Mathematics Olympiad: A Guidebook

Ryan Joo Rui An

You don't have to be a mathematician to have a feel for numbers.

— John Forbes Nash, Jr. (1928–2015)
American mathematician

This is (still!) an incomplete draft. Please send corrections and comments to ryanjooruian18@gmail.com, or pull-request at https://github.com/Ryanjoo18/latex.

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Preface

About the Author

Ryan Joo Rui An is a high school student currently completing his A Levels in Singapore. He has consistently achieved decent results in numerous Mathematics Olympiad competitions, including the Singapore Mathematics Olympiad (SMO) and IMO National Selection Test (IMONST), which are the two selection tests for the prestigious International Mathematics Olympiad (IMO) by Singapore and Malaysia respectively.

About the Book

The motivation of the writer to write this book was to the author for Mathematics Olympiad competitions by summarising important topics in Mathematics Olympiad, with a focus on SMO, which feature rather challenging problems.

Do reach out to the author at ryanjooruian18@gmail.com if you wish to point out any (glaring) mistakes, or provide comments and suggestions regarding the content and/or formatting of the book.

Resources

Here are some notes and materials that you might find useful when learning Mathematics Olympiad:

- Unofficial syllabus for math olympiads, by Evan Chen
- Notes on proof-writing style
- AoPS Contest Collections
- Yufei Zhao's olympiad handouts
- Evan Chen's olympiad handouts
- Olympiad Problems and Solutions, by Evan Chen
- Canadian Mathematical Society Resources

- Mathematical Reflections
- Selected Problems of the Vietnamese Mathematical Olympiad (1962–2009)
- 102 Combinatorial Problems, by Titu Andreescu & Zuming Feng
- 3Blue1Brown

Problem Solving

Solve mathematical problems using problem solving heuristics

- Working backwards
- Uncovering pattern and structure
- Solving a simpler/similar problem
- Considering cases
- Restating the problem (e.g. contrapositive)

Contents

Acronyms

AUSTRALIA Australian Mathematical Olympiad. 32, 58, 127

CANADA Canadian Mathematical Olympiad. 33

CHINA China Mathematical Olympiad. 99

H3M H3 Mathematics. 37, 85, 87, 89, 90

IMO International Mathematics Olympiad. 30, 38, 93, 97, 98, 114, 115

ITALY Italian Mathematical Olympiad. 34

MAT Oxford Maths Admissions Test. 161

PUTNAM William Lowell Putnam Mathematics Competition. 189

SMO Singapore Mathematics Olympiad. 30, 55–57, 71, 92, 107–113, 122, 126, 145, 146, 154–158, 187, 188, 209

TRIPOS Mathematical Tripos. 60

USAMO United States of America Mathematical Olympiad. 35, 96, 195

Part I

Number Theory

1 Modular Arithmetic

- Olympiad Number Theory Through Challenging Problems
- A Concise Introduction to Pure Mathematics, Fourth Edition
- NT notes

§1.1 Divisibility

Definition 1.1.1. Let n and d be integers with $d \neq 0$. We say n is **divisible** by d if and only if there exists an integer k such that $n = dk$, denoted by $d \mid n$; otherwise $d \nmid n$.

Remark. If a is a divisor of b, then b is also divisible by $-a$, so the divisors of an integer always occur in pairs. To find all the divisors of a given integer, it is sufficient to obtain the positive divisors and then adjoin them to the corresponding negative integers. For this reason, we usually limit ourselves to the consideration of the positive divisors.

Using the definition above, for integers a, b, c, the following properties hold:

(i) $a \mid 0$, $1 \mid a$, $a \mid a$

(ii) $a \mid 1$ if and only if $a = \pm 1$

(iii) If $a \mid b$ and $c \mid d$, then $ac \mid bd$

(iv) If $a \mid b$ and $b \mid c$, then $a \mid c$

(v) $a \mid b$ and $b \mid a$ if and only if $a = \pm b$

(vi) If $a \mid b$ and $b \neq 0$, then $a \leq b$

(vii) If $a \mid b$ and $a \mid c$, then $a \mid (bx + cy)$ for arbitrary integers x and y.

Remark. Property (vii) extends by induction to sums of more than two terms. That is, if $a \mid bk$ for $k = 1, 2, \ldots, n$, then

$$a \mid (b_1x_1 + b_2x_2 + \cdots + b_nx_n)$$

for all integers x_i.

Exercise 1.1.1

Let $a, b \in \mathbb{Z}$. Prove that if $a, b > 0$ and $a \mid b$, then $a \leq b$.

Proof. Suppose $a, b > 0$ and $a \mid b$. Then there exists an integer k such that $b = ak$.

So $k > 0$ as a and b are positive.

It follows that $1 \leq k$, as every positive integer ≥ 1.

Then $a \leq ak$, as multiplying both sides of an inequality by a positive number preserves the inequality.

Hence $a \leq b$. □

Exercise 1.1.2

Let $a, b, c \in \mathbb{Z}$. Prove that if $a \mid b$ and $a \mid c$, then $a \mid (b + c)$ and $a \mid (b - c)$.

§1.1.1 Divisibility Rules

These divisibility rules help determine when positive integers are divisible by particular other integers. All of these rules apply for base-10 only – other bases have their own, different versions of these rules.

Integer	Divisibility
2 and powers of 2	A number is divisible by 2^n if and only if the last n digits of the number are divisible by 2^n. Thus, in particular, a number is divisible by 2 if and only if its units digit is divisible by 2, i.e. if the number ends in 0, 2, 4, 6 or 8.
3 and 9	A number is divisible by 3 or 9 if and only if the sum of its digits is divisible by 3 or 9, respectively. Note that this does not work for higher powers of 3. For instance, the sum of the digits of 1899 is divisible by 27, but 1899 is not itself divisible by 27.
5 and powers of 5	A number is divisible by 5^n if and only if the last n digits are divisible by that power of 5.
7	Partition N into 3 digit numbers from the right $(d_3d_2d_1, d_6d_5d_4, \dots)$. The alternating sum $(d_3d_2d_1 - d_6d_5d_4 + d_9d_8d_7 - \dots)$ is divisible by 7 if and only if N is divisible by 7.
10 and powers of 10	If a number is power of 10, define it as a power of 10. The exponent is the number of zeros that should be at the end of a number for it to be divisible by that power of 10. Example: A number needs to have 6 zeroes at the end of it to be divisible by 1,000,000 because $1,000,000 = 10^6$.
11	A number is divisible by 11 if and only if the alternating sum of the digits is divisible by 11.
13	Partition N into 3 digit numbers from the right $(d_3d_2d_1, d_6d_5d_4, \dots)$. The alternating sum $(d_3d_2d_1 - d_6d_5d_4 + d_9d_8d_7 - \dots)$ is divisible by 13 if and only if N is divisible by 13.
17	Truncate the last digit, multiply it by 5 and subtract from the remaining leading number. The number is divisible if and only if the result is divisible. The process can be repeated for any number.
19	Truncate the last digit, multiply it by 2 and add to the remaining leading number. The number is divisible if and only if the result is divisible. This can also be repeated for large numbers.
29	Truncate the last digit, multiply it by 3 and add to the remaining leading number. The number is divisible if and only if the result is divisible. This can also be repeated for large numbers.

§1.2 Congruence

Definition 1.2.1. Let a, b, n be integers with $n > 0$. Then we say a and b are **congruent** modulo n, or a is congruent to b modulo n, denoted by $a \equiv b \pmod{n}$, if and only if $n \mid a - b$ (or $a = b + nk$ for some integer k).

Properties of congruence[1]

- **Reflexive**: for every integer a, $a \equiv a \pmod{n}$.
- **Symmetric**: for all integers a and b, if $a \equiv b \pmod{n}$ then $b \equiv a \pmod{n}$.
- **Transitive**: for all integers a, b and c, if $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$ then $a \equiv c \pmod{n}$.

§1.3 Modular Arithmetic

For all integers a, b, c, d and n, with $n > 1$, if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then

- $a + c \equiv b + d \pmod{n}$ (preserve addition)
- $ac \equiv bd \pmod{n}$ (preserve multiplication)
- $a + k \equiv b + k \pmod{n}$ for every $k \in \mathbb{Z}$
- $ka \equiv kb \pmod{n}$ for every $k \in \mathbb{Z}$
- $a^m \equiv b^m \pmod{n}$ for every $m \in \mathbb{Z}^+$ (preserve power)

Exercise 1.3.1

Prove that if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$.

Proof. Given $a \equiv b \pmod{n}$. So $a = b + nk$ for some integer k.

Given $c \equiv d \pmod{n}$. So $c = d + nh$ for some integer h.

Hence

$$ac = (b + nk)(d + nh) = bd + n(dk + bh + nkh)$$

Let $q = dk + bh + nkh$. Since $ac = bd + nq$ for some integer q, hence $ac \equiv bd \pmod{n}$. □

[1] these properties can be proven during the definition of congruence.

§1.4 Greatest Common Divisor and Lowest Common Multiple

§1.4.1 Greatest Common Divisor (GCD)

Definition 1.4.1. Let a, b be integers, not both 0, and $d \in \mathbb{Z}^+$. d is the **greatest common divisor** of a and b, denoted by $d = \gcd(a, b)$, if and only if

(i) $d \mid a$ and $d \mid b$

(ii) for all $k \in \mathbb{Z}^+$, if $k \mid a$ and $k \mid b$ then $k \leq d$.

Remark. (ii) simply means all common divisors are less than or equal to the *greatest* common divisor.

Basic properties of GCD:

- $\gcd(a, b) > 0$, whether a and b are positive or negative
- $\gcd(a, a) = |a|$
- $\gcd(a, 0) = |a|$
- $\gcd(a, b) = \gcd(-a, b) = \gcd(a, -b) = \gcd(-a, -b)$

Theorem 1.4.1

Let c and d be integers, not both 0. If q and r are integers such that $c = dq + r$ then $\gcd(c, d) = \gcd(d, r)$.

Proof. Let $m = \gcd(c, d)$ and $n = \gcd(d, r)$. To prove $m = n$, we will show $m \leq n$ and $n \leq m$.

We first show $n \leq m$. Since $n = \gcd(d, r)$ then $n \mid d$ and $n \mid r$. There exists integers x and y such that $d = nx$ and $r = ny$.

From $c = dq + r$ we have
$$c = (nx)q + ny = n(xq + y)$$

Hence $n \mid c$. Since n is a common divisor of c and d, $n \leq \gcd(c, d)$ so $n \leq m$.

The proof of $m \leq n$ is similar and shall be left as an exercise. □

Let a and b be two non-zero integers. Then a and b are said to be **relatively prime** (or coprime) if and only if $\gcd(a, b) = 1$.

Lemma 1.4.1 (Euclid's Lemma)**.** Let a, b, c be any integers. If $a \mid bc$ and $\gcd(a, b) = 1$ then $a \mid c$.

Proof. Since $a \mid bc$, $bc = ak$ for some $k \in \mathbb{Z}$.

Since $\gcd(a, b) = 1$ then $ax + by = 1$ for some $x, y \in \mathbb{Z}$.

$$\begin{aligned} cax + cby &= c \\ acx + aky &= c \\ acx + aky &= c \\ a(cx + ky) &= c \end{aligned}$$

Hence $a \mid c$. □

§1.4.2 Lowest Common Multiple (LCM)

Definition 1.4.2. The **lowest common multiple** of a and b, denoted by $\operatorname{lcm}(a, b)$, is the *smallest* positive integer m where $a \mid m$ and $b \mid m$.

Theorem 1.4.2

For positive integers a and b,

$$\gcd(a, b) \times \operatorname{lcm}(a, b) = ab.$$

§1.4.3 Primes

Definition 1.4.3. An integer $n > 1$ is **prime** if and only if for all positive integers r and s, if $n = rs$ then either $r = n$ or $s = n$ (equivalently, the only positive divisors of n are 1 and n).

Conversely, an integer $n > 1$ is **composite** if and only if $n = rs$ for some positive integers r and s such that $1 < r < n$ and $1 < s < n$ (equivalently, n has divisor d such that $1 < d < n$).

Theorem 1.4.3: Prime Number Theorem

The **Riemann Zeta Function** describes the distribution of prime numbers. For a positive real x, the function $\pi(x)$ denotes the number of primes less than or equal to x.
Then the number of primes not exceeding x is asymptotic to $\dfrac{x}{\ln x}$; that is,

$$\pi(x) \sim \frac{x}{\ln x}.$$

Also, regarding prime numbers,

Theorem 1.4.4: Euler

There are infinitely many primes.

Proof. We first prove the following lemma.

Lemma 1.4.2. Every integer greater than 1 has a prime factor.

Proof. We argue by (strong) induction that each integer $n > 1$ has a prime factor. For the base case $n = 2$, 2 is prime and is a factor of itself.

Now assume $n > 2$ all integers greater than 1 and less than n have a prime factor. To show n has a prime factor, we take cases.

Case 1: n is prime.

Since n is a factor of itself, n has a prime factor when n is prime.

Case 2: n is not prime.

Since n is not prime, it has a factorisation $n = ab$ where $1 < a, b < n$. Then by the strong inductive hypothesis, a has a prime factor, say p. Since $p \mid a$ and $a \mid n$, also $p \mid n$ and thus n has prime factor p. □

To show there are infinitely many primes, we will show that every finite list of primes is missing a prime number, so the list of all primes cannot be finite.

To begin, there are prime numbers such as 2. Suppose $p_1, \ldots, p_r$ is a finite list of prime numbers. We want to show this is not the full list of the primes. Consider the number

$$N = p_1 \cdots p_r + 1.$$

Since $N > 1$, it has a prime factor p by lemma 1.4.2. The prime p cannot be any of $p_1, \ldots, p_r$ since N has remainder 1 when divided by each p_i. Therefore p is a prime not on our list, so the set of primes cannot be finite. □

Remark. Some people misunderstand this proof to be saying that if $p_1, \ldots, p_r$ are prime then $p_1 \cdot p_r + 1$ is prime. That is not generally true.

Theorem 1.4.5: Fundamental Theorem of Arithmetic

Every integer $n > 1$ can be expressed as a product of primes in a unique way apart from the order of the prime factors; that is,

$$n = {p_1}^{a_1} {p_2}^{a_2} \cdots {p_k}^{a_k}$$

where p_i are prime numbers and a_i are positive integers.

Proof. We prove this by strong induction. Consider some integer $n > 1$. Either it is prime or it is composite.

Case 1: If n is prime, we are done.

Case 2: If n is composite, then there exists an integer $d \mid n$ and $1 < d < n$. Among all such integers d, choose the smallest, say p_1. Then p_1 must be prime. Hence we can write $n = p_1 n_1$ for some integer n_1 satisfying $1 < n_1 < n$. This completes the induction. □

§1.5 Division Algorithm

Theorem 1.5.1: Division Algorithm

For all integers n and d with $d > 0$, there exists unique integers q and r, known as **quotient** and **remainder** respectively, such that

$$n = dq + r, \qquad \text{where } 0 \le r < d.$$

§1.6 Euclidean Algorithm

The **Euclidean Algorithm** is an efficient method to determine the gcd of two numbers. It makes use of the following properties:

- $\gcd(a, 0) = a$
- $\gcd(a, b) = \gcd(a \mod b, b)$

The Euclidean Algorithm may be described as follows: Let a and b be two integers whose greatest common divisor is desired. WLOG $a \ge b > 0$.

The first step is to apply the Division Algorithm to a and b to obtain

$$a = q_1 b + r_1 \quad 0 \le r_1 < b$$

In the case where $r_1 = 0$, then $b \mid a$ and $\gcd(a, b) = b$. When $r_1 \ne 0$, divide b by r_1 to obtain integers q_2 and r_2 satisfying

$$b = q_2 r_1 + r_2 \quad 0 \le r_2 < r_1$$

If $r_2 = 0$, we stop; otherwise, proceed as before to obtain

$$r_1 = q_3 r_2 + r_3 \quad 0 \le r_3 < r_2$$

This division process continues until some zero remainder appears, say at the $(n+1)$-th step where r_{n+1} is divided by r_n Note that this process will stop eventually as the strictly decreasing sequence $b > r_1 > r_2 > \cdots > r_{n+1} > r_{n+2} = 0$ cannot contain more than b integers.

The result is the following sequence of equations (or operations):

$$\begin{aligned}
a &= q_1 b + r_1 & 0 &\le r_1 < b \\
b &= q_2 r_1 + r_2 & 0 &\le r_2 < r_1 \\
r_1 &= q_3 r_2 + r_3 & 0 &\le r_3 < r_2 \\
&\vdots \\
r_{n-2} &= q_n r_{n-1} + r_n & 0 &\le r_n < r_{n-1} \\
r_{n-1} &= q_{n+1} r_n + 0
\end{aligned}$$

Hence $\gcd(a,b) = \gcd(b,r_1) = \cdots = \gcd(r_n,0) = r_n$.

Exercise 1.6.1

Find $\gcd(682, 264)$.

Solution.

$$\begin{aligned} 682 &= 3 \times 264 - 110 \\ 264 &= 2 \times 110 + 44 \\ 110 &= 2 \times 44 + 22 \\ 44 &= 2 \times 22 + 0 \end{aligned}$$

Hence $\gcd(682, 264) = \boxed{22}$. $\square$

§1.7 Modular Inverse

For coprime a and n, there exists an inverse of $a \pmod n$. In other words, there exists an integer b such that

$$ab \equiv 1 \pmod n.$$

b is known as the *inverse* of a modulo n.

Proof. Since $\gcd(a,n) = 1$, we have $ab + nm = 1$ for some $b, m \in \mathbb{Z}$.
So $ab + nm \equiv 1 \pmod n$. Since $nm \equiv 0 \pmod n$, we get $ab \equiv 1 \pmod n$. □

We can find the modular inverse by reversing the Euclidean algorithm.

Exercise 1.7.1

Solve the following modular equation.

$$7x \equiv 1 \pmod{26}$$

Solution. Compute GCD and keep the tableau:

$$\gcd(26,7) = \gcd(7,5) = \gcd(5,2) = \gcd(2,1) = \gcd(1,0) = 1$$

Solve the equations for r in the tableau:

$$\begin{aligned} 26 &= 3(7) + 5 \\ 7 &= 1(5) + 2 \\ 5 &= 2(2) + 1 \end{aligned}$$

Back substitute the equations:

$$\begin{aligned} 1 &= 5 - 2 \times (7 - 1 \times 5) \\ &= (-2) \times 7 + 3 \times 5 \\ &= (-2) \times 7 + 3 \times (26 - 3 \times 7) \\ &= 3 \times 26 + (-11) \times 7 \end{aligned}$$

Modular inverse of $7 \pmod{26}$ is $-11 \pmod{26} = 15$. Hence $x = 26k + 15$ for $k \in \mathbb{Z}$. □

§1.8 Important Theorems

For $n = {p_1}^{a_1}{p_2}^{a_2}\dots{p_k}^{a_k}$,

- Number of factors:
$$\tau(n) = \prod_{i=1}^{k}(a_i + 1)$$
- Sum of factors:
$$d(n) = \prod_{i=1}^{k}(1 + p_i + \cdots + {p_i}^{a_i})$$
- Number of positive integers less than n which are coprime to n:
$$\phi(n) = n\prod_{i=1}^{k}\left(1 - \frac{1}{p_i}\right)$$
  This is known as the **totient function**.

Theorem 1.8.1: Fermat's Little Theorem

For prime p and $p \nmid a$,
$$a^{p-1} \equiv 1 \pmod{p} \tag{1.1}$$

Proof. The idea is that if we write down the sequence of numbers
$$\{a, 2a, 3a, \dots, (p-1)a\}$$
and reduce each one modulo p, the resulting sequence turns out to be a rearrangement of
$$\{1, 2, 3, \dots, p-1\}.$$

To show this, we just need to show the elements in the first sequence are all different mod p: we want to show $k_1 \not\equiv k_2 \pmod{p} \implies k_1 a \not\equiv k_2 a \pmod{p}$.

We prove by contrapositive: suppose $k_1 a \equiv k_2 a \pmod{p}$. Since $\gcd(a, p) = 1$, we can do cancellation to give $k_1 \equiv k_2 \pmod{p}$. Hence proven.

Thus if we multiply together the numbers in each sequence, the results must be identical modulo p:
$$a \times 2a \times 3a \times \cdots \times (p-1)a \equiv 1 \times 2 \times 3 \times \cdots \times (p-1) \pmod{p}$$
Collecting together the a terms yields
$$a^{p-1}(p-1)! \equiv (p-1)! \pmod{p}.$$

Since $\gcd(p, (p-1)!) = 1$, we can cancel $(p-1)!$ on both sides to give $a^{p-1} \equiv 1 \pmod{p}$. □

Exercise 1.8.1

If $n \in \mathbb{N}$ and $\gcd(n, 35) = 1$, prove that $n^{12} \equiv 1 \pmod{35}$.

Proof. By Fermat's Little Theorem,

$$n^4 \equiv 1 \pmod 5 \iff n^{12} \equiv 1 \pmod 5$$

$$n^6 \equiv 1 \pmod 7 \iff n^{12} \equiv 1 \pmod 7$$

$\therefore n^{12} \equiv 1 \pmod{35}$ □

The following theorem generalises Fermat's Little Theorem:

Theorem 1.8.2: Euler's Totient Theorem

For coprime a and n,

$$a^{\phi(n)} \equiv 1 \pmod n \tag{1.2}$$

Theorem 1.8.3: Wilson's Theorem

For odd prime p,

$$(p-1)! \equiv -1 \pmod p \tag{1.3}$$

Theorem 1.8.4: Chinese Remainder Theorem

Given k pairwise coprime positive integers n_i and arbitrary integers a_i, the system of simultaneous congruences

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ x &\equiv a_2 \pmod{n_2} \\ &\vdots \\ x &\equiv a_k \pmod{n_k} \end{aligned}$$

has a unique solution modulo $n_1 n_2 \cdots n_k$.

Proof. We first prove the case where $i = 2$. Let $n_1 = p, n_2 = q$.

Let $p_1 \equiv p^{-1} \pmod q$ and $q_1 \equiv q^{-1} \pmod p$. These must exist since p and q are coprime.

We claim that if y is an integer such that

$$y \equiv aqq_1 + bpp_1 \pmod{pq}$$

then y satisfies

$$\begin{aligned} y &\equiv aqq_1 \pmod p \\ y &\equiv a \pmod p \end{aligned}$$

Similarly,

$$\begin{aligned} y &\equiv bpp_1 \pmod q \\ y &\equiv b \pmod q \end{aligned}$$

Since $y \equiv a \pmod p$ and $y \equiv b \pmod q$, then y is a solution for x. QED.

To prove the general case, we define

$$b_i = \frac{N}{n_i}$$

where $N = n_1 \cdots n_k$ and

$$b_i{}' \equiv b_i{}^{-1} \pmod{n_i}$$

By a similar argument as before, $x = \sum_{i=1}^{n} a_i b_i b_i{}' \pmod{N}$ is a unique solution. □

Exercise 1.8.2

Find the set of values of n that satisfy the following system of modular equations.

$$\begin{cases} n \equiv 4 \pmod{5} \\ n \equiv 5 \pmod{9} \\ n \equiv 3 \pmod{11} \end{cases}$$

Solution. From the first equation, let $n = 4+5x$. Substituting this into the second equation gives $4 + 5x \equiv 5 \pmod{9}$, which reduces to $x \equiv 2 \pmod{9}$.

Let $x = 2 + 9y$. Substituting this into the third equation gives $14 + 45y \equiv 3 \pmod{11}$, which reduces to $y \equiv 0 \pmod{11}$.

Let $y = 0 + 11z$. Substituting expressions for x and y into $n = 4 + 5x$ gives $n = 14 + 495z$. Hence the set of values are $\{z \in \mathbb{Z} \mid 14 + 495z\}$.

Remark. To check our answer, using the Chinese Remainder Theorem, we can see that there is indeed a unique solution modulo $5 \times 9 \times 11 = 495$.

□

Exercise 1.8.3

Solve the following linear system of congruences:

$$\begin{cases} x \equiv 1 \pmod{2} \\ x \equiv 2 \pmod{3} \\ x \equiv 3 \pmod{5} \\ x \equiv 4 \pmod{7} \end{cases}$$

Solution. $N = 2 \times 3 \times 5 \times 7 = 210$.

□

§1.9 Orders Modulo A Prime

§1.9.1 Order

Definition 1.9.1. Let p be a prime and take $a \not\equiv 0 \pmod p$. The **order** of $a \pmod p$ is defined to be the smallest positive integer m such that

$$a^m \equiv 1 \pmod p.$$

Remark. This order is clearly finite because Fermat's Little Theorem tells us $a^{p-1} \equiv 1 \pmod p$, id est, the order of a is at most $p-1$.

Example 1.9.1

Here are some examples of each $a \pmod{11}$ and $a \pmod{13}$.

a	mod 11	mod 13
1	1	1
2	10	12
3	5	3
4	5	6
5	5	4
6	10	12
7	10	12
8	10	4
9	5	3
10	2	6
11		12
12		2

One observation you might make about this is that it seems that the orders all divide $p-1$. Obviously if $m \mid p-1$, then $a^{p-1} \equiv 1 \pmod p$ as well. The miracle of orders is that the converse of this statement is true in an even more general fashion.

Theorem 1.9.1: Fundamental Theorem of Orders

Suppose $a^N \equiv 1 \pmod p$. Then the order of $a \pmod p$ divides N.

§1.9.2 Primitive Roots

§1.10 Quadratic Residues

Definition 1.10.1. For prime p and integer a, a is a **quadratic residue** modulo p if there is a perfect square in the congruence class $a \pmod p$:

$$x^2 \equiv a \pmod p$$

Otherwise, a is a non-quadratic residue modulo p.

Some examples include

- $n^2 \equiv 0/1 \pmod 3$
- $n^2 \equiv 0/1 \pmod 4$
- $n^2 \equiv 0/1/4 \pmod 5$
- $n^2 \equiv 0/1/4 \pmod 8$

In general, we have the following fact:

Proposition 1.10.1. If p is an odd prime, the residue classes of $0^2, 1^2, \ldots, (\frac{p-1}{2})^2$ are distinct and give a complete list of the quadratic residues modulo p. So there are $\frac{p-1}{2}$ residues and $\frac{p-1}{2}$ non-residues.

Proof. They give a complete list because x^2 and $(p-x)^2$ are congruent mod p. To see that they are distinct, note that

$$\begin{aligned} x^2 \equiv y^2 \pmod p &\iff p \mid x^2 - y^2 \\ &\iff p \mid (x+y)(x-y) \\ &\iff p \mid x+y \text{ or } p \mid x-y \end{aligned}$$

which is impossible if x and y are two different members of the set $\{0, 1, \ldots, \frac{p-1}{2}\}$. □

§1.10.1 Legendre symbol

Definition 1.10.2. For any integer a and prime p, we define the **Legendre symbol** as such:

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & p \mid a \\ 1 & p \nmid a \text{ and } x^2 \equiv a \pmod p \text{ has solutions} \\ -1 & p \nmid a \text{ and } x^2 \equiv a \pmod p \text{ has no solutions} \end{cases} \tag{1.4}$$

In other words, the Legendre symbol returns the following values:

- 0 if n is divisible by p;
- 1 if n is a non-zero quadratic residue modulo p; and
- -1 if a is a non-quadratic residue modulo p.

We will discuss some properties of the Legendre Symbol:

Theorem 1.10.1: Euler's Criterion

Let p be an odd prime and let a be an integer not divisible by p, then

$$\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod p \tag{1.5}$$

Proof. By Fermat's little theorem, $a^(p-1) \equiv 1 \pmod p$, so $\left(a^{\frac{p-1}{2}}\right)^2 - 1 \equiv 0 (modp)$.

The LHS can be factorised, and one of them must be divisible by p due to the above.

If a is a non-zero quadratic residue modulo p, then $a \equiv x^2 \pmod p$ where $x \neq 0$; taking the $\frac{p-1}{2}$-th power on both sides give $a^{\frac{p-1}{2}} \equiv 1 \pmod p$.

We claim that for all non-quadratic residues a we have $a^{\frac{p-1}{2}} \equiv -1 \pmod p$. This is sufficient as its contrapositive gives the following statement: if $a^{\frac{p-1}{2}} \equiv 1 \pmod p$, then a is a nonzero quadratic residue modulo p. So the iff case is derived automatically.

For this one we need the following fact from abstract algebra:

Lemma 1.10.1. Let $P(x)$ be a polynomial of degree n, then $P(x) \equiv 0 \pmod p$ has at most n roots modulo p.

Now consider the modular equation $x^{\frac{p-1}{2}} \equiv 1 \pmod p$.

As we've proven above, all non-zero quadratic residues modulo p are roots to the above equation.

However, we can actually show that there are at least $\frac{p-1}{2}$ quadratic residues by direct constructions: Consider the set of congruence classes modulo p denoted by

Now we consider $\{1^2, 2^2, \ldots, (p-1)^2\}$. If the congruence class $a \pmod p$ lies in the above set, then by construction we know that a is a quadratic residue modulo p. Conversely, if a is a quadratic residue modulo p, then some of the elements in $\{1^2, 2^2, \dots, (p-1)^2\}$ correspond to a.

However, there are in fact at most two of them that are actually a. This is because the element must satisfy $x^2 \equiv a \pmod{p}$, and $x^2 - a$ is a polynomial of degree 2. So there are at most two roots, meaning at most two elements. In fact we know exactly which of the elements are the same: $x^2 \equiv (p-x)^2 \pmod{p}$. So the set of quadratic residues is precisely the set $\{1^2, 2^2, \dots, (\frac{p-1}{2})^2\}$.

And thus, there are at least $\frac{p-1}{2}$ quadratic residues modulo p (Note that at this point we still haven't proven the fact that $\{1^2, 2^2, \dots, (\frac{p-1}{2})^2\}$ accounts for all quadratic residues; we're very close though)

The final step is to combine the two: 1. $x^{\frac{p-1}{2}} \equiv 1 \pmod{p}$ has at most $\frac{p-1}{2}$ roots 2. Since $x^2 \equiv a$ has at most 2 roots, there are at least $\frac{p-1}{2}$ quadratic residues, each satisfying $x^{\frac{p-1}{2}} \equiv 1 \pmod{p}$.

Therefore, if a is a non-quadratic residue modulo p, the congruence class a (mod p) is distinct from all elements in $\{1^2, \dots, (p-1)^2\}$ and thus together with the at least $\frac{p-1}{2}$ quadratic residues in $\{1^2, \dots, (p-1)^2\}$, there will be more than (p-1)/2 distinct roots in $x^{\frac{p-1}{2}} \equiv 1 \pmod{p}$, a contradiction.

This finally proves that $a^{\frac{p-1}{2}} \equiv -1 \pmod{p}$, and everything falls into place due to this. □

Proposition 1.10.2 (Multiplicity)**.** For prime p and integers a, b not divisible by p,

$$\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right)$$

Proof. This follows from Euler's Criterion.

1b) is a direct consequence of 1a), although there is in fact a way to prove this directly : The nontrivial deduction is that the product of two nonquadratic residues must be a quadratic residue : First note that the proof in 1a) actually provided a proof for 1c) : Now that I think about it, this feels strange because we are still referring to the proof of 1a) The thing is that, if we know 1c), then we can show 1b) □

Proposition 1.10.3. For odd prime p, then there are an equal number of non-zero quadratic residues and non-quadratic residues modulo p.

Theorem 1.10.2: Gauss' Lemma

For odd prime p and integer a where $p \nmid a$,

$$\left(\frac{a}{p}\right) = (-1)^n$$

where n is the number of integers $0 < k < \frac{p}{2}$ such that $k \cdot a$ belongs to the congruence class $m \pmod{p}$ where $\frac{p}{2} < m < p$.

Theorem 1.10.3: Second Supplementary Law

For odd prime p,

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$$

Theorem 1.10.4: Eisenstein's Lemma

For distinct odd primes p and q,

$$\left(\frac{q}{p}\right) = (-1)^{\alpha} \tag{1.6}$$

where

$$\alpha = \sum_{k=1}^{\frac{p-1}{2}} \left\lfloor \frac{kq}{p} \right\rfloor$$

Theorem 1.10.5: Law of Quadratic Reciprocity

For distinct odd primes p and q,

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}} \tag{1.7}$$

Proof. Evaluate the product $a \cdot 2a \cdots \frac{p-1}{2}a \pmod p$ in two different ways. By rearranging terms, we get

$$a^{\frac{p-1}{2}}\left(\frac{p-1}{2}\right)!$$

But the product can also be evaluated by noticing that each of the distinct integers in Gauss's lemma is either x or $p-x$ for $1 \leq x \leq \frac{p-1}{2}$, and showing that each of the x's is distinct. Multiplying them together modulo p gives $(\frac{p-1}{2})!$ multiplied by n minus signs due to the number of $p-x$ terms of which there are n, hence the sign is $(-1)^n$. The result follows by Euler's criterion and cancelling the $(\frac{p-1}{2})!$. □

Special cases

- $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$
- $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$
- $\left(\frac{-3}{p}\right) = \begin{cases} 1 & p = 1 \pmod 6 \\ -1 & p = 5 \pmod 6 \end{cases}$
- $\left(\frac{5}{p}\right) = \begin{cases} 1 & p = 1, 9 \pmod{10} \\ -1 & p = 3, 7 \pmod{10} \end{cases}$

§1.11 Exercises

SMO Open 2018 Q18

Problem 1 (SMO Open 2018 Q21)**.** Determine the largest value of the expression $2^{k_1} + 2^{k_2} + \cdots + 2^{k_{498}}$, where for each $i = 1, 2, \ldots, 498$, k_i is an integer, $1 \le k_i \le 507$, and $k + 1 + k_2 + \cdots + k_{498} = 507$.

Solution. □

SMO Open 2017 Q23

SMO Open 2016 Q18

SMO Open 2016 Q19

SMO Open 2013 Q15

SMO Open 2013 Q17

Problem 2 (SMO Open 2006 Q18)**.** Find the largest integer n such that n is a divisor of $a^5 - a$ for all integers n.

Solution. Factorising,

$$a^5 - a = a(a-1)(a+1)(a^2+1).$$

It is clear that $2 \mid a^5 - a$ and $3 \mid a^5 - a$. We can show that $5 \mid a^5 - a$ by considering the five cases of $a \equiv i \pmod 5$, $i = 0, 1, 2, 3, 4$. Thus $30 \mid a^5 - a$. When $a = 2$, we have $a^5 - a = 30$. Thus the maximum n is 30. □

Problem 3 (SMO Open 2005 Q1)**.** Find the last three digits of $9^{100} - 1$.

Solution.

$$9^{100} - 1 = (1-10)^{100} - 1 = 1 - \binom{100}{1}10^1 + \cdots + \binom{100}{100}10^{100} - 1 = 1000k$$

for some integer k. Thus the last three digits are 000. □

Problem 4 (IMO 2023 P1)**.** Determine all composite integers $n > 1$ that satisfy the following property: if $d_1, d_2, \ldots, d_k$ are all the positive divisors of n with $1 = d_1 < d_2 < \cdots < d_k = n$, then d_i divides $d_{i+1} + d_{i+2}$ for every $1 \le i \le k - 2$.

Solution. If n has at least 2 prime divisors, WLOG let $p < q$ be the smallest two of these primes. Then the ordered tuple of divisors is of the form $(1, p, p^2, \ldots, p^a, q \ldots, n)$ for some integer $a \ge 1$.

To prove this claim, note that p is the smallest prime that divides n, so it is the smallest divisor not equal to 1, meaning the first 2 divisors are 1 and p. Furthermore, the smallest divisor of n that is not equal to a power of p (i.e. not equal to $(1, p, p^2 \ldots)$ is equal to q. This is because all other divisors either include a prime z different from both q and p, which is larger than q (since q and p are the smallest two prime divisors of n), or don't include a different prime z. In the first case, since $z > q$, the divisor is larger than q. In the second case, all divisors divisible by q^2 are also larger than q, and otherwise are of the form $p^x \cdot q^1$ or p^x for some non-negative integer x. If the divisor is of the form p^x, then it is a power of p. If it is of the form $p^x \cdot q^1$, the smallest of these factors is $p^0 \cdot q^1 = q$. Therefore, (in the case where 2 or more primes divide n) the ordered tuple of divisors is of the form $(1, p, p^2 \ldots, p^a, q \ldots, n)$ for some integer $a \geq 1$, since after each divisor p^x, the next smallest divisor is either p^{x+1} or simply q.

If $a \geq 2$, the condition fails. This is because $p^{a-1} \nmid p^a + q$, since p^a is divisible by p^{a-1}, but q is not since it is a prime different from p. If $a = 1$, then $p^{a-1} = p^0 = 1$, which does divide q. Therefore a must equal 1 for the condition to be satisfied in this case. However, we know that the ordered list of divisors satisfies $d_i \cdot d_{k+1-i} = n$, meaning since the first 3 divisors are $(1, p, q)$, then the last 3 divisors are $(\frac{n}{q}, \frac{n}{p}, n)$, so $(\frac{n}{q})$ must divide $(\frac{n}{p} + n)$. The fraction $\frac{(\frac{n}{p}+n)}{(\frac{n}{q})} = \frac{(\frac{1}{p}+1)}{(\frac{1}{q})} = (\frac{q}{p}) + q$ which is clearly not an integer since q is an integer, but $\frac{q}{p}$ is not an integer, so $(\frac{q}{p}) + q$ is not an integer. Therefore the condition fails specifically for the final 3 divisors in the list in this case, meaning n can never have 2 or more prime divisors.

When $n = p^x$, it is easy to verify this works for all primes p and all $x \geq 2$, since $p^y \mid (p^{y+1} + p^{y+2})$, and the divisors are ordered as $1, p, p^2, \ldots, p^x$. $\square$

Problem 5 (AUSTRALIA 2020 Q2)**.** Amy and Ben play the following game. Initially, there are three piles, each containing 2020 stones. The players take turns to make a move, with Amy going first. Each move consists of choosing one of the piles available, removing the unchosen pile(s) from the game, and then dividing the chosen pile into 2 or 3 non-empty piles. A player loses the game if they are unable to make a move.

Prove that Ben can always win the game, no matter how Amy plays.

Proof. Call a pile *perilous* if the number of stones in it is one more than a multiple of three, and *safe* otherwise. Ben has a winning strategy by ensuring that he only leaves Amy perilous piles. Ben wins because the number of stones is strictly decreasing, and eventually Amy will be left with two or three piles each with just one stone.

To see that this is a winning strategy, we prove that Ben can always leave Amy with only perilous piles, and that under such circumstances, Amy must always leave Ben with at least one safe pile.

On Amy's turn, whenever all piles are perilous it is impossible to choose one such perilous pile and divide it into two or three perilous piles by virtue of the fact that $1+1 \not\equiv 1 \pmod 3$ and $1+1+1 \not\equiv 1 \pmod 3$. Thus Amy must leave Ben with at least one safe pile.

On Ben's turn, whenever one of the piles is safe, he can divide it into two or three piles, each of which are safe, by virtue of the fact that $2 \equiv 1+1 \pmod 3$ and $0 \equiv 1+1+1 \pmod 3$. □

Problem 6 (CANADA 1969 P7)**.** Show that there are no integers a, b, c for which

$$a^2 + b^2 - 8c = 6.$$

Proof. Using quadratic residues, all perfect squares are equivalent to $0, 1, 4 \pmod 8$. Hence, the problem statement is equivalent to $a^2 + b^2 \equiv 6 \pmod 8$. It is impossible to obtain a sum of 6 with two of $0, 1, 4$, so our proof is complete. $\square$

Problem 7 (ITALY 2011)**.** Given that p is a prime number, find integer solutions to

$$n^3 = p^2 - p - 1.$$

Solution. It is easy to see that $n < p$.

$$\begin{aligned} p^2 - p &= n^3 + 1 \\ p(p-1) &= (n+1)(n^2 - n + 1) \end{aligned}$$

Since p is prime, $p \mid n+1$ or $p \mid n^2 - n + 1$.

Case 1: $p \mid n+1$

Since $n < p$, thus $n + 1 \leq p$. Hence, $n + 1 = p$. Substituting this into the original equation gives us

$$n^3 = n^2 + n - 1$$

$$(n-1)^2(n+1) = 0$$

$$n = 1$$

$\therefore (n, p) = (1, 2)$.

Case 2: $p \mid n^2 - n + 1$

Let $n^2 - n + 1 = kp$ where k is a positive integer. Then

$$\begin{aligned} p(p-1) &= (n+1)(n^2 - n + 1) \\ &= kp(n+1) \\ p - 1 &= k(n+1) \\ p &= kn + k + 1 \\ n^2 - n + 1 &= k(kn + k + 1) \\ n^2 - n(1 + k^2) - (k^2 + k - 1) &= 0 \end{aligned}$$

Taking discriminant,

$$\Delta = (1 + k^2)^2 + 4(k^2 + k - 1) = k^4 + 6k^2 + 4k - 3$$

which is a perfect square.

Let $f(k) = k^4 + 6k^2 + 4k - 3$.

We find that $k = 3$ via trial and error, then $n = 11$, $p = 37$.

For $k \geq 4$, we can prove that $f(k)$ is not a perfect square; in fact, $f(k)$ lies between two consecutive perfect squares, as shown below:

$$(k^2 + 3)^2 < f(k) < (k^2 + 4)^2$$

which can be easily shown by expanding the terms.

$\therefore (n, p) = (11, 37)$ □

Problem 8 (USAMO 2003)**.** Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.

Proof. This is immediate by induction on n. For $n = 1$ we take 5; moving forward if M is a working n-digit number then exactly one of

$$\begin{aligned} N_1 &= 10^n + M \\ N_3 &= 3 \cdot 10^n + M \\ N_5 &= 5 \cdot 10^n + M \\ N_7 &= 7 \cdot 10^n + M \\ N_9 &= 9 \cdot 10^n + M \end{aligned}$$

is divisible by 5^{n+1}; as they are all divisible by 5^n and $\frac{N_k}{5^n}$ are all distinct. □

Problem 9 (ALBANIA/2009)**.** Find all the natural numbers m, n such that $1+5\cdot 2^m = n^2$.

Solution. We have $5 \cdot 2^m = (n-1)(n+1) \implies n-1 = 2^k$ or $n+1 = 2^k$

Case 1: $n-1 = 2^k$

This implies $n+1 = 2^k + 2$

But $5 \mid 2^k + 2$, $2^k + 2 = 2^t \cdot 5 \implies t = 1, k = 3 \implies n = 9, m = 4$

Case 2: $n+1 = 2^k$

This implies $n-1 = 2^k - 2$. But $5 \mid 2^k - 2$, $2^k - 2 = 2^t \cdot 5 \implies t = 1, 2^k = 12$ which has no integer solution for k.

$\therefore (m, n) = (4, 9)$ is a unique solution. $\square$

Problem 10 (NJC H3M 2019 Prelim Q4)**.** Let p be a prime number. Show that

$$\binom{2p}{p} \equiv 2 \pmod{p}$$

Proof. We first express $\binom{2p}{p}$ as

$$\binom{2p}{p} = \frac{(2p)!}{p!p!} = \frac{(2p)(2p-1)(2p-2)\cdots(p+1)}{(p)(p-1)(p-2)\cdots 1}$$

Note that $2p$ and p will cancel each other out to give 2. We hence need to prove the remaining thing is congruent to $1 \pmod{p}$.

$$\frac{(2p-1)(2p-2)\cdots(p+1)}{(p-1)(p-2)\cdots 1} = \binom{2p-1}{p-1}$$

which is an integer, so

$$(p-1)! \mid (2p-1)(2p-2)\cdots(p+1)$$

We can hence write

$$(2p-1)(2p-2)\cdots(p+1) = k(p-1)! \quad k \in \mathbb{Z}^+$$

Note that since p is prime,

$$(p+1)(p+2)\cdots(2p-1) \equiv (1)(2)\cdots(p-1) = (p-1)! \pmod{p}$$

Hence,

$$p \mid (2p-1)(2p-2)\cdots(p+1) - (p-1)! \implies p \mid k(p-1)! - (p-1)! = (k-1)(p-1)!$$

Since p is prime, $\gcd(p, (p-1)!) = 1$ which implies $p \mid k-1$ or $k \equiv 1 \pmod{p}$.

The rest follows easily. □

Problem 11 (IMO 1988 P6)**.** Let a and b be positive integers such that $ab + 1$ divides $a^2 + b^2$. Show that $\dfrac{a^2 + b^2}{ab + 1}$ is the square of an integer.

Proof. We proceed by way of contradiction, using a method known as **Vieta Jumping**.

WLOG, let $a \geq b$ and fix c to be the nonsquare positive integer such that such that $\frac{a^2+b^2}{ab+1} = c$, or $a^2 + b^2 = c(ab + 1)$. Choose a pair (a, b) out of all valid pairs such that $a + b$ is minimized. Expanding and rearranging,

$$P(a) = a^2 + a(-bc) + b^2 - c = 0.$$

This quadratic has two roots, r_1 and r_2, such that

$$(a - r_1)(a - r_2) = P(a) = 0.$$

WLOG, let $r_1 = a$. By Vieta's, **(1)** $r_2 = bc - a$, and **(2)** $r_2 = \frac{b^2-c}{a}$. From **(1)**, r_2 is an integer, because both b and c are integers.

From **(2)**, r_2 is nonzero since c is not square, from our assumption.

We can plug in r_2 for a in the original expression, because $P(r_2) = P(a) = 0$, yielding $c = \frac{r_2^2+b^2}{r_2 b+1}$. If $c > 0$, then $r_2 b + 1 > 0$, and $r_2 b + 1 \neq 0$, and because $b > 0, r_2$ is a positive integer.

We construct the following inequalities: $r_2 = \frac{b^2-c}{a} < a$, since c is positive. Adding b, $r_2 + b < a + b$, contradicting the minimality of $a + b$. $\square$

2 Diophantine Equations

Definition 2.0.1. A **Diophantine equation** is a polynomial equation with 2 or more integer unknowns.

§2.1 Linear Diophantine Equations

A **linear Diophantine equation** is an equation with 2 or more integer unknowns and the integer unknowns are each to at most degree of 1. Linear Diophantine equation in two variables takes the form of

$$ax + by = c$$

where $x, y \in \mathbb{Z}$ and a, b, c are integer constants. x and y are unknown variables.

Such equations can be solved completely, and the first known solution was constructed by Brahmagupta, which makes use of the Euclidean algorithm, as we will see later.

§2.1.1 Homogeneous Linear Diophantine Equations

A special case of linear Diophantine equations is **homogeneous linear Diophantine equations**, i.e. when $c = 0$, which take the form

$$ax + by = 0$$

where $x, y \in \mathbb{Z}$. Note that $(x, y) = (0, 0)$ is a solution, known as the *trivial solution* for this equation.

Homogeneous linear Diophantine equations can be easily solved: If $d = gcd(a, b)$, then the complete family of solutions to the above equation is

$$x = \frac{b}{d}k, \quad y = -\frac{a}{d}k$$

for $k \in \mathbb{Z}$.

Exercise 2.1.1

Solve the homogeneous linear Diophantine equation

$$6x + 9y = 0$$

where $x, y \in \mathbb{Z}$.

Solution. Note that $\gcd(6, 9) = 3$. Hence the solutions are

$$x = \frac{9k}{3} = 3k \quad \text{and} \quad y = -\frac{6k}{3} = -2k$$

with $k \in \mathbb{Z}$. □

§2.1.2 Non-homogeneous Linear Diophantine Equations

We can use the following steps to solve non-homogeneous linear Diophantine equations (of the form $ax + by = c$, $x, y \in \mathbb{Z}$).

1. Determine $\gcd(a, b)$.

   Let $d = \gcd(a, b)$. For smaller a, b you can simply determine d by manually checking factors of a and b; for larger a, b use the Euclidean algorithm to determine d.

2. Check that $d \mid c$.

   If YES, continue on. If NO, stop as there are no solutions.

3. Find a particular solution to $ax + by = c$.

   by first finding x_0 and y_0 such that $ax + by = d$. Suppose $x = cdx0$ and $y = cdy0$.

4. Use a change of variables.

   Let $u = x - cdx0$ and $v = y - cdy0$, then we will see that $au + bv = 0$ (important to check your results).

5. Solve $au + bv = 0$.

   That is: $u = -\frac{b}{d}m$ and $v = \frac{a}{d}m$, $m \in \mathbb{Z}$.

6. Substitute for u and v.

   Thus the general solutions are $x - cdx0 = -bdmandy - cdy0 = adm$, or $(x, y) = ()$, $m \in \mathbb{Z}$.

One famous problem is the McNugget Numbers. McNugget Numbers (Henri Picciotto, 1980s) I Original boxes had 6, 9, and 20 nuggets. I Worked out the largest non-McNugget number on a napkin. I g(6, 9, 20) = 43.

A Diophantine equation in the form $ax + by = c$ is known as a linear combination. There will always be an infinite number of solutions when $\gcd(a, b) = 1$ and $\gcd(a, b) \mid c$.

Theorem 2.1.1: Bezout's Lemma

For non-zero integers a and b, let $d = \gcd(a,b)$. Then there exists integers s and t that satisfy
$$sa + tb = d.$$

An important case of Bezout's Lemma is when a, b are coprime:

$$ax + by = 1, x, y \in \mathbb{Z} \iff a, b \text{ coprime}$$

Exercise 2.1.2

Find integers x and y that satisfy

$$102x + 38y = 2.$$

Solution. Apply the extended Euclidean algorithm on a and b to calculate $\gcd(a,b)$:

$$\begin{aligned} 102 &= 2 \times 38 + 26 \\ 38 &= 1 \times 26 + 12 \\ 26 &= 2 \times 12 + 2 \\ 12 &= 6 \times 2 + 0 \\ 6 &= 3 \times 2 + 0 \end{aligned}$$

Work backwards and substitute the numbers from above:

$$\begin{aligned} 2 &= 26 - 2 \times 12 \\ &= 3 \times 26 - 2 \times 38 \\ &= 3 \times 102 - 8 \times 38 \end{aligned}$$

Hence $x = 3$, $y = -8$. □

All solutions of linear Diophantine equations:

Theorem 2.1.2

If (x_0, y_0) is a solution of $ax + by = n$, then all solutions are given by

$$\{(x,y) \mid x = x_0 + bt, y = y_0 - at, t \in \mathbb{Z}\}$$

For three variables in the equation $ax + by + cz = d$, this is the equation of a plane, instead of a line.

§2.1.3 Chicken Mcnugget Theorem

Mcdonalds once offered Chicken Mcnuggets in sets of 9 and 20 only. A question prompted from this is, assuming you only buy sets of 9 and 20 Chicken Mcnuggets and do not eat/add any during this process, what is the largest amount of Mcnuggets that is impossible to make? It turns out the answer is 151, which we will explore in this section.

Theorem 2.1.3: Chicken Mcnugget Theorem

For relatively prime naturals m and n, the largest impossible sum of m and n (i.e. largest number not expressable in the form $mx + ny$ for non-negative integer x and y) is $mn - m - n$.

Proof. Because the equation is symmetric, WLOG assume that $n \geq m$.

Assume that $mn - m - n = mx + ny$. Taking $\mod m$, we arrive at

$$ny \equiv -n \pmod m \implies y \equiv -1(modm)$$

This implies that $y \geq m - 1$, however, this gives

$$mx + ny \geq mx + mn - m > mn - m - n$$

which is a contradiction. Now, we prove that

$$mn - m - n + k = mx + ny, \quad k \in \{1, 2, 3, \dots, m\}.$$

The reason for this is that for $k = k_1 > m$, then we can repeatedly add m to the reduced value of $k_1 \mod m$ until we reach k_1. Our goal is to prove that for every k, there exists an x such that

- x is an integer.
- x is a non-negative integer.

Taking the equation $\mod m$ brings us to

$$n(y + 1) \equiv k \pmod m \implies y \equiv kn^{-1} - 1 \pmod m$$

Using this value of y produces the first desired outcome. For the second, we must have $mn - m - n + k - ny \geq 0$. For $y = m - y_0$ and $m \geq y_0 \geq 2$, we get

$$mn - m - n + k - ny = (y_0 - 1)n - m + k > 0.$$

For $y_0 = 1$, we have

$$y \equiv -1 \pmod m \implies kn^{-1} - 1 \equiv -1 \pmod m \implies k \equiv 0 \pmod m$$

Therefore, $k = m$, and we get

$$mn - m - n + k - ny = mn - m - n + m - n(m - 1) = 0 \implies x = 0$$

Therefore, we have proven our desired statement, and we are done.

□

§2.2 Pythagorean Triples

Definition 2.2.1. A **Pythagorean triple** is a triplet (a, b, c), where a, b, c are positive integers that satisfy $a^2 + b^2 = c^2$.

The smallest and best-known Pythagorean triple is $(a, b, c) = (3, 4, 5)$. The right triangle having these side lengths is sometimes called the $3 - 4 - 5$ triangle.

In fact, all Pythagorean triples can be expressed in the form of

$$a = k(m^2 - n^2) \quad b = k(2mn) \quad c = k(m^2 + n^2)$$

Exercise 2.2.1

Prove that there is one and only one Pythagorean triple (a, b, c) such that a, b, c are consecutive integers.

Proof. We need to prove two parts: existence ("one") and uniqueness ("only one").

Existence:

Take $(a, b, c) = (3, 4, 5)$, which indeed satisfies $a^2 + b^2 = c^2$.

Uniqueness:

Suppose a, b, c are consecutive. Then $b = a + 1$ and $c = a + 2$. By Pythagoras' theorem we have

$$a^2 + (a + 1)^2 = (a + 2)^2$$

which simplifies down to $(a - 3)(a + 1) = 0$, so $a = 3$ or $a = -1$. Since Pythagorean triple consists of positive integers, a can only be 3. □

Exercise 2.2.2

Prove that there are infinitely many Pythagorean triples.

Proof. This is an existential statement. We prove this by construction.

Suppose (a, b, c) is a Pythagorean triple (Pythagorean triples exist, one example being $(a, b, c) = (3, 4, 5)$ as shown earlier). Then $a^2 + b^2 = c^2$.

Let k be any positive integer. Note that

$$(ka)^2 + (kb)^2 = k^2a^2 + k^2b^2 = k^2(a^2 + b^2) = k^2c^2 = (kc)^2.$$

Thus (ka, kb, kc) is also a Pythagorean triple.

Since there are infinitely many k, we have infinitely many Pythagorean triples (ka, kb, kc). □

§2.3 Pell's Equation

Theorem 2.3.1: Pell's Equation

If $n > 0$ is not a perfect square, then the equation

$$x^2 - ny^2 = 1$$

has infinitely many solutions.

Note that $(x, y) = (1, 0)$ is a trivial solution.

Steps:

1. Find one non-trivial solution (x, y).
2. Let $\alpha^n = (x + y\sqrt{d})^n$ where $n = 2, 3, \ldots$. The coefficients of the integer and square root give us the values of x and y respectively.

Exercise 2.3.1

Find positive integers x and y that satisfy

$$x^2 - 2y^2 = 1.$$

Solution. We first observe that $(x, y) = (3, 2)$ is a solution.

$$\alpha = (3 + 2\sqrt{2})$$
$$\alpha^n = (3 + 2\sqrt{2})^n$$

For $n = 2$,

$$\alpha^2 = (3 + 2\sqrt{2})^2$$
$$= 17 + 12\sqrt{2}$$

From this, we deduce that another solution is $(x, y) = (17, 12)$.

Simply repeat the above method to find further solutions. □

Theorem 2.3.2: Frobenius coin problem

What is the largest number n such that $ax + by = n$ has no solutions for $x, y \geq 0$?
Let the Frobenius Number be $n = g(a, b)$.
The claim is

$$g(a, b) = ab - a - b.$$

Proof. □

Theorem 2.3.3: Fermat's Last Theorem

For $n > 2$, there are no non-zero solutions to

$$a^n + b^n = c^n.$$

The proof is quite complicated and can be found here.

Part II

Algebra

3 Basic Algebra

§3.1 Alegbraic Manipulation

This book assumes the reader should be familiar with basic algebraic manipulation. Some common factorisations include

- Difference of squares:
$$a^2 - b^2 = (a+b)(a-b)$$
- Sum of squares:
$$a^2 + b^2 = (a+b)^2 - 2ab$$
- Sum of cubes:
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$
- Difference of cubes:
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$
- Generalised formulae:
$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}) \quad \forall n \in \mathbb{N}$$
$$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + \cdots - ab^{n-2} + b^{n-1}) \quad \forall \text{ odd } n \in \mathbb{N}$$
- Generalised expansion of square:
$$(a_1+a_2+\cdots+a_n)^2 = (a_1^2+a_2^2+\cdots+a_n^2)+(2a_1a_2+\cdots+2a_1a_n)+(2a_2a_3+\cdots+2a_2a_n)+\cdots+2a_{n-1}a_n$$
- Useful ones:
$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

Exercise 3.1.1

Evaluate the expression
$$(2+1)(2^2+1)(2^4+1)\cdots(2^{2^{10}}+1)+1.$$

Solution. By using the formula $(a-b)(a+b) = a^2 - b^2$ repeatedly, we have

$$\begin{aligned}
&(2+1)(2^2+1)(2^4+1)\cdots(2^{2^{10}}+1)+1 \\
&= (2-1)(2+1)(2^2+1)(2^4+1)\cdots(2^{2^{10}}+1)+1 \\
&= ((2^{2^{10}})^2 - 1) + 1 = \boxed{2^{2048}}
\end{aligned}$$

Note the trick here: to multiply the expression by $(2-1)$ □

Exercise 3.1.2

Given that the real numbers x, y and z satisfy the system of equations

$$\begin{cases} x+y+z = 6 \\ x^2+y^2+z^2 = 26 \\ x^3+y^3+z^3 = 90 \end{cases}$$

Find the values of xyz and $x^4+y^4+z^4$.

Solution. $(x+y+z)^2 = (x^2+y^2+z^2) + 2(xy+yz+zx)$ implies that $xy+yz+zx = 5$. Since $x^3+y^3+z^3-3xyz = (x+y+z)[(x^2+y^2+z^2)-(xy+yz+zx)]$, $xy+yz+zx = \boxed{-12}$. Further, by completing squares,

$$\begin{aligned}
x^4+y^4+z^4 &= (x^2+y^2+z^2)^2 - 2(x^2y^2+y^2z^2+z^2x^2) \\
&= (x^2+y^2+z^2)^2 - 2[(xy+yz+zx)^2 - 2(xy^2z+yz^2x+x^2yz)] \\
&= (x^2+y^2+z^2)^2 - 2[(xy+yz+zx)^2 - 2xyz(x+y+z)] = \boxed{338}
\end{aligned}$$

□

Since $(a-b)^2 \geq 0$, we have $a^2+b^2 \geq 2ab$. Adding a^2+b^2 to both sides gives us

$$2(a^2+b^2) \geq (a+b)^2 \tag{3.1}$$

Expanding this to three and four variables,

$$3(a+b+c)^2 \geq (a+b+c)^2 \tag{3.2}$$

and

$$4(a^2+b^2+c^2+d^2) \geq (a+b+c+d)^2 \tag{3.3}$$

Remark. This can be easily seen by applying Titu's Lemma.

The above equations are useful in finding the minimum value of a^2+b^2 given $a+b$, or finding the maximum value of $a+b$ given a^2+b^2.

§3.2 Polynomials

A **polynomial**, in terms of the variable x, takes the form of

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where a_i are the **coefficients**, a_0 is the **constant term**. The highest power n is the **degree** of the polynomial denoted by $\deg P(x)$.

§3.2.1 Finding Roots of Polynomials

Theorem 3.2.1: Quadratic formula

For $a, b, c \in \mathbb{R}$, $a \neq 0$, the quadratic equation $ax^2 + bx + c = 0$ has solutions

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.4}$$

Proof. The proof is quite simple; it can be done by completing the square. □

Let Δ denote the **discriminant**, then $\Delta = b^2 - 4ac$.

- For $\Delta < 0$, the 2 roots are complex and conjugates to each other.
- For $\Delta = 0$, the 2 roots are real and repeated.
- For $\Delta > 0$, the 2 roots are real and distinct.

Theorem 3.2.2: Vieta's relations

For polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ with complex coefficients with roots $r_1, r_2, \ldots, r_n$,

$$\begin{aligned} r_1 + r_2 + \cdots + r_n &= -\frac{a_{n-1}}{a_n} \\ r_1 r_2 + r_2 r_3 + \cdots + r_{n-1} r_n &= \frac{a_{n-2}}{a_n} \\ &\vdots \\ r_1 r_2 \cdots r_n &= (-1)^n \frac{a_0}{a_n} \end{aligned} \tag{3.5}$$

The important ones are

- Sum of roots: $-\dfrac{a_{n-1}}{a_n}$
- Product of roots: $(-1)^n \dfrac{a_0}{a_n}$

For a polynomial with integer coefficients a_i, $x = \frac{p}{q}$ is a rational root, where $p \mid a_0$ and $q \mid a_n$.

Theorem 3.2.3: Division Algorithm

If $P(x)$ and $D(x)$ are polynomials where $Q(x) \neq 0$ and $\deg D(x) < \deg P(x)$, then

$$P(x) = D(x)Q(x) + R(x) \tag{3.6}$$

Theorem 3.2.4: Remainder Theorem

If the polynomial $P(x)$ is divided by $x - c$, then the remainder is $P(c)$.

Theorem 3.2.5: Factor Theorem

Let $P(x)$ be a polynomial. $f(c) = 0$ if and only if $x - c$ is a factor of $f(x)$.

Theorem 3.2.6: Linear Factorisation Theorem

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $n \geq 1, a_n \neq 0$, then

$$f(x) = a_n(x - c_1)(x - c_2)\ldots(x - c_n)$$

where $c_1, c_2, \ldots c_n$ are complex numbers.

A polynomial equation of degree n has n roots, counting multiple roots (multiplicities) separately.

Imaginary roots occur in conjugate pairs; if $a + bi$ is a root $(b \neq 0)$, then the imaginary number $a - bi$ is also a root.

Theorem 3.2.7: Fundamental Theorem of Algebra

A polynomial of degree $n > 0$, with real or complex coefficients, has n complex roots (some may be repeated).

Theorem 3.2.8: Lagrange interpolation formula

The polynomial Q that solves $Q(a_0) = 1, Q(a_1) = 0, Q(a_2) = 0, \ldots, Q(a_n) = 0$ is

$$\frac{(x - a_1)(x - a_2)\cdots(x - a_n)}{(a_0 - a_1)(a_0 - a_2)\cdots(a_0 - a_n)}.$$

We can take a linear combination of these types of polynomials to create a polynomial where $P(a_0) = b_0, P(a_1) = b_1, \ldots, P(a_n) = b_n$.

Theorem 3.2.9: Gauss's lemma

If a polynomial with integer coefficients can be factored with rational coefficients, then it can also be factored with integer coefficients.

§3.2.2 Combinatorial Nullstellensatz

Let

$$P(x) = cx^d + \cdots$$

be a polynomial with degree d. For any set S such that $|S| > d$, there will be an element $a \in S$ such that

$$P(a) \neq 0.$$

More generally, let

$$P(x_1, x_2, \ldots, x_k) = cx_1^{d_1} x_2^{d_2} \cdots x_k^{d_k} + \cdots$$

where c is non-zero be a polynomial with total degree $d_1 + d_2 + \cdots + d_k$. Then for any sets $S_1, \ldots, S_k$ such that $|S_1| > d_1, \ldots, |S_k| > d_k$, there exists $a_1 \in S_1, \ldots, a_k \in S_k$ such that

$$P(a_1, \ldots, a_k) \neq 0.$$

§3.3 Absolute Value Equations

Common problem-solving techniques:

- Squaring both sides
- Casework: solve each case for x
- Sketching graph

Exercise 3.3.1

Find all real values of x such that

$$|x+2|+|2x+6|+|3x-3|=12.$$

Solution. The three turning points are at $x=-2$, $x=-3$ and $x=1$. Hence we just need to check the following cases:

- $x \leq 3$
- $-3 < x \leq -2$
- $-2 < x \leq 1$
- $x > 1$

Solving all the cases gives us $\boxed{x=-\frac{5}{2},\ x=\frac{7}{6}}$ □

Exercise 3.3.2

How many real solutions x are there to the equation $x|x|+1=3|x|$?

Solution. Two cases: either $x \geq 0$ and $x^2+1=3x$, or $x<0$ and $-x^2+1=-3x$. The first case has solutions $\frac{3 \pm \sqrt{5}}{2}$ which are both positive. The second case has solutions $\frac{3 \pm \sqrt{13}}{2}$ and only one of these is negative. So there are $\boxed{3}$ solutions in total. □

§3.4 Logarithms

Properties of logarithms

1. $\log_a b^n = n \log_a b$
2. $\log_a b + \log_a c = \log_a bc$
3. $\log_a b - \log_a c = \log_a \frac{b}{c}$
4. $\frac{\log_a b}{\log_a c} = \log_c b$
5. $\log_{a^n} b^n = \log_a b$

One useful identity is the chain rule for logarithms:

$$(\log_a b)(\log_b c) = \log_c a$$

which can be easily proven by changing of base.

Exercise 3.4.1

Find the sum

$$\log \frac{1}{2} + \log \frac{2}{3} + \log \frac{3}{4} + \cdots + \frac{99}{100}.$$

Solution. Using the sum of logarithms,

$$S = \log\left(\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{99}{100}\right) = \log \frac{1}{100} = \log 10^{-2} = \boxed{-2}$$

□

Another useful property is

$$x^{\log_x y} = y$$

§3.5 Exercises

Problem 12. Let a, b, c be distinct non-zero real numbers such that

$$a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a}.$$

Prove that $|abc| = 1$.

Proof. From the given conditions it follows that

$$a - b = \frac{b-c}{bc} \quad b - c = \frac{c-a}{ca} \quad c - a = \frac{a-b}{ab}.$$

Multiplying the above equations gives $(abc)^2 = 1$, from which the desired result follows. □

Problem 13 (SMO 2014 (Junior) Q16)**.** If m and n are positive real numbers satisfying the equation $m + 4\sqrt{mn} - 2\sqrt{m} - 4\sqrt{n} + 4n = 3$, find the value of $\dfrac{\sqrt{m} + 2\sqrt{n} + 2014}{4 - \sqrt{m} - 2\sqrt{n}}$.

Solution. Grouping the expression as $m + 4\sqrt{mn} + 4n - 2\left(\sqrt{m} + 2\sqrt{n}\right) - 3 = 0$ becomes

$$\left(\sqrt{m} + 2\sqrt{n}\right)^2 - 2\left(\sqrt{m} + 2\sqrt{n}\right) - 3 = 0$$

which is factorised as $\left(\sqrt{m} + 2\sqrt{n} - 3\right)\left(\sqrt{m} + 2\sqrt{n} + 1\right) = 0$ which gives $\sqrt{m} + 2\sqrt{n} = 3$ or -1 (rejected). Hence $\sqrt{m} + 2\sqrt{n} = 3$.

Substituting this into the given expression gives $\boxed{2017}$. $\square$

Problem 14 (SMO 2005 (Open) Q19)**.** Let x and y be positive integers such that

$$\frac{100}{151} < \frac{y}{x} < \frac{200}{251}.$$

What is the minimum value of x?

Solution. The inequality can be transformed to

$$\frac{302}{200}y > x > \frac{251}{200}y.$$

The minimum y such that $\left(\frac{251}{200}y, \frac{302}{200}y\right)$ contains an integer is $y = 2$ and when $y = 2$, the only integer it contains is 3. Hence the answer is 3. $\square$

Problem 15 (SMO 2006 (Open) Q2)**.** Given that p and q are integers that satisfy the equation

$$36x^2 - 4(p^2 + 11)x + 135(p + q) + 576 = 0,$$

find the value of $p + q$.

Solution. By Vieta's, we have

$$p + q = \frac{p^2 + 11}{9} \quad \text{and} \quad pq = \frac{135(p + q) + 576}{36}.$$

Solving simultaneously gives $p = 13$ and $p + q = 20$. □

Problem 16 (AUSTRALIA 2020 Q1)**.** Determine all pairs (a, b) of non-negative integers such that

$$\frac{a+b}{2} - \sqrt{ab} = 1.$$

Solution. Reorganising the equation we get

$$2\sqrt{ab} = a + b - 2$$

thus after squaring:

$$4ab = a^2 + b^2 + 4 + 2ab - 4a - 4b.$$

Hence

$$0 = a^2 + b^2 + 4 - 2ab - 4a - 4b = (a-b)^2 + 4(1-a-b),$$

so 4 divides $(a-b)^2$. It follows that $a - b$ is even and hence $1 - a - b$ is odd. Thus $a - b = 4k + 2$ for some k. Then $(2k+1)^2 = a + b - 1 = 2b + 4k + 1$, and so $b = 2k^2$ and $a = 2k^2 + 4k + 2 = 2(k+1)^2$. □

Problem 17. Let $\alpha, \beta, \gamma, \delta$ be the roots of $x^4 - 8x^3 + 24x^2 - 42x + 16 = 0$. Given

$$\left(\frac{2}{\sqrt[4]{\alpha}+\sqrt[4]{\beta}+\sqrt[4]{\gamma}}+\frac{2}{\sqrt[4]{\beta}+\sqrt[4]{\gamma}+\sqrt[4]{\delta}}+\frac{2}{\sqrt[4]{\alpha}+\sqrt[4]{\beta}+\sqrt[4]{\delta}}+\frac{2}{\sqrt[4]{\delta}+\sqrt[4]{\gamma}+\sqrt[4]{\alpha}}\right)^2 = \frac{a\sqrt{b}}{c}$$

where a, b, c are pairwise coprime. Find the value of $a + b + c$.

Hint: Vieta's Relation

Solution. The given quartic equation is simply $(x-2)^4 = 10x$. so we get the stuff as $\left(\sum_{\text{cyc}} \frac{2\cdot\sqrt[4]{10}}{2-\alpha}\right)^2$ this is just $4\sqrt{10}\cdot\left(\frac{f'(2)}{f(2)}\right)^2$, where $f(x)$ is the given polynomial and $\frac{f'(2)}{f(2)} = \frac{1}{2}$.

Hence our answer is $\boxed{\sqrt{10}}$. □

Problem 18 (TRIPOS 1878)**.** If $x + y + z = 0$, show that

$$\left(\frac{y-z}{x} + \frac{z-x}{y} + \frac{x-y}{z}\right)\left(\frac{x}{y-z} + \frac{y}{z-x} + \frac{z}{x-y}\right) = 9$$

Proof. We have

$$\begin{aligned}
\left(\frac{y-z}{x} + \frac{z-x}{y} + \frac{x-y}{z}\right)\frac{x}{y-z} &= 1 + \frac{x}{y}\cdot\frac{z-x}{y-z} + \frac{x}{z}\cdot\frac{x-y}{y-z} \\
&= 1 + \frac{xz(z-x) + xy(x-y)}{yz(y-z)} \\
&= 1 + \frac{x(z^2 - zx + xy - y^2)}{yz(y-z)} \\
&= 1 + \frac{x}{yz}(x-y-z) \\
&= 1 + \frac{2x^2}{yz} \qquad \because y + z = -x
\end{aligned}$$

Therefore

$$\left(\frac{y-z}{x} + \frac{z-x}{y} + \frac{x-y}{z}\right)\left(\frac{x}{y-z} + \frac{y}{z-x} + \frac{z}{x-y}\right) = 3 + 2\frac{x^3+y^3+z^3}{xyz} = 3 + 6 = 9$$

for, since $x + y + z = 0$, $x^3 + y^3 + z^3 - 3xyz = 0$. □

Problem 19 (SSSMO 2000)**.** For any real numbers a, b and c, find the smallest possible value that the following expression can take:

$$3a^2 + 27b^2 + 5c^2 - 18ab - 30c + 237$$

Proof. By completing squares, the above expression can be rewritten as

$$3(a - 3b)^2 + 5(c - 3)^2 + 192 \geq 192$$

The value 192 is obtainable when $a = 3b$, $c = 3$.

Hence the smallest possible value is $\boxed{192}$. □

Remark. The technique of completing squares is an important tool in determining the extreme values of polynomials.

Problem 20 (GERMANY)**.** Given that $m^{15} + m^{16} + m^{17} = 0$, solve for m^{18}.

Solution. Factorising gives us
$$m^{15}(m^2 + m + 1) = 0$$

Case 1: $m^{15} = 0$

Then $m = 0$, thus $m^{18} = 0$.

Case 2: $m^2 + m + 1 = 0$

Multiplying $m - 1$ on both sides,

$$(m-1)(m^2 + m + 1) = 0 \implies m^3 - 1 = 0 \implies m^3 = 1$$

Hence $m^{18} = (m^3)^6 = \boxed{1}$. □

Problem 21. Given that for positive real number a,

$$\left(\frac{5}{x}\right)^{\log_a 25} = \left(\frac{3}{x}\right)^{\log_a 9}$$

Solution. Taking log base a at both sides,

$$\begin{aligned}
\log_a \left(\frac{5}{x}\right)^{\log_a 25} &= \log_a \left(\frac{3}{x}\right)^{\log_a 9} \\
\log_a 25 \times \log_a \left(\frac{5}{x}\right) &= \log_a 9 \times \log_a \left(\frac{3}{x}\right) \\
2\log_a 5 \times (\log_a 5 - \log_a x) &= 2\log_a 3 \times (\log_a 3 - \log_a x)
\end{aligned}$$

Let $p = \log_a 5$, $q = \log_a 3$. The equation can be rewritten as

$$2(p-q)(p+q) = 2(p-q) \times \log_a x$$

Dividing $2(p-q)$ for both sides, $p + q = \log_a x \implies \boxed{x = 15}$. $\square$

Problem 22. Consider the following expression. Find all possible real roots.

$$\frac{1}{x^2-10x-29}+\frac{1}{x^2-10x-45}-\frac{2}{x^2-10x-69}=0$$

Solution. Since the given equation looks quite complicated to solve, we try the substitution method. Let $a=(x-5)^2$, then the equation can be rewritten as

$$\frac{1}{a-54}+\frac{1}{a-70}-\frac{2}{a-94}=0$$

Solving quadratically gives us $\boxed{x=13}$ or $\boxed{x=-3}$. □

Problem 23 (DOKA)**.** Given that $f(x) = \dfrac{x}{x-3}$, $f^8(x) = \dfrac{x}{ax+b}$, where a and b are integers. Find a and b.

Solution. Observe that

$$\begin{aligned} f(x) &= \frac{x}{x-p} \\ f^2(x) &= \frac{x}{(1-p)x+p^2} \\ f^3(x) &= \frac{x}{(1-p+p^2)x-p^3} \\ f^4(x) &= \frac{x}{(1-p+p^2-p^3)x+p^4} \end{aligned}$$

Suppose $f^n(x) = \dfrac{x}{a_n x - b_n}$,

$$a_n = (-3)^0 + (-3)^1 + \cdots + (-3)^{n-1}$$

$$b_n = (-3)^n$$

Hence $\boxed{a = -1640}$ and $\boxed{b = 6561}$. $\square$

Problem 24 (Oxford MAT 2022)**.** Find the constant term of the expression

$$\left(x + 1 + \frac{1}{x}\right)^4$$

Solution. We calculate the square of $(x + 1 + x^{-1})$ first;

$$x^2 + 2x + 1 + 2x^{-1} + x^{-2}$$

Now if we were to square this expression, the constant term independent of x would be

$$2(x^2)(x^{-2}) + 2(2x)(2x^{-1}) + 32$$

Most of the terms have a factor of 2 because they occur in either order. This sum is $2 + 8 + 9 = \boxed{19}$. □

Problem 25 (Oxford MAT)**.** How many real solutions x are there to the following equation?

$$\log_2(2x^3 + 7x^2 + 2x + 3) = 3\log_2(x + 1) + 1$$

Solution. We can use laws of logarithms to write the right-hand side of the given equation as

$$\log_2(2x^3 + 6x^2 + 6x + 2).$$

Since $\log_2 x$ is an increasing function for $x > 0$, we can compare the arguments of the logarithms, provided that both are positive. This gives the polynomial equation

$$2x^3 + 7x^2 + 2x + 3 = 2x^3 + 6x^2 + 6x + 2$$

which rearranges to $x^2 - 4x + 1 = 0$, which has $\boxed{2}$ real solutions. We should check that $2x^3 + 7x^2 + 2x + 3$ is positive for these roots, but it definitely is because the roots of the quadratic are both positive and all the coefficients of the cubic are positive. $\square$

Problem 26 (CHINA 1979)**.** Given that $x^2 - x + 1 = 0$, find the value of $x^{2015} - x^{2014}$.

Solution. Multiplying both sides by $x + 1$,

$$(x^2 - x + 1)(x + 1) = 0 \implies x^3 + 1 = 0 \implies x^3 = -1$$

Substituting this into the given expression,

$$x^{2015} - x^{2014} = x^{2014}(x - 1) = x^{2014} \cdot x^2 = x^{2016} = (x^3)^{672} = (-1)^6 = \boxed{1}$$

□

Problem 27 (DOKA)**.** For a positive integer n, we have the polynomial

$$\left(2+\frac{x}{2}\right)\left(2+\frac{2x}{2}\right)\left(2+\frac{3x}{2}\right)\cdots\left(2+\frac{nx}{2}\right)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$$

where $a_0,\ldots,a_n$ are the coefficients of the polynomial. Find the smallest possible value of n if $2a_0+4a_1+8a_2+\cdots+2^{n+1}a_n-(n+1)!$ is divisible by 2020.

Solution. By comparing the coefficients of x^n,

$$\left(\frac{1}{2}\right)\left(\frac{2}{2}\right)\left(\frac{3}{2}\right)\cdots\left(\frac{n}{2}\right)=\frac{n!}{2^n}=a_n.$$

Now if we let $x=2$, notice that

$$\begin{aligned}(2+1)(2+2)(2+3)\cdots(2+n) &= a_0+a_1(2)+a_2(2)^2+a_3(2)^3+\cdots+a_n(2)^n\\ (3)(4)(5)\cdots(n+2) &= a_0+2a_1+4a_2+8a_3+\cdots+2^na_n\\ \frac{(n+2)!}{2} &= a_0+2a_1+4a_2+8a_3+\cdots+2^na_n\\ (n+2)! &= 2a_0+4a_1+8a_2+\cdots+2^{n+1}a_n\end{aligned}$$

Therefore

$$2a_0+4a_1+8a_2+\cdots+2^{n+1}a_n-(n+1)!=(n+2)!-(n+1)!=(n+1)!(n+1)$$

which is divisible by 2020.

Smallest possible n is when $(n+1)!$ is divisible by 2020. Hence $n+1=101$ gives $\boxed{n=100}$.

□

Problem 28. Consider the sequence of real numbers $\{a_n\}$ is defined by $a_1 = \dfrac{1}{1000}$ and $a_{n+1} = \dfrac{a_n}{na_n - 1}$ for $n \geq 1$. Find the value of $\dfrac{1}{a_{2020}}$.

Solution.

$$\frac{1}{a_{n+1}} = \frac{na_n - 1}{n} = n - \frac{1}{a_n}$$

Listing out terms,

$$\begin{aligned}
\frac{1}{a_{2020}} &= 2019 - \frac{1}{a_{2019}} \\
-\frac{1}{a_{2019}} &= -2018 + \frac{1}{a_{2018}} \\
\frac{1}{a_{2018}} &= 2017 - \frac{1}{a_{2017}} \\
&\vdots \\
\frac{1}{a_4} &= 3 - \frac{1}{a_3} \\
-\frac{1}{a_3} &= -2 + \frac{1}{a_2} \\
\frac{1}{a_2} &= 1 - \frac{1}{a_1} = 1 - 1000
\end{aligned}$$

Thus $\dfrac{1}{a_{2020}} = 2019 - 2018 + 2017 - 2016 + \cdots + 3 - 2 + 1 - 1000 = \boxed{10}$. ☐

Problem 29 (SMO 2014 (Senior) Q2)**.** Find, with justification, all positive real numbers a, b, c satisfying the system of equations

$$a\sqrt{b} = a + c, \quad b\sqrt{c} = b + a, \quad c\sqrt{a} = c + b.$$

Solution. Adding the three equations, we have

$$a\sqrt{b} + b\sqrt{c} + c\sqrt{a} = 2a + 2b + 2c$$

and factorising yields

$$a\left(\sqrt{b} - 2\right) + b\left(\sqrt{c} - 2\right) + c\left(\sqrt{a} - 2\right) = 0.$$

Thus $a = b = c = 0$ (rejected) or $\sqrt{b} - 2 = \sqrt{c} - 2 = \sqrt{a} - 2$ which gives $\boxed{a = b = c = 4}$. □

Problem 30. Let x, y, z be positive real numbers, such that $x + y + z = 1$ and $xy + yz + zx = \dfrac{1}{3}$. Find the value of

$$\frac{4x}{y+1} + \frac{16y}{z+1} + \frac{64z}{x+1}.$$

Solution. Dividing the two equations, we have

$$\frac{x+y+z}{xy+yz+zx} = \frac{1}{\frac{1}{3}}$$

and after cross multiplication

$$3xy + 3yz + 3zx = x + y + z.$$

Upon factorising

$$x(3y-1) + y(3z-1) + z(3x-1) = 0.$$

We arrive at $x = y = z = \dfrac{1}{3}$. Substituting these values into the above expression gives $\boxed{21}$. □

Problem 31 (SMO 2018 (Open) Q11)**.** Find the shortest distance from the point $(22, 21)$ to the graph with equation $x^3 + 1 = y(3x - y^3)$.

Solution. The equation can be expanded as $x^3 + 1 - 3xy + y^3 = 0$ which by observation is "symmetric". Substituting $x = y$ we have $x^3 + 1 - 3x^2 + x^3 = 0$ which upon solving gives $(x, y) = (1, 1)$ or $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$.

Since $(1, 1)$ is nearer to $(22, 21)$ than $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$, the shortest distance is

$$\sqrt{(22-1)^2 + (21-1)^2} = \boxed{29}.$$

□

Problem 32. Let a, b, c be non-zero real numbers such that $a + b + c = 0$ and

$$28(a^4 + b^4 + c^4) = a^7 + b^7 + c^7.$$

Find $a^3 + b^3 + c^3$.

Proof. We use two lemmas.

Lemma 3.5.1. For $a + b + c = 0$,

$$2(a^4 + b^4 + c^4) = (a^2 + b^2 + c^2)^2 \tag{1}$$

Proof. We prove this by direct expansion.

$$\begin{aligned} 2(a^4 + b^4 + c^4) &= 2[a^4 + b^4 + (a + b)^4] \\ &= 4(a^4 + 2a^3b + 3a^2b^2 + 2ab^3 + b^4) \\ &= 4(a^2 + ab + b^2)^2 \\ &= (a^2 + b^2 + c^2)^2 \end{aligned}$$

□

Lemma 3.5.2. For $a + b + c = 0$,

$$4(a^7 + b^7 + c^7) = 7abc(a^2 + b^2 + c^2)^2$$

Proof. This is a restatement of $(a + b)^7 - a^7 - b^7 = 7ab(a + b)(a^2 + ab + b^2)^2$. □

Dividing (2) by (1) gives us

$$\frac{2(a^7 + b^7 + c^7)}{a^4 + b^4 + c^4} = 7abc \implies abc = 8$$

Hence $a^3 + b^3 + c^3 = 3abc = \boxed{24}$. □

Problem 33. Find all the real solution pairs (x, y) that satisfy the system

$$\frac{1}{\sqrt{x}} + \frac{1}{2\sqrt{y}} = (x + 3y)(3x + y) \tag{1}$$

$$\frac{1}{\sqrt{x}} - \frac{1}{2\sqrt{y}} = 2(y^2 - x^2) \tag{2}$$

Solution. Let $a = \sqrt{x}$ and $b = \sqrt{y}$. The two equations become

$$\frac{1}{a} + \frac{1}{2b} = (a^2 + 3b^2)(3a^2 + b^2) \tag{3}$$

$$\frac{1}{a} - \frac{1}{2b} = 2(b^4 - a^4) \tag{4}$$

Adding the two equations above, we obtain

$$\frac{2}{a} = (3a^4 + 10a^2b^2 + 3b^4) + (2b^4 - 2a^4).$$

Thus we get

$$1 = 5a^4b + 10a^2b^3 + b^5. \tag{5}$$

By adding (5) and (6), we obtain $3 = (a + b)^5$. Similarly, by subtracting (6) from (5), we obtain $1 = (a - b)^5$. It is now easy to deduce that

$$a = \frac{1 + \sqrt[5]{3}}{2} \text{ and } b = \frac{\sqrt[5]{3} - 1}{2}.$$

Consequently, we obtain

$$x = \left(\frac{1 + \sqrt[5]{3}}{2}\right)^2 \text{ and } y = \left(\frac{\sqrt[5]{3} - 1}{2}\right)^2.$$

$\square$

4 Inequalities

For Mathematics Olympiad competitions, you are only required to apply these inequalities; hence the proofs of the following inequalities will not be included in this book.

§4.1 AM–GM Inequality

The most well-known and frequently used inequality is the Arithmetic mean–Geometric mean inequality, also known as the AM–GM inequality. The inequality simply states that the arithmetic mean is greater than or equal to the geometric mean.

§4.1.1 General AM–GM Inequality

The **arithmetic mean** (AM) of n non-negative real variables is given by

$$A(n) = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

The **geometric mean** (GM) of n non-negative real variables is given by

$$G(n) = \sqrt[n]{a_1 a_2 \cdots a_n}.$$

Theorem 4.1.1: AM–GM Inequality

$$A(n) \geq G(n),$$

or

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}. \tag{4.1}$$

Equality holds iff $a_1 = a_2 = \cdots = a_n$.

Exercise 4.1.1

For real numbers a, b, c prove that

$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$

Proof. By AM–GM inequality, we have

$$a^2 + b^2 \geq 2ab$$
$$b^2 + c^2 \geq 2bc$$
$$c^2 + a^2 \geq 2ca$$

Adding the three inequalities and then dividing by 2 we get the desired result. Equality holds if and only if $a = b = c$. $\square$

Exercise 4.1.2

Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n = 1$. Prove that

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq 2^n.$$

Proof. By AM–GM,

$$1 + a_1 \geq 2\sqrt{a_1}$$
$$1 + a_2 \geq 2\sqrt{a_2}$$
$$\vdots$$
$$1 + a_n \geq 2\sqrt{a_n}$$

Multiplying the above inequalities and using the fact $a_1 a_2 \cdots a_n = 1$ we get our desired result. Equality holds if and only if $a_i = 1$ for $i = 1, 2, \ldots, n$. $\square$

Exercise 4.1.3

Let a, b, c be non-negative real numbers. Prove that

$$(a + b)(b + c)(c + a) \geq 8abc.$$

Proof. By AM–GM,

$$a + b \geq 2\sqrt{ab}$$
$$b + c \geq 2\sqrt{bc}$$
$$c + a \geq 2\sqrt{ca}$$

Multiplying the above inequalities gives us our desired result. Equality holds if and only if $a = b = c$. $\square$

Exercise 4.1.4

Let $a, b, c > 0$. Prove that

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c.$$

Proof. By AM–GM we deduce that

$$\frac{a^3}{bc} + b + c \geq 3a$$
$$\frac{b^3}{ca} + c + a \geq 3b$$
$$\frac{c^3}{ab} + a + b \geq 3c$$

Adding the three inequalities we get our desired result. □

Exercise 4.1.5

Let a, b, c be positive real numbers. Prove that

$$ab(a+b) + bc(b+c) + ca(c+a) \geq \sum_{\text{cyc}} ab\sqrt{\frac{a}{b}(b+c)(c+a)}.$$

Proof. By AM–GM,

$$\begin{aligned}
&2ab(a+b) + 2ac(a+c) + 2bc(b+c)\\
&= ab(a+b) + ac(a+c) + bc(b+c) + ab(a+b) + ac(a+c) + bc(b+c)\\
&= a^2(b+c) + b^2(a+c) + c^2(a+b) + (a^2b + b^2c + ca^2) + (ab^2 + bc^2 + ca^2)\\
&\geq a^2(b+c) + b^2(a+c) + c^2(a+b) + (a^2b + b^2c + c^2a) + 3abc\\
&= a^2(b+c) + b^2(a+c) + c^2(a+b) + ab(a+c) + bc(a+b) + ac(b+c)\\
&= \left(a^2(b+c) + ab(a+c)\right) + \left(b^2(a+c) + bc(a+b)\right) + \left(c^2(a+b) + ac(b+c)\right)\\
&\geq 2\sqrt{a^3b(b+c)(c+a)} + 2\sqrt{b^3c(c+a)(a+b)} + 2\sqrt{c^3a(a+b)(b+c)}\\
&= 2ab\sqrt{\frac{a}{b}(b+c)(c+a)} + 2bc\sqrt{\frac{b}{c}(c+a)(a+b)} + 2ca\sqrt{\frac{c}{a}(a+b)(b+c)}
\end{aligned}$$

Equality holds if and only if $a = b = c$. □

Exercise 4.1.6: Nesbitt's Inequality

For positive real numbers a, b, c prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

Proof. Our inequality is equivalent to

$$1 + \frac{a}{b+c} + 1 + \frac{b}{c+a} + 1 + \frac{c}{a+b} \geq \frac{9}{2}$$

or

$$(a+b+c)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) \geq \frac{9}{2}$$

Using AM–GM, we have

$$\frac{(b+c)+(c+a)+(a+b)}{3} \geq \sqrt[3]{(b+c)(c+a)(a+b)}$$

and

$$\frac{\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}}{3} \geq \frac{1}{\sqrt[3]{(b+c)(c+a)(a+b)}}$$

$\square$

Exercise 4.1.7

For positive real numbers p, q, r, find the minimum of

$$\frac{p+q}{r}+\frac{q+r}{p}+\frac{r+p}{q}.$$

Solution. We split the expression up first into

$$\frac{p}{r}+\frac{q}{r}+\frac{r}{p}+\frac{q}{p}+\frac{p}{q}+\frac{r}{q}$$

Now we pair up terms as such:

$$\left(\frac{p}{r}+\frac{r}{p}\right)+\left(\frac{q}{r}+\frac{r}{q}\right)+\left(\frac{q}{p}+\frac{p}{q}\right)$$

Applying AM–GM to each pair, the result follows easily. $\square$

Exercise 4.1.8

Show that A, the maximum area of a triangle XYZ with a fixed perimeter P, happens when the triangle is equilateral. You may use the Heron's formula which states that A, the area of a triangle is

$$A=\sqrt{\frac{P}{2}\left(\frac{P}{2}-x\right)\left(\frac{P}{2}-y\right)\left(\frac{P}{2}-z\right)}$$

where x, y, z are the lengths of the sides of the triangle.

Proof. Using the AM–GM inequality,

$$\left(\frac{P}{2}-x\right)\left(\frac{P}{2}-y\right)\left(\frac{P}{2}-z\right) \leq \left(\frac{\frac{P}{2}-x+\frac{P}{2}-y+\frac{P}{2}-z}{3}\right)^3 = \left(\frac{P}{6}\right)^3.$$

Hence the maximum area is $A=\sqrt{\frac{P}{2}\left(\frac{P}{6}\right)^3}=\frac{P^2}{12\sqrt{3}}$ and occurs when $\frac{P}{2}-x=\frac{P}{2}-y=\frac{P}{2}-z$, i.e. $x=y=z$ which means that the triangle is equilateral. $\square$

§4.1.2 Weighted AM–GM Inequality

The weighted version of the AM–GM inequality follows from the original AM–GM inequality.

Theorem 4.1.2: Weighted AM–GM Inequality

For positive real $a_1, a_2, \dots, a_n$ and positive integers $\omega_1, \omega_2, \dots, \omega_n$, by AM–GM,

$$\frac{\omega_1 a_1 + \omega_2 a_2 + \cdots + \omega_n a_n}{\omega_1 + \omega_2 + \cdots + \omega_n} \geq \left(a_1^{\omega_1} a_2^{\omega_2} \cdots a_n^{\omega_n}\right)^{\frac{1}{\omega_1 + \omega_2 + \cdots + \omega_n}} \tag{4.2}$$

Or equivalently in symbols

$$\frac{\sum \omega_i a_i}{\sum \omega_i} \geq \left(\prod a_i^{\omega_i}\right)^{\frac{1}{\sum \omega_i}}$$

Exercise 4.1.9

Let a, b, c be positive real numbers such that $a + b + c = 3$. Show that

$$a^b b^c c^a \leq 1$$

Proof. Notice that

$$1 = \frac{a+b+c}{3} \geq \frac{ab+bc+ca}{a+b+c} \geq \left(a^b b^c c^a\right)^{\frac{1}{a+b+c}}$$

which implies $a^b b^c c^a \leq 1$. □

§4.1.3 Other mean quantities

For non-negative real $a_1, a_2, \dots, a_n$, the **harmonic mean** (HM) is given by

$$H(n) = \frac{1}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

For non-negative real $a_1, a_2, \dots, a_n$, the **quadratic mean** (QM) is given by

$$Q(n) = \frac{\sqrt{{a_1}^2 + {a_2}^2 + \cdots + {a_n}^2}}{n}$$

Relating AM, GM, HM, and SM, we have the following inequality:

$$Q(n) \geq A(n) \geq G(n) \geq H(n)$$

§4.2 Cauchy–Schwarz and Hölder's Inequalities

Theorem 4.2.1: Cauchy–Schwarz's Inequality

For real $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$,

$$({a_1}^2 + {a_2}^2 + \cdots + {a_n}^2)({b_1}^2 + {b_2}^2 + \cdots + {b_n}^2) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \tag{4.3}$$

Written compactly,

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} {a_i}^2\right)\left(\sum_{i=1}^{n} {b_i}^2\right)$$

Equality holds iff $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \cdots = \dfrac{a_n}{b_n}$.

Proof. Let $\mathbf{v} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ for real $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$. Note that the inequality holds trivially for $\mathbf{v} = \mathbf{0}$.

Let $\lambda \in \mathbb{R}$. Since $|\mathbf{u} - \lambda\mathbf{v}| \geq 0$,

$$\begin{aligned} (\mathbf{u} - \lambda\mathbf{v}) \cdot (\mathbf{u} - \lambda\mathbf{v}) &\geq 0 \\ \mathbf{u}\cdot\mathbf{u} - \lambda\mathbf{v}\cdot\mathbf{u} - \lambda\mathbf{u}\cdot\mathbf{v} + \lambda^2\mathbf{v}\cdot\mathbf{v} &\geq 0 \\ (\mathbf{v}\cdot\mathbf{v})\lambda^2 - (2\mathbf{u}\cdot\mathbf{v})\lambda + (\mathbf{u}\cdot\mathbf{u}) &\geq 0 \end{aligned}$$

This is a quadratic function in $\lambda \in \mathbb{R}$, hence the discriminant must be less than or equal to zero. Thus

$$\begin{aligned} (2\mathbf{u}\cdot\mathbf{v})^2 - 4(\mathbf{v}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{u}) &\leq 0 \\ (\mathbf{u}\cdot\mathbf{v})^2 &\leq |\mathbf{u}|^2|\mathbf{v}|^2 \\ \left(\sum_{k=1}^{n} a_k b_k\right)^2 &\leq \left(\sum_{k=1}^{n} {a_k}^2\right)\left(\sum_{k=1}^{n} {b_k}^2\right) \end{aligned}$$

□

Theorem 4.2.2: Titu's Lemma

For positive real $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$,

$$\frac{{a_1}^2}{b_1} + \frac{{a_2}^2}{b_2} + \cdots + \frac{{a_n}^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n} \tag{4.4}$$

Equality holds iff $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \cdots = \dfrac{a_n}{b_n}$.

Proof. Substitute $a_i = \dfrac{x_i}{\sqrt{y_i}}$ and $b_i = \sqrt{y_i}$. □

Applying Cauchy–Schwarz to integrals,

$$\left(\int_a^b f(t)g(t)\,\mathrm{d}t\right)^2 \le \left(\int_a^b [f(t)]^2\,\mathrm{d}t\right)\left(\int_a^b [g(t)]^2\,\mathrm{d}t\right) \tag{4.5}$$

Proof. For all $x \in \mathbb{R}$,

$$\left(xf(t) + g(t)\right)^2 \ge 0$$

□

Theorem 4.2.3: Hölder's Inequality

For positive real $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, and positive real p, q that satisfy $\frac{1}{p} + \frac{1}{q} = 1$,

$$\left(\sum_{i=1}^n {a_i}^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n {b_i}^q\right)^{\frac{1}{q}} \ge \sum_{i=1}^n a_i b_i \tag{4.6}$$

Remark. Cauchy–Schwarz is a special case of Hölder, when $p = q = 2$.

§4.3 Rearrangement and Chebyshev's Inequalities

Theorem 4.3.1: Chebyshev's Inequality

For real $a_1 \ge \cdots \ge a_n$ and $b_1 \ge \cdots \ge b_n$,

$$\frac{a_1 b_1 + \cdots + a_n b_n}{n} \ge \frac{a_1 + \cdots + a_n}{n} \frac{b_1 + \cdots + b_n}{n} \ge \frac{a_1 b_n + \cdots + a_n b_1}{n} \tag{4.7}$$

Theorem 4.3.2: Rearrangement Inequality

For real $a_1 \ge \cdots \ge a_n$ and $b_1 \ge \cdots \ge b_n$, for any permutation σ of $\{1, \ldots, n\}$,

$$\sum_{i=1}^n a_i b_i \ge \sum_{i=1}^n a_i b_{\sigma(i)} \ge \sum_{i=1}^n a_i b_{n+1-i} \tag{4.8}$$

§4.4 Other Useful Strategies

Theorem 4.4.1: Triangle Inequality

For real $a_1, a_2, \ldots, a_n$,

$$|a_1| + |a_2| + \cdots + |a_n| \ge |a_1 + a_2 + \cdots + a_n| \tag{4.9}$$

Equality holds iff $a_1, a_2, \cdots, a_n$ are all non-negative.

§4.4.1 Schur's Inequality

Theorem 4.4.2: Schur's Inequality

For non-negative real a, b, c and $n > 0$,

$$a^n(a-b)(a-c) + b^n(b-c)(b-a) + c^n(c-a)(c-b) \geq 0 \tag{4.10}$$

Equality holds iff either $a = b = c$ or when two of a, b, c are equal and the third is 0.

Proof. WLOG, let $a \geq b \geq c$. Note that

$$\begin{aligned} & a^n(a-b)(a-c) + b^n(b-a)(b-c) \\ & = a^n(a-b)(a-c) - b^n(a-b)(b-c) \\ & = (a-b)(a^n(a-c) - b^n(b-c)) \end{aligned}$$

Clearly, $a^n \geq b^n \geq 0$, and $a - c \geq b - c \geq 0$. Thus,

$$(a-b)(a^n(a-c) - b^n(b-c)) \geq 0 \implies a^n(a-b)(a-c) + b^n(b-a)(b-c) \geq 0$$

However, $c^n(c-a)(c-b) \geq 0$, and thus the proof is complete. ☐

When $n = 1$, we have the well-known inequality:

$$a^3 + b^3 - c^3 + 3abc \geq a^2b + a^2c + b^2a + b^2c + c^2a + c^2b$$

When $n = 2$, an equivalent form is:

$$a^4 + b^4 + c^4 + abc(a+b+c) \geq a^3b + a^3c + b^3a + b^3c + c^3a + c^3b$$

§4.4.2 Jensen's Inequality

A real function defined on (a, b) is said to be **convex** if for all $x, y \in (a, b)$,

$$f\left(\frac{x+y}{2}\right) \leq f(x) + f(y).$$

If the opposite inequality holds, then f is said to be **concave**.

Properties

- If $f(x)$ and $g(x)$ are convex functions on (a, b), then so are $h(x) = f(x) + g(x)$ and $M(x) = \max\{f(x), g(x)\}$.
- If $f(x)$ and $g(x)$ are convex functions on (a, b) and if $g(x)$ is non-decreasing on (a, b), then $h(x) = g(f(x))$ is convex on (a, b).

Theorem 4.4.3: Jensen's Inequality

Let a real-valued function f be convex on the interval I. Let $x_1, \ldots, x_n \in I$ and $\omega_1, \ldots, \omega_n \geq 0$. Then we have

$$\frac{\omega_1 f(x_1) + \omega_2 f(x_2) + \cdots + \omega_n f(x_n)}{\omega_1 + \omega_2 + \cdots + \omega_n} \geq f\left(\frac{\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_n x_n}{\omega_1 + \omega_2 + \cdots + \omega_n}\right) \tag{4.11}$$

If f is concave, the direction of the inequality is flipped.
In particular, if we take the weights $\omega_1 = \omega_2 = \cdots = \omega_n = 1$, we get the inequality

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \geq f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right).$$

If f is concave, the inequality is reversed.

Exercise 4.4.1

Let $a, b, c > 0$. Prove that

$$a^a b^b c^c \geq \left(\frac{a+b+c}{3}\right)^{a+b+c}.$$

Proof. Consider the function $f(x) = x \ln x$. Verify that $f''(x) = \dfrac{1}{x} > 0$ for all $x \in \mathbb{R}^+$. Thus f is convex in $\mathbb{R}^+$ and by Jensen's inequality we conclude that

$$f(a) + f(b) + f(c) \geq 3f\left(\frac{a+b+c}{3}\right) \iff \ln a^a + \ln b^b + \ln c^c \geq 3\ln\left(\frac{a+b+c}{3}\right)^{a+b+c},$$

which is equivalent to

$$\ln\left(a^a b^b c^c\right) \geq \ln\left(\frac{a+b+c}{3}\right)^{a+b+c}$$

as desired. □

§4.4.3 Minkowski's Inequality

Theorem 4.4.4: Minkowski's Inequality

For positive real $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ and $p > 1$,

$$\left(\sum_{i=1}^{n} {a_i}^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} {b_i}^p\right)^{\frac{1}{p}} \geq \left(\sum_{i=1}^{n} (a_i + b_i)^p\right)^{\frac{1}{p}} \tag{4.12}$$

Theorem 4.4.5: Generalised Minkowski's Inequality

Let $a_{ij} \geq 0$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ and let $p > 1$, then

$$\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{m} a_{ij}\right)\right]^{\frac{1}{p}} \leq \sum_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ij}{}^{p}\right)^{\frac{1}{p}} \tag{4.13}$$

Theorem 4.4.6: Bernoulli's Inequality

For $x > -1, x \neq 0$ and integer $n > 1$,

$$(1+x)^n > 1 + nx \tag{4.14}$$

§4.4.4 Ravi Transformation

Suppose that a, b, c are the side lengths of a triangle. Then positive real numbers x, y, z exist such that $a = x + y$, $b = y + z$ and $c = z + x$.

To verify this, let s be the semi-perimeter. Then denote $z = s - a$, $x = s - b$, $y = s - c$ and the conclusion is obvious since $s - a = \dfrac{b + c - a}{2} > 0$ and similarly for the others.

Exercise 4.4.2

Let a, b, c be the lengths of the sides of a triangle. Prove that

$$\sqrt{3\left(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}\right)} \geq \sqrt{a + b - c} + \sqrt{b + c - a} + \sqrt{c + a - b}.$$

Proof. Let $x, y, z > 0$ such that $a = x + y$, $b = y + z$, $c = z + x$. Then the above inequality is equivalent to

$$3\sum_{\text{cyc}}\sqrt{(x+y)(x+z)} \geq 2\left(\sum_{\text{cyc}}\sqrt{x}\right)^2.$$

From Cauchy,

$$\begin{aligned}
3\sum_{\text{cyc}}\sqrt{(x+y)(x+z)} &\geq 3\sum_{\text{cyc}}\left(y + \sqrt{zx}\right) \\
&\geq 2\sum_{\text{cyc}} y + 4\sum_{\text{cyc}}\sqrt{zx} \\
&= 2\left(\sum_{\text{cyc}}\sqrt{x}\right)^2
\end{aligned}$$

□

§4.4.5 Normalisation

Homogeneous inequalities can be **normalised** e.g. applied restrictions with homogeneous expressions in the variables

For example, to show that $a^3+b^3+c^3-3abc \geq 0$, assume that, WLOG, $abc = 1$ or $a+b+c = 1$ etc. The reason is explained below.

Suppose that $abc = k^3$. Let $a = ka'$, $b = kb'$, $c = kc'$. This implies $a'b'c' = 1$, and our inequality becomes $a'^3 + b'^3 + c'^3 - 3a'b'c' \geq 0$, which is the same as before. Therefore the restriction $abc = 1$ doesn't change anything of the inequality. Similarly one might also assume $a + b + c = 1$.

§4.4.6 Homogenisation

Homogenisation is the opposite of normalisation. It is often useful to substitute $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$, when the condition $abc = 1$ is given. Similarly when $a + b + c = 1$ we can substitute $a = \frac{x}{x} + y + z$, $b = \frac{y}{x} + y + z$, $c = \frac{z}{x} + y + z$ to homogenise the inequality.

Exercise 4.4.3

If $a, b, c > 0$ and $a + b + c = 1$, prove that $a^2 + b^2 + c^2 + 1 \geq 4(ab + bc + ca)$.

Proof. So all the terms except for the 1 are of the second degree. We substitute $a + b + c$ for 1. The inequality still gives a non-homogeneous inequality. So instead we square the condition to make it second degree and get

$$a^2 + b^2 + c^2 + 2(ab + bc + ca) = 1$$

Now plugging this for 1 in the inequality and simplifying gives $a^2 + b^2 + c^2 \geq ab + bc + ca$, which is well-known by the Rearrangement Inequality. $\square$

§4.5 Exercises

Problem 34 (H3M 2018 Q3)**.** A triangle has sides of length a, b, c units. In each of the following cases, prove that there is a triangle having sides of the given lengths.

(a) $\dfrac{a}{1+a}$, $\dfrac{b}{1+b}$ and $\dfrac{c}{1+c}$ units.

(b) $\sqrt{a}$, $\sqrt{b}$ and $\sqrt{c}$ units.

(c) $\sqrt{a(b+c-a)}$, $\sqrt{b(c+a-b)}$ and $\sqrt{c(a+b-c)}$ units.

Proof. To prove that there is a triangle having sides of the given lengths, prove that the sum of lengths of any two sides is greater than length of third side.

(a) WLOG, assume that $a \le b \le c$. By monotonicity of the function $y = \dfrac{x}{1+x}$, $x > 0$, we have $\dfrac{a}{1+a} \le \dfrac{b}{1+b} \le \dfrac{c}{1+c}$.

It remains to be shown that $\dfrac{a}{1+a} + \dfrac{b}{1+b} > \dfrac{c}{1+c}$, as the remaining other triangle inequalities are trivially true.

$$\begin{aligned}\frac{a}{1+a} + \frac{b}{1+b} &\ge \frac{a}{1+b} + \frac{b}{1+b}\\ &= \frac{a+b}{1+b}\\ &> \frac{c}{1+b} \quad \text{[using triangle inequality on triangle with sides } a,b,c]\\ &\ge \frac{c}{1+c}\end{aligned}$$

(b) WLOG, assume that $a \le b \le c$. By monotonicity of the function $y = \sqrt{x}$, $x > 0$, we have $\sqrt{a} \le \sqrt{b} \le \sqrt{c}$.

It remains to be shown that $\sqrt{a} + \sqrt{b} > \sqrt{c}$, as the remaining other triangle inequalities are trivially true.

$$\begin{aligned}\left(\sqrt{a} + \sqrt{b}\right)^2 &= a + b + 2\sqrt{ab}\\ &> c + 2\sqrt{ab} \quad \text{[using triangle inequality on triangle with sides } a,b,c]\\ &> c + 0 = c\end{aligned}$$

(c) WLOG, we only need to show that $\sqrt{a(b+c-a)} + \sqrt{b(c+a-b)} > \sqrt{c(a+b-c)}$ due to the symmetricity of the terms involved.

$$\begin{aligned}& a(b+c-a) + b(c+a-b) > c(a+b-c)\\ \iff & ab + ac - a^2 + bc + ab - b^2 > ac + bc - c^2\\ \iff & c^2 - a^2 + 2ab - b^2 > 0\\ \iff & c^2 - (a-b)^2 > 0\\ \iff & (c+a-b)(c-a+b) > 0\end{aligned}$$

By triangle inequality, $c + a - b > 0$ and $c - a + b > 0$.

□

Problem 35 (H3M 2016 Q3).

(i) For some positive integer n, let $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots y_n$ be real numbers. By considering the sum of all n^2 terms of the form $(x_i - x_j)(y_i - y_j)$, prove that

$$\sum_{i=1}^{n} x_i y_i \geq \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right).$$

[5]

(ii) Let a triangle have angles A, B, C and let the lengths of the opposite sides be a, b and c.

By applying the result of part (i), prove that

$$aA + bB + cC \geq \frac{1}{3}\pi(a+b+c).$$

[4]

(iii) Let a, b, c be three positive numbers such that $a^2 + b^2 + c^2 = 1$. By applying the result of part (i) with $\{x_i\} = \left\{\frac{a+b}{c}, \frac{c+a}{b}, \frac{b+c}{a}\right\}$, find the minimum possible value of

$$\frac{(a+b)(a^2+b^2)}{c} + \frac{(c+a)(c^2+a^2)}{b} + \frac{(b+c)(b^2+c^2)}{a}.$$

[7]

Proof.

(i) We have

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_j)(y_i - y_j) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}(x_i y_i - x_j y_i - x_i y_j + x_j y_j) \\
&= \sum_{i=1}^{n}\left(n x_i y_i - y_i \sum_{j=1}^{n} x_j - x_i \sum_{j=1}^{n} y_j + \sum_{j=1}^{n} x_j y_j\right) \\
&= 2n\sum_{i=1}^{n} x_i y_i - 2\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i
\end{aligned}$$

Also,

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_j)(y_i - y_j) &= \sum_{i=j}(x_i - x_j)(y_i - y_j) + \sum_{i\neq j}(x_i - x_j)(y_i - y_j) \\
&= 0 + \sum_{i<j}(x_i - x_j)(y_i - y_j) + \sum_{i>j}(x_i - x_j)(y_i - y_j) \\
&= 0 + \sum(\leq 0)(\leq 0) + \sum(\geq 0)(\geq 0) \\
&\geq 0
\end{aligned}$$

Combining both results,

$$
\begin{aligned}
2n\sum_{i=1}^{n} x_i y_i - 2\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i - 2\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i &\geq 0 \\
2n\sum_{i=1}^{n} x_i y_i &\geq 2\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i \\
\sum_{i=1}^{n} x_i y_i \geq \frac{1}{n}\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i
\end{aligned}
$$

(ii) Let $x_1 = a, x_2 = b, x_3 = c$, and $y_1 = A, y_2 = B, y_3 = C$, and without loss of generality let $a \leq b \leq c$.

Otherwise or in short we assign the x's in ascending length order, the objective is to show that (i) can be applied and apply it to show the result in (ii).

Case 1:

(iii)

□

Problem 36 (H3M)**.** If a, b, c are sides of a triangle, show that

$$\frac{a}{b+c-a} + \frac{b}{a+c-b} + \frac{c}{a+b-c} \geq 3.$$

Proof. Let $x = a + b - c$, $y = a + c - b$, $z = b + c - a$, which are all positive by triangle inequality.

Thus we have $a = \dfrac{x+y}{2}$, $b = \dfrac{x+z}{2}$, $c = \dfrac{y+z}{2}$.

Substituting these in gives us

$$\begin{aligned}
\frac{a}{b+c-a} + \frac{b}{a+c-b} + \frac{c}{a+b-c} &= \frac{x+y}{2z} + \frac{x+z}{2y} + \frac{y+z}{2x} \\
&= \frac{1}{2}\left(\frac{x}{z} + \frac{y}{z} + \frac{x}{y} + \frac{z}{y} + \frac{y}{x} + \frac{z}{x}\right) \\
&\geq \frac{1}{2} \cdot 6\sqrt[6]{1} = 3
\end{aligned}$$

□

Problem 37 (HCI H3M Prelim 2020 Q7)**.**

(a) Given positive reals a, b, c, prove that

$$\frac{(a+1)^3}{b}+\frac{(b+1)^3}{c}+\frac{(c+1)^3}{a}\geq\frac{81}{4}.$$

(b) Given $x, y, z \geq 1$, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$, prove that

$$\sqrt{x+y+z}\geq\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}.$$

(c) Given that $a+b+c+d=3$ and $a^2+2b^2+3c^2+6d^2=5$, prove that $1\leq a\leq 2$.

Proof.

(a) Applying AM-GM gives us

$$\frac{(a+1)^3}{b}+\frac{(b+1)^3}{c}+\frac{(c+1)^3}{a}\geq\frac{3(a+1)(b+1)(c+1)}{\sqrt[3]{abc}}$$

We note the denominator is a cube root and hence we want to get the numerator to be a product of cube roots containing a, b and c so that they cancel out. To do so, split each term in the bracket into three terms and apply AM-GM on the numerator:

$$\begin{aligned}\frac{3(a+1)(b+1)(c+1)}{\sqrt[3]{abc}}&=\frac{3(a+\frac{1}{2}+\frac{1}{2})(b+\frac{1}{2}+\frac{1}{2})(c+\frac{1}{2}+\frac{1}{2})}{\sqrt[3]{abc}}\\&\geq\frac{81\sqrt[3]{\frac{a}{4}}\sqrt[3]{\frac{b}{4}}\sqrt[3]{\frac{c}{4}}}{\sqrt[3]{abc}}=\frac{81}{4}\end{aligned}$$

(b) The expression is certainly odd, as we would usually expect some squares in a Cauchy. We note that if we square both sides, we get something that better resembles a Cauchy inequality.

We construct the inequality

$$(x+y+z)\left(\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}\right)\geq\left(\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}\right)^2$$

where

$$\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}=3-\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=3-2=1$$

so applying Cauchy gives us the desired result.

(c) Since the desired result concerns the variable a, we naturally try to express the remaining variables in terms of a.

We hence yield

$$b+c+d=3-a$$

and

$$2b^2 + 3c^2 + 6d^2 = 5 - a^2$$

Note that since both the sum of variables and the sum of the squares of said variables are given, it is an indication that Cauchy is the way to go for this question.

Applying Cauchy,

$$(2b^2 + 3c^2 + 6d^2)(18 + 12 + 6) \geq (6b + 6c + 6d)^2$$

Substituting in the relevant expressions gives us $36(5 - a^2) \geq (18 - 6a)^2$ which is a fairly easy inequality to solve.

$\square$

Problem 38 (SMO Open 2020 Q24)**.** Let x, y, z and w be real numbers such that $x+y+z+w=5$. Find the minimum value of $(x+5)^2+(y+10)^2+(z+20)^2+(w+40)^2$.

Solution. By Cauchy–Schwarz,

$$4(a^2+b^2+c^2+d^2) \geq (a+b+c+d)^2.$$

Substituting $a=x+5$, $b=y+10$, $c=z+20$, $d=w+40$ gives us

$$4\left((x+5)^2+(y+10)^2+(z+20)^2+(w+40)^2\right) \geq (x+5+y+10+z+20+w+40)^2$$

from which we can work out the answer of $\boxed{1600}$. □

Problem 39 (IMO 1995)**.** Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

Proof. Let $x = \frac{1}{a}$, $y = \frac{1}{b}$, $z = \frac{1}{c}$. Then by the given condition we obtain $xyz = 1$. Note that

$$\sum_{\text{cyc}} \frac{1}{a^3(b+c)} = \sum_{\text{cyc}} \frac{1}{\frac{1}{x^3}\left(\frac{1}{y} + \frac{1}{z}\right)} = \sum_{\text{cyc}} \frac{x^2}{y+z}$$

Now by Cauchy–Schwarz,

$$\sum_{\text{cyc}} \frac{x^2}{y+z} \geq \frac{(x+y+z)^2}{2(x+y+z)} = \frac{x+y+z}{2}$$

and by AM–GM,

$$\frac{1}{2}(x+y+z) \geq \frac{3}{2}\sqrt[3]{xyz} = \frac{3}{2}$$

Hence proven. $\square$

Problem 40. Prove that for all $n \in \mathbb{N}$,

$$\sqrt{1^2+1} + \sqrt{2^2+1} + \cdots + \sqrt{n^2+1} \geq \frac{n}{2}\sqrt{n^2+2n+5}.$$

Proof. Define the function $f(x) = \sqrt{x^2+1}$. Observe that for $x \to \infty$, we have $f(x) \to |x|$, so the graph is convex.

Applying Jensen's inequality with reals $x_1 = 1, x_2 = 2, \ldots, x_n = n$,

$$\begin{aligned}
\frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n} &\geq f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right) \\
\frac{f(1)+f(2)+\cdots+f(n)}{n} &\geq f\left(\frac{1+2+\cdots+n}{n}\right) \\
\frac{\sqrt{1^2+1}+\sqrt{2^2+1}+\cdots+\sqrt{n^2+1}}{n} &\geq f\left(\frac{n(n+1)/2}{n}\right) \\
&= f\left(\frac{n+1}{2}\right) = \frac{1}{2}\sqrt{n^2+2n+5}
\end{aligned}$$

□

Problem 41. Given that $a, b > 0$ and $ab(a+b) = 2000$, find the minimum value of

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{a+b}.$$

Solution. Using AM–GM,

$$\begin{aligned} a+b &\geq 2\sqrt{ab} \\ \left(\frac{a+b}{2}\right)^2 &\geq ab \\ \left(\frac{a+b}{2}\right)^2 (a+b) &\geq ab(a+b) \\ \frac{(a+b)^3}{4} &\geq 2000 \\ a+b &\geq 20 \end{aligned}$$

Using AM–GM,

$$\begin{aligned} \frac{1}{a} + \frac{1}{b} + \frac{1}{a+b} &= \frac{1}{2a} + \frac{1}{2a} + \frac{1}{2b} + \frac{1}{2b} + \frac{1}{a+b} \\ &\geq 5\sqrt[5]{\frac{1}{16}\frac{a+b}{a^2b^2(a+b)^2}} \\ &\geq 5\left(\frac{1}{20}\right) = \boxed{\frac{1}{4}} \end{aligned}$$

Equality holds if and only if $\frac{1}{2a} = \frac{1}{2b} = \frac{1}{a+b}$, or $a = b = 10$. □

Problem 42 (USAMO 2012). Find all integers $n \geq 3$ such that among any n positive real numbers $a_1, a_2, \ldots, a_n$ with

$$\max(a_1, a_2, \ldots, a_n) \leq n \cdot \min(a_1, a_2, \ldots, a_n),$$

there exist three that are the side lengths of an acute triangle.

Solution. We claim that $n \geq 13$ are all the satisfying positive integers.

WLOG, let $a_1 \leq a_2 \leq \cdots \leq a_n$. Three positive real numbers $a \leq b \leq c$ are the side lengths of an acute triangle iff $a^2 + b^2 > c^2$.

Thus, if our n positive real numbers contain no such triple, we must have $a_i^2 + a_j^2 \leq a_k^2$ for all $i < j < k$.

We have the following claim:

Lemma 4.5.1. Let $S = \{a_1, a_2, \cdots, a_n\}$ be a set of $n \geq 3$ positive real numbers, where $a_1 \leq a_2 \leq \cdots \leq a_n$. If S contains no three numbers that are side lengths of an acute triangle, we have $a_i \geq F_i \cdot a_1^2$ for all $1 \leq i \leq n$, where F_i is the i-th Fibonacci number.

Proof. If $n = 3$, we must have $a_1^2 + a_2^2 \leq a_3^2$. And since $a_1^2 \leq a_2^2$ and $a_3^2 \geq a_1^2 + a_2^2 \geq 2a_2^2$, the claim holds for $n = 3$.

Assume that the claim holds for all $t \leq n$. Consider a set S of $n + 1$ real numbers such that $a_1 \leq a_2 \leq \cdots \leq a_{n+1}$ and S contains no three numbers that are side lengths of an acute triangle. Then, we must have

$$\begin{gathered} a_1^2 + a_2^2 \leq a_3^2 \\ a_2^2 + a_3^2 \leq a_4^2 \\ \vdots \\ a_{n-1}^2 + a_n^2 \leq a_{n+1}^2. \end{gathered}$$

Since the statement holds for all $t \leq n$, we have $a_i \geq F_i \cdot a_1^2$ for all $1 \leq i \leq n$. Thus, $a_{n+1}^2 \geq a_{n-1}^2 + a_n^2 \geq F_{n-1} \cdot a_1^2 + F_n \cdot a_1^2 = F_{n+1} \cdot a_1^2$. QED □

Now, if $n \geq 13$, we have $a_n \geq F_n \cdot a_1^2$. However, since $\max(a_1, a_2, \ldots, a_n) \leq n \cdot \min(a_1, a_2, \ldots, a_n)$, we have $n \cdot a_1 \geq a_n$, or $a_n^2 \geq n^2 \cdot a_1^2$. But for all $n \geq 13$, we have $n^2 < F_n$, hence $a_n \geq F_n \cdot a_1^2 > n^2 a_1^2 \geq a_n$, which is absurd. Thus for all $n \geq 13$, we will always have three numbers that are side lengths of an acute triangle.

For $n \leq 12$, the set $S = \{\sqrt{F_i} t \mid 1 \leq i \leq n, t \in \mathbb{R}^+, F_i \text{ is the } i\text{-th Fibonacci number}\}$ satisfies that it contains no three numbers that are side lengths of an acute triangle. □

Problem 43 (IMO 2000)**.** Let a, b, c be positive real numbers so that $abc = 1$. Prove that

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) \le 1.$$

Proof. Let $a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$, then

$$\prod_{cyc}(x-y+z) \le xyz \iff (x^3+y^3+z^3) + 3xyz \ge \sum_{cyc} x^2y + \sum_{cyc} x^2z$$

This holds by Schur's inequality. □

Problem 44 (IMO 1964)**.** Let a, b, c be the side lengths of a triangle. Prove that

$$a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c) \le 3abc.$$

Proof. The inequality can be written as

$$(a+b-c)(b+c-a)(c+a-b) \le abc.$$

Let $a = x+y$, $b = y+z$ and $c = z+x$. Then the above inequality becomes $8xyz \le (x+y)(y+z)(z+x)$ which is obvious by AM–GM. □

Problem 45 (CHINA 2007 Q1)**.** Given complex numbers a, b, c, let $|a+b| = m$, $|a-b| = n$, and suppose $mn \neq 0$. Prove that

$$\max\{|ac+b|, |a+bc|\} \geq \frac{mn}{\sqrt{m^2+n^2}}.$$

Proof. We have

$$\begin{aligned}
\max\{|ac+b|, |a+bc|\} &\geq \frac{|b||ac+b| + |a||a+bc|}{|b|+|a|} \\
&\geq \frac{|b(ac+b) - a(a+bc)|}{|a|+|b|} \\
&= \frac{|b^2 - a^2|}{|a|+|b|} \\
&\geq \frac{|b+a||b-a|}{\sqrt{2\left(|a|^2+|b|^2\right)}}.
\end{aligned}$$

Since

$$m^2 + n^2 = |a-b|^2 + |a+b|^2 = 2\left(|a|^2+|b|^2\right),$$

we get

$$\max\{|ac+b|, |a+bc|\} \geq \frac{mn}{\sqrt{m^2+n^2}}$$

as desired. □

Problem 46. Show that

$$\sum_{k=1}^{n} a_k^2 \geq a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1$$

Proof. Multiply both sides by 2,

$$2\sum_{k=1}^{n} a_k^2 \geq 2\left(a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1\right)$$

Subtracting each side by the RHS,

$$\left(a_1 - a_n\right)^2 + \left(a_2 - a_1\right)^2 + \left(a_3 - a_2\right)^2 + \cdots + \left(a_n - a_{n-1}\right)^2 \geq 0$$

which is always true. □

Problem 47 (MACEDONIA 2016)**.** For $n \geq 3$, $a_1, a_2, \ldots, a_n \in \mathbb{R}^+$ satisfy

$$\frac{1}{1+{a_1}^4} + \frac{1}{1+{a_2}^4} + \cdots + \frac{1}{1+{a_n}^4} = 1.$$

Prove that

$$a_1 a_2 \cdots a_n \geq (n-1)^{\frac{n}{4}}.$$

Proof. Let $b_i = {a_i}^4$ where $b_i \geq 0$. The given condition becomes

$$\frac{1}{1+b_1} + \frac{1}{1+b_2} + \cdots + \frac{1}{1+b_n} = 1.$$

and we want to prove

$$b_1 b_2 \cdots b_n \geq (n-1)^n.$$

Let $t_i = \dfrac{1}{1+b_i}$. Rewriting the given condition gives us

$$t_1 + t_2 + \cdots + t_n = 1.$$

and we want to prove

$$\frac{1-t_1}{t_1}\frac{1-t_2}{t_2}\cdots\frac{1-t_n}{t_n} \geq (n-1)^n.$$

Using AM-GM, we have

$$\begin{aligned}
\frac{1-t_1}{t_1}\frac{1-t_2}{t_2}\cdots\frac{1-t_n}{t_n} &= \frac{t_2+t_3+\cdots+t_n}{t_1}\frac{t_1+t_3+\cdots+t_n}{t_2}\cdots\frac{t_1+t_2+\cdots+t_{n-1}}{t_n} \\
&\geq \frac{(n-1)(t_2t_3\cdots t_n)^{\frac{1}{n-1}}}{t_1}\frac{(n-1)(t_1t_3\cdots t_n)^{\frac{1}{n-1}}}{t_2}\cdots\frac{(n-1)(t_1t_2\cdots t_{n-1})^{\frac{1}{n-1}}}{t_n} \\
&= (n-1)^n
\end{aligned}$$

□

Problem 48 (CANADA/1969)**.** Show that if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$ and p_1, p_2, p_3 are not all zero, then

$$\left(\frac{a_1}{b_1}\right)^n = \frac{p_1{a_1}^n + p_2{a_2}^n + p_3{a_3}^n}{p_1{b_1}^n + p_2{b_2}^n + p_3{b_3}^n}$$

for every positive integer n.

Proof. Instead of proving the two expressions equal, we prove that their difference equals zero.

Subtracting the LHS from the RHS,

$$\frac{p_1{a_1}^n + p_2{a_2}^n + p_3{a_3}^n}{p_1{b_1}^n + p_2{b_2}^n + p_3{b_3}^n} - \frac{{a_1}^n}{{b_1}^n} = 0$$

Finding a common denominator, the numerator becomes

$$\begin{aligned} &{b_1}^n\left(p_1{a_1}^n + p_2{a_2}^n + p_3{a_3}^n\right) - {a_1}^n\left(p_1{b_1}^n + p_2{b_2}^n + p_3{b_3}^n\right) \\ &= p_2\left({a_2}^n{b_1}^n - {a_1}^n{b_2}^n\right) + p_3\left({a_3}^n{b_1}^n - {a_1}^n{b_3}^n\right) = 0 \end{aligned}$$

(The denominator is irrelevant since it never equals zero)

From $\frac{a_1}{b_1} = \frac{a_2}{b_2}$, we have

$${a_1}^n{b_2}^n = {a_2}^n{b_1}^n$$

Similarly, from $\frac{a_1}{b_1} = \frac{a_3}{b_3}$, we have

$${a_1}^n{b_3}^n = {a_3}^n{b_1}^n$$

Hence, ${a_2}^n{b_1}^n - {a_1}^n{b_2}^n = {a_3}^n{b_1}^n - {a_1}^n{b_3}^n = 0$ and our proof is complete. □

5 Functional Equations

Definition 5.0.1. Let X and Y be sets. A function $f : X \to Y$ is an assignment of a value in Y for each $x \in X$; we denote this value $f(x) \in Y$.

Let $f : X \to Y$ be a function. The set X is called the **domain**, and Y the **codomain**. A couple definitions which will be useful:

Definition 5.0.2. A function $f : X \to Y$ is **injective** if $f(x) = f(y) \iff x = y$. (Sometimes also called *one-to-one*.)

Definition 5.0.3. A function $f : X \to Y$ is surjective if for all $y \in Y$, there is some $x \in X$ such that $f(x) = y$. (Sometimes also called *onto*.)

Definition 5.0.4. A function is **bijective** if it is both injective and surjective.

An equation containing an unknown function is called a **functional equation**. A typical functional equation problem will ask you to find all functions satisfying a certain property. For such problems, you must prove *both* directions. In fact, I recommend structuring the opening lines of your solution as follows:

Solution. The answer is $f(x) = kx, k \in \mathbb{R}$. It's easy to see that these functions satisfy the given equation.

We now show these are the only solutions ... □

§5.1 Heuristics

At the beginning of a problem:

- Figure out what the answer is. For many problems, plug in $f(x) = kx + c$ and find which k and c work. It may also be worth trying general polynomial functions.
- Make obvious optimisations (like scaling or shifting).
- Plug in $x = y = 0$, $x = 0$ into the givens, et cetera. See what the most simple substitutions give first.

Once you have done these obvious steps, some other things to try:

- The battle cry "DURR WE WANT STUFF TO CANCEL". Plug in things that make lots of terms cancel or that make lots of terms vanish (think $x = y = 0$).
- Watch for opportunities to prove injectivity or surjectivity, for example using isolated parts.
- Watch for bumps in symmetry and involutions.
- For equations over $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{Q}$, induction is often helpful. It can also be helpful over $\mathbb{R}$ as well. The triggers for induction are the same as any other olympiad problem: you can pin down new values to previous ones.
- It may help to rewrite the function in terms of other functions.

Here are three more tricks that are frequently useful.

- Tripling an involution.

  If you know something about $f(f(x))$, try applying it $f(f(f(x)))$ in different ways. For example, if we know that $f(f(x)) = x + 2$, then we obtain $f^3(x) = f(x+2) = f(x) + 2$.

- Isolated parts.

  When trying to obtain injective or surjective, watch for "isolated" variables or parts of the equation. For example, suppose you have a condition like

  $$f(x + 2xf(y)^2) = yf(x) + f(f(y) + 1)$$

  (I made that up). Noting that $f \equiv 0$ works, assume f is not zero everywhere. Then by taking x_0 with $f(x_0) \neq 0$, one obtains f is injective. (Try putting in y_1 and y_2.)

  Proving surjectivity can often be done in similar spirit. For example, suppose

  $$f(f(y) + xf(x)) = y + f(x)^2.$$

  By varying y with x fixed we get that f is surjective, and thus we can pick x_0 so that $f(x_0) = 0$ and go from there. Surjectivity can be especially nice if every y is wrapped in an f, say; then each f(y) just becomes replace by an arbitrary real.

- Exploiting "bumps" in symmetry.

  If some parts of an equation are symmetric and others are not, swapping x and y can often be helpful. For example, suppose you have a condition like

  $$f(x + f(y)) + f(xy) = f(x+1)f(y+1) - 1$$

  (again I made that up). This equation is "almost symmetric", except for a "bump" on the far left where $f(x + f(y))$ is asymmetric. So if we take the equation with x and y flipped and then eliminate the common terms, we manage to obtain

  $$f(x + f(y)) = f(y + f(x)).$$

  If we have shown f is injective, we are even done! So often these "bumps" are what let you solve a problem. (In particular, don't get rid of the bumps!)

Some other small tricks I should mention:

- Often, you'll get something like $f(x)^2 = x^2$. When this happens, make sure you do not automatically assume $f(x) = x$ for each x; this type of equality holds only for each individual x.
- Check the solutions work! Don't get a 6 unnecessarily after solving the problem just because you forget this trivial step.

Exercise 5.1.1

If $f\left(x^2 f(y)\right) = x^2 y$, find $f(2000)$.

Solution. We guess $f(x) = x$, which in fact works. Hence $f(2000) = \boxed{2000}$. □

Exercise 5.1.2

Let $a \neq 1$. Solve the equation

$$af(x) + f\left(\frac{1}{x}\right) = ax$$

where the domain of f is the set of all non-zero real numbers.

Solution. Replacing x by x^{-1}, we get

$$af\left(\frac{1}{x}\right) + f(x) = \frac{a}{x}$$

We therefore have

$$(a^2 - 1)f(x) = a^2 x - \frac{a}{x}$$

and hence

$$\boxed{f(x) = \frac{a^2 x - \frac{a}{x}}{a^2 - 1}}.$$

□

§5.2 Cauchy's Functional Equation Over $\mathbb{Q}$

For this section, all functions are $f : \mathbb{Q} \to \mathbb{Q}$.

Exercise 5.2.1: Cauchy's functional equation

Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x + y) = f(x) + f(y)$$

holds for each $x, y \in \mathbb{Q}$.

Solution. First put $y = 0$:

$$f(x+0) = f(x) + f(0) \implies f(0) = 0$$

Then put $y = -x$:

$$f(x-x) = f(x) + f(-x) \implies f(-x) = -f(x) \quad \forall x \in \mathbb{Q}$$

Then, by repeated application of the original equation to expand the right side of $f(nx) = f(x + x + \cdots + x)$ we get

$$f(nx) = nf(x) \quad \forall x \in \mathbb{Q}, \forall n \in \mathbb{N}$$

By substituting $x = \frac{1}{n}$:

$$f\left(\frac{1}{n}\right) = \frac{1}{n} f(1) \quad \forall n \in \mathbb{N}$$

Combining the two equations above with $x = \frac{1}{m}$, we get:

$$f\left(\frac{n}{m}\right) = nf\left(\frac{1}{m}\right) = \frac{n}{m} f(1) \quad \forall m, n \in \mathbb{N}$$

Using $f(-x) = -f(x)$ and multiplying the equation above by -1, we get

$$-f\left(\frac{n}{m}\right) = -\frac{n}{m} f(1)$$
$$f\left(-\frac{n}{m}\right) = \left(-\frac{n}{m}\right) f(1) \quad \forall m, n \in \mathbb{N}$$
$$f(q) = qf(1) \quad \forall q \in \mathbb{Q}$$

Thus, we have found that $f(x) = cx \forall x \in \mathbb{Q}$ and some constant $c \in \mathbb{R}$. It is obvious that this family of functions is indeed a solution of $f(x+y) = f(x) + f(y)$ for rational x and y. More generally, it is easy to show that $\boxed{f(\alpha q) = qf(\alpha) \forall q \in \mathbb{Q}, \alpha \in \mathbb{R}}$. □

§5.3 Cauchy's Functional Equation Over $\mathbb{R}$

§5.4 Exercises

Problem 49 (SMO 2005 (Open) Q9)**.** The function $f(n)$ is defined for all positive integer n and take on non-negative integer values such that $f(2) = 0$, $f(3) > 0$ and $f(9999) = 3333$. Also, for all m and n,

$$f(m+n) - f(m) - f(n) = 0 \text{ or } 1.$$

Determine $f(2005)$.

Solution. The given relation implies that

$$f(m+n) \geq f(m) + f(n).$$

Putting $m = n = 1$ we obtain $0 = f(2) \geq 2f(1) \geq 0$. Thus $f(1) = 0$. Next,

$$f(3) = f(2+1) = f(2) + f(1) + x = x, \quad \text{where } x = 0 \text{ or } 1.$$

Since $f(3) > 0$, it follows that $f(3) = 1$. Since

$$f(\big(3(m+1)\big) \geq f(3m) + f(3),$$

it follows, by induction, that $f(3n) \geq n$ for all n. Also, it follows that $f(3k) > k$ for some k, then $f(3m) > m$ for all $m \geq k$.

So since $f(3\times3333) = f(9999) = 3333$, it follows that $f(3n) = n$ for $n \leq 3333$. In particular $f(3 \times 2005) = 2005$. Consequently,

$$2005 = f(3 \times 2005) \geq f(2 \times 2005) + f(2005) \geq 3f(2005)$$

and so $f(2005) \leq \frac{2005}{3} < 669$. On the other hand.

$$f(2005) \geq f(2004) + f(1) = f(3 \times 668) = 668.$$

Therefore $f(2005) = 668$. $\square$

Problem 50 (SMO 2005 (Open) Q12)**.** A function $f : \mathbb{N} \to \mathbb{N}$ satisfies

$$f(m+n) = f\left(f(m)+n\right)$$

for all $m, n \in \mathbb{N}$, and $f(6) = 2$. Also, no two of the values $f(6)$, $f(9)$, $f(12)$ and $f(15)$ coincide. How many three-digit positive integers n satisfy $f(n) = f(2005)$?

Solution. Since

$$f(6+n) = f\left(f(6)+n\right) = f(2+n) \quad \text{for all } n,$$

the function f is periodic with period 4 starting from 3 onwards.

Now $f(6), f(5), f(4), f(3)$ are four distinct values. Thus in every group of 4 consecutive positive integers greater than 3, there is exactly one that is mapped by f to $f(2005)$.

Since the collection of three-digit positive integers can be divided into exactly 225 groups of 4 consecutive integers each, there are 225 three-digit positive integer n that satisfies $f(n) = f(2005)$. □

Problem 51 (SMO 2005 (Open) Q13)**.** Let f be a real-valued function so that

$$f(x,y) = f(x,z) - 2f(y,z) - 2z$$

for all real numbers x, y and z. Find $f(2005,1000)$.

Solution. Setting $x = y = z$, we see that

$$f(x,x) = f(x,x) - 2f(x,x) - 2x \implies f(x,x) = -x \quad \forall x.$$

Setting $y = x$ gives

$$f(x,x) = f(x,z) - 2f(x,z) - 2x \implies f(x,z) = x - 2z \quad \forall x,z.$$

Therefore, $f(2005,1000) = 5$. □

Problem 52 (SMO 2006 (Open) Q3)**.** A function f is such that $f : \mathbb{R} \to \mathbb{R}$ where

$$f(xy+1) = f(x)f(y) - f(y) - x + 2$$

for all $x, y \in \mathbb{R}$. Find $10f(2006) + f(0)$.

Solution. By interchanging x and y, we have

$$f(yx+1) = f(y)f(x) - y + 2 \implies f(x) = y = f(y) + x.$$

Let $y = 0$. Then $f(x) = f(0) + x$. Putting $x = y = 0$, we get

$$f(0) + 1 = f(1) = f(0)f(0) - f(0) + 2 \implies \big(f(0) - 1\big)^2 = 0 \implies f(0) = 1.$$

Thus $100f(2006) + f(0) = 20071$. $\square$

Problem 53 (SMO 2006 (Open) Q9)**.** Suppose f is a function satisfying $f(x+x^{-1}) = x^6 + x^{-6}$, for all $x \neq 0$. Determine $f(3)$.

Solution. Let $y = x + x^{-1}$. Factoring the RHS,

$$\begin{aligned} f(y) &= x^6 + \frac{1}{x^6} \\ &= \left(x^2 + \frac{1}{x^2}\right)\left(x^4 - 1 + \frac{1}{x^4}\right) \\ &= \left(\left(x + \frac{1}{x}\right)^2 - 2\right)\left(\left(x^2 + \frac{1}{x^2}\right)^2 - 3\right) \\ &= \left(\left(x + \frac{1}{x}\right)^2 - 2\right)\left(\left(\left(x + \frac{1}{x}\right)^2 - 2\right)^2 - 3\right) \\ &= (y^2 - 2)\left((y^2 - 2)^2 - 3\right) \end{aligned}$$

Therefore $f(3) = 322$. $\square$

Problem 54 (SMO 2020 (Open) Q7)**.** Given that $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(a^2 - b^2) = (a - b)\big(f(a) + f(b)\big).$$

For all numbers a and b and that $f(1) = \frac{1}{101}$. Find the value of $\sum_{k=1}^{100} f(k)$.

Solution. Our first guess is $f(x)$, since $a^2 - b^2 = (a - b)(a + b)$. But upon looking at $f(1) = \frac{1}{101}$, we have $f(x) = \frac{x}{101}$.

Hence we evaluate

$$\sum_{k=1}^{100} f(k) = \frac{1}{101} + \frac{2}{101} + \cdots + \frac{100}{101} = \frac{1}{101}\left(\frac{100 \times 101}{2}\right) = \boxed{50}.$$

$\square$

Problem 55 (SMO 2021 (Open) Q6)**.** Consider all the polynomials $P(x, y)$ in two variables such that $P(0,0) = 2020$ and for all x and y, $P(x, y) = P(x+y, y-x)$. Find the largest possible value of $P(1,1)$.

Solution. Starting with $P(x, y) = P(x+y, y-x)$, $P(0,1) = P(1,1)$.

Then $P(0,1) = P\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $P\left(-\frac{1}{2}, \frac{1}{2}\right) = P\left(-\frac{1}{2}, 0\right)$ and $P\left(-\frac{1}{2}, 0\right) = P\left(-\frac{1}{4}, -\frac{1}{4}\right)$.

Then $P\left(-\frac{1}{4}, -\frac{1}{4}\right) = P\left(-\frac{1}{4}, 0\right)$ and $P\left(-\frac{1}{4}, 0\right) = P\left(-\frac{1}{8}, -\frac{1}{8}\right)$ and it will converge to $P(0,0)$. Thus $P(1,1) = P(0,0) = \boxed{2020}$. □

Problem 56 (IMO 2019)**.** Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b,

$$f(2a) + 2f(b) = f(f(a+b)).$$

Solution. First, we substitute $a = 0$ to get

$$f(0) + 2f(b) = f(f(b)).$$

It follows that

$$f(f(a+b)) = 2f(a+b) + f(0),$$

so we have

$$2f(a+b) + f(0) = f(2a) + 2f(b).$$

Substituting $a = 1$ (the motivation is that, since $f(x)$ takes the integers to the integers, it might be useful to relate $f(x+1)$ with $f(x)$) yields

$$2f(b+1) + f(0) = f(2) + 2f(b).$$

Rearranging this a little bit, we get

$$f(b+1) - f(b) = \frac{f(2) - f(0)}{2}.$$

Clearly, $\frac{f(2)-f(0)}{2}$ is constant, so it follows that $f(x)$ is linear.

Now, we let $f(x) = gx + h$. Substituting this back, we find that either $g = h = 0$ or $g = 2$.

Hence, we have $\boxed{f(x) = 2x + h \text{ for some constant h}}$, or $\boxed{f \equiv 0}$. □

Problem 57 (IMO 2015)**.** Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the equation

$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$$

for all real x and y.

Solution. Let $P(x,y)$ denote the assertion. Then, $P(0,y)$ gives $f(f(y)) + f(0) = f(y) + yf(0)$. Therefore, $y = 0$ gives $f(f(0)) = 0$ and $y = f(0)$ gives $2f(0) = f(0)^2$. This implies $f(0) = 0$ or $f(0) = 2$.

Case 1: $f(0) = 2$ Then, $f(2) = 0$ and $f(f(y)) = f(y) + 2y - 2$. This implies f is injective and $f(y) = y$ if and only if $y = 1$. Now, $P(x,1)$ gives $f(x + f(x + 1)) = x + f(x + 1)$, so $f(x + 1) = 1 - x$. Therefore, $f(x) = 2 - x$. This works because both sides are equal to $y + 2 - xy$.

Case 2: $f(0) = 0$ Then, $f(f(y)) = f(y)$. Now, $P(f(k), k - f(k))$ gives

$$f(2f(k)) + f(f(k)(k - f(k))) = 2f(k) + (k - f(k))f(k)$$

and $P(f(k),0)$ gives

$$f(2f(k)) = 2f(k).$$

This means that $f(f(k)(k - f(k))) = (k - f(k))f(k)$. Therefore, $P(k - f(k), f(k))$ gives

$$f(k) + f(f(k)(k - f(k))) = f(k) + (k - f(k))f(k) = k - f(k)f(k - f(k)),$$

so

$$(k - f(k))(f(k) - 1) = -f(k - f(k)).$$

Therefore, if $f(a) - a = f(b) - b \neq 0$, then $f(a) = f(b)$, so $a = b$. Since $P(1,-1)$ gives $f(1) + f(-1) = 1 - f(1)$ and $P(-1,1)$ gives $f(-1) + f(-1) = -1 + f(-1)$, we get $f(-1) = -1$ and $f(1) = 1$. Now, $P(1,y)$ gives $f(1 + f(1 + y)) - (1 + f(1 + y)) + f(y) - y = 0$, so if $g(x) = f(x) - x$, then $g(y) = -g(1 + f(1 + y))$. If $g(y) \neq 0$, then $g(y) = -g(1 + f(1 + y)) = g(1+f(1+1+f(1+y)))$, so $y-1 = f(f(y+1)+2)$. Therefore, $f(y-1) = y-1$. If $f(y+1) \neq y+1$, then $f(y) = y$, contradiction. Therefore, $f(y + 1) = y + 1$, so $f(y + 3) = y - 1$, which implies $f(y + 2) = y + 2$. However, $P(1, y + 2)$ gives $f(y) - y + f(y + 2) - (y + 2) = 0$, contradiction since $f(y + 2) = y + 2$ but $f(y) \neq y$. Therefore, we must have $f(y) = y$ for all y, which works since both sides are equal to $2x + y + xy$.

Therefore, the only solutions are $\boxed{f(x) = x}$ and $\boxed{f(x) = 2 - x}$. □

Problem 58 (CHINA 2016)**.** Find all functions $f : \mathbb{Z} \to \mathbb{Z}$, such that for $\forall m, n \in \mathbb{Z}$,

$$f(f(m+n)) = f(m) + f(n).$$

Solution. Let $a = f(0)$ and $c = f(1) - f(0)$ $f(m) + f(1) = f(f(m+1)) = f(m+1) + f(0)$ and so $f(m+1) = f(m) + c$ and so $f(x) = cx + a$

Plugging this back into the original equation, we get $\boxed{\text{S1: } f(x) = 0 \quad \forall x \in \mathbb{Z}}$, which indeed fits

$\boxed{\text{S2: } f(x) = x + a \quad \forall x \in \mathbb{Z}}$, which indeed fits, whatever is $a \in \mathbb{Z}$ □

Problem 59 (Jensen's Inequality)**.** Let f be a convex function over an interval I. Given $x_i \in I$ and real values $t_i \geq 0$ with $\sum_{i=1}^{n} t_i = 1$, for $1, 2, \ldots, n$, show that

$$f\left(\sum_{i=1}^{n} t_i x_i\right) \leq \sum_{i=1}^{n} t_i f(x_i).$$

Proof. Prove by induction. Let $P(n)$ be the statement that $f\left(\sum_{i=1}^{n} t_i x_i\right) \leq \sum_{i=1}^{n} t_i f(x_i)$ with the given conditions supplied by the question.

Base cases: $P(1)$ is trivial with $f(x_1) \leq f(x_1)$, and $P(2)$ is true by the definition of convexity for f [recall that f being convex is defined as $f\left(tx_1 + (1-t)x_2\right) \leq tf(x_1) + (1-t)f(x_2)$].

Assume the relation holds for $n = k$ for some positive integer k. Then we have

$$\begin{aligned}
f\left(\sum_{i=1}^{k+1} t_i x_i\right) &= f\left(t_{k+1}x_{k+1} + \sum_{i=1}^{k} t_i x_i\right) \\
&= f\left(t_{k+1}x_{k+1} + (1-t_{k+1})\frac{1}{1-t_{k+1}}\sum_{i=1}^{k} t_i x_i\right) \\
&\leq t_{k+1}f(x_{k+1}) + (1-t_{k+1})f\left(\frac{1}{1-t_{k+1}}\sum_{i=1}^{n} t_i x_i\right) \quad \text{[use result for } n = 2\text{]} \\
&= t_{k+1}f(x_{k+1}) + (1-t_{k+1})f\left(\sum_{i=1}^{k} \frac{t_i}{1-t_{k+1}} x_i\right) \\
&\leq t_{k+1}f(x_{k+1}) + (1-t_{k+1})\sum_{i=1}^{k} \frac{t_i}{1-t_{k+1}} f(x_i) \quad \text{[use induction hypothesis]} \\
&= t_{k+1}f(x_{k+1}) + \sum_{i=1}^{k} t_i f(x_i) = \sum_{i=1}^{k+1} t_i f(x_i)
\end{aligned}$$

□

6 Sequences and Series

§6.1 Summation Series

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \tag{6.1}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \tag{6.2}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2 \tag{6.3}$$

Proof. These can be proven using mathematical induction. □

§6.2 Arithmetic and Geometric Progressions

§6.2.1 Arithmetic Progression

An **arithmetic progression** (AP) is a sequence where the next term is a constant addition/subtraction of the previous term.

For a particular term,

$$u_n = a + (n-1)d$$

where a is the first term, d is the common difference.

The series is given by

$$S_n = \frac{[a + a + (n-1)d]n}{2} = \frac{n}{2}[2a + (n-1)d].$$

§6.2.2 Geometric Progression

A **geometric progression** (GP) is a sequence where the next term is a constant product of the previous term.

For a particular term,

$$u_n = ar^{n-1}$$

where a is the first term, r is the common ratio.

The series is given by

$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{or} \quad \frac{a(r^n-1)}{r-1}$$

where the former formula is preferred for $|r| < 1$.

If the sequence is convergent with $|r| < 1$, then the sum to infinity exists, given by

$$S_\infty = \frac{a}{1-r}.$$

§6.3 Telescoping Sums

A *telescoping sum* is a sum in which subsequent terms cancel each other, leaving only initial and final terms. For example,

$$\begin{aligned} S &= \sum_{i=1}^{n-1}(a_i - a_{i+1}) \\ &= (a_1 - a_2) + (a_2 - a_3) + \ldots + (a_{n-2} - a_{n-1}) + (a_{n-1} - a_n) \\ &= a_1 - a_n \end{aligned}$$

Exercise 6.3.1

Evaluate the following sum:

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}}.$$

Solution.

$$\begin{aligned} \frac{1}{\sqrt{n+1}+\sqrt{n}} &= \frac{\sqrt{n+1}-\sqrt{n}}{(\sqrt{n+1}+\sqrt{n})(\sqrt{n+1}-\sqrt{n})} \\ &= \sqrt{n+1}-\sqrt{n} \end{aligned}$$

Doing this for each fraction gives us

$$(\sqrt{2}-\sqrt{1}) + (\sqrt{3}-\sqrt{2}) + \cdots + (\sqrt{100}-\sqrt{99}) = \sqrt{100}-\sqrt{1} = 9$$

□

Exercise 6.3.2

Evaluate the following sum:

$$\sum_{n=1}^{2015} \frac{1}{n^2+3n+2}.$$

Solution. A common method is to use partial fractions which will cancel each other out.

$$\frac{1}{n^2+3n+2} = \frac{1}{n+1} - \frac{1}{n+2}$$

$$\begin{aligned}\sum_{n=1}^{2015}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) &= \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2016} - \frac{1}{2017}\right) \\ &= \frac{1}{2} - \frac{1}{2017} = \frac{2015}{4034}\end{aligned}$$

□

§6.4 Power Series

The *Taylor series* is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{6.4}$$

The *Maclaurin series* is a special case of Taylor Series, where $a = 0$; that is,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \tag{6.5}$$

The power series below can be easily computed:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{6.6}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{6.7}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{6.8}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{6.9}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \tag{6.10}$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots \tag{6.11}$$

§6.5 Generating Functions

Let $\{a_n\}$ be an infinite sequence of numbers.

The *ordinary generating function* (OGF) of the sequence is defined by the formal series

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \tag{6.12}$$

The *exponential generating function* (EGF) of the sequence is defined by the formal series

$$f(x) = a_0 + a_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + \cdots + a_n \frac{x^2}{n!} + \cdots \tag{6.13}$$

We are never required to consider the convergence of these functions. In fact, we can always substitute such a formal series with functions whose Taylor series is equal to the formal series.

§6.5.1 Properties of OGFs

Proposition 6.5.1. If $f(x)$ is the OGF of $\{a_n\}$, then the OGF of $\{a_{n+1}\}$ is

$$\frac{f(x) - a_0}{x}.$$

In general, the OGF of $\{a_{n+k}\}$ is given by

$$\frac{f(x) - a_0 - a_1 x - \cdots - a_{k-1} x^{k-1}}{x^k}.$$

Proof. If we let $g(x)$ to be the OGF of $\{a_{n+1}\}$ then

$$g(x) = a_1 + a_2 x + a_3 x^2 + \cdots$$

Thus

$$f(x) = a_0 + x\left(a_1 + a_2 x + \cdots\right) = a_0 + x g(x).$$

□

Proposition 6.5.2. If $f(x)$ is the OGF of $\{a_n\}$, then the OGF of $\{n a_n\}$ is $x\mathrm{D}f = x f'(x)$ where D is the differential operator $\frac{\mathrm{d}}{\mathrm{d}x}$. In general, the OGF of $\{n^k a_n\}$ is given by $(x\mathrm{D})^n f$. More generally, if P is a polynomial, then the OGF of $\{P(n) a_n\}$ is $P(x\mathrm{D}) f$.

Proposition 6.5.3. If $f(x)$ and $g(x)$ are the OGFs of $\{a_n\}$ and $\{b_n\}$ respectively, then $f(x)g(x)$ is the OGF of

$$\left\{ \sum_{k=0}^{n} a_k b_{n-k} \right\}.$$

§6.5.2 Properties of EGFs

§6.6 Exercise

Problem 60 (SMO Open 2013 Q1)**.** Evaluate the sum

$$\frac{1}{1\times2\times3}+\frac{1}{2\times3\times4}+\cdots+\frac{1}{100\times101\times102}.$$

Solution. □

SMO Open 2011 Q25

7 Recurrence Relations

• solution of (i) first order linear (homogeneous and nonhomogeneous) recurrence relations with constant coefficients (ii) second order linear homogeneous recurrence relations with constant coefficients

A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the *previous terms* $a_0, a_1, \ldots, a_{n-1}$.

We will study more closely linear homogeneous recurrence relations of degree k with constant coefficients:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers and $c_k \neq 0$.

Remark. linear=previous terms appear with exponent 1 (not squares, cubes, etc),

homogeneous=no term other than the multiples of a_i

degree k=expressed in terms of previous k terms

constant coefficients=coefficients in front of the terms are constants, instead of general functions.

Relevant examples:

- $a_{n+1} = a_n + k$

  Solution: $a_n = a_1 + (k-1)n$

- $a_{n+1} = a_n + n$

  Solution: $a_n = a_1 + 1 + 2 + \cdots + (n-1) = a_1 + \frac{n(n-1)}{2}$

- $a_{n+1} = a_n \cdot k$

  Solution: $a_n = a_1 k^{n-1}$

- $a_{n+1} = a_n \cdot n$

  Solution: $a_n = a_1 (n-1)!$

- Fibonacci sequence: $a_{n+1} = a_n + a_{n-1}$ (for $n \geq 2$), $a_1 = a_2 = 1$

- Cauchy equation: $f(x+y) = f(x) + f(y)$

  Solution: $f(x) = f(1)x$ for rational x

§7.1 First-order Recurrence Relations

The homogeneous case can be written in the following way:

$$x_n = rx^{n-1}, x_0 = A$$

Its general solution is

$$x_n = Ar^n$$

which is a geometric sequence with ratio r.

§7.2 Second-order Recurrence Relations

$$c_0x_n + c_1x_{n-1} + c_2x_{n-2} = 0$$

We first look for solutions in the form of $x_n = cr^n$. Plugging in the equation, we get

$$c_0cr^n + c_1cr^{n-1} + c_2cr^{n-2} = 0.$$

This simplifies to

$$c_0r^2 + c_1r + c_2 = 0$$

which is known as the *characteristic equation* of the recurrence. The roots r_1, r_2 of the above equation are known as the *characteristic roots*.

In the case of distinct real roots, the general solution is

$$x_n = c_1{r_1}^n + c_2{r_2}^n$$

where c_1 and c_2 are constants to be found.

Exercise 7.2.1: Fibonacci Equation

Let

$$f(n+2) = f(n+1) + f(n)$$

where $f(0) = 0, f(1) = 1$. Find a general formula for the sequence.

Solution. Consider the solution of the form

$$f(n) = \alpha^n$$

for some real number α. Then we have

$$\alpha^{n+2} = \alpha^{n+1} + \alpha^n$$

from which we conclude that $\alpha^2 - \alpha - 1$. Solving quadratically, we have

$$\alpha_1 = \frac{1+\sqrt{5}}{2}, \alpha_2 = \frac{1-\sqrt{5}}{2}.$$

Hence, a general solution of the sequence can be written as

$$f(n) = c_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + c_2 \left(\frac{1-\sqrt{5}}{2} \right)^n$$

where c_1 and c_2 are coefficients to be determined using the initial values.

By the initial conditions, we have

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 \left(\frac{1+\sqrt{5}}{2} \right) + c_2 \left(\frac{1-\sqrt{5}}{2} \right) &= 1 \end{aligned}$$

Thus we have

$$c_1 = \frac{1}{\sqrt{5}},\ c_2 = -\frac{1}{\sqrt{5}}.$$

Hence this gives us

$$\boxed{f(n) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n}.$$

$\square$

§7.3 Exercise

SMO Open 2017 Q24

SMO Open 2015 Q3

SMO Open 2012 Q21

Problem 61 (SMO Open 2006 Q20)**.** Let $a_1, a_2, \dots$ be a sequence satisfying the condition that

$$a_1 = 1 \quad \text{and} \quad a_n = 10a_{n-1} - 1 \quad \text{for all } n \geq 2.$$

Find the minimum n such that $a_n > 10^{100}$.

Solution. Note that $a_n - \frac{1}{9} = 10\left(a_{n-1} - \frac{1}{9}\right)$ for all $n \geq 2$. Let $b_n = a_n - \frac{1}{9}$. Then

$$b_n = 10b_{n-1} = \cdots = 10^{n-1}b_1 = \frac{8 \cdot 10^{n-1}}{9}.$$

Therefore $a_n = \dfrac{1 + 8 \cdot 10^{n-1}}{9}$. For $n \geq 2$, $8 \cdot 10^{n-2} < a_n < 10^{n-1}$. Thus $a_{101} < 10^{101} < a_{102}$ and the answer is 102. □

Problem 62 (SMO Open 2005 Q14)**.** Let $a_1 = 2006$, and for $n \geq 2$, $a_1 + a_2 + \cdots + a_n = n^2 a_n$. What is the value of $2005a_{2005}$?

Solution.

$$a_n = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} a_i = n^2 a_n - (n-1)^2 a_{n-1}.$$

This gives $a_n = \dfrac{n-1}{n+1} a_{n-1}$. Thus

$$a_{2005} = \frac{2004}{2006} a_{2004} = \cdots = \frac{2004 \times 2003 \times \cdots \times 1}{2006 \times 2005 \times \cdots \times 3} a_1 = \frac{2}{2005}.$$

□

Problem 63 (AUSTRALIA 2020 Q4)**.** Define the sequence $A_1, A_2, A_3, \ldots$ by $A_1 = 1$ and for $n = 1, 2, 3, \ldots$

$$A_{n+1} = \frac{A_{n+2}}{A_n + 1}.$$

Define the sequence $B_1, B_2, B_3, \ldots$ by $B_1 = 1$ and for $n = 1, 2, 3, \ldots$

$$B_{n+1} = \frac{{B_n}^2 + 2}{2B_n}.$$

Prove that $B_{n+1} = A_{2^n}$ for all non-negative integers n.

Proof. □

Part III

Geometry

8 Synthetic Geometry

Otherwise mentioned, let $\triangle ABC$ denote a triangle.

§8.1 Angles

The reader should be familiar with terms such as **segment**, **endpoint**, **midpoint**, **ray**, **origin**, **line**, **collinear**.

§8.1.1 Phantom Points

§8.2 Triangle

§8.2.1 Triangle centres

Definitions

Although there are many triangle centres, we will only be concerned with the following more important ones.

1. **Centroid** G: intersection of medians
2. **circumcentre** O: centre of the circle which passes through vertices of the triangle
3. **Incentre** I: intersection of the internal angle bisectors, which is also the centre of the unique circle inside the triangle tangent to all three sides
4. **Excentre** J: centre of the circle which is tangent to sides of the triangle but lies outside the triangle
5. **Orthocentre** H: intersection of altitudes

Along with triangle centres, there are some associated inscribed triangles.

1. **Intouch triangle**: the triangle whose vertices are the contact points of the incircle with the sides of ABC
2. **Medial triangle**: the triangle whose vertices are the midpoints of the sides of ABC
3. **Orthic triangle**: the triangle whose vertices are the feet of the altitudes

Properties

Centroid medians cut triangle into 6 parts of equal area G divides medians into segments of ratio 2:1

circumcentre R = a/2sinA = b/2sinB = c/2sinC

Incentre r=area/s (s is semiperimeter)

Orthocentre cyclic quadrilaterals H,A,B,C are incentre/excentres of orthic triangle

Point where angle bisector of A meets perp bisector of BC lies on circumcircle of ABC. This point is also the centre of circle passing through B, C, I, J_A

§8.2.2 Right triangle

Theorem 8.2.1: Pythagoras' Theorem

Given $\triangle ABC$ where the right angle is at C,

$$a^2 + b^2 = c^2 \tag{8.1}$$

Using similar triangles, we can deduce

$$\frac{1}{h^2} = \frac{1}{a^2} + \frac{1}{b^2} \tag{8.2}$$

where h is the perpendicular distance from C to a.

§8.2.3 Congruency and Similarity

Triangle similarity

1. Two angles match.
2. Two sides have lengths in the same ratio, and the angle between those two sides matches.
3. Three sides have lengths in the same ratio.

4. The triangles are right-angled and the ratio between one side and hypotenuse is the same.

Triangle congruence Same conditions but check that one corresponding side matches.

§8.2.4 Triangle Laws

We first begin with the well-known sine rule and cosine rule.

Theorem 8.2.2: Sine rule

Given $\triangle ABC$,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \tag{8.3}$$

where R denotes the circumradius of $\triangle ABC$.

Theorem 8.2.3: Cosine rule

Given $\triangle ABC$,

$$c^2 = a^2 + b^2 - 2ab\cos C \tag{8.4}$$

Exercise 8.2.1: Pythagorean Inequality

Use the law of cosines to justify the statement that if a, b and c are the sides of triangle $\triangle ABC$ and $a \leq b \leq c$, then $\triangle ABC$ is acute if $a^2 + b^2 > c^2$ and obtuse if $a^2 + b^2 < c^2$.

Proof. From the law of cosines,

$$\cos C = \frac{c^2 - a^2 - b^2}{-2ab}.$$

If $c^2 < a^2 + b^2$, then the numerator of RHS is negative, so $\cos C$ is positive and $\angle C$ is acute. Similarly, if $c^2 > a^2 + b^2$, $\cos C$ is negative and $\angle C$ is obtuse. □

Theorem 8.2.4: Viviani's Theorem

The sum of the distances from any interior point to the sides of an equilateral triangle equals the length of the triangle's altitude.

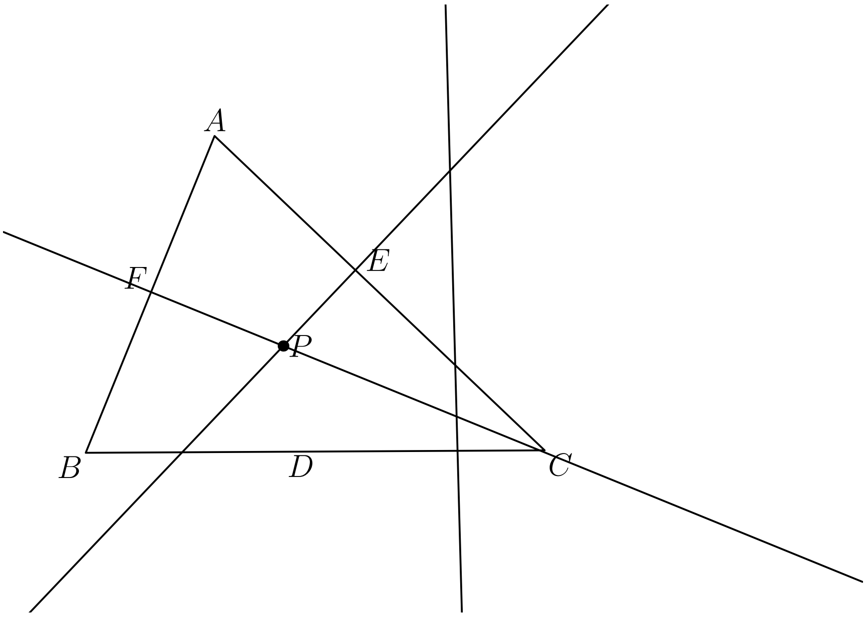
Proof. The core idea of the proof is to find the relationship between the area and the height of the triangle. Given $\triangle ABC$ is an equilateral triangle with height h and side length a. Let P denote an arbitrary point within the triangle.

Let s, t and u be the distances from P to BC, CA and AB respectively. □

Theorem 8.2.5: Carnot's Theorem

Given $\triangle ABC$. P lies in the interior of the triangle. D, E and F denote the foots of perpendicular from P to sides BC, CA and AB respectively. Then we have

$$AE^2 + BD^2 + CF^2 = AF^2 + BE^2 + CD^2 \tag{8.5}$$



Proof. By Pythagoras' Theorem, we have

$$PC^2 - CF^2 = PF^2$$
$$PA^2 - AF^2 = PF^2$$

This gives us

$$PC^2 - PA^2 = CF^2 - AF^2 \tag{1}$$

Similarly, we have

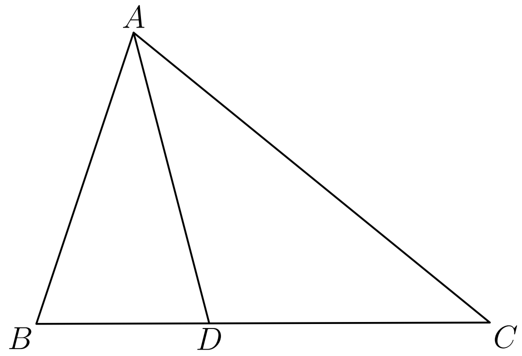
$$PA^2 - PB^2 = AE^2 - BE^2 \tag{2}$$
$$PB^2 - PC^2 = BD^2 - CD^2 \tag{3}$$

Adding (1), (2) and (3) gives us our desired outcome. □

Theorem 8.2.6: Angle bisector theorem

Given $\triangle ABC$. Let the angle bisector of $\angle A$ intersect BC at point D. Then we have

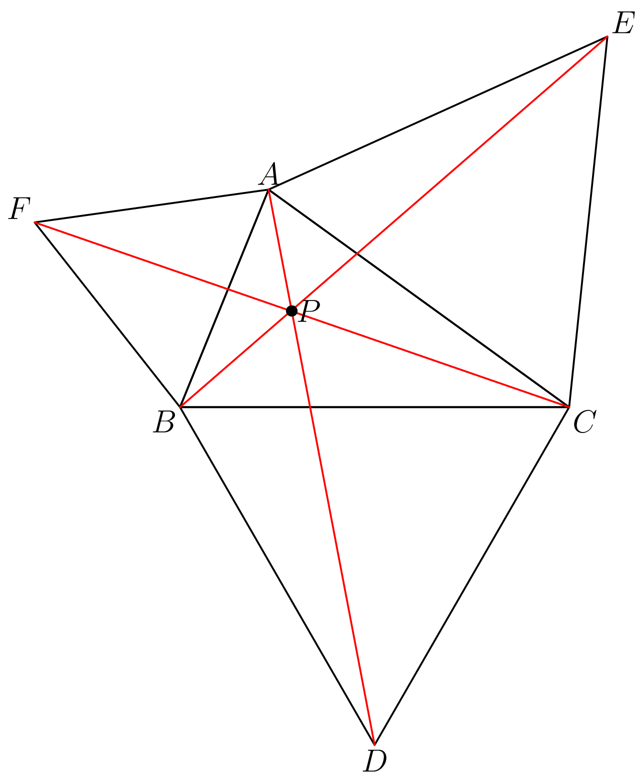
$$\frac{BD}{CD} = \frac{AB}{AC} \tag{8.6}$$



Definition 8.2.1. A **Fermat point** is a point in a triangle (with no angle greater than 120°) such that the sum of the distances from the vertices to the point is minimal.

One of the methods to construct the Fermat point – which is provided by Fermat – is as follows:

- Draw an equilateral triangle on any of the 2 sides of the triangle.
- For each new vertex of the equilateral triangles draw a line from the new vertex joining to the original triangles opposite vertex.
- The points intersect at the Fermat point.



§8.2.5 Area of triangle

Let $[ABC]$ denote the area of $\triangle ABC$.

We have the familiar formula

$$[ABC] = \frac{ah_a}{2} \tag{8.7}$$

Observe that using trigonometry, we have $h_a = b\sin C$. Hence this gives us

$$[ABC] = \frac{1}{2}ab\sin C \tag{8.8}$$

From sine rule, we have $\sin C = \frac{c}{2R}$. Substituting this into the above equation gives us

$$[ABC] = \frac{abc}{4R} \tag{8.9}$$

$$[ABC] = sr \tag{8.10}$$

where s denotes the semiperimeter, r denotes the radius of incircle.

Theorem 8.2.7: Heron's formula

Given $\triangle ABC$,

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)} \tag{8.11}$$

where s denotes the semiperimeter.

Proof. Our objective is to find the area of a triangle from just its three sides.

Since we know how to relate the angles of a triangle to the sides, we use $[ABC] = \frac{1}{2}ab\sin C$. Since $\sin C = \sqrt{1-\cos^2 C}$, we have

$$\begin{aligned}
[ABC] &= \frac{ab}{2}\sqrt{1-\cos^2 C} \\
&= \frac{ab}{2}\sqrt{1-\frac{(c^2-a^2-b^2)^2}{4a^2b^2}} \quad \text{[cosine rule]} \\
&= \sqrt{\frac{4a^2b^2-(c^2-a^2-b^2)^2}{16}} \\
&= \sqrt{\frac{(2ab-c^2+a^2+b^2)(2ab+c^2-a^2-b^2)}{16}} \\
&= \sqrt{\frac{[(a+b)^2-c^2][c^2-(a-b)^2]}{16}} \\
&= \sqrt{\frac{(a+b-c)(a+b+c)(a-b+c)(-a+b+c)}{16}} \\
&= \sqrt{s(s-a)(s-b)(s-c)}
\end{aligned}$$

□

$$[ABC] = 2R^2\sin A\sin B\sin C \tag{8.12}$$

Proof. □

Theorem 8.2.8: Stewart's Theorem

Given $\triangle ABC$. If cevian AD with length d divides BC into segments of lengths m and n, we have

$$cnc + bmb = dad + man. \tag{8.13}$$

You will usually use this theorem to find the lengths of angle bisectors and medians.

Proof. Since $\cos \angle ADB = -\cos \angle ADC$ (because the angles are supplementary), we can relate all the given lengths using cosine rule:

$$\cos \angle ADB = \frac{c^2 - d^2 - m^2}{-2dm} = -\frac{b^2 - d^2 - n^2}{-2dn} = -\cos \angle ADC$$

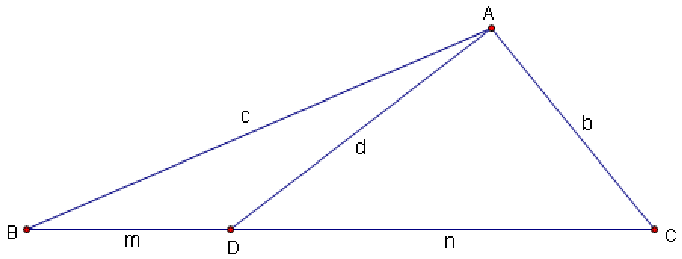
Multiplying both sides by $-2mnd$,

$$c^2n - d^2n - nm^2 = -b^2m + d^2m + n^2m.$$

Rearranging this, we have

$$c^2n + b^2m = d^2(m+n) + mn(m+n)$$

and notice that $m + n = BC = a$. □



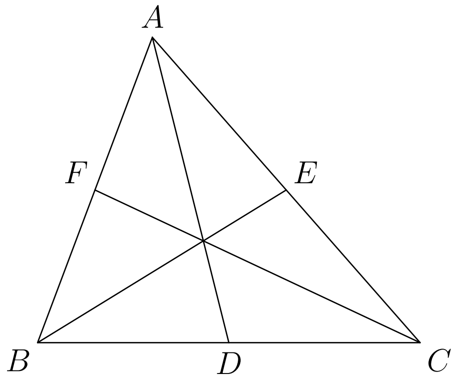
§8.2.6 Ceva and Menelaus

We will specifically look at these two theorems because they are of utmost importance in olympiad.

Theorem 8.2.9: Ceva's Theorem

Given $\triangle ABC$. Point P inside the triangle. Continue lines AP, BP and CP to intersect BC, CA and AB at D, E and F respectively. Then we have

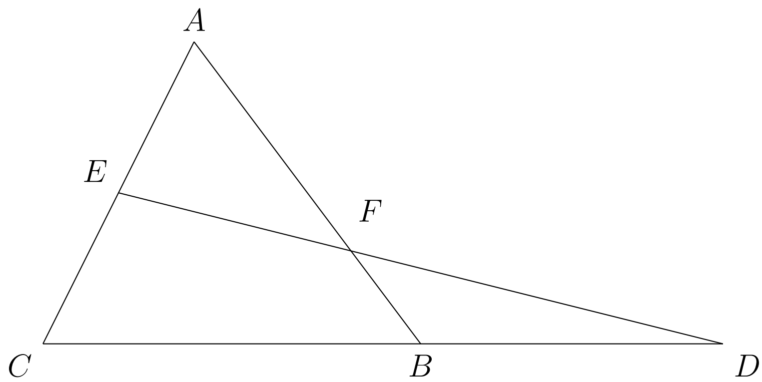
$$\frac{AF}{FB}\frac{BD}{DC}\frac{CE}{EA} = 1 \tag{8.14}$$



Theorem 8.2.10: Menelaus' Theorem

Given $\triangle ABC$. A transversal intersects BC, AC and AB at points D, E and F respectively. Then we have

$$\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = 1 \tag{8.15}$$

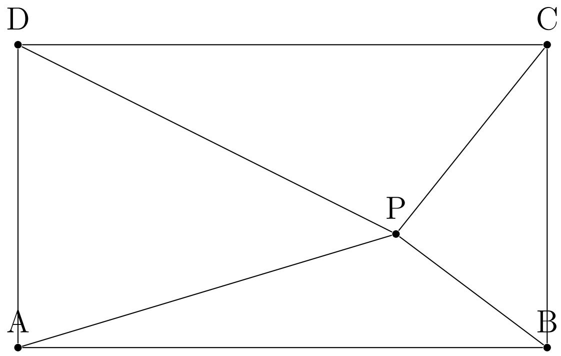


§8.3 Quadrilaterals

Theorem 8.3.1: British Flag Theorem

If $ABCD$ is a rectangle and P is a point inside of it, then we have

$$PA^2 + PC^2 = PB^2 + PD^2$$



Proof. This can be easily proven using Pythagoras' Theorem. □

§8.4 Circle

The reader should be familiar with basic circle terminology, such as **centre**, **radius**, **chord**, **diameter**, **tangent**, **secant**, **arc**, **sector**, **segment**, and **circumference**.

§8.4.1 Angles

- Angle subtended at the centre of a circle by a chord is twice the angle subtended on the circumference.
- All angles subtended by a fixed chord in the same segment of a circle are equal.

§8.4.2 Cyclic Quadrilaterals

Unlike triangles, not all quadrilaterals can be inscribed in a circle. Those which can be are known as **cyclic quadrilaterals**. Such quadrilaterals have the following special properties:

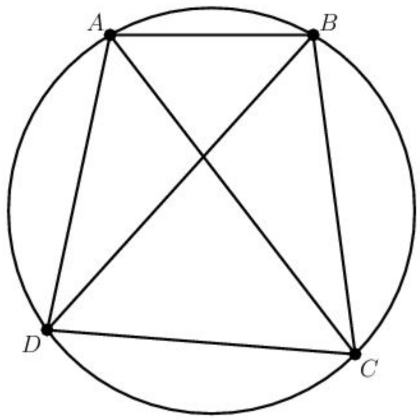
- Sum of opposite angles in a cyclic quadrilaterals is always 180°.
- When we draw the diagonals of a cyclic quadrilaterals, we form four pairs of equal inscribed angles.

Angle chasing and cyclic quadrilaterals

Theorem 8.4.1: Ptolemy's Theorem

Given a cyclic quadrilateral $ABCD$, the product of lengths of diagonals is equal to the sum of products of lengths of the pairs of the opposite sides:

$$AC \cdot BD = AB \cdot CD + AD \cdot BC \tag{8.16}$$



Theorem 8.4.2: Ptolemy's Inequality

For four points A, B, C, D in the plane,

$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD \tag{8.17}$$

where equality holds if and only if $ABCD$ is a cyclic quadrilateral with diagonals AC and BD (or a trivial case where A, B, C and D are collinear.

Proof. We construct a point P such that the triangles APB and DCB are similar and have the same orientation. This means that

$$BD = \frac{BA \cdot DC}{AP} \tag{1}$$

But since this is a spiral similarity, we also know that the triangles ABD and PBC are also similar, which implies that

$$BD = \frac{BC \cdot AD}{PC} \tag{2}$$

By the triangle inequality, we have $AP + PC \geq AC$. Multiplying both sides of the inequality by BD and using equations (1) and (2) gives us

$$BA \cdot DC + BC \cdot AD \geq AC \cdot BD$$

which is the desired inequality. Equality holds iff A, P, C are collinear. But since the triangles BAP and BDC are similar, this would imply that the angles BAC and BDC are congruent, i.e. that $ABCD$ is a cyclic quadrilateral. □

Theorem 8.4.3: Brahmagupta's Formula

Given a cyclic quadrilateral $ABCD$,

$$[ABCD] = \sqrt{(s-a)(s-b)(s-c)(s-d)} \tag{8.18}$$

where s denotes the semiperimeter.

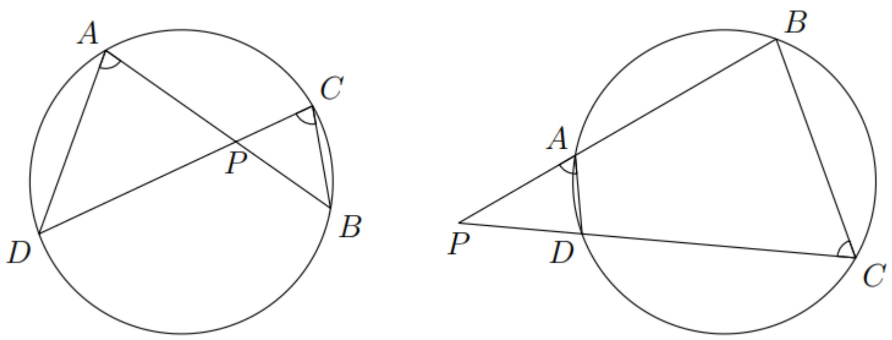
§8.4.3 Power of a Point

Power of a point is a frequently used tool in Olympiad geometry.

Theorem 8.4.4: Power of a point

Let Γ be a circle, and P be a point. Let a line through P meet Γ at points A and B, and another line through P meet Γ at points C and D. Then

$$PA \cdot PB = PC \cdot PD \tag{8.19}$$



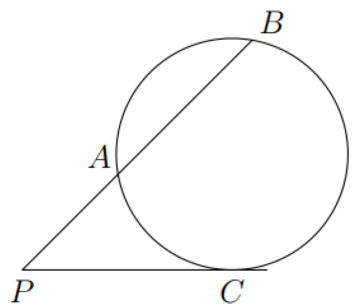
Proof. There are two configurations to consider, depending on whether P lies inside the circle or outside the circle.

When P lies inside the circle, we have $\angle PAD = \angle PCB$ and $\angle APD = \angle CPB$, so triangles PAD and PCB are similar. Hence $\dfrac{PA}{PD} = \dfrac{PC}{PB}$. Rearranging, we get $PA \cdot PB = PC \cdot PD$.

When P lies outside the circle, we have $\angle PAD = \angle PCB$ and $\angle APD = \angle CPB$, so again triangles PAD and PCB are similar. We get the same result in this case. $\square$

As a special case, when P lies outside the circle and $C = D$ (PC is a tangent), we have

$$PA \cdot PB = PC^2 \tag{8.20}$$



Theorem 8.4.5: Converse to Power of a Point

Let A, B, C, D be four distinct points. Let lines AB and CD intersect at P. Assume that either (1) P lies on both line segments AB and CD, or (2) P lies on neither line segments. Then A, B, C, D are concyclic if and only if $PA \cdot PB = PC \cdot PD$.

Proof. The expression $PA \cdot PB = PC \cdot PD$ can be rearranged as $\frac{PA}{PD} = \frac{PC}{PB}$. In both configurations described in the statement of the theorem, we have $\angle APD = \angle CPB$. It follows by angles and ratios that triangles APD and CPB are similar.

Thus $\angle PAD = \angle PCB$. In both cases this implies that A, B, C and D are concyclic. □

Definition 8.4.1. Suppose that Γ has centre O and radius r. We say that the **power** of point P with respect to Γ is

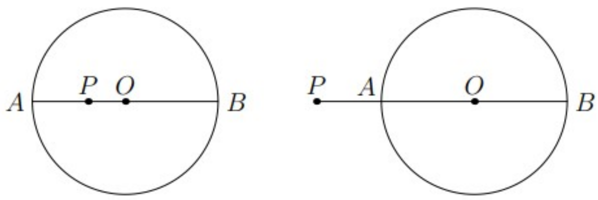
$$OP^2 - r^2.$$

Let line PO meet Γ at points A and B, so that AB is a diameter. We will use *directed lengths*, meaning that for collinear points P, A, B, an expression such as $PA \cdot PB$ is assigned a positive value if PA and PB point in the same direction, and a negative value if they point in opposite directions. Then

$$PA \cdot PB = (PO + OA)(PO + OB) = (PO - r)(PO + r) = PO^2 - r^2,$$

which is the power of P. So the power of a point theorem says that this quantity equals to $PC \cdot PD$, where C and D are the intersections with Γ of any line through P.

By convention, the power of P is negative when P is inside the circle, and positive when P is outside the circle. When P is outside the circle, the power equals to the square of the length of the tangent from P to the circle.



Let Γ_1 and Γ_2 be two circles with different centres O_1 and O_2, and radii r_1 and r_2 respectively.

Definition 8.4.2. The **radical axis** of Γ_1 and Γ_2 is the set of points with equal powers with respect to both circles.

$${PO_1}^2 - {r_1}^2 = {PO_2}^2 - {r_2}^2$$

which can be represented as

$$pow(P, \Gamma_1) = pow(P, \Gamma_2).$$

Lemma 8.4.1. The radical axis is a line perpendicular to the line connecting the circles' centres.

Proof. We first prove two lemmas.

Lemma 8.4.2. Let P be a point in the plane, and let P' be the foot of the perpendicular from P to O_1O_2. Then

$$pow(P,\Gamma_1) - pow(P,\Gamma_2) = pow(P',\Gamma_1) - pow(P',\Gamma_2).$$

The proof of the lemma is an easy application of the Pythagorean Theorem.

Lemma 8.4.3. There is a unique point P on line O_1O_2 such that $pow(P,O_1) = pow(P,O_2)$.

Proof: First show that P lies between O_1 and O_2 via proof by contradiction, by using a bit of inequality theory and the fact that $O_1O_2 > r_1 + r_2$. Then, use the fact that $O_1P + PO_2 = O_1O_2$ (a constant) to prove the lemma.

The first lemma shows that every point on the plane can be equivalently mapped to a line on O_1O_2. The second lemma shows that only one point in this mapping satisfies the given condition. Combining these two lemmas shows that the radical axis is a line perpendicular to ℓ, hence proved. □

When Γ_1 and Γ_2 intersect, the intersection points A and B both have a power of 0 with respect to either circle, so A and B must lie on the radical axis. This shows that the radical axis *coincides with the common chord* when the circles intersect.

To show that some point lies on the radical axis or the common chord, we can show that the point has *equal powers with respect to the two circles.*

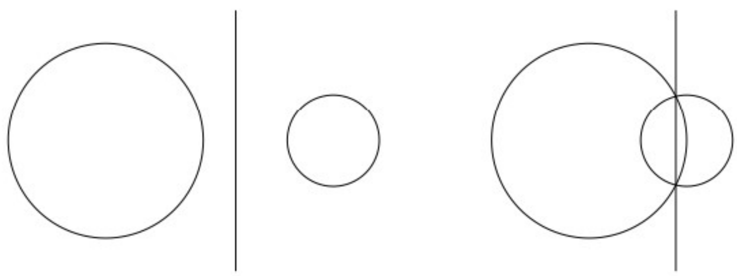


Figure 8.1: Radical axis

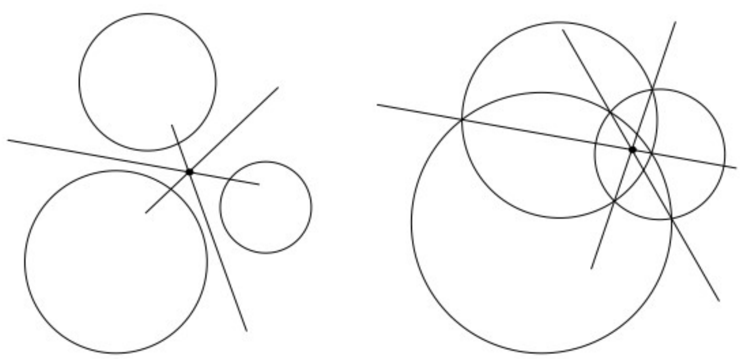


Figure 8.2: Radical centre

Theorem 8.4.6: Radical Axis Theorem

Given three circles, no two concentric, the three pairwise radical axes (which are non-parallel) are concurrent, at a point known as the radical centre.

Proof. Denote the three circles by $\Gamma_1, \Gamma_2, \Gamma_3$, and denote the radical axes of Γ_i and Γ_j by ℓ_{ij}.

Suppose that the radical axes are not all parallel. Let ℓ_{12} and ℓ_{13} meet at X. Since X lies on ℓ_{12}, it has equal powers with respect to Γ_1 and Γ_2. Since X lies on ℓ_{13}, it has equal powers with respect to Γ_1 and Γ_3. Therefore, X has equal powers with respect to all three circles, and hence it must lie on ℓ_{23} as well. □

§8.4.4 Euler's Line and Nine-point Circle

Theorem 8.4.7: Hamilton's theorem

For $\triangle ABC$ with circumcentre O and orthocentre H,

$$\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$$

Theorem 8.4.8: Euler line

The circumcentre O, the centroid G and the orthocentre H are collinear.
In fact, we have

$$GH = 2OG$$

and

$$OI^2 = R^2 - 2Rr$$

Theorem 8.4.9: Nine-point circle

The midpoints of each side of the triangle, the feet of each altitude, and the midpoints of the line segments from each vertex to the orthocentre, all lie on a single circle.

The centre of this circle lies on the Euler line, at the midpoint between the orthocentre and circumcentre.

The radius of this circle is half the circumradius of the triangle.

§8.4.5 Simson Line

§8.4.6 Miquel's Theorem

Theorem 8.4.10: Miquel's theorem

Let ABC be a triangle, and let X, Y, Z be points on lines BC, CA, AB respectively. Assume that the six points A, B, C, X, Y, Z are all distinct. Then the circumcircles of triangles AYZ, BZX, CXY pass through a common point.

Proof. The proof involves angle chasing. □

The circumcircles of the four triangles in a complete quadrilateral are concurrent.

§8.5 Exercises

Problem 64 (SMO 2005 (Open) Q2)**.** Circles C_1 and C_2 have radii 3 and 7 respectively. The circles intersect at distinct points A and B. A point P outside C_2 lies on the line determined by A and B at a distance of 5 from the center of C_1. Point Q is chosen on C_2 so that PQ is tangent to C_2 at Q. Find the length of the segment PQ.

Solution. The point P lies on the radical axis of C_1 and C_2, namely AB, and hence has equal power with respet to both circles. Thus $PQ^2 = 5^2 - 3^2 = 16$. □

Problem 65 (SMO 2005 (Open) Q10)**.** It is known that the three sides of a triangle are consecutive positive integers and the largest angle is twice the smallest angle. Find the perimeter of this triangle.

Solution. Let $\angle C = 2\angle A$ and CD the bisector of $\angle C$. Let $BC = x-1$, $CA = x$, $AB = x+1$. Then $\triangle ABC \sim \triangle CBD$, which implied

$$\frac{BD}{BC} = \frac{BC}{AB} \implies BD = \frac{(x-1)^2}{x+1}$$

and

$$\frac{CD}{AC} = \frac{CB}{AB} \implies AD = CD = \frac{x(x-1)}{x+1}.$$

Since $AB = AD + BD$, we have

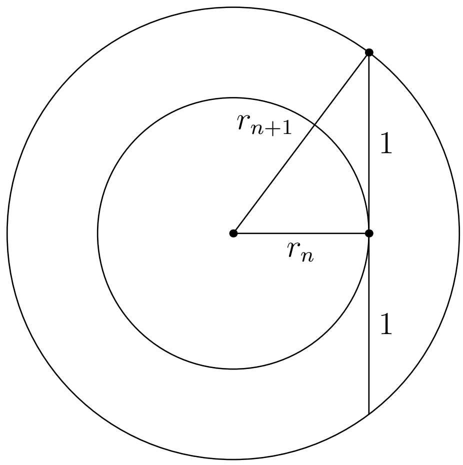
$$\frac{x(x-1)}{x+1} + \frac{(x-1)^2}{x+1} = x+1.$$

The only positive solution to this equation is $x = 5$. Hence the perimeter is 15. □

Problem 66 (Oxford MAT 2022)**.** 100 circles all share the same centre, the n-th circle named as C_n. For each whole number n between 1 and 99 inclusive, a tangent to circle C_n intersects circle C_{n+1} at two points, separated by a distance of 2.

Given that C_1 has radius 1, what is the radius of C_{100}?

Solution. The relationship between circle C_n of radius r_n and circle C_{n+1} of radius r_{n+1} is shown below.



The tangent is perpendicular to the radius, so there is a right-angled triangle with hypotenuse r_{n+1} and other sides r_n and 1. By Pythagoras, we have

$$r_{n+1}{}^2 = r_n{}^2 + 1.$$

Since $r_1{}^2 = 1$, we have $r_2{}^2 = 2$ and $r_3{}^2 = 3$ and so on, up to $r_{100}{}^2 = 100$, so the radius of C_{100} is 10. $\square$

9 Trigonometry

§9.1 Basic Definitions

Trigonometric functions describe how the ratio of lengths vary according to the angle between the lengths.

For a triangle $\triangle ABC$ with $\angle C = 90°$, we define **sine** as

$$\sin A = \frac{a}{c} \tag{9.1}$$

and **cosine** as

$$\cos A = \frac{b}{c}. \tag{9.2}$$

We define **tangent** as the ratio of sine and cosine; that is,

$$\tan A = \frac{\sin A}{\cos A} = \frac{a}{b}. \tag{9.3}$$

We also define the reciprocals of the above trigonometric functions: **cosecant**, **secant** and **cotangent** are the reciprocals of sine, cosine and tangent respectively.

$$\operatorname{cosec} A = \frac{1}{\sin A} \tag{9.4}$$

$$\sec A = \frac{1}{\cos A} \tag{9.5}$$

$$\cot A = \frac{1}{\tan A} \tag{9.6}$$

§9.2 Formulae

§9.2.1 Pythagorean identities

The following equation is known as the **Pythagorean identity**.

$$\sin^2 A + \cos^2 A = 1 \tag{9.7}$$

Proof.

$$\sin^2 A + \cos^2 A = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = \frac{a^2 + b^2}{c^2}$$

By Pythagoras' Theorem, for a right-angled triangle, $a^2 + b^2 = c^2$. Hence we have our desired result. □

The following two equations are corollaries of eq. (9.7); they can be easily derived and shall be left as an exercise for the reader.

$$\tan^2 A + 1 = \sec^2 A$$

$$1 + \cot^2 A = \operatorname{cosec}^2 A$$

Exercise 9.2.1

Compute $\sin^2 5° + \sin^2 10° + \cdots + \sin^2 90°$.

Solution.

$$\begin{aligned}
&\sin^2 5° + \sin^2 10° + \cdots + \sin^2 90° \\
&= (\sin^2 5° + \sin^2 85°) + \cdots + (\sin^2 40° + \sin^2 50°) + \sin^2 45° + \sin^2 90° \\
&= (\sin^2 5° + \cos^2 5°) + \cdots + (\sin^2 40° + \cos^2 40°) + \frac{1}{2} + 1 \\
&= 8(1) + \frac{1}{2} + 1 \\
&= \boxed{9\frac{1}{2}}
\end{aligned}$$

□

Exercise 9.2.2

Evaluate

$$\tan 10° \tan 20° \tan 30° \cdots \tan 80°.$$

Solution. Writing this in terms of sines and cosines, we have

$$\frac{\sin 10° \sin 20° \sin 30° \cdots \sin 80°}{\cos 10° \cos 20° \cos 30° \cdots \cos 80°}.$$

Applying $\sin x = \cos(90° - x)$ to each term in the numerator, we get

$$\frac{\cos 80° \cos 70° \cos 60° \cdots \cos 10°}{\cos 10° \cos 20° \cos 30° \cdots \cos 80°} = \boxed{1}$$

□

§9.2.2 Addition formulae

$$\begin{aligned}
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\
\tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}
\end{aligned}$$

Double-angle formulae

$$\begin{aligned}
\sin 2A &= 2 \sin A \cos A \\
\cos 2A &= \cos^2 A - \sin^2 A \\
&= 2\cos^2 A - 2 \\
&= 2 - 2\sin^2 A \\
\tan 2A &= \frac{2 \tan A}{1 - \tan^2 A}
\end{aligned}$$

Triple-angle formulae

$$\begin{aligned}
\sin 3A &= 3 \sin A - 4 \sin^3 A \\
\cos 3A &= 4 \cos^3 A - 3 \cos A \\
\tan 3A &= \frac{3 \tan A - \tan^3 A}{1 - 3\tan^2 A}
\end{aligned}$$

Half-angle formulae

The following formulae are corollaries of the double angle formulae:

$$\sin \frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{2}}$$

$$\cos \frac{A}{2} = \pm\sqrt{\frac{1 + \cos A}{2}}$$

Multiple-angle formulae

To generalise, multiple-angle formulae are given by

$$\sin nA = \sum_{k=0}^{n} \binom{n}{k} \cos^k A \sin^{n-k} A \sin \frac{n-k}{2}\pi$$

$$\cos nA = \sum_{k=0}^{n} \binom{n}{k} \cos^k A \sin^{n-k} A \cos \frac{n-k}{2}\pi$$

Sum to product

The sum-to-product formulae can be derived from the addition formulae.

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

Product to sum

The product-to-sum formulae can be, in turn, simply observed from the sum-to-product formulae.

$$\sin A\cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$\sin A\sin B = -\frac{1}{2}\left[\cos(A+B) - \cos(A-B)\right]$$

§9.2.3 R-formula

It is rather difficult to immediately the amplitude of some function of a combination of trigonometric functions, given by $f(x) = a\sin\theta + b\cos\theta$. To do so, we combine the two trigonometric functions using the **R-formula**.

$$a\sin\theta \pm b\cos\theta = \sin(\theta \pm \alpha)$$

$$a\cos\theta \mp b\cos\theta = \cos(\theta \pm \alpha)$$

where $R = \sqrt{a^2+b^2}$, $\alpha = \tan^{-1}\dfrac{b}{a}$ where $0 < \alpha < \frac{\pi}{4}$.

§9.3 Hyperbolic Functions

§9.3.1 Basics

The three main hyperbolic functions are:

$$\sinh x = \frac{e^x - e^{-x}}{2} \tag{9.8}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \tag{9.9}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} \tag{9.10}$$

It is easy to see that

$$\sinh x + \cosh x = e^x$$

§9.3.2 Reciprocals and Inverses

Again these functions all have their inverse functions:

$$\sinh^{-1} x = \operatorname{arsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right) \tag{9.11}$$

$\operatorname{arsinh} x$ has domain $x \geq 1$.

$$\cosh^{-1} x = \operatorname{arcosh} x = \ln\left(x + \sqrt{x^2 - 1}\right) \tag{9.12}$$

$\operatorname{arcosh} x$ has domain $x \geq 1$.

$$\tanh^{-1} x = \operatorname{artanh} x = \frac{1}{2}\ln(1 + x) - \frac{1}{2}\ln(1 - x) \tag{9.13}$$

$\operatorname{artanh} x$ has domain $-1 < x < 1$.

As well as their reciprocal functions:

$$\operatorname{cosech} x = \frac{1}{\sinh x} \tag{9.14}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} \tag{9.15}$$

$$\coth x = \frac{1}{\tanh x} \tag{9.16}$$

§9.3.3 Identities

Hyperbolic function identities have very similar forms to the trigonometric identities.

However there is one key difference outlined in **Osborn's rule**: all the identities for the hyperbolic functions are exactly the same as the trigonometric identities, except whenever a product of two sinh functions is present we put a minus sign in front. For example if a trigonometric formula involved a $\sin^2 x$, then the corresponding hyperbolic formula would contain a $-\sinh^2 x$ instead.

Trigonometric	Hyperbolic
$\cos^2\theta + \sin^2\theta = 1$	$\cosh^2 x - \sinh^2 x = 1$
$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sinh(A \pm B) = \sinh A \cosh B \pm \cosh A \sinh B$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\cosh(A \pm B) = \cosh A \cosh B \mp \sinh A \sinh B$
$\cos 2\theta = \cos^2\theta - \sin^2\theta$	$\cosh 2\theta = \cosh^2\theta + \sinh^2\theta$
$\sin 2\theta = 2\sin\theta\cos\theta$	$\sinh 2\theta = 2\sinh\theta\cosh\theta$
$1 + \tan^2\theta = \sec^2\theta$	$1 - \tanh^2\theta = \mathrm{sech}^2\,\theta$
$1 + \cot^2\theta = \mathrm{cosec}^2\,\theta$	$1 - \coth^2\theta = -\,\mathrm{cosech}^2\,\theta$

§9.4 Exercises

Problem 67 (SMO Open 2023 Q11)**.** Let ABC be a triangle satisfying the following conditions that $\angle A + \angle C = 2\angle B$ and $\dfrac{1}{\cos A} + \dfrac{1}{\cos C} = \dfrac{-\sqrt{2}}{\cos B}$. Determine the value of $\cos \dfrac{A-C}{2}$.

Solution. Given that $\angle A + \angle C = 2\angle B$, we can easily see that $\angle B = 60° \implies \cos B = \dfrac{1}{2}$ and $\angle A + \angle C = 120°$.

$$\begin{aligned}\frac{1}{\cos A} + \frac{1}{\cos C} &= -2\sqrt{2} \\ \cos A + \cos C &= -2\sqrt{2}\cos A \cos C\end{aligned}$$

Applying sum-to-product to the LHS and product-to-sum to the RHS gives

$$2\cos\frac{A+C}{2}\cos\frac{A-C}{2} = -\sqrt{2}\left[\cos(A+C) + \cos(A-C)\right]$$

Let $x = \cos\dfrac{A-C}{2}$. Then since $A + C = 120°$,

$$\begin{aligned}x &= -\sqrt{2}\left(-\frac{1}{2} + 2x^2 - 1\right) \\ 2\sqrt{2}x^2 + x - \frac{3}{2}\sqrt{2} &= 0 \\ x = -\frac{3}{\sqrt{2}} \text{ (rej.)} \quad &\text{or} \quad x = \frac{1}{\sqrt{2}}\end{aligned}$$

□

Problem 68 (SMO Open 2023 Q19)**.** Let ABC be a triangle with $AB = c$, $AC = b$ and $BC = a$, and satisfies the conditions $\tan C = \dfrac{\sin A + \sin B}{\cos A + \cos B}$, $\sin(B - A) = \cos C$ and area of triangle ABC is $3 + \sqrt{3}$. Determine the value of $a^2 + c^2$.

Solution.

$$\tan C = \frac{\sin A + \sin B}{\cos A + \cos B} = \frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}} = \frac{\sin\frac{A+B}{2}}{\cos\frac{A+B}{2}} = \tan\frac{A+B}{2} \iff C = \frac{A+B}{2}$$

Since $A + B = 180° - C$, we then have $C = 60°$.

$$\sin(B - A) = \cos C = \cos(180° - A - B) = \sin(A + B - 90°)$$

which implies that $B - A = A + B - 90°$. Thus $A = 45°, B = 75°$.

Given the area of the triangle,

$$\begin{aligned}3+\sqrt{3} &= \frac{1}{2}ac\sin B\\ &= \frac{1}{2}ac\sin(45^\circ+30^\circ)\\ &= \frac{1}{2}ac(\sin 45^\circ\cos 30^\circ+\cos 45^\circ\sin 30^\circ)\\ &= \frac{1}{2}ac\left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right)\end{aligned}$$

Hence

$$ac = \frac{4\sqrt{2}(3+\sqrt{3})}{1+\sqrt{3}} = 4\sqrt{6}.$$

By Sine Rule, we have $\frac{a}{\sin A} = \frac{c}{\sin C}$, giving us $\frac{a}{\sqrt{2}} = \frac{c}{\sqrt{3}}$. Hence

$$a\left(\frac{\sqrt{3}a}{\sqrt{2}}\right) = 4\sqrt{6} \implies a^2 = 8 \quad \text{and} \quad \left(\frac{\sqrt{2}c}{\sqrt{3}}\right) = 4\sqrt{6} \implies c^2 = 12.$$

This gives us $a^2+c^2=20$. □

Problem 69 (SMO Open 2021 Q1)**.** Given that $\frac{\pi}{2} < \beta < \alpha < \frac{3\pi}{4}$, $\cos(\alpha-\beta) = \frac{12}{13}$ and $\sin(\alpha+\beta) = -\frac{3}{5}$. Find $\sin 2\alpha$.

Solution. Note that $\alpha-\beta$ is in the first quadrant and $\alpha+\beta$ is in the third quadrant.

$$\begin{aligned}\sin 2\alpha &= \sin[(\alpha+\beta)+(\alpha-\beta)]\\ &= \sin(\alpha+\beta)\cos(\alpha-\beta)+\cos(\alpha+\beta)\sin(\alpha-\beta)\\ &= \left(-\frac{3}{5}\right)\left(\frac{12}{13}\right)+\left(-\frac{4}{5}\right)\left(\frac{5}{13}\right) = \boxed{-\frac{56}{65}}\end{aligned}$$

□

Problem 70 (SMO Open 2018 Q13)**.** Let $\triangle ABC$ be a triangle with $a = BC$, $b = AC$ and $c = AB$. Given that $a+c = 2b$, $\angle A - \angle C = \frac{\pi}{3}$, find $\sin B$.

Solution. From Sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Since $a+c=2b$, we have

$$2\sin B = \sin A + \sin C$$

or

$$2\left(2\sin\frac{B}{2}\cos\frac{B}{2}\right) = 2\sin\frac{A+C}{2}\cos\frac{A-C}{2}.$$

Since $A+B+C=\pi$,

$$2\sin\frac{A+C}{2}\cos\frac{A-C}{2} = 2\sin\frac{\pi-B}{2}\cos\frac{\frac{\pi}{3}}{2} = 2\cos\frac{B}{2}\frac{\sqrt{3}}{2}.$$

Hence we have $\sin\frac{B}{2}=\frac{\sqrt{3}}{4}$ and thus $\cos\frac{B}{2}=\frac{\sqrt{13}}{4}$. Therefore

$$\sin B = 2\sin\frac{B}{2}\cos\frac{B}{2} = \boxed{\frac{\sqrt{39}}{8}}.$$

□

Problem 71 (SMO Open 2013 Q9)**.** Let $A=\cos^2 10°+\cos^2 50°-\sin 40°\sin 80°$. Determine the value of A.

Solution. Let $B=\sin^2 10°+\sin^2 50°-\cos 40°\cos 80°$.

Then

$$\begin{aligned}
A+B &= 2-\cos 40° \\
A-B &= (\cos^2 10° - \sin 10°) + (\cos^2 50° - \sin^2 50°) + (\cos 40°\cos 80° - \sin 40°\sin 80°) \\
&= \cos 20° + \cos 100° + \cos(40°+80°) \\
&= \cos 20° + \cos 100° + \cos 120° \\
&= 2\cos 60°\cos 40° - \cos 60° \\
&= \cos 40° - \frac{1}{2}
\end{aligned}$$

Adding up the two equations gives us $2A=\frac{3}{2}$. Hence $\boxed{A=\frac{3}{4}}$. □

Problem 72 (SMO Open 2011 Q16)**.** Determine the value of $\frac{3}{\sin^2 20°}-\frac{1}{\cos^2 20°}+64\sin^2 20°$.

Solution.

$$\begin{aligned}
&\frac{3}{\sin^2 20°}-\frac{1}{\cos^2 20°}+64\sin^2 20° \\
&= \frac{6}{1-\cos 40°}-\frac{2}{1+\cos 40°}+32(1-\cos 40°) \quad \text{[double angle formula]} \\
&= \frac{6(1+\cos 40°)}{1-\cos^2 40°}-\frac{2(1-\cos 40°)}{1-\cos^2 40°}-32\cos 40°+32 \\
&= \frac{4(1-6\cos 40°+8\cos^3 40°)}{1-\cos^2 40°}+32 \\
&= \frac{4(1+2(4\cos^3 40°-3\cos 40°))}{1-\cos^2 40°}+32 \\
&= \frac{4\left(1+2\cos 3(40°)\right)}{1-\cos^2 40°}+32 \quad \text{[triple angle formula]} \\
&= \frac{4\left(1-2\left(\frac{1}{2}\right)\right)}{1-\cos^2 40°} = \boxed{32}
\end{aligned}$$

□

Problem 73 (SMO Open 2009 Q1)**.** Evaluate the expression $\sin 10° \cos 20° \cos 30° \cos 40°$.

Solution. □

Problem 74 (SMO Open 2009 Q6)**.** Find the value of $\dfrac{\sin 80°}{\sin 20°} - \dfrac{\sqrt{3}}{2\sin 80°}$.

Solution. □

Problem 75 (SMO Open 2009 Q22)**.** Evaluate $\displaystyle\sum_{k=0}^{\infty} \frac{2}{\pi} \tan^{-1} \frac{2}{(2k+1)^2}$.

Solution. We can rewrite the fraction as

$$\frac{2}{(2k+1)^2} = \frac{2}{4k^2+4k+1} = \frac{2k+2-2k}{1+2k(2k+2)}$$

which is in the same form as $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A \tan B}$. Thus

$$\tan\left(\tan^{-1}(2k+2) - \tan^{-1}(2k)\right) = \frac{2k+2-2k}{1+2k(2k+2)}.$$

Therefore

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{2}{\pi} \tan^{-1} \frac{2}{(2k+1)^2} &= \frac{2}{\pi} \sum_{k=0}^{\infty} \tan^{-1} \tan\left(\tan^{-1}(2k+2) - \tan^{-1}(2k)\right) \\
&= \frac{2}{\pi} \sum_{k=0}^{\infty} \left(\tan^{-1}(2k+2) - \tan^{-1}(2k)\right) \\
&= \frac{2}{\pi} \lim_{n\to\infty} \sum_{k=0}^{n} \left(\tan^{-1}(2k+2) - \tan^{-1}(2k)\right) \\
&= \frac{2}{\pi} \lim_{n\to\infty} \left(-\tan^{-1} 0 + \tan^{-1}(2n+2)\right) \quad \text{[telescoping sum]} \\
&= \frac{2}{\pi} \lim_{n\to\infty} \tan^{-1}(2n+2) \\
&= \frac{2}{\pi}\left(\frac{\pi}{2}\right) = \boxed{1}
\end{aligned}$$

□

Problem 76 (SMO Open 2006 Q16)**.** Find the value of

$$\frac{400(\cos^5 15° + \sin^5 15°)}{\cos 15° + \sin 15°}.$$

Solution.

$$\begin{aligned}
&\frac{400(\cos^5 15° + \sin^5 15°)}{\cos 15° + \sin 15°} \\
&= 400(\cos^4 15° - \cos^3 15° \sin 15° + \cos^2 15° \sin^2 15° - \cos 15° \sin^3 15° + \sin^4 15°) \\
&= 400(\cos^4 15° + \sin^4 15° - \cos 15° \sin 15° + \cos^2 15° \sin^2 15°) \\
&= 400\left((\cos^2 15° + \sin^2 15°)^2 - \cos^2 15° \sin^2 15° - \cos 15° \sin 15°\right) \\
&= 400\left[1 - \left(\frac{1}{2}\sin 30°\right)^2 - \left(\frac{1}{2}\sin 30°\right)\right] = \boxed{275}
\end{aligned}$$

□

Problem 77 (SMO (Open))**.** Find the value of

$$\frac{\tan 40° \tan 60° \tan 80°}{\tan 40° + \tan 60° + \tan 80°}.$$

Solution. We can show, more generally, that an acute $\triangle ABC$,

$$\frac{\tan A \tan B \tan C}{\tan A + \tan B + \tan C} = 1.$$

We see that

$$\begin{aligned}
\tan A + \tan B + \tan &= \tan A + \tan B + \tan[180° - (A+B)] \\
&= \tan A + \tan B - \tan(A+B) \\
&= \tan A + \tan B - \frac{\tan A + \tan B}{1 - \tan A \tan B} \\
&= (\tan A + \tan B)\left(1 - \frac{1}{1 - \tan A \tan B}\right) \\
&= (\tan A + \tan B)\left(-\frac{\tan A \tan B}{1 - \tan A \tan B}\right) \\
&= \tan A \tan B \left(-\frac{\tan A + \tan B}{1 - \tan A \tan B}\right) \\
&= \tan A \tan B[-\tan(A+B)] \\
&= \tan A \tan B \tan[180° - (A+B)] \\
&= \tan A \tan B \tan C
\end{aligned}$$

Hence proven. $\square$

Problem 78. Angles of $\triangle ABC$ satisfies

$$\frac{\sin A + \sin B + \sin C}{\cos A + \cos B + \cos C} = \frac{12}{7}$$

and

$$\sin A \sin B \sin C = \frac{12}{15}.$$

Given that $\sin C$ takes on three possible values s_1, s_2, s_3, find the value of $s_1 s_2 s_3$.

Solution.

$$\begin{aligned}
\sin A + \sin B + \sin C &= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} + \sin C \\
&= 2\sin\left(90^\circ - \frac{C}{2}\right)\cos\frac{A-B}{2} + 2\sin\frac{C}{2}\cos\frac{C}{2} \\
&= 2\cos\frac{C}{2}\cos\frac{A-B}{2} + 2\sin\frac{C}{2}\cos\frac{C}{2} \\
&= 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right) \\
&= 2\cos\frac{C}{2}\left(2\cos\frac{A}{2}\cos\frac{B}{2}\right) = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}
\end{aligned}$$

and

$$\begin{aligned}
\cos A + \cos B + \cos C &= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} + \cos C \\
&= 2\cos\left(90^\circ - \frac{C}{2}\right)\cos\frac{A-B}{2} + \cos 2\left(\frac{C}{2}\right) \\
&= 2\sin\frac{C}{2}\cos\frac{A-B}{2} + \left(1 - 2\sin^2\frac{C}{2}\right) \\
&= 1 + 2\sin\frac{C}{2}\left(\cos\frac{A-B}{2} - \sin\frac{C}{2}\right) \\
&= 1 + 2\sin\frac{C}{2}\left(\cos\frac{A-B}{2} - \sin\left(90^\circ - \frac{A+B}{2}\right)\right) \\
&= 1 + 2\sin\frac{C}{2}\left(\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right) \\
&= 1 + 2\sin\frac{C}{2}\left(2\sin\frac{A}{2}\sin\frac{B}{2}\right) = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}
\end{aligned}$$

This gives us the following simultaneous equations.

$$\begin{cases}
\dfrac{\sin A + \sin B + \sin C}{\cos A + \cos B + \cos C} = \dfrac{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \dfrac{12}{7} \\
\sin A\sin B\sin C = 8\left(\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}\right)\left(\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}\right) = \dfrac{12}{15}
\end{cases}$$

Solving, we get

$$\begin{cases} \sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2} = \dfrac{1}{10} \\ \cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2} = \dfrac{3}{5} \end{cases}$$

We see that

$$\begin{aligned} \sin\frac{C}{2} &= \cos\frac{A+B}{2} = \cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2} \\ \sin^2\frac{C}{2}\cos\frac{C}{2} &= \frac{3}{5}\sin\frac{C}{2} - \frac{1}{10}\cos\frac{C}{2} \end{aligned}$$

Let $t = \cos\dfrac{C}{2}$. Then we get a quadratic equation. Solving it gives us

$$t = \sqrt{\frac{1}{2}}, \quad t = \sqrt{\frac{4}{5}}, \quad t = \sqrt{\frac{3}{10}}.$$

Hence $\boxed{s_1 = 1, s_2 = \dfrac{4}{5}, s_3 = \dfrac{3}{5}}$. □

Problem 79 (MAT)**.** Evaluate

$$\sin^2 1^\circ + \sin^2 2^\circ + \cdots + \sin^2 89^\circ + \sin^2 90^\circ.$$

Solution. Recall the Pythagorean Identity $\sin^2 x + \cos^2 x = 1$.

Rewriting and pairing up terms gives us

$$\begin{aligned}
&\sin^2 1^\circ + \sin^2 2^\circ + \cdots + \cos^2 2^\circ + \cos^2 1^\circ + 1 \\
&= (\sin^2 1^\circ + \cos^2 1^\circ) + \cdots + (\sin^2 44^\circ + \cos^2 44^\circ) + \sin^2 45^\circ + 1 \\
&= 44(1) + \frac{1}{2} + 1 = \boxed{45\frac{1}{2}}
\end{aligned}$$

□

Problem 80. Evaluate $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ$.

Solution.

$$\begin{aligned}
\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ &= \frac{1}{4}\sin 10^\circ (2\sin 70^\circ \sin 50^\circ) \\
&= \frac{1}{4}\sin 10^\circ (\cos 20^\circ - \cos 120^\circ) \\
&= \frac{1}{4}\sin 10^\circ \left(\cos 20^\circ + \frac{1}{2}\right) \\
&= \frac{1}{8}(2\sin 10^\circ \cos 20^\circ) + \frac{1}{8}\sin 10^\circ \\
&= \frac{1}{8}(\sin 30^\circ - \sin 10^\circ) + \frac{1}{8}\sin 10^\circ \\
&= \frac{1}{16}
\end{aligned}$$

□

Problem 81. Let A, B, C be angles of a triangle. Determine the maximum value of $\sin A + \sin B + \sin C$.

Solution. WLOG, assume $A \le B \le C$, then $A \le 60°$. $\sin 3A \ge 0$, $\sin 3B \ge -1$, $\sin 3C \ge -1$ thus

$$\sin 3A + \sin 3B + \sin 3C \ge -2.$$

Let $B = C$, then $B = C = 90° - \frac{A}{2}$. If A is very small, B and C are close to $90°$, thus $\sin 3A + \sin 3B + \sin 3C$ is close to -2.

Now we want to find an upper bound. When $A = 20°, B = 20°, C = 140°$. Let $X = 3A, Y = 3B, Z = 3(C - 120°)$, then $X + Y + Z = 180°$ and

$$\sin 3A + \sin 3B + \sin 3C = \sin X + \sin Y + \sin Z.$$

Suppose that X, Y, Z satisfy the condition that $X + Y + Z = 180°$ such that $\sin X + \sin Y + \sin Z$ has max value. We can then show that $X = Y = Z$.

Assume that $X \le Y \le Z$. If $X \le Z$, then

$$\sin X + \sin Z = 2 \sin \frac{X + Z}{2} \cos \frac{X - Z}{2} < 2 \sin \frac{X + Z}{2}$$

implying that

$$\sin X + \sin Y + \sin Z < \sin \frac{X + Z}{2} + \sin Y + \sin \frac{X + Z}{2}$$

which contradicts the assumption that $\sin X + \sin Y + \sin Z$ has max value.

Hence $X = Y = Z = 60°$, $A = 20°, B = 20°, C = 140°$. Thus $\sin 3A + \sin 3B + \sin 3C = \frac{3\sqrt{3}}{2}$. □

10 Coordinate Systems

§10.1 Cartesian Coordinates

§10.1.1 Basics

The coordinate plane is determined by two **axes** – a horizontal x-axis and a vertical y-axis; both axes intersect at a point called the **origin**. Each point in the coordinate plane can be specified by an ordered pair of numbers (x, y).

The **gradient** of a line with points (x_1, y_1) and (x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{10.1}$$

Given gradient m and y-intercept c, a line can be represented in the point-slope form:

$$y = mx + c \tag{10.2}$$

Given gradient m and a point on the line (x_1, y_1), a line also can be represented as

$$y - y_1 = m(x - x_1) \tag{10.3}$$

For two parallel lines, they have the same gradients.

For two perpendicular lines, the product of the gradients is -1.

Distance between two points (x_1, y_1) and (x_2, y_2) is given by

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \tag{10.4}$$

which follows easily from Pythagoras' Theorem.

The distance from a point (m, n) to the line $Ax + By + C = 0$ is given by

$$d = \frac{Am + Bn + C}{\sqrt{A^2 + B^2}}$$

The distance between two parallel lines $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ is given by

$$d = \frac{|C_2 - C_1|}{\sqrt{A^2 + B^2}}$$

Using the **shoelace formula**, the area of a polygon is given by

The reflection of point/line about line

§10.1.2 Conic Sections

Conic sections are the family of curves obtained by intersecting a cone with a plane. This intersection can take different forms according to the angle the intersecting plane makes with the side of the cone.

The standard conic sections are[1]

1. **circle**
2. **parabola**
3. **ellipse**
4. **hyperbola**

Conic	Cartesian equation	Parametric equation
Circle	$x^2 + y^2 = a^2$	$x = \cos t, y = \sin t$
Parabola	$x = 4ay^2$	$x = 4at^2, y = t$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$x = a\cos t, y = b\sin t$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$x = b\tan t, y = a\sec t$

Remark. The circle is a special case of the ellipse, where $a = b$.

Parabolas

Given a line l and a point P in a plane, a **parabola** is defined as the set of points S in the plane such that the length SP equals the distance from S to l. The point P is known as the **focus**, and the line l is known as the **directrix**.

[figure]

The minimum point on the curve is known as the **vertex**, denoted by $X(h,k)$.

If we let the distance from X to P be a, we have $P(h,k+a)$. Similarly, l is a below X (since X is equidistant from P and l) and thus can be described by $y = k-a$. (Remember, l is a horizontal line.) If we choose any point $S = (x,y)$ on the parabola, we have

[1]There are also special cases, such as a point or a line, however these are trivial (sometimes called degenerate) so we shall not cover them.

$SP = \sqrt{(x-h)^2 + (y-k-a)^2}$ and the distance from S to l is simply $y-(k-a)$, or $y-k+a$. Hence from our definition of a parabola we have

$$\sqrt{(x-h)^2 + (y-k-a)^2} = y-k-a.$$

Through some algebraic manipulation we have

$$y-k = \frac{1}{4a}(x-h)^2 \tag{10.5}$$

which is the general form of a parabola with a horizontal directrix.

Similarly, if the directrix is vertical, the equation is

$$x-h = \frac{1}{4a}(y-k)^2$$

The **axis of symmetry** is the line through the focus and the vertex.

Ellipses

The general equation for a circle with radius R is

$$(x-h)^2 + (y-k)^2 = R^2 \tag{10.6}$$

Dividing both sides by R^2, we can write this as

$$\frac{(x-h)^2}{R^2} + \frac{(y-k)^2}{R^2} = 1.$$

Notice that we can "stretch" a circle to form an ellipse. Let a denote the "radius" in the x-direction and b denote the "radius" in the y-direction. Then the equation of an ellipse is given by

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \tag{10.7}$$

Notice we have two different "diameters" in the x and y directions. These are known as the **major axis** and **minor axis**, where the major axis is the longer of the two.

Taking two points F_1 and F_2, known as **foci**, we can define an **ellipse** as the set of points Z such that $ZF_1 + ZF_2$ is constant.

We measure the amount an ellipse is stretched away from a circle by its **eccentricity** e. Let c denote distance from centre of ellipse to either focus.

$$e = \frac{c}{a} \tag{10.8}$$

The area enclosed in an ellipse is $ab\pi$.

Hyperbolas

The general form of the equation for a hyperbola is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \tag{10.9}$$

With each hyperbola we can associate a pair of **foci** F_1 and F_2 so that the hyperbola is the set of all points S where $|SF_1 - SF_2|$ is constant.

§10.2 Polar Coordinates

You will already be familiar with coordinates in the form (x, y) meaning that we move x units in the x-direction (along the x-axis) and y in the y-direction (along the y-axis). These are Cartesian coordinates on the xy-plane. Although Cartesian coordinates are very useful, there are sometimes situations where it is much easier to use another coordinate system called **polar coordinates**. These are coordinates in the form (r, θ) where r is the distance to the point from the origin and θ is the angle in radians between the positive x-axis and the line formed by r.

From trigonometry and Pythagoras' theorem there are the following relationships:

$$x = r\cos\theta \quad y = r\sin\theta \quad r = \sqrt{x^2 + y^2}$$

We can use the formulae above to allow us to convert between polar and Cartesian coordinates.

Polar Coordinates, Parametric Equations and Vector Functions: polar coordinate system. Parametric equations are introduced. Derivatives and integrals of polar, parametric and vector functions will also be taught.

§10.3 Barycentric Coordinates

§10.3.1 Normalised Coordinates

In this chapter, $\triangle ABC$ is a fixed non-degenerate reference triangle with vertices in counterclockwise order. The lengths will be abbreviated $a = BC$, $b = CA$, $c = AB$. These correspond with points in the vector plane $\vec{A}, \vec{B}, \vec{C}$.

Definition 10.3.1. Each point in the plane is assigned an ordered triple of real numbers $P = (x, y, z)$ such that

$$\vec{P} = x\vec{A} + y\vec{B} + z\vec{C} \quad \text{and} \quad x + y + z = 1.$$

These are called the **barycentric coordinates** of the point.

The most important result about these coordinates is that one can recover signed areas using them. (For this reason, they are sometimes called *areal coordinates* instead.) We state this in the following theorem. To avoid technicalities about how to define the notion of "area" rigorously, we do not include the proof.

Theorem 10.3.1

Let $P = (x_P, y_P, z_P)$, $Q = (x_Q, y_Q, z_Q)$, $R = (x_R, y_R, z_R)$. The signed area of a triangle $[PQR]$ is given by

$$[PQR] = \begin{vmatrix} x_P & y_P & z_P \\ x_Q & y_Q & z_Q \\ x_R & y_R & z_R \end{vmatrix}.$$

This theorem right away gives an equivalent, useful definition for the coordinates of point P.

Corollary 10.3.1. For each point P,

$$P = \left(\frac{[BPA]}{[ABC]}, \frac{[CPB]}{[ABC]}, \frac{[APC]}{[ABC]} \right).$$

Another important corollary is that equations of a line take the form of linear equations.

Theorem 10.3.2

Let u, v, w be real numbers not all equal. Then the locus of points (x, y, z) satisfying

$$ux + vy + wz = 0$$

is a straight line, and moreover all lines are of this form.

Proof. One direction follows by fixing two points Q and R in Theorem 7.1.1 noting that the locus of points lying on line QR is precisely those points with $[PQR] = 0$.

Conversely, suppose WLOG that $u \neq v$ and $u \neq w$. Then one may take $Q = \left(\frac{-v}{u-v}, \frac{u}{u-v}, 0\right)$ and $R = \left(\frac{-w}{u-w}, 0, \frac{u}{u-w}\right)$ and line QR will have the desired form. $\square$

§10.3.2 Coordinates of Triangle Centers

We now work out the coordinates of some common triangle centers.

Proposition 10.3.1. The barycentric coordinates of the centroid are $G = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

§10.3.3 Homogenised Coordinates

§10.3.4 Vector Plane and Dot Product

§10.3.5 Displacement Vector

§10.3.6 Equation of Circle

§10.4 Exercises

11 Advanced Techniques

§11.1 Inversive Geometry

In what follows, we consider the usual Euclidean plane $\mathbb{R}^2$ with an additional "point at infinity", which we denote as ∞. We consider every line to pass through this point ∞.

§11.1.1 Definition and First Properties

Definition 11.1.1. For a circle Γ with center O and radius $r > 0$, an **inversion** around Γ is a map that sends each point P to be a point P^* as follows:

- If $P = \infty$, then $P^* = O$.
- If $P = O$, then $P^* = \infty$.
- For any other point P, we choose P^* to be the unique point satisfying

$$OP \cdot OP^* = r^2.$$

We immediately identify some properties of the inversion.

Proposition 11.1.1. Inversion is an involution: $(P^*)^* = P$.

Proposition 11.1.2. The point P lies on Γ if and only if Γ lies on Γ.

In the case where P lies outside the circle, there is also a geometric interpretation.

Theorem 11.1.1

Let P be a point outside Γ and suppose PA, PB are tangents to the circle. Then P^* coincides with the midpoint of AB.

§11.1.2 Generalised Lines and Circles

We continue to fix a circle Γ with center O and radius r, through which we will perform inversions.

If ℓ is a line, by its inverse ℓ^* we mean the set

$$\ell^* = \{P^* \mid P \in \ell\}.$$

Similarly for a circle γ its inverse is the set

$$\gamma^* = \{P^* \mid P \in \gamma\}.$$

The main result of this section is that inverses of lines and circles are themselves lines and circles. We check this using the following propositions.

§11.1.3 Inversion Distance Formula

§11.2 Projective Geometry

Cross ratios Projective transformations O Projective geometry, e.g. cross ratios, harmonic bundles, poles and polars, Pascal's theorem, and so on

§11.2.1 Homothety

One way to capture at once a lot of information that normally would form a similar triangles argument is through the notion of homothety.

Definition 11.2.1. A **homothety** h is a transformation defined by a center O and a nonzero real number k (not necessarily positive). It sends a point P to another point $h(P)$, multiplying the distance from O by k.

Remark. k can be negative; in that case, O will lie between P and $h(P)$.

It is easy to see that homothety preserves similarity. Homothety also preserves many things, including but not limited to tangency, angles (both vanilla and directed), circles, and so on. They do not preserve length, but they work well enough: the lengths are simply all multiplied by k.

Proposition 11.2.1. Given non-congruent parallel segments AB and XY , there is a unique homothety sending A to X and B to Y.

Proof. Since $AB \neq XY$, the quadrilateral $ABYX$ is not a parallelogram, and we may take O to be the intersection of lines AX and BY. Then $\triangle OAB \sim \triangle OXY$.

The common scale factor is then the signed quotient $\dfrac{OX}{OA} = \dfrac{OY}{OB}$. □

This is often used with triangles. A consequence of this is the following useful lemma.

Corollary 11.2.1. Let ABC and XYZ be non-congruent triangles such that $AB \parallel XY$, $BC \parallel YZ$, and $CA \parallel ZX$. Then lines AX, BY, CZ concur at some point O, and O is a center of a homothety mapping $\triangle ABC$ to $\triangle XYZ$.

§11.3 Complete Quadrilaterals

§11.3.1 Spiral Similarity

§11.3.2 Miquel Point of Cyclic Quadrilateral

§11.4 Exercises

Part IV

Combinatorics

basics of counting; the inclusion-exclusion principle; the pigeonhole principle; permutations and combinations; the binomial theorem; recurrence relations and linear recurrence relations;

12 Combinatorics

- 102 Combinatorial Problems From The Training of The USA IMO Team

" Recursion and recurrence relations " Definition of sets and functions O Elementary probability O Expected value and linearity of expectation O Basic properties and definitions from graph theory, e.g. connectedness and degree of a vertex O Definition and existence of the convex hull of a finite set of points !! Nontrivial results from graph theory, such as Hall's marriage lemma or Turan's theorem

§12.1 Permutations and Combinations

A **permutation** is an arrangement of objects in a specific order. Number of ways to permute k of n items:

$$^{n}\mathrm{P}_k = \frac{n!}{(n-k)!}$$

A **combination** is a selection of objects without regard to the order. Number of ways to choose k ok n items:

$$^{n}\mathrm{C}_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Number of subsets of a set with n elements is 2^n.

Number of ways to choose k objects from n objects if repetition is allowed $= \binom{n+k-1}{k}$.

Number of paths from $(0,0)$ to (m,n) going 1 unit rightwards or upwards $= \binom{m+n}{n}$.

Number of k-tuples of positive integers which sum equals n is $\binom{n-1}{k-1}$.

Number of k-tuples of non-negative integers which sum equals n is $\binom{n+k-1}{k-1}$.

§12.1.1 Stars and Bars

The setup is the following: suppose there are three children c_1, c_2, c_3, and we distribute 10 identical candies among these three children. Each child can receive any number of candies, including 0. For example, one possible distribution is $(4,3,3)$: in this case, c_1 receives 4 candies, c_2 receives 3, and c_3 receives 3. How many ways can we distribute the candies?

The key observation is the following: we can distribute the candies by arranging them in a line, and then placing two "bars" somewhere along the line. For example, the $(4,3,3)$ described above can be modeled by the following:

$$* * * * \mid * * * \mid * * *$$

Each $*$ represents a candy, and the two location of the two bars determines the distribution of the candies. Notice that the following distribution is also possible:

$$\mid\mid * * * * * * * * * *$$

The above diagram corresponds to the distribution $(0,0,10)$. In general, c_1 receives the candies left of the first bar, c_2 receives the candies between the two bars, and c_3 receives the candies right of the second bar.

So we can see that distributing candies is identical to choosing the location of the two bars to place in $10+2=12$ empty slots, hence $\binom{12}{2}$ ways.

In general, if there are n candies and k children, then there are $n+k-1$ slots, and we must place $k-1$ bars. The remaining n candies, interspersed among the bars, represent a istribution. Thus, the number of distributions is

$$\binom{n+k-1}{k-1}.$$

§12.2 Combinatorial Identities

§12.2.1 Pascal's Triangle

We can observe that by means of expansion,

$$\binom{n}{k} = \binom{n}{n-k} \tag{12.1}$$

Each number in the Pascal's triangle is a binomial coefficient. Pascal's and hockey-stick identities:

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} \tag{12.2}$$

$$\sum_{r=k}^{n} \binom{r}{k} = \binom{n+1}{k+1} \tag{12.3}$$

$$\sum_{r=0}^{n} \binom{k+r}{r} = \binom{n+k+1}{n} \tag{12.4}$$

We also have

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m} \tag{12.5}$$

which can be easily proven via expansion.

Theorem 12.2.1: Vandermonde's Identity

$$\sum_{r=0}^{k} \binom{m}{r}\binom{n}{k-r} = \binom{m+n}{k} \tag{12.6}$$

§12.2.2 Binomial Theorem

Theorem 12.2.2: Binomial Theorem

For $n \in \mathbb{Z}^+$ and $a, b \in \mathbb{R}$,

$$\begin{aligned}(a+b)^n &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \\ &= \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \cdots + \binom{n}{n} b^n\end{aligned} \tag{12.7}$$

Proof. This can be proven using mathematical induction. □

Corollary 12.2.1. For all $n \in \mathbb{Z}^+$, the following equality holds:

$$2^n = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} \tag{12.8}$$

Remark. The above identity simply follows by $a = b = 1$.

We give an alternate proof below that relates this identity to the set of subsets of a set.

Proof. Let A be a set with n elements, and let A_k denote the subset of the power set 2_A containing the subsets of A of size k. Then the sets $A_0, A_1, \ldots, A_n$ partition 2^A, which means the following equalities hold:

$$\begin{aligned} 2^n = |2^A| &= |A_0| + |A_1| + \cdots + |A_n| \\ &= \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} \end{aligned}$$

In other words, every subset of A has a size in $\{0, 1, \ldots, n\}$, so to count the number of subsets of A, we can count the number of subsets of each size over all possible sizes. □

Sums:

$$\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1} \tag{12.9}$$

$$\sum_{k=0}^{n} k^2 \binom{n}{k} = n(n+1)2^{n-2} \tag{12.10}$$

§12.3 Cardinality Rules and Principles

In this section, we will see the formalisation of counting strategies that we often take for granted: the product rule, the sum rule, and the pigeonhole principle.

§12.3.1 Product Rule

Examples of counting

Counting number of rectangles:

For a $m \times n$ grid, to form a rectangle, choose 2 points from the $m+1$ points along the column, and choose 2 points from the $n+1$ points along the row. Hence the number of rectangles we can form is

$$\binom{m+1}{2}\binom{n+1}{2}$$

§12.3.2 Principle of Inclusion-Exclusion

The **principle of inclusion and exclusion** is a counting technique that computes the number of elements that satisfy at least one of several properties while guaranteeing that elements satisfying more than one property are not counted twice.

The idea behind this principle is that summing the number of elements that satisfy at least one of two categories and subtracting the overlap prevents double counting.

For two sets,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

where $|S|$ denotes the cardinality (i.e. number of elements) of set S.

For three sets,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

More generally, if A_i are finite sets, then

$$\begin{aligned} |\bigcup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i \le j \le n} |A_i \cap A_j| + \sum_{1 \le i \le j \le k \le n} |A_i \cap A_j \cap A_k| \\ - \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|. \end{aligned} \tag{12.11}$$

Example 12.3.1

Find the number of integers from the set $\{1, 2, \ldots, 1000\}$ which are divisible by 3 or 5.

Solution. Let

$$\begin{aligned} S &= \{1, 2, \ldots, 1000\} \\ A &= \{x \in S \mid x \text{ is divisible by } 3\} \\ B &= \{x \in S \mid x \text{ is divisible by } 5\} \end{aligned}$$

It follows that

$$A \cap B = \{x \in S \mid x \text{ is divisible by } 15\}$$

Observe that

> for any two natural numbers n and k with $n \geq k$, the number of integers in the set $\{1, 2, \ldots, n\}$ which are divisible by k is $\left\lfloor \frac{n}{k} \right\rfloor$

Hence we have

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= \left\lfloor \frac{1000}{3} \right\rfloor + \left\lfloor \frac{1000}{5} \right\rfloor - \left\lfloor \frac{1000}{15} \right\rfloor \\ &= 333 + 200 - 66 = \boxed{467} \end{aligned}$$

□

§12.3.3 Pigeonhole Principle

Theorem 12.3.1: Pigeonhole Principle

If $k + 1$ objects are placed into k boxes, then at least one box contains two or more objects.

Proof. We use a proof by contraposition.

Suppose none of the k boxes has more than one object. Then the total number of objects would be at most k. This contradicts the statement that we have $k + 1$ objects. □

Theorem 12.3.2: Generalised Pigeonhole Principle

If n objects are placed into k boxes, then there is at least one box containing at least $\left\lceil \frac{n}{k} \right\rceil$ objects.

Proof. We use a proof by contradiction.

Suppose that none of the boxes contains more than $\left\lceil \frac{n}{k} \right\rceil - 1$ objects.

Then the total number of objects is

$$k\left(\left\lceil \frac{n}{k} \right\rceil - 1\right)$$

but

$$k\left(\left\lceil \frac{n}{k} \right\rceil - 1\right) < k\left[\left(\frac{n}{k} + 1\right) - 1\right] = n$$

where the inequality $\left\lceil \frac{n}{k} \right\rceil < \frac{n}{k} + 1$ was used.

This is a contradiction, because there are a total of n objects. $\square$

§12.4 Catalan Numbers

Theorem 12.4.1: Catalan numbers

The Catalan numbers are given by the formula

$$C_n = \frac{1}{n+1}\binom{2n}{n} \tag{12.12}$$

§12.4.1 Dyck Paths and Acceptable Sequences

The number of valid parenthesis expressions that consist of n right parentheses and n left parentheses is equal to the n-th Catalan number.

For example, $C_3 = 5$ and there are 5 ways to create valid expressions with 3 sets of parenthesis:

- ( ) ( ) ( )
- (( )) ( )
- ( ) (( ))
- ((( )))
- (( ) ( ))

Considering right parenthesis to be +1s, and left −1s, we can write this more formally as follows:

The number of sequences $a_1, \ldots, a_n$ of $2n$ terms that can be formed using n copies of +1s and n copies of −1s whose partial sums satisfy

§12.4.2 Recurrence Relation; Generating Function

§12.5 Stirling Numbers

Before we begin solving problems with Stirling numbers involved, we give a short background behind these numbers. Firstly, these numbers are named after James Stirling, who introduced Stirling numbers in a purely algebraic setting in his book from 1730. They were then rediscovered and given a combinatorial meaning by Masanobu Saka in 1782. There are two different sets of numbers that bear Stirling's name: Stirling numbers of the first kind and the Stirling numbers of the second kind. In this section, we will focus on Stirling numbers of the second kind.

Stirling numbers of the second kind, notated $\left\{ n \atop k \right\}$, are the number of ways to partition n objects into k disjoint, nonempty sets.

Example 12.5.1. Given the set $\{a, b, c, d\}$, if we wanted to find the number of ways to distribute the elements into 3 nonempty subsets, such as:

$$\begin{gathered}
\{a, b\}, \{c\}, \{d\} \\
\{a, c\}, \{b\}, \{d\} \\
\{a, d\}, \{b\}, \{c\} \\
\{a\}, \{b, c\}, \{d\} \\
\{a\}, \{b, d\}, \{c\} \\
\{a\}, \{b, c\}, \{d\}
\end{gathered}$$

we would obtain $\left\{ 4 \atop 3 \right\} = 6$.

§12.6 Derangements

A derangement is a permutation with no fixed points. That is, a derangement of a set leaves no element in its original place. For example, the derangements of $\{1, 2, 3\}$ are $\{2, 3, 1\}$ and $\{3, 1, 2\}$, but $\{3, 2, 1\}$ is not a derangement of $\{1, 2, 3\}$ because 2 is a fixed point.

The number of derangements of an n-element set is denoted D_n. This number satisfies the recurrences

$$D_n = n \cdot D_{n-1} + (-1)^n$$

and

$$D_n = (n - 1) \cdot (D_{n-1} + D_{n-2})$$

and is given by the formula

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

§12.7 Probability

§12.8 Exercises

Problem 82 (SMO 2005 (Open) Q6)**.** From the first 2005 natural numbers, k of them are arbitrarily chosen. What is the least value of k to ensure that there is at least one pair of numbers such that one of them is divisible by the other?

Solution. Take any set A of 1004 numbers. Write each number in the form

$$2^{a_i}b_i, \quad \text{where } a_i \geq 0 \text{ and } b_i \text{ is odd.}$$

Thus there are 1004 odd numbers $b_1, \dots, b_{1004}$. Since there are 1003 odd numbers among the first 2005 positive integers, at least two of these odd numbers are equal, say $b_i = b_j$. Then $2^{a_i}b_i \mid 2^{a_j}b_j$ if $2^{a_i}b_i < 2^{a_j}b_j$. So the answer is 1004. □

Problem 83 (SMO 2006 (Open) Q1)**.** How many integers are there between 0 and 10^5 having the digit sum equal to 8?

Solution. Each integer can be written as $\overline{x_1x_2x_3x_4x_5}$ where each $x_i = 0, 1, \dots, 9$ and $x_1 + \cdots + x_5 = 8$. The number of non-negative integer solutions to the above equation is $\binom{8+4}{4} = 495$. So there are 495 such integers. □

Problem 84 (SMO 2006 (Open) Q15)**.** Let $X = \{1, 2, 3, \dots, 17\}$. Find the number of subsets Y of X with odd cardinalities.

Solution. We have $|A|$ odd if and only if $|A^c|$ is even. Thus the number of subsets of odd cardinality is the same as the number of subsets with even cardinality. Hence the answer is $\frac{1}{2} \times 2^{17} = 2^{16} = 65536$. □

Problem 85 (SMO 2006 (Open) Q25)**.** Let

$$S = \sum_{r=0}^{n} \binom{3n+r}{r}.$$

Evaluate $\dfrac{S}{23 \times 38 \times 41 \times 43 \times 47}$ when $n = 12$.

Solution. By using the fact that $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$, and writing $\binom{3n}{0}$ as $\binom{3n+1}{0}$, we have

$$\begin{aligned}
S &= \left[\binom{3n+1}{0} + \binom{3n+1}{1}\right] + \binom{3n+2}{2} + \cdots + \binom{3n+n}{n} \\
&= \binom{3n+2}{1} + \binom{3n+2}{2} + \cdots + \binom{3n+n}{n} \\
&= \cdots \\
&= \binom{3n+n}{n-1} + \binom{3n+n}{n} = \binom{4n+1}{n}
\end{aligned}$$

Thus when $n = 12$,

$$\frac{S}{23 \times 38 \times 41 \times 43 \times 47} = 1274.$$

□

Problem 86 (SMO 2007 (Open) Q2)**.** Determine the number of those 0-1 binary sequences of ten 0's and ten 1's which do not contain three 0's together.

Solution. In such a binary sequence, 0's either appear singly or in blocks of two. If the sequence has exactly m blocks of double 0's, then there are $10-2m$ single 0's. The number of such binary sequences is $\binom{11}{m} \times \binom{11-m}{10-2m}$. Thus the answer is

$$\sum_{m=0}^{5} \binom{11}{m} \times \binom{11-m}{10-2m} = 24068.$$

□

Problem 87 (PUTNAM 1992 A6)**.** Four points are chosen independently and at random on the surface of a sphere (using the uniform distribution). What is the probability that the center of the sphere lies inside the resulting tetrahedron?

Solution. Having placed 3 points A, B and C, the fourth point D will enclose the center in the tetrahedron iff it lies in the spherical triangle $A'B'C'$, where P' is directly opposite to P (so that the center lies on PP').

The probability of this is the area of ABC divided by the area of the sphere. So taking the area of the sphere as 1, we want to find the expected area of ABC. But the 8 triangles $ABC, A'BC, AB'C, ABC', A'B'C, AB'C', A'BC', A'B'C'$ are all equally likely and between them partition the surface of the sphere. So the expected area of ABC, and hence the required probability, is just $\boxed{\frac{1}{8}}$. □

Problem 88 (Moser's circle problem)**.** Determine the number of regions into which a circle is divided if n points on its circumference are joined by chords with no three internally concurrent.

Solution. Circle division: chords divide a circle number of chords = n choose 2 where there are n points number of intersection points = n choose 4 (any 4 points forms 2 chords, thus gives a unique intersection point)

The lemma asserts that the number of regions is maximal if all "inner" intersections of chords are simple (exactly two chords pass through each point of intersection in the interior). This will be the case if the points on the circle are chosen "in general position". Under this assumption of "generic intersection", the number of regions can also be determined in a non-inductive way, using the formula for the Euler characteristic of a connected planar graph (viewed here as a graph embedded in the 2-sphere S^2).

A planar graph determines a cell decomposition of the plane with F faces (2-dimensional cells), E edges (1-dimensional cells) and V vertices (0-dimensional cells). As the graph is connected, the Euler relation for the 2-dimensional sphere S 2

$$V - E + F = 2$$

holds. View the diagram (the circle together with all the chords) above as a planar graph. If the general formulas for V and E can both be found, the formula for F can also be derived, which will solve the problem.

Its vertices include the n points on the circle, referred to as the exterior vertices, as well as the interior vertices, the intersections of distinct chords in the interior of the circle. The "generic intersection" assumption made above guarantees that each interior vertex is the intersection of no more than two chords.

Thus the main task in determining V is finding the number of interior vertices. As a consequence of the lemma, any two intersecting chords will uniquely determine an interior vertex. These chords are in turn uniquely determined by the four corresponding endpoints of the chords, which are all exterior vertices. Any four exterior vertices determine a cyclic quadrilateral, and all cyclic quadrilaterals are convex quadrilaterals, so each set of four exterior vertices have exactly one point of intersection formed by their diagonals (chords). Further, by definition all interior vertices are formed by intersecting chords.

Therefore, each interior vertex is uniquely determined by a combination of four exterior vertices, where the number of interior vertices is given by

$$V_{\text{interior}} = \binom{n}{4},$$

and so

$$V = V_{\text{exterior}} + V_{\text{interior}} = n + \binom{n}{4}.$$

The edges include the n circular arcs connecting pairs of adjacent exterior vertices, as well as the chordal line segments (described below) created inside the circle by the collection of chords. Since there are two groups of vertices: exterior and interior, the chordal line segments can be further categorized into three groups:

1. Edges directly (not cut by other chords) connecting two exterior vertices. These are chords between adjacent exterior vertices, and form the perimeter of the polygon. There are n such edges.

2. Edges connecting two interior vertices.

3. Edges connecting an interior and exterior vertex.

To find the number of edges in groups 2 and 3, consider each interior vertex, which is connected to exactly four edges. This yields

$$4\binom{n}{4}$$

edges. Since each edge is defined by two endpoint vertices, only the interior vertices were enumerated, group 2 edges are counted twice while group 3 edges are counted once only.

Every chord that is cut by another (i.e., chords not in group 1) must contain two group 3 edges, its beginning and ending chordal segments. As chords are uniquely determined by two exterior vertices, there are altogether

$$2\left(\binom{n}{2} - n\right)$$

group 3 edges. This is twice the total number of chords that are not themselves members of group 1.

The sum of these results divided by two gives the combined number of edges in groups 2 and 3. Adding the n edges from group 1, and the n circular arc edges brings the total to

$$E = \frac{4\binom{n}{4} + 2\left(\binom{n}{2} - n\right)}{2} + n + n = 2\binom{n}{4} + \binom{n}{2} + n.$$

Substituting V and E into the Euler relation solved for F, $F = E - V + 2$, we then obtain

$$F = \binom{n}{4} + \binom{n}{2} + 2.$$

Since one of these faces is the exterior of the circle, the number of regions r_G inside the circle is $F - 1$, or

$$r_G = \binom{n}{4} + \binom{n}{2} + 1.$$

□

Problem 89 (Langford's Problem $L(n)$)**.** Given the multiset[1] of positive integers:

$$\{1, 1, 2, 2, 3, 3, \ldots, n, n\},$$

can they be arranged in a sequence such that for $1 \leq i \leq n$ there are i numbers between the two occurrences of i?

Proof. $\square$

[1] A multiset is like a set except that there may be more than one occurrence of an element.

Problem 90. Use a combinatorial proof to show that

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}.$$

Proof. For combinatorial proofs, we begin with a story. Consider a group of $2n$ animals, where n are dogs and n are cats.

RHS: Number of ways to pick n animals from a group of $2n$ animals.

For LHS, we try to understand what's going on in the summation:

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots$$

We see that each term looks like a case. For example, for the first term, pick 0 items from the first group, and pick n items from the second group. This shows that if we want to pick n animals, we can pick k dogs and $n-k$ cats.

LHS: Consider all cases where we pick k dogs and $n-k$ cats.

$\therefore$ LHS is the same as RHS as they both count the same number of things. Hence proven. $\square$

Problem 91. Evaluate

$$S = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}.$$

Solution. Writing the sum backwards yields

$$\begin{aligned} S &= n\binom{n}{n} + (n-1)\binom{n}{n-1} + \cdots + \binom{n}{1} \\ &= n\binom{n}{0} + (n-1)\binom{n}{1} + \cdots + \binom{n}{n-1} \end{aligned}$$

Add this to the original series gives us

$$\begin{aligned} &2S = n\left[\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}\right] \\ &2S = n2^n \\ &\boxed{S = n2^{n-1}} \end{aligned}$$

□

This is the proof of the sum

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$$

Problem 92 (USAMO 2005)**.** Legs L_1, L_2, L_3, L_4 of a square table each have length n, where n is a positive integer. For how many ordered 4-tuples (k_1, k_2, k_3, k_4) of non-negative integers can we cut a piece of length k_i from the end of leg L_i $(i = 1, 2, 3, 4)$ and still have a stable table?

(The table is stable if it can be placed so that all four of the leg ends touch the floor. Note that a cut leg of length 0 is permitted.)

Solution. The table is stable if $k_1 + k_3 = k_2 + k_4$. Let this common value be k such that that $k_1 + k_3 = k_2 + k_4 = k$. Let c_k be the number of ways to make the table stable for each value of k. We want to find $\sum_{k=0}^{2n} c_k$.

Note that each table leg is at least 0 and at most n, hence we'll break this into two sums so that it's easier to handle:

$$\sum_{k=0}^{n} c_k + \sum_{k=n+1}^{2n} c_k$$

Case 1: If $0 \le k \le n$, there are $k + 1$ ways to partition k_1 and k_3, and another $k + 1$ ways to partition k_2 and k_4. There are $(k + 1)^2$ ways to partition k_i in this interval. Hence

$$\sum_{k=0}^{n} (k + 1)^2$$

Case 2: If $n+1 \le k \le 2n$, each of the k_i is at most n and at least 0. There are $(2n-k+1)^2$ ways to partition the k_i in this interval. Hence

$$\sum_{k=n+1}^{2n} (2n - k + 1)^2$$

Evaluating the sum gives us $\boxed{\dfrac{(n + 1)(2n^2 + 4n + 3)}{3}}$. □

13 Graph Theory

§13.1 Introduction, Definitions and Notations

- A **graph** is a pair of sets $G = (V, E)$ where V is a set of vertices and E is a collection of edges whose endpoints are in V. It is possible that a graph can have infinitely many vertices and edges. Unless stated otherwise, we assume that all graphs are simple.[1]
- Two vertices v, w are said to be **adjacent** if there is an edge joining v and w. An edge and a vertex are said to be **incident** if the vertex is an endpoint of the edge.
- Given a vertex v, the **degree** of v is defined to be the number of edges containing v as an endpoint.
- A **path** in a graph G is defined to be a finite sequence of distinct vertices $v_0, v_1, \ldots, v_t$ such that v_i is adjacent to v_{i+1}. (A graph itself can also be called a path.) The **length** of a path is defined to be the number of edges in the path.
- A **cycle** in a graph G is defined to be a finite sequence of distinct vertices $v_0, v_1, \ldots, v_t$ such that v_i is adjacent to v_{i+1} where the indices are taken modulo $t+1$. (A graph itself can also be called a cycle.) The **length** of a cycle is defined to be the number of vertices (or edges) in the path.
- A graph is said to be **connected** if for any pair of vertices, there exists a path joining the two vertices. Otherwise, a graph is said to be **disconnected**.
- The **distance** between two vertices u, v in a graph is defined to be the length of the shortest path joining u and v. (In the case the graph is disconnected, this may not be well-defined.)
- Let $G = (V, E)$ be a graph. The **complement** $\bar{G}$ of G is a graph with the same vertex set as G and $E(\bar{G}) = \{e \notin E(G)\}$. i.e. $\bar{G}$ has edges exactly where there are no edges in G.

[1] An edge whose endpoints are the same is called a **loop**. A graph where there is more than one edge joining a pair of vertices is called a **multigraph**. A graph without loops and is not a multigraph is said to be **simple**.

- Let $G = (V, E)$ be a finite graph. A graph G is said to be **complete** if every pair of vertices in G is joined by an edge. A complete graph on n vertices is denoted by K_n.
- A graph G is said to be **bipartite** if $V(G)$ can be partitioned into two non-empty disjoint sets A, B such that no edge has both endpoints in the same set. A graph is said to be **complete bipartite** if G is bipartite and all possible edges between the two sets A, B are drawn. In the case where $|A| = m$, $|B| = n$, such a graph is denoted by $K_{m,n}$.
- Let $k \geq 2$. A graph G is said to be k-partite if $V(G)$ can be partitioned into k pairwise disjoint sets $A_1, \ldots, A_k$ such that no edge has both endpoints in the same set. A complete k-partite graph is defined similarly as a complete bipartite. In the case where $|A_i| = n_i$, such a graph is denoted by $K_{n_1,n_2,\ldots,n_k}$. (Note that a 2-partite graph is simply a bipartite graph.)

§13.2 Trees and Balancing

A **tree** is defined to be a connected graph that does not contain any cycles. We will first give characterisations to such graphs.

Lemma 13.2.1.

§13.3 Friends, Strangers and Cliques

§13.4 Directed Graphs, Lots of Arrows and Tournaments

§13.5 Matchings: Pair Them Up

§13.6 Hamiltonian and Eulerian Paths and Cycles

In a graph with E edges, the sum of degrees of each vertex is $2E$.

A Eulerian path is a path that visits each edge exactly once. For a connected path, a Eulerian path exists if and only if the number of vertices with odd degree is 0 or 2. If there are no vertices with odd degree, the path will return to the start forming a circuit.

For a connected planar graph with V vertices, E edges and F faces,

$$V - E + F = 2.$$

Turan's theorem: Let G be a graph with V vertices and E edges and let $2 \leq r \leq V$. If there are no r-cliques,

$$E \leq \frac{V^2}{2}\left(1 - \frac{1}{r-1}\right).$$

Berge's lemma: An augmenting path is a path that starts and ends on unmatched vertices, and alternates between edges in and not in the matching. A matching is maximuum if and only if there is no augmenting path.

Four colour theorem: A map can be coloured with 4 colours such that adjacent regions have different colours.

Part V

Other Topics

14 Proofs

§14.1 Induction

Theorem 14.1.1 (Principle of Mathematical Induction (PMI))**.** Let $P(n)$ be a family of statements indexed by $\mathbb{Z}^+$. Suppose that

(i) (**base case**) $P(1)$ is true and

(ii) (**inductive step**) for all $k \in \mathbb{Z}^+$, $P(k) \implies P(k+1)$.

Then $P(n)$ is true for all $n \in \mathbb{Z}^+$.

Exercise 14.1.1

Prove that for all positive integers n,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof. The statement is true for $n = 1$ (base case) because with $n = 1$, LHS = RHS = 1.

Assume that the statement is true for some $n = k$, where $k \in \mathbb{Z}^+$. By our induction hypothesis, we have $1 + 2 + 3 + \cdots + k = \dfrac{k(k+1)}{2}$.

To show that the statement is true for $k + 1$,

$$\begin{aligned} 1 + 2 + 3 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{(k+1)(k+2)}{2} \\ &= \frac{(k+1)[(k+1)+1]}{2} \end{aligned}$$

Since $P(1)$ is true and $P(k) \implies P(k+1)$, by mathematical induction, $P(n)$ is true for all positive integers n. □

Another variant on induction is when the inductive step relies on some earlier case(s) but not necessarily the immediately previous case. This is known as *strong induction*:

Theorem 14.1.2 (Strong Form of Induction)**.** Let $P(n)$ be a family of statements indexed by the natural numbers. Suppose that

(i) (**base case**) $P(1)$ is true and

(ii) (**inductive step**) for all $m \in \mathbb{Z}^+$, if for integers k with $1 \leq k \leq m$, $P(k)$ is true then $P(m+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Theorem 14.1.3 (Cauchy Induction)**.** Let $P(n)$ be a family of statements indexed by $\mathbb{Z}^+_{\geq 2}$. Suppose that

(i) (**base case**) $P(2)$ is true and

(ii) (**inductive step**) for all $k \in \mathbb{Z}^+$, $P(k) \implies P(2k)$ and $P(k) \implies (k-1)$.

Then $P(n)$ is true for all $n \in \mathbb{Z}^+_{\geq 2}$.

15 Riemann Integrals

§15.1 Riemann Sums

Riemann sums are an infinite sequence of sums that converges to an integral.

Given $y = f(x)$, we want to find the integral on the x-interval $[0, 1]$.

We first split the interval $[0, 1]$ into n equal subintervals

$$\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \cdots, \left[\frac{n-1}{n}, 1\right]$$

as shown in the figure below.

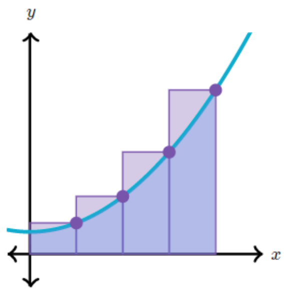


Figure 15.1: Riemann sum

Considering the height of the rectangles, there is not much difference between choosing the left and right values, but the right value is usually chosen because it's simpler for calculation. Hence for the k-th subinterval $\left[\frac{k-1}{n}, \frac{k}{n}\right]$ where $k = 1, \dots, n$, the height of rectangle is $f\left(\frac{k}{n}\right)$. Thus the area of the k-th rectangle is given by

$$\frac{1}{n} f\left(\frac{k}{n}\right).$$

Therefore, the integral is obtained by summing up the area of n rectangles; that is,

$$\int_0^1 f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right). \tag{15.1}$$

Exercise 15.1.1

Find the values of the following expressions:

(a) $\displaystyle\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)$

(b) $\displaystyle\lim_{n\to\infty}\frac{1}{n}\left(e^{\frac{1}{n}}+e^{\frac{2}{n}}+\cdots+e^{1}\right)$

(c) $\displaystyle\lim_{n\to\infty}\frac{\sqrt{n+1}+\sqrt{n+2}+\cdots+\sqrt{2n}}{n\sqrt{n}}$

(d) $\displaystyle\lim_{n\to\infty}\left(\frac{1}{\sqrt{n(n+1)}}+\frac{1}{\sqrt{n(n+2}}+\cdots+\frac{1}{\sqrt{n(4n)}}\right)$

(e) $\displaystyle\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+1}}+\cdots+\frac{1}{\sqrt{2n^2}}\right)$

Solution.

(a)

$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)\\
&=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\cdots+\frac{1}{1+\frac{n}{n}}\right)\\
&=\int_0^1\frac{1}{1+x}\,\mathrm{d}x=\Big[\ln(1+x)\Big]_0^1=\boxed{\ln 2}
\end{aligned}$$

Note that in this case $f(x)=\dfrac{1}{1+x}$.

(b)

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}\left(e^{\frac{1}{n}}+e^{\frac{2}{n}}+\cdots+e^{1}\right)\\
&=\int_0^1 e^x\,\mathrm{d}x=[e^x]_0^1=\boxed{e-1}
\end{aligned}$$

Note that in this case $f(x)=e^x$.

(c)

$$\begin{aligned}
&\lim_{n\to\infty}\frac{\sqrt{n+1}+\sqrt{n+2}+\cdots+\sqrt{2n}}{n\sqrt{n}}\\
&=\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}+\cdots+\sqrt{1+\frac{n}{n}}\right)\\
&=\int_0^1\sqrt{1+x}\,\mathrm{d}x=\left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^1=\boxed{\frac{2}{3}(2\sqrt{2}-1)}
\end{aligned}$$

Note that in this case $f(x)=\sqrt{1+x}$.

(d)

$$
\begin{aligned}
&\lim_{n\to\infty}\left(\frac{1}{\sqrt{n(n+1)}}+\frac{1}{\sqrt{n(n+2}}+\cdots+\frac{1}{\sqrt{n(4n)}}\right)\\
&=\lim_{n\to\infty}\sum_{k=1}^{3n}\frac{1}{n}\left(\frac{1}{\sqrt{1+\frac{k}{n}}}\right)\\
&=\int_0^3\frac{1}{\sqrt{1+x}}\,\mathrm{d}x=\left[2\sqrt{1+x}\right]_0^3=\boxed{2}
\end{aligned}
$$

Note that the sum itself is the sum of $3n$ rectangles, with their heights determined by $f(x)$ from $x=0$ to $x=3$.

(e)

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+1}}+\cdots+\frac{1}{\sqrt{2n^2}}\right)\\
&=\lim_{n\to\infty}\frac{1}{n^2}\left(\frac{1}{\sqrt{1+\frac{1}{n^2}}}+\frac{1}{\sqrt{1+\frac{2}{n^2}}}+\cdots+\frac{1}{\sqrt{1+\frac{n^2}{n^2}}}\right)\\
&=\int_0^1\frac{1}{\sqrt{1+x}}\,\mathrm{d}x=\left[2\sqrt{1+x}\right]_0^1=\boxed{2\left(\sqrt{2}-1\right)}
\end{aligned}
$$

Note that the interval $[0,1]$ is divided into n^2 subintervals.

$\square$

Exercise 15.1.2

(a) Prove that for any positive integer $n\ge 2$,

$$2\sqrt{n}-2<1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}<2\sqrt{n}-1.$$

(b) Compare the sizes between $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}$ and $2\sqrt{n}-\frac{3}{2}$.

(c) Find the integer part of $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{2023}}$.

Solution.

(a) The sum $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}$ is relevant area under the graph $y=\frac{1}{x}$, in the interval $[1,n]$. By integration, this area is $2\sqrt{n}-2$.

On the other hand, consider the area to be approximated by rectangles under the graph, then this approximation is $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$. Because all these rectangles are under the graph, we have

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n} - 2$$

or

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n} - 1.$$

Consider these rectangles to cover the area under the graph. The same area can be covered if we consider the slightly higher rectangles which use the f-value of each of the left endpoints. So we have $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n-1}} > 2\sqrt{n} - 2$.

(b) Consider the sum of trapeziums instead of just rectangles.
For each interval $[k, k+1]$, we use $f(k)$ and $f(k+1)$ for the two heights of the trapezium. Thus the area of one such trapezium is $\frac{1}{2}\left(\frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}\right)$. Hence the total area over the interval $[1, n]$ is

$$\frac{1}{2} \cdot 1 + \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n-1}}\right) + \frac{1}{2} \cdot \frac{1}{\sqrt{n}}.$$

Since the graph $y = \frac{1}{\sqrt{x}}$ is convex, the trapeziums do in fact cover the area under its graph. Thus

$$\frac{1}{2} \cdot 1 + \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n-1}}\right) + \frac{1}{2} \cdot \frac{1}{\sqrt{n}} > 2\sqrt{n} - 2$$

or

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n-1}} + \frac{1}{2} \cdot \frac{1}{\sqrt{n}} > 2\sqrt{n} - \frac{3}{2}.$$

(c) From (b) we have

$$2\sqrt{n} - \frac{3}{2} < 1 + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n} - 1.$$

Substituting $n = 2025$ gives

$$\begin{aligned} 1 + \cdots + \frac{1}{\sqrt{2023}} &< 1 + \cdots + \frac{1}{\sqrt{2025}} < 2(45) - 1 = 89 \\ 1 + \cdots + \frac{1}{\sqrt{2023}} &= 1 + \cdots + \frac{1}{\sqrt{2025}} - \frac{1}{\sqrt{2024}} - \frac{1}{\sqrt{2025}} \\ &> 2(45) - \frac{3}{2} - \frac{1}{\sqrt{2024}} - \frac{1}{\sqrt{2025}} \end{aligned}$$

The only thing left to do is to show that $\frac{1}{\sqrt{2024}} + \frac{1}{\sqrt{2025}} < \frac{1}{2}$, which is obviously true.
Hence $1 + \cdots + \frac{1}{\sqrt{2023}} > 2(45) - 2 = 88$ so the integer part is 88.

□

§15.2 Double Integrals

Exercise 15.2.1

Find the value of the sum

$$\lim_{n\to\infty}\sum_{k,l=1}\frac{1}{n^2+kl}.$$

Solution. Consider the double integral

$$\iint_{[0,1]\times[0,1]} f(x,y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

We approximate using cuboids under the graph; split this region up by slicing horizontally and vertically into n slices each. Each cuboid has side length $\frac{1}{n}$, n^2 cuboids in total.

Hence

$$\begin{aligned}
&\iint_{[0,1]\times[0,1]} f(x,y)\,\mathrm{d}x\,\mathrm{d}y\\
&= \lim_{n\to\infty}\frac{1}{n^2}\sum_{k,l=1}^{n}\frac{1}{1+\frac{k}{n}\frac{l}{n}}\\
&= \iint_{[0,1]\times[0,1]}\frac{1}{1+xy}\,\mathrm{d}x\,\mathrm{d}y\\
&= \iint_{[0,1]\times[0,1]}\left(1-xy+x^2y^2-x^3y^3+\cdots\right)\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

Note that

$$\iint_{[0,1]\times[0,1]} x^k y^k\,\mathrm{d}x\,\mathrm{d}y = \int_{[0,1]} x^k\,\mathrm{d}x\int_{[0,1]} y^k\,\mathrm{d}y = \frac{1}{(k+1)^2}.$$

Hence

$$\begin{aligned}
&\iint_{[0,1]\times[0,1]}\left(1-xy+x^2y^2-x^3y^3+\cdots\right)\mathrm{d}x\,\mathrm{d}y\\
&= 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\\
&= \left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right)-2\left(\frac{1}{2^2}+\frac{1}{4^2}+\cdots\right)\\
&= \frac{\pi^2}{6}-2\cdot\frac{1}{2^2}\cdot\frac{\pi^2}{6}=\boxed{\frac{\pi^2}{12}}
\end{aligned}$$

where we have applied the solution for Basel's problem. □

Double integrals are usually calculated by integrating the terms one by one.

Exercise 15.2.2

Find the value of the following integral:

$$\iint_{[-2,2]\times[-1,1]} x^2|y|^3\,\mathrm{d}x\,\mathrm{d}y$$

Solution.

$$\begin{aligned}
&\iint_{[-2,2]\times[-1,1]} x^2|y|^3 \,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{[-2,2]} \left(\int_{[-1,1]} x^2|y|^3 \,\mathrm{d}y \right) \mathrm{d}x \\
&= \int_{[-2,2]} \mathrm{d}x \int_{[-1,1]} x^2|y|^3 \,\mathrm{d}y \\
&= 4 \int_0^2 \mathrm{d}x \int_0^1 x^2y^3 \,\mathrm{d}y \quad x^2|y|^3 \text{ is even wrt to both } x \text{ and } y \\
&= 4 \int_0^2 \left[\frac{x^2y^4}{4} \right]_0^1 \mathrm{d}x \\
&= \int_0^2 x^2 \,\mathrm{d}x = \boxed{\frac{8}{3}}
\end{aligned}$$

□

One other simplication method: $x^2|y|^3$ is separable because it is an expression in the form $f(x)g(y)$, and the region of integration is a product set $A \times B$, then

$$\iint_{A\times B} f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y = \int_A f(x)\,\mathrm{d}x \cdot \int_B g(y)\,\mathrm{d}y$$

Proof.

$$\begin{aligned}
&\iint_{A\times B} f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_A \left(\int_B f(x)g(y)\,\mathrm{d}y \right) \mathrm{d}x \\
&= \int_A \left(f(x) \int_B g(y)\,\mathrm{d}y \right) \mathrm{d}x \\
&= \int_A kf(x)\,\mathrm{d}x \ \text{ where } k = \int_B g(y)\,\mathrm{d}y \ \text{ is constant wrt } x \\
&= k \int_A f(x)\,\mathrm{d}x \\
&= \int_A f(x)\,\mathrm{d}x \cdot \int_B g(y)\,\mathrm{d}y
\end{aligned}$$

□

Exercise 15.2.3

Find the values of the following integrals:

(a) $\displaystyle\int_0^{\sqrt{3}} \mathrm{d}x \int_0^1 \frac{8x}{(x^2+y^2+1)^2}\,\mathrm{d}y$

(b) $\displaystyle\iint_D (x+y)\,\mathrm{d}x\,\mathrm{d}y$, where D is the region enclosed by $y = e^x$, $y = 1$, $x = 0$ and $x = 1$.

(c) $\displaystyle\iint_D x^2y^2\,\mathrm{d}x\,\mathrm{d}y$, where D is the region enclosed by $x^2 + y^2 = 1$.

Exercise 15.2.4

Find the volume of the region enclosed by the surfaces $z = x^2+y^2$ and $z = 2x-y+2$.

Exercise 15.2.5

Two numbers a and b are chosen randomly from the interval $[0,2]$. Find the expected value of their product.

Exercise 15.2.6

Two points P and Q are chosen randomly on the line segment AB. Find the expected value of the volume of a cuboid of side lengths AP, PQ and QB.

§15.3 Exercises

Problem 93 (SMO 2020 (Open) Q10)**.** Find the value of

$$S = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n(n+k)}}$$

Solution.

$$S = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\sqrt{\frac{1}{1+\frac{k}{n}}} = \int_0^1 \frac{1}{\sqrt{1+x}}\,\mathrm{d}x = \boxed{2\sqrt{2}-2}$$

□

Problem 94 (SMO 2018 (Open) Q12)**.** Given that

$$S = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k}$$

Find the value of S.

Solution.

$$S = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}} = \int_0^1 \frac{1}{1+x}\,\mathrm{d}x = \boxed{\ln 2}$$

□