + \lambda} \tag{43}$$

Midpoint theorem is a special case where P is the midpoint:

$$\overrightarrow{OX} = \frac{\overrightarrow{OA} + \overrightarrow{OB}}{2}$$

Equal vectors:

$$\mathbf{a} = \mathbf{b} \iff \text{same magnitude and direction}$$

Parallel vectors:

$$\mathbf{a} \parallel \mathbf{b} \iff \exists \lambda \in \mathbb{R}, \lambda \neq 0 \text{ s.t. } \mathbf{a} = \lambda \mathbf{b}$$

That is, one vector is a *scalar multiple* of the other.

Collinear points:

$$A, B, C \text{ collinear} \iff \overrightarrow{AB} \parallel \overrightarrow{AC}$$

(and the two vectors share a common point A)

Coplanar vectors:

$$\mathbf{a}, \mathbf{b}, \mathbf{c} \text{ coplanar} \iff \exists \lambda, \mu \in \mathbb{R} \text{ s.t. } \mathbf{a} = \lambda \mathbf{b} + \mu \mathbf{c}$$

where $\mathbf{b}$ and $\mathbf{c}$ are non-parallel, non-zero vectors. That is, one of the vectors can be expressed as a *unique linear combination* of the other two vectors.

Parallelogram:

$$OACB \text{ is a parallogram} \iff \overrightarrow{BC} = \overrightarrow{OA}$$

That is, two opposite sides are equal.

- **Dot product** (or scalar product) is defined as

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \cdots + a_n b_n. \tag{44}$$

For 3D vectors, it can be shown that

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta \tag{45}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $\theta \in [0, \pi]$.

> **Remark.** The angle between two vectors which either *both* point outwards or inwards.

> **Remark.** Geometrically, dot product measures the **alignment** between the two vectors.

Properties:

- Commutative law: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- Distributive law: $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{c} + \mathbf{d}) = \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{d}$
  Distributive property of scalar multiplication: $(\lambda \mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\lambda \mathbf{b}) = \lambda(\mathbf{a} \cdot \mathbf{b})$
- Associative law: $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$

Applications

- Find length of vector

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$$

- Show perpendicular vectors

$$\mathbf{a} \perp \mathbf{b} \iff \mathbf{a} \cdot \mathbf{b} = 0$$

  where $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors.

- Show parallel vectors

$$\mathbf{a} \cdot \mathbf{b} = \pm |\mathbf{a}||\mathbf{b}|$$

  where positive sign implies same direction, negative sign implies opposite directions.

- Angle between two vectors

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

  If $\mathbf{a} \cdot \mathbf{b} > 0$, then angle between $\mathbf{a}$ and $\mathbf{b}$ is acute.
  If $\mathbf{a} \cdot \mathbf{b} < 0$, then angle between $\mathbf{a}$ and $\mathbf{b}$ is obtuse.

- Length of projection of $\mathbf{a}$ onto $\mathbf{b}$

$$|\mathbf{a} \cdot \hat{\mathbf{b}}|$$

> **Remark.** Modulus sign, since length must be positive.

- Vector projection of $\mathbf{a}$ onto $\mathbf{b}$

$$(\mathbf{a} \cdot \hat{\mathbf{b}})\hat{\mathbf{b}}$$

> **Remark.** No modulus sign, since vectors can take on both positive and negative values for direction.

- **Cross product** is defined as

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \, \hat{\mathbf{n}} \tag{46}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $\theta \in [0, \pi]$. $\hat{\mathbf{n}}$ denotes unit normal vector, perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

For 3D vectors, it can be shown that

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}. \tag{47}$$

> **Remark.** Geometrically, cross product produces a **new vector** perpendicular (orthogonal) to the two vectors.

Properties:

- Not commutative; in fact $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$. Also note that $|\mathbf{a} \times \mathbf{b}| = |\mathbf{b} \times \mathbf{a}|$.
- Distributive law: $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$
  Distributive property of scalar multiplication: $(\lambda \mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\lambda \mathbf{b}) = \lambda(\mathbf{a} \times \mathbf{b})$
- Not associative: $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

Applications

- Find normal vector (perpendicular to two given vectors $\mathbf{a}$ and $\mathbf{b}$)

$$\mathbf{c} = \mathbf{a} \times \mathbf{b}$$

> **Remark.** Normal vectors are not unique; any vector parallel to $\mathbf{a} \times \mathbf{b}$ is a normal to the plane containing $\mathbf{a}$ and $\mathbf{b}$.

- Show parallel vectors

$$\mathbf{a} \parallel \mathbf{b} \iff \mathbf{a} \times \mathbf{b} = 0$$

- Show perpendicular vectors

$$\mathbf{a} \perp \mathbf{b} \iff |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|$$

- Shortest distance from point to vector or line

$$|\mathbf{a} \times \hat{\mathbf{b}}|$$

- Area of parallelogram

$$|\mathbf{a} \times \mathbf{b}|$$

  where $\mathbf{a}$ and $\mathbf{b}$ are two adjacent sides of the parallelogram.

> **Remark.** Modulus sign, since area must be positive.

- Area of triangle

$$\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$$

  where $\mathbf{a}$ and $\mathbf{b}$ are two adjacent sides of the triangle.

> **Remark.** Modulus sign, since area must be positive.

- **Lines**

  Equation of a line

  1. **Vector form**

$$\ell : \mathbf{r} = \mathbf{a} + \lambda \mathbf{m}, \lambda \in \mathbb{R} \tag{48}$$

  where $\mathbf{r}$ is general position vector of any point on the line; $\mathbf{a}$ is position vector of fixed point; $\mathbf{m}$ is *direction vector*.

  2. **Cartesian form**

$$\frac{x - a_1}{m_1} = \frac{y - a_2}{m_2} = \frac{z - a_3}{m_3} = \lambda \tag{49}$$

  where $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\mathbf{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\mathbf{m} = m_1\hat{i} + m_2\hat{j} + m_3\hat{k}$, $m_1, m_2, m_3 \neq 0$

  3. **Parametric form**

$$x = a_1 + \lambda m_1, \quad y = a_2 + \lambda m_2, \quad z = a_3 + \lambda m_3 \tag{50}$$

Relationships between two lines ($\ell_1 : \mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{m}_1$ and $\ell_2 : \mathbf{r} = \mathbf{a}_2 + \lambda \mathbf{m}_2$)

- **Parallel** (coplanar)
  Direction vectors are parallel to each other: $\mathbf{m}_1 \parallel \mathbf{m}_2$.
- **Intersecting** (coplanar)
  Direction vectors are not parallel to each other: $\mathbf{m}_1 \nparallel \mathbf{m}_2$.
  Solving simultaneously (by equating the two lines) gives one unique solution (λ, μ).
- **Skew**[4] (non-coplanar)
  The lines do not satisfy the conditions in both cases above, i.e. no unique solution (λ, μ).

Applications

- Determine whether a point P lies on line ℓ
  Check if $\mathbf{p}$ into equation for ℓ.
- Acute angle between two lines
  For two lines with direction vectors $\mathbf{m}_1$ and $\mathbf{m}_2$, using dot product,

$$\cos\theta = \frac{|\mathbf{m}_1 \cdot \mathbf{m}_2|}{|\mathbf{m}_1||\mathbf{m}_2|}.$$

> **Remark.** Take absolute value of $|\mathbf{m}_1 \cdot \mathbf{m}_2|$ so that $\cos\theta$ is positive.

- Foot of perpendicular from point to line
  Let F be foot of perpendicular from P to ℓ. Since F lies on ℓ, $\overrightarrow{OF} = \mathbf{a} + \lambda \mathbf{m}$ for some λ (which we want to determine). Find $\overrightarrow{PF}$, in terms of λ. Since $\overrightarrow{PF} \perp \mathbf{m}$, using dot product,

$$\overrightarrow{PF} \cdot \mathbf{m} = 0,$$

  solve for λ. Then substitute the value of λ into the equation of $\overrightarrow{OF}$.
- Perpendicular distance from point to line
  Let h be perpendicular distance from P to ℓ. Using cross product,

$$h = |\overrightarrow{AP} \times \hat{\mathbf{m}}|.$$

[4] Do not intersect and are not parallel

– Length of projection of vector onto line

For line with direction vector $\mathbf{m}$ and vector $\mathbf{v}$,

$$|\mathbf{v}\cdot\hat{\mathbf{m}}|.$$

– Point of reflection of point in line

Use ratio theorem

- **Planes**

Equation of a plane

1. **Vector equation (parametric form)**

$$\pi:\mathbf{r}=\mathbf{a}+\lambda\mathbf{m}_1+\mu\mathbf{m}_2,\lambda,\mu\in\mathbb{R} \tag{51}$$

where $\mathbf{a}$ is position vector of fixed point, $\mathbf{m}_1$ and $\mathbf{m}_2$ are non-zero, non-parallel vectors that are parallel to the plane.

2. **Vector equation (scalar product form)**

$$\pi:\mathbf{r}\cdot\mathbf{n}=D \text{ where } D=\mathbf{a}\cdot\mathbf{n} \tag{52}$$

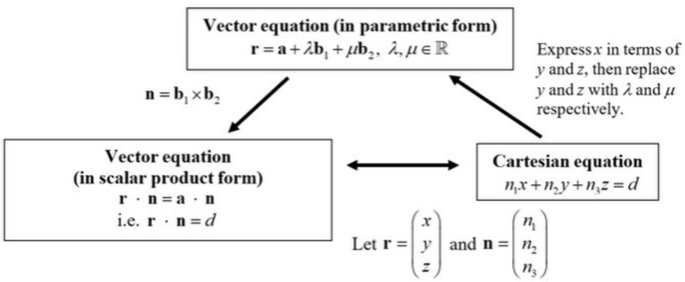
where $\mathbf{r}$ is position vector of any point, $\mathbf{a}$ is position vector of fixed point, $\mathbf{n}$ is normal vector of plane.[5]

$D=0$ if plane passes through origin (since dot product of $\mathbf{0}$ with any vector is 0).

3. **Cartesian equation**

$$\pi:n_1x+n_2y+n_3z=D \tag{53}$$

where $\mathbf{n}=n_1\hat{i}+n_2\hat{j}+n_3\hat{k}$, $\mathbf{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$.[6]



Relationships between a line and a plane ($\ell:\mathbf{r}=\mathbf{a}+\lambda\mathbf{m}$ and $\pi:\mathbf{r}\cdot\mathbf{n}=D$)

– ℓ and π **do not intersect**

$\mathbf{m}\cdot\mathbf{n}=0$ and $\mathbf{a}\cdot\mathbf{n}\neq D$; that is, ℓ and π are parallel, and line and plane have no common point.

When solving line and plane simultaneously, no solution.

– ℓ **lies on** π

$\mathbf{m}\cdot\mathbf{n}=0$ and $\mathbf{a}\cdot\mathbf{n}=D$; that is, ℓ and π are parallel, and line and plane have infinitely many common points.

When solving line and plane simultaneously, infinitely many solutions.

– ℓ and π **intersect**

$\mathbf{m}\cdot\mathbf{n}=0$; that is, ℓ and π are not parallel, and line and plane have one common point.

When solving line and plane simultaneously, one solution.

Applications

– Form vector equation in parametric form

Find position vector of a fixed point, and two direction vectors.

– Convert from vector equation in parametric form to scalar product form

To find $\mathbf{n}$, given two vectors parallel to the plane, take cross product.

– Convert from Cartesian equation to vector equation (parametric)

Express x in terms of y and z (to reduce number of variables to 2). Then replace y and z with λ and μ.

– Point lying on a plane

A point lies on a plane if its position vector satisfies equation of plane.

For example, unique (λ,μ) satisfies position vector of a point.

– Perpendicular distance from point to plane

Let F denote foot of perpendicular from Q to $\pi:\mathbf{r}\cdot\mathbf{n}=D$. Using length of projection,

$$QF=|\overrightarrow{QF}|=|\overrightarrow{QA}\cdot\hat{\mathbf{n}}|$$

given position vector $\mathbf{q}$, and position vector $\mathbf{a}$ or D.

Alternative: Find foot of perpendicular, find vector, take magnitude

– Foot of perpendicular from point to plane

Consider line ℓ_{QF} passing through Q and F. $\mathbf{n}$ can be used as its direction vector. Hence

$$\begin{cases}\pi:\mathbf{r}\cdot\mathbf{n}=D\\ \ell_{QF}:\mathbf{r}=\overrightarrow{OQ}+\lambda\mathbf{n}\end{cases}$$

Since line and plane intersect at P, solve simultaneously to find λ, substitute back in to find $\overrightarrow{OF}$.

– Acute angle between line and plane

– Acute angle between two planes

– Foot of perpendicular

– Relationship among three planes

Reduce to relationship between a line and a plane.

[5]If you rearrange it, it is $(\mathbf{r}-\mathbf{a})\cdot\mathbf{n}=0$. Since $\mathbf{r}-\mathbf{a}$ is the vector joining $\mathbf{a}$ to $\mathbf{r}$, this equation says that $\mathbf{n}$ is orthogonal (perpendicular) to the vector joining $\mathbf{a}$ to $\mathbf{r}$ for any $\mathbf{r}$ on the plane. If you draw a diagram, you should be able to intuitively see that this is the case.

[6]This can be derived by expressing vectors in column form, then taking dot product.

Problem 11. Given three points P_1, P_2, P_3 in 3-dimensional space (with position vectors $\mathbf{p}_i$ for point P_i). Given two points C_1 and C_2 in the same space (with position vectors $\mathbf{c}_i$ for point C_i), determine if C_1 and C_2 are on the different sides of the plane formed by points P_1, P_2, P_3.

Solution. Firstly find the plane passing through the three points P_1, P_2, P_3. It can be done by taking the normal to the plane to be $(\mathbf{p}_1 - \mathbf{p}_2) \times (\mathbf{p}_1 - \mathbf{p}_3) = \mathbf{n}$.

Notice that the normal $\mathbf{n}$ lies on one of the two sides.

Then find the two vectors connecting the plane to C_1 and C_2, i.e. $\mathbf{p}_1 - \mathbf{c}_1$ and $\mathbf{p}_1 - \mathbf{c}_2$.

Taking the scalar product with the normal indicates how much angle is in between the normal and the two directions. Hence if $\mathbf{n} \cdot (\mathbf{p}_1 - \mathbf{c}_1)$ and $\mathbf{n} \cdot (\mathbf{p}_1 - \mathbf{c}_2)$ have different signs, they lie on different sides of the plane. $\square$

§11 Complex Numbers

- The **imaginary number** is denoted by $i = \sqrt{-1}$. A **complex number** is of the form $a + bi$ where a and b are real numbers. The set of complex numbers is denoted by $\mathbb{C} = \{z \mid z = a + bi, a, b \in \mathbb{R}\}$.

  a is the **real part** of z, denoted by $\operatorname{Re}(z)$; b is the **imaginary part** of z, denoted by $\operatorname{Im}(z)$.

- **Cartesian form**

  $a + bi$ is the **Cartesian form** of complex number z.

  - **Compare real and imaginary parts**
  - Equality: equal if and only if corresponding real and imaginary parts are equal

    $$a + bi = c + di \iff a = c \text{ and } b = d$$

  - Addition and subtraction: add or subtract real and imaginary parts
  - Multiplication: expand brackets in the usual fashion and remember that $i^2 = -1$
  - The **conjugate** is $\bar{z} = a - bi$. Multiplying a complex number with its conjugate eliminates the imaginary part:

    $$z\bar{z} = a^2 + b^2$$

    A very useful property is

    $$z\bar{z} = |z|^2$$

  - Division: multiply numerator and denominator by the conjugate of denominator
  - **Argand diagram**: consists of **real axis** and **imaginary axis**. The point (a, b) represents complex number $a + bi$.

    Geometric representation of complex numbers:

    * **Addition and subtraction**: similar to that of vectors.
    * **Conjugate**: reflection about the real axis.
    * **Multiplication by** i: rotate 90° anti-clockwise about origin.
    * **Scalar multiplication**: scaling

- Complex roots of polynomial equations

  - **Fundamental Theorem of Algebra**

> **Theorem 11.1** (Fundamental Theorem of Algebra)
> Let $P(z) = a_n z^n + \cdots + a_1 z + a_0$ be a polynomial of degree $n \geq 1$ with (complex) coefficients a_k.
> Then the roots of $P(z) = 0$ are complex; there are n (not necessarily distinct) complex numbers $\gamma_1, \ldots, \gamma_n$ such that
>
> $$P(z) = a_n(z - \gamma_1)(z - \gamma_2)\cdots(z - \gamma_n).$$
>
> The theorem shows that a degree n polynomial has n roots in $\mathbb{C}$ (including repetitions).

  - **Conjugate Root Theorem**

> **Theorem 11.2** (Conjugate Root Theorem)
> Complex roots of a polynomial equation with <u>real</u> coefficients occur in conjugate pairs.

- **Polar form** expresses a point in space by distance from origin r and angle measured anticlockwise from real axis θ.

  - Trigonometric form

    $$z = r(\cos\theta + i\sin\theta) \tag{54}$$

    where r is the **modulus** $|z|$:

    $$|z| = r = \sqrt{a^2 + b^2}$$

    and θ is the **argument** $\arg(z)$:

    $$\arg(z) = \theta = \tan^{-1}\frac{b}{a}$$

    where $\theta \in (-\pi, \pi]$.
  - Exponential form

> **Theorem 11.3** (Euler's Formula)
> For a complex number z with modulus r and argument θ, z can be expressed as
>
> $$z = r(\cos\theta + i\sin\theta) = re^{i\theta}. \tag{55}$$

> **Theorem 11.4** (Euler's Identity)
> A special case of Euler's Formula, in which $r = 1$ and $\theta = \pi$:
>
> $$e^{i\pi} + 1 = 0. \tag{56}$$

NOT IN SYLLABUS

- **Triangle inequality**

  For complex numbers z_1 and z_2,

$$|z_1| + |z_2| \geq |z_1 + z_2| \tag{57}$$

  with equality only if one of them is 0 or $\arg(z_1) = \arg(z_2)$ i.e. z_1 and z_2 are on the same ray from the origin.

- Proof of **Conjugate Root Theorem**

  Proof. Consider the equation

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$$

  where for $a_0, \ldots, a_n \in \mathbb{R}$, $a_n \neq 0$, $n \in \mathbb{Z}^+$.

  Suppose β is a non-real root of the equation, then

$$a_n \beta^n + a_{n-1} \beta^{n-1} + \cdots + a_1 \beta + a_0 = 0.$$

  Taking conjugates on both sides of the equation,

$$a_n \overline{\beta^n} + a_{n-1} \overline{\beta^{n-1}} + \cdots + a_1 \overline{\beta} + a_0 = 0$$

  Note that $\overline{\beta^k} = \overline{\beta}^k$. Thus we have

$$a_n \bar{\beta}^n + a_{n-1} \bar{\beta}^{n-1} + \cdots + a_1 \bar{\beta} + a_0 = 0$$

  which means that the complex conjugate $\bar{\beta}$ is also a non-real root of the given equation. □

- Proof of **Euler's Formula**

  Proof. Recall that the Taylor Series for e^x is given by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

  Substituting ix for x gives us

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \cdots$$

  Separating real and imaginary parts,

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

  Observe that the real part is the Taylor series of $\cos x$, while the imaginary part is the Taylor series of $\sin x$. Hence proven. □

- **de Moivre's Theorem**

  Theorem 11.5 (de Moivre's Theorem)

  For a real number θ and integer n we have that

$$\cos n\theta + i \sin n\theta = (\cos\theta + i \sin\theta)^n.$$

  Proof. The proof easily follows from mathematical induction. □

  to derive trigonometric identities

- **Roots of unity**

  The n-th roots of unity, that is the solutions of the equation $z^n = 1$, are

$$z = e^{i\frac{2k\pi}{n}}, \quad k = 0, 1, 2, \ldots, n-1.$$

  When plotted these n-th roots of unity form a regular n-gon inscribed within the unit circle with a vertex at 1. More generally, for $c \neq 0$, the n solutions of $z^n = c$ make a regular n-gon inscribed in the circle $|z| = |c|^{\frac{1}{n}}$.

- Loci of simple equations and inequalities

  - Circle: $|z - c| \leq r$, where $r \in \mathbb{R}^+, c \in \mathbb{C}$

- Perpendicular bisector of line segment AB: $|z-a| = |z-b|$, where $a, b \in \mathbb{C}$
- Half-line with end point A (excluding A), inclined at angle α to positive real axis: $\arg(z-a) = \alpha$
- Apollonius circle: $|z-a| = k|z-b|$, where $k > 0, k \neq 1$[7]
- Part of a circle with end points A and B: $\arg(z-a) - \arg(z-b) = \alpha$

- **Extended complex plane**

  Let $A, C \in \mathbb{R}$, $B \in \mathbb{C}$, with A, B not both zero. Then

$$Az\bar{z} + B\bar{z} + \bar{B}z + C = 0 \tag{1}$$

represents

(a) a line in direction iB when $A = 0$;

(b) a circle, if $A \neq 0$ and $|B|^2 \geq AC$, with centre $-\dfrac{B}{A}$ and radius $|A|^{-1}\sqrt{|B|^2 - AC}$;

and otherwise has no solutions. Moreover every circle and line can be represented in the form of (1).

Proof. If $A \neq 0$ then we can rearrange (1) as

$$\begin{aligned} z\bar{z} + \frac{B}{A}\bar{z} + \frac{\bar{B}}{A}z + \frac{C}{A} &= 0 \\ \left(z + \frac{B}{A}\right)\overline{\left(z + \frac{B}{A}\right)} &= \frac{B\bar{B}}{A^2} - \frac{C}{A} \\ \left|z + \frac{B}{A}\right|^2 &= \frac{|B|^2 - AC}{A^2}. \end{aligned}$$

If $|B|^2 \geq AC$ then this is a circle with centre $-\dfrac{B}{A}$ and radius $|A|^{-1}\sqrt{|B|^2 - AC}$ and otherwise there are no solutions to (1). Conversely, note that the equation of a general circle is $|z-a| = r$ where r where $a \in \mathbb{C}, r \geq 0$. This is can be rearranged as

$$z\bar{z} - a\bar{z} - \bar{a}z + \left(|a|^2 - r^2\right) = 0$$

which is of the form of (1) with $A = 1$, $B = -a$, and $C = |a|^2 - r^2$.

If $A = 0$ then we have the equation $B\bar{z} + \bar{B}z + C = 0$. If we write $B = u + iv$ and $z = x + yi$ then

$$(u+iv)(x-yi) + (u-vi)(x+yi) + c = 0$$

which rearranges to

$$2ux + 2vy + C = 0 \tag{2}$$

which is the equation of a line. Moreover we see that every line appears in this form by choosing u, v, C appropriately. The line is paralell to the vector $(v, -u)$ or equivalently $v - ui = i(u+iv) = iB$. □

[7]You can prove this by putting $z = x + iy$ and squaring both sides.

Problem 12 (STEP 2 2020 Q7)**.** In this question, $w = \dfrac{2}{z-2}$.

(i) Let z be the complex number $3 + ti$, where $t \in \mathbb{R}$. Show that $|w-1|$ is independent of t. Hence show that, if z is a complex number on the line $\mathrm{Re}(z) = 3$ in the Argand diagram, then w lies on a circle in the Argand diagram with centre 1.

Let V be the line $\mathrm{Re}(z) = p$, where p is a real constant not equal to 2. Show that, if z lies on V, then w lies on a circle whose centre and radius you should give in terms of p. For which z on V is $\mathrm{Im}(w) > 0$?

(ii) Let H be the line $\mathrm{Im}(z) = q$, where q is a non-zero real constant. Show that, if z lies on H, then w lies on a circle whose centre and radius you should give in terms of q. For which z on H is $\mathrm{Re}(w) > 0$?

Solution.

(i)

$$|w-1|^2 = \left|\frac{1-ti}{1+ti}\right|^2 = \frac{(1-ti)(1+ti)}{(1+ti)(1-ti)} = 1,$$

which is independent of t.

Points on the line $\mathrm{Re}(z) = 3$ have the form $z = 3+ti$ and the points satisfying $|w-1| = 1$ lie on a circle with centre 1.

If $z = p + ti$, then

$$|w-c|^2 = \left|\frac{2-(p-2)c-cti}{(p-2)+ti}\right|^2 = \frac{\big(2-(p-2)c\big)^2 + c^2t^2}{(p-2)^2+t^2}$$

which is independent of t when $\big(2-(p-2)c\big)^2 = c^2(p-2)^2$, i.e., $c = \dfrac{1}{p-2}$. Thus the circle has centre at $\dfrac{1}{p-2}$ and radius $\dfrac{1}{|p-2|}$.

$$w = \frac{2}{(p-2)+ti} = \frac{2(p-2)-2ti}{(p-2)^2+t^2},$$

so $\mathrm{Im}(w) > 0$ when $t < 0$; that is, for those z on V with negative imaginary part.

(ii) If $z = t + qi$ then

$$|w-ci|^2 = \left|\frac{2+cq-(t-2)ci}{(t-2)+qi}\right|^2 = \frac{c^2(t-2)^2+(cq+2)^2}{(t-2)^2+q^2}$$

which is independent of t when $(cq+2)^2 = c^2q^2$, i.e., $c = -\dfrac{1}{q}$, so the circle has centre $-\dfrac{1}{q}i$ and radius $\sqrt{c^2} = \dfrac{1}{|q|}$.

$$w = \frac{2}{(t-2)+qi} = \frac{2(t-2)-2qi}{(t-2)^2+q^2},$$

so $\mathrm{Re}(w) > 0$ when $t > 2$; that is, for those z on H with real part greater than 2.

□

Part II

Statistics

§12 Permutation and Combination

- Counting principles
  - **Addition principle**: If there are r choices for performing a particular task, and the number of ways to carry out the i-th choice is n_i, then the total number of ways of performing the particular task is equal to the sum of the number of ways for all the r different choices, i.e.,
  $$n_1 + n_2 + n_3 + \cdots + n_r.$$
  - **Multiplication principle**: If one task can be performed in m ways, and following this, a second task can be performed in n ways (regardless of which way the first task was performed), then the number of ways of performing the 2 tasks in succession is $m \times n$.
- **Permutation**: ordered arrangement of objects
  - Arrange n distinct objects in a row:
  $$n!$$
  - Arrange (in a row) n objects, of which n_1 of them are of the first kind, n_2 of them of the second kind, and so on for k kinds of objects:
  $$\frac{n!}{n_1!n_2!\cdots n_k!}$$

  > **Remark.** We are essentially dividing away the order created by the identical objects had they been considered to be distinct.

  - Arrange r of n distinct objects in a row:
  $$\binom{n}{r} \times r! = ^n\mathrm{P}_r$$
  - Arrange n distinct objects in a circle:
  $$\frac{n!}{n} = (n-1)!$$
  if positions are distinguishable.
  If positions are distinguishable,
  $$n \times (n-1)! = n!$$
  where there are n ways to number the positions.

  > **Remark.** Positions become distinguishable if they are numbered, different colour and different shapes/sizes.

- **Combination**: unordered selection of a number of objects
  - Choose r objects from n distinct objects without replacement:
  $$\binom{n}{r} = ^n\mathrm{C}_r = \frac{n!}{(n-r)!r!}$$
  - Select at least 1 object from n distinct objects:
  $$2^n - 1$$

> **Remark.** When restrictions are given, they must be satisfied first before the number of combinations is selected. This can be done by breaking down into different cases, calculating separately and adding/subtracting them.

Some useful techniques:

1. Complementary technique
2. Grouping technique (order must be taken into account)
3. Insertion technique (order must be taken to account – use for 3 or more objects)

> **Remark.** When subdividing into groups of equal number, remember to divide by the number of groups.

Problem 13 (RI 2023 Prelim Q6). A group of 5 boys and 3 girls sit at random at a round table. Find the number of arrangements so that

(a) no 2 girls are adjacent to each other, [3]

(b) all 3 girls are seated together, [3]

(c) exactly 2 of the 3 girls are adjacent to each other. [2]

Solution.

(a) Arrange the 5 boys in $(5-1)! = 24$ ways.

Then slot in each of the 3 girls into the 5 spaces between the boys in ${}^{5}P_3 = 60$ ways.

Total number of arrangements with no 2 girls being adjacent to each other is $24 \times 60 = \boxed{1440}$.

(b) Arrange the 3 girls within a unit in $3! = 6$ ways.

Then arrange the unit of 3 girls with the 5 boys in $(6-1)! = 120$ ways.

Total number of arrangements with all 3 girls seated together is $6 \times 120 = \boxed{720}$.

(c) Total number of arrangements with exactly 2 of the 3 girls adjacent to each other is $(8-1)! - 1440 - 720 = \boxed{2880}$.

□

Problem 14 (HCI 2022 Prelim Q8)**.** A school canteen committee consists of 4 parents, 2 student leaders and 4 teachers, chosen from 10 parents, 5 student leaders and 8 teachers.

(a) There is a married couple amongst the 10 parents. How many different canteen committees can be formed if the couple cannot serve on the committee together? [3]

The school canteen committee of 10 members has been formed.

(b) All members are to stand in a row to take a group photo with the Vice-Principal. Find the number of arrangements such that the Vice-Principal stands at the centre, both ends of the row are occupied by the students' leaders, and no two parents stand next to each other. [3]

(c) The committee members, together with the Vice-Principal, are seated at a round table with 11 chairs during lunch time. Find the probability that the parents are seated together and the teachers are separated. [3]

Solution.

(a) Total number of committees formed $= \binom{5}{2} \times \binom{10}{4} \times \binom{8}{4} = 147000$

Number of committees with the couple serving together $= \binom{5}{2} \times \binom{8}{2} \times \binom{8}{4} = 19600$

Required number of committees formed $= 147000 - 19600 = \boxed{127400}$

(b) Number of arrangements if no two parents are to stand next to each other $= 2! \times 4! \times 4! \times 3 \times 3 = \boxed{10368}$

(c) No. of circular arrangements if all parents are together and teachers are separated $= 3! \times 4! \times 4! = 3456$

Required probability $= \dfrac{3456}{10!} = \boxed{\dfrac{1}{1050}}$

□

Problem 15 (NYJC 2022 Prelim Q6)**.** Jean has forgotten the six-character login password for her laptop. She remembers that the password consists of four distinct letters from the twenty-six letters of the alphabet A-Z and two distinct digits from the ten digits 0-9.

(i) Assuming that Jean keys in a six-character password for all her login attempts and she never repeats the same incorrect password, find the largest number of unsuccessful login attempts. [2]

(ii) Find the number of possible six-character passwords if the first four characters are distinct letters in alphabetical order. [2]

(iii) Given that the first four characters are distinct letters, and the last two characters are distinct digits, find the probability that exactly one of the four letters is a vowel. [3]

Solution.

(i) Largest number of unsuccessful login attempts $= {}^{26}C_4 \times {}^{10}C_2 \times 6! - 1 = \boxed{484379999}$

(ii) Number of passwords =

(iii)

□

Problem 16 (N2017/II/6)**.**

Problem 17 (N2016/II/7)**.**

Problem 18 (N2015/II/11)**.**

§13 Probability

- Probability is concerned with the outcome of **trials**. Trial refers to an event whose outcome is unknown.

  Sample space S: set of all possible elementary outcomes of a trial. $P(S) = 1$.

  Event $E \subseteq S$: specification of the outcome of a trial. An event can consist of a single outcome or a set of outcomes.

  Complement of an event is everything in the sample space that is not that event.

  The probability of an event is always between 0 and 1.

- Operations on events
  - **Union** of A and B (denoted by $A \cup B$): the event that either A or B occurs (or both).
  - **Intersection** of A and B (denoted by $A \cap B$): the event that both A and B occurs.
  - **Complement** of A (denoted by A'): the event that A does not occur.

  > **Remark.** $A \cap A' = \varnothing$ and $A \cup A' = S$.

  A and B are **mutually exclusive** (or disjoint) if they cannot occur simultaneously, i.e. they do not share any common outcomes:

  $$A \cap B = \varnothing.$$

- **Probability** of an event:

  $$P(A) = \frac{n(A)}{n(S)}$$

  where $n(A)$ is number of outcomes in A, $n(S)$ is total number of outcomes in sample space S.
  - $0 \le P(A) \le 1$
  - Combined events: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  - Mutually exclusive events: $P(A \cap B) = 0 \implies P(A \cup B) = P(A) + P(B)$
  - Complement: $P(A') = 1 - P(A)$

- **Conditional probability** of A given B, denoted by $P(A \mid B)$: probability of A occurring given that B has occurred.

  $$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \tag{58}$$

  A and B are **independent** if probability that A occurs is not affected by occurrence of B:

  $$P(A \mid B) = P(A)$$

  or

  $$P(A \cap B) = P(A) \times P(B).$$

- Useful techniques
  - **Tree diagram**: few stages with few outcomes
  - **Venn diagram**: different probabilities for different events
  - **Table of outcomes**: small sample space
  - **P&C method**: only WITHOUT replacement
  - **Sequences and series**: for turn-by-turn situations

Problem 19. For two independent events A and B, it is given that $P(A) = a$ and $P(B) = b$.

(a) Show that A' and B' are independent events. [2]

It is given that $P(A \cup B) = 0.7$ and $b = 0.5$.

(b) Show that $a = 0.4$. [2]

For a third event C, it is given that $P(C) = 0.3$ and events A and C are independent.

(c) If events B and C are mutually exclusive, find $P(A' \cap B' \cap C')$. [2]

(d) If events B and C are not mutually exclusive, find the greatest possible value of $P(A' \cap B' \cap C')$, showing your working clearly. [3]

Solution.

(a) Given that A and B are independent events,

$$P(A \cap B) = P(A)P(B) = ab.$$

Then

$$
\begin{aligned}
P(A' \cap B') &= 1 - P(A \cup B) \\
&= 1 - [P(A) + P(B) - P(A \cap B)] \\
&= 1 - (a + b - ab) \\
&= (1 - a)(1 - b) \\
&= P(A')P(B')
\end{aligned}
$$

Hence A' and B' are independent events.

(b)

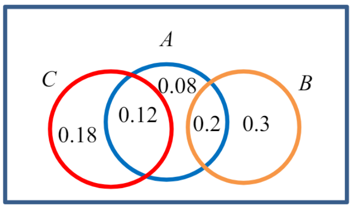
$$
\begin{aligned}
P(A \cup B) &= 0.7 \\
P(A) + P(B) - P(A \cap B) &= 0.7 \\
a + b - ab &= 0.7 \\
a + 0.5 - 0.5a &= 0.7 \\
0.5a &= 0.2 \\
a &= 0.4
\end{aligned}
$$

(c) Given $P(A) = 0.4$, $P(B) = 0.5$, $P(C) = 0.3$.

Since A and B are independent events, $P(A \cap B) = 0.4 \times 0.5 = 0.2$

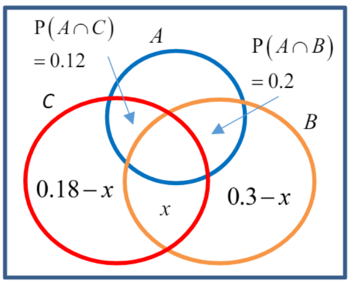
Since A and C are independent events, $P(A \cap C) = 0.4 \times 0.3 = 12$

Events B and C are mutually exclusive:



$$
\begin{aligned}
P(A' \cap B' \cap C') &= 1 - P(A) - 0.18 - 0.3 \\
&= 1 - 0.4 - 0.18 - 0.3 \\
&= 0.12
\end{aligned}
$$

(d) Given events B and C are not mutually exclusive,



$$
\begin{aligned}
P(A' \cap B' \cap C') &= 1 - P(A) - 0.18 - (0.3 - x) \\
&= 0.12 + x
\end{aligned}
$$

Since probabilities are all non-negative, $0 \leq x \leq 0.18$.

When x is the greatest, i.e. when $x = 0.18$, $P(A' \cap B' \cap C')$ is greatest.

Hence greates possible value of $P(A' \cap B' \cap C')$ is $0.12 + 0.18 = \boxed{0.3}$.

□

Problem 20 (N2023/II/5).

Problem 21 (N2023/II/6).

Problem 22 (N2022/II/6). Interesting problem involving geometric progression.

Problem 23 (N2022/II/7c).

Problem 24 (N2022/II/9).

Problem 25 (N2020/II/8).

Problem 26 (N2018/II/7).

Problem 27 (N2015/II/9).

§14 Discrete Random Variables

- A **random variable** X for an experiment is one where the event that X takes on a certain value x, corresponds exactly to a possible event E of the experiment. We express the event E as $\{X = x\}$ in this case.

  > **Remark.** The convention is to use capital letters, e.g. X and Y, to denote random variables, while corresponding lower case letters, e.g. x and y, are used to represent one of the values it can take.

  > **Remark.** A random variable can be either discrete or continuous.

  A **discrete random variable** can only take certain numerical values in an interval.

- Let X be a discrete random variable taking values $x_1, x_2, \dots, x_n$. Then the **probability distribution function** of X is the function f that maps each value x_k to the *probability* that $X = x_k$, i.e.

$$f(x) = P(X = x) \quad \text{for } x = x_1, x_2, \dots, x_n$$

- Let X be a discrete random variable with probability distribution function $P(X = x)$ for $x = x_1, x_2, \dots$, then the **cumulative distribution function** of X is given by

$$P(X \le x) = \sum_{r \le x} P(X = r)$$

- **Expectation** μ:

$$\mathrm{E}(X) = \sum_{\forall x} x P(X = x) \tag{59}$$

  In general, the expectation of $g(X)$ is given by

$$\mathrm{E}\big(g(X)\big) = \sum_{\forall x} g(x) P(X = x).$$

  For constants a and b,

  - $\mathrm{E}(a) = a$ (average of a constant is itself)
  - $\mathrm{E}(aX) = a\,\mathrm{E}(X)$
  - $\mathrm{E}(aX \pm b) = a\,\mathrm{E}(X) \pm b$
  - $\mathrm{E}(aX \pm bY) = a\,\mathrm{E}(X) \pm b\,\mathrm{E}(Y)$

- **Variance**: measures how widely spread the values of the discrete random variable are

$$\mathrm{Var}(X) = \sum (x_i - \mu)^2 p_i \tag{60}$$

  However this definition is not useful; we introduce a more useful equation.

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 \tag{61}$$

  Standard deviation is $\sigma = \sqrt{\mathrm{Var}(X)}$.
  For constants a and b,

  - $\mathrm{Var}(a) = 0$ (variance of a constant is zero)
  - $\mathrm{Var}(aX) = a^2\,\mathrm{Var}(X)$
  - $\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$

Problem 28 (N2020/II/5).

Problem 29 (N2018/II/8).

§15 Binomial Distribution

- $X \sim B(n, p)$, where n is number of trials, p is probability of success.

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \tag{MF26}$$

  To find **mode**, use TABLE function in G.C. to get the value of x that gives the highest value of $P(X = x)$.

- Conditions for binomial distribution:
  1. Finite number of trials
  2. Outcome of each trial is termed a "success" or "failure"
  3. *Probability of success is same for each trial
  4. *Trials are independent of each other
- Expectation: $\mathrm{E}(X) = np$
- Variance: $\mathrm{Var}(X) = np(1-p)$

NOT IN SYLLABUS: Poisson distribution

- Conditions for poisson distribution:
  1. The events occur at random and are independent of each other in a given interval of time or space*,
  2. The average number of events per interval is constant throughout the interval*
  3. The average number of events per interval is proportional to the size of the interval.
- $X \sim Po(\lambda)$, where λ is the mean number of occurences.

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, 3, \ldots$$

- Expectation: $\mathrm{E}(X) = \lambda$
- Variance: $\mathrm{Var}(X) = \lambda$

Problem 30 (N2023/II/10)**.**

Problem 31 (N2021/II/11)**.**

Problem 32 (N2020/II/9)**.**

Problem 33 (N2019/II/7)**.**

Problem 34 (N2018/II/6)**.**

Problem 35 (N1982/I/13)**.** The random variable X is the number of successes in n independent trials of an experiment in which the probability of success in any one trial is p.

Show that

$$\frac{P(X = k+1)}{P(X = k)} = \frac{(n-k)p}{(k+1)(1-p)}, \quad k = 0, 1, 2, \ldots, n-1.$$

Find the most probable number of success when $n = 10$ and $p = \frac{1}{4}$.

Solution. Given that $X \sim B(n, p)$, thus

$$\begin{aligned}
\frac{P(X = k+1)}{P(X = k)} &= \frac{(n-k)p}{(k+1)(1-p)} \\
&= \frac{\binom{n}{k+1} p^{k+1} (1-p)^{n-(k+1)}}{\binom{n}{k} p^k (1-p)^{n-k}} \\
&= \frac{\frac{n!}{(k+1)(n-k-1)!}}{\frac{n!}{k!(n-k)!}} p^{k+1-k} (1-p)^{n-k-1-(n-k)} \\
&= \frac{k!(n-k)!}{(k+1)!(n-k-1)!} p^1 (1-p)^{-1} \\
&= \frac{(n-k)p}{(k+1)(1-p)}
\end{aligned}$$

Given $n = 10$, $p = \frac{1}{4}$. If k is the mode,

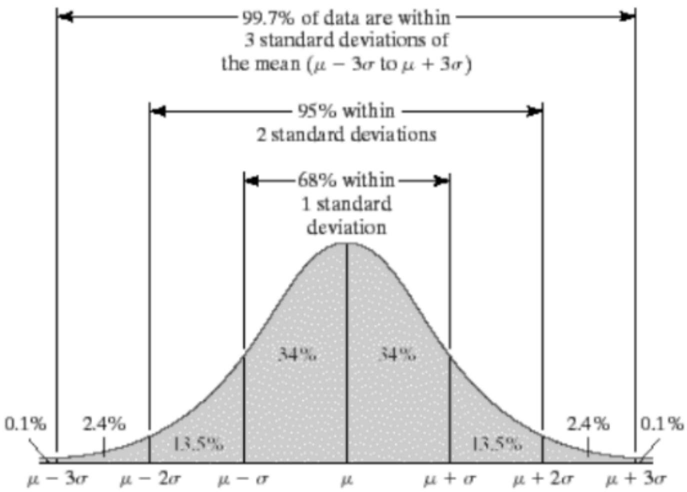
$$P(X = k+1) < P(X = k)$$
$$\frac{P(X = k+1)}{P(X = k)} < 1$$
$$\frac{(10-k)\frac{1}{4}}{(k+1)\frac{3}{4}} < 1$$
$$10 - k < 3(k+1)$$
$$7 < 4k$$
$$\frac{7}{4} < k$$

$$P(X = k-1) < P(X = k)$$
$$1 < \frac{P(X = k)}{P(X = k-1)}$$
$$1 < \frac{(11-k)\frac{1}{4}}{(k)\frac{3}{4}}$$
$$3k < 11 - k$$
$$4k < 11$$
$$k < \frac{11}{4}$$

Thus $\frac{7}{4} < k < \frac{11}{4}$ so mode is $\boxed{k = 2}$. □

§16 Normal Distribution

- Normal distribution: $X \sim N(\mu, \sigma^2)$, where μ is mean, σ^2 is variance.
- Normal curve is symmetrical:



It is easy to see that

  - $P(X > a) = 1 - P(X < a)$
  - $P(X < \mu - a) = P(X > \mu + a)$
  - $P(X < \mu + a) = P(X > \mu - a)$

- Use `normalcdf` to evaluate $P(X < a)$.

  Use `invNorm` to find the value of a such that $P(X < a) = p$.

- Standard normal distribution: $Z \sim N(0,1)$, where $\mu = 0$, $\sigma = 1$. To standardise,

$$Z = \frac{X - \mu}{\sigma}$$

Standardising to solve problems involving finding μ or σ^2:

> **Example.** $X \sim N(\mu, 5^2)$. Given that $P(X < 18) = 0.9032$, find μ.

Solution. Standardising,

$$P(X < 18) = P\left(Z < \frac{18 - \mu}{5}\right) = 0.9032.$$

Using invNorm, we find that

$$\frac{18 - \mu}{5} = 1.30 \implies \boxed{\mu = 11.5}$$

□

> **Example.** $X \sim N(\mu, \sigma^2)$. Given that $P(X < 17) = 0.8159$ and $P(X < 25) = 0.9970$, find μ and σ.

Solution. Standardising,

$$P(X < 17) = P\left(Z < \frac{17 - \mu}{\sigma}\right) = 0.8159$$
$$P(X < 25) = P\left(Z < \frac{25 - \mu}{\sigma}\right) = 0.9970$$

Using invNorm, we find that

$$\frac{17 - \mu}{\sigma} = -0.8998 \implies 17 - \mu = -0.893\sigma$$
$$\frac{25 - \mu}{\sigma} = 2.748 \implies 25 - \mu = 2.748\sigma$$

Solving the simultaneous equations gives $\boxed{\mu = 19.0, \sigma = 2.20}$. □

- Properties of expectation and variance
  - If $X \sim N(\mu_X, {\sigma_X}^2)$ and $Y \sim N(\mu_Y, {\sigma_Y}^2)$, then $X + Y \sim N(\mu_X + \mu_Y, {\sigma_X}^2 + {\sigma_Y}^2)$.
  - $\mathrm{E}(X_1 + \cdots + X_n) = \mathrm{E}(X_1) + \cdots + \mathrm{E}(X_n) = n\,\mathrm{E}(X)$

    $\mathrm{Var}(X_1 + \cdots + X_n) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n) = n\,\mathrm{Var}(X)$
  - $\mathrm{E}(aX \pm bY) = a\,\mathrm{E}(X) \pm b\,\mathrm{E}(Y)$

    $\mathrm{Var}(aX \pm bY) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y)$

§17 Sampling

- **Random sampling**: every member of population has equal chance of being selected, and the selections are independent of each other.

  One type of sampling is **simple random sampling**: each possible sample of size n has the same chance of being chosen from population of size N.

  1. Create a list of the population (sampling frame), number members of population from 1 to N.
  2. Make a selection of n distinct members by using random number generator.

  Advantage: Free from bias

  Disadvantage: Not able to get access to some members chosen from the sample

- If $X \sim N(\mu, \sigma^2)$, then
$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$
  where sample mean $\bar{X} = \dfrac{X_1 + \cdots + X_n}{n}$.

- If $X_1, \dots, X_n$ is random sample of size n taken from non-normal or unknown distribution with mean μ, variance σ^2, then for sufficiently large n (at least 50), by **Central Limit Theorem** (CLT),
$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \text{ approximately.}$$

  A common question:

  > **Remark.** It is not necessary to assume that XXX follows a normal distribution in part.
  >
  > Since the sample size is large, Central Limit Theorem can be applied such that the distribution of the **sample mean** XXX is approximately normal.

- Estimation

  $\bar{x}$ is unbiased estimate of μ:
$$\bar{x} = \frac{\sum x}{n} \tag{62}$$
  s^2 is unbiased estimate of σ^2:
$$\begin{aligned} s^2 &= \frac{n}{n-1} \times \text{sample variance} \\ &= \frac{n}{n-1}\left(\frac{\sum(x-\bar{x})^2}{n}\right) \\ &= \frac{1}{n-1}\left(\sum x^2 - \frac{(\sum x)^2}{n}\right) \end{aligned} \tag{63}$$
  Unbiased estimate of common population variance from two samples:
$$s^2 = \frac{\sum(x_1-\bar{x}_1)^2 + \sum(x_2-\bar{x}_2)^2}{n_1+n_2-2} \tag{MF26}$$
  If data is represented in the form $x - c$,
$$\bar{x} = \frac{\sum(x-c)}{n} + c$$
  and
$$s^2 = \frac{1}{n-1}\left(\sum(x-c)^2 - \frac{\left(\sum(x-c)\right)^2}{n}\right)$$

§18 Hypothesis Testing

- **Null hypothesis** H_0: particular claim for a *value* for the population mean (status quo claim).

  Alternative hypothesis H_1: range of values that excludes the value specified by null hypothesis (suspicion).

  Test statistic: random variable whose value is calculated from sample data, usually sample mean $\bar{X}$. (Thus we consider the distribution of the test statistic.)

  Test value: value of the test statistic, usually observed sample mean $\bar{x}$.

  Probability value (p-value): probability of getting a test statistic as extreme or more extreme than the observed value.

  p-value is also the lowest significance level at which H_0 is rejected. (try to graph this out!)

  $$\begin{aligned} p\text{-value} \le \alpha &\implies \text{reject } H_0 \\ p\text{-value} > \alpha &\implies \text{do not reject } H_0 \end{aligned}$$

  Critical region: range of values of test statistic that leads to the rejection of H_0. The value of c which determines the critical region is known as the **critical value**.

  $$\begin{aligned} \bar{x} \text{ lies in critical region} &\implies \text{reject } H_0 \\ \bar{x} \text{ does not lie in critical region} &\implies \text{do not reject } H_0 \end{aligned}$$

  Significance level α: probability of rejecting H_0 when it is actually true (i.e. wrongly rejecting null hypothesis).

- **1-tail test**: H_1 looks for increase/decrease in μ.

  - For an increase, $H_1 : \mu > \mu_0$, critical region and p-value are in **right tail**.

    $$p\text{-value} = P(\bar{X} \ge \bar{x}), \quad \alpha = P(\bar{X} \ge c)$$

  - For a decrease, $H_1 : \mu < \mu_0$, critical region and p-value are in **left tail**.

    $$p\text{-value} = P(\bar{X} \le \bar{x}), \quad \alpha = P(\bar{X} \le c)$$

  2-tail test: $H_1 : \mu \ne \mu_0$ looks for a *change* in μ, without specifying whether it is an increase or decrease.

  $$\alpha = P(\bar{X} \le c_1) + P(\bar{X} \ge c_2) = 2P(\bar{X} \le c_1) = 2P(\bar{X} \ge c_2),$$

  $$p\text{-value} = \begin{cases} 2P(\bar{X} \le \bar{x}) & \text{if } \bar{x} < \mu_0, \\ 2P(\bar{X} \ge \bar{x}) & \text{if } \bar{x} > \mu_0. \end{cases}$$

- **Answering format for hypothesis test**:

  1. State null hypothesis $H_0 : \mu = \mu_0$ and alternative hypothesis H_1 (depends if left-tail, right-tail, or 2-tail)
  2. State significance level α (provided in question).
  3. Consider distribution of test statistic (if we are testing sample mean, then test statistic is $\bar{X}$).

  Critical value approach:

  4. Calculate critical value based on significance level, and test value based on sample data.
  5. If test value falls in critical region, reject H_0; otherwise, do not reject H_0.
     Write down conclusion in the context of question.

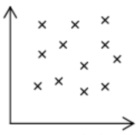
  p-value approach:

  6. Calculate p-value based on sample data.
  7. If $p \le \alpha$, reject H_0; otherwise, do not reject H_0.
     Write down conclusion in the context of question.

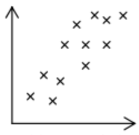
> **Remark.** When concluding a hypothesis test, it is either "reject H_0" or "do not reject H_0".
>
> Note that "do not reject H_0" is not equivalent to "accept H_0". This is because under the framework of hypothesis testing, there is no way to prove H_0 is true; we can only assess whether there is sufficient evidence *against* it.

§19 Correlation and Regression

- **Scatter diagram**: a sketch where each axis represents a variable, each point represents an observation.
  - Need not start from $(0,0)$.
  - Label axes according to context.
  - Indicate range of data values (minimum and maximum values).
  - Relative position of points should be accurate.



no correlation



positive correlation

negative correlation

Interpreting scatter diagram

1. Direction: positive / negative direction
2. Form: points lie on straight line (linear) / curve

> **Example.** Positive linear relationship, negative linear relationship, curvilinear relationship, no clear relationship.

- **Product moment correlation coefficient** r: measures strength and direction of a linear correlation between two variables.

$$r = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sqrt{\left(\sum(x-\bar{x})\right)\left(\sum(y-\bar{y})\right)}} = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sqrt{\left(\sum x^2 - \frac{(\sum x)^2}{n}\right)\left(\sum y^2 - \frac{(\sum y)^2}{n}\right)}} \qquad \text{(MF26)}$$

It can be shown that $-1 \le r \le 1$.

  - $r \approx 1$: strong positive linear correlation
  - $r \approx -1$: strong negative linear correlation
  - $r = 0$: no linear correlation

> **Remark.** r is independent of units of measurement.

Identify **outlier**(s) from scatter diagram, remove them to calculate more accurate value of r.

Correlation does not imply causation.

- Regression line

Least squares method

  - Regression line of y on x: line which minimises sum of squares of vertical distances from points to line

$$y - \bar{y} = b(x-\bar{x}), \quad b = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2} \qquad \text{(MF26)}$$

  - Regression line of x on y: line which minimises sum of squares of horizontal distances from points to line

$$x - \bar{x} = d(y-\bar{y}), \quad d = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(y-\bar{y})^2}$$

replace x with y and vice versa.

> **Remark.** Regression lines pass through mean point $(\bar{x}, \bar{y})$.

> **Remark.** The stronger the linear correlation, the closer the two regression lines are to each other.

Choice of regression line

Case	Estimate y given x	Estimate x given y
x independent, y dependent	y on x	
y independent, x dependent	x on y	
not specified	y on x	x on y

Interpolation: estimate within given range of values

Extrapolation: estimate outside given range of values

Reliability of estimate*

1. Appropriateness of regression line used
2. Strength of linear correlation: $|r|$ should be close to 1 for the estimate to be reliable
3. Interpolation or extrapolation: interpolation is likely to give a more reliable estimate than extrapolation

> **Example.** Since $x = 10$ lies outside the given data range ($31 \le x \le 98$), it is an extrapolation, so the linear model might not hold out of this range. Hence estimated value is unreliable.
>
> Since $y = 75$ lies within the given data range ($54 \le y \le 96$), it is an interpolation. Hence estimated value is reliable.

- **Transformation to linearity**
  - square transformation: $y = a + bx^2$
  - reciprocal transformation: $y = ab^x \implies \ln y = \ln a + x \ln b$
  - logarithmic transformation: $y = ax^b \implies \ln y = \ln a + b \ln x$