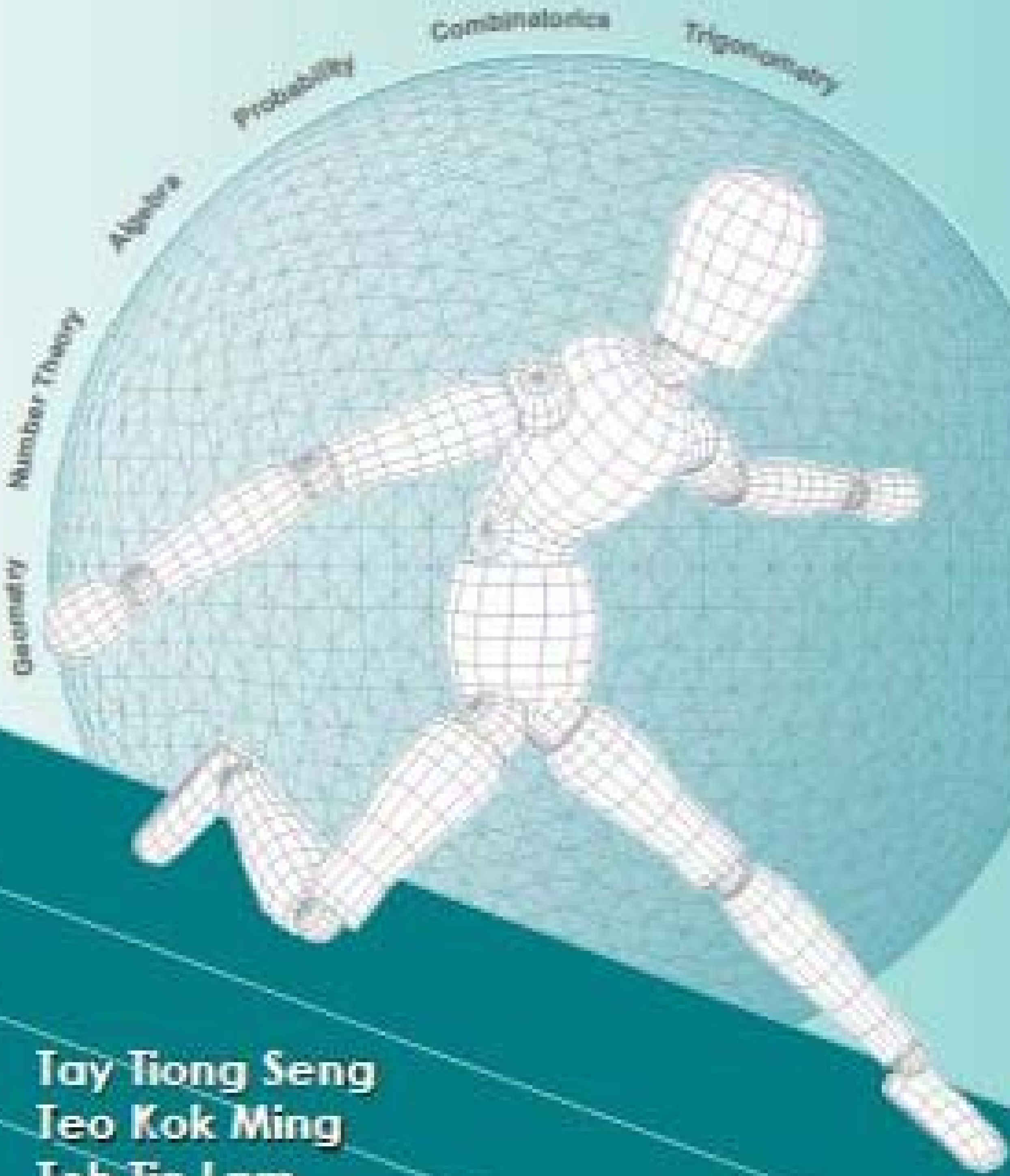
SINGAPORE MATHEMATICAL OLYMPIADS 2010



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(Junior Section)

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Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.

For the other short questions, write your answer on the answer sheet and shade the appropriate bubbles below your answer.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1. Among the five real numbers below, which one is the smallest?

   (A) $\sqrt[2009]{2010}$; (B) $\sqrt[2010]{2009}$; (C) 2010; (D) $\dfrac{2010}{2009}$; (E) $\dfrac{2009}{2010}$.

2. Among the five integers below, which one is the largest?

   (A) 2009^{2010}; (B) 20092010^2; (C) 2010^{2009}; (D) $3^{(3^{(3^3)})}$; (E) $2^{10} + 4^{10} + \cdots + 2010^{10}$.

3. Among the four statements on real numbers below, how many of them are correct?

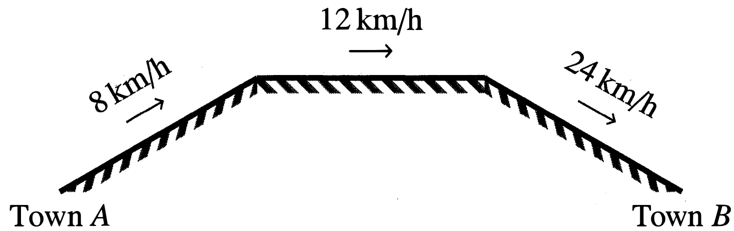
   "If $a < b$ and $a, b \neq 0$ then $\frac{1}{b} < \frac{1}{a}$"; "If $a < b$ then $ac < bc$";
   "If $a < b$ then $a + c < b + c$"; "If $a^2 < b^2$ then $a < b$".

   (A) 0; (B) 1; (C) 2; (D) 3; (E) 4.

4. What is the largest integer less than or equal to $\sqrt[3]{(2010)^3 + 3 \times (2010)^2 + 4 \times 2010 + 1}$?

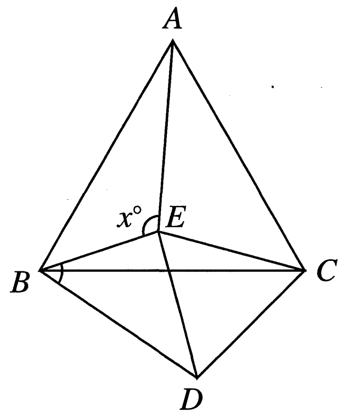
   (A) 2009; (B) 2010; (C) 2011; (D) 2012; (E) None of the above.

5. The conditions of the road between Town A and Town B can be classified as up slope, horizontal or down slope and total length of each type of road is the same. A cyclist travels from Town A to Town B with uniform speeds 8 km/h, 12 km/h and 24 km/h on the up slope, horizontal and down slope road respectively. What is the average speed of his journey?



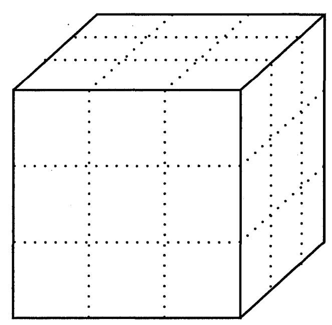
   (A) 12 km/h; (B) $\frac{44}{3}$ km/h; (C) 16 km/h; (D) 17 km/h; (E) 18 km/h.

6. In the diagram, ΔABC and ΔCDE are equilateral triangles. Given that $\angle EBD = 62°$ and $\angle AEB = x°$, what is the value of x?



(A) 100; (B) 118; (C) 120; (D) 122; (E) 135.

7. A carpenter wishes to cut a wooden $3 \times 3 \times 3$ cube into twenty seven $1 \times 1 \times 1$ cubes. He can do this easily by making 6 cuts through the cube, keeping the pieces together in the cube shape as shown:



What is the minimum number of cuts needed if he is allowed to rearrange the pieces after each cut?

(A) 2; (B) 3; (C) 4; (D) 5; (E) 6.

8. What is the last digit of $7^{(7^7)}$?

(A) 1; (B) 3; (C) 5; (D) 7; (E) 9.

9. Given that n is an odd integer less than 1000 and the product of all its digits is 252. How many such integers are there ?

(A) 3; (B) 4; (C) 5; (D) 6; (E) 7.

10. What is the value of

$$(\sqrt{11}+\sqrt{5})^8+(\sqrt{11}-\sqrt{5})^8?$$

(A) 451856; (B) 691962; (C) 903712; (D) 1276392; (E) 1576392.

Short Questions

11. Let x and y be real numbers satisfying

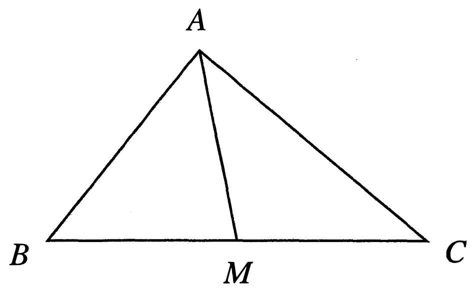
$$y=\sqrt{\frac{2008x+2009}{2010x-2011}}+\sqrt{\frac{2008x+2009}{2011-2010x}}+2010.$$

Find the value of y.

12. For integers $a_1,\ldots,a_n\in\{1,2,3,\ldots,9\}$, we use the notation $\overline{a_1a_2\ldots a_n}$ to denote the number $10^{n-1}a_1+10^{n-2}a_2+\cdots+10a_{n-1}+a_n$. For example, when $a=2$ and $b=0$, $\overline{ab}$ denotes the number 20. Given that $\overline{ab}=b^2$ and $\overline{acbc}=(\overline{ba})^2$. Find the value of $\overline{abc}$.

13. Given that $(m-2)$ is a positive integer and it is also a factor of $3m^2-2m+10$. Find the sum of all such values of m.

14. In triangle ABC, $AB=32$ cm, $AC=36$ cm and $BC=44$ cm. If M is the midpoint of BC, find the length of AM in cm.



15. Evaluate

$$\frac{678 + 690 + 702 + 714 + \cdots + 1998 + 2010}{3 + 9 + 15 + 21 + \cdots + 327 + 333}.$$

16. Esther and Frida are supposed to fill a rectangular array of 16 columns and 10 rows, with the numbers 1 to 160. Esther chose to do it row-wise so that the first row is numbered $1, 2, \ldots, 16$ and the second row is $17, 18, \ldots, 32$ and so on. Frida chose to do it column-wise, so that her first column has $1, 2, \ldots, 10$, and the second column has $11, 12, \ldots, 20$ and so on. Comparing Esther's array with Frida's array, we notice that some numbers occupy the same position. Find the sum of the numbers in these positions.

1	2	3	$\cdots$	$\cdots$	16
17	18	19	$\cdots$	$\cdots$	32
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
145	146	147	$\cdots$	$\cdots$	160

Esther

1	11	21	$\cdots$	$\cdots$	151
2	12	22	$\cdots$	$\cdots$	152
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
10	20	30	$\cdots$	$\cdots$	160

Frida

17. The sum of two integers A and B is 2010. If the lowest common multiple of A and B is 14807, write down the larger of the two integers A or B.

18. A sequence of polynomials $a_n(x)$ are defined recursively by

$$\begin{aligned} a_0(x) &= 1, \\ a_1(x) &= x^2 + x + 1, \\ a_n(x) &= (x^n + 1)a_{n-1}(x) - a_{n-2}(x), \text{ for all } n \geq 2. \end{aligned}$$

For example,

$$\begin{aligned} a_2(x) &= (x^2 + 1)(x^2 + x + 1) - 1 = x^4 + x^3 + 2x^2 + x, \\ a_3(x) &= (x^3 + 1)(x^4 + x^3 + 2x^2 + x) - (x^2 + x + 1) \\ &= x^7 + x^6 + 2x^5 + 2x^4 + x^3 + x^2 - 1. \end{aligned}$$

Evaluate $a_{2010}(1)$.

19. A triangle ABC is inscribed in a semicircle of radius 5. If $AB = 10$, find the maximum value of s^2 where $s = AC + BC$.

20. Find the last two digits of $2011^{(2010^{2009})}$.

21. Your national football coach brought a squad of 18 players to the 2010 World Cup, consisting of 3 goalkeepers, 5 defenders, 5 midfielders and 5 strikers. Midfielders are versatile enough to play as both defenders and midfielders, while the other players can only play in their designated positions. How many possible teams of 1 goalkeeper, 4 defenders, 4 midfielders and 2 strikers can the coach field?

22. Given that $169(157-77x)^2+100(201-100x)^2 = 26(77x-157)(1000x-2010)$, find the value of x.

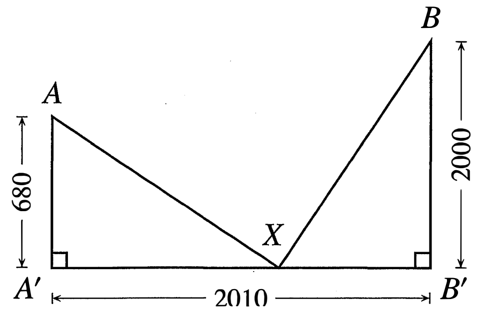
23. Evaluate

$$\frac{(2020^2-20100)(20100^2-100^2)(2000^2+20100)}{2010^6-10^6}.$$

24. When 15 is added to a number x, it becomes a square number. When 74 is subtracted from x, the result is again a square number. Find the number x.

25. Given that x and y are positive integers such that $56 \le x+y \le 59$ and $0.9 < \frac{x}{y} < 0.91$, find the value of y^2-x^2.

26. Let AA' and BB' be two line segments which are perpendicular to $A'B'$. The lengths of AA', BB' and $A'B'$ are 680, 2000 and 2010 respectively. Find the minimal length of $AX + XB$ where X is a point between A' and B'.

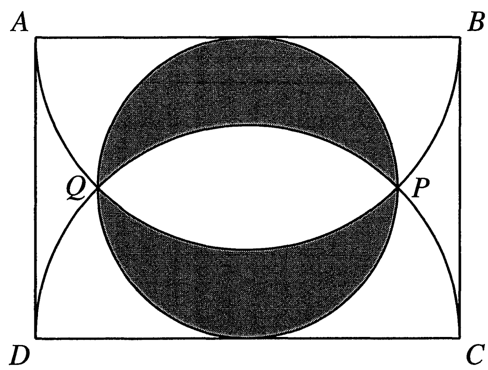


27. The product $1 \times 2 \times 3 \times \cdots \times n$ is denoted by $n!$. For example $4! = 1 \times 2 \times 3 \times 4 = 24$. Let $M = 1! \times 2! \times 3! \times 4! \times 5! \times 6! \times 7! \times 8! \times 9!$. How many factors of M are perfect squares?

28. Starting from any of the L's, the word *LEVEL* can be spelled by moving either up, down, left or right to an adjacent letter. If the same letter may be used twice in each spell, how many different ways are there to spell the word *LEVEL*?

		L		
	L	*E*	*L*	
L	*E*	V	*E*	*L*
	L	*E*	*L*	
		L		

29. Let $ABCD$ be a rectangle with $AB = 10$. Draw circles C_1 and C_2 with diameters AB and CD respectively. Let P, Q be the intersection points of C_1 and C_2. If the circle with diameter PQ is tangent to AB and CD, then what is the area of the shaded region?



30. Find the least prime factor of

$$1\underbrace{0000\cdots00}_{2010\text{-many}}1.$$

31. Consider the identity $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$. If we set $P_1(x) = \frac{1}{2}x(x+1)$, then it is the unique polynomial such that for all positive integer n, $P_1(n) = 1 + 2 + \cdots + n$. In general, for each positive integer k, there is a unique polynomial $P_k(x)$ such that

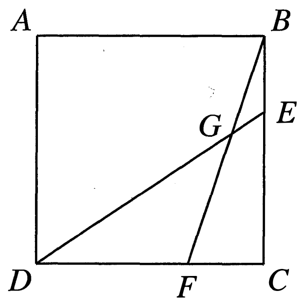
$$P_k(n) = 1^k + 2^k + 3^k + \cdots + n^k \quad \text{for each } n = 1, 2, \ldots.$$

Find the value of $P_{2010}(-\frac{1}{2})$.

32. Given that $ABCD$ is a square. Points E and F lie on the side BC and CD respectively. such that $BE = CF = \frac{1}{3}AB$. G is the intersection of BF and DE. If

$$\frac{\text{Area of } ABGD}{\text{Area of } ABCD} = \frac{m}{n}$$

is in its lowest term, find the value of $m + n$.



33. It is known that there is only one pair of positive integers a and b such that $a \le b$ and $a^2 + b^2 + 8ab = 2010$. Find the value of $a + b$.

34. The digits of the number 123456789 can be rearranged to form a number that is divisible by 11. For example, 123475869, 459267831 and 987453126. How many such numbers are there?

35. Suppose the three sides of a triangular field are all integers, and its area equals the perimeter (in numbers). What is the largest possible area of the field?

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(Junior Section Solutions)

1. Ans: (E)

   It is the only number less than 1.

2. Ans: (D)

   Other than (D), all numbers are less than 2010^{2010}. Now $3^7 > 2010$. Thus $2010^{2010} < (3^7)^{3^7} < 3^{(3^9)}$, the result follows.

3. Ans: (B)

   Only the third statement is correct: $a < b$ implies $a + c < b + c$. For other statements, counterexamples can be taken as $a = -1, b = 1; c = 0$ and $a = 0, b = -1$ respectively.

4. Ans: (C)

   Since $(2010 + 1)^3 = 2010^3 + 3 \times 2010^2 + 3 \times 2010 + 1$. The result follows.

5. Ans: (A)

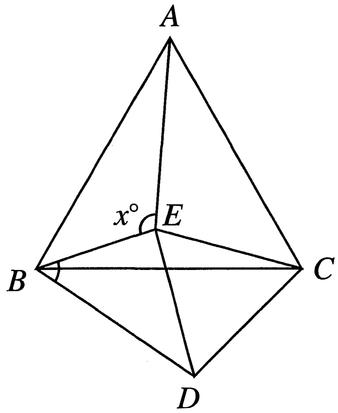
   Let the distance between Town A and Town B be $3s$. The total time taken for up slope, horizontal and down slope road are $\frac{s}{8}$, $\frac{s}{12}$ and $\frac{s}{24}$ respectively. His average speed for the whole journey is $\dfrac{3s}{\frac{s}{8} + \frac{s}{12} + \frac{s}{24}} = 12$ km/h.

6. Ans: (D)

   Observe that ΔBCD is congruent to ΔACE (using SAS, $BC = AC, CD = CE$ and $\angle ACE = 60^\circ - \angle ECB = \angle BCD$). Thus $\angle AEC = \angle BDC$. We get

$$\begin{aligned} x^\circ &= 360^\circ - \angle AEC - \angle BEC \\ &= 360^\circ - \angle BDC - \angle BEC \\ &= \angle EBD + \angle ECD \\ &= 62^\circ + 60^\circ = 122^\circ \end{aligned}$$

   Thus $x = 122$.



7. Ans: (E)

   There is no way to reduce the cuts to fewer than 6: Just consider the middle cube (the one which has no exposed surfaces in the beginning), each of the its sides requires at least one cut.

8. Ans: (B)

   The last digit of 7^k is $1, 7, 9, 3$ respectively for $k \equiv 0, 1, 2, 3 \pmod 4$.
   Since $7^7 \equiv (-1)^7 \equiv 3 \pmod 4$, the last digit of $7^{(7^7)}$ is 3.

9. Ans: (C)

   $252 = 2 \times 2 \times 3 \times 3 \times 7$. We can have: $667, 497, 479, 947, 749$.

10. Ans: (C)

    We only need a rough estimate to rule out the wrong answers. $\sqrt{11} \approx 3.3$ and $\sqrt{5} \approx 2.2$, so the sum is $\approx 5.5^8 + (1.1)^8 \approx 5.5^8 = 30.25^4 \approx 30^4 \approx 810000$. Thus (C). Of course the exact answer can be obtained by calculations, for example,

$$\sum_{i=0}^{8} \binom{8}{i} (\sqrt{11})^i (\sqrt{5})^{8-i} + \sum_{i=0}^{8} \binom{8}{i} (\sqrt{11})^i (-\sqrt{5})^{8-i}$$
$$= 2 \sum_{i \text{ even}}^{8} \binom{8}{i} (\sqrt{11})^i (\sqrt{5})^{8-i}$$
$$= 2 \sum_{j=0}^{4} \binom{8}{2j} (11)^j (5)^{4-j}.$$

11. Ans: 2010.

Let

$$a = \frac{2008x + 2009}{2010x - 2011}.$$

Then $a \geq 0$ and $-a \geq 0$ since they are under the square root. Hence $a = 0$. Thus $y = 2010$.

12. Ans: 369.

It is easy to see that $b = 5$, $\overline{ab} = 25$ or $b = 6$, $\overline{ab} = 36$. Upon checking, $b = 6$, $a = 3$ and so $(\overline{ba})^2 = 63^2 = 3969$. Therefore $c = 9$. Hence $\overline{abc} = 369$.

13. Ans: 51.

Since

$$\frac{3m^2 - 2m + 10}{m - 2} = 3m + 4 + \frac{18}{m - 2}$$

is an integer. Thus $m - 2$ is a factor of 18. $m - 2 = 1, 2, 3, 6, 9, 18$, thus $m = 3, 4, 5, 8, 11, 20$. The required sum is 51.

14. Ans: 26.

Using the Median Formula, $AM^2 = \dfrac{2AB^2 + 2AC^2 - BC^2}{4}$. Thus $AM = 26$ cm.

15. Ans: 16.

Note that both numerator and denominator are arithmetic progressions. Removing common factors gives

$$2 \times \frac{113 + 115 + 117 + \ldots + 333 + 335}{1 + 3 + 5 + \ldots + 109 + 111}$$

$$= \frac{2 \times \frac{112}{2}(335 + 113)}{\frac{56}{2}(111 + 1)} = 16.$$

16. Ans: 322.

The number in the r-th row and c-th column has value $16(r - 1) + c$ in Esther's array and value $10(c - 1) + r$ in Frida's array. So we need to solve

$$16(r - 1) + c = 10(c - 1) + r \implies 5r = 3c + 2.$$

There are exactly four solutions

$$(r, c) = \{(1, 1), (4, 6), (7, 11), (10, 16)\}.$$

17. Ans: 1139.

A direct way to solve the problem is to factor 14807 directly. Alternatively, one may hope for A and B to have common factors to simplify the problem. This is a good strategy because of the following fact:

> "The greatest common divisor of A and B, equals the greatest common divisor of $A + B$ and $\operatorname{lcm}(A, B)$."

2010 is easily factored as $2 \times 3 \times 5 \times 67$. Checking that 67 is also a factor of 14807, we can conclude that 67 is also a factor of A and B. The problem is reduced to finding a and b such that

$$a + b = \frac{2010}{67} = 30 \qquad \text{and} \qquad ab = \frac{14807}{67} = 221.$$

Since 221 can be factored easily, a and b must be 13 and 17. So the answer is $17 \times 67 = 1139$.

18. Ans: 4021.

$a_n(1)$ is the simple recurrence relation $f_n = 2f_{n-1} - f_{n-2}$, $f_0 = 1$ and $f_1 = 3$. Using standard technique or simply guess and verify that $f_n = 2n + 1$. So $a_{2010}(1) = f_{2010} = 2(2010) + 1$.

19. Ans: 200.

ABC must be a right-angled triangle. Let $x = AC$ and $y = BC$, by Pythagoras theorem $x^2 + y^2 = 10^2$.

$$s^2 = (x + y)^2 = x^2 + y^2 + 2xy = 100 + 2 \times \text{ area of } ABC.$$

Maximum area occurs when $x = y$, i.e. $\angle CAB = 45°$. So $x = y = \sqrt{50}$.

20. Ans: 01.

Note that $2011 \equiv 11 \pmod{100}$ and $11^2 \equiv 21, 11^3 \equiv 31 \pmod{100}$ etc. So $11^{10} \equiv 1 \pmod{100}$. Since 2010^{2009} is divisible by 10,

$$2011^{2010^{2009}} \equiv 11^{10} \times \cdots \times 11^{10} \equiv 1 \pmod{100}.$$

21. Ans: 2250.

$\binom{3}{1} \times \binom{5}{2} \times \binom{5}{4}$ choices for goalkeepers, strikers and midfielders respectively. The remaining midfielder and defenders can all play as defenders, hence total number of possibilities are

$$3 \times 10 \times 5 \times \binom{6}{4} = 2250.$$

22. Ans: 31.

Let $a = 1001x - 2041$ and $b = 1000x - 2010$.

Then the equation becomes $a^2 + b^2 = 2ab$. Thus $(a - b)^2 = 0$. The result follows.

23. Ans: 100.

Let $x = 2010$ and $y = 10$. The numerator becomes

$$\begin{aligned} & [(x+y)^2 - xy] \cdot (x^2y^2 - y^2y^2) \cdot [(x-y)^2 + xy] \\ = \; & (x^2 + xy + y^2) \cdot y^2(x-y)(x+y) \cdot (x^2 - xy + y^2) \\ = \; & y^2(x^3 - y^3)(x^3 + y^3) \\ = \; & y^2(x^6 - y^6). \end{aligned}$$

Hence the answer is 100.

24. Ans: 2010.

Let $15 + x = m^2$ and $x - 74 = n^2$. We have $m^2 - n^2 = 89 = 1 \times 89$. $(m-n)(m+n) = 1 \times 89$. Let $m - n = 1$ and $m + n = 89$. Solving $m = 45$ and $n = 44$. Thus the number x is $45^2 - 15 = 2010$.

25. Ans: 177.

From $0.9y < x < 0.91y$, we get $0.9y + y < x + y < 0.91y + y$. Thus $0.9y + y < 59$ and $0.91y + y > 56$. It follows that $y < 31.05$ and $y > 29.3$. Thus $y = 30$ or 31. If $y = 30$, then $27 < x < 27.3$, no integer value of x. If $y = 31$, then $27.9 < x < 28.21$, thus $x = 28$. Thus $y^2 - x^2 = (31 + 28)(31 - 28) = 177$.

26. Ans: 3350.

Take the reflection with respect to $A'B'$. Let A'' be the image of A. Then the minimal length is equal to the length of $A''B = \sqrt{2010^2 + (2000 + 680)^2} = 3350$.

27. Ans: 672.

$$\begin{aligned} M &= 1! \times 2! \times 3! \times 4! \times 5! \times 6! \times 7! \times 8! \times 9! \\ &= 2^8 \times 3^7 \times 4^6 \times 5^5 \times 6^4 \times 7^3 \times 8^2 \times 9 \\ &= 2^{30} \times 3^{13} \times 5^5 \times 7^3 \end{aligned}$$

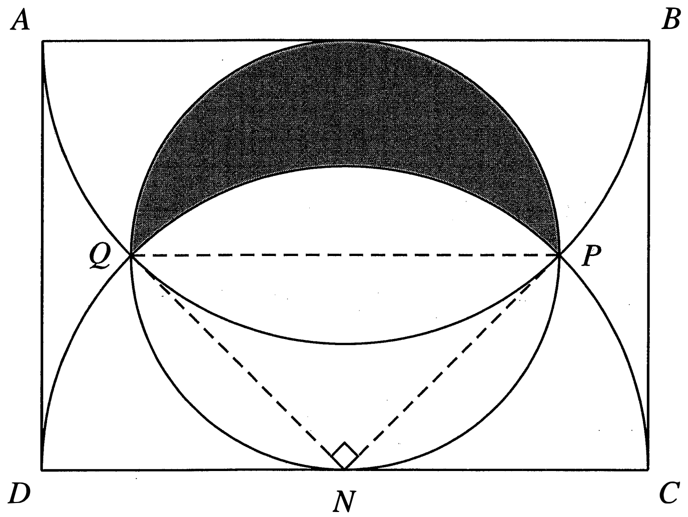
A perfect square factor of M must be of the form $2^{2x} \times 3^{2y} \times 5^{2z} \times 7^{2w}$, where x, y, z and w are whole numbers such that $2x \leq 30$, $2y \leq 13$, $2z \leq 5$, $2w \leq 3$. Hence, the number of perfect square factors of M is $16 \times 7 \times 3 \times 2 = 672$.

28. Ans: 144.

There are total 12 ways starting from any of the L's to reach the middle V. Hence the total number of ways to spell the word *LEVEL* is $12^2 = 144$.

29. Ans: 25.

Let N be the midpoint of CD. Then $\angle PNQ = 90°$. So $PQ = 5\sqrt{2}$.



Then the area of the shaded region is

$$A = 2\left[\frac{1}{2}\pi\left(\frac{PQ}{2}\right)^2 + \frac{1}{2}(PN)^2 - \frac{1}{4}\pi(PN)^2\right]$$
$$= 2\left[\frac{1}{2}\pi\left(\frac{5\sqrt{2}}{2}\right)^2 + \frac{1}{2}\cdot 5^2 - \frac{1}{4}\pi\cdot 5^2\right] = 25.$$

30. Ans: 11.

Clearly 11 is a factor and $2, 3, 5$ are not. We only need to rule out 7. $10^{2011} + 1 \equiv 4 \pmod 7$ because $10^3 \equiv -1 \pmod 7$.

31. Ans: 0.

Let k be a positive even number.

Define $f(x) = P_k(x) - P_k(x-1)$. Then $f(n) = n^k$ for all integer $n \geq 2$. Note that f is a polynomial. We must have $f(x) = x^k$. In particular, for integers $n \geq 2$,

$$\begin{aligned}
P_k(-n+1) - P_k(-n) &= f(-n+1) = (n-1)^k, \\
P_k(-n+2) - P_k(-n+1) &= f(-n+2) = (n-2)^k, \\
&\cdots\cdots\cdots \\
P_k(0) - P_k(-1) &= f(0) = 0^k, \\
P_k(1) - P_k(0) &= f(1) = 1^k.
\end{aligned}$$

Summing these equalities, $P_k(1) - P_k(-n) = 1^k + 0^k + 1^k + \cdots + (n-1)^k$. That is,

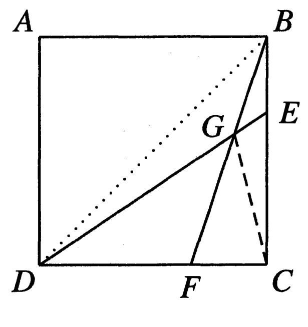
$$P_k(-n) + P_k(n-1) = 0.$$

Define $g(x) = P_k(-x) + P_k(x-1)$. Then $g(n) = 0$ for all integers $n \geq 2$. Since g is a polynomial, $g(x) = 0$.

In particular, $P_k(-\frac{1}{2}) + P_k(-\frac{1}{2}) = 0$, i.e., $P_k(-\frac{1}{2}) = 0$.

32. Ans: 23.

Join BD and CG and note that $\dfrac{DF}{FC} = 2$.



Assume the length of AB is 1. Let the area of ΔBGE and ΔFGC be x and y respectively. Then the areas of ΔEGC and ΔDGF are $2x$ and $2y$. Since the area of ΔBFC is $\frac{1}{3}$, we have $3x + y = \frac{1}{6}$. Similarly, $3y + 2x =$ the area of $\Delta DEC = \frac{1}{3}$. Solve $x = \frac{1}{42}$ and $y = \frac{2}{21}$. Thus

$$\frac{\text{Area of } ABGD}{\text{Area of } ABCD} = 1 - 3(x+y) = 1 - \frac{15}{42} = \frac{9}{14}.$$

So $m = 9$ and $n = 14$. The result follows.

33. Ans: 42.

Since $a \geq 1, 2010 = a^2+b^2+8ab \geq 1+b^2+8b$. $b^2+8b-2009 \leq 0$. However $b^2+8b-2009 = 0$ has an integer solution 41. So $a = 1$ and $b = 41$. The result follows.

34. Ans: 31680.

Let X and Y be the sum of the digits at even and odd positions respectively. Note that $1+2+3+\cdots+9 = 45$. We have $X + Y = 45$ and 11 divides $|X - Y|$. It's easy to see $X = 17$ and $Y = 28$; or $X = 28$ and $Y = 17$. Hence we split the digits into 2 sets whose sum is 17 and 28 respectively.

There are 9 ways for 4 digits to sum to 17: $\{9,5,2,1\}$, $\{9,4,3,1\}$, $\{8,6,2,1\}$, $\{8,5,3,1\}$, $\{8,4,3,2\}$, $\{7,6,3,1\}$, $\{7,5,4,1\}$, $\{7,5,3,2\}$, $\{6,5,4,2\}$. There are 2 ways for 4 digits to sum to 28: $\{9,8,7,4\}$, $\{9,8,6,5\}$. Thus the total number of ways is $11 \times 4! \times 5! = 31680$.

35. Ans. 60.

Let the three sides of the triangle be a, b, c respectively. Then

$$\sqrt{s(s-a)(s-b)(s-c)} = a + b + c = 2s,$$

where $s = \dfrac{a+b+c}{2}$. Note that s is an integer; otherwise $s(s-a)(s-b)(s-a)$ is a non-integer. Let $x = s - a$, $y = s - b$ and $z = s - c$. Then x, y, z are positive integers satisfying

$$xyz = 4(x + y + z).$$

Assume that $x \geq y \geq z$. Then $xyz \leq 12x$, i.e., $yz \leq 12$, and thus $z \leq 3$.

If $z = 1$, $xy = 4(x + y + 1)$ implies $(x - 4)(y - 4) = 20 = 20 \cdot 1 = 10 \cdot 2 = 5 \cdot 4$. So $(x, y) = (24, 5), (14, 6), (9, 8)$.

If $z = 2$, $2xy = 4(x+y+2)$ implies $(x-2)(y-2) = 8 = 8 \cdot 1 = 4 \cdot 2$. So $(x, y) = (10, 3), (6, 4)$.

If $z = 3$, $3xy = 4(x + y + 3)$ implies $(3x - 4)(3y - 4) = 52$, which has no solution $x \geq y \geq 3$.

The area is 60, 42, 36, 30, 24, respectively; and the largest possible value is 60.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2010

(Junior Section, Round 2)

Saturday, 25 June 2010 **0930-1230**

INSTRUCTIONS TO CONTESTANTS

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

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1. Let the diagonals of the square $ABCD$ intersect at S and let P be the midpoint of AB. Let M be the intersection of AC and PD and N the intersection of BD and PC. A circle is incribed in the quadrilateral $PMSN$. Prove that the radius of the circle is $MP - MS$.

2. Find the sum of all the 5-digit integers which are not multiples of 11 and whose digits are 1, 3, 4, 7, 9.

3. Let $a_1, a_2, \ldots, a_n$ be positive integers, not necessarily distinct but with at least five distinct values. Suppose that for any $1 \leq i < j \leq n$, there exist k, ℓ, both different from i and j such that $a_i + a_j = a_k + a_\ell$. What is the smallest possible value of n?

4. A student divides an integer m by a positive integer n, where $n \leq 100$, and claims that

$$\frac{m}{n} = 0 \cdot 167a_1a_2 \cdots.$$

Show the student must be wrong.

5. The numbers $\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{2010}$ are written on a blackboard. A student chooses any two of the numbers, say x, y, erases them and then writes down $x + y + xy$. He continues to do this until only one number is left on the blackboard. What is this number?

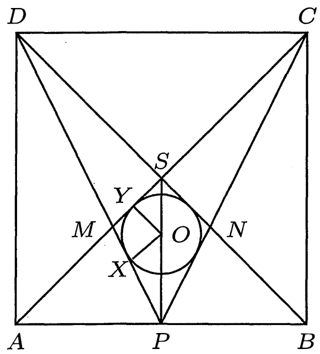
Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2010

(Junior Section, Round 2 solutions)

1. Let O be the centre and r the radius of the circle. Let X, Y be its points of contact with the sides PM, MS, respectively.

Since $OY \perp MS$ and $\angle YSO = \angle ASP = 45°$, $SY = YO = r$. Also $\angle OPX = \angle PDA$ (since $OP \parallel DA$) and $\angle OXP = \angle PAD = 90°$. Therefore $\triangle OXP \simeq \triangle PAD$. Hence $OX/XP = PA/AD = 1/2$. Hence $PX = 2r$. Therefore $PM - MS = 2r + MX - MY - r = r$.



2. First note that an integer is divisible by 11 if and only if the alternating sum of the digits is divisible by 11. In our case, these are the integers where 1,4 and 7 are at the odd positions. Let S be the sum of all the 5-digit integers formed by 1, 3, 4, 7, 9 and let T be the sum of those which are multiples of 11. Then

$$\begin{aligned} S &= 4!(1+3+4+7+9)(1+10+100+1000+10000) \\ &= 6399936 \\ T &= 2!2!(1+4+7)(1+100+10000) + 3!(3+9)(10+1000) = 557568. \end{aligned}$$

Thus the sum is $6399936 - 557568 = 5842368$.

3. $a_1 \le a_2 \le \cdots \le a_n$. Suppose $x < y$ are the two smallest values. Then $a_1 = x$ and let s be the smallest index such that $a_s = y$. Now there are two other terms whose sum is $x + y$. Thus we have $a_2 = x$ and $a_{s+1} = y$. Since $a_1 + a_2 = 2x$, we must have $a_3 = a_4 = x$. Similarly, by considering the largest two values $w < z$, we have $a_n = a_{n-1} = a_{n-2} = a_{n-3} = z$ and another two terms equal to w. Since there is one other value, there are at least $4 + 2 + 4 + 2 + 1 = 13$ terms. The following 13 numbers satisfy the required property: $1,1,1,1,2,2,3,4,4,5,5,5,5$. Thus the smallest possible value of n is 13.

4. We have
$$0\cdot 167 \le \frac{m}{n} < 0\cdot 168 \quad\Rightarrow\quad 167n \le 1000m < 168n.$$
Multiply by 6, we get
$$1002n \le 6000m < 1008n \quad\Rightarrow\quad 6000m - 1000n < 8n \le 800.$$
But $6000m - 1000n \ge 2n > 0$. Thus $6000m - 1000n \ge 1000$ since it is a multiple of 1000. We thus get a contradiction.

5. We shall prove by induction that if the original numbers are $a_1, \ldots, a_n$, $n \ge 2$, then the last number is $(1+a_1)\cdots(1+a_n)-1$.

The assertion is certainly true for $n=2$, the base case. Now suppose it is true for $n=k\ge 2$. Consider $k+1$ numbers $a_1,\ldots,a_{k+1}$ written on the board. After one operation, we are left with k numbers. Without loss of generality, we can assume that the student erases a_k and a_{k+1} and writes $b_k = a_k+a_{k+1}+a_ka_{k+1} = (1+a_k)(1+a_{k+1})-1$. After a further k operations, we are left with the number
$$(1+a_1)\cdots(1+a_{k-1})(1+b_k)-1 = (1+a_1)\cdots(1+a_{k-1})(1+a_k)(1+a_{k+1})-1.$$
This completes the proof of the inductive step. Thus the last number is
$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\cdots\left(1+\frac{1}{2010}\right)-1 = 2010$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2010

(Senior Section)

Tuesday, 1 June 2010 0930 – 1200 hrs

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D *or* E) *corresponding to the correct answer.*

For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Multiple Choice Questions

1. Find the value of $\dfrac{(1\times2\times3)+(2\times4\times6)+(3\times6\times9)+\cdots+(335\times670\times1005)}{(1\times3\times6)+(2\times6\times12)+(3\times9\times18)+\cdots+(335\times1005\times2010)}$.

(A) $\dfrac{1}{3}$

(B) $\dfrac{2}{3}$

(C) $\dfrac{1}{6}$

(D) $\dfrac{1}{2}$

(E) $\dfrac{4}{9}$

2. If a, b, c and d are real numbers such that

$$\frac{b+c+d}{a}=\frac{a+c+d}{b}=\frac{a+b+d}{c}=\frac{a+b+c}{d}=r,$$

find the value of r.

(A) 3
(B) 1
(C) –1
(D) 3 or 1
(E) 3 or –1

3. If $0<x<\dfrac{\pi}{2}$ and $\sin x-\cos x=\dfrac{\pi}{4}$ and $\tan x+\dfrac{1}{\tan x}=\dfrac{a}{b-\pi^c}$, where a, b and c are positive integers, find the value of $a+b+c$.

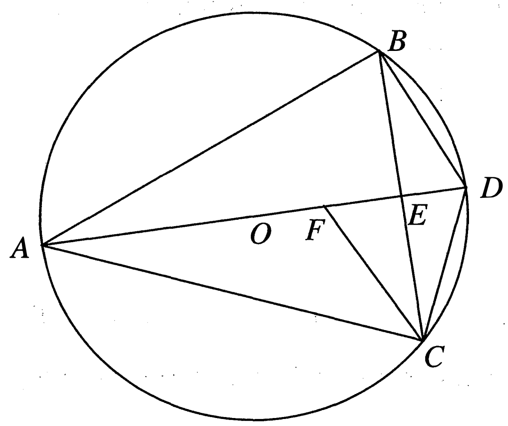
(A) 8
(B) 32
(C) 34
(D) 48
(E) 50

4. Find the value of $\sqrt{14^3+15^3+16^3+\cdots+24^3+25^3}$.

(A) 104
(B) 224
(C) 312
(D) 336
(E) 676

5. In the figure below, ABC is an isosceles triangle inscribed in a circle with centre O and diameter AD, with $AB = AC$. AD intersects BC at E, and F is the midpoint of OE. Given that BD is parallel to FC and $BC = 2\sqrt{5}$ cm, find the length of CD in cm.

(A) $\frac{3\sqrt{5}}{2}$
(B) $\sqrt{6}$
(C) $2\sqrt{3}$
(D) $\sqrt{7}$
(E) $2\sqrt{6}$



6. Find the number of ordered pairs (x, y), where x is an integer and y is a perfect square, such that $y = (x - 90)^2 - 4907$.

(A) 0
(B) 1
(C) 2
(D) 3
(E) 4

7. Let $S = \{1, 2, 3, \ldots, 9, 10\}$. A non-empty subset of S is considered "Good" if the number of even integers in the subset is more than or equal to the number of odd integers in the same subset. For example, the subsets $\{4, 8\}$, $\{3, 4, 7, 8\}$ and $\{1, 3, 6, 8, 10\}$ are "Good". How many subsets of S are "Good"?

(A) 482
(B) 507
(C) 575
(D) 637
(E) 667

8. If the graph of a quadratic function $f(x) = ax^2 + bx + c$ $(a \neq 0)$ passes through two distinct points (r, k) and (s, k), what is $f(r+s)$?

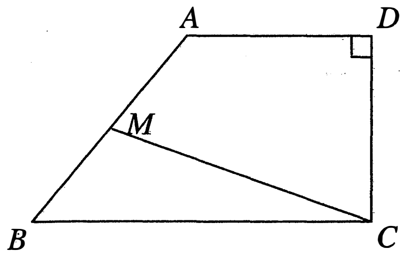
(A) $2k$
(B) c
(C) $k - c$
(D) $2k - c$
(E) None of the above

9. Find the number of positive integers $k < 100$ such that $2(3^{6n}) + k(2^{3n+1}) - 1$ is divisible by 7 for any positive integer n.

(A) 10
(B) 12
(C) 13
(D) 14
(E) 16

10. Let $ABCD$ be a trapezium with AD parallel to BC and $\angle ADC = 90^\circ$, as shown in the figure below. Given that M is the midpoint of AB with $CM = \dfrac{13}{2}$ cm and $BC + CD + DA = 17$ cm, find the area of the trapezium $ABCD$ in cm^2.

(A) 26
(B) 28
(C) 30
(D) 33
(E) 35



Short Questions

11. The area of a rectangle remains unchanged when either its length is increased by 6 units and width decreased by 2 units, or its length decreased by 12 units and its width increased by 6 units. If the perimeter of the original rectangle is x units, find the value of x.

12. For $r = 1, 2, 3, \ldots$, let $u_r = 1+2+3+\ldots+r$. Find the value of

$$\frac{1}{\left(\frac{1}{u_1}\right)}+\frac{2}{\left(\frac{1}{u_1}+\frac{1}{u_2}\right)}+\frac{3}{\left(\frac{1}{u_1}+\frac{1}{u_2}+\frac{1}{u_3}\right)}+\cdots+\frac{100}{\left(\frac{1}{u_1}+\frac{1}{u_2}+\cdots+\frac{1}{u_{100}}\right)}.$$

13. If $2010! = M \times 10^k$, where M is an integer not divisible by 10, find the value of k.

14. If $a > b > 1$ and $\frac{1}{\log_a b}+\frac{1}{\log_b a}=\sqrt{1229}$, find the value of $\frac{1}{\log_{ab} b}-\frac{1}{\log_{ab} a}$.

15. For any real number x, let $\lceil x \rceil$ denote the smallest integer that is greater than or equal to x and $\lfloor x \rfloor$ denote the largest integer that is less than or equal to x (for example, $\lceil 1.23 \rceil = 2$ and $\lfloor 1.23 \rfloor = 1$). Find the value of

$$\sum_{k=1}^{2010}\left[\frac{2010}{k}-\left\lfloor\frac{2010}{k}\right\rfloor\right].$$

16. Let $f(x)=\frac{x^{2010}}{x^{2010}+(1-x)^{2010}}$. Find the value of

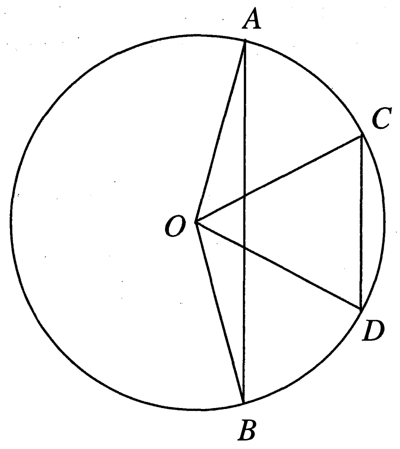
$$f\left(\frac{1}{2011}\right)+f\left(\frac{2}{2011}\right)+f\left(\frac{3}{2011}\right)+\ldots+f\left(\frac{2010}{2011}\right).$$

17. If a, b and c are positive real numbers such that

$$ab + a + b = bc + b + c = ca + c + a = 35,$$

find the value of $(a + 1)(b + 1)(c + 1)$.

18. In the figure below, AB and CD are parallel chords of a circle with centre O and radius r cm. It is given that $AB = 46$ cm, $CD = 18$ cm and $\angle AOB = 3 \times \angle COD$. Find the value of r.



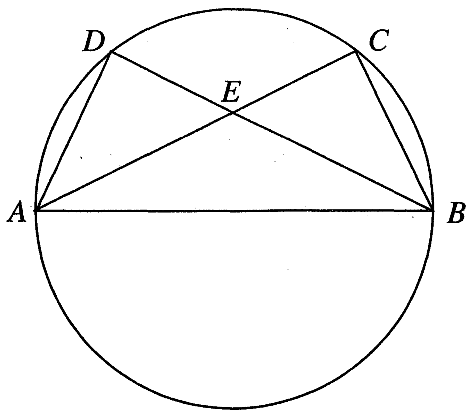
19. Find the number of ways that 2010 can be written as a sum of one or more positive integers in non-decreasing order such that the difference between the last term and the first term is at most 1.

20. Find the largest possible value of n such that there exist n consecutive positive integers whose sum is equal to 2010.

21. Determine the number of pairs of positive integers n and m such that

$$1! + 2! + 3! + \cdots + n! = m^2.$$

22. The figure below shows a circle with diameter AB. C and D are points on the circle on the same side of AB such that BD bisects $\angle CBA$. The chords AC and BD intersect at E. It is given that $AE = 169$ cm and $EC = 119$ cm. If $ED = x$ cm, find the value of x.



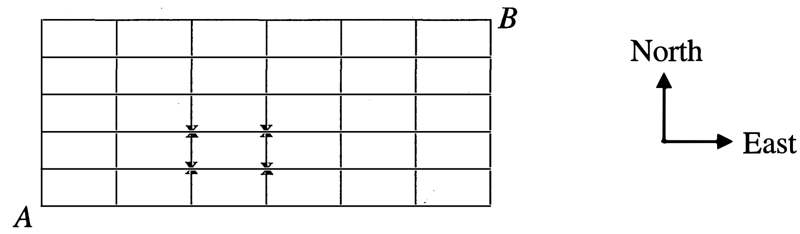
23. Find the number of ordered pairs (m, n) of positive integers m and n such that $m + n = 190$ and m and n are relatively prime.

24. Find the least possible value of $f(x) = \dfrac{9}{1+\cos 2x} + \dfrac{25}{1-\cos 2x}$, where x ranges over all real numbers for which $f(x)$ is defined.

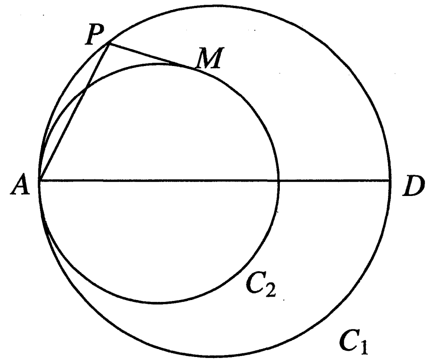
25. Find the number of ways of arranging 13 identical blue balls and 5 identical red balls on a straight line such that between any 2 red balls there is at least 1 blue ball.

26. Let $S = \{1, 2, 3, 4, \ldots, 100000\}$. Find the least possible value of k such that any subset A of S with $|A| = 2010$ contains two distinct numbers a and b with $|a - b| \leq k$.

27. Find the number of ways of traveling from A to B, as shown in the figure below, if you are only allowed to walk east or north along the grid, and avoiding all the 4 points marked x.



28. Two circles C_1 and C_2 of radii 10 cm and 8 cm respectively are tangent to each other internally at a point A. AD is the diameter of C_1 and P and M are points on C_1 and C_2 respectively such that PM is tangent to C_2, as shown in the figure below. If $PM = \sqrt{20}$ cm and $\angle PAD = x°$, find the value of x.



29. Let a, b and c be integers with $a > b > c > 0$. If b and c are relatively prime, $b + c$ is a multiple of a, and $a + c$ is a multiple of b, determine the value of abc.

30. Find the number of subsets $\{a, b, c\}$ of $\{1, 2, 3, 4, \ldots, 20\}$ such that $a < b - 1 < c - 3$.

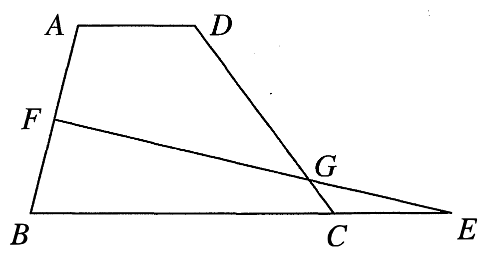
31. Let $f(n)$ denote the number of 0's in the decimal representation of the positive integer n. For example, $f(10001123) = 3$ and $f(1234567) = 0$. Let

$$M = f(1)\times 2^{f(1)} + f(2)\times 2^{f(2)} + f(3)\times 2^{f(3)} + \cdots + f(99999)\times 2^{f(99999)}.$$

Find the value of $M - 100000$.

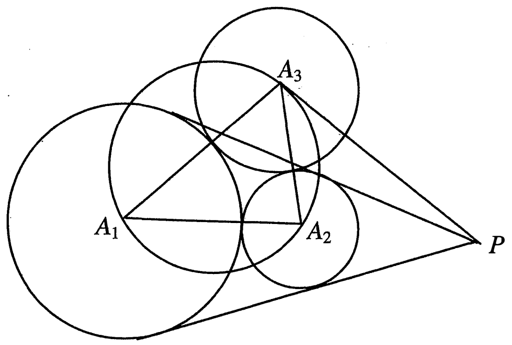
32. Determine the odd prime number p such that the sum of digits of the number $p^4 - 5p^2 + 13$ is the smallest possible.

33. The figure below shows a trapezium $ABCD$ in which $AD \parallel BC$ and $BC = 3AD$. F is the midpoint of AB and E lies on BC extended so that $BC = 3CE$. The line segments EF and CD meet at the point G. It is given that the area of triangle GCE is 15 cm^2 and the area of trapezium $ABCD$ is k cm^2. Find the value of k.



34. Let $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ be a polynomial in x where the coefficients $a_0, a_1, a_2, \ldots, a_n$ are non-negative integers. If $P(1) = 25$ and $P(27) = 1771769$, find the value of $a_0 + 2a_1 + 3a_2 + \cdots + (n+1)a_n$.

35. Let three circles Γ_1, Γ_2, Γ_3 with centres A_1, A_2, A_3 and radii r_1, r_2, r_3 respectively be mutually tangent to each other externally. Suppose that the tangent to the circumcircle of the triangle $A_1A_2A_3$ at A_3 and the two external common tangents of Γ_1 and Γ_2 meet at a common point P, as shown in the figure below. Given that $r_1 = 18$ cm, $r_2 = 8$ cm and $r_3 = k$ cm, find the value of k.



Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2010

(Senior Section Solutions)

1. Answer: (A)

$$\frac{(1\times 2\times 3)+(2\times 4\times 6)+(3\times 6\times 9)+\cdots+(335\times 670\times 1005)}{(1\times 3\times 6)+(2\times 6\times 12)+(3\times 9\times 18)+\cdots+(335\times 1005\times 2010)}$$

$$=\frac{(1\times 2\times 3)[1^3+2^3+3^3+\cdots+335^3]}{(1\times 3\times 6)[1^3+2^3+3^3+\cdots+335^3]}$$

$$=\frac{1\times 2\times 3}{1\times 3\times 6}=\frac{1}{3}.$$

2. Answer: (E)

From the given equations, we obtain

$$a+b+c+d=a(r+1),\ \ a+b+c+d=b(r+1),$$
$$a+b+c+d=c(r+1),\ \ a+b+c+d=d(r+1).$$

Adding these four equations gives

$$4(a+b+c+d)=(a+b+c+d)(r+1),$$

that is,

$$(3-r)(a+b+c+d)=0.$$

Thus $r = 3$, or $a+b+c+d=0$. If $a+b+c+d=0$, then we see from the original given equations that $r = -1$. Hence the value of r is either 3 or -1.

3. Answer: (E)

We have $(\sin x-\cos x)^2=\dfrac{\pi^2}{16}$, which implies that $\sin x\cos x=\dfrac{16-\pi^2}{32}$.

Therefore we obtain

$$\tan x+\frac{1}{\tan x}=\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}=\frac{1}{\sin x\cos x}=\frac{32}{16-\pi^2}.$$

Hence $a + b + c = 32 + 16 + 2 = 50$.

4. Answer: (C)

First, we note that $4n^3 = [n(n+1)]^2-[n(n-1)]^2$. Thus

$$n^3 = \left(\frac{n(n+1)}{2}\right)^2 - \left(\frac{n(n-1)}{2}\right)^2.$$

Therefore

$$\begin{aligned}
&14^3 + 15^3 + \ldots + 24^3 + 25^3 \\
&= \frac{14^2 \times 15^2}{4} - \frac{13^2 \times 14^2}{4} + \\
&\quad \frac{15^2 \times 16^2}{4} - \frac{14^2 \times 15^2}{4} + \ldots. + \\
&\quad \frac{24^2 \times 25^2}{4} - \frac{23^2 \times 24^2}{4} + \\
&\quad \frac{25^2 \times 26^2}{4} - \frac{24^2 \times 25^2}{4} \\
&= \frac{25^2 \times 26^2}{4} - \frac{13^2 \times 14^2}{4} = (25 \times 13 + 13 \times 7)(25 \times 13 - 13 \times 7) \\
&= (32 \times 13)(18 \times 13) = 9 \times 64 \times 13^2.
\end{aligned}$$

Thus $\sqrt{14^3 + 15^3 + 16^3 + \ldots + 24^3 + 25^3} = 3 \times 8 \times 13 = 312.$

5. Answer: (B)
Since the diameter AD perpendicularly bisects the chord BC,

$$BE = EC = \sqrt{5}.$$

Also, given that $BD \parallel FC$, we have $\angle DBE = \angle FCE$. Thus ΔBDE is congruent to ΔCFE, so $DE = FE$. As F is the midpoint of OE, we have $OF = FE = ED$.
Let $OF = x$. Then $AE = 5x$.
Using Intersection Chord Theorem, we have

$$AE \times ED = BE \times EC,$$

which leads to $5x^2 = 5$. Consequently we obtain $x = 1$. Now $CD^2 = CE^2 + ED^2$ gives $CD = \sqrt{5+1} = \sqrt{6}$.

6. Answer: (E)
Let $y = m^2$ and $(x - 90)^2 = k^2$, where m and k are positive integers. Then we obtain $k^2 - m^2 = 4907 = 7 \times 701 = 1 \times 4907$, which gives

$$(k-m)(k+m) = 7 \times 701 \quad \text{or} \quad (k-m)(k+m) = 1 \times 4907.$$

It follows that

$$k - m = 7 \text{ and } k + m = 701, \quad \text{or} \quad k - m = 1 \text{ and } k + m = 4907.$$

Solving these two pairs of equations gives

$$(k, m) = (354, 347) \text{ and } (k, m) = (2454, 2453).$$

Therefore the ordered pairs (x, y) that satisfy the given equation are:

$(444, 347^2)$, $(-264, 347^2)$, $(2544, 2453^2)$, $(-2364, 2453^2)$.

Hence the answer is 4.

7. Answer: (D)

Let the number of even integers in a "Good" subset of S be i, where i = 1, 2, 3, 4, 5, and the number of odd integers in that subset be j, where j = 0, 1, 2, … , i. Then the number of "Good" subsets of S is

$$\sum_{i=1}^{5}\left(\binom{5}{i}\sum_{j=0}^{i}\binom{5}{j}\right)=\binom{5}{1}\left(\binom{5}{0}+\binom{5}{1}\right)+\binom{5}{2}\left(\binom{5}{0}+\binom{5}{1}+\binom{5}{2}\right)+\cdots$$

$$+\binom{5}{5}\left(\binom{5}{0}+\binom{5}{1}+\cdots+\binom{5}{5}\right)$$

$$= 5(1 + 5) + 10(1 + 5 + 10) + \cdots + (1 + 5 + 10 + 10 + 5 + 1)$$

$$= 30 + 160 + 260 + 155 + 32 = 637.$$

8. Answer: (B)

Let $g(x) = f(x) - k$. Then $g(r) = f(r) - k = k - k = 0$. Similarly, $g(s) = 0$. Therefore r and s are roots of the quadratic equation $g(x) = ax^2 + bx + c - k = 0$, from which we deduce that $r + s = -\frac{b}{a}$. Hence

$$f(r+s)=f(-\frac{b}{a})=a(-\frac{b}{a})^2+b(-\frac{b}{a})+c=c.$$

9. Answer: (D)

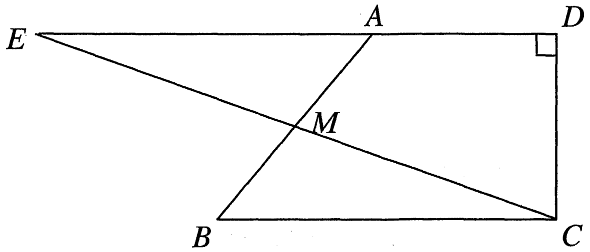
We have

$$2(3^{6n})+k(2^{3n+1})-1=2(27^{2n})+2k(8^n)-1\equiv 2(-1)^{2n}+2k(1^n)-1 \pmod 7$$

$$\equiv 2k+1 \pmod 7.$$

Thus, for any positive integer n, $2(3^{6n})+k(2^{3n+1})-1$ is divisible by 7 if and only if $2k+1\equiv 0 \pmod 7$. As $k < 100$, it is clear that the congruence holds for k = 3, 10, 17, …, 94. Thus the required number of positive integers k is 14.

10. Answer: (C)



Extend DA and CM to meet at E as shown in the figure above. Since $AM = MB$, $\angle AEM = \angle BCM$ and $\angle AME = \angle BMC$, we conclude that ΔAEM is congruent to ΔBCM. Therefore $AE = BC$ and $CM = EM$. Thus $CE = 2CM = 13$.

Let the area of trapezium $ABCD$ be S cm^2. Then $S = \frac{1}{2}(DE)(DC)$, and we have

$$(DE+DC)^2 = DE^2 + DC^2 + 2(DE)(DC) = CE^2 + 4S = 13^2 + 4S = 169 + 4S.$$

Now $DE + DC = DA + AE + DC = DA + BC + DC = 17$. Hence $17^2 = 169 + 4S$, and it follows that $S = 30$.

11. Answer: 132

Let the length and width of the original rectangle be L and W respectively. Then

$$LW = (L+6)(W-2) \quad \text{and} \quad LW = (L-12)(W+6).$$

Simplifying the above equations, we obtain

$$L - 2W = 12 \quad \text{and} \quad 3W - L = 6.$$

Solving the simultaneous equations, we get $L = 48$ and $W = 18$. Hence the perimeter of the rectangle is 132 units.

12. Answer: 2575

As $u_r = 1 + 2 + 3 + \ldots + r = \dfrac{r(r+1)}{2}$, we have

$$\sum_{r=1}^{i} \frac{1}{u_r} = \sum_{r=1}^{i} \frac{2}{r(r+1)} = \sum_{r=1}^{i} \left(\frac{2}{r} - \frac{2}{r+1} \right) = 2 - \frac{2}{i+1} = \frac{2i}{i+1}.$$

Hence,

$$S_n := \sum_{i=1}^{n} \frac{i}{\sum_{r=1}^{i} \frac{1}{u_r}} = \sum_{i=1}^{n} \left(\frac{i}{\frac{2i}{i+1}} \right) = \sum_{i=1}^{n} \left(\frac{i+1}{2} \right) = \frac{1}{2} \left(\frac{n(n+1)}{2} + n \right) = \frac{n}{4}(n+3).$$

In particular, $S_{100} = 2575$.

13. Answer: 501

The number k is the number of the factor 10 that occurs in 2010!. This number is given by the number of pairs of prime factors 2 and 5 in 2010!.
Now between 1 and 2010, there are:

- 402 integers with 5 as a factor;
- 80 integers with 25 as a factor;
- 16 integers with 125 as a factor;
- 3 integers with 625 as a factor.

Therefore the total number of prime factor 5 in 2010! is 402 + 80 + 16 + 3 = 501. As there are clearly more than 501 prime factor 2 in 2010!, we obtain $k = 501$.

14. Answer: 35

First note that since $a > b > 1$, $\frac{1}{\log_{ab} b} - \frac{1}{\log_{ab} a} > 0$. Then

$$\begin{aligned}
\frac{1}{\log_{ab} b} - \frac{1}{\log_{ab} a} &= \log_b ab - \log_a ab \\
&= (\log_b a + 1) - (\log_a b + 1) \\
&= \log_b a - \log_a b \\
&= \frac{1}{\log_a b} - \frac{1}{\log_b a} \\
&= \sqrt{\left(\frac{1}{\log_a b} - \frac{1}{\log_b a}\right)^2} \\
&= \sqrt{\left(\frac{1}{\log_a b} + \frac{1}{\log_b a}\right)^2 - 4\left(\frac{1}{\log_a b}\right)\left(\frac{1}{\log_b a}\right)} \\
&= \sqrt{1229 - 4} \\
&= \sqrt{1225} \\
&= 35.
\end{aligned}$$

15. Answer: 1994

Consider $k = 1, 2, \ldots, 2010$. If $k \mid 2010$, then $x := \frac{2010}{k} - \left\lfloor \frac{2010}{k} \right\rfloor = 0$, so $\lceil x \rceil = 0$. If $k \nmid 2010$, then $0 < y := \frac{2010}{k} - \left\lfloor \frac{2010}{k} \right\rfloor < 1$, so $\lceil y \rceil = 1$.

Since the prime factorization of 2010 is $2 \times 3 \times 5 \times 67$, we see that 2010 has 16 distinct divisors. Hence

$$\begin{aligned}
\sum_{k=1}^{2010} \left\lceil \frac{2010}{k} - \left\lfloor \frac{2010}{k} \right\rfloor \right\rceil &= \sum_{k \mid 2010} \lceil x \rceil + \sum_{k \nmid 2010} \lceil y \rceil \\
&= \text{number of non-divisor of 2010 among } k \\
&= 2010 - 16 = 1994.
\end{aligned}$$

16. Answer: 1005

Observe that $f(x) + f(1-x) = \frac{x^{2010}}{x^{2010} + (1-x)^{2010}} + \frac{(1-x)^{2010}}{(1-x)^{2010} + x^{2010}} = 1$.

It follows that

$$f\left(\frac{1}{2011}\right)+f\left(\frac{2}{2011}\right)+f\left(\frac{3}{2011}\right)+\ldots+f\left(\frac{2010}{2011}\right)$$
$$=\left\{f\left(\frac{1}{2011}\right)+f\left(\frac{2010}{2011}\right)\right\}+\left\{f\left(\frac{2}{2011}\right)+f\left(\frac{2009}{2011}\right)\right\}+\ldots$$
$$+\left\{f\left(\frac{1005}{2011}\right)+f\left(\frac{1006}{2011}\right)\right\}$$
$= 1005$.

17. Answer: 216

Adding 1 to both sides of the given equation $ab + a + b = 35$, we obtain

$$(a + 1)(b + 1) = 36.$$

Likewise, adding 1 to the other two given equations gives

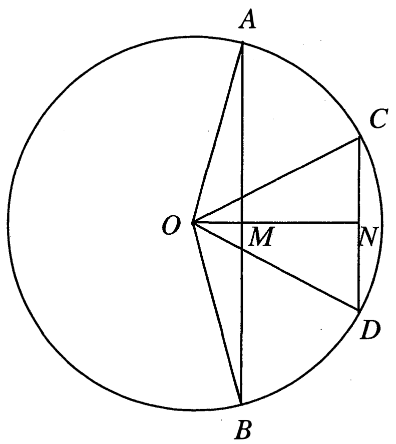
$$(b + 1)(c + 1) = 36 \text{ and } (c + 1)(a + 1) = 36.$$

Now multiplying the three resulting equations above leads to

$$[(a+1)(b+1)(c+1)]^2 = 36^3 = 6^6.$$

It follows that $(a+1)(b+1)(c+1) = 6^3 = 216$.

18. Answer: 27



Let M and N be the midpoints of AB and CD respectively and let $\angle CON = x$. Then $\angle AON = 3x$ and

$$\frac{23}{9}=\frac{AM}{CN}=\frac{r\sin 3x}{r\sin x}=\frac{3\sin x-4\sin^3 x}{\sin x}=3-4\sin^2 x.$$

Thus $\sin^2 x=\frac{1}{4}\left(3-\frac{23}{9}\right)=\frac{1}{9}$, and so $\sin x=\frac{1}{3}$.

Hence $r=\dfrac{CN}{\sin x}=27$.

19. Answer: 2010

Consider any integer k where $1 \le k \le 2010$. By division algorithm, there exists unique pair of integers (q, r) such that $2010 = kq + r$ with $0 \le r \le k - 1$. We rewrite this as $2010 = (k - r)q + r(q + 1)$. That is, $k - r$ copies of q and r copies of $q + 1$ add up to 2010. Thus there is one desired expression for each value of k, which is clearly unique. Hence there are 2010 such expressions in all.

20. Answer: 60

Let a be a positive integer such that the sum of n consecutive integers $a, a + 1, \ldots, a + (n - 1)$ is 2010, that is,

$$a+(a+1)+\cdots+(a+n-1)=2010.$$

This gives $\dfrac{n(2a+n-1)}{2}=2010$, or

$$n(2a+n-1)=4020=2^2\times3\times5\times67. \qquad (1)$$

Since $n < 2a + n - 1$, we have

$$n<\sqrt{2^2\times3\times5\times67}=2\sqrt{1005}<2\times32=64. \qquad (2)$$

Now (1) and (2) imply that $n\in\{1,2,3,4,5,6,10,12,15,20,30,60\}$. Since n and $2a + n - 1$ have different parities, it follows that n and $\dfrac{4020}{n}$ have different parities. Consequently, we have

$$n\in\{1,3,4,5,12,15,20,60\}.$$

If $n = 60$, then $2a+n-1=\dfrac{4020}{60}=67$, so $a = 4$. Thus the largest possible value of n is 60.

21. Answer: 2

First note that if $n \ge 4$, then

$$\begin{aligned}1!+2!+3!+\cdots+n! &\equiv 1!+2!+3!+4! \pmod 5\\ &\equiv 1+2+1+4\equiv 3 \pmod 5).\end{aligned}$$

Since the square of any integer is congruent to either 1 or 4 modulo 5, it follows that $1! + 2! + 3! + \cdots + n! \neq m^2$ for any integer m in this case. So we consider $n < 4$.
Now we have

$$1! = 1^2,\ \ 1! + 2! = 3,\ \ 1! + 2! + 3! = 3^2.$$

Hence we conclude that there are two pairs of positive integers (n, m), namely, $(1, 1)$ and $(3, 3)$, such that $1! + 2! + 3! + \cdots + n! = m^2$.

22. Answer: 65

Since BE bisects $\angle CBA$, we have $\frac{BC}{BA} = \frac{EC}{EA} = \frac{119}{169}$. Thus we can let $BC = 119y$ and $BA = 169y$ for some real number y. Since $\angle BCA = 90^\circ$, we have

$$\begin{aligned}
AB^2 &= AC^2 + BC^2 \\
(169y)^2 &= (169+119)^2 + (119y)^2 \\
y^2(169-119)(169+119) &= (169+119)^2 \\
y^2 &= \frac{169+119}{169-119} = \frac{144}{25} \\
y &= \frac{12}{5}
\end{aligned}$$

Hence, from triangle BCE, we have $BE = \sqrt{119^2 + (119y)^2} = 119 \times \frac{13}{5}$.

Finally, note that ΔADE and ΔBCE are similar, so we have

$$ED = \frac{AE \times CE}{BE} = \frac{169 \times 119}{119 \times \frac{13}{5}} = 65\,\text{cm}.$$

23. Answer: 72

First we find the number of ordered pairs (m, n) of positive integers m and n such that $m + n = 190$ and m and n are not relatively prime.

To this end, write $m = ka$ and $n = kb$, where k, a and b are positive integers with $k > 1$. Since $m + n = 190$, we see that k is a factor of $190 = 2 \times 5 \times 19$ with $k \neq 190$. We consider six cases:

(i) $k = 2$. Then $a + b = 95$, and there are 94 such pairs (a, b) of a and b such that the equation holds.

(ii) $k = 5$. Then $a + b = 38$, and there are 37 such pairs (a, b) of a and b such that the equation holds.

(iii) $k = 19$. Then $a + b = 10$, and there are 9 such pairs (a, b) of a and b such that the equation holds.

(iv) $k = 10$. Then $a + b = 19$, and there are 18 such pairs (a, b) of a and b such that the equation holds.

(v) $k = 38$. Then $a + b = 5$, and there are 4 such pairs (a, b) of a and b such that the equation holds.

(vi) $k = 95$. Then $a + b = 2$, and there is 1 such pair (a, b) of a and b such that the equation holds.

It follows from the above cases that the number of ordered pairs (m, n) of positive integers m and n such that $m + n = 190$ and m and n are not relatively prime is $94 + 37 + 9 - 18 - 4 - 1 = 117$.

Since the total number of ordered pairs (m, n) such that $m + n = 190$ is 189, we conclude that the required number of ordered pairs (m, n) where m and n are relatively prime is 189 − 117 = 72.

24. Answer: 32

For all real values of x for which $f(x)$ is defined, we have

$$\begin{aligned}
f(x) &= \frac{9}{1+\cos 2x} + \frac{25}{1-\cos 2x} = \frac{9}{2\cos^2 x} + \frac{25}{2\sin^2 x} \\
&= \frac{1}{2}\left(9\tan^2 x + 9\right) + \frac{1}{2}\left(25\cot^2 x + 25\right) \\
&= 17 + \frac{1}{2}\left(9\tan^2 x + 25\cot^2 x\right) \\
&\geq 17 + \frac{1}{2}\left(2\sqrt{(9\tan^2 x)(25\cot^2 x)}\right) \quad \text{(AM-GM Inequality)} \\
&= 17 + \frac{1}{2}\left(2\sqrt{9\times 25}\right) \\
&= 32.
\end{aligned}$$

Note that $f(\tan^{-1}\sqrt{\frac{5}{3}}) = 32$. Thus the least possible value of $f(x)$ is 32.

25. Answer: 2002

First we place the 5 red balls on a straight line and then place 1 blue ball between 2 adjacent red balls. With this arrangement fixed, the condition of the question is satisfied. We are now left with 9 blue balls. We can place the remaining 9 blue balls into the spaces before, after or in between the 5 red balls. The number of ways that this can be done is the answer to the question. Including the two ends, there are 4 + 2 = 6 spaces into which these 9 blue balls can be placed. The number of ways of distributing the 9 blue balls into the 6 spaces is

$$\binom{9+6-1}{9} = \binom{14}{9} = \binom{14}{5} = 2002.$$

26. Answer: 49

Consider the following subsets of S:

$$\begin{aligned}
S_1 &= \{1, 2, 3, \ldots, 49, 50\}, \\
S_2 &= \{51, 52, 53, \ldots, 99, 100\}, \\
S_3 &= \{101, 102, 103, \ldots, 149, 150\}, \\
&\vdots \\
S_{2000} &= \{99951, 99952, 99953, \ldots, 99999, 100000\}.
\end{aligned}$$

In other words, $S_i = \{50i - 49, 50i - 48, \ldots, 50i\}$ for $i = 1, 2, \ldots, 2000$. Note that S is partitioned into these subsets $S_1, S_2, S_3, \ldots, S_{2000}$.

By Pigeonhole Principle, for any subset A of S with $|A| = 2010$, there exists i, where $1 \le i \le 2000$, such that $|A \cap S_i| \ge 2$.

Let $a,b \in A \cap S_i$. It is clear that $|a - b| \le 49$.

To show that 49 is the least possible value of k, we find a subset $A \subseteq S$ with $|A| = 2010$ such that $|a - b| \ge 49$ for any distinct $a,b \in A$. Let

$$A = \{49j+1 : j = 0,1,2, ..., 2009\} = \{1,50,99,148,..., 98442\}.$$

Then A is a subset of S with $|A| = 2010$ and $|a - b| \ge 49$ for any distinct $a,b \in A$.

27. Answer: 112

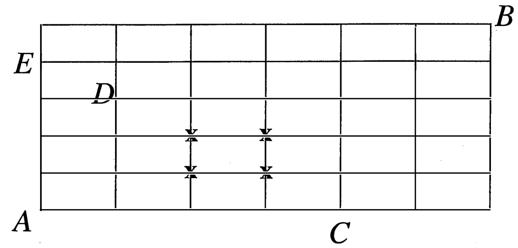


Figure 1

We observe that to avoid the four points marked x, the path must cross either C, D or E as shown in Figure 1 above. Further, the paths that cross C, D or E are exclusive, that is, no path can cross both C and D or D and E, or C and E.

There is only 1 way to get from A to C and from A to E. It is easy to see that there are 4 ways to get from A to D.

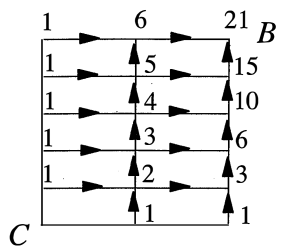


Figure 2

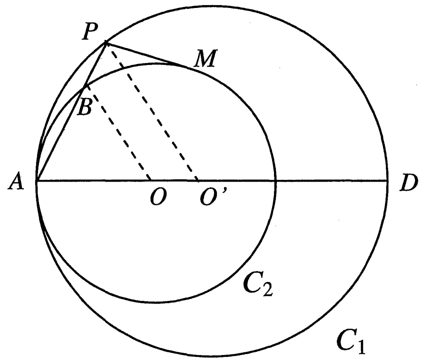
To count the number of ways to get from either C, D or E to B, we note that the number of ways to get to a certain junction is the sum of the numbers of ways to get to the two junctions immediately preceding it from the left or from below (as shown in Figure 2). Therefore there are 21 ways to get from C to B. Similarly, there are 21 ways to get from D to B and 7 ways to get from E to B.

Hence the number of ways to get from A to B that

- pass through C: number of ways from A to C × number of ways from C to $B = 1 \times 21 = 21$;
- pass through D: number of ways from A to D × number of ways from D to $B = 4 \times 21 = 84$;
- pass through E: number of ways from A to E × number of ways from E to $B = 1 \times 7 = 7$.

It follows that the total number of ways from A to B is $21 + 84 + 7 = 112$.

28. Answer: 60



Let O be the centre of C_2 and let PA intersect C_2 at B. The homothety centred at A mapping C_2 to C_1 has similitude ratio $\frac{8}{10}$. It maps B to P. Thus $\frac{AB}{AP} = \frac{8}{10}$. (This can also be seen by connecting P to the centre O' of C_1 so that the triangles ABO and APO' are similar.) The power of P with respect to C_2 is $PM^2 = 20$. Thus $PB \cdot PA = 20$, or equivalently $(PA - AB)PA = 20$. Together with $\frac{AB}{AP} = \frac{8}{10}$, we obtain $AB = 8$ and $AP = 10$. Consequently, the triangle ABO is equilateral, and hence $\angle PAD = \angle BAO = 60°$.

29. Answer: 6

We shall show that $a = 3$, $b = 2$ and $c = 1$.
Note that $2a > b + c$. As $b + c$ is a multiple of a, it follows that $a = b + c$.
Let $a + c = kb$. Then $kb = a + c = b + c + c$, so $2c = (k - 1)b$. Since $c < b$, we must have $k = 2$ and therefore $b = 2c$. Since b and c are relatively prime, this implies that $c = 1$ and $b = 2$. Thus $a = 3$. Hence $abc = 6$.

30. Answer: 680

For any 3-element subset $\{a, b, c\}$, define a mapping f by

$$f(\{a, b, c\}) = \{a, b - 1, c - 3\}.$$

Now observe that $\{a, b, c\}$ is a subset of $\{1, 2, 3, 4, \ldots, 20\}$ with $a < b - 1 < c - 3$ if and only if $f(\{a, b, c\})$ is a 3-element subset of $\{1, 2, 3, \ldots, 17\}$. Hence the answer is $\binom{17}{3} = 680$.

31. Answer: 2780

Note that $0 \le f(n) \le 4$ for $1 \le n \le 99999$. For $k = 0, 1, 2, 3, 4$, let a_k denote the number of integers n, where $1 \le n \le 99999$, such that $f(n) = k$. Then

$$M = \sum_{k=0}^{4} k a_k 2^k = \sum_{k=1}^{4} k a_k 2^k.$$

By considering the number of 2-digit, 3-digit, 4-digit and 5-digit positive integers with exactly one 0 in their decimal representation, we obtain

$$a_1 = 9 + 9\times 9\times 2 + 9\times 9\times 9\times 3 + 9\times 9\times 9\times 9\times 4 = 28602.$$

Similarly, we have

$$a_2 = 9 + 9\times 9\times \binom{3}{2} + 9\times 9\times 9\times \binom{4}{2} = 4626,$$

$$a_3 = 9 + 9^2\times 4 = 333,$$

$$a_4 = 9.$$

Hence

$$M = 1\times 28602\times 2 + 2\times 4626\times 2^2 + 3\times 333\times 2^3 + 4\times 9\times 2^4 = 102780,$$

and it follows that $M - 100000 = 2780$.

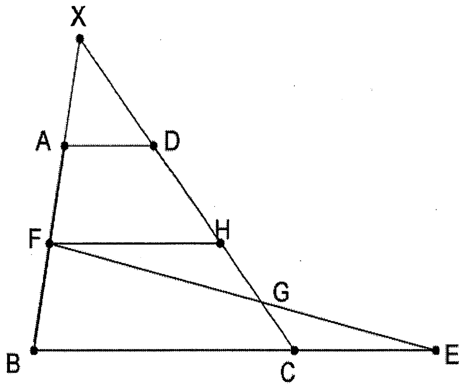
32. Answer: 5

Let $n = p^4 - 5p^2 + 13$. When $p = 3$, we have $n = 49$ and so the sum of digits is 13. When $p = 5$, we have $n = 513$ and the resulting sum of digits is 9.
Now let $p > 5$ be a prime. We have

$$n = p^4 - 5p^2 + 13 = (p-2)(p-1)(p+1)(p+2) + 9.$$

Since $p - 2, p - 1, p, p + 1$ and $p + 2$ are five consecutive integers, at least one of them is divisible by 5. Since $p \neq 5$, we must have 5 divides one of the numbers $p - 2, p - 1, p + 1$ and $p + 2$, so 5 divides the product $(p - 2)(p - 1)(p + 1)(p + 2)$. Observe that at least one of the numbers $p + 1$ and $p + 2$ is even. Therefore we see that $(p - 2)(p - 1)(p + 1)(p + 2)$ is divisible by 10. It follows that for any prime $p > 5$, the number n has at least two digits and the units digit is 9, so the sum of the digits is greater than 9. Consequently the smallest possible sum of digits is 9, and this value is attained when $p = 5$.

33. Answer: 360



Extend BA and CD to meet at X. Let H be the point on CD such that $FH \,//\, BC$.

Let $AD = CE = a$. Then $BC = 3a$, and $FH = \frac{1}{2}(AD + BC) = 2a$.

By the similarity of triangles FHG and ECG, we have

(i) area of $\Delta FHG = \left(\frac{FH}{CE}\right)^2 \times$ area of $\Delta ECG = 60$ cm^2;

(ii) $\frac{HG}{CG} = \frac{FH}{CE} = 2$, so that $HG = 2CG$ and $DH = HC = \frac{3}{2}HG$.

It follows from (i) and (ii) that the area of triangle $FDH = \frac{3}{2} \times 60 = 90$ cm^2.

Now, let area of triangle XAD be y cm^2. By the similarity of triangles XAD and XFH, we have

(iii) $\frac{XA}{XF} = \frac{AD}{FH} = \frac{1}{2}$, so that $XA = AF$ and hence
area of $\Delta XDF = 2 \times$ area of $\Delta XAD = 2y$ cm^2;

(iv) area of $\Delta XFH = \left(\frac{FH}{AD}\right)^2 y = 4y$.

It follows from (iii) and (iv) that the area of triangle $FDH = 4y - 2y = 2y$ cm^2.

Since the area of triangle FDH is 90 cm^2, we get $y = 45$.

Finally, by the similarity of triangles XAD and ABC,

$$\text{area of } \Delta XBC = \left(\frac{BC}{AD}\right)^2 y = 9y.$$

Hence the area of trapezium $ABCD = 8y = 360$ cm^2.

34. Answer: 75

First we note that every positive integer m can be written uniquely (in base 27) as

$$m = b_0 + b_1 \times 27 + b_2 \times 27^2 + \cdots + b_r \times 27^r,$$

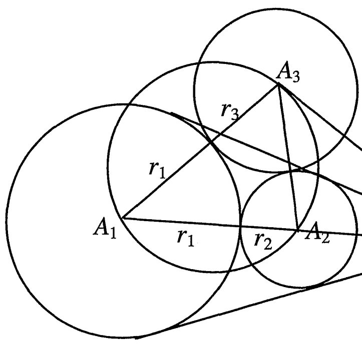
where r and $b_0, b_1, b_2, \ldots, b_r$ (depending on m) are non-negative integers with $b_i < 27$ for $i = 0, 1, 2, \ldots, r$.

Since the coefficients of $P(x)$ are non-negative integers and $P(1) = 25$, we see that $a_k \le 25 < 27$ for $0 \le k \le n$. Thus by the above remark, the polynomial $P(x)$ is uniquely determined by the value $P(27) = 1771769$. Writing 1771769 in base 27, we obtain $1771769 = 2 + 11 \times 27 + 9 \times 27^3 + 3 \times 27^4$. Therefore

$$P(x) = 2 + 11x + 9x^3 + 3x^4.$$

Hence $a_0 + 2a_1 + 3a_2 + \cdots + (n+1)a_n = 2 + 2 \times 11 + 4 \times 9 + 5 \times 3 = 75.$

35. Answer: 12



First we shall show that $r_3 = \sqrt{r_1 r_2}$. Let P be the point of concurrence of the tangent to the circumcircle of the triangle $A_1A_2A_3$ at A_3 and the two external common tangents of Γ_1 and Γ_2. Note that the line joining A_1 and A_2 also passes through P.

First we have $PA_3^2 = PA_2 \cdot PA_1$, so $\left(\dfrac{PA_3}{PA_2}\right)^2 = \dfrac{PA_1}{PA_2} = \dfrac{r_1}{r_2}$. That is, $\dfrac{PA_3}{PA_2} = \sqrt{\dfrac{r_1}{r_2}}$.

On the other hand,

$$\frac{PA_3}{PA_2} = \frac{\sin \angle A_3A_2P}{\sin \angle A_2A_3P} = \frac{\sin A_2}{\sin A_1} = \frac{r_1 + r_3}{r_2 + r_3}.$$

Thus $\sqrt{\dfrac{r_1}{r_2}} = \dfrac{r_1 + r_3}{r_2 + r_3}$. Solving for r_3, we obtain $r_3 = \sqrt{r_1 r_2}$. Substituting $r_1 = 18$, $r_2 = 8$, we get $r_3 = 12$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2010

(Senior Section, Round 2)

Saturday, 25 June 2010 **0930-1230**

INSTRUCTIONS TO CONTESTANTS

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

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1. In the triangle ABC with $AC > AB$, D is the foot of the perpendicular from A onto BC and E is the foot of the perpendicular from D onto AC. Let F be the point on the line DE such that $EF \cdot DC = BD \cdot DE$. Prove that AF is perpendicular to BF.

2. The numbers $\frac{1}{1}, \frac{1}{2}, \cdots, \frac{1}{2010}$ are written on a blackboard. A student chooses any two of the numbers, say x, y, erases them and then writes down $x + y + xy$. He continues to do this until only one number is left on the blackboard. What is this number?

3. Given $a_1 \geq 1$ and $a_{k+1} \geq a_k + 1$ for all $k = 1, 2, \cdots, n$, show that

$$a_1^3 + a_2^3 + \cdots + a_n^3 \geq (a_1 + a_2 + \cdots + a_n)^2.$$

4. An infinite sequence of integers, $a_0, a_1, a_3 \ldots$, with $a_0 > 0$, has the property that for any $n \geq 0$, $a_{n+1} = a_n - b_n$, where b_n is the number having the same sign as a_n, but having the digits written in the reverse order. For example if $a_0 = 1210$, $a_1 = 1089$ and $a_2 = -8712$, etc. Find the smallest value of a_0 so that $a_n \neq 0$ for all $n \geq 1$.

5. Let p be a prime number and let $a_1, a_2, \ldots, a_k$ be distinct integers chosen from $1, 2, \ldots, p-1$. For $1 \leq i \leq k$, let $r_i^{(n)}$ denote the remainder of the integer na_i upon division by p, so $0 \leq r_i^{(n)} < p$. Define

$$S = \{n : 1 \leq n \leq p-1,\ r_1^{(n)} < \cdots < r_k^{(n)}\}.$$

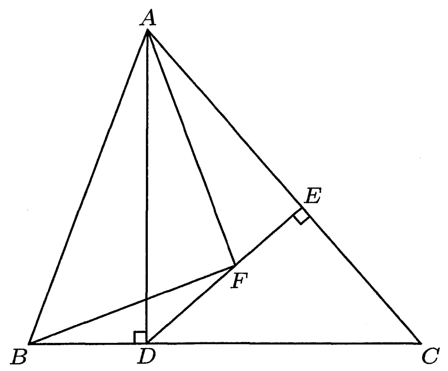
Show that S has less than $\frac{2p}{k+1}$ elements.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2010

(Senior Section, Round 2 solutions)

1. Since we are supposed to prove $\angle AFB = 90°$, it means that the 4 points A, B, D, F are concyclic. Note that $AC > AB$ implies that $\angle B > \angle C$. If TD is the tangent to the circumcircle ω of the triangle ABD with B and T lying opposite sides of the line AD, then $\angle ADT = \angle B > \angle C = \angle ADE$ so that ω intersects the interior of DE at F. Therefore F can only be in the interior of DE. Now observe that the triangles ADE and DCE are similar so that $AD/AE = DC/DE$. By the given condition, this can be written as $AD/AE = BD/EF$. This means the triangles ABD and AFE are similar. Thus $\angle ABD = \angle AFE$. This shows that A, B, D, F are concyclic. Therefore $\angle AFB = \angle ADB = 90°$.



2. See Junior Section Question 5.

3. We will prove it by induction. First, it is clear that $a_1^3 \geq a_1^2$ since $a_1 \geq 1$. Next, suppose it is true for n terms. Then

$$\begin{aligned}\sum_{k=1}^{n+1} a_k^3 \geq a_{n+1}^3 + \sum_{k=1}^{n} a_k^3 \geq a_{n+1}^3 + \Big(\sum_{k=1}^{n} a_k\Big)^2 \\ = \Big(\sum_{k=1}^{n+1} a_k\Big)^2 + a_{n+1}^3 - a_{n+1}^2 - 2a_{n+1}\sum_{k=1}^{n} a_k.\end{aligned}$$

To complete the induction, we'll now show that $a_{n+1}^3 - a_{n+1}^2 - 2a_{n+1}\sum_{k=1}^n a_k \geq 0$. Since $a_{k+1} - a_k \geq 1$, we have $a_{k+1}^2 - a_k^2 \geq a_{k+1} + a_k$. Summing up over $k = 1, \ldots, n$, and using $a_1^2 - a_1 \geq 0$, we have

$$a_{n+1}^2 - a_1^2 \geq a_{n+1} + 2\sum_{k=1}^n a_k - a_1 \quad \Rightarrow \quad a_{n+1}^3 - a_{n+1}^2 - 2a_{n+1}\sum_{k=1}^n a_k \geq 0.$$

4. If a_0 has a single digit, then $a_1 = 0$. Thus a_0 has at least 2 digits. If $a_0 = \overline{ab} = 10a+b$, then $a_1 = 9(a-b)$ which is divisible by 9. it follows that all subsequent terms are divisible by 9. Checking all 2-digit multiples of 9 shows that eventually 9 appears (Note that $\overline{ab}$ and $\overline{ba}$ give rise to the same sequence, but with opposite signs):

$$81 \rightarrow 63 \rightarrow 27 \rightarrow 45 \rightarrow 9.$$

If $a_0 = \overline{abc}$, then $a_1 = 99(a-c)$. Thus if suffices to investigate 3-digit multiples of 99, i.e., $198, \ldots, 990$. Here we find that 99 will eventually appear:

$$990 \rightarrow 891 \rightarrow 693 \rightarrow 297 \rightarrow -495 \rightarrow 99.$$

If $a_0 = \overline{abcd}$, then $a_1 = 999(a-d)+90(b-c)$. If b, c are both 0, then a_1 and all subsequent terms are multiples of 999. However, if such numbers appear in the sequence, eventually 999 will appear:

$$9990 \rightarrow 8991 \rightarrow 6993 \rightarrow 2997 \rightarrow -4995 \rightarrow 999.$$

For 1010, we get 909 and for 1011 we get -90. For 1012, we get

$$1012 \rightarrow -1089 \rightarrow -8712 \rightarrow 6534 \rightarrow 2178 \rightarrow -6534$$

and the sequence becomes periodic thereafter. Thus the smallest $a_0 = 1012$.

5. Let $r_0^{(n)} = 0$ and $r_{k+1}^{(n)} = p$. Set

$$S' = \{n : 1 \leq n \leq p-1, \sum_{i=0}^{k} |r_{i+1}^{(n)} - r_i^{(n)}| = p\}.$$

Note that

$$\sum_{i=0}^{k} |r_{i+1}^{(n)} - r_i^{(n)}| = p \quad \text{iff} \quad r_0^{(n)} \leq r_1^{(n)} \leq \cdots \leq r_{k+1}^{(n)}.$$

Thus $|S| = |S'|$. Since for $n \in S'$, $|r_{i+1}^{(n)} - r_i^{(n)}| = r_{i+1}^{(n)} - r_i^{(n)} \equiv n(a_{i+1} - a_i) \pmod p$ and $p \nmid (a_{i+1} - a_i)$, the numbers $r_{i+1}^{(n)} - r_i^{(n)}$, $1 \leq n \leq p-1$, are all distinct. Therefore

$$p|S'| = \sum_{n\in S'}\sum_{i=0}^{k} |r_{i+1}^{(n)} - r_i^{(n)}| \geq \sum_{i=0}^{k}\sum_{n\in S'} |r_{i+1}^{(n)} - r_i^{(n)}| \geq \sum_{i=0}^{k}\sum_{j=1}^{|S'|} j \geq \frac{(k+1)|S'|(|S'|+1)}{2}.$$

Therefore $|S| < \frac{2p}{k+1}$.

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2010
(Open Section, Round 1)

Wednesday, 2 June 2010 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

1. Let S be the set of all integers n such that $\dfrac{8n^3-96n^2+360n-400}{2n-7}$ is an integer. Find the value of $\sum_{n\in S}|n|$.

2. Determine the largest value of x for which

$$\left|x^2-4x-39601\right| \geq \left|x^2+4x-39601\right|.$$

3. Given that

$$x = \lfloor 1^{1/3} \rfloor + \lfloor 2^{1/3} \rfloor + \lfloor 3^{1/3} \rfloor + \cdots + \lfloor 7999^{1/3} \rfloor,$$

find the value of $\lfloor \frac{x}{100} \rfloor$, where $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y. (For example, $\lfloor 2.1 \rfloor = 2$, $\lfloor 30 \rfloor = 30$, $\lfloor -10.5 \rfloor = -11$.)

4. Determine the smallest positive integer C such that $\dfrac{6^n}{n!} \leq C$ for all positive integers n.

5. Let CD be a chord of a circle Γ_1 and AB a diameter of Γ_1 perpendicular to CD at N with $AN > NB$. A circle Γ_2 centred at C with radius CN intersects Γ_1 at points P and Q, and the segments PQ and CD intersect at M. Given that the radii of Γ_1 and Γ_2 are 61 and 60 respectively, find the length of AM.

6. Determine the minimum value of $\sum_{k=1}^{50} x_k$, where the summation is done over all possible positive numbers $x_1, \ldots, x_{50}$ satisfying $\sum_{k=1}^{50} \dfrac{1}{x_k} = 1$.

7. Find the sum of all positive integers p such that the expression $(x-p)(x-13)+4$ can be expressed in the form $(x+q)(x+r)$ for distinct integers q and r.

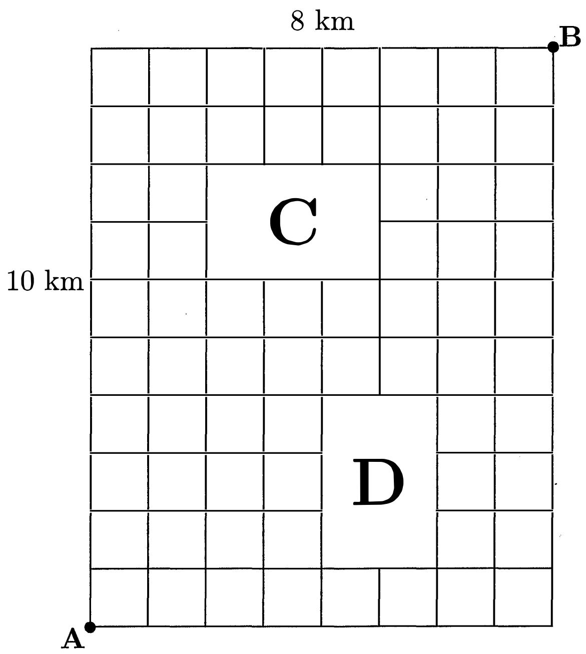
8. Let $p_k = 1 + \dfrac{1}{k} - \dfrac{1}{k^2} - \dfrac{1}{k^3}$, where k is a positive integer. Find the least positive integer n such that the product $p_2 p_3 \cdots p_n$ exceeds 2010.

9. Let B be a point on the circle centred at O with diameter AC and let D and E be the circumcentres of the triangles OAB and OBC respectively. Given that $\sin\angle BOC = \frac{4}{5}$ and $AC = 24$, find the area of the triangle BDE.

10. Let f be a real-valued function with the rule $\mathrm{f}(x) = x^3 + 3x^2 + 6x + 14$ defined for all real value of x. It is given that a and b are two real numbers such that $\mathrm{f}(a) = 1$ and $\mathrm{f}(b) = 19$. Find the value of $(a+b)^2$.

11. If $\cot\alpha + \cot\beta + \cot\gamma = -\dfrac{4}{5}$, $\tan\alpha + \tan\beta + \tan\gamma = \dfrac{17}{6}$ and $\cot\alpha\cot\beta + \cot\beta\cot\gamma + \cot\gamma\cot\alpha = -\dfrac{17}{5}$, find the value of $\tan(\alpha+\beta+\gamma)$.

12. The figure below shows a road map connecting two shopping malls A and B in a certain city. Each side of the smallest square in the figure represents a road of distance 1km. Regions C and D represent two large residential estates in the town. Find the number of shortest routes to travel from A to B along the roads shown in the figure.



13. Let $a_1 = 1$, $a_2 = 2$ and for all $n \geq 2$, $a_{n+1} = \frac{2n}{n+1}a_n - \frac{n-1}{n+1}a_{n-1}$. It is known that $a_n > 2 + \frac{2009}{2010}$ for all $n \geq m$, where m is a positive integer. Find the least value of m.

14. It is known that

$$\sqrt{9 - 8\sin 50^\circ} = a + b\sin c^\circ$$

for exactly one set of positive integers (a, b, c), where $0 < c < 90$. Find the value of $\frac{b+c}{a}$.

15. If α is a real root of the equation $x^5 - x^3 + x - 2 = 0$, find the value of $\lfloor \alpha^6 \rfloor$, where $\lfloor x \rfloor$ is the least positive integer not exceeding x.

16. If a positive integer cannot be written as the difference of two square numbers, then the integer is called a "cute" integer. For example, $1, 2$ and 4 are the first three "cute" integers. Find the 2010^{th} "cute" integer.

(Note: A *square number* is the square of a positive integer. As an illustration, $1, 4, 9$ and 16 are the first four square numbers.)

17. Let $\mathrm{f}(x)$ be a polynomial in x of degree 5. When $\mathrm{f}(x)$ is divided by $x - 1$, $x - 2$, $x - 3$, $x - 4$ and $x^2 - x - 1$, $\mathrm{f}(x)$ leaves a remainder of $3, 1, 7, 36$ and $x - 1$ respectively. Find the square of the remainder when $\mathrm{f}(x)$ is divided by $x + 1$.

18. Determine the number of ordered pairs of positive integers (a, b) satisfying the equation

$$100(a+b) = ab - 100.$$

(Note: As an illustration, $(1,2)$ and $(2,1)$ are considered as two distinct ordered pairs.)

19. Let $p = a^b + b^a$. If a, b and p are all prime, what is the value of p?

20. Determine the value of the following expression:

$$\left\lfloor \frac{11}{2010} \right\rfloor + \left\lfloor \frac{11 \times 2}{2010} \right\rfloor + \left\lfloor \frac{11 \times 3}{2010} \right\rfloor + \left\lfloor \frac{11 \times 4}{2010} \right\rfloor + \cdots + \left\lfloor \frac{11 \times 2009}{2010} \right\rfloor,$$

where $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y.

21. Numbers $1, 2, \ldots, 2010$ are placed on the circumference of a circle in some order. The numbers i and j, where $i \neq j$ and $i, j \in \{1, 2, \ldots, 2010\}$ form a *friendly* pair if

(i) i and j are not neighbours to each other, and

(ii) on one or both of the arcs connecting i and j along the circle, all numbers in between them are greater than both i and j.

Determine the minimal number of *friendly* pairs.

22. Let S be the set of all non-zero real-valued functions f defined on the set of all real numbers such that

$$\mathrm{f}(x^2 + yf(z)) = x\mathrm{f}(x) + z\mathrm{f}(y)$$

for all real numbers x, y and z. Find the maximum value of f(12345), where $\mathrm{f} \in S$.

23. All possible 6-digit numbers, in each of which the digits occur in non-increasing order from left to right (e.g., 966541), are written as a sequence in increasing order (the first three 6-digit numbers in this sequence are 100000, 110000, 111000 and so on). If the 2010th number in this sequence is denoted by p, find the value of $\lfloor \frac{p}{10} \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

24. Find the number of permutations $a_1a_2a_3a_4a_5a_6$ of the six integers from 1 to 6 such that for all i from 1 to 5, a_{i+1} does not exceed a_i by 1.

25. Let

$$A = \left(\binom{2010}{0} - \binom{2010}{-1}\right)^2 + \left(\binom{2010}{1} - \binom{2010}{0}\right)^2 + \left(\binom{2010}{2} - \binom{2010}{1}\right)^2 + \cdots + \left(\binom{2010}{1005} - \binom{2010}{1004}\right)^2.$$

Determine the minimum integer s such that

$$sA \geq \binom{4020}{2010}.$$

(Note: For a given positive integer n, $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ for $r = 0, 1, 2, 3, \cdots, n$; and for all other values of r, define $\binom{n}{r} = 0$.)

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1. Answer: 50

   Note that $\dfrac{8n^3-96n^2+360n-400}{2n-7}=4n^2-34n+61+\dfrac{27}{2n-7}$. Since $4n^2-34n+61$ is an integer for all integers n, we must have that 27 divisible by $2n-7$. Hence, $2n-7=-1,1,-3,3,-9,9,-27,27$, so that $n=3,4,2,5,-1,8,-10,17$. Hence the required sum equals 50.

2. Answer: 199

   By direct computation,

$$\begin{aligned}
&\left|x^2-4x-39601\right|\geq\left|x^2+4x-39601\right|\\
\iff&(x^2-4x-39601)^2-(x^2+4x-39601)^2\geq 0\\
\iff&2(x^2-39601)(-8x)\geq 0\\
\iff&x(x+199)(x-199)\leq 0,
\end{aligned}$$

   we conclude that the largest possible value of x is 199.

3. Answer: 1159

   Note that

$$\begin{aligned}
x &= \lfloor 1^{1/3}\rfloor+\lfloor 2^{1/3}\rfloor+\lfloor 3^{1/3}\rfloor+\cdots+\lfloor 7999^{1/3}\rfloor\\
&= \sum_{1^3\leq k<2^3}\lfloor k^{1/3}\rfloor+\sum_{2^3\leq k<3^3}\lfloor k^{1/3}\rfloor+\sum_{3^3\leq k<4^3}\lfloor k^{1/3}\rfloor+\cdots+\sum_{19^3\leq k<20^3}\lfloor k^{1/3}\rfloor\\
&= \sum_{1^3\leq k<2^3}1+\sum_{2^3\leq k<3^3}2+\sum_{3^3\leq k<4^3}3+\cdots+\sum_{19^3\leq k<20^3}19\\
&= \left(2^3-1^3\right)+2\left(3^3-2^3\right)+3\left(4^3-3^3\right)+\cdots+18\left(19^3-18^3\right)+19\left(20^3-19^3\right)\\
&= 19(8000)-\sum_{k=1}^{19}k^3\\
&= 19(8000)-\left(\frac{19\times 20}{2}\right)^2\\
&= 115900
\end{aligned}$$

   $\therefore \lfloor \frac{x}{100}\rfloor = 1159.$

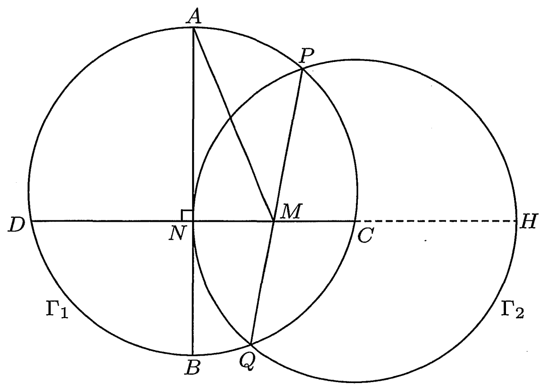
4. Answer: 65

Define $f(n) = \dfrac{6^n}{n!}$ for $n = 1, 2, 3, \cdots$. It is clear that $f(1) = 6, f(2) = 18, f(3) = 36, f(4) = 54$ and $f(5) = 64.8$. For all $n \geq 6$,

$$f(n) = \frac{6^n}{n!} \leq \frac{6}{1} \times \frac{6}{2} \times \frac{6}{3} \times \frac{6}{4} \times \frac{6}{5} \times \frac{6}{6} = 6 \times 3 \times 2 \times 1.8 = 64.8$$

5. Answer: 78

Extend DC meeting Γ_2 at H. Note that $DN = NC = CH = 60$. Since M is of equal power with respect to Γ_1 and Γ_2. Thus $MN \cdot MH = MC \cdot MD$. That is $MN(MC+60) = MC(MN+60)$ giving $MN = MC$. Thus $MN = 30$.



The power of N with respect to Γ_1 is $DN \cdot NC = 60^2$, and is also equal to $NA \cdot NB = NA \cdot (AB - NA) = NA \cdot (122 - NA)$. Thus $NA \cdot (122 - NA) = 60^2$. Solving this quadratic equation, we get $NA = 72$ or 50. Since $NA > NB$, we have $NA = 72$. Consequently $AM = \sqrt{NA^2 + MN^2} = \sqrt{72^2 + 30^2} = 78$.

6. Answer: 2500

By Cauchy-Schwarz inequality,

$$\sum_{k=1}^{50} 1 \leq \sqrt{\sum_{k=1}^{50} x_k \sum_{k=1}^{50} \frac{1}{x_k}},$$

and equality holds if and only if $x_k = 50$ for $k = 1, \ldots, 50$. Therefore the required value is 2500.

7. Answer: 26

Let $(x-p)(x-13)+4 = (x+q)(x+r)$. Substituting $x = -q$ into the above identity yields $(-q-p)(-q-13) = -4$, which becomes $(q+p)(q+13) = -4$. Since p and q are integers, we must have the following cases:

(a) $q + p = 4, q + 13 = -1$;

(b) $q + p = -4, q + 13 = 1$;

(c) $q + p = 2, q + 13 = -2$; or

(d) $q + p = -2, q + 13 = 2$

For case (a), we obtain $q = -14, p = 8$ and hence $(x-p)(x-13)+4 = (x-14)(x-17)$;

For case (b), we obtain $q = -12, p = 8$ and hence $(x-p)(x-13)+4 = (x-9)(x-12)$;

For case (c), we obtain $q = -15, p = 17$ and hence $(x-p)(x-13)+4 = (x-15)^2$; which is NOT what we want;

For case (d), we obtain $q = -11, p = 9$ and hence $(x-p)(x-13)+4 = (x-11)^2$; which is also NOT what we want. Hence the two possible values of p are 8 and 18, the sum of which is 26.

8. Answer: 8038

First, note that

$$p_k = 1 + \frac{1}{k} - \frac{1}{k^2} - \frac{1}{k^3} = \left(1 - \frac{1}{k}\right)\left(1 + \frac{1}{k}\right)^2 = \frac{(k-1)(k+1)^2}{k^3}.$$

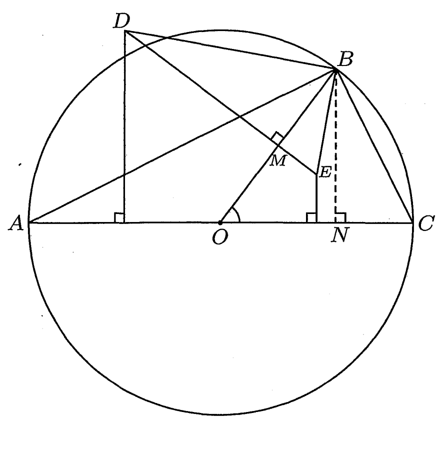
Therefore,

$$p_2 p_3 \cdots p_n = \frac{1\cdot 3^2}{2^3} \cdot \frac{2\cdot 4^2}{3^3} \cdot \frac{3\cdot 5^2}{4^3} \cdots \frac{(n-1)(n+1)^2}{n^3} = \frac{(n+1)^2}{4n}.$$

Next, observe that $\dfrac{(n+1)^2}{4n} > 2010$ is equivalent to $n^2 - 8038n + 1 > 0$, which is equivalent to $n(n-8038) > -1$. Since n is a positive integer, the last inequality is satisfied if and only if $n \geq 8038$. Consequently, the least n required is 8038.

9. Answer: 45

Let $d = AC = 24$. First, it is not difficult to see that $\angle DEB = \angle ACB$ and $\angle EDB = \angle CAB$, so that the triangles DBE and ABC are similar.



Let M and N be the feet of the perpendiculars from B onto DE and AC respectively. As M is the midpoint of OB, we have $BM = \frac{d}{4}$. Also $BN = BO \sin\angle BOC = \frac{d}{2} \times \frac{4}{5} = \frac{2d}{5}$. Therefore $DE = AC \times \frac{BM}{BN} = d \times \frac{5d}{8d} = \frac{5d}{8}$. Thus the area of the triangle BDE is $\frac{1}{2} \times BM \times DE = \frac{1}{2} \times \frac{d}{4} \times \frac{5d}{8} = \frac{5d^2}{64}$. Substituting $d = 24$, the area of the triangle BDE is 45.

10. Answer: 4

We note that $f(x) = (x+1)^3 + 3(x+1) + 10$. Let $g(y) = y^3 + 3y$, which is a strictly increasing odd function. Hence $f(a) = 1$ implies $g(a+1) = -9$ and $f(b) = 19$ implies $g(b+1) = 9$. Thus, $a+1 = -(b+1)$, implying $a+b = -2$, so that $(a+b)^2 = 4$.

11. Answer: 11

Let $x = \tan\alpha$, $y = \tan\beta$ and $z = \tan\gamma$. Then

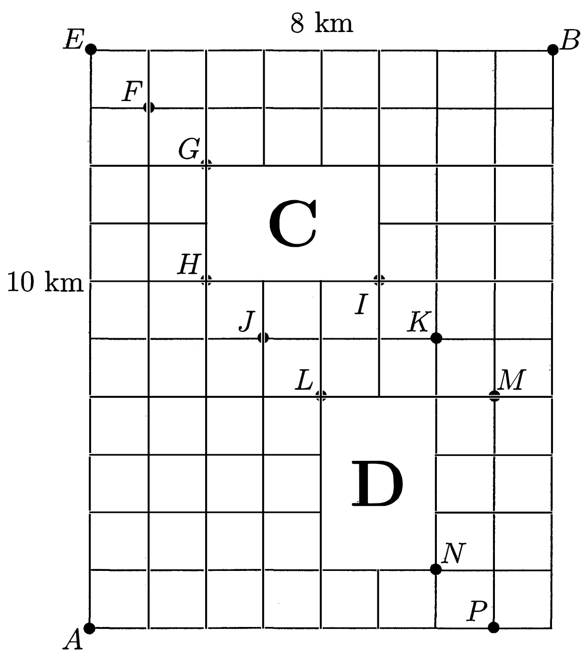
$$\begin{aligned} \frac{xy+yz+zx}{xyz} &= -\frac{4}{5} \\ x+y+z &= \frac{17}{6} \\ \frac{x+y+z}{xyz} &= -\frac{17}{5} \end{aligned}$$

From the above three equations, $xyz = -\dfrac{5}{6}$ and $xy+yz+zx = \dfrac{2}{3}$. It follows that

$$\tan(\alpha+\beta+\gamma) = \frac{x+y+z-xyz}{1-(xy+yz+zx)} = \frac{17/6-(-5/6)}{1-2/3} = 11.$$

12. Answer: 22023

Include the points $E, F, G, H, I, J, K, L, M, N$ and P in the diagram as shown and consider all possible routes:



For the route $A \to E \to B$, there is 1 way.

For the route $A \to F \to B$, there are $\dbinom{10}{1} \cdot \dbinom{8}{1} = 80$ ways.

For the route $A \to G \to B$, there are $\dbinom{10}{2} \cdot \dbinom{8}{2} = 1260$ ways.

For the route $A \to H \to I \to B$, there are $\binom{8}{2} \cdot \binom{7}{3} = 980$ ways.

For the route $A \to J \to I \to B$, there are $\binom{8}{3} \cdot \binom{3}{1} \cdot \binom{7}{3} = 5880$ ways.

For the route $A \to J \to K \to B$, there are $\binom{8}{3} \cdot \binom{7}{2} = 1176$ ways.

For the route $A \to L \to I \to B$, there are $\binom{8}{4} \cdot \binom{3}{1} \cdot \binom{7}{3} = 7350$ ways.

For the route $A \to L \to K \to B$, there are $\binom{8}{4} \cdot \binom{3}{1} \cdot \binom{7}{2} = 4410$ ways.

For the route $A \to L \to M \to B$, there are $\binom{8}{4} \cdot \binom{7}{1} = 490$ ways.

For the route $A \to N \to B$, there are $\binom{7}{1} \cdot \binom{11}{2} = 385$ ways.

For the route $A \to P \to B$, there are $\binom{11}{1} = 11$ ways.

Hence, by adding up, there are altogether 22023 ways.

13. Answer: 4021

Rearranging the recurrence relation yields $a_{n+1} - a_n = \frac{n-1}{n+1}(a_n - a_{n-1})$ for $n \geq 2$. Thus, for $n \geq 3$, we have

$$\begin{aligned} a_n - a_{n-1} &= \frac{n-2}{n}(a_{n-1} - a_{n-2}) = \frac{n-2}{n} \cdot \frac{n-3}{n-1}(a_{n-2} - a_{n-3}) \\ &= \cdots = \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdots \frac{2}{4} \cdot \frac{1}{3}(a_2 - a_1) \\ &= \frac{2}{n(n-1)} = \frac{2}{n-1} - \frac{2}{n} \end{aligned}$$

for $n \geq 3$. Using the method of difference, we obtain $a_n = 3 - \frac{2}{n}$ for $n \geq 3$. Given that $a_n > 2 + \frac{2009}{2010}$, we have

$$3 - \frac{2}{n} > 2 + \frac{2009}{2010} = 3 - \frac{1}{2010},$$

yielding $n > 4020$. Hence the least value of m is 4021.

14. Answer: 14

We have

$$\begin{aligned} 9 - 8\sin 50^\circ &= 8 + 8\sin 10^\circ - 8\sin 10^\circ - 8\sin 50^\circ \\ &= 9 + 8\sin 10^\circ - 8\left(2\sin 30^\circ \cos 20^\circ\right) \\ &= 9 + 8\sin 10^\circ - 8(1 - 2\sin^2 10^\circ) \\ &= 16\sin^2 10^\circ + 8\sin 10^\circ + 1 \\ &= (1 + 4\sin 10^\circ)^2 \\ \therefore \sqrt{9 - 8\sin 50^\circ} &= 1 + 4\sin 10^\circ. \end{aligned}$$

Thus, $a = 1$, $b = 4$ and $c = 10$, and hence $\frac{b+c}{a} = 14$.

15. Answer: 3

It can be easily seen that the given equation has exactly one real root α, since (1) all polynomial equations of degree 5 has at least one real root, and (2) the function $f(x) = x^5 - x^3 + x - 2$ is strictly increasing since $f'(x) = 5x^4 - 3x^2 + 1 > 0$ for all real values of x. It can also be checked that $f(\frac{1}{2}) < 0$ and $f(2) > 0$, so that $\dfrac{1}{2} < \alpha < 2$. This is equivalent to $\alpha^6 < 4$ since

$$\begin{aligned}\alpha^6 < 4 \quad &\Longleftrightarrow \alpha^4 - \alpha^2 + 2\alpha < 4\\ &\Longleftrightarrow \alpha^5 - \alpha^3 + 2\alpha^2 < 4\alpha\\ &\Longleftrightarrow 2\alpha^2 - 5\alpha + 2 < 0\\ &\Longleftrightarrow \frac{1}{2} < \alpha < 2.\end{aligned}$$

In addition, we claim that $\alpha^6 \geq 3$ since

$$\begin{aligned}\alpha^6 \geq 3 \quad &\Longleftrightarrow \quad \alpha^4 - \alpha^2 + 2\alpha \geq 3\\ &\Longleftrightarrow \quad \alpha^5 - \alpha^3 + 2\alpha^2 - 3\alpha \geq 0\\ &\Longleftrightarrow \quad 2\alpha^2 - 4\alpha + 2 \geq 0,\end{aligned}$$

the last inequality is always true. Hence $3 \leq \alpha^6 < 4$, thereby showing that $\lfloor \alpha^6 \rfloor = 3$.

16. Answer: 8030

Any odd number greater than 1 can be written as $2k+1$, where $2k+1 = (k+1)^2 - k^2$. Hence all odd integers greater than 1 are not "cute" integers. Also, since $4m = (m+1)^2 - (m-1)^2$, so that all integers of the form $4m$, where $m > 1$, are not "cute". We claim that all integers of the form $4m + 2$ are "cute". Suppose $4m + 2$ (for $m \geq 1$) is not "cute", then

$$4m + 2 = x^2 - y^2 = (x - y)(x + y)$$

for some integers positive integers x and y. However, $x + y$ and $x - y$ have the same parity, so that $x - y$ and $x + y$ must all be even since $4m + 2$ is even. Hence $4m + 2$ is divisible by 4, which is absurd. Thus, $4m + 2$ is "cute". The first few "cute" integers are $1, 2, 4, 6, 10 \cdots$. For $n > 3$, the n^{th} "cute" integer is $4n - 10$. Thus, the 2010^{th} "cute" integer is $4(2010) - 10 = 8030$.

17. Answer: 841

We have $f(1) = 3$, $f(2) = 1$, $f(3) = 7$, $f(4) = 36$ and

$$f(x) = (x^2 - x - 1)g(x) + (x - 1),$$

where $g(x)$ is a polynomial in x of degree 3. Hence $g(1) = -3$, $g(2) = 0$, $g(3) = 1$, and $g(4) = 3$. Thus

$$\begin{aligned}f(x) \quad = \quad (x^2 - x - 1)\Big[&(-3) \cdot \frac{(x-2)(x-3)(x-4)}{(-1)(-2)(-3)} + (1) \cdot \frac{(x-1)(x-2)(x-4)}{(2)(1)(-1)}\\ &+ (3) \cdot \frac{(x-1)(x-2)(x-3)}{(3)(2)(1)}\Big] + (x - 1).\end{aligned}$$

Thus, $f(-1) = -29$, so that its square is 841.

18. Answer: 18

Solving for b, we get $b = \frac{100a+100}{a-100}$. Since a and b are positive, we must have $a > 100$. Let $a = 100 + m$, where m is a positive integer. Thus $b = \frac{100(100+m)+100}{m} = 100 + \frac{10100}{m}$. Therefore m must be a factor of $10100 = 101 \times 2^2 \times 5^2$. Conversely, each factor r of m determines a unique solution $(a, b) = (100 + r, 100 + \frac{10100}{r})$ of the equation $100(a + b) = ab - 100$. There are $18 = (1 + 1)(2 + 1)(2 + 1)$ factors of 10100. Consequently there are 18 solutions of the given equation. In fact, these 18 solutions can be found to be $(a, b) = (101, 10200)$, $(102, 5150)$, $(104, 2625)$, $(105, 2120)$, $(110, 1110)$, $(120, 605)$, $(125, 504)$, $(150, 302)$, $(200, 201)$, $(201, 200)$, $(302, 150)$, $(504, 125)$, $(605, 120)$, $(1110, 110)$, $(2120, 105)$,
$(2625, 104)$, $(5150, 102)$, $(10200, 101)$.

19. Answer: 17

If both a, b are odd prime numbers, then p is even and $p \geq 4$, contradicting the condition that p is prime. Thus $a = 2$ or $b = 2$.

Assume that $a = 2$. Then $b \neq 2$; otherwise, $p = 8$ which is not prime.

Thus b is an odd prime number. Let $b = 2k+1$, where k is an integer greater than 1. Thus

$$p = 2^{2k+1} + (2k + 1)^2 = 2 \times 4^k + (2k + 1)^2.$$

We shall show that $k = 1$.

Suppose that $k \geq 2$. If $k \equiv 1 (\text{mod } 3)$, then $b > 3$ and

$$b = 2k + 1 \equiv 0 (\text{mod } 3),$$

contradicting the condition that b is prime. If $k \not\equiv 1 (\text{mod } 3)$, then

$$p = 2 \times 4^k + (2k + 1)^2 \equiv 2 + 4k^2 + 4k + 1 \equiv 4k(k + 1) \equiv 0 (\text{mod } 3),$$

a contradiction too. Thus $k = 1$ and $b = 3$. Therefore

$$p = 2^3 + 3^2 = 8 + 9 = 17.$$

20. Answer: 10045

The number of grid points (x, y) inside the rectangle with four corner vertices $(0, 0)$, $(11, 0)$, $(0, 2010)$ and $(11, 2010)$ is

$$(11 - 1) \times (2010 - 1) = 20090.$$

There are no grid points on the diagonal between $(0, 0)$ and $(11, 2010)$ excluding these two points, since for any point (x, y) on this segment between $(0, 0)$ and $(11, 2010)$, we have

$$y = \frac{2010x}{11}.$$

But for an integer x with $1 \leq x \leq 10$, $\frac{2010x}{11}$ is not an integer.

Hence the number of grid points (x, y) inside the triangle formed by corner points $(0, 0)$, $(0, 2010)$ and $(11, 2010)$ is

$$(11 - 1) \times (2010 - 1)/2 = 20090/2 = 10045.$$

For any grid point (x, y) inside the triangle formed by corner points $(0,0)$, $(0,2010)$ and $(11,2010)$, we have

$$1 \le y \le 2009, \quad 1 \le x < \frac{11y}{2010}.$$

Thus, for any fixed positive integer y, there are the number of grid points satisfying the condition that $1 \le x < \frac{11y}{2010}$ is

$$\left\lfloor \frac{11y}{2010} \right\rfloor,$$

as $\frac{11y}{2010}$ is not an integer when $1 \le y \le 2009$. Thus the number of grid points (x, y) inside the triangle formed by corner points $(0,0)$, $(0,2010)$ and $(11,2010)$ is

$$\left\lfloor \frac{11}{2010} \right\rfloor + \left\lfloor \frac{11 \times 2}{2010} \right\rfloor + \left\lfloor \frac{11 \times 3}{2010} \right\rfloor + \left\lfloor \frac{11 \times 4}{2010} \right\rfloor + \cdots + \left\lfloor \frac{11 \times 2009}{2010} \right\rfloor.$$

Therefore the answer is 10045.

21. Answer: 2007

Consider n distinct numbers where $n \ge 3$. We shall show by induction that the number of friendly pairs is always $n - 3$ for $n \ge 3$. Hence the minimal number of friendly pairs is also $n - 3$.

If $n = 3$, then there are no friendly pairs as each number is adjacent to the other two. Thus the number of friendly pairs is 0. Assume that the number of friendly pairs for any arrangement of n distinct numbers on the circle is $n - 3$ for some $n \ge 3$. Consider $n + 1$ distinct numbers on the circle. Let m be the largest number. Now consider the n numbers on circle after deleting m. The two numbers adjacent to m which originally form a friendly pair do not form a friendly pair anymore. Any other friendly pair remains a friendly pair when m is deleted. On the other hand, any friendly pair after m is deleted was originally a friendly pair. By induction hypothesis, there are $n - 3$ friendly pairs after m is deleted. Therefore, there are $(n + 1) - 3$ friendly pairs originally.

22. Answer: 12345

We are given the equation

$$\mathrm{f}(x^2 + y\mathrm{f}(z)) = x\mathrm{f}(x) + z\mathrm{f}(y). \tag{1}$$

Substituting $x = y = 0$ into (1), we get $z\mathrm{f}(0) = \mathrm{f}(0)$ for all real number z. Hence $\mathrm{f}(0) = 0$. Substituting $y = 0$ into (1), we get

$$\mathrm{f}(x^2) = x\mathrm{f}(x) \tag{2}$$

Similarly, substituting $x = 0$ in (1) we get

$$\mathrm{f}(y\mathrm{f}(z)) = z\mathrm{f}(y). \tag{3}$$

Substituting $y = 1$ into (3) we get

$$\mathrm{f}(\mathrm{f}(z)) = z\mathrm{f}(1) \tag{4}$$

for all real z. Now, using (2) and (4), we obtain

$$\mathrm{f}(x\mathrm{f}(x)) = \mathrm{f}(\mathrm{f}(x^2)) = x^2\mathrm{f}(1). \tag{5}$$

Substituting $y = z = x$ into (3) also yields

$$\mathrm{f}(x\mathrm{f}(x)) = x\mathrm{f}(x). \tag{6}$$

Comparing (5) and (6), it follows that $x^2\mathrm{f}(1) = x\mathrm{f}(x)$, so that if x is non-zero, then $\mathrm{f}(x) = cx$ for some constant c. Since $\mathrm{f}(0) = 0$, we also have $\mathrm{f}(x) = cx$ for all real values of x. Substituting this into (1), we have that $c(x^2 + cyz) = cx^2 + cyz$. This implies that $c^2 = c$, forcing $c = 0$ or $c = 1$. Since f is non-zero, we must have $c = 1$, so that $\mathrm{f}(x) = x$ for all real values of x. Hence $\mathrm{f}(12345) = 12345$.

23. Answer: 86422

The number of ways of choosing r objects from n different types of objects of which repetition is allowed is $\binom{n+r-1}{r}$. In particular, if we write r-digit numbers using n digits allowing for repetitions with the condition that the digits appear in a non-increasing order, there are $\binom{n+r-1}{r}$ ways of doing so. Grouping the given numbers into different categories, we have the following tabulation. In order to track the enumeration of these elements, the cumulative sum is also computed:

Numbers Beginning with	Digits used other than the fixed part	n	r	$\binom{n+r-1}{r}$	Cumulative
1	$1, 0$	2	5	$\binom{6}{5} = 6$	6
2	$2, 1, 0$	3	5	$\binom{7}{5} = 21$	27
3	$3, 2, 1, 0$	4	5	$\binom{8}{5} = 56$	83
4	$4, 3, 2, 1, 0$	5	5	$\binom{9}{5} = 126$	209
5	$5, 4, 3, 2, 1, 0$	6	5	$\binom{10}{5} = 252$	461
6	$6, 5, 4, 3, 2, 1, 0$	7	5	$\binom{11}{5} = 462$	923
7	$7, 6, 5, 4, 3, 2, 1, 0$	8	5	$\binom{12}{5} = 792$	1715
From 800000 to 855555	$5, 4, 3, 2, 1, 0$	6	5	$\binom{10}{5} = 252$	1967
From 860000 to 863333	$3, 2, 1, 0$	4	4	$\binom{7}{4} = 35$	2002

The next 6-digit numbers are:

$864000, 864100, 864110, 864111, 864200, 864210, 864211, 864220$. Hence, the 2010^{th} 6-digit number is 864220. Therefore, $x = 864220$ so that $\lfloor \frac{x}{10} \rfloor = 86422$.

24. Answer: 309

Let S be the set of permutations of the six integers from 1 to 6. Then $|S| = 6! = 720$. Define $P(i)$ to be the subset of S such that the digit $i+1$ follows immediately i, $i = 1, 2, 3, 4, 5$.

Then

$$\sum_{i}|P(i)| = \binom{5}{1}5!$$

$$\sum_{i_1<i_2}\left|P(i_1)\bigcap P(i_2)\right| = \binom{5}{2}4!$$

$$\sum_{i_1<i_2<i_3}\left|P(i_1)\bigcap P(i_2)\bigcap P(i_3)\right| = \binom{5}{3}3!$$

$$\sum_{i_1<i_2<i_3<i_4}\left|P(i_1)\bigcap P(i_2)\bigcap P(i_3)\bigcap P(i_4)\right| = \binom{5}{4}2!$$

$$\left|P(1)\bigcap P(2)\bigcap P(3)\bigcap P(4)\bigcap P(5)\right| = \binom{5}{5}1!.$$

By the Principle of Inclusion and Exclusion, the required number is

$$6! - \binom{5}{1}5! + \binom{5}{2}4! - \binom{5}{3}3! + \binom{5}{4}2! - \binom{5}{5}1! = 309.$$

25. Answer: 2011

Note that

$$A = \left(\binom{2010}{0} - \binom{2010}{-1}\right)^2 + \left(\binom{2010}{1} - \binom{2010}{0}\right)^2 + \left(\binom{2010}{2} - \binom{2010}{1}\right)^2$$
$$+\cdots+\left(\binom{2010}{1005} - \binom{2010}{1004}\right)^2 = \frac{1}{2}\sum_{k=0}^{2011}\left(\binom{2010}{k} - \binom{2010}{k-1}\right)^2.$$

Observe that

$$\sum_{k=0}^{2011}\left(\binom{2010}{k} - \binom{2010}{k-1}\right)^2$$

is the coefficient of x^{2010} in the expansion of the following expression:

$$\left(\sum_{k=0}^{2011}\left(\binom{2010}{k} - \binom{2010}{k-1}\right)x^k\right)\left(\sum_{k=-1}^{2010}\left(\binom{2010}{k} - \binom{2010}{k+1}\right)x^k\right).$$

We also have

$$\sum_{k=0}^{2011}\left(\binom{2010}{k} - \binom{2010}{k-1}\right)x^k = \sum_{k=0}^{2011}\binom{2010}{k}x^k - \sum_{k=0}^{2011}\binom{2010}{k-1}x^k$$
$$= (x+1)^{2010} - x(x+1)^{2010} = (1-x)(x+1)^{2010}$$

and

$$\sum_{k=-1}^{2010}\left(\binom{2010}{k} - \binom{2010}{k+1}\right)x^k = \sum_{k=0}^{2010}\binom{2010}{k}x^k - \frac{1}{x}\sum_{k=-1}^{2009}\binom{2010}{k+1}x^{k+1}$$
$$= (x+1)^{2010} - \frac{1}{x}(x+1)^{2010} = (1-1/x)(x+1)^{2010}.$$

The coefficient of x^{2010} in the expansion of the following expression:

$$(1-x)(x+1)^{2010}(1-1/x)(x+1)^{2010} = (2-1/x-x)(x+1)^{4020}$$

is

$$2\binom{4020}{2010} - \binom{4020}{2011} - \binom{4020}{2009} = 2\binom{4020}{2010} - 2\binom{4020}{2009}.$$

Hence

$$A = \binom{4020}{2010} - \binom{4020}{2009}.$$

Consider the inequality:

$$sA \geq \binom{4020}{2010},$$

$$s(1-2010/2011) \geq 1,$$

$$s \geq 2011.$$

Hence the answer is 2011.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2010

(Open Section, Round 2)

Saturday, 26 June 2010 **0900-1330**

INSTRUCTIONS TO CONTESTANTS

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

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1. Let CD be a chord of a circle Γ_1 and AB a diameter of Γ_1 perpendicular to CD at N with $AN > NB$. A circle Γ_2 centred at C with radius CN intersects Γ_1 at points P and Q. The line PQ intersects CD at M and AC at K; and the extension of NK meets Γ_2 at L. Prove that PQ is perpendicular to AL.

2. Let a_n, b_n, $n = 1, 2, \ldots$ be two sequences of integers defined by $a_1 = 1, b_1 = 0$ and for $n \geq 1$,

$$a_{n+1} = 7a_n + 12b_n + 6$$
$$b_{n+1} = 4a_n + 7b_n + 3.$$

Prove that a_n^2 is the difference of two consecutive cubes.

3. Suppose that $a_1, \ldots, a_{15}$ are prime numbers forming an arithmetic progression with common difference $d > 0$. If $a_1 > 15$, prove that $d > 30,000$.

4. Let n be a positive integer. Find the smallest positive integer k with the property that for any colouring of the squares of a $2n \times k$ chessboard with n colours, there are 2 columns and 2 rows such that the 4 squares in their intersections have the same colour.

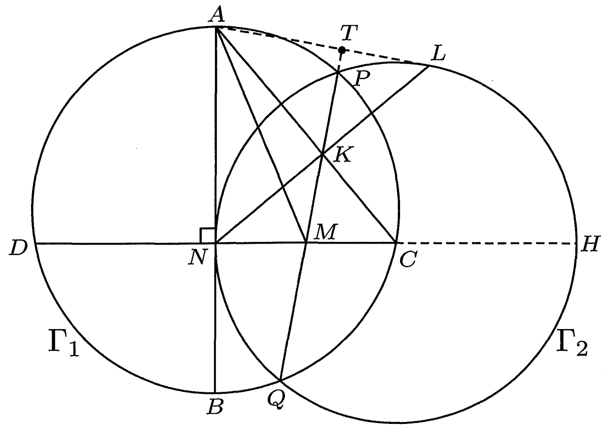
5. Let p be a prime number and let x, y, z be positive integers so that $0 < x < y < z < p$. Suppose that x^3, y^3 and z^3 have the same remainder when divided by p, show that $x^2 + y^2 + z^2$ is divisible by $x + y + z$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2010

(Open Section, Round 2 solutions)

1. Extend DC meeting Γ_2 at H. Let the radius of Γ_2 be r. Note that $DN = NC = CH = r$. Since M is of equal power with respect to Γ_1 and Γ_2. Thus $MN \cdot MH = MC \cdot MD$. That is $MN(MC + r) = MC(MN + r)$ giving $MN = MC$. Thus M is the midpoint of NC.



As K lies on the radical axis of Γ_1 and Γ_2, the points C, N, A, L are concyclic. Thus $\angle ALC = \angle ANC = 90^\circ$ so that AL is tangent to Γ_2. It follows that AC is perpendicular to NL at K, and hence $MN = MC = MK$.

Now let PQ intersect AL at T. We have $\angle TAK = \angle KNM = \angle NKM = \angle LKT$ and similarly $\angle TLK = \angle AKT$. Consequently, $2\angle KTL = 2(\angle TAK + \angle AKT) = \angle TAK + \angle AKT + \angle LKT + \angle TLK = 180^\circ$, which means $\angle KTL = 90^\circ$.

2. First we shall prove that a_n^2 is the difference of two consecutive cubes. To do so, we shall prove by induction that $a_n^2 = (b_n + 1)^3 - b_n^3$. When $n = 1$, this is true. Suppose for $n \geq 1$, this is true. We have

$$\begin{aligned}
(b_{n+1} + 1)^3 - b_{n+1}^3 &= 3b_{n+1}^2 + 3b_{n+1} + 1 \\
&= 3(4a_n + 7b_n + 3)^2 + 3(4a_n + 7b_n + 3) + 1 \\
&= 48a_n^2 + 147b_n^2 + 168a_nb_n + 84a_n + 147b_n + 37 \\
&= (7a_n + 12b_n + 6)^2 + (3b_n^2 + 3b_n + 1) - a_n^2 \\
&= a_{n+1}^2 + (b_n + 1)^3 - b_n^3 - a_n^2 = a_{n+1}^2
\end{aligned}$$

where the last equality is by induction hypothesis.

3. Lemma: Suppose p is prime and $a_1, \ldots, a_p$ are primes forming an A.P. with common difference d. If $a_1 > p$, we claim that $p \mid d$.

Proof: Since p is prime and every a_i is a prime $> p$, p does not divide a_i for any i. By the pigeonhole principle, there exist $1 \leq i < j \leq p$ so that $a_i \equiv a_j \pmod{p}$. Now $a_j - a_i = (j-i)d$, and p does not divide $j - i$. So p must divide d.

Apply the Lemma to the sequences $a_1, \ldots, a_k$ for $k = 2, 3, 5, 7, 11$ and 13. Then all such k's are factors of d. So $d > 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 > 30,000$.

4. The answer is $2n^2 - n + 1$.

Consider an n-colouring of the $2n \times k$ chessboard. A *vertical-pair* is a pair of squares in the same column that are coloured the same. In every column there are at least n vertical-pairs. Let P be the total number of vertical-pairs and P_i be the number of vertical-pairs with colour i. Then $P = P_1 + \cdots + P_n \geq nk$. Thus there is colour i with $P_i \geq k$. There are $\binom{2n}{2} = 2n^2 - n$ pairs of rows. Thus if $k \geq 2n^2 - n + 1$, there is a pair of rows that contains two vertical-pairs with colour i.

Next for $k = 2n^2 - n$, exhibit an n-colouring where no such sets of 4 squares exists. Note that it suffices to find such an n-colouring for the $2n \times (2n-1)$ board. We can then rotate the colours to obtain n of these boards which can then be put together to obtain the requiring n-colouring of the $2n \times (2n^2 - n)$ board. For each $i = 1, 2, \ldots, 2n-1$, let $A_i = \{(i, 2n-1+i), (i+1, 2n-2+i), \ldots, (n-1+i, n+i)\}$, where $2n+k$, $k > 0$, is taken to be k. Note that the pairs in each A_i give a partition of $\{1, 2, \ldots, 2n\}$. Moreover, each pair of elements appears in exactly one A_i. Now colour the squares of column i using n colours so that the two squares in each pair of A_i receive the same colour and the colours the $2n$ pairs are mutually distinct. This gives an n-colouring of the $2n \times 2n - 1$ board with the required property and we are done.

5. It is given that

$$p \mid x^3 - y^3 = (x-y)(x^2 + xy + y^2).$$

Moreover, $-p < x - y < 0$ and so $p \nmid x - y$. Thus $p \mid x^2 + xy + y^2$. Similarly, $p \mid y^2 + yz + z^2$ and $p \mid x^2 + xz + z^2$. Hence

$$p \mid x^2 + xy - yz - z^2 = (x-z)(x+y+z).$$

Since $p \nmid x - z$, $p \mid x + y + z$. Note that $0 < x + y + z < 3p$. So $x + y + z = p$ or $2p$. We will show that $p \mid x^2 + y^2 + z^2$. Assuming this for the moment, the proof is complete if $x + y + z = p$. If $x + y + z = 2p$, then $x + y + z$ is even and hence $x^2 + y^2 + z^2$ is even. So both 2 and p divide $x^2 + y^2 + z^2$. Moreover, $p > 2$ and so 2 and p are relatively prime. Thus $2p$ divides $x^2 + y^2 + z^2$ and the proof is also complete in this case.