Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2022

Junior Section (Round 1)

Wednesday, 1 June 2022 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*
8. *Throughout this paper, let* $\lfloor x \rfloor$ *denote the greatest integer less than or equal to* x*. For example,* $\lfloor 2.1 \rfloor = 2$, $\lfloor 3.9 \rfloor = 3$, $\lfloor 5 \rfloor = 5$.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Co-organizer
Department of Mathematics, NUS

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Multiple Choice Questions

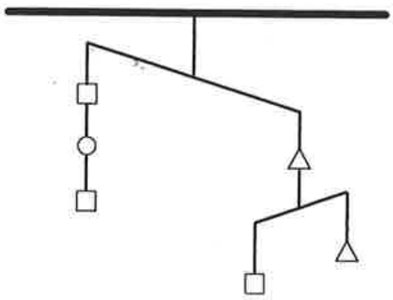
1. Which of the five numbers, 30^2, 10^3, 5^4, 4^5 or 3^6, is the largest?

   (A) 30^2 (B) 10^3 (C) 5^4 (D) 4^5 (E) 3^6

2. Adele has a fruit basket containing one apple, two bananas and three oranges, while Betty has a fruit basket containing two apples and three oranges. Assuming that Adele and Betty each randomly picked a fruit from her own basket, what is the probability that *together* they picked an apple and an orange?

   (A) $\frac{1}{10}$ (B) $\frac{3}{10}$ (C) $\frac{1}{5}$ (D) $\frac{6}{11}$ (E) None of the above

3. The following diagram shows a system of balances hanging from the ceiling with three types of weights. The balances tip down to the heavier side. If we use $\square < \triangle$ to represent $\square$ is lighter than $\triangle$, which of the following is true?



   (A) $\circ < \square < \triangle$ (B) $\circ < \triangle < \square$ (C) $\triangle < \square < \circ$ (D) $\triangle < \circ < \square$
   (E) Impossible to tell

4. A shop sells two types of buns, with either cream or jam filling, which are indistinguishable until someone bites into the buns. Four mathematicians visited the shop and ordered (not necessarily in that sequence): three cream buns, two cream buns and one jam bun, one cream and two jam buns, and three jam buns. Each knew precisely what the others had ordered. Unfortunately, the shop owner mixed up the orders and gave each mathematician the *wrong order*!

   The mathematicians started eating, *all still unaware of the mixup*, until the shop owner ran over to inform them of the mistake. Mathematician A said: "I ate two buns and both had cream filling. So, if my order was wrong, I now know what type my third bun is." Mathematician B then said: "I only ate one bun and it had cream filling. Based on what A said and since I remember A's order, I now know what type my other two buns are." Finally, Mathematician C said: "I have not started eating but I must have received three jam buns." Which of the following statements about Mathematician D is correct?

   (A) D ordered two cream and one jam but received three jam buns.
   (B) D ordered one cream and two jam but received two cream and one jam buns.
   (C) D ordered three cream but received one cream and two jam buns.
   (D) D ordered three jam but received three cream buns.
   (E) None of the above

5. If n can take the value of any positive integer, how many distinct values can the expression

$$\left\lfloor \frac{n}{4} \right\rfloor - \left\lfloor \frac{n}{7} \right\rfloor - \left\lfloor \frac{n}{14} \right\rfloor - \left\lfloor \frac{n}{28} \right\rfloor$$

take?

(A) 1 (B) 2 (C) 3 (D) 4 (E) infinitely many

Short Questions

6. Let n be a positive integer. If the lowest common multiple of n and 12 is 60, and the highest common factor of n and 36 is 4, what is the value of n?

7. The digits $1, 2, 3, 4, 5$ and 6 are arranged to form two positive integers with each digit appearing exactly once. How many ways can this be done if the sum of the two integers is 570?

8. Consider the following product of two *mixed* fractions

$$m\frac{6}{7} \times n\frac{1}{3} = 23,$$

where m and n are positive integers. What is the value of $m + n$?

9. What are the last four digits of the sum

$$222 + 2022 + 20022 + \cdots + 2\underbrace{0000000000}_{\text{ten 0s}}22\,?$$

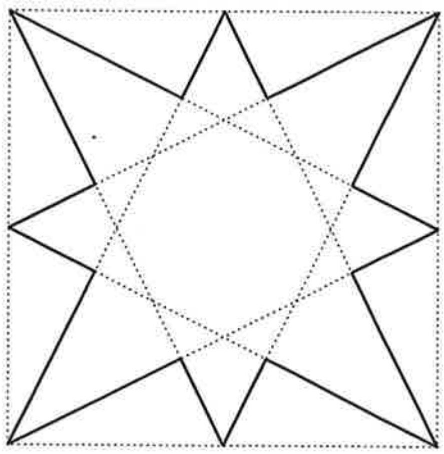
Give your answer as a 4-digit number.

10. If a and b are distinct solutions to the equation

$$x^2 + 10x + 20 = 0,$$

what is the value of $a^4 + b^4$?

11. The following diagram shows a star that is cut out from a square with sides of length 30. What is the area enclosed by the star?



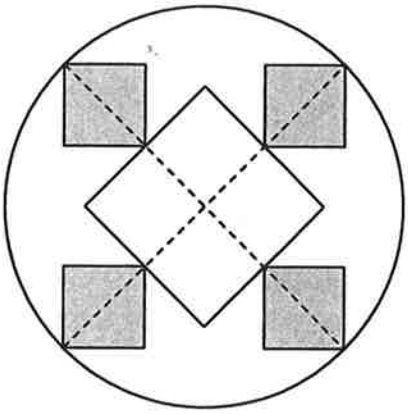
12. If x is a real number, how many solutions are there to the equation

$$(3x+2)^{x+5} = 1?$$

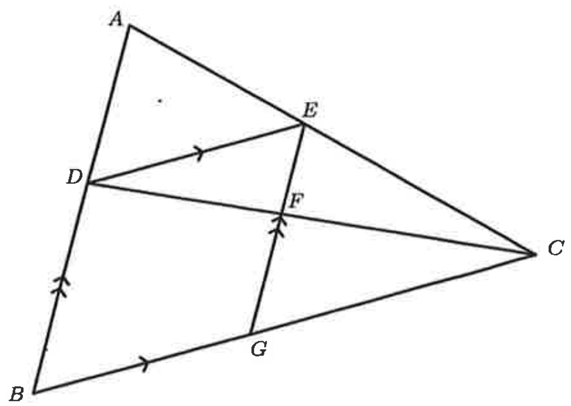
13. In the figure below, each distinct letter represents a unique distinct digit such that the arithmetic holds. If W represents 5, what number does TROOP represent?

$$\begin{array}{cccccc} & T & R & O & O & P \\ + & T & O & O & T & S \\ \hline N & E & W & S & O & N \end{array}$$

14. In the following diagram, a white square and four grey squares of equal size are drawn in a circle such that both dashed lines form diameters of the circle. If the diameter has length 60, find the smallest possible value for the total area of the five squares.



15. In the following diagram, ABC is a triangle and D lies on AB such that the ratio of $|AD| : |DB| = 3 : 4$. Points E and G lie on AC and BC respectively such that $DBGE$ is a parallelogram and F is the intersection of DC and GE. If the ratio of the area of $\triangle EDF$ to the area of $\triangle ABC$ is $m : n$, where the highest common factor of m and n is 1, what is the value of $m + n$?

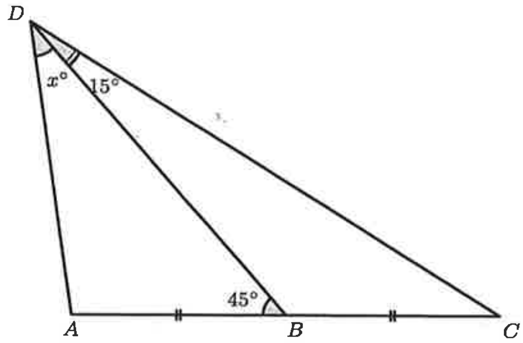


16. Eggs in a certain supermarket are sold *only* in trays containing exactly 10, 12 or 30 eggs per tray. It is thus impossible to buy exactly 14 eggs or any odd number of eggs. However, it is possible to buy exactly 78 eggs using four trays of 12 and one tray of 30. What is the largest even number of eggs that is impossible to be bought from this supermarket?

17. How many integers n are there in $\{1, 2, \ldots, 2022\}$ such that at least one of the digits of n is '2'?

18. What is the smallest k such that every subset of $\{1, 2, \ldots, 99\}$ with exactly k integers contains at least two distinct integers x and y, such that $|x - y| \leqslant 2$?

19. In the following diagram, ACD is a triangle such that $|AB| = |BC|$, $\angle ABD = 45°$ and $\angle BDC = 15°$. If $\angle ADB = x°$, what is the value of x?



20. What is the area of a triangle with side lengths

$$\sqrt{6^2 + 7^2}, \sqrt{12^2 + 7^2} \text{ and } \sqrt{6^2 + 14^2}\,?$$

21. If $x = \sqrt[3]{4} + \sqrt[3]{2} + 1$, what is the value of

$$2022 + \frac{3}{x} + \frac{3}{x^2} + \frac{1}{x^3}\,?$$

22. If we have

$$\frac{\sqrt{15} + \sqrt{35} + \sqrt{21} + 5}{\sqrt{3} + 2\sqrt{5} + \sqrt{7}} = \frac{a\sqrt{7} + b\sqrt{5} + c\sqrt{3}}{2}$$

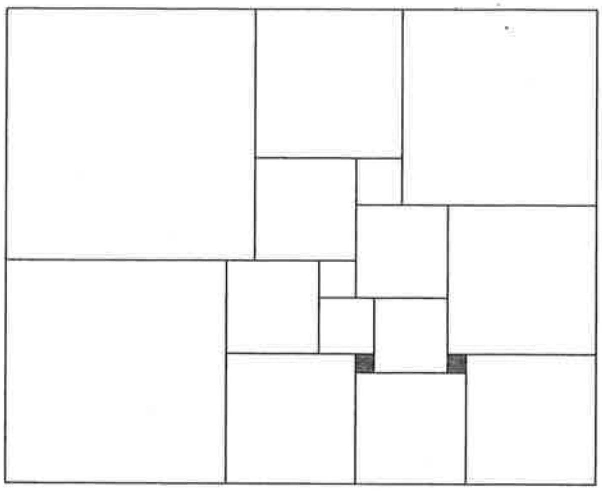
for some integers a, b, c. What is the value of $a + b + c$?

23. How many integers n are there in $\{1, 2, \ldots, 2022\}$ such that $\lfloor \sqrt[3]{n} \rfloor$ is a factor of n?

24. If x and y can take any real values, what is the smallest possible value of the expression

$$2x^2 + 4xy + 5y^2 + 4x + 10y + 13\,?$$

25. The following diagram shows a rectangle that is partitioned into 17 squares. If the two smallest squares, shaded in grey, have sides of length 2, what is the area of the rectangle?



SMO 2022 (Junior Section) Answers

1. D
2. B
3. B
4. E
5. C
6. 20
7. 4
8. 11
9. 2442
10. 2800
11. 540
12. 3
13. 92334
14. 1200
15. 379
16. 38
17. 565
18. 34
19. 30
20. 63
21. 2023
22. 2
23. 267
24. 8
25. 3264

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2022

Senior Section (Round 1)

Wednesday, 1 June 2022 **0930 – 1200 hrs**

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Multiple Choice Questions

1. Suppose the roots of $\frac{x^2}{2} + mx + n = 0$ are $\frac{m}{2}$ and $\frac{n}{3}$. Find the smallest value of mn.

   (A) -1080 (B) -90 (C) 0 (D) 90 (E) 1080

2. Which of the following is true?

   (A) $\sqrt[6]{\frac{1}{333}} < \sqrt[3]{\frac{1}{18}} < \sqrt{\frac{1}{7}}$ (B) $\sqrt[3]{\frac{1}{18}} < \sqrt[6]{\frac{1}{333}} < \sqrt{\frac{1}{7}}$ (C) $\sqrt[3]{\frac{1}{18}} < \sqrt{\frac{1}{7}} < \sqrt[6]{\frac{1}{333}}$
   (D) $\sqrt{\frac{1}{7}} < \sqrt[6]{\frac{1}{333}} < \sqrt[3]{\frac{1}{18}}$ (E) None of the above.

3. Suppose $\sqrt{(\log_{377\times 377} 2022)(\log_{377} 2022)} = \log_k 2022$. Find k.

   (A) $\sqrt{337}$ (B) $337^{\sqrt{2}}$ (C) $337\sqrt{2}$ (D) $\sqrt{337}^{\sqrt{2}}$ (E) $\sqrt{337 \times 2}$

4. Suppose $y = \cos^2 x - 7\cos x + 25$, where x is any real number. Find the range of y.

   (A) $17 \le y \le 33$ (B) $18 \le y \le 33$ (C) $19 \le y \le 33$ (D) $20 \le y \le 33$
   (E) None of the above

5. Suppose $\sin(180^\circ + x) = -\frac{7}{9}$, where $450^\circ < x < 540^\circ$. Find $\sin(2x)$.

   (A) $\frac{49}{81}\sqrt{2}$ (B) $\frac{56}{81}\sqrt{2}$ (C) $-\frac{56}{81}$ (D) $-\frac{49}{81}\sqrt{2}$ (E) $-\frac{56}{81}\sqrt{2}$

Short Questions

6. Find the value of

$$\left(\frac{\cos 10^\circ + \cos 50^\circ + \cos 70^\circ + \cos 110^\circ}{\cos 20^\circ}\right)^8.$$

7. Suppose $x^{20} + \frac{x^{10}}{2} - \frac{3^{2x}}{9} + \frac{1}{16} = 0$ for some positive real number x. Find the value of $4 \cdot 3^x - 12x^{10}$.

8. How many positive integers less than or equal to 2022 **cannot** be expressed as

$$\lfloor 2x+1 \rfloor + \lfloor 5x+1 \rfloor$$

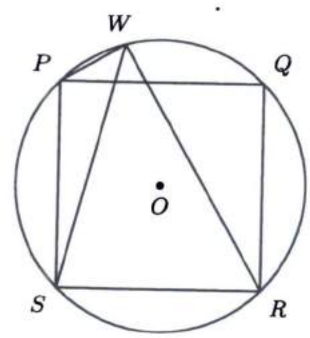
   for some real number x? Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. For example, $\lfloor -2.1 \rfloor = -3$, $\lfloor 3.9 \rfloor = 3$.

9. Suppose

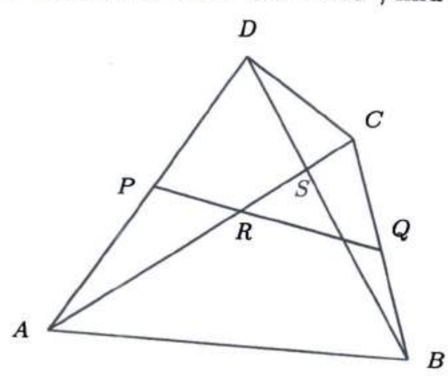
$$y = \frac{\tan^2 x - \tan x + \sqrt{33}}{\tan^2 x + \tan x + 1},$$

where $-90^\circ < x < 90^\circ$. Find the maximum possible value of $\sqrt{33}(y-5)$.

10. In the figure below, $PQRS$ is a square inscribed in a circle. Let W be a point on the arc PQ such that $WS = \sqrt{20}$. Find $(WP + WR)^2$.



11. The figure below shows a quadrilateral $ABCD$ such that $AC = BD$ and P and Q are the midpoints of the sides AD and BC respectively. The lines PQ and AC meet at R and the lines BD and AC meet at S. If $\angle PRC = 130^\circ$, find the angle $\angle DSC$ (in °).



12. How many distinct terms are there if $(x^2+y^2)^{11}(x^{11}+y^{11})^9$ is algebraically expanded and simplified?

13. If $\sqrt{x^2+7x-4} + \sqrt{x^2-x+4} = x-1$, find the value of $3x^2+14x$.

14. Let $k = -1 + \sqrt{2022^{1/5}-1}$, and let $f(x) = (k^2+2k+2)^{10x}$. Find the value of $\log_{2022} f(2022)$.

15. Find the smallest **odd** integer N, where $N > 2022$, such that when 1808, 2022 and N are each divided by a positive integer p, where $p > 1$, they all leave the same remainder.

16. If $\dfrac{12}{x}+\dfrac{48}{y}=1$, where x and y are positive real numbers, find the smallest possible value of $x+y$.

17. Find the largest value of $40x+60y$ if $x-y\leq 2$, $5x+y\geq 5$ and $5x+3y\leq 15$.

18. Suppose

$$\begin{aligned}\cos x-\cos y &= \frac{1}{2},\\ \sin x-\sin y &= -\frac{1}{3}.\end{aligned}$$

If $\sin(x+y)=\dfrac{m}{n}$, where $\dfrac{m}{n}$ is expressed as a fraction in its lowest terms, find the value of $m+n$.

19. For some positive integer n, the number n^3-3n^2+3n has a units digit of 6. Find the product of the last two digits of the number $7(n-1)^{12}+1$.

20. Find the largest positive integer n for which $\dfrac{20n+2020}{3n-6}$ is a positive integer.

21. In the xy-coordinate system, there are two circles passing through the point $(11,3\sqrt{3})$, and each of these circles is tangent to both the x-axis and the line $y=\sqrt{3}x$. Let S be the sum of the radii of the two circles. Find $\sqrt{3}S$.

22. Let P and Q be the points $(20(\sqrt{5}-1),0)$ and $(0,10(\sqrt{5}-1))$ on the xy-plane. Let R be the point (a,b). If $\angle PRQ$ is a right angle, find the maximum possible value of b.

23. How many positive integers n **do not** satisfy the inequality

$$n^{\frac{1}{3}\log_{20}n}>\sqrt{n}\ ?$$

24. Let $f(x)$ be a function such that

$$3f(x^2)+f(13-4x)=3x^2-4x+293$$

for all real number x. Find the value of $f(1)$.

25. Find the largest positive integer M such that

$$\cos^2 x-\sin^2 x+\sin x=\frac{M}{888}$$

has a real solution.

SMO 2022 (Senior Section) Answers

1. A
2. D
3. B
4. C
5. E
6. 81
7. 3
8. 289
9. 22
10. 40
11. 80
12. 112
13. 65
14. 4044
15. 2129
16. 108
17. 300
18. 25
19. 42
20. 2062
21. 28
22. 20
23. 89
24. 71
25. 999

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2022
(Open Section, Round 1)

Thursday, 2 June 2022 **0930-1200 hrs**

Instructions to contestants

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In this paper, let $\mathbb{R}$ denote the set of all real numbers, and $\lfloor x \rfloor$ denote the greatest integer not exceeding x. For examples, $\lfloor 5 \rfloor = 5$, $\lfloor 2.8 \rfloor = 2$, and $\lfloor -2.3 \rfloor = -3$.

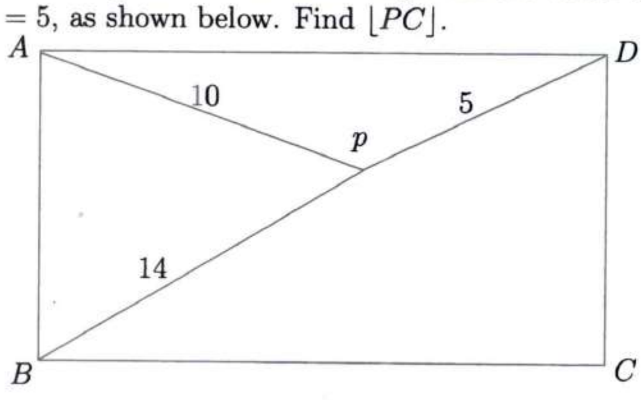
1. If $S = \displaystyle\sum_{k=-2021}^{2021} \frac{1}{10^k + 1}$, find $\lfloor 2S \rfloor$.

2. All the positive integers $1, 2, 3, 4, \cdots$, are grouped in the following way: $G_1 = \{1, 2\}$, $G_2 = \{3, 4, 5, 6\}$, $G_3 = \{7, 8, 9, 10, 11, 12, 13, 14\}$, that is, the set G_n contains the next 2^n positive integers listed in ascending order after the set G_{n-1}, $n > 1$. If S is the sum of all the positive integers from G_1 to G_8, find $\left\lfloor \dfrac{S}{100} \right\rfloor$.

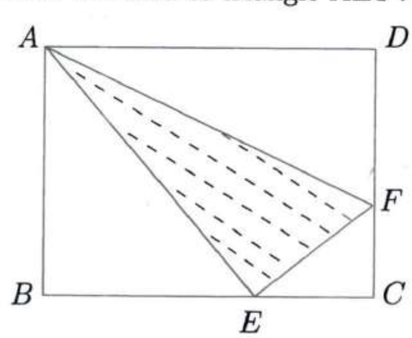
3. A sequence of one hundred positive integers $x_1, x_2, x_3, \cdots, x_{100}$ are such that
$$(x_1)^2 + (2x_2)^2 + (3x_3)^2 + (4x_4)^2 + \cdots + (100x_{100})^2 = 338350.$$
Find the largest possible value of $x_1 + x_2 + x_3 + \cdots + x_{100}$.

4. Let a and b be two real numbers satisfying $a < b$, and such that for each real number m satisfying $a < m < b$, the circle $x^2 + (y - m)^2 = 25$ meets the parabola $4y = x^2$ at four distinct points in the Cartesian plane. Let S be the maximum possible value of $b - a$. Find $\lfloor 4S \rfloor$.

5. Let P be a point within a rectangle $ABCD$ such that $PA = 10$, $PB = 14$ and $PD = 5$, as shown below. Find $\lfloor PC \rfloor$.



6. In the diagram below, the rectangle $ABCD$ has area 180 and both triangles ABE and ADF have areas 60. Find the area of triangle AEF.

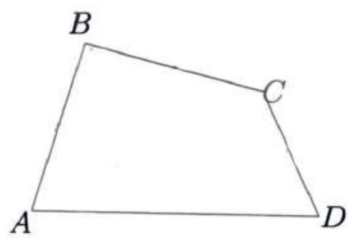


7. A tetrahedron in $\mathbb{R}^3$ has one vertex at the origin O and the other vertices at the points $A(6,0,0)$, $B(4,2,4)$ and $C(3,2,6)$. If x is the height of the tetrahedron from O to the plane ABC, find $\lfloor 5x^2 \rfloor$.

8. Let x and y be real numbers such that $(x-2)^2 + (y-3)^2 = 4$. If S is the largest possible value of $x^2 + y^2$, find $\lfloor (S-17)^2 \rfloor$.

9. Let S be the maximum value of $w^3 - 3w$ subject to the condition that $w^4 + 9 \leq 10w^2$. Find $\lfloor S \rfloor$.

10. In the quadrilateral $ABCD$ below, it is given that $AB = BC = CD$ and $\angle ABC = 80^\circ$ and $\angle BCD = 160^\circ$. Suppose $\angle ADC = x^\circ$. Find the value of x.



11. Let a, b, c be integers with $ab + c = 49$ and $a + bc = 50$. Find the largest possible value of abc.

12. Find the largest possible value of $|a| + |b|$, where a and b are coprime integers (i.e., a and b are integers which have no common factors larger than 1) such that $\frac{a}{b}$ is a solution of the equation below:

$$\sqrt{4x + 5 - 4\sqrt{x+1}} + \sqrt{x + 2 - 2\sqrt{x+1}} = 1.$$

13. Let S be the set of real solutions (x, y, z) of the following system of equations:

$$\begin{cases} \dfrac{4x^2}{1+4x^2} = y, \\ \dfrac{4y^2}{1+4y^2} = z, \\ \dfrac{4z^2}{1+4z^2} = x. \end{cases}$$

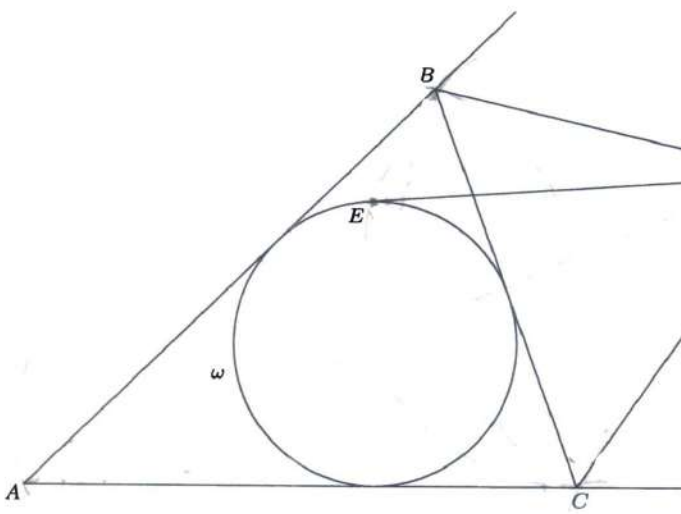
For each $(x, y, z) \in S$, define $m(x, y, z) = 2000(|x| + |y| + |z|)$. Determine the maximum value of $m(x, y, z)$ over all $(x, y, z) \in S$.

14. Assume that t is a positive solution to the equation

$$t = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+t}}}}.$$

Determine the value of $t^4 - t^3 - t + 10$.

15. In the triangle ABC shown in the diagram below, the external angle bisectors of $\angle B$ and $\angle C$ meet at the point D. The tangent from D to the incircle ω of the triangle ABC touches ω at E, where E and B are on the same side of the line AD. Suppose $\angle BEC = 112°$. Find the size of $\angle A$ in degrees.



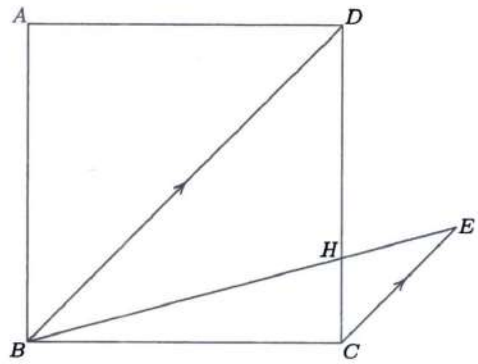
16. Find the largest integer n such that $n^2 + 5n - 9486 = 10s(n)$, where $s(n)$ is the product of all digits of n in the decimal representation of n.

(For example, $s(481) = 4 \times 8 \times 1 = 32$.)

17. Find the number of integer solutions to the equation $19x + 93y = 4xy$.

18. Find the number of integer solutions to the equation $x_1 + x_2 - x_3 = 20$ with $x_1 \geq x_2 \geq x_3 \geq 0$.

19. In the diagram below, E is a point outside a square $ABCD$ such that CE is parallel to BD, $BE = BD$, and BE intersects CD at H. Given $BE = \sqrt{6} + \sqrt{2}$, find the length of DH.



20. The diagram below shows the region $R = \{(x, y) \in \mathbb{R}^2 \mid y \geq \frac{1}{2}x^2\}$ on the xy-plane bounded by the parabola $y = \frac{1}{2}x^2$. Let C_1 be the largest circle lying inside R with its lowest point at the origin. Let C_2 be the largest circle lying inside R and resting on top of C_1. Find the sum of radii of C_1 and C_2.

21. Find the smallest positive integer x such that $3x^2 + x = 4y^2 + y$ for some positive integer y.

22. A group of students participate in some sports activities among 6 different types of sports. It is known that for each sports activity there are exactly 100 students in the group participating in it; and the union of all the sports activities participated by any two students is NOT the entire set of 6 sports activities. Determine the minimum number of students in the group.

23. Let p and q be positive prime integers such that $p^3 - 5p^2 - 18p = q^9 - 7q$. Determine the smallest value of p.

24. Given that a, b, c are positive real numbers such that $a+b+c=9$, find the maximum value of $a^2b^3c^4$.

25. Let $\mathbb{R}^+$ be the set of all positive real numbers. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying

$$xyf(x)\,(f(y) - f(yf(x))) = 1$$

for all $x, y \in \mathbb{R}^+$. Find $f(\frac{1}{2022})$.

SMO 2022 (Open Section) Answers

1. 4043
2. 1303
3. 100
4. 9
5. 11
6. 50
7. 144
8. 208
9. 18
10. 40
11. 544
12. 17
13. 3000
14. 11
15. 44
16. 99
17. 8
18. 121
19. 2
20. 4
21. 30
22. 200
23. 29
24. 27648
25. 2023