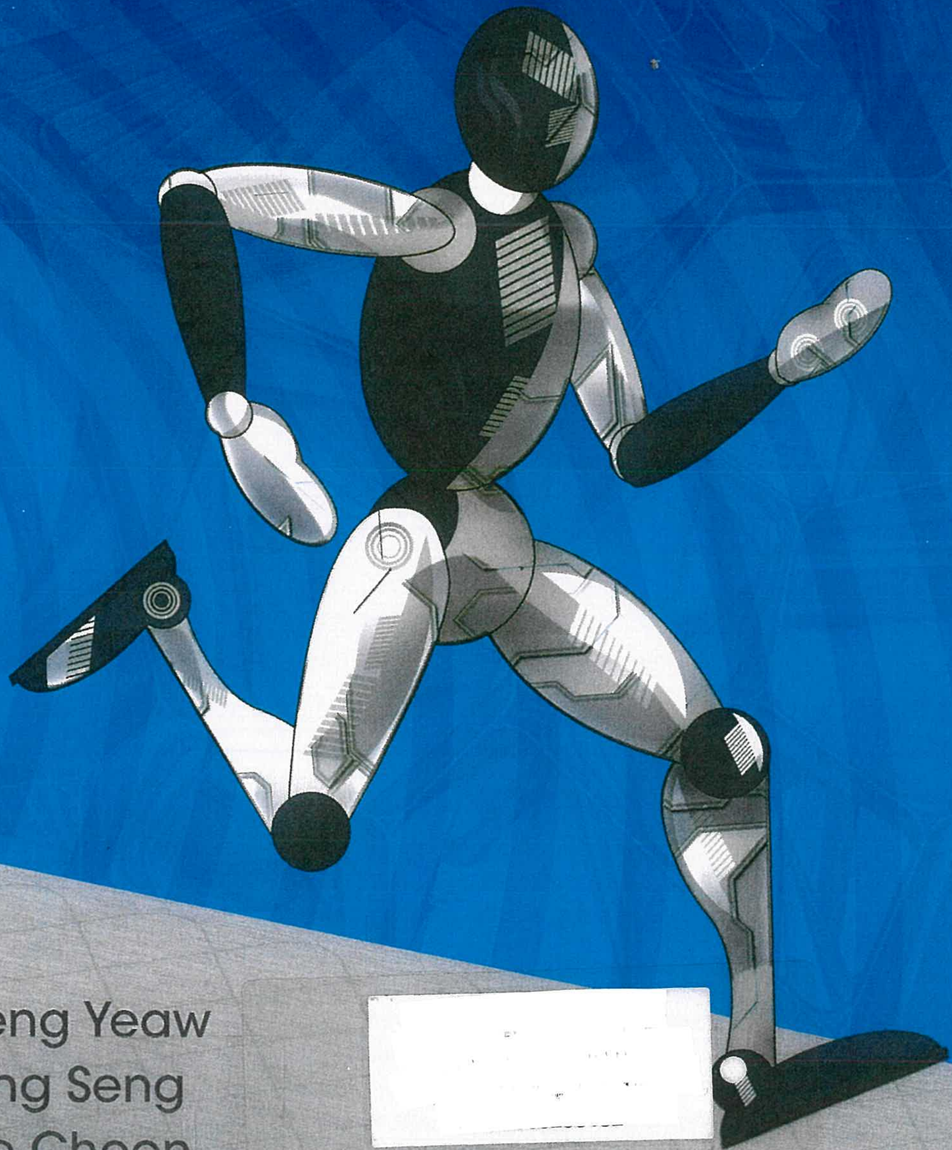
Singapore Mathematical Olympiads 2018



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Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2018

Junior Section (Round 1)

Wednesday, 30 May 2018 0930-1200 hrs

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Multiple Choice Questions

1. Which of the five numbers

$$\frac{9}{11},\ \frac{19}{22},\ \frac{29}{33},\ \frac{39}{44}\ \text{and}\ \frac{49}{55},$$

has the largest value?

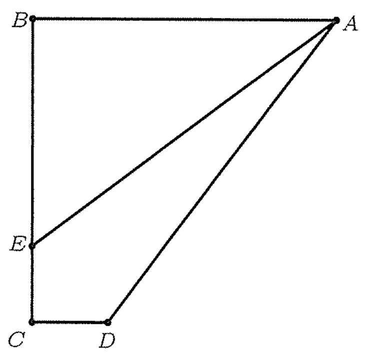
(A) $\frac{9}{11}$ (B) $\frac{19}{22}$ (C) $\frac{29}{33}$ (D) $\frac{39}{44}$ (E) $\frac{49}{55}$

2. Find the value of

$$\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2018}\right)\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2017}\right)$$
$$-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2018}\right)\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2017}\right).$$

(A) $\frac{1}{2017}$ (B) $\frac{1}{2018}$ (C) $\frac{2017}{2018}$ (D) 1 (E) $\frac{2019}{2018}$

3. Given that $ABCD$ is a right-angled trapezium with $AB = BC$, $\angle ABC = \angle BCD = 90°$ and E is a point on BC such that $AE = AD$. If $AD = 10$ and $BE = 6$, find the length of DE.



(A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) 2 (D) $2\sqrt{2}$ (E) $2\sqrt{3}$

4. In a strange island, there are only two types of inhabitants: truth-tellers who only tell the truth and liars who only tell lies. One day, you meet two such inhabitants A and B. A said: "At least one of us is a liar." B kept silent. Which of the following must be true?

(A) Both A and B are truth-tellers
(B) Both A and B are liars
(C) A is a truth-teller and B is a liar
(D) A is a liar and B is a truth-teller
(E) Not enough information to decide

5. Suppose $ax + 5 = 0$ is satisfied by some x, where $-5 < x < 5$. Which one of the following conditions describes completely the range of a?

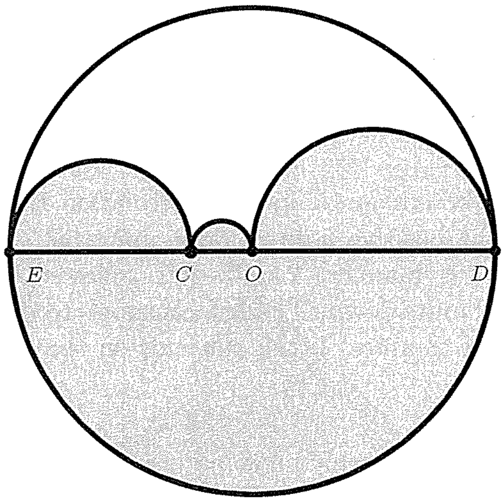
(A) $0 < a < 1$ (B) $a > 1$ or $a < -1$ (C) $a > 1$
(D) $a < -1$ (E) $a > 1$ or $-1 < a < 0$

Short Questions

6. There are 24 identical containers, each with a capacity of 12 litres. The containers contain some amount of water. If half of the containers are half full, a quarter of the containers are a quarter full, one-sixth of the containers are one-sixth full and one-twelfth of the containers are one-twelfth full, find the total amount of water (in litres) that are in the containers.

7. A quadruple (a, b, c, d) of positive integers is *balanced* if the mean, median and mode of a, b, c, d are equal. How many balanced quadruples (a, b, c, d) of positive integers are there that satisfy $a \leq b \leq c \leq d$ and $a + b + c + d = 44$?

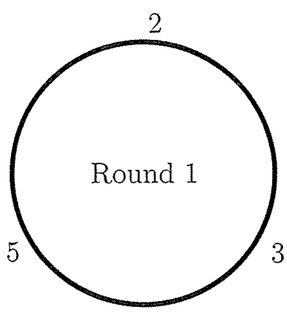
8. In the following diagram, ED is a diameter of the circle, EC, CO and OD are respective diameters of the three semi-circles. If $EC = 12$, $CO = 4$, $OD = 16$ and the area of the shaded region is $k\pi$, what is the value of k?

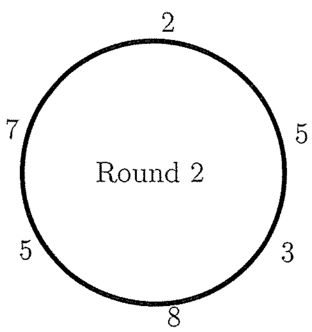


9. If $m^2 - 5m - 300 = 0$, find the value of

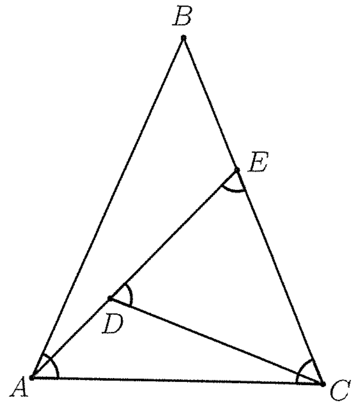
$$m^3 - 325m + 1503.$$

10. In Round 1 of a game, three numbers $2, 3$ and 5 are written around a circle. In Round 2, the sum of every two adjacent numbers with respect to the circle is written between them. The following diagram shows the numbers after Rounds 1 and 2. Find the sum of all the numbers written around the circle after Round 5.





11. In the following diagram, if $AD = 2$, $AC = 6$, $\angle BAC = \angle BCA = \angle DEC = \angle CDE$, find the value of AB^2.

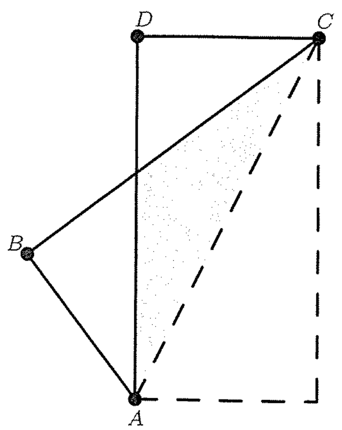


12. A business man bought p identical mobile phones at a total cost of m dollars, where p is a prime and m is a positive integer. He sold 2 of the mobile phones each at half of the cost price to a charitable organisation and sold the rest of the mobile phones in his shop at a profit of \$200 per phone. If his total profit from selling all these mobile phones was \$1800, find the least possible value of m.

13. Consider the x-y plane, where we adopt the convention that the positive x-direction is towards the right and the positive y-direction is upwards. An ant starts from the origin and crawls in the following manner: 2 units left, followed by 3 units up, followed by 4 units right, followed by 2 units down. The ant continues to move in the same pattern until it has moved a total of 2018 units. If the ant ends on the coordinate (m, n), what is the value of $m + n$?

14. A three-digit positive integer has the following properties:
(i) if its tens and ones digits are swapped, the integer would increase by 36;
(ii) if its hundreds and ones digits are swapped, the integer would decrease by 495.
Find the three-digit integer.

15. Let $ABCD$ be a rectangular sheet of paper with $AB = 12$ and $BC = 24$. If we fold the sheet of paper along the diagonal AC, there will be a overlapping region as shown in the diagram. Find the area of this overlapping region.

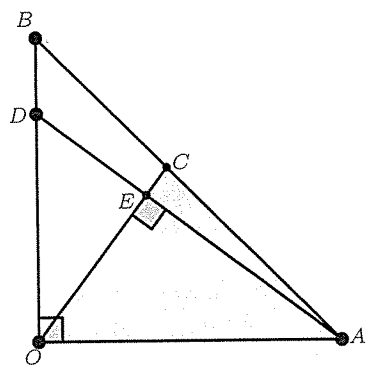


16. If $a - b = 3 + \sqrt{8}$ and $b - c = 3 - \sqrt{8}$, find the value of

$$a^2 + b^2 + c^2 - ab - bc - ac.$$

17. A teacher organised an outing for her students. The outing costs T dollars, regardless of how many students participated. This cost will be shared equally among the participants. If all her students were to participate, each student would pay a whole number of dollars. After the teacher announced how much each student would pay, there were 4 students who decided to withdraw. Without these four students, each of the remaining students would still pay a whole number of dollars. However, on the day of the outing, another 2 students were sick and absent. As these 2 sick students did not participate, each student who participated in the outing had to fork out an additional 3 dollars. If the teacher has between 24 and 40 students (inclusive), find T, the total cost of the outing in dollars.

18. In the following diagram, $\triangle AOB$ is an isosceles right-angled triangle with $OA = OB = 40$. Point D lies on the side OB such that $OD : DB = 3 : 1$. Point E is a point on AD such that OE is perpendicular to AD and OE extended meets AB at C. If $[OAC]$ denotes the area of $\triangle OAC$, find the value of $7[OAC]$.



19. Find the value of

$$\sqrt{9^2 + 19} + \sqrt{19^2 + 39} + \sqrt{29^2 + 59} + \sqrt{39^2 + 79} + \cdots + \sqrt{639^2 + 1279}.$$

20. If x is a positive real number that satisfies $x^2 = \sqrt{3}$, evaluate

$$\frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2+1}.$$

21. If x is a real number such that $x^3 + \dfrac{1}{x^3} = 18$, determine the value of $\left(x + \dfrac{1}{x}\right)^2$.

22. When expressed as a fraction in the lowest terms,

$$\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \cdots + \frac{1}{99 \times 100 \times 101} = \frac{m}{n}.$$

Find the value of $m + n$.

23. If α, β and γ satisfy

$$(x-\alpha)(x-\beta)(x-\gamma) = x^3 - 18x^2 - 122x + 161,$$

find the value of $\alpha^2 + \beta^2 + \gamma^2$.

24. How many integers x satisfy

$$x^2 + 2017x \leq 2018x + 2019?$$

25. Determine the coefficient of x in the following expansion:

$$1 - (1-x) + (1-x)^2 - (1-x)^3 + (1-x)^4 - \cdots - (1-x)^{2017}.$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2018

Junior Section (Round 1 Solutions)

Multiple Choice Questions

1. Answer: (E)

   If $0 < x < y$ then $\dfrac{10x-1}{11x} < \dfrac{10y-1}{11y}$. So the sequence is increasing from (A) to (E).

2. Answer: (B)

   Let $x = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2017}$, then the expression becomes

   $$\left(x + \frac{1}{2018}\right)(1+x) - \left(1 + x + \frac{1}{2018}\right)x = \frac{1}{2018}.$$

3. Answer: (D)

   Given $AB = BC$ and $\angle ABC = \angle BCD = 90°$, we can extend CD to a point F so that ABCF is a square. $\triangle ABE$ is congruent to $\triangle AFD$ (by RHS), so $BE = FD$ and $EC = DC$. Using Pythagoras Theorem, $AB = \sqrt{10^2 - 6^2} = 8 = BC$ and thus $EC = DC = 8 - 6 = 2$. Using Pythagoras Theorem again, $DE = \sqrt{2^2 + 2^2} = 2\sqrt{2}$.

4. Answer: (C)

   Suppose A is a liar, then his statement would be true and not a lie. So A cannot be a liar. Thus A must a truth-teller. Since A tells the truth, B must be a liar.

5. Answer: (B)

   If $a = 0$, then $ax + 5 = 0$ is not satisfied by any x. Hence $a \neq 0$ and the only solution is $x = -5/a$. By the given condition,

   $$\begin{aligned} -5 < -5/a < 5 &\iff |-5/a| < 5 \\ &\iff |a| > 1. \end{aligned}$$

   Therefore $a > 1$ or $a < -1$.

Short Questions

6. Answer: 100

   Note that $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{12}=1$ and so there is sufficient information to solve the problem. The total amount of water is

   $$12\times 6+6\times 3+4\times 2+2\times 1=100.$$

7. Answer: 11

   Since the mean, median and mode must be equal to 11, we have $b=c=11$. It remains to count all possibilities of $a+d=22$ subject to $1\le a\le 11$ and $11\le d\le 21$. Fixing a determines d and there are exactly 11 possibilities.

8. Answer: 180

   The area of the shaded region is

   $$\frac{1}{2}\pi(16)^2+\frac{1}{2}\pi(8)^2+\frac{1}{2}\pi(2)^2+\frac{1}{2}\pi(6)^2=180\pi.$$

9. Answer: 3003

   $$\begin{aligned} m^3-325m+1503 &= m(m^2-5m-300)+5m^2-25m+1503\\ &= 0+5(300)+1503\\ &= 3003. \end{aligned}$$

10. Answer: 810

    Suppose $x_1,\ldots,x_k$ are the numbers around the the circle. In the next round, each x_i will be used twice to create new numbers adjacent to it. In other words, the sum total will now be

    $$x_1+\cdots+x_k+2\left(x_1+\cdots+x_k\right).$$

    More generally if we let S_r denote the sum of the numbers after round r, then

    $$S_1=x_1+x_2+x_3=2+3+5=10;\qquad S_{r+1}=3S_r.$$

    So $S_n=3^{n-1}\times 10$ and $S_5=810$.

11. Answer: 54

    Since $\triangle AEC$ is isosceles, $AC=AE=6$, thus $DE=6-2=4$. $\triangle CEA$ is similar to $\triangle DEC$, so $CE:AE=DE:CE$, that is $CE:6=4:CE$. Thus $CE^2=24$. We also have $\triangle ACB$ is similar to $\triangle CEA$, so $AB:AC=CA:CE$, that is $AB:6=6:CE$. Thus

    $$AB^2=\frac{36^2}{24}=54.$$

12. Answer: 5200

His cost price per phone is $\frac{m}{p}$ dollars, so he incurred losses of $2 \times \frac{m}{2p}$ dollars on the two phones he sold to the charitable organisation. Thus his total profit in dollars is

$$200(p-2) - \frac{m}{p} = 1800 \implies m = 200p(p-11).$$

In order for m to be least (but positive) and p prime, p must be 13 and $m = 5200$.

13. Answer: 550

After each cycle of 4 moves, the ant would have moved 11 units, but the resultant displacement is only 2 units to the right and 1 unit up. Now $2013 = 183 \times 11$, so after moving 2013 units, the ant would be at $(366, 183)$. It then moves 2 units left followed by 3 units up to arrive at $(364, 186)$. Finally $364 + 186 = 550$.

14. Answer: 904

Let a, b and c respectively be the hundreds, tens and ones digit. The required number is $100a + 10b + c$. Property (i) means

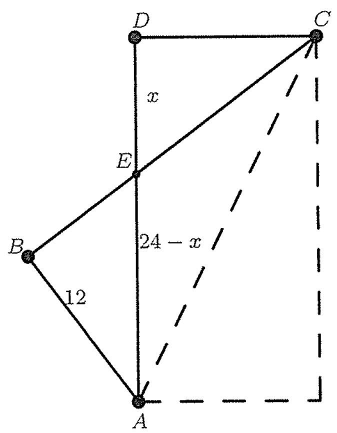
$$10(c-b) + (b-c) = 36 \implies c - b = 4.$$

Property (ii) means

$$100(a-c) + (c-a) = 495 \implies a - c = 5.$$

From the above, $a - b = 9$. Since the smallest possible value for b is 0, we must have $a = 9$ and thus $c = 4$.

15. Answer: 90



Note that $AB = CD$, $\angle ABE = \angle CDE = 90°$ and $\angle AEB = \angle CED$. So $\triangle ABE$ is congruent to $\triangle CDE$ (AAS). Thus $BE = DE$. Let $BE = x$ then $AE = 24 - x$. Using Pythagoras Theorem,

$$12^2 + x^2 = (24 - x)^2 \implies 144 = 576 - 48x.$$

Thus $x = 9$ and the area of the shaded triangle is

$$\frac{1}{2} \times 12 \times (24 - x) = 90.$$

16. Answer: 35

First observe that

$$(a-b)(b-c) = 9-8 = 1 \implies 1 = ab - b^2 - ac + bc.$$

Secondly, $a - c = 6$, and the required expression can be written as

$$(a-c)^2 - (ab - b^2 - ac + bc) = 36 - 1 = 35.$$

17. Answer: 1440

Let n be the number of students in her class. So $n-6$ went for the outing and each paid 3 dollars extra to cover the cost for the two sick students. So

$$3(n-6) = 2\left(\frac{T}{n-4}\right) \implies 3n^2 - 30n + 72 = 2T.$$

From the above, $3n^2$ must be even which means n is even and we can write $n = 2k$ to simplify the equation to

$$T = 6k^2 - 30k + 36 = n(3k - 15) + 36.$$

Since originally $\frac{T}{n}$ is a whole number, n must be a factor of T and thus n must also be a factor of 36. The only factor of 36 between 24 and 40 is 36. So we conclude $n = 36$ and

$$T = 36(3 \times 18 - 15) + 36 = 1440.$$

18. Answer: 3200

We have $OD = 30$ and $AD = 50$ by Pythagoras Theorem. Next, observe that $\angle DOE = \angle DAO$ and hence $\triangle DOE$ is similar to $\triangle DAO$. Thus

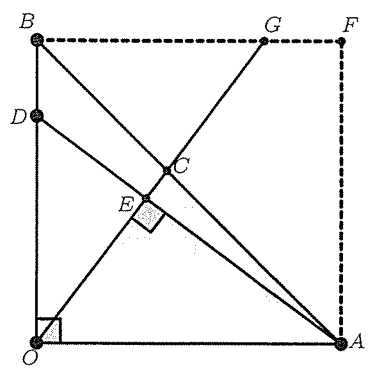
$$DE = \frac{DO^2}{DA} = 18.$$

Consequently $AE = 32$. We now form a square $OAFB$ as shown in the diagram and extend OC to meet BF at G. We have $\angle GOB = \angle DOE = \angle DAO$ and by ASA, $\triangle GOB$ and $\triangle DAO$ are congruent. We can now conclude $BG = 30$ and $OG = 50$. Finally, we observe that $\triangle BGC$ and $\triangle AOC$ are similar. Moreover,

$$\frac{BG}{AO} = \frac{30}{40} \implies OC = \frac{4}{7} \times OG = \frac{200}{7}.$$

We then conclude

$$7[OAC] = 7 \times \frac{1}{2} \times 32 \times \frac{200}{7} = 3200.$$



19. Answer: 20800

Note that for any $k \geq 1$,

$$(10k-1)^2 + 20k - 1 = 100k^2 - 20k + 1 + 20k - 1 = (10k)^2.$$

So the required sum is

$$\sum_{k=1}^{64} 10k = \frac{10(64)(65)}{2} = 20800.$$

20. Answer: 2

$$\begin{aligned}
\frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2+1} &= \frac{x+1}{x^2-1} - \frac{x-1}{x^2-1} - \frac{2}{x^2+1} \\
&= \frac{2}{x^2-1} - \frac{2}{x^2+1} \\
&= \frac{2(x^2+1)}{x^4-1} - \frac{2(x^2-1)}{x^4-1} \\
&= \frac{4}{3-1} = 2.
\end{aligned}$$

21. Answer: 9

We have

$$\left(x + \frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right).$$

If we set $y = x + \frac{1}{x}$, we obtain the equation

$$y^3 = 18 + 3y.$$

Observing that $y = 3$ is a possible solution, the equation factors into

$$(y-3)(y^2 + 3y + 6) = 0,$$

with $y = 3$ being the only real solution. Furthermore $x + \frac{1}{x} = 3$ has real solutions in x. Hence $\left(x + \frac{1}{x}\right)^2 = 9$.

22. Answer: 25249

Observe that

$$\frac{2}{k \times (k+1) \times (k+2)} = \frac{1}{k \times (k+1)} - \frac{1}{(k+1) \times (k+2)}.$$

and thus

$$\begin{aligned}
\sum_{k=1}^{99} \frac{1}{k \times (k+1) \times (k+2)} &= \frac{1}{2} \sum_{k=1}^{99} \left(\frac{1}{k \times (k+1)} - \frac{1}{(k+1) \times (k+2)} \right) \\
&= \frac{1}{2} \left(\frac{1}{2} - \frac{1}{100 \times 101} \right) \\
&= \frac{5049}{20200}.
\end{aligned}$$

Hence $m + n = 5049 + 20200 = 25249$.

23. Answer: 568

By expanding the left side of the equation and comparing coefficients of x, we have

$$\alpha+\beta+\gamma=18 \quad \text{and} \quad \alpha\beta+\alpha\gamma+\beta\gamma=-122.$$

Thus

$$\begin{aligned}\alpha^2+\beta^2+\gamma^2&=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)\\&=18^2+244=568.\end{aligned}$$

24. Answer: 90

The inequality is equivalent to

$$x^2-x+\frac{1}{4}\leq 2019+\frac{1}{4} \iff \left(x-\frac{1}{2}\right)^2\leq 2019+\frac{1}{4}.$$

Thus we have

$$\frac{1}{2}-\sqrt{2019+\frac{1}{4}}\leq x\leq \frac{1}{2}+\sqrt{2019+\frac{1}{4}}.$$

Since x is an integer, we deduce from

$$\left(44+\frac{1}{2}\right)^2=1980+\frac{1}{4} \quad \text{and} \quad \left(45+\frac{1}{2}\right)^2=2070+\frac{1}{4}$$

that $-44\leq x\leq 45$, giving us a total of 90 possible values.

25. Answer: 1009

First observe that $(1-x)^k=1-kx+\cdots$. So we are required to determine

$$\begin{aligned}1-2+3-4+\cdots-2016+2017&=\sum_{i=1}^{2017}-(-1)^i i\\&=\sum_{i=1}^{2017}i-2\sum_{i=1}^{1008}2i\\&=\frac{(2017)(2018)}{2}-4\frac{(1008)(1009)}{2}=1009.\end{aligned}$$

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Singapore Mathematical Olympiad (SMO) 2018

(Junior Section, Round 2)

Saturday, 23 June 2018 **0900-1200**

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1. Consider the integer $30x070y03$ where x, y are unknown digits. Find all possible values of x, y so that the given integer is a multiple of 37.

2. In $\triangle ABC$, $AB = AC = 14\sqrt{2}$, D is the midpoint of CA and E is the midpoint of BD. Suppose $\triangle CDE$ is similar to $\triangle ABC$. Find the length of BD.

3. One hundred balls labelled 1 to 100 are to be put into two identical boxes so that each box contains at least one ball and the greatest common divisor of the product of the labels of all the balls in one box and the product of the labels of all the balls in the other box is 1. Determine the number of ways that this can be done.

4. Determine all positive integers n with at least 4 factors such that n is the sum of the squares of its 4 smallest factors.

5. You are given some equilateral triangles and squares, all with side length 1, and asked to form convex n sided polygons using these pieces. If both types must be used, what are the possible values of n, assuming that there is sufficient supply of the pieces?

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Singapore Mathematical Olympiad (SMO) 2018

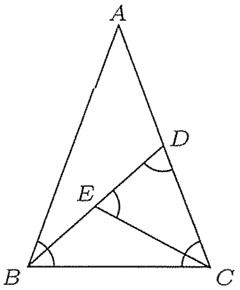
(Junior Section, Round 2 solutions)

1. We have

$$30x070y03 = 300070003 + 10^6x + 10^2y = 37(8110000 + 27027x + 3y) + (3 + x - 11y).$$

Therefore $3 + x - 11y$ is a multiple of 37. Since x, y lie between 0 and 9, $-96 \leq 3 + x - 11y \leq 12$. Therefore $3 + x - 11y = 0, -37, -74$. This gives the following three solutions $(x, y) = (8, 1), (4, 4), (0, 7)$.

2. Let $\ell = AB = AC = 14\sqrt{2}$ and $BC = x$. The 4 angles marked in the figure are all equal. This implies that $\triangle ABC, \triangle BCD, \triangle CDE$ are all similar. Thus $BD = BC = x$. Also $AB/BC = BC/CD$. That is $\ell/x = x/(\ell/2)$. From this we get $x = \ell/\sqrt{2}$. Since $\ell = 14\sqrt{2}$, we have $x = 14$.



3. The answer is 2047.

First let's assume the two boxes are different. The balls labelled 1 and 2 can be put into either box. All balls with even labels must then be put into the same box that the ball labelled 2 goes. Thus any ball with label which has a common factor greater than 1 with anyone of those even labels must be put into the same box too. Therefore, we only need to consider those balls with labels greater than 50. That leaves with 10 balls labelled by the primes: 53, 59, 61, 67, 71,73, 79, 83, 89, 97. Thus we have twelve entities, namely the ten prime numbers, the number 1 and all the balls that must go with 2. Each can be put into any of the two boxes. Taking away the 2 cases where one of the boxes is empty, the number ways is $2^{12} - 2 = 4094$. Since the two boxes are identical, we need to divide this number by 2. Therefore the answer is 2047.

4. Let $a < b < c < d$ be the 4 smallest factors of n. Then $n = a^2 + b^2 + c^2 + d^2$. If n is odd, then a, b, c, d are all odd, impossible. Thus n is even. Hence $a = 1, b = 2$. If

$4 \mid n$, then one of c, d is 4 and the other is odd. By taking mod 4, we see the LHS is 0 while the RHS is 2, a contradiction. Thus 4 is not a factor. Note that c cannot be even otherwise $c/2$ is also a factor which means c has to be 2. Thus c is odd and d is even. Therefore $d = 2c$. So we have $n = 1+4+c^2+4c^2 = 5+5c^2$. Hence 5 is a factor. If 3 is a factor, then $c = 3$ and $d = 5$, a contradiction. Therefore $c = 5$ and $n = 5(1+5^2) = 130$. It is easily checked that the smallest 4 factors are 1, 2, 5, 10 and 130 is the sum o their squares.

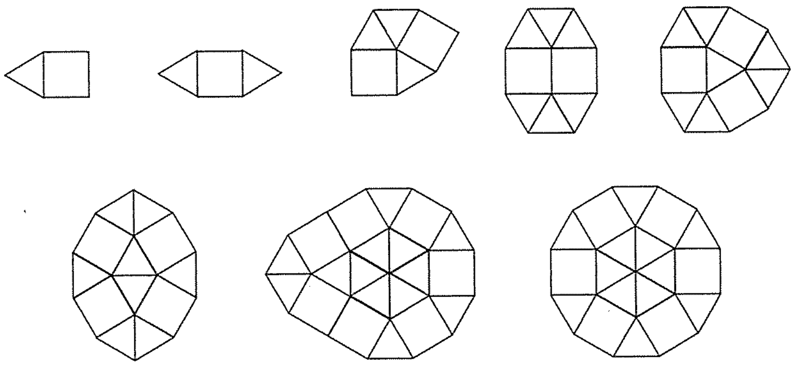
5. The possible internal angles are $60°, 90°, 120°, 150°$. Let Their respective numbers be a, b, c, d. Then we have

$$
\begin{aligned}
&a + b + c + d = n \\
&60a + 90b + 120c + 150d = (n-2)180
\end{aligned}
$$

Eliminating n, we obtain

$$4a + 3b + 2c + d = 12.$$

Thus $n \leq 12$. Since both types must be used $n \geq 5$. It's easy to construct such polygons with 5-12 sides.



Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2018

Senior Section (Round 1)

Wednesday, 30 May 2018 0930 – 1200 hrs

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1. The roots of the quadratic equation $x^2 - 3x + 1 = 0$ are α and β. Find a quadratic equation whose roots are $\alpha^2 + \alpha$ and $\beta^2 + \beta$.

   (A) $x^2 + 10x - 5 = 0$ (B) $x^2 - 10x + 5 = 0$ (C) $x^2 - 5x - 10 = 0$

   (D) $x^2 + 5x - 10 = 0$ (E) $x^2 - 5x - 5 = 0$

2. Simplify

$$\sqrt{2+\sqrt{3}} + \sqrt{2-\sqrt{3}}.$$

   (A) $\sqrt{5}$ (B) $\sqrt{6}$ (C) $\sqrt{7}$ (D) $\sqrt{8}$ (E) $\sqrt{10}$

3. Which of the following is true?

   (A) $10^{30} > 40^{15} > 30^{20}$ (B) $30^{20} > 10^{30} > 40^{15}$ (C) $10^{30} > 30^{20} > 40^{15}$

   (D) $30^{20} > 40^{15} > 10^{30}$ (E) $40^{15} > 30^{20} > 10^{30}$

4. Which of the following is the largest?

   (A) $\log_5 7 - \log_5 6$ (B) $\log_6 8 - \log_6 7$ (C) $\log_7 9 - \log_7 8$

   (D) $\log_8 10 - \log_8 9$ (E) $\log_9 11 - \log_9 10$

5. In the expansion of $\left(x^2 + \dfrac{1}{x}\right)^9$, find the coefficient of x^{15}.

   (A) 9 (B) 36 (C) 84 (D) 126 (E) None of the above

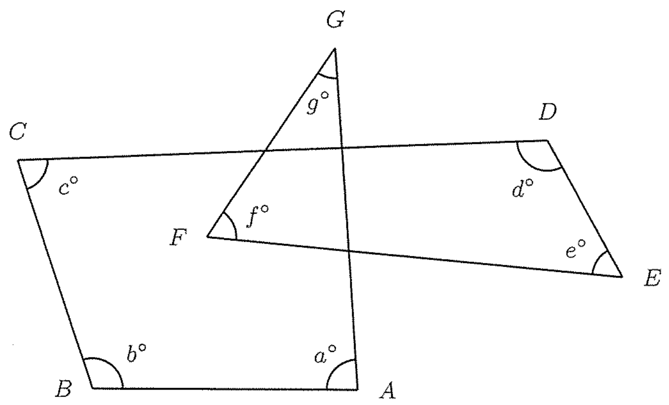
Short Questions

6. The line $y = 5x - 10$ meets the curve $x^2 - 4x + y^2 - 32 = 0$ at two points P and Q. Find the length of the line segment PQ in meters, assuming that both the x and y axis are measured in meters.

7. Suppose n is a positive integer, and x is measured in radians. If the equation $2\pi \sin nx = 2\pi - x$, where $0 \leq x < 2\pi$, has exactly 2018 different solutions, find n.

8. The polynomial $p(x) = x^3 + Ax^2 + Bx - 3$ has a factor of $x^2 + 7x + 1$. Find the remainder when the polynomial $p(x)$ is divided by $x - 4$.

9. In the figure below, the angles indicated at the vertex A, B, C, D, E, F, G are given by $a°$, $b°$, $c°$, $d°$, $e°$, $f°$, $g°$ respectively. Find $a + b + c + d + e + f + g$.



10. Find x, where $0° \leq x° \leq 90°$, such that $\cos x° = \cos 49° + \cos 71°$.

11. Let x and y be real numbers. Find the maximum value of $2x^2 - 3xy - 2y^2$ subject to the condition that

$$25x^2 - 20xy + 40y^2 = 36.$$

12. Suppose $\sin 2x = \frac{7}{9}$. Find $108(\sin^6 x + \cos^6 x)$.

13. Find the sum of all the positive integers x satisfying

$$(4 \log_2(\log_{16} x))\,(3 \log_{16}(\log_2 x) - 1) = 1.$$

14. Consider the function $f(x) = ax^2 - c$, where a and c are some constants. Suppose that $-3 \leq f(1) \leq -2$ and $1 \leq f(2) \leq 6$. Find the largest possible value of $f(4)$.

15. Suppose a is the smallest number satisfying the inequality

$$\left| \frac{|x+9|}{10} - \frac{|x-9|}{10} \right| \leq x - 1.$$

If $a = \frac{m}{n}$, where m and n are positive integers having no common factors larger than 1, find the value of $m + n$.

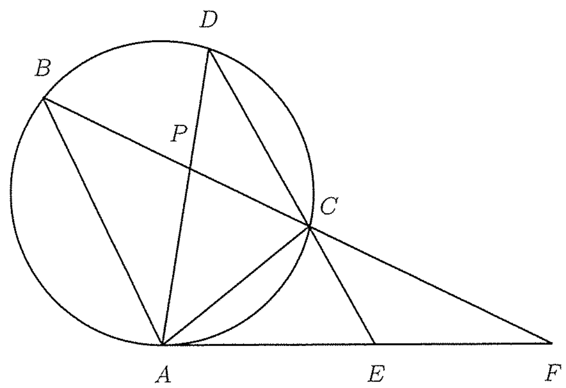
16. Let $P(x)$ be a polynomial of degree 4 such that $P(n) = \frac{120}{n}$ for $n = 1, 2, 3, 4, 5$. Determine the value of $P(6)$.

17. Let
$$L = \sum_{k=7}^{16} \left(1 + \tan(15k^\circ + 15^\circ)\tan 15k^\circ\right).$$
Find the largest integer which is smaller than or equal to L.

18. Find the smallest three-digit positive integer whose square ends in the digits 129.

19. In the figure below, the points A, B, C and D lie on the circle such that the lines AD and BC intersect at the point P. The line AF is tangent to the circle. The point E lies on the line AF such that the lines DE and PF intersect at C.

If the line AP bisects the angle $\angle BAC$, and $\angle CEA = (22 + y)^\circ$ where $y^\circ = \angle BAP$, find the angle $\angle APC$ (in $^\circ$).



20. Eleven identical boxes are arranged in a row. In how many ways can eight identical balls be put into the boxes if each box can hold at most one ball and no three empty boxes can appear consecutively next to each other?

21. Find the minimum positive integer N such that among any N distinct positive integers, there always exist two distinct positive integers such that either their sum or their difference is a multiple of 2018.

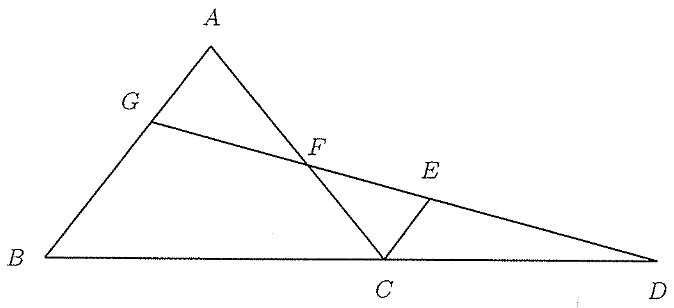
22. Consider a function f such that $2f(x - x^{-1}) + f(x^{-1} - x) = 3(x + x^{-1})^2$ for all $x \neq 0$. Find $f(99)$.

23. Suppose x, y and z are positive real numbers satisfying the following system of equations:

$$\frac{\sqrt{xyz}}{x+y} = 3,$$
$$\frac{\sqrt{xyz}}{y+z} = \frac{5}{2},$$
$$\frac{\sqrt{xyz}}{z+x} = \frac{15}{7}.$$

If $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{N}{900}$, find N.

24. In the figure below, the lines AB and CE are parallel to each other. The point F is the midpoint of the line AC, and $AB = 3AG$. If the area of the triangle $\triangle ABC$ is 666 cm^2, find the area of the triangle $\triangle CDE$ (in cm^2).



25. For any real number x, let $\lfloor x \rfloor$ denote the largest integer smaller than or equal to x. For example, $\lfloor 3 \rfloor = 3$, $\lfloor 2.8 \rfloor = 2$, $\lfloor -2.3 \rfloor = -3$. Suppose that R is a real number such that

$$\left\lfloor R - \frac{1}{200} \right\rfloor + \left\lfloor R - \frac{2}{200} \right\rfloor + \left\lfloor R - \frac{3}{200} \right\rfloor + \cdots \left\lfloor R - \frac{99}{200} \right\rfloor = 2018.$$

Find $\lfloor 20R \rfloor$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2018

Senior Section (Round 1 Solutions)

Multiple Choice Questions

1. **Answer.** (B)

   Solution. Note that $\alpha+\beta=3$, $\alpha\beta=1$.

$$\begin{aligned}(\alpha^2+\alpha)+(\beta^2+\beta) &= \alpha^2+\beta^2+\alpha+\beta\\ &= (\alpha+\beta)^2-2\alpha\beta+(\alpha+\beta)\\ &= 3^2-2(1)+3=10.\\ (\alpha^2+\alpha)(\beta^2+\beta) &= \alpha^2\beta^2+\alpha^2\beta+\alpha\beta^2+\alpha\beta\\ &= (\alpha\beta)^2+\alpha\beta(\alpha+\beta)+\alpha\beta\\ &= 1^2+1(3)+1=5.\end{aligned}$$

   A quadratic equation whose roots are $\alpha^2+\alpha$ and $\beta^2+\beta$ is

$$x^2-10x+5=0.$$

2. **Answer.** (B)

   Solution. Let $A=\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}}$. Then

$$\begin{aligned}A^2=\left(\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}}\right)^2 &= (2+\sqrt{3})+(2-\sqrt{3})+2\sqrt{2^2-3}\\ &= 6.\end{aligned}$$

   So $A=\sqrt{6}$.

3. **Answer.** (C)

   Solution. Note that $10^{30}>30^{20}\iff 10^{10}10^{20}>3^{20}10^{20}\iff 10^{10}>3^{20}$, which is true since $10>3^2\implies 10^{10}>(3^2)^{10}=3^{20}$. On the other hand, $30^{20}>40^{15}$ $\iff 3^{20}10^{20}>4^{15}10^{15}\iff 3^{20}10^5>4^{15}\iff 3^{20}(10^{1/3})^{15}>4^{15}$, which is true since $10^{1/3}>2\implies 3^{20}(10^{1/3})^{15}>3^{20}2^{15}>3^{15}2^{15}=6^{15}>4^{15}$.

4. **Answer.** (A)

   Solution. Since $\ln x$ is an increasing function, we have

$$\ln\left(1+\frac{1}{6}\right)>\ln\left(1+\frac{1}{7}\right)>\ln\left(1+\frac{1}{8}\right)>\ln\left(1+\frac{1}{9}\right)>\ln\left(1+\frac{1}{10}\right).$$

This implies that $\frac{\ln(\frac{7}{6})}{\ln 5} > \frac{\ln(\frac{8}{7})}{\ln 6} > \frac{\ln(\frac{9}{8})}{\ln 7} > \frac{\ln(\frac{10}{9})}{\ln 8} > \frac{\ln(\frac{11}{10})}{\ln 9}$, which is equivalent to

$$\log_5 7 - \log_5 6 > \log_6 8 - \log_6 7 > \log_7 9 - \log_7 8 > \log_8 10 - \log_8 9 > \log_9 11 - \log_9 10.$$

Thus, the largest number is $\log_5 7 - \log_5 6$.

5. **Answer.** (A)

   Solution. By the Binomial Theorem, we have

$$\left(x^2 + \frac{1}{x}\right)^9 = \sum_{i=0}^{9} \binom{9}{i} (x^2)^i \left(\frac{1}{x}\right)^{9-i} = \sum_{i=0}^{9} \binom{9}{i} x^{3i-9}.$$

   Set $3i - 9 = 15$. Then $i = 8$. Thus, the coefficients of x^{15} is $\binom{9}{8} = 9$.

Short Questions

6. **Answer.** 12

   Solution. The curve $x^2 - 4x + y^2 - 32 = (x-2)^2 + y^2 - 36 = 0$ is a circle with radius 6 centred at the point $(2, 0)$. Since the line $y = 5x - 10$ passes through the point $(2, 0)$, the length of the line segment PQ is just the diameter of the circle. Thus, $|PQ| = 2 \times 6 = 12$.

7. **Answer.** 1009

   Solution. Consider the curve $\mathcal{C}$ given by $y = \sin nx$ and the line $\mathcal{L}$ given by $y = 1 - \frac{x}{2\pi}$ on the xy-plane. The number of different solutions to the equation $2\pi \sin nx = 2\pi - x$ is the number of distinct intersection points of the curve $\mathcal{C}$ and the line $\mathcal{L}$. Since the function $\sin nx$ has a period of $\frac{2\pi}{n}$, the line $\mathcal{L}$ only intersects the curve $\mathcal{C}$ at exactly two distinct points over each of the interval

$$\left[\frac{2\pi}{n}k, \frac{\pi}{n} + \frac{2\pi}{n}k\right], \quad 0 \leq k \leq n-1,$$

   where k is an integer. Thus, the total number of distinct intersection points is $2n$. It follows that $n = \frac{2018}{2} = 1009$.

8. **Answer.** 45

   Solution. By long division, we have

$$p(x) = x^3 + Ax^2 + Bx - 3 = (x^2 + 7x + 1)(x + A - 7) + (B - 7A + 48)x + (4 - A).$$

   Since $x^2 + 7x + 1$ is a factor of the polynomial, we deduce that $B - 7A + 48 = 0$ and $4 - A = 0$. It follows that $A = 4$ and $B = -20$. Hence, $p(x) = (x^2 + 7x + 1)(x - 3)$. The remainder when $p(x)$ is divided by $x - 4$ is $p(4) = (4^2 + 7(4) + 1)(4 - 3) = 45$.

9. **Answer.** 540

Solution. Let P be the intersection point of the lines AG and CD. Let $x^\circ = \angle CPA$, $y^\circ = \angle CPG = \angle APD$. Since the interior angles of the quadrilateral $ABCP$ sum to 360°, we have $a + b + c + x = 360$. Since the interior angles of the 5-side polygon $DEFGP$ sum to 540°, we have $d + e + f + g + x + 2y = 540$. Together, we have $(a+b+c+d+e+f+g)+2(x+y) = 900 \Longrightarrow a+b+c+d+e+f+g = 900-2(x+y) = 900 - 2(180) = 540$.

10. **Answer.** 11

Solution.

$$\begin{aligned}
\cos 49^\circ + \cos 71^\circ &= \cos(60^\circ - 11^\circ) + \cos(60^\circ + 11^\circ) \\
&= \cos 60^\circ \cos 11^\circ + \sin 60^\circ \sin 11^\circ + \cos 60^\circ \cos 11^\circ - \sin 60^\circ \sin 11^\circ \\
&= \cos 11^\circ.
\end{aligned}$$

Thus, $x = 11$.

11. **Answer.** 3

$$\begin{gathered}
25x^2 - 20xy + 40y^2 = 36 \\
(16x^2 + 16xy + 4y^2) + (9x^2 - 36xy + 36y^2) = 36 \\
4(2x + y)^2 + 9(x - 2y)^2 = 36 \\
\left(\frac{2x + y}{3}\right)^2 + \left(\frac{x - 2y}{2}\right)^2 = 1
\end{gathered}$$

Denote $\frac{2x+y}{3} = \cos\theta$ and $\frac{x-2y}{2} = \sin\theta$, we have $2x + y = 3\cos\theta$ and $x - 2y = 2\sin\theta$. Solving this, we have

$$x = \frac{6\cos\theta + 2\sin\theta}{5}, \quad y = \frac{3\cos\theta - 4\sin\theta}{5}.$$

So $2x^2 - 3xy - 2y^2 = (2x + y)(x - 2y) = 6\sin\theta\cos\theta = 3\sin 2\theta$. Hence, the maximum value of $2x^2 - 3xy - 2y^2$ is 3 when $\theta = \frac{\pi}{4}$, $x = \frac{4\sqrt{2}}{5}$ and $y = -\frac{\sqrt{2}}{10}$.

12. **Answer.** 59

Solution. Note that

$$\begin{aligned}
\sin^6 x + \cos^6 x &= (\sin^2 x + \cos^2 x)^3 - 3\sin^4 x \cos^2 x - 3\sin^2 x \cos^4 x \\
&= 1 - 3\sin^2 x \cos^2 x(\sin^2 x + \cos^2 x) \\
&= 1 - \frac{3}{4}(\sin 2x)^2 \\
&= \frac{59}{108}.
\end{aligned}$$

Therefore, $108(\sin^6 x + \cos^6 x) = 59$.

23

13. **Answer.** 4

Solution. Let $a = \log_2 \log_{16} x$. Then

$$\begin{aligned}\log_{16}\log_2 x &= \log_{16}\left(\frac{\log_{16} x}{\log_{16} 2}\right)\\ &= \log_{16}(\log_2 16 \log_{16} x)\\ &= \log_{16}(4\log_{16} x)\\ &= \log_{16} 4 + \log_{16}\log_{16} x\\ &= \frac{1}{2} + \frac{\log_2 \log_{16} x}{\log_2 16}\\ &= \frac{1}{2} + \frac{a}{4}.\end{aligned}$$

Thus, we have

$$4a\left(3\left(\frac{1}{2}+\frac{a}{4}\right)-1\right) = 1,$$
$$\left(a-\frac{1}{3}\right)(a+1) = 0$$
$$x = 16^{2^{1/3}}, 16^{2^{-1}} = 4.$$

Since $16^{2^{1/3}}$ is not an integer, we deduce that the only positive integer x satisfying the condition is 4. Thus, the required sum is also 4.

14. **Answer.** 42

Note that

$$-3 \le f(1) = a - c \le -2 \Longrightarrow 2 \le c - a \le 3 \tag{1}$$
$$1 \le f(2) = 4a - c \le 6. \tag{2}$$

Adding (1) and (2), we have

$$3 \le 3a \le 9 \Longrightarrow 1 \le a \le 3 \Longrightarrow 12 \le 12a \le 36. \tag{3}$$

Adding (2) and (3), we have

$$13 \le f(4) = 16a - c \le 42.$$

The upper bound is attainable if we set $c - a = 3$ and $4a - c = 6$, whence $a = 3$, $c = 6$. Thus, 42 is the largest possible value for $f(4)$.

15. **Answer.** 9

Solution. Let $f(x) = |x+9| - |x-9|$. Note that $f(x) \ge 0$ if and only if $x \ge 0$. Thus, the inequality $|f(x)| \le 10x - 10$ becomes

$$|x+9| - |x-9| \le 10x - 10 \quad \text{if } x \ge 0 \tag{4}$$
$$|x-9| - |x+9| \le 10x - 10 \quad \text{if } x < 0. \tag{5}$$

It follows that

$$
\begin{aligned}
(x+9)-(9-x) &\leq 10x-10, && \text{if } 0 \leq x < 9 && (6)\\
(x+9)-(x-9) &\leq 10x-10, && \text{if } x \geq 9 && (7)\\
(9-x)-(x+9) &\leq 10x-10, && \text{if } -9 < x < 0 && (8)\\
(9-x)+(x+9) &\leq 10x-10, && \text{if } x \leq -9. && (9)
\end{aligned}
$$

From (6), we have $10x-10 \geq 2x \Longrightarrow x \geq \frac{5}{4}$. So (6) holds for all $\frac{5}{4} \leq x < 9$.

From (7), we have $10x-10 \geq 18 \Longrightarrow x \geq \frac{28}{10}$. Since $\frac{28}{10} < 9$, (7) holds for all $x \geq 9$.

From (8), we have $10x-10 \geq -2x \Longrightarrow x \geq \frac{10}{12}$. Since $-9 < x < 0$, there is no solution to (8).

From (9), we have $10x-10 \geq 18 \Longrightarrow x \geq \frac{28}{10}$. Since $x \leq -9$, there is no solution to (9).

We deduce that the smallest number a satisfying the inequality is $a = \frac{5}{4}$. Thus, $m+n=9$.

16. **Answer.** 40

Solution. Let $Q(x) = xP(x) - 120$. Then $Q(x)$ is a polynomial of degree 5 such that $Q(1) = Q(2) = Q(3) = Q(4) = Q(5) = 0$. So $Q(x) = k(x-1)(x-2)(x-3)(x-4)(x-5)$ for some constant k. Comparing the constant term on both sides, we have $-120 = -120k$, whence $k=1$. Thus, we have the following:

$$
\begin{gathered}
xP(x) - 120 = (x-1)(x-2)(x-3)(x-4)(x-5),\\
6P(6) - 120 = 5(4)(3)(2)(1)\\
P(6) = 240/6 = 40.
\end{gathered}
$$

17. **Answer.** 27

Solution. Let $L = \sum_{k=7}^{16}(1+\tan(15k^\circ + 15^\circ)\tan 15k^\circ)$. Multiplying L by $\tan 15^\circ$, we have

$$
\begin{aligned}
\tan 15^\circ \cdot L &= \sum_{k=7}^{16}(1+\tan(15k^\circ+15^\circ)\tan 15k^\circ)(\tan 15^\circ)\\
&= \sum_{k=7}^{16}(\tan(15k^\circ+15) - \tan 15k^\circ)\\
&= (\tan 120^\circ - \tan 105^\circ) + (\tan 135^\circ - \tan 120^\circ) + (\tan 150^\circ - \tan 135^\circ)\\
&\quad + \cdots + (\tan 255^\circ - \tan 240^\circ))\\
&= \tan 255^\circ - \tan 105^\circ\\
&= 2\tan 75^\circ.
\end{aligned}
$$

Hence,

$$L = \frac{2\tan 75^\circ}{\tan 15^\circ}.$$

Recall that $\frac{1}{\sqrt{3}} = \tan 30° = \frac{2\tan 15°}{1-\tan^2 15°}$. Let $a = \tan 15°$. Then $\frac{1}{\sqrt{3}} = \frac{2a}{1-a^2}$. Solving for a, we have $a = \pm 2 - \sqrt{3}$. Since $a > 0$, we deduce that $a = 2 - \sqrt{3}$. It follows that

$$L \quad = \quad \frac{2\tan 75°}{\tan 15°} = 2 \cdot \frac{\frac{1}{2-\sqrt{3}}}{2-\sqrt{3}} = 14 + 8\sqrt{3}.$$

Now, $13^2 = 169 < (8\sqrt{3})^2 < 196 = 14^2 \implies 13 < 8\sqrt{3} < 14 \implies 27 = 14 + 13 < 14 + 8\sqrt{3} = L < 14 + 14 = 28$. Thus, the largest integer N such that $N \leq L$ is 27.

18. **Answer.** 123

Solution. Let n be a three-digit positive integer whose square ends in the digits 129. We can write $n = 10^2 a + 10b + c$, where $a \neq 0$, b, $c \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Note that

$$n^2 = 10^4 a^2 + 10^3(2ab) + 10^2(2ac + b^2) + 10(2bc) + c^2.$$

Since n^2 ends in the digits 129, we have $c^2 \equiv 9 \pmod{10}$ (so c is either 3 or 7), and $20bc + c^2 \equiv 29 \pmod{100}$. If $c = 3$, the last equation becomes $60b + 9 \equiv 29 \pmod{100}$ whence $60b \equiv 20 \pmod{100}$, so b is either 2 or 7. Since $123^2 = 15129$, we conclude that 123 is the smallest such integer.

19. **Answer.** 79

Solution. Let $z° = \angle ABP$. Then $\angle CAE = z°$, and so $\angle PAE = \angle PAC + \angle CAE = (y + z)°$. On the other hand, $\angle APC = \angle ABP + \angle BAP = (y + z)°$. Thus, the triangle $\triangle APF$ is an isosceles triangle with $|FP| = |FA|$.

Note that $\angle ADC = \angle ABC = \angle ABP = z°$. For the triangle $\triangle ADE$, the sum of its inner angles is $22 + 2(y + z) = 180$.

For the isosceles triangle $\triangle APF$, the sum of its inner angles is $2(y+z) + \angle AFC = 180$, whence $\angle AFC = 180 - 2(y + z) = 22$.

This implies that $\angle APC = \frac{1}{2}(180 - 22) = 79°$.

20. **Answer.** 156

Solution. Exactly eight of the boxes will contain exactly one ball each. Suppose these eight boxes are represented as follows in a row.

$$\boxed{\text{o}}\ \boxed{\text{o}}\ \boxed{\text{o}}\ \boxed{\text{o}}\ \boxed{\text{o}}\ \boxed{\text{o}}\ \boxed{\text{o}}\ \boxed{\text{o}}.$$

The number of ways is given by the number of ways we can insert the remaining three empty boxes into the empty spaces (including the ones on the far left and far right) between the boxes above such that these three empty boxes cannot all go into the same empty space. There are 9 empty spaces altogether.

- If 3 of these empty spaces are chosen, then the empty boxes can only be inserted in one way. E.g.

$$\boxed{\text{O}}\ \boxed{\text{O}}\ \square\ \boxed{\text{O}}\ \boxed{\text{O}}\ \square\ \boxed{\text{O}}\ \square\ \boxed{\text{O}}\ \boxed{\text{O}}\ \boxed{\text{O}}.$$

  So the total number of ways is $\binom{9}{3} = 84$.
- If 2 of these empty spaces are chosen, then the empty boxes can be inserted in two ways. E.g.

$$\boxed{\text{O}}\ \boxed{\text{O}}\ \square\ \square\ \boxed{\text{O}}\ \boxed{\text{O}}\ \boxed{\text{O}}\ \square\ \boxed{\text{O}}\ \boxed{\text{O}}\ \boxed{\text{O}},$$

$$\boxed{\text{O}}\ \boxed{\text{O}}\ \square\ \boxed{\text{O}}\ \boxed{\text{O}}\ \boxed{\text{O}}\ \square\ \square\ \boxed{\text{O}}\ \boxed{\text{O}}\ \boxed{\text{O}}.$$

  So the total number of ways is $2 \times \binom{9}{2} = 72$.

Hence, the total number of ways is $84 + 72 = 156$.

21. **Answer.** 1011

Solution. Consider the following set of 1010 positive integers:

$$\{2018, 1, 2, 3, 4, \ldots, 1009\}.$$

Since $2018 \equiv 0 \pmod{2018}$ and $i \equiv i \pmod{2018}$ for all $i = 1, 2, \ldots, 1009$, it is obvious that the sum (or difference) of any two distinct numbers from this set is never a multiple of 2018. This implies that $N \geq 1011$. We shall prove that $N = 1011$ as follows.

The remainders of a number when divided by 2018 are 0, 1, 2, ..., 2017. We distribute these remainders into the following 1010 sets:

$$A_0 = \{0\}, A_1 = \{1, 2017\}, A_2 = \{2, 2016\}, A_3 = \{3, 2015\}, \ldots,$$

$$A_{1008} = \{1008, 1010\}, A_{1009} = \{1009\}.$$

Consider a set of 1011 distinct positive integers. By the pigeonhole principle, there exist two distinct numbers a, b from this set whose remainders belong to the same A_i for some i. If these remainders are different then $a + b$ is a multiple of 2018; otherwise $a - b$ (or $b - a$) is a multiple of 2018.

22. **Answer.** 9805

Solution. Let $p = f(x - x^{-1})$ and $q = f(x^{-1} - x)$. Then $2p + q = 3(x + x^{-1})^2$. Substituting x by x^{-1}, we have $2q + p = 3(x + x^{-1})^2$. Thus, $p = q = (x + x^{-1})^2$, in particular, $f(x - x^{-1}) = (x + x^{-1})^2$. For any c, there exists x such that $c = x - x^{-1}$ (since $x^2 - cx - 1 = 0 \iff c^2 + 4 \geq 0$). Then $c^2 = x^2 - 2 + x^{-2}$. Thus

$$f(c) = (x + x^{-1})^2 = x^2 + 2 + x^{-2} = c^2 + 4.$$

Hence, $f(99) = 99^2 + 4 = 9805$.

23. **Answer.** 104

Solution. We have

$$\begin{aligned} 5\sqrt{xyz} &= 15(x+y), & (10)\\ 6\sqrt{xyz} &= 15(y+z), & (11)\\ 7\sqrt{xyz} &= 15(z+x). & (12) \end{aligned}$$

Adding (10), (11) and (12), we have

$$18\sqrt{xyz} = 30(x+y+z). \qquad (13)$$

Subtracting twice of (10), (11), (12) respectively from (13),

$$\begin{aligned} 8\sqrt{xyz} &= 30z & (14)\\ 6\sqrt{xyz} &= 30x & (15)\\ 4\sqrt{xyz} &= 30y. & (16) \end{aligned}$$

Multiplying (14), (15) and (16),

$$(8\times 6\times 4)(xyz)\sqrt{xyz} = 30^3 xyz \implies \sqrt{xyz} = \frac{30^3}{8\times 6\times 4},$$

since $x, y, z > 0$. Substituting this back into (14), (15), (16), we have

$$\begin{aligned} z &= \frac{8}{30}\sqrt{xyz} = \frac{30^2}{6\times 4} & (17)\\ x &= \frac{6}{30}\sqrt{xyz} = \frac{30^2}{8\times 4} & (18)\\ y &= \frac{4}{30}\sqrt{xyz} = \frac{30^2}{8\times 6} & (19) \end{aligned}$$

Hence,

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = \frac{8(4)+8(6)+6(4)}{30^2} = \frac{104}{900} \implies N = 104.$$

24. **Answer.** 222

Solution. The triangles $\triangle AGF$ and $\triangle CEF$ are similar. Since $\angle AFG = \angle EFC$ and $AF = FC$, we deduce that $AG = CE$. Also, $AB = 3AG \implies BG = 2AG = 2CE \implies \frac{CE}{BG} = \frac{1}{2}$.

Since AB is parallel to CE, the triangles $\triangle BGD$ and $\triangle CED$ are similar, and so $\frac{CD}{BD} = \frac{CE}{BG} = \frac{1}{2}$.

Let $[ABC]$, $[CED]$, $[BGD]$ denote the area of the triangle $\triangle ABC$, $\triangle CED$, $\triangle BGD$ respectively. Also, let $[BGEC]$ denote the area of the quadrilateral $BGEC$. Then

$$\frac{[CED]}{[BGD]} = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Since $[BGD] = [BGEC] + [CED]$ and $[ABC] = [BGEC]$, we deduce that

$$\frac{[CED]}{[BGEC]+[CED]} = \frac{1}{4} \implies [CED] = \frac{1}{3}[BGEC] = \frac{1}{3}[ABC] = \frac{1}{3}(666) = 222.$$

25. **Answer.** 423

Solution. For each $i = 1, \ldots, 99$, let $a_i = \left\lfloor R - \frac{i}{200} \right\rfloor = \left\lfloor (R-1) + \frac{200-i}{200} \right\rfloor$. Note that a_i is either $\lfloor R - 1 \rfloor$ or $\lfloor R - 1 \rfloor + 1$.

There are 99 terms in the sum. Since $1980 = 20 \cdot 99 < 2018 < 21 \cdot 99 = 2079$, and $2018 = 20(99) + 38$, each of the first 38 terms in the sum equals 21 and each of the last 61 terms equals 20:

$$a_1 = a_2 = \cdots = a_{38} = 21, \quad a_{39} = a_{40} = \cdots = a_{99} = 20.$$

In particular, we have

$$a_{38} = \left\lfloor R - \frac{38}{200} \right\rfloor = 21, \quad a_{39} = \left\lfloor R - \frac{39}{200} \right\rfloor = 20.$$

This implies that

$$21 + \frac{38}{200} \leq R < 21 + \frac{39}{200}.$$

Hence,

$$423.8 = 20\left(21 + \frac{38}{200}\right) \leq 20R < 20\left(21 + \frac{39}{200}\right) = 423.9,$$

whence $\lfloor 20R \rfloor = 423$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2018

(Senior Section, Round 2)

Saturday, 23 June 2018 **0900-1300**

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1. You are given some equilateral triangles and squares, all with side length 1, and asked to form convex n sided polygons using these pieces. If both types must be used, what are the possible values of n, assuming that there is sufficient supply of the pieces?

2. In a convex quadrilateral $ABCD$, $\angle A < 90°$, $\angle B < 90°$ and $AB > CD$. Points P and Q are on the segments BC and AD respectively. Suppose the triangles APD and BQC are similar. Prove that AB is parallel to CD.

3. Find the largest positive integer n such that there exist n real polynomials where the sum of any two has no real roots but the sum of any three does.

4. Let a, b, c, d be positive integers such that $a + c = 20$ and $\frac{a}{b} + \frac{c}{d} < 1$. Find the maximum possible value of $\frac{a}{b} + \frac{c}{d}$.

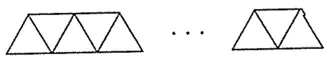
5. Starting with any n-tuple R_0, $n > 1$, of symbols from A, B, C, we define a sequence $R_0, R_1, R_2, \ldots$, according to the following rule: If $R_j = (x_1, x_2, \ldots, x_n)$, then $R_{j+1} = (y_1, y_2, \ldots, y_n)$, where $y_i = x_i$ if $x_i = x_{i+1}$ (taking $x_{n+1} = x_1$) and y_i is the symbol other than x_i, x_{i+1} if $x_i \neq x_{i+1}$. Find all positive integers $n > 1$ for which there exists some integer $m > 0$ such that $R_m = R_0$.

Singapore Mathematical Society

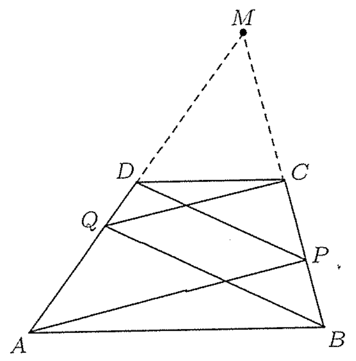
Singapore Mathematical Olympiad (SMO) 2018

(Senior Section, Round 2 solutions)

1. (*See Junior Section Round 2 Question 5 for the earlier part.*) To complete the proof, we now show that n cannot be 3 or 4. First note when a triangle and a square meet at the boundary, they meet at a vertex of the polygon and the angle at the vertex is $150°$. Thus when $n = 4$, the boundary of the polygon formed consists entirely of squares. Removing these squares either leaves nothing or another rectangle. This means that no triangle is used, a contradiction. When $n = 3$, the boundary of the triangle formed contains only triangles. The bottom level of the triangle formed consists of triangles arranged as shown in the figure below. Removing these triangles leaves a smaller triangle. We conclude that no square is used. Therefore n cannot be 3 or 4.



2.



Let the lines BC and AD intersect at M. The conditions $\angle A < 90°$, $\angle B < 90°$ and $AB > CD$ imply that C lies between MB and D lies between MA. Given the triangles APD and BQC are similar, we have $\angle BCQ = \angle ADP$ so that $\angle MCQ = \angle MDP$. From this, we see that the triangles MCQ and MDP are similar. Thus $MC/MD = MQ/MP$. Using the given condition that the triangles APD and BQC are similar, we have $\angle PAD = \angle QBC$ so that $\angle PAM = \angle QBM$. It follows that the triangles PAM and QBM are similar. Thus $MQ/MP = MB/MA$. Therefore, $MB/MA = MC/MD$. This implies that AB is parallel to CD.

3. We claim that the answer is $n = 3$. When $n = 3$, we can take the constant polynomials $f, g, h \equiv -1, -2, 3$ which clearly satisfy the problem conditions. Now assume that $n = 4$ and let our polynomials be f_1, f_2, f_3, f_4.

Note that for any i, j, we must have either $f_i(x) + f_j(x) > 0$ or $f_i(x) + f_j(x) < 0$ for all x as otherwise it must have a real root. If there exist indices i, j, k such that $f_i(0) + f_j(0)$, $f_i(0) + f_k(0)$, $f_j(0) + f_k(0)$ all have the same sign, say positive, then for all x

$$f_i(x) + f_j(x) > 0, \quad f_i(x) + f_k(x) > 0, \quad f_j(x) + f_k(x) > 0$$
$$\therefore \quad f_i(x) + f_j(x) + f_k(x) > 0.$$

which is a contradiction as the sum $f_i(x) + f_j(x) + f_k(x)$ would have no real roots. We shall show that such a triple of indices must exist.

WLOG let $|f_1(0)| \geq |f_i(0)|$ for $i = 2, 3, 4$ and that $f_1(0) > 0$. Then $f_1(0) + f_i(0) > 0$ for all i. If there exist i, j chosen from $2, 3, 4$ such that $f_i(0) + f_j(0) > 0$, then $1, i, j$ is such a triple. If not, then $2, 3, 4$ is such a triple. Thus n cannot be four.

4. We shall use the following result. Let p, q, k be positive integers. Then $\frac{p}{p+k} < \frac{q}{q+k}$ iff $p < q$.

First fix a, c. Without loss of generality, we assume that $a \leq c$. Let $b = a + x$ and $d = c + y$ where x, y are positive integers. Then it is easy to see that

$$\frac{a}{a+x} + \frac{c}{c+y} \geq \frac{a}{a+y} + \frac{c}{c+x}$$

if $y \leq x$. Thus we may assume that $y \leq x$. Also

$$1 > \frac{a}{b} + \frac{c}{d} = \frac{a}{a+x} + \frac{c}{c+y} = \frac{ac + ay + ac + cx}{ac + ay + xy + cx}.$$

Thus $ac < xy$ and $xy - ac$ is the difference of the denominator and numerator of the fraction on the RHS. Thus if x', y' are positive integers such that $y' \leq x'$ and $xy = x'y'$, then $|y - y'| \leq |x - x'|$. Therefore

$$\begin{aligned} & \frac{a}{a+x} + \frac{c}{c+y} \leq \frac{a}{a+x'} + \frac{c}{c+y'} \\ \iff \quad & (a+x)(c+y) \leq (a+x')(c+y') \\ \iff \quad & a(y-y') \leq c(x'-x) \\ \iff \quad & y \geq y \quad \text{and} \quad x' \geq x \end{aligned}$$

Therefore the maximum value of the sum of the fractions is attained when y is minimum, which is 1. When $y = 1$, since $ac < xy$, the optimum value for x is $ac + 1$ (we want to choose x so that it is as small as possible). Thus for fixed a, c, the maximum value is $\frac{a}{a+ac+1} + \frac{c}{c+1} = 1 - \frac{1}{(a+ac+1)(c+1)}$. By checking all pairs $(a, c) = (1, 19), (2, 18), \ldots, (10, 10)$, we find that the maximum value of $(a + ac + 1)(c + 1)$ is 1386, attained when $(a, c) = (7, 13)$. Therefore the maximum value is $1 - \frac{1}{1386} = \frac{1385}{1386}$.

5. Replace A, B, C by 0, 1, 2. Then $y_i \equiv -x_i - x_{i+1} \pmod 3$. We first show that such an m does not exist for even n as seen from the following n-tuple

$$(1, 0, 1, 0, \ldots, 1, 0).$$

Next we show that m exists when n is odd. Since the total number of tuples that can be formed is 3^n, there exist indices $s < t$ such that $R_s = R_t$. Without loss of generality, we may assume that R_s is the only term in the sequence from R_0 to R_t that is the same as R_t. If we can show that for any $i \geq 0$, R_i can be obtained from R_{i+1}, then the sequence going backwards from R_s is the same as the sequnce going backwards from R_t. Since going backwards from R_s, we can get R_0, then going backwards from R_t, before reaching R_s,we can get an n-tuple which is the same as R_0. In other words, there is an index m, $s < m < t$, such that $R_0 = R_m$ and we are done.

To show that the sequence can be reversed, all we need to show is if two n-tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ give rise to the same n-tuple, then they are in fact the same. We have from the given rule:

$$\begin{aligned}
-a_1 - a_2 &= -b_1 - b_2 \\
a_2 + a_3 &= b_2 + b_3 \\
&\cdots \\
-a_{n-2} - a_{n-1} &= -b_{n-2} - b_{n-1} \\
a_{n-1} + a_n &= b_{n-1} + b_n \\
-a_n - a_1 &= -b_n - b_1
\end{aligned}$$

Summing up, we get $2a_1 = 2b_1$ and therefore $a_1 = b_1$. Repeat this starting with the index pair $(2, 3)$, we can show that $a_2 = b_2$. Continuing this way, we have $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2018

(Open Section, Round 1)

Thursday, 31 May 2018 — 0930-1200 hrs

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

In this paper, let $\lfloor x \rfloor$ denote the greatest integer not exceeding x. For examples, $\lfloor 5 \rfloor = 5$, $\lfloor 2.8 \rfloor = 2$, and $\lfloor -2.3 \rfloor = -3$.

1. Find the area of the region R on the xy-plane consisting of the points (x, y) satisfying the equation $\lfloor x \rfloor + \lfloor y \rfloor = 10$, where $0 \le x \le 3$.

   (Note: If you think that the region R does not have any area, enter your answer as "0"; if you think that the region R is unbounded, enter your answer as "99999".)

2. Find the sum

$$1+1+2+1+2+3+1+2+3+4+\cdots+1+2+3+\cdots+20.$$

3. Let a and b be the largest and smallest values of x that satisfy the equation

$$|x-1|+|6-2x|=|5-x|.$$

   Find $a-b$.

4. Let $S_n = \displaystyle\sum_{k=1}^{n} \frac{k}{(k+1)!}$. Find the value of $2019! \times (1 - S_{2018})$.

5. A rectangular table has two chairs on each of the longer sides and one chair on each of the shorter sides. In how many ways can six people be seated?

   (Note: Any two arrangements are the same up to 'rotation' of the rectangular table.)

6. Find the smallest positive integer such that the sum of the fifth power of its digits is not divisible by the sum of its digits.

7. Three of the four integers between 100 and 1000 which are equal to the sum of the cubes of their digits are 153, 371, and 407 (For example, $1^3 + 5^3 + 3^3 = 153$.). Determine the fourth integer.

8. Find the minimum value of the function $f(x) = \dfrac{x^2 + x + 2018}{x - 2017}$ for $x > 2017$.

9. Let $p(x) = x^3 + ax^2 + bx + c$ be a polynomial where a, b, c are distinct non-zero integers. Suppose $p(a) = a^3$ and $p(b) = b^3$. Find $p(13)$.

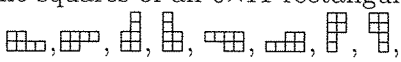
10. Find the smallest integer r such that

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\cdots+\frac{1}{\sqrt{2018}} \le r\sqrt{2018}.$$

11. Find the shortest distance (rounded off to the nearest whole number if necessary) from the point $(22, 21)$ to the graph with equation $x^3 + 1 = y(3x - y^2)$.

12. Given that $S = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{n+k}$, find the value of e^S, where e is the base of the natural logarithm.

13. Let $\triangle ABC$ be a triangle with $a = BC, b = AC$ and $c = AB$. Assume that $a + c = 2b$, $\angle A - \angle C = \frac{\pi}{3}$ and $\sin B = \frac{\sqrt{m}}{n}$ for some positive integers m, n. What is the least possible value of $m + n$?

14. Ah Meng is going to shade some squares of an 6×11 rectangular board in such a way that in every L-shape of the forms , at least half of the squares in it are shaded. Determine the smallest number of squares in the board that Ah Meng must shade in order to fulfill this condition.

15. In the triangle ABC, $AB = 7, BC = 10$ and $CA = \sqrt{73}$, M is the midpoint of AC, and P is the point on BC such that AP intersects BM at Q and $BP = BQ$. Find the length of AP.

16. A triangle $A'B'C'$ is formed with sides whose lengths are the lengths of the medians of a triangle ABC. Suppose the product of the lengths of the three sides of ABC is 640. Find the product of the lengths of the medians of the triangle $A'B'C'$.

17. Let $f_0(x) = \frac{x}{3x+2}$ and for any integer $n \geq 1$, $f_n(x) = f_0(f_{n-1}(x))$. Assume that $f_{2018}(x) = \frac{x}{Ax+B}$. What is the value of $3B - A$?

18. How many 3-element subsets $\{a, b, c\}$ of $\{1, 2, \cdots, 100\}$ have the property that $a + b + c$ is a multiple of 6? For example, $\{1, 2, 3\}$ and $\{2, 6, 10\}$ are examples of such sets.

(Note: $\{a, b, c\}$, $\{a, c, b\}$ and $\{b, a, c\}$ are considered as the same set.)

19. Assume that $a_1 < 2$, and for any integer $n \geq 2$, $a_n = 1 + a_{n-1}(a_{n-1} - 1)$. If

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_m} = 1$$

for some integer m, and that $16a_1 - a_{m+1} \leq N$ for some integer N, what is the least possible value? (Note that N may not be attainable).

20. Let a, b, c be real numbers such that $a + bc = b + ca = c + ab = 870$. Determine the maximum value of $ab + bc + ca$.

21. Determine the largest value of the expression

$$2^{k_1} + 2^{k_2} + \cdots + 2^{k_{498}},$$

where for each $i = 1, \ldots, 498$, k_i is an integer, $1 \le k_i \le 507$, and $k_1 + \cdots + k_{498} = 507$.

22. Some persons in a party shake hands with each other. The following information is known.

- Each person shakes hands with exactly 20 persons.
- For each pair of persons who shake hands with each other, there is exactly 1 other person who shake hands with both of them.
- For each pair of persons who do not shake hands with each other, there are exactly 6 other persons who shake hands with both of them.

Determine the number of persons in the party.

23. In a rectangle $ABCD$, E is a point on AD and F is a point on CD such that the line through the midpoint of EF and the centre of the rectangle is perpendicular to AC. Given $AB = 100, BC = 60$ and $AE = 40$, find the area of the triangle BEF.

24. Let a, b, c be positive numbers such that $a + b + c = 2$. If the minimum value of

$$(a + \frac{1}{a})^2 + (b + \frac{1}{b})^2 + (c + \frac{1}{c})^2$$

is $\frac{m}{n}$, where m and n have no common factors larger than 1, find the value of $m + n$.

25. In a triangle ABC, $AB = 21, BC = 27$ and $CA = 24$, a circle ω is tangent to the sides AB and AC and is also tangent internally to the circumcircle of ABC. Let the inradius of the triangle ABC be r and the radius of ω be k. Find the value of rk.

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2018
(Open Section, Round 1)
(Solution)

1. **Answer.** 3.

   Solution. Note that the graph in the region $0 \leq x \leq 3$ represents the area represented by the three squares

$$R_k = \{(x, y) : k \leq x < k+1, 10-k \leq y < 11-k\},$$

   where $k = 0, 1, 2$. Thus the total area is 3. $\square$

2. **Answer.** 1540

   Solution. The series can be expressed and evaluated as

$$\sum_{k=1}^{20} \frac{k(k+1)}{2} = \frac{1}{2}\sum_{k=1}^{20} k(k+1) = \frac{1}{2}\left(\frac{20(21)(22)}{3}\right) = 1540.$$

   $\square$

3. **Answer.** 2

   Solution. Solving the given equation is equivalent to solving the inequality

$$(x-1)(6-2x) \geq 0,$$

   which means $1 \leq x \leq 3$. Hence the difference between the largest and the smallest value is 2. $\square$

4. **Answer.** 1

   Solution. By the method of difference,

$$\sum_{k=1}^{n} \frac{k}{(k+1)!} = \sum_{k=1}^{n}\left(\frac{1}{k!} - \frac{1}{(k+1)!}\right) = 1 - \frac{1}{(n+1)!},$$

   so that $2019! \times (1 - S_{2018}) = 1$. $\square$

5. **Answer.** 360

   Solution. Fix one person, say P. Note that there are three positions to consider: (1) P is seated on the shorter side; (2) P is seated on the left seat of the longer side; and (3) P is seated on the right seat of the longer side. In each case there are $5! = 120$ ways, so that in total there are 360 ways. $\square$

6. **Answer.** 112

Note that since $a^5 + b^5 = (a+b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$, such an integer must have at least three non-zero digits. It is clear that 111 does not satisfy the criterion but 112 does. Hence it must be the smallest. $\square$

7. **Answer.** 370

Since 1 is the last digit of 371 and $1^3 = 1$, the fourth integer must be 370. $\square$

8. **Answer.** 8071

Solution. Note that $f(x) = \dfrac{x^2 + x + 2018}{x - 2017} = x + 2018 + \dfrac{2018^2}{x - 2017}$, so that by equating $f'(x) = 0$ we obtain $x = 4035$. By second derivative test it is easy to see that it will give a minimum value. Hence the minimum value is $f(4035) = 8071$.

Alternatively,

$$f(x) = \frac{x^2 + x + 2018}{x - 2017} = x + 2018 + \frac{2018^2}{x - 2017} = 2018 + 2017 + x - 2017 + \frac{2018^2}{x - 2017}.$$

When $x > 2017$, we have

$$x - 2017 + \frac{2018^2}{x - 2017} \geq 2\sqrt{(x - 2017) \times \frac{2018^2}{x - 2017}} = 2 \times 2018,$$

where the lower bound is attained when $x - 2017 = \frac{2018^2}{x-2017}$, i.e., $x = 2018 + 2017 = 4035$. Thus the minimum value of $f(x)$ for $x > 2017$ is

$$2018 + 2017 + 2 \times 2018 = 8071.$$

Hence the answer is 8071. $\square$

9. **Answer.** 1927

Solution. Let $q(x) = p(x) - x^3$. Then $q(a) = q(b) = 0$. Thus $q(x) = a(x-a)(x-b)$. Thus $p(x) = x^3 + a(x-a)(x-b) = x^3 + ax^2 - a(a+b)x + a^2b$. This implies $b = -a(a+b)$ and $c = a^2b$. From $b = -a(a+b)$, we obtain $b = 1 - a - \frac{1}{a+1}$. Since b is an integer $a + 1 = \pm 1$. Since $a \neq 0$ we must have $a = -2$. From this we get $b = 4$ and $c = 16$. Thus $p(x) = x^3 - 2x^2 + 4x + 16$ so that $p(13) = 1927$. $\square$

10. **Answer.** 2

Solution. We first show that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots + \frac{1}{\sqrt{2018}} > \sqrt{2018}.$$

Clearly,

$$1 > \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{3}} > \frac{1}{\sqrt{4}} > \cdots > \frac{1}{\sqrt{2018}} = \frac{\sqrt{2018}}{2018}.$$

Thus

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\cdots+\frac{1}{\sqrt{2018}}>2018\times\frac{\sqrt{2018}}{2018}=\sqrt{2018}.$$

Next we shall prove by induction that for any integer $n\geq 1$,

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}\leq 2\sqrt{n}.$$

The above inequality is obvious when $n=1$. Assume that it holds when $n=k$, where $k\geq 1$. Now consider the case that $n=k+1$. By inductive assumption,

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{k+1}}\leq 2\sqrt{k}+\frac{1}{\sqrt{k+1}}.$$

$$=\frac{2\sqrt{k}\sqrt{k+1}+1}{\sqrt{k+1}}<\frac{k+k+1+1}{\sqrt{k+1}}=2\sqrt{k+1}.$$

Hence for any integer $n\geq 1$, the following inequality holds:

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}\leq 2\sqrt{n}.$$

Therefore the answer for this question is 2. □

11. **Answer.** 29

Solution. From the factorization

$$x^3+y^3-3xy+1=\frac{1}{2}(x+y+1)\left((x-1)^2+(y-1)^2+(x-y)^2\right),$$

we see that the points satisfying the equation $x^3+y^3-3xy+1=0$ consist of the points on the straight line $x+y+1=0$ and the single point $(1,1)$. The distance from the point $(22,21)$ to the line $x+y+1=0$ is equal to $\dfrac{22+21+1}{\sqrt{1^2+1^2}}=\dfrac{44}{\sqrt{2}}$. The distance from the point $(22,21)$ to the point $(1,1)$ is $\sqrt{(22-1)^2+(21-1)^2}=29$ which is less than $\frac{44}{\sqrt{2}}$. Therefore, the shortest distance from the point $(22,21)$ to the given curve is 29. □

12. **Answer.** 2

Solution. Note that $S=\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\dfrac{1}{n+k}$ can be interpreted as $\displaystyle\int_0^1\frac{1}{x+1}\,dx$, which can be evaluated to be $\ln 2$. Hence $e^S=2$. □

13. **Answer.** 47

Solution. Since $a+c=2b$, by sine rule,

$$\sin A+\sin C=2\sin B.$$

By the Factor Formula and Double Angle Formula, we have

$$\sin\frac{A+C}{2}\cos\frac{A-C}{2}=2\sin\frac{B}{2}\cos\frac{B}{2}.$$

Since $A+C=\pi-B$, we have $\dfrac{A+C}{2}=\dfrac{\pi}{2}-\dfrac{B}{2}$, implying that

$$\begin{aligned}\sin(\frac{\pi}{2}-\frac{B}{2})\cos\frac{A-C}{2} &= 2\sin\frac{B}{2}\cos\frac{B}{2},\\ \cos\frac{B}{2}\cos\frac{A-C}{2} &= 2\sin\frac{B}{2}\cos\frac{B}{2},\\ \cos\frac{A-C}{2} &= 2\sin\frac{B}{2},\end{aligned}$$

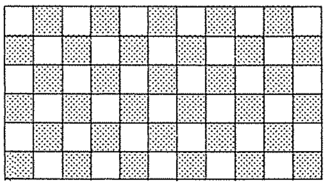
since $\cos\dfrac{B}{2}\neq 0$. As we are given $A-C=\dfrac{\pi}{3}$, we have

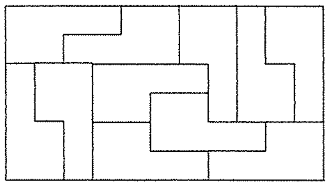
$$\cos\frac{A-C}{2}=\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}.$$

Thus $\sin\dfrac{B}{2}=\dfrac{\sqrt{3}}{4}$, and we have $\cos\dfrac{B}{2}=\sqrt{1-\dfrac{3}{16}}=\dfrac{\sqrt{13}}{4}$, and $\sin B=2\sin\dfrac{B}{2}\cos\dfrac{B}{2}=\dfrac{\sqrt{39}}{8}$. Hence the answer is $39+8=47$. □

14. **Answer.** 33

Solution. If the board is shaded in a checkerboard manner, then every L-shape contains 3 shaded squares. Thus the minimum number of shaded squares is less than or equal to 33. The bound 33 is tight as it is possible to tile the 6×11 rectangular board by eleven L-shapes.

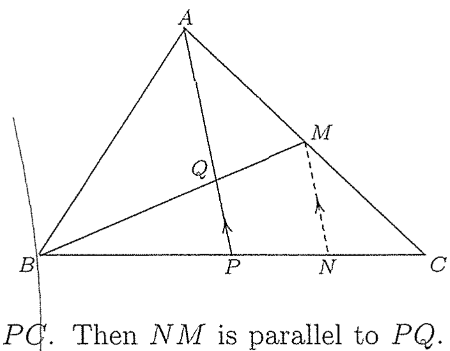




□

15. **Answer.** 6

Solution.



Let N be the midpoint of PC. Then NM is parallel to PQ. Since the triangle BPQ is isosceles with $BP=BQ$, the triangle BNM is isosceles with $BN=BM$. Since M is the midpoint of AC, we can find its length using Apollonius' theorem. We have $BM^2=\frac{1}{2}(10^2+7^2)-\frac{1}{4}(\sqrt{73})^2=\frac{225}{4}$. Thus $BN=BM=\frac{15}{2}$. Therefore, $NC=BC-BN=10-\frac{15}{2}=\frac{5}{2}$. Since N is the midpoint of PC, we have $PC=5$. This also implies that P is the midpoint of BC. Using Apollonius' theorem, $AP^2=\frac{1}{2}(7^2+73)-\frac{1}{4}(10^2)=\frac{144}{4}=36$. That is $AP=6$. □

16. **Answer.** 270

Solution. Let the side lengths of the triangle ABC be $BC = a, CA = b, AB = c$. By Apollonius' theorem, the length of the median from the vertex A is $\frac{1}{2}\sqrt{2b^2+2c^2-a^2}$. We may take this as $B'C' = p$. Similarly, $C'A' = q = \frac{1}{2}\sqrt{2c^2+2a^2-b^2}$ and $A'B' = r = \frac{1}{2}\sqrt{2a^2+2b^2-c^2}$. The length of the median of the triangle $A'B'C'$ from the vertex A' is

$$\begin{aligned}&\tfrac{1}{2}\sqrt{2q^2+2r^2-p^2}\\ &= \tfrac{1}{2}\sqrt{\tfrac{1}{2}(2c^2+2a^2-b^2)+\tfrac{1}{2}(2a^2+2b^2-c^2)-\tfrac{1}{4}(2b^2+2c^2-a^2)}\\ &= \tfrac{3a}{4}.\end{aligned}$$

Similarly, the lengths of the other two medians of the triangle $A'B'C'$ are $\frac{3b}{4}$ and $\frac{3c}{4}$. Therefore, the product of the lengths of the medians of the triangle $A'B'C'$ is $\frac{27abc}{64} = \frac{27\times 640}{64} = 270$. □

17. **Answer.** 3

Observe that for $n \geq 1$,

$$\frac{1}{f_n(x)} = \frac{3f_{n-1}(x)+2}{f_{n-1}(x)} = \frac{2}{f_{n-1}(x)} + 3 = 2\left(\frac{1}{f_{n-1}(x)}+3\right) - 3,$$

i.e.,

$$\frac{1}{f_n(x)} + 3 = 2\left(\frac{1}{f_{n-1}(x)}+3\right).$$

Thus

$$\frac{1}{f_n(x)} + 3 = 2^n\left(\frac{1}{f_0(x)}+3\right) = 2^n\left(\frac{3x+2}{x}+3\right),$$

implying that

$$\frac{1}{f_n(x)} = 2^n\left(\frac{3x+2}{x}+3\right) - 3 = \frac{3\times(2^{n+1}-1)x+2^{n+1}}{x}.$$

So

$$f_n(x) = \frac{x}{3\times(2^{n+1}-1)x+2^{n+1}}.$$

For $n = 2018$,

$$f_{2018}(x) = \frac{x}{3\times(2^{2019}-1)x+2^{2019}} = \frac{x}{Ax+B},$$

where $A = 3\times(2^{2019}-1)$ and $B = 2^{2019}$.

Hence the answer is

$$3B - A = 3.$$

□

18. **Answer.** 26961

We first divide the set $\{1, 2, \cdots, 100\}$ into six subsets: $S_0, S_1, \cdots, S_5$, where

$$S_i = \{1 \le k \le 100 : k - i \text{ is a multiple of } 6\}.$$

Thus $S_0 = \{6, 12, \cdots, 96\}$, $S_1 = \{1, 7, \cdots, 97\}$, $S_2 = \{2, 8, \cdots, 98\}, \cdots, S_5 = \{5, 11, \cdots, 95\}$, and

$$|S_5| = |S_0| = 16, \quad |S_1| = |S_2| = |S_3| = |S_4| = 17.$$

Note that for any three integers a, b, c, the number $a+b+c$ is a multiple of 3 if and only if either a, b, c have the same remainder when divided by 3 or a, b, c have pairwisely different remainders when divided by 3.

Thus, for any $a \in S_i, b \in S_j, c \in S_k$, where $0 \le i \le j \le k \le 5$, $a+b+c$ is a multiple of 6 if and only if (i, j, k) is one of the following triples:

$$(0,0,0), (0,3,3), (1,1,4), (4,4,4), (2,2,2), (2,5,5),$$
$$(0,1,5), (0,2,4), (1,2,3), (3,4,5).$$

Hence the number of 3-element subsets $\{a, b, c\}$ of $\{1, 2, \cdots, 100\}$ have the property P that $a+b+c$ is a multiple of 6 is equal to

$$\begin{aligned}
&\binom{|S_0|}{3} + |S_0|\binom{|S_3|}{2} + \binom{|S_1|}{2}|S_4| + \binom{|S_4|}{3} + \binom{|S_2|}{3} + |S_2|\binom{|S_5|}{2} \\
&+|S_0||S_1||S_5| + |S_0||S_2||S_4| + |S_1||S_2||S_3| + |S_3||S_4||S_5| \\
&= \binom{16}{3} + 16\binom{17}{2} + 17\binom{17}{2} + \binom{17}{3} + \binom{17}{3} + 17\binom{16}{2} \\
&+16 \times 17 \times 16 + 16 \times 17^2 + 17^3 + 17^2 \times 16 = 26961.
\end{aligned}$$

□

19. **Answer.** 24

Solution. For $n \ge 2$, as $a_n = 1 + a_{n-1}(a_{n-1} - 1)$, we have

$$\frac{1}{a_n - 1} = \frac{1}{(a_{n-1} - 1)(a_{n-1})} = \frac{1}{(a_{n-1} - 1)} - \frac{1}{a_{n-1}},$$

i.e.,

$$\frac{1}{a_{n-1}} = \frac{1}{(a_{n-1} - 1)} - \frac{1}{a_n - 1}.$$

Thus

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_m} = \frac{1}{a_1 - 1} - \frac{1}{a_{m+1} - 1}.$$

By the given condition,

$$\frac{1}{a_1 - 1} - \frac{1}{a_{m+1} - 1} = 1.$$

Solving this equation gives that $a_{m+1} = \frac{-1}{a_1 - 2}$. Thus

$$\begin{aligned}
16a_1 - a_{m+1} &= 16a_1 + \frac{1}{a_1 - 2} = 16(a_1 - 2) + 32 + \frac{1}{a_1 - 2} \\
&= 32 - \left[16(2 - a_1) + \frac{1}{2 - a_1}\right] \\
&\le 32 - 2\sqrt{16} = 24,
\end{aligned}$$

where the equality is attained when $16(2 - a_1) = \frac{1}{2 - a_1}$, i.e., $a_1 = \frac{7}{4}$. Note however that in this problem the maximum value is not attainable. □

20. **Answer.** 2700

Solution. We have $a-b=c(a-b)$ so that either $c=1$ or $a=b$. If $c=1$, then $a+b=870$ while $ab=869$. Hence $(a,b)=(1,869)$ or $(a,b)=(869,1)$, and we have $ab+bc+ca=1739$. If $a=b$, we have $a(1+c)=c+a^2=870$ so that $a(1+870-a^2)=870$. This factors into $(a-1)(a^2+a-870)=0$. If $a=1$, then $b=1$ and $c=869$ and we have $ab+bc+ca=1739$. again. Otherwise, $a=870-a^2=870-ab=c$ so that $a=b=c$. As $a^2+a-870=(a+30)(a-29)=0$, we have $a=b=c=29,-30$. Thus $ab+bc+ca=2700,2523$. Therefore, the maximum value of $ab+bc+ca=2700$. □

21. **Answer.** 2018

Solution. For any integers $p,q\geq 1$,

$$2^{p+q-1}+2-2^p-2^q=2(2^{p-1}-1)(2^{q-1}-1)\geq 0.$$

Thus $2^p+2^q\leq 2+2^{p+q-1}$. Therefore,
$2^{k_1}+2^{k_2}+\cdots+2^{k_{498}}\leq 2+2^{k_1+k_2-1}+2^{k_3}+\cdots+2^{k_{498}}\leq\cdots\leq 497\times 2+2^{k_1+\cdots+k_{498}-497}=497\times 2+2^{507-497}=994+2^{10}=2018.$

The value 2018 is attained when $k_1=k_2=\cdots=k_{497}=1$ and $k_{498}=10$. □

22. **Answer.** 81

Solution. Let n be the number of persons in the party. Fix a person P. Consider the set

$$S=\{(A,B)\mid A\neq P, A \text{ shakes hands with } P \text{ and } B, B \text{ does not shake hands with } P\}.$$

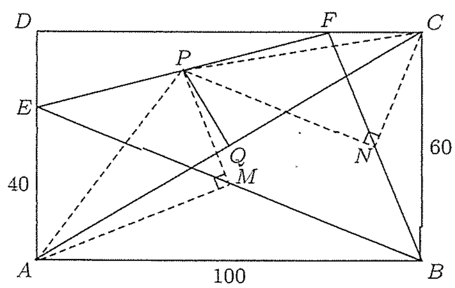
We shall count the number of elements in S in 2 different ways.

There are 20 persons who shake hands with P. For each person A who shakes hands with P, there are 20 persons who shake hands with A and one of them is P. Therefore, for each A, A shakes hands with 19 persons other than P. By the second information, among these 19 persons, there is exactly one person who shades hands with P and A; and for the rest of 18 persons, they all do not shake hands with P. Thus $|S|=20\times 18=360$.

On the other hand, there are $n-21$ persons ($\neq P$) who do not shake hands with P. For each such person B, by the last information, there are exactly 6 other persons who shake hands with both of them. Thus $|S|=6(n-21)$. Therefore, $6(n-21)=360$ so that $n=81$. □

23. **Answer.** 2520

Solution.



We first prove that the triangles ABE and CBF are similar. Let M and N be the midpoints of BE and BF respectively. First note that in the right-angled triangles ABE and CBF, we have $AM = EM = BM$ and $CN = BN = FN$. Since PQ is the perpendicular bisector AC, we have $PA = PC$. As $AM = \frac{1}{2}EB = PN$ and $PM = \frac{1}{2}FB = CN$, we conclude that the triangles PMA and CNP are congruent. Thus $\angle PMA = \angle CNP$. Also $\angle PME = \angle FBE = \angle FNP$. Therefore, $\angle EMA = \angle PMA - \angle PME = \angle CNP - \angle FNP = \angle CNF$. It follows that the triangles ABE and CBF are similar. Thus $CF = \frac{40\times 60}{100} = 24$. Then $DE = 20$ and $DF = 76$. Therefore the area of the triangle BEF is $100 \times 60 - \frac{1}{2} \times 100 \times 40 - \frac{1}{2} \times 60 \times 24 - \frac{1}{2} \times 76 \times 20 = 2520$. □

24. **Answer.** 181

Note that when $a = b = c = \dfrac{2}{3}$, we have

$$(a + \frac{1}{a})^2 + (b + \frac{1}{b})^2 + (c + \frac{1}{c})^2 = \frac{169}{12}.$$

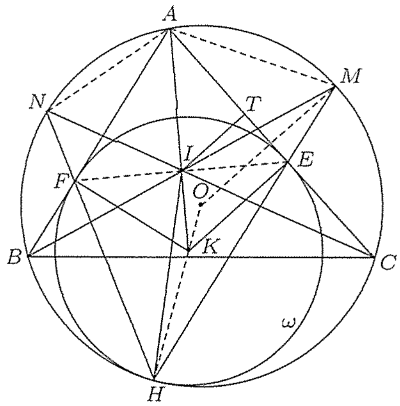
For any positive numbers a, b, c with $a + b + c = 2$, we have

$$\begin{aligned}
& (a + \frac{1}{a})^2 + (b + \frac{1}{b})^2 + (c + \frac{1}{c})^2 \\
= \; & \frac{1}{3}(1^2 + 1^2 + 1^2)\left[(a + \frac{1}{a})^2 + (b + \frac{1}{b})^2 + (c + \frac{1}{c})^2\right] \\
\geq \; & \frac{1}{3}\left[1 \times (a + \frac{1}{a}) + 1 \times (b + \frac{1}{b}) + 1 \times (c + \frac{1}{c})\right]^2 \\
= \; & \frac{1}{3}\left[5 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right]^2 \\
= \; & \frac{1}{3}\left[5 + \frac{1}{2}(a + b + c)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})\right]^2 \\
\geq \; & \frac{1}{3}\left[2 + \frac{1}{2}(\sqrt{a} \times \frac{1}{\sqrt{a}} + \sqrt{b} \times \frac{1}{\sqrt{b}} + \sqrt{c} \times \frac{1}{\sqrt{c}})^2\right]^2 \\
= \; & \frac{1}{3} \times \frac{169}{4} = \frac{169}{12}.
\end{aligned}$$

Thus the minimum value of $(a + \frac{1}{a})^2 + (b + \frac{1}{b})^2 + (c + \frac{1}{c})^2$ is $\frac{169}{12}$, and so $m = 169, n = 12$. Hence the answer is $m + n = 181$. □

25. **Answer.** 70.

Solution. Let ω touch the side AB at F, CA at E and the circumcircle of ABC at H. Let I and K be the incenter of ABC and the center of ω respectively. Clearly A, I, K are collinear. Join HI. It is a well-known fact that E, I, F are collinear. We shall prove this fact in the last part of the solution.



Since $AE = AF$, the segment EF is perpendicular to AK. Also KE is perpendicular to AE. Thus $\angle KEI = \angle EAI = \frac{1}{2}\angle A$. Let T be the foot of the perpendicular from I onto AC. Then $IT = r$ and IT is parallel to KE. Thus $k = KE = \frac{IE}{\cos\frac{A}{2}} = \frac{IT}{\cos^2\frac{A}{2}} = \frac{r}{\cos^2\frac{A}{2}} = \frac{2r}{1+\cos A}$.

By Heron's formula, $(ABC) = 108\sqrt{5}$, the semi-perimeter is 36. Thus $r = 3\sqrt{5}$. By cosine rule $\cos A = \frac{2}{7}$. Thus $k = \frac{6\sqrt{5}}{1+\frac{2}{7}} = \frac{14\sqrt{5}}{3}$. Consequently, $rk = 70$.

Lastly, we complete the proof by showing that E, I, F are collinear. Denote the circumcircle of the triangle ABC by Ω with centre O. Let the line HE intersect Ω at M. The homothety centred at H sending ω to Ω maps K to O and E to M. Thus KE is parallel to OM. Since KE is perpendicular to AC, OM is also perpendicular to AC. This implies that M is the midpoint of the arc AC. Therefore, B, I, M are collinear. Similarly, C, I, N are collinear. Now $\angle MCE = \angle MAC = \angle MHC$ so that the triangles MCE and MHC are similar. Thus $MC^2 = ME \cdot MH \Rightarrow MI^2 = ME \cdot MH \Rightarrow \angle MIE = \angle MHI$. Similarly, $\angle NIF = \angle NHI$. Next it is well-known that $NA = NI$ and $MA = MI$. Consequently, $\angle NIF + \angle AIN + \angle MIA + \angle MIE = \angle NHI + \angle NAI + \angle MAI + \angle MHI = \angle MAN + \angle MHN = 180°$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2018

(Open Section, Round 2)

Saturday, 30 June 2018 **0900-1300**

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1. Consider a regular cube with side length 2. Let A and B be two vertices that are furthest apart. Construct a sequence of points on the surface of the cube $A_1, A_2, \ldots, A_k$ so that $A_1 = A$, $A_k = B$ and for any $i = 1, \ldots, k-1$, the distance from A_i to A_{i+1} is 3. Find the minimum value of k.

2. Let O be a point inside a triangle ABC such that $\angle BOC = 90°$ and $\angle BAO = \angle BCO$. Prove that $\angle OMN = 90°$, where M, N are the midpoints of AC and BC respectively.

3. Let n be a positive integer. Show that there exists an integer m such that

$$2018m^2 + 20182017m + 2017$$

is divisible by 2^n.

4. Each of the squares in a 2×2018 grid of squares is to be coloured black or white such that in any 2×2 block, at least one of the 4 squares is white. Let P be the number of ways colouring the grid. Find the largest k so that 3^k divides P.

5. Consider a polynomial $P(x, y, z)$ in three variables with integer coefficients such that for any real numbers a, b, c,

$$P(a, b, c) = 0 \;\Leftrightarrow\; a = b = c.$$

Find the largest integer r such that for all such polynomials $P(x, y, z)$ and integers n, m,

$$m^r \mid P(n, n+m, n+2m).$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2018

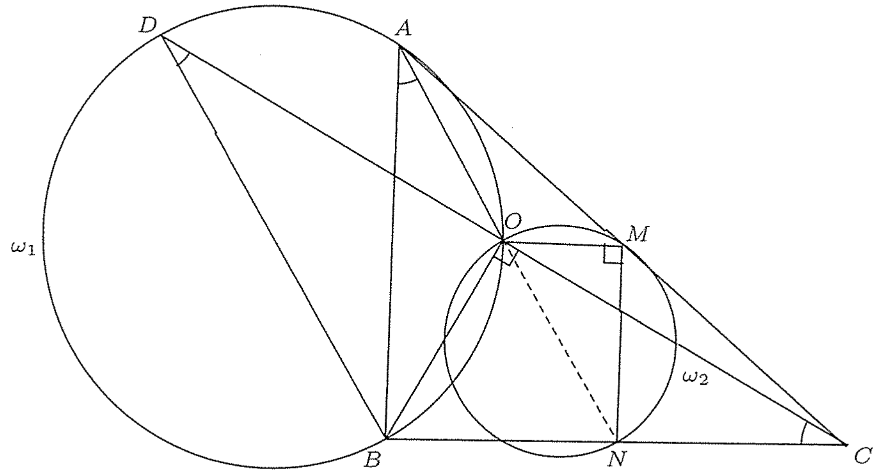
(Open Section, Round 2 solutions)

1. The sphere with centre A and radius 3 intersects the three edges at B in three points K, L, N. By straightforward calculation using Pythagoras Theorem, tt's easy to show that they are the midpoints of the edges. Let M be an interior point of any of the arcs KL, LN, KN arising from the intersection withe the sphere. The sphere with centre M and radius 3 contains the point A. All the other points of the cube are in the interior of this sphere since the distance from M to all the other vertices are < 3. Thus A_2 must be one of K, L, N.

From each of K, L, N we can reach one of the vertices adjacent to A. Thus A_3 is a vertex connected to A by a single edge. Now A_4 must a midpoint of a side and A_5 is a vertex connected to A by at most 2 edges. Since $A$is connected to B by a sequence of 3edges, we see that $k \geq 7$.

It is easy to construct a sequence of length 7 that works.

2.



Extend CO to a point D such that $CO = OD$. Since $\angle BOC = 90°$, the triangles $\angle BOC$ and BOD are congruent. Let ω_1 the circumcircle of the triangle BOD. Since $\angle BDO = \angle BCO = \angle BAO$, the point A lies on ω_1. As M, O, N are the midpoints of CA, CD, CB respectively, the homothety centered at C with similitude ratio $\frac{1}{2}$ maps

ω_1 to the cicumcircle ω_2 of the triangle MON. As DB is a diameter of ω_1, ON is a diameter of ω_2. Thus $\angle OMN = 90°$.

3. We shall show more generally that $am^2 + bm + c \equiv 0 \pmod{2^n}$ has a solution for all n whenever b is odd and a or c is even. For $n = 1$, take $m = 0$ if c is even and $m = 1$ if c is odd. Now suppose the claim is true for n. If c is even, then by assumption, the congruence $2at^2 + bt + c/2 \equiv 0 \pmod{2^n}$ has a solution in t. Letting $m = 2t$, we get $am^2 + bm + c = 2(2at^2 + bt + c/2) \equiv 0 \pmod{2^{n+1}}$. If c is odd, then a is even and thus $a + b + c$ is even; hence by assumption, the congruence $2at^2 + (2a + b)t + (a + b + c)/2 \equiv 0 \pmod{2^n}$ has a solution in t. Letting $m = 2t + 1$, we get $am^2 + bm + c = 2\left(2at^2 + (2a + b)t + (a + b + c)/2\right) \equiv 0 \pmod{2^{n+1}}$. This completes the proof of the claim by induction.

4. Let P_n be the number of colourings of a $2 \times n$ grid, A_n be the number of colourings in which the first two squares are coloured black and B_n be the number of colourings in which at least one of the first two squares is coloured white. Then $P_1 = 4$, $P_2 = 15$ and for $n \geq 3$,

$$\begin{aligned} P_n &= A_n + B_n \\ A_n &= B_{n-1} \\ B_n &= 3(B_{n-1} + A_{n-1}) \\ \therefore\quad A_n &= 3(B_{n-2} + A_{n-2}) \\ \therefore\quad B_n &= 3(B_{n-1} + B_{n-2}) \\ \therefore\quad P_n &= 3(P_{n-1} + P_{n-2}) \end{aligned}$$

Let $v(n)$ denote the highest power of 3 that divides P_n. We have that $v(1) = 0$, $v(2) = 1$, $v(3) = 1$, $v(4) = 3$, $v(5) = 2$, $v(6) = 3$, etc. It is easy to see by induction that $v(2k - 1) \geq k - 1$, $v(2k) \geq k$. That is $v(n) \geq \lfloor \frac{n}{2} \rfloor$.

Let $Q_n = \frac{P_n}{3^{\lfloor \frac{n}{2} \rfloor}}$. Then Q_n satisfies the recurrence relations:

$$\begin{aligned} Q_{2k} &= Q_{2k-1} + Q_{2k-2}, \\ Q_{2k-1} &= 3Q_{2k-2} + Q_{2k-3}. \end{aligned}$$

From this we have $Q_1 = 4, Q_2 = 5, Q_3 = 19, Q_4 = 24, Q_5 = 91$, etc. Taking mod 3 of the above relations, we have

$$\begin{aligned} Q_{2k} &\equiv Q_{2k-1} + Q_{2k-2} \pmod{3}, \\ Q_{2k-1} &\equiv Q_{2k-3} \pmod{3}. \end{aligned}$$

The second equation gives $Q_n \equiv Q_1 \equiv 1 \pmod{3}$ for all n odd. Therefore $Q_{2k} \equiv 1 + Q_{2k-2} \pmod{3}$. Inductively, $Q_{2k} \equiv k - 1 + Q_2 \pmod{3}$. Since $Q_2 = 5$, we have

$Q_{2k} \equiv k+1 \pmod 3$. Thus $Q_{2018} \equiv 1010 \equiv 2 \pmod 3$. Therefore $P_{2018}/3^{1009} = Q_{2018}$ is not divisible by 3. In other words, $v(2018) = 1009$.

5. Consider $P(x,y,z) = (y-x)^2 + (z-y)^2$. Then, $P(n, n+m, n+2m) = 2m^2$. Clearly, $r > 2$ does not work. We show that $r = 2$. If there exist $(a_1, b_1, c_1), (a_2, b_2, c_2)$ such that $P(a_1, b_1, c_1) > 0$ and $P(a_2, b_2, c_2) < 0$, draw a continuous path from (a_1, b_1, c_1) to (a_2, b_2, c_2), not passing through the line $x = y = z$.

Since P is continuous, there exists (a_3, b_3, c_3) on the path such that $P(a_3, b_3, c_3) = 0$. This is a contradiction as the condition that $a_3 = b_3 = c_3$ is not satisfied. Hence, either all $P \le 0$ or $P \ge 0$.

Let $u = y - x$, $v = z - y$. $P(x,y,z) = Q(x,u,v) = uvQ_1(x,u,v) + u^2Q_2(x,u,v) + v^2Q_3(x,u,v) + uR_1(x) + vR_2(x) + R_3(x)$. Substituting $u, v = 0$, $x = a$, we have $R_3(a) = 0$ for all real $a \Rightarrow R_3(x) = 0$. Substituting $v = 0$, $x = a$, we have $Q(a,u,0) = u^2Q_2(a,u,0) + uR_1(a)$. However, since $P \ge 0$ or $P \le 0$, for any fixed real a, there must be a double root at $u = 0$. Hence, $R_1(a) = 0$ for all $a \Rightarrow R_1(x) = 0$. Similarly, $R_2(x) = 0$. Since $m \mid u, v$, we have that m^2 divides $P(n, n+m, n+2m) = Q(n,m,m) = m^2(Q_1(n,m,m) + Q_2(n,m,m) + Q_3(n,m,m))$.

Singapore Mathematical Olympiad 2018 Results

School Award (Category 1)

Within each category of award, the schools are arranged according to alphabetical order.

Gold Award

Hwa Chong Institution
NUS High School of Mathematics and Science
Raffles Girls' School (Secondary)
Raffles Institution

Silver Award

Anglo-Chinese School (Independent)
Catholic High School
Maris Stella High School
Nanyang Girls' High School
River Valley High School
St Joseph's Institution
Victoria School

Bronze Award

Anderson Secondary School
Cedar Girls' Secondary School
CHIJ St Nicholas Girls' School
Dunman High School
Nan Chiau High School
Nan Hua High School
National Junior College
Ngee Ann Secondary School
NPS International School, Singapore

Commendation Award

Anglican High School
Bukit Panjang Govt High School
Chung Cheng High School (Main)
Clementi Town Secondary School

Crescent Girls' School
Dunman Secondary School
Fairfield Methodist Secondary School
Fuhua Secondary School
Methodist Girls' School
Riverside Secondary School
Singapore Chinese Girls' School
St Andrew's Secondary School
Swiss Cottage Secondary School
Tanjong Katong Girls' School
Tanjong Katong Secondary School
Temasek Junior College
Temasek Secondary School
Xinmin Secondary School
Yishun Town Secondary School
Zhonghua Secondary School

Singapore Mathematical Olympiad 2018 Results

School Award (Category 2)

Within each category of award, the schools are arranged according to alphabetical order.

Gold Award

Hwa Chong Institution
National Junior College
NUS High School of Mathematics and Science
Raffles Institution

Silver Award

Anderson Junior College
Anglo-Chinese School (Independent)
Dunman High School
Nanyang Junior College
Temasek Junior College

Bronze Award

Anderson Secondary School
Victoria Junior College

Commendation Award

Anglo-Chinese Junior College
Bukit Panjang Govt High School
Catholic Junior College
Eunoia Junior College
Hwa Chong International School
Meridian Junior College
Nan Hua High School
Nanyang Girls' High School
Pioneer Junior College
River Valley High School
St Joseph's Institution
St Joseph's Institution International

Singapore Mathematical Olympiad 2018 Results

Individual Award (Junior Section)

Rank	Name	School Name
1	SHEVONNE CHIA	NUS High School of Mathematics and Science
2	JAMIE LIM JIA SIN	NUS High School of Mathematics and Science
3	TAN YUN HUI	NUS High School of Mathematics and Science
4	TOH JING EN DANIEL	NUS High School of Mathematics and Science
5	MARIO TANIJAYA	NUS High School of Mathematics and Science
6	Wu Xinyue	Nanyang Girls' High School
7	Hu Xinghui	Raffles Girls' School (Secondary)
8	KRISHNAN AKASH	NUS High School of Mathematics and Science
8	SHEN XING YANG	NUS High School of Mathematics and Science
10	DREW MICHAEL TERREN RAMIREZ	- Independent -
11	YU QINXI	- Independent -
12	Zhang Chenxi	Raffles Girls' School (Secondary)
12	ANDREW YAR KWOK WING	Raffles Institution
14	Brandon Ong Jing Jie	Hwa Chong Institution
14	DYLON WONG YEE KIN	NUS High School of Mathematics and Science
16	OH ZHI YUAN	NUS High School of Mathematics and Science
17	AU HENG HOI, JOEL	NUS High School of Mathematics and Science
18	ASHLEY ARAGORN KHOO	NUS High School of Mathematics and Science
19	LAI PENGCHONG	Raffles Institution
20	CHANG CHIA WEI	Raffles Institution
21	LIM JIN TAO	NUS High School of Mathematics and Science
22	Arunav MAHESHWARI	United World College of South East Asia
23	Lee Xianrui Ryan	- Independent -
24	CHEONG SIK FENG	NUS High School of Mathematics and Science
25	LIU HAO	Raffles Institution
26	SU ZIXUAN	Raffles Institution
27	RAHUL KUMAR SINGH	Raffles Institution
28	DANIEL NG CHUN KIT	NUS High School of Mathematics and Science
29	KWAN KAIEE	Raffles Institution
29	YE XINTAI	Raffles Institution

Singapore Mathematical Olympiad 2018 Results

Individual Award (Senior Section)

Rank	Name	School Name
1	BOO TSE YANG LUCAS	Raffles Institution
2	NG YANGYI, ALOYSIUS	Raffles Institution
3	TAN XU CHEN	Raffles Institution
4	Ang Boon Han Nathaniel	Anglo-Chinese School (Independent)
5	Benson Lin Zhan Li	Hwa Chong Institution
6	HO LI XIONG, TIMOTHY	NUS High School of Mathematics and Science
7	TAN WEE KEAN	Raffles Institution
8	NGUYEN KHOI NGUYEN	St Joseph's Institution
9	LOW CHOO RAY	Raffles Institution
10	Chen Xinyi	Raffles Girls' School (Secondary)
11	SHEN XINYI	Raffles Institution
12	LEONG EU-SHAUN	Raffles Institution
13	HAN JIATAO JERRY	Raffles Institution
14	DAVID TOH HUI KAI	Raffles Institution
15	Gao Jiquan	Hwa Chong Institution
16	JOVAN LIAU YI JUN	NUS High School of Mathematics and Science
16	GOH CHEE KIAT, ETHAN	Raffles Institution
18	WANG HAOJI	St Joseph's Institution
19	CLAUDEON REINARD SUSANTO	Raffles Institution
20	Wang Yuhan	Hwa Chong Institution
21	WANG YUNRUI	Raffles Institution
22	GABRIEL TAN JIAXU	NUS High School of Mathematics and Science
23	HUANG KEYI	Anderson Secondary School
24	LI HANTAO	Raffles Institution
25	Guan Yangchen	Hwa Chong Institution
25	Yu Shuhuai	Hwa Chong Institution
25	ONG EE SHUA, ETIENNE	Raffles Institution
28	Xu Ruobei	Hwa Chong Institution
29	CHEN RUIMIN	Raffles Institution
29	TANG YETENG	Raffles Institution

Singapore Mathematical Olympiad 2018 Results

Individual Award (Open Section)

Rank	Name	School Name
1	JOEL TAN JUNYAO	NUS High School of Mathematics and Science
2	BOO TSE YANG LUCAS	Raffles Institution
3	DYLAN TOH SHAN HONG	NUS High School of Mathematics and Science
4	CHENG PUHUA	Raffles Institution
5	Wang Jianzhi	Raffles Institution
6	TAN XU CHEN	Raffles Institution
7	GABRIEL GOH KHENG LIN (WU ZHENNING)	NUS High School of Mathematics and Science
8	LEE KER YANG	Raffles Institution
9	DAVID TOH HUI KAI	Raffles Institution
10	Benson Lin Zhan Li	Hwa Chong Institution
11	NG YANGYI, ALOYSIUS	Raffles Institution
12	DANIEL LEONG ZHI MING (LIANG ZHIMING)	NUS High School of Mathematics and Science
13	LEONG EU-SHAUN	Raffles Institution
14	HO LI XIONG, TIMOTHY	NUS High School of Mathematics and Science
15	LE VIET HUNG	Anderson Secondary School
16	Shi Cheng	Hwa Chong Institution
17	LOW CHOO RAY	Raffles Institution
18	Ang Boon Han Nathaniel	Anglo-Chinese School (Independent)
19	Ng Yu Peng	Hwa Chong Institution
20	Ding Tian Yao	St Joseph's Institution International
21	LIU SIYONG	Raffles Institution
21	ZHANG YU CHI	Raffles Institution
23	Huang Tianjie	Hwa Chong Institution
24	Li Chenxu	Raffles Institution
25	Yang Yue	Hwa Chong Institution
26	WU ZHUOLIN	Anglo-Chinese Junior College
27	LI YUELIN	NUS High School of Mathematics and Science
28	DONG LEYI	Nanyang Girls' High School
29	NGUYEN KHOI NGUYEN	St Joseph's Institution
30	Tan Xiaochen	Dunman High School