H2 Physics

Ryan Joo Rui An

Last updated: February 7, 2024

Abstract

This book is written with the intention to provide readers with a brief summary of each topic in the Singapore GCE A-Level Physics at the H2 Level. Some sample problems are provided at the end of each topic for the reader to try out.

Contents

Part I

Measurement

§1 Measurement

§1.1 Physical quantities and SI units

Definition 1.1: Base quantity

Physical quantity that cannot be defined in terms of other quantities.

Definition 1.2: Base unit

Unit which is defined without referring to other units.

Remark. Base units are not to be confused with **SI units**, which refer to the set of standard units that are commonly used. For instance, the SI unit for frequency is Hz, but the base unit is s^{-1}.

Quantity	Unit
mass	kilogram (kg)
length	metre (m)
time	second (s)
current	ampere (A)
temperature	kelvin (K)
amount of substance	mole (mol)

Definition 1.3: Derived quantity

Physical quantity derived from base quantities, can be expressed in terms of product and/or quotient of base quantities.

Definition 1.4: Derived unit

Unit derived from base units, can be expressed in terms of products and/or quotients of base units.

List of **prefixes**:

Prefix	Symbol	Factor
tera	T	10^{12}
giga	G	10^{9}
mega	M	10^{6}
kilo	k	10^{3}
deci	d	10^{-1}
centi	c	10^{-2}
milli	m	10^{-3}
micro	μ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}

These are some reasonable estimates of physical quantities.

Quantity	Estimatation
Frequency of audible sound wave	20 Hz to 20 kHz
Wavelength of UV radiation	1×10^{-7} to 1×10^{-8} nm
Mass of 30cm plastic ruler	30 g to 50 g
Density of atmospheric air	1 kg.m^{-3}

§1.2 Dimensional analysis

A **homogeneous equation** is an equation where all quantities have the *same units*. Use SI base units to check the homogeneity of physical equations.

A physically correct equation must be homogeneous; a homogeneous equation may not be physically correct. Some reasons include:

- Value of dimensionless factor may be incorrect.
- Missing or extra terms that may have the same unit.

§1.3 Scalars and vectors

Definition 1.5: Scalar quantity

Only has magnitude but no direction.

Examples: distance, speed, energy

Definition 1.6: Vector quantity

Has both magnitude and direction.

Examples: displacement, velocity, force

§1.4 Vectors

Use of trigonometry, Sine Rule and Cosine Rule is relevant.

§1.4.1 Vector addition

Vectors can be added via

1. **Triangle method**
2. **Parallelogram method**

§1.4.2 Vector subtraction

Used to determine the *change* in a certain vector quantity.

§1.4.3 Resolving vector

Represent a vector as two perpendicular components

§1.5 Errors

Definition 1.7: Systematic error

Error where repeating the measurement under the same conditions yields all measurements bigger or smaller than true value.

Definition 1.8: Random error

Error where repeating the measurement under the same conditions yields all measurements scattered about mean value.

Systematic error	Random error
Same magnitude and sign	**Different** magnitudes and signs
Can be eliminated by careful design of experiment, good experimental techniques.	Cannot be eliminated, but can be reduced by repeating measurements and averaging readings by plotting a best fit line for data points.
Examples: poorly calibrated instrument, instrumental zero error, human reaction time, parallax error	Examples: non-uniformity of wires, instrument sensitivity, fluctuations in the testing environment (temperature, wind), irreproducible readings (repeat timing for 20 oscillations)

Definition 1.9: Accuracy

Degree of agreement between measurements and true value.

Definition 1.10: Precision

Degree of agreement among a series of measurements.

Accuracy	Precision
High accuracy is associated with small systematic error; mean value is close to true value.	High precision is associated with small random error; small scattering of readings about mean value.
Graphically, line of best fit does not pass through the origin.	Graphically, data points do not lie on a straight line, but scattered around the line of best fit.

§1.6 Uncertainties

Given a measurement R.

- **Actual uncertainty** is denoted as ΔR.
- **Fractional uncertainty** is given by $\frac{\Delta R}{R}$.
- **Percentage uncertainty** is given by $\frac{\Delta R}{R} \times 100\%$.

When there are more quantities, uncertainty increases.

Given the measurements R, A, B, and coefficients m, n.

- For addition and subtraction where $R = mA + nB$, add or subtract **actual** uncertainties:
$$\Delta R = |m|\Delta A + |n|\Delta B \tag{1}$$
- For multiplication and division where $R = A^m B^n$, add or subtract **fractional** uncertainties:
$$\frac{\Delta R}{R} = |m|\frac{\Delta A}{A} + |n|\frac{\Delta B}{B} \tag{2}$$
- Use the **First Principle** to deal with complex expressions
$$\Delta R = \frac{R_{\max} - R_{\min}}{2} \tag{3}$$
from which we can derive
$$\Delta R = R_{\max} - R = R - R_{\min}$$

§1.7 Problems

Problem 1.1. check homogeneity

Problem 1.2. State why, by drawing a line of best fit for the data points, the effect of random error is reduced.

Answer. Random errors have different signs and magnitudes in repeated measurements, causing readings to be scattered.

Since the line of best fit has on average an equal number of readings on both sides, errors that cause overestimation of experimental result will partially cancel the errors that cause underestimation, thus reducing the effect of random errors. □

Problem 1.3. Uncertainty calculation

Problem 1.4. Precision and accuracy

Part II

Newtonian Mechanics

§2 Kinematics

Definition 2.1: Displacement s

Distance moved in a specific direction.

Graphically, change in displacement is the area under a velocity-time graph.

Definition 2.2: Velocity v

Rate of change of displacement with respect to time.

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t} \tag{4}$$

Graphically, velocity is the gradient of a displacement-time graph; change in displacement is the area under a velocity-time graph.

Definition 2.3: Acceleration a

Rate of change of velocity with respect to time.

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \tag{5}$$

Graphically, acceleration is the gradient of a velocity-time graph; change in velocity is the area under an acceleration-time graph.

§2.1 Rectilinear motion

The following **equations of motion** only hold for uniformly accelerated motion in a straight line.

$$v = u + at \tag{6}$$

$$s = \frac{1}{2}(u + v)t \tag{7}$$

$$s = ut + \frac{1}{2}at^2 \tag{8}$$

$$v^2 = u^2 + 2as \tag{9}$$

See Appendix for the derivation.

§2.2 Projectile motion

Definition 2.4: Projectile motion

Motion due to

- **uniform velocity** in one direction, and
- **uniform acceleration** in a perpendicular direction.

Analyse horizontal motion in the x-direction, and vertical motion in the y-direction separately. Resolving velocity into components,

$$v_x = v\cos\theta, \qquad v_y = v\sin\theta$$

$$v = \sqrt{{v_x}^2 + {v_y}^2} \qquad \theta = \tan^{-1}\frac{v_y}{v_x}$$

§2.2.1 Horizontal motion

Horizontal motion does not undergo acceleration; hence, horizontal velocity remains constant.

$$v_x = u_x$$

$$s_x = v_x t$$

§2.2.2 Vertical motion

Vertical motion undergoes acceleration due to gravity g; hence, vertical velocity changes.

$$v_y = u_y - gt$$

$$s_y = \frac{1}{2}(u_y + v_y)t$$

$$s_y = u_y t - \frac{1}{2}gt^2$$

$${v_y}^2 = {u_y}^2 - 2gs_y$$

§2.2.3 Relevant quantities

The following quantities should be derived and not memorised.

Time of flight:

$$v_y = u_y - gt$$
$$0 = u \sin\theta - gt$$
$$t = \frac{u \sin\theta}{g}$$

$$\boxed{t_{\text{flight}} = \frac{2u \sin\theta}{g}}$$

Maximum height:

$$v_y{}^2 = u_y{}^2 - 2gs_y$$
$$0 = (u \sin\theta)^2 - 2gh$$

$$\boxed{h = \frac{u^2 \sin^2\theta}{2g}}$$

Range:

$$R = u_x t_{\text{flight}}$$
$$R = u \cos\theta \cdot \frac{2u \sin\theta}{g}$$

$$\boxed{R = \frac{u^2 \sin 2\theta}{g}}$$

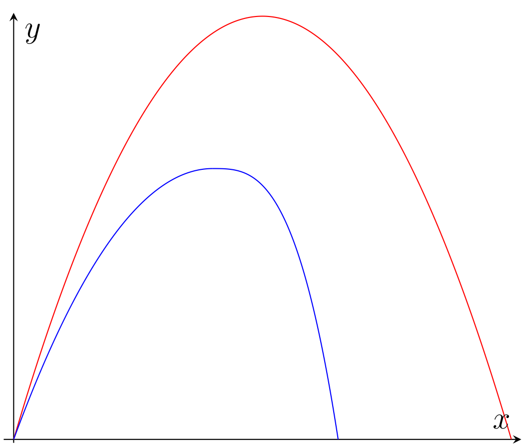
Range is maximum when $\theta = 45°$, then maximum range $R = \frac{u^2}{g}$.

§2.2.4 Effect of air resistance

Air resistance is negligible	Air resistance is not negligible
Horizontal velocity remains unchanged, since no horizontal deceleration.	Horizontal velocity: decreases at a decreasing rate, since air resistance becomes smaller with time due to diminishing horizontal velocity; horizontal velocity asymptotically approaches zero.
Vertical velocity increases from zero with constant rate downward at the acceleration of free fall, since resultant force on the stone is only its weight alone, which is constant, hence constant vertical deceleration.	Vertical velocity increases from zero at a decreasing rate, since acceleration decreases due to reduction of resultant downward force (air resistance opposing motion increases as speed increases).
	Time of flight for downward motion is longer than upward motion, because net downward acceleration (weight - air resistance) is smaller than net upward deceleration (weight + air resistance).

Characteristics of path of object with non-negligible air resistance:

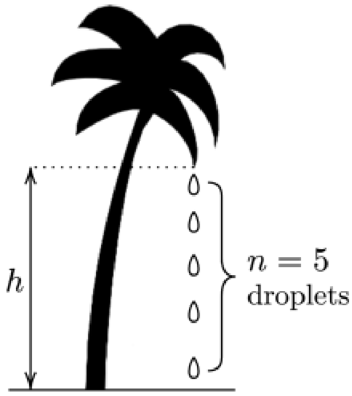
1. Lower maximum height, displaced to the left
2. Asymmetrical shape
3. Shorter range



§2.3 Problems

Problem 2.1. A tree is wet after a rain and slowly drips water, with one droplet falling from rest every $t = 1$ s. At any time, exactly $n = 5$ droplets can be observed mid-air. Determine the height h of the tree. Neglect air resistance.

Leave your answer to 2 *significant figures in units of* m.



Solution. Consider the falling motion of a single droplet. Let the time taken for the droplet to reach the ground be T. From kinematics, we have:

$$h = \frac{1}{2}gT^2$$

Throughout the duration of its fall, an additional n droplets must have fallen from the tree, such that the n-th additional droplet falls exactly when the initial droplet hits the ground. This condition is necessary to ensure that there are always n droplets falling mid-air. Hence, T must be related to t by:

$$T = nt$$

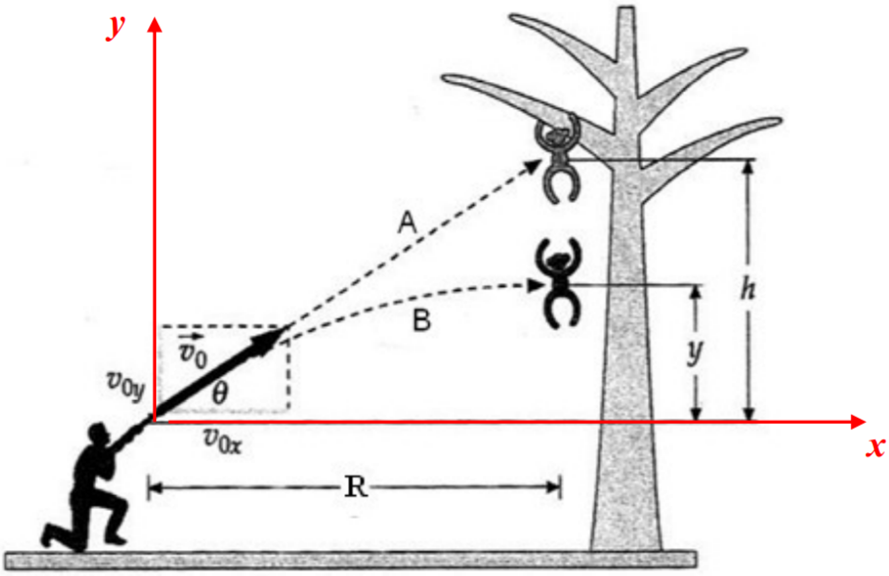
We can then solve for h:

$$h = \frac{1}{2}g(nt)^2$$

$$\boxed{h \approx 120 \text{ m}}$$

□

Problem 2.2 (The Monkey and the Hunter Problem)**.** A student fires a dart at a stuff monkey held by an electromagnet a distance h vertically above the dart gun and a distance R horizontally away from the dart gun. The student aims directly at the monkey and fires, but as the student fires, the power of the electromagnet is turned off, causing the monkey to drop simultaneously. Will the dart hit the monkey?



Solution. It takes time t the dart to travel a horizontal distance R:

$$t = \frac{R}{v_0 \cos\theta}$$

At time t,

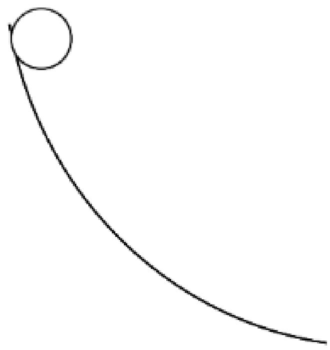
$$y_d = v_0 \sin\theta t - \frac{1}{2}gt^2, \; y_m = h - \frac{1}{2}gt^2$$

Difference in y-position at time t:

$$\begin{aligned} y_d - y_m &= \left(v_0 \sin\theta t - \frac{1}{2}gt^2\right) - \left(h - \frac{1}{2}gt^2\right) \\ &= v_0 \sin\theta t - h \\ &= v_0 \sin\theta \frac{R}{v_0 \cos\theta} - h \\ &= R\tan\theta - h \\ &= R\left(\frac{h}{R}\right) - h = 0 \end{aligned}$$

Therefore, the dart will hit the monkey. □

Problem 2.3. As the ball here rolls down the hill as shown in the figure below, describe the variation in its speed and acceleration.



Answer. Slope of the hill gets gentler as the ball rolls down, so **acceleration decreases**.

Though acceleration decreases, it is always acting downwards, so **speed increases** due to the conversion of gravitational potential energy to kinetic energy (conservation of energy). □

§3 Dynamics

When drawing a **free body diagram**:

- Draw all external forces acting only on the chosen system.
- Do not draw in the resultant force.

§3.1 Newton's Laws of Motion

Definition 3.1: Newton's 1st Law of Motion

A body at rest will remain at rest, a body in motion will remain in motion at constant velocity, in absence of external resultant force.

Definition 3.2: Newton's 2nd Law of Motion

Rate of change of momentum is directly proportional to external resultant force acting on it, and occurs in the direction of the external resultant force.

$$\sum \mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \tag{10}$$

(The constant of proportionality is found experimentally to be 1.)

Remark. For questions, you will usually use the form $\sum \mathbf{F} = \dfrac{\Delta \mathbf{p}}{\Delta t}$.

Remark. Using product rule, we have

$$\sum \mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \frac{\mathrm{d}(m\mathbf{v})}{\mathrm{d}t} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \mathbf{v}\frac{\mathrm{d}m}{\mathrm{d}t}$$

Hence $\sum F = ma$ holds only when mass is constant.

Remark. Resultant force and acceleration act in the same direction.

Definition 3.3: Newton's 3rd Law of Motion

When body A exerts a force on body B, body B exerts force of the same type, equal in magnitude, opposite in direction on body A.

$$\mathbf{F}_{AB} = -\mathbf{F}_{BA} \tag{11}$$

Definition 3.4: Inertia

Reluctance of a body to change its state of rest or uniform motion in a straight line, due to mass.

Remark. Mass is the property of a body which resists change in motion (inertia).

§3.2 Linear momentum and its conservation

Definition 3.5: Linear momentum

Product of a body's mass and velocity.

$$\mathbf{p} = m\mathbf{v} \tag{12}$$

Definition 3.6: Impulse

Product of a constant force F and the time interval t for which the constant force acts.

$$\mathbf{J} = \int \mathbf{F}\,\mathrm{d}t = \mathbf{F}_{\text{avg}}\Delta t \tag{13}$$

Graphically, impulse is the area under a force-time graph.

For collisions, the area under both graphs (representing impulse of force on each object by the other) must be the same, as linear momentum is conserved.

Definition 3.7: Impulse-Momentum Theorem

The impulse applied to a body is equal to the body's change in momentum.

$$\mathbf{J} = \Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i \tag{14}$$

Definition 3.8: Principle of Conservation of Momentum

Total momentum of a system of bodies is constant, provided no external resultant force acts on the system.

$$\sum \mathbf{F} = 0 \implies \mathbf{J} = 0 \implies \mathbf{p}_i = \mathbf{p}_f \tag{15}$$

§3.3 Collisions

§3.3.1 1D collision

Head-on collision: contact forces between the two objects act radially along a line joining their centres of mass, with no component tangential to their circumference.

Types of collisions:

1. **Elastic** collision
2. **Inelastic** collision
3. **Perfectly inelastic** collision (coalescence)
4. **Super elastic** collision

Problem solving:

- **Linear momentum** is always conserved for every type of collision (except super elastic collisions).
$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$
- **Kinetic energy** is conserved for elastic collisions, from which we can derive: relative speed of approach = relative speed of separation (r.s.a. = r.s.s.)
$$u_1 - u_2 = v_2 - v_1$$
Kinetic energy is not conserved for inelastic collisions, is lost to surroundings, hence relative speed of approach > relative speed of separation (r.s.a > r.s.s.)
$$u_1 - u_2 > v_2 - v_1$$
Kinetic energy is not conserved for super elastic collisions, is gained due to stored forms of energy (potential energy) e.g. bullet shot at a stationary hand grenade and a hand grenade explodes, where the additional energy comes from chemical energy of explosives stored in the grenade.

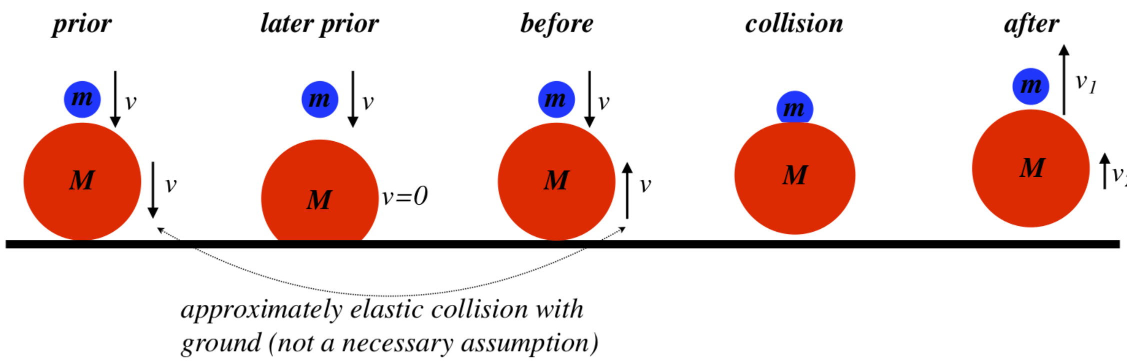
§3.3.2 2D collision

Oblique collision: If an object obliquely collides with another stationary object of equal mass, they travel off at an angle of 90° relative to one another (for elastic collision).

§3.4 Problems

Problem 3.1 (Stacked ball drop)**.** Two balls are dropped to the floor, with the lighter ball atop the heavier one. The balls collide approximately elastically with each other and with the floor. The observation is the small ball flies up to a height higher than it was dropped.

Solution.



Conservation of momentum gives us

$$Mv - mv = mv_1 + Mv_2$$

Conservation of kinetic energy gives us

$$\frac{1}{2}mv^2 + \frac{1}{2}Mv^2 = \frac{1}{2}m{v_1}^2 + \frac{1}{2}M{v_2}^2$$

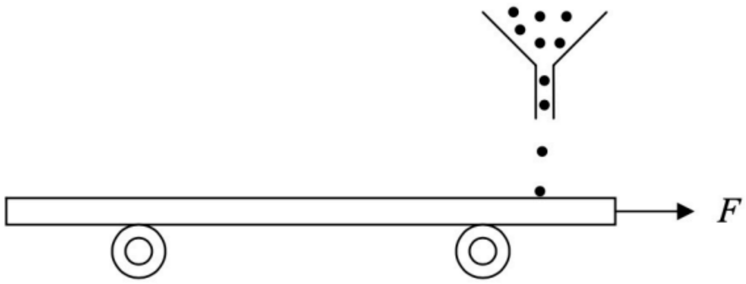
We can now solve for v_1 and v_2 in terms of v, which we can determine from the height that the balls are dropped from.

$$v_1 = \left(\frac{3M - m}{M + m}\right)v$$

$$v_2 = \left(\frac{M - 3m}{M + m}\right)v$$

We see that the small ball must rise to a height greater than that from which it was dropped, because the fraction in front of v is always greater than 1. □

Problem 3.2. A flatcar of mass m moves towards the right from rest due to a constant horizontal force F. At the same time, sand spills on the flatcar from a stationary hopper at a constant rate of μ kg.s^{-1}.



What is the dependence of the velocity v of the flatcar with respect to time t, assuming that friction is negligibly small?

Solution. Recall that force is the rate of change of momentum.

$$F = \frac{\mathrm{d}p}{\mathrm{d}t} \implies \Delta p = F \Delta t$$

The momentum of the flatcar due to from the additional mass of the sand which is μt, as well as from the increase in velocity v.

At time $t = 0$, $p = 0$. At time $t = t$, the momentum is

$$p = (m + \mu t)v = Ft$$

Hence we get

$$\boxed{v = \frac{Ft}{m + \mu t}}$$

□

§4 Forces

§4.1 Types of force

Definition 4.1: Hooke's Law

Force is directly proportional to extension of a spring, provided that the elastic limit has not been exceeded; that is, $\mathbf{F} \propto \mathbf{x}$.

$$\mathbf{F} = k\mathbf{x} \tag{16}$$

where k is the **spring constant**.

- For springs in **parallel**,
$$k_{\text{eff}} = \sum_i k_i \tag{17}$$
- For springs in **series**,
$$\frac{1}{k_{\text{eff}}} = \sum_i \frac{1}{k_i} \tag{18}$$

Elastic potential energy is stored in an object when it undergoes deformation (e.g. when a spring is extended or compressed).

$$U = \frac{1}{2}Fx = \frac{1}{2}kx^2 \tag{19}$$

Graphically, elastic potential energy is the area under a force-extension graph.

$$W = \int F \, \mathrm{d}x$$

§4.2 Upthrust

Pressure P of a liquid column is given by

$$P = \rho g h \tag{20}$$

See Appendix for the derivation.

Definition 4.2: Upthrust U

Vertical upward force exerted by the surrounding fluid when a body is submerged, fully or partially, in a fluid.

Remark. Origin of upthrust: Upthrust is the resultant force due to the difference in pressure exerted by fluid at the top and bottom surfaces of the body.

Definition 4.3: Archimedes' Principle

Upthrust is equal in magnitude, opposite in direction to the weight of fluid displaced by the body.

$$\mathbf{U} = \mathbf{W}_{\text{displaced}} = \rho_{\text{fluid}} V_{\text{displaced}}\, g \tag{21}$$

See Appendix for the derivation.

For an object floating in equilibrium, upthrust is equal in magnitude, opposite in direction to weight of the object.

$$\mathbf{U} = \mathbf{W}_{\text{object}} \tag{22}$$

§4.3 Centre of gravity

Definition 4.4: Centre of gravity

A single point where the entire weight of the object may be taken as acting at.

§4.4 Turning effects of forces

Definition 4.5: Moment of a force

Product of magnitude of the force and perpendicular distance of the *line of action* of the force from the pivot point.

$$M = F \times \perp d \tag{23}$$

Definition 4.6: Couple

A pair of forces of equal magnitude but acting in opposite directions whose lines of action are parallel but separate.

Remark. A couple is a pair of forces which tends to produce *rotation* only.

Definition 4.7: Torque of a couple

Product of one of the forces and the perpendicular distance between the forces.

$$\tau = F \times \perp d \tag{24}$$

Definition 4.8: Principle of Moments

When a system is in equilibrium, sum of clockwise moments about any axis must be equal to sum of anticlockwise moments about the same axis.

$$\sum \text{clockwise moments} = \sum \text{anticlockwise moments} \tag{25}$$

§4.5 Equilibrium of forces

A system is in equilibrium when there is

- no resultant force, and
- no resultant torque

Definition 4.9: Translational equilibrium

Net force is zero in any direction.

$$\sum \mathbf{F} = \mathbf{0}$$

Definition 4.10: Rotational equilibrium

Net torque is zero about any axis of rotation.

$$\sum \tau = \mathbf{0}$$

For multiple non-parallel forces, this is illustrated by:

- Forces form a **closed vector polygon**.
- Lines of actions of forces **intersect at one point**, so that there is no resultant moment about their point of intersection.

§4.6 Problems

Problem 4.1. Explain why the upthrust acting on a human body when in air is normally ignored.

Answer. The average person weighs about 600 N. Upthrust by air is about 1 N, less than 0.2% of the weight of the person. □

Problem 4.2. Why can a lump of plasticine moulded into the shape of a bowl float in water?

Answer. Bowl is able to displace a greater volume of water.

If the plasticine floats,

$$W = U$$

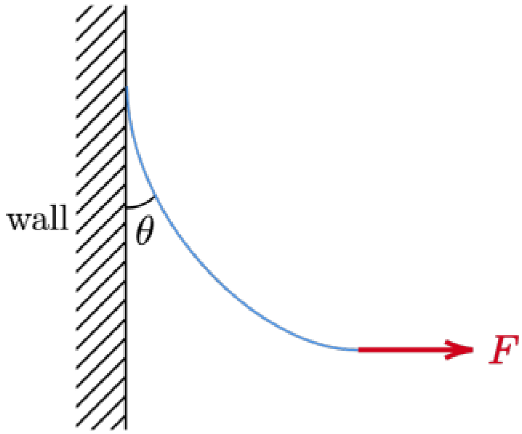
$$\rho_{\text{plasticine}} g V_{\text{plasticine}} = p_{\text{water}} g V_{\text{water displaced}}$$

Since $p_{\text{plasticine}} > \rho_{\text{water}}$, in order for the plasticine to float, $V_{\text{water displaced}} > V_{\text{plasticine}}$.

Hence plasticine must be able to displace a larger volume of water than its own volume. □

Problem 4.3. A rope is connected to a vertical wall at one end, and a horizontal external force $F = 15.0$ N pulls on the other end. The rope is in equilibrium and makes an angle $\theta = 25.0°$ with the wall. What is the weight W of the rope?

Leave your answer to 3 *significant figures in units of* N.



Solution. Since the rope makes an angle θ with the wall, and the tension in the rope acts along the rope, we can write the following force balance equations for the horizontal and vertical axes respectively:

$$T \sin\theta = F$$

$$T \cos\theta = W$$

Hence solving for W,

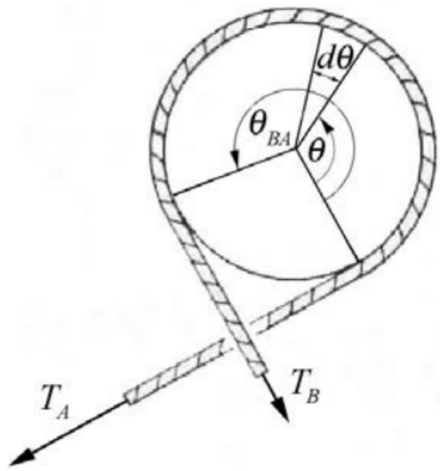
$$W = \frac{F\cos\theta}{\sin\theta}$$

$$W = F\cot\theta$$

$$\boxed{W \approx 32.2 \text{ N}}$$

□

Problem 4.4 (Sailor capstan). A capstan is a device used aboard ships in order to control a rope that is under great tension. The rope is wrapped around a fixed drum of radius R, usually for several turns. The load on the rope pulls it with a force T_A, and the sailor holds the other end of the rope with a much smaller force T_B. The coefficient of static friction between the rope and the drum is μ_s. The sailor is holding the rope so that it is just about to slip.

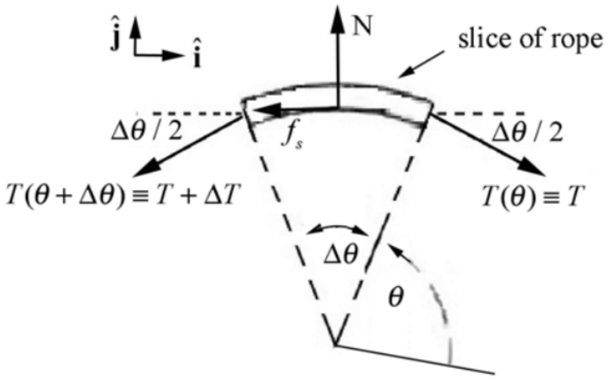


Show that

$$T_B = T_A e^{-\mu_s \theta_{BA}}$$

where θ_{BA} is the angle subtended by the rope on the drum.

Solution. Analysing forces on a small slice of rope of arc length $R\Delta\theta$:



In the horizontal direction,

$$T \cos\frac{\Delta\theta}{2} - (T + \Delta T)\cos\frac{\Delta\theta}{2} - f_s = 0$$

In the vertical direction,

$$N - T\sin\frac{\Delta\theta}{2} - (T + \Delta T)\sin\frac{\Delta\theta}{2} = 0$$

Solving the two equations simultaneously gives us

$$\frac{\Delta T}{\Delta \theta} = \mu_s T$$

As $\Delta\theta \to 0$,

$$\frac{\mathrm{d}T}{\mathrm{d}\theta} = -\mu_s T$$

Solving the differential equation,

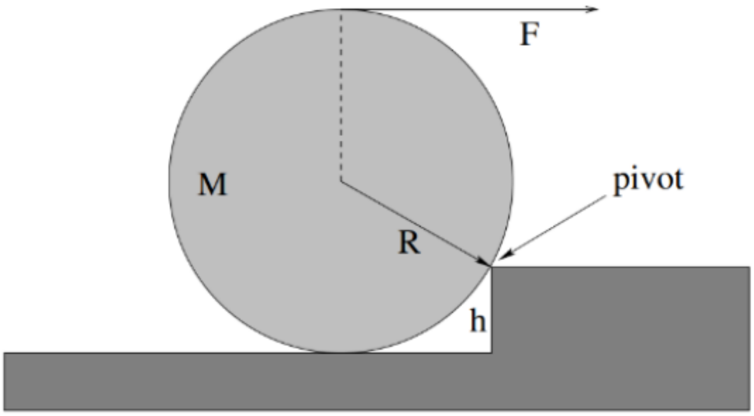
$$\begin{aligned}\int_{T_A}^{T_B} \frac{1}{T}\,\mathrm{d}T &= -\mu_s \int_{\theta_A}^{\theta_B} \mathrm{d}\theta \\ \ln\frac{T_B}{T_A} &= -\mu_s\left(\theta_B - \theta_A\right) \\ &\boxed{T_B = T_A e^{-\mu_s \theta_{BA}}}\end{aligned}$$

The exponential dependence suggests that the coefficient $e^{-\mu_s \theta_{BA}}$ becomes very small when θ_{BA} increases. Note that for n turns of the rope, $\theta_{BA} = 2\pi n$ rad. □

Problem 4.5 (Yoyo problem)**.**

Solution. □

Problem 4.6. What is the minimum force F that must be applied to cause the cylinder to barely lift up off of the bottom step and rotate up around the corner of the next one? Assume that the cylinder does not slip on the corner of the next step.



Solution. Computing the torques of F and Mg gives us

$$\boxed{F = \frac{Mg\sqrt{R^2 - (R-h)^2}}{2R-h}}$$

□

§5 Work, Energy and Power

§5.1 Work

Definition 5.1: Work done (by constant force)

Product of the force and displacement <u>in the direction of the force</u>.

$$W = Fs\cos\theta \tag{26}$$

In vector form, this is given by

$$W = \mathbf{F}\cdot\mathbf{s}$$

- Work done by a **variable force** is given by

$$W = \int \mathbf{F}\,\mathrm{d}\mathbf{s} \tag{27}$$

  Graphically, work done is area under force-displacement graph.

- Work done to deform (stretch/compress) a material is stored as elastic potential energy in the material.

$$U = \frac{1}{2}Fx = \frac{1}{2}kx^2 \tag{28}$$

- Work done by a **gas** which is expanding against a constant external pressure:

$$W = p\Delta V \tag{29}$$

  Graphically, work done is the area under a pressure-volume graph.

$$W = \int p\,\mathrm{d}V$$

§5.2 Energy conversion and conservation

Definition 5.2: Principle of Conservation of Energy

Energy can neither be created nor destroyed, but can be transformed from one form to another, and transferred from one body to another. Total energy in a closed system is always constant.

$$(E_k + E_p)_i + W = (E_k + E_p)_f \tag{30}$$

Remark. Work done by dissipative forces is *negative* as the forces act in opposite direction to displacement.

Gravitational potential energy is energy stored due to height raised.

$$\text{GPE} = mgh \tag{31}$$

See Appendix for the derivation.

Kinetic energy is energy possessed by an object due to motion.

$$\text{KE} = \frac{1}{2}mv^2 = \frac{p^2}{2m} \tag{32}$$

See Appendix for the derivation.

Elastic potential energy is energy stored in an object when it is deformed.

$$\text{EPE} = \frac{1}{2}Fx = \frac{1}{2}kx^2 \tag{33}$$

Definition 5.3: Work-Energy Theorem

Net work done by a force on a body is equal to the change in kinetic energy of the body.

$$W = \Delta\text{KE} \tag{34}$$

The relationship between conservative force F and potential energy U is

$$\mathbf{F} = -\frac{\mathrm{d}U}{\mathrm{d}\mathbf{x}} \tag{35}$$

Remark. A conservative force is one where work done by the force is independent of its path.

§5.3 Power

Definition 5.4: Power

Rate at which work is done; rate at which energy is transferred.

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} \tag{36}$$

Instantaneous power P when a constant force F acts on an object with velocity v is given by

$$P = Fv \tag{37}$$

Remark. This means power is the product of a force and velocity in the direction of the force.

Average power P_{avg} when a constant force F acts on an object with average velocity v_{avg} is given by

$$P_{\text{avg}} = Fv_{\text{avg}} \tag{38}$$

§5.4 Efficiency

Efficiency η is given by

$$\eta = \frac{\text{useful power/energy output}}{\text{total power/energy input}} \times 100\% \tag{39}$$

Problems

Problem 5.1. A hydroelectric dam has a water height of 50 m (as measured from the bottom of the dam where water is let out). What is the rate at which water is let out to produce 50 MW of electrical power? You should assume an energy conversion efficiency of the dam (from mechanical to electrical) to be 30%, and the density of water as $997\,\mathrm{kg\,m^{-3}}$.

Assume that the dam is large enough so that the water height does not substantially change during power generation.

Solution. By conservation of energy, the kinetic energy of water that leaves the bottom of the dam is equal to gravitational potential energy at the top of the dam.

Power generated is (efficiency) × (rate of flow) × (density) × g × (50 metres). Equating this to 50 MW produces a rate of flow of $341\ \mathrm{m^3\,s^{-1}}$. □

§6 Circular motion

§6.1 Uniform circular motion

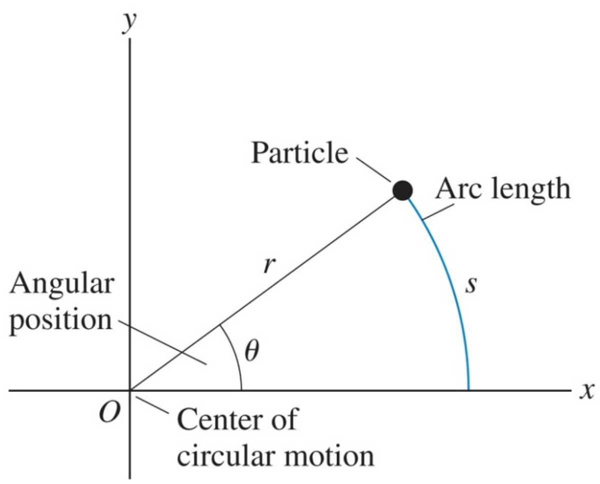
Uniform circular motion is a type of motion in which an object moves at a *constant speed* in a circular path.

Definition 6.1: Radian

Unit of angular measure, defined as the angle subtended at the centre of a circle by an arc of a length equal to the radius of the circle.

Angular displacement θ refers to the angle in radians through which a point is rotated.

$$\theta = \frac{s}{r} \tag{40}$$



Angular velocity ω refers to rate of change of angular displacement.

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{41}$$

Relating angular velocity to period T and frequency f,

$$\omega = \frac{2\pi}{T} = 2\pi f \tag{42}$$

Linear velocity v is given by

$$v = r\omega \tag{43}$$

Centripetal acceleration a is given by

$$a = \frac{v^2}{r} = r\omega^2 \tag{44}$$

Centripetal force F_c is given by

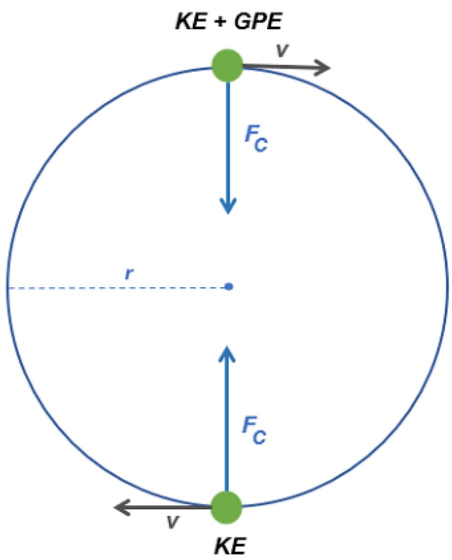
$$F_c = \frac{mv^2}{r} = mr\omega^2 \tag{45}$$

Remark. Centripetal force is a **resultant force**; it can be provided by gravitational force, friction force, normal force, etc.

Why does a resultant force exist in a uniform circular motion?
Velocity changes due to change in direction, hence the object undergoes acceleration. By Newton's 2nd Law, a resultant force acts on the object.
Since the force does not change the speed of the object, it does no work to accelerate the object, thus centripetal force acts perpendicularly to motion, towards the centre of the circle.

§6.2 Non-uniform circular motion

Consider an object rotating vertically in a circle of radius r.



Tension in the spring reaches maximum at the bottom and minimum at the top.

At the top, if the string is just taut, mg provides centripetal force completely, $T = 0$ N.

$$mg = \frac{m{v_{\text{top}}}^2}{r}$$
$$v_{\text{top}} = \sqrt{gr}$$

At the bottom, by conservation of mechanical energy,

$$\frac{1}{2}m{v_{\text{bottom}}}^2 = mg(2r) + \frac{1}{2}m{v_{\text{top}}}^2$$
$$v_{\text{bottom}} = \sqrt{5gr}$$

Problems

2014 P1 Q11, Q12, 2015 P1 Q10, 2016 P1 Q13, 2019 P1 Q10, 2013 P2 Q3 (pg 7(7))

§7 Gravitational Field

§7.1 Gravitational force

Definition 7.1: Newton's Law of Gravitation

Gravitational force of attraction between two point masses is directly proportional to the product of their masses and inversely proportional to the square of separation between their centres.

$$F_g = -\frac{GMm}{r^2}\hat{r} \tag{46}$$

where gravitational constant $G = 6.67 \times 10^{-11}\ \mathrm{kg^{-1}\,m^3\,s^{-2}}$.
In vector form this is given by

$$\mathbf{F}_g = -\frac{GMm}{|\mathbf{r}|^2}\frac{\mathbf{r}}{|\mathbf{r}|}$$

- The sign is negative due to the attractive nature of gravitational force. (The negative sign is ignored when only the magnitude of the force is required.)
- The gravitational forces between two masses are an action-reaction pair; they are equal in magnitude, opposite in direction, and act along the line joining the two point masses.
- **Point masses** have their masses concentrated at one point. Two objects can be considered point masses when they are placed *sufficiently far apart* such that their *dimensions* become negligible compared to the *distance* separating them.

§7.2 Gravitational field

Definition 7.2: Gravitational field

Region of space where a mass experiences gravitational force.

- A field of force is a region of space where they may be a *non-contact* force acting on an object placed in that field due to interaction between the field's property and the object's property.
- Field lines are used to indicate the direction of a field of force. Density of field lines corresponds to strength of field. (Field lines never touch or cross.)
- Gravitational field around Earth is non-uniform (field strength is stronger near Earth, weaker further away from Earth). Field lines are drawn radially pointing towards the centre of Earth.

- Gravitational field near Earth's surface is uniform (field strength is the same at all points). Field lines are drawn parallel to each other and of equal spacing.

Definition 7.3: Gravitational field strength g

Gravitational force per unit mass exerted on a *small test mass* placed at that point.

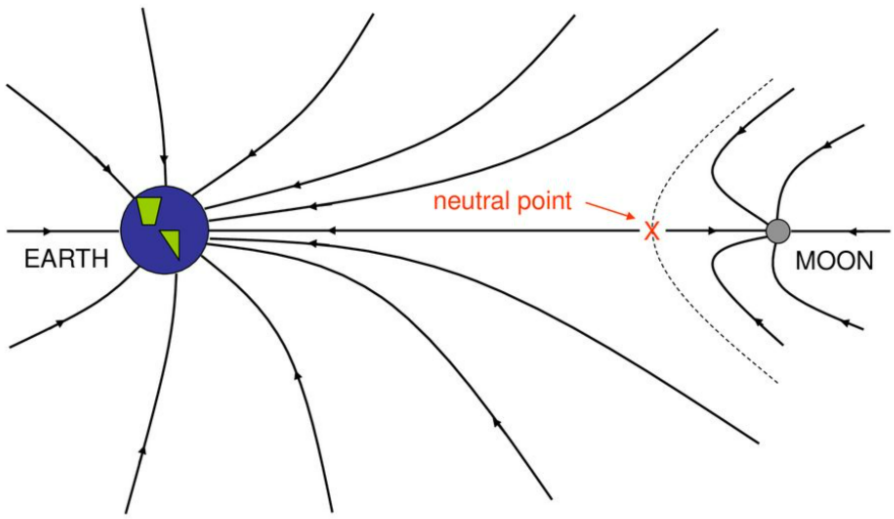
$$\vec{\mathbf{g}} = \frac{\vec{\mathbf{F}}}{m} = -\frac{GM}{r^2}\hat{r} \tag{47}$$

This means that the gravitational field is a inverse square field.

See Appendix for the derivation.

This expression refers to the gravitational field created by mass M in its surrounding region of space, where a gravitational force acts on mass m.

- The sign is negative due to the attractive nature of gravitational force acting on a mass in the field. (The negative sign is ignored when only the magnitude is required.)
- A **neutral point** refers to the point at which the resultant gravitational field due to surrounding masses is zero.



Near the surface of Earth, g can be approximated to have a constant value of 9.81 N.kg^{-1}, equal to the acceleration of free fall.

§7.3 Gravitational potential energy

Definition 7.4: Gravitational potential energy U

Work done by an external force in bringing a small test mass from infinity to that point.

$$U = -\frac{GMm}{r} \tag{48}$$

- Note that the negative sign cannot be omitted.
- Maximum U is defined to be 0 at $r = \infty$, hence U is negative.
- Work done is negative as force and displacement act in opposite directions, hence U is negative.

Definition 7.5: Gravitational potential ϕ

Work done per unit mass by an external force in bringing a small test mass from infinity to that point.

$$\phi = \frac{U}{m} = -\frac{GM}{r} \tag{49}$$

- Note that the negative sign cannot be omitted.
- For the same reasons as above, ϕ is negative.

§7.4 Relationship summary

		At a point distant r from M	For an object (m)	
Vectors	$\propto \frac{1}{r^2}$	Gravitational Field Strength $\vec{g} = -\frac{GM}{r^2}\hat{r}$	Gravitational Force $\vec{F} = -\frac{GMm}{r^2}\hat{r}$	$\vec{F} = m\vec{g}$
Scalars	$\propto \frac{1}{r}$	Gravitational Potential $\phi = -\frac{GM}{r}$	Gravitational Potential Energy $U = -\frac{GMm}{r}$	$U = m\phi$
		$\vec{g} = -\frac{d\phi}{dr}\hat{r}$	$\vec{F} = -\frac{dU}{dr}\hat{r}$	

Relationship between $\vec{g}$ and ϕ

When the equation $\vec{F} = -\frac{dU}{dr}\hat{r}$ is divided by m, i.e.

$$\vec{F} = -\frac{dU}{dr}\hat{r}$$

$$\rightarrow \quad m\vec{g} = -\frac{d(m\phi)}{dr}\hat{r}$$

$$\rightarrow \quad \vec{g} = -\frac{d\phi}{dr}\hat{r}$$

Relationship between $\vec{F}$ and U

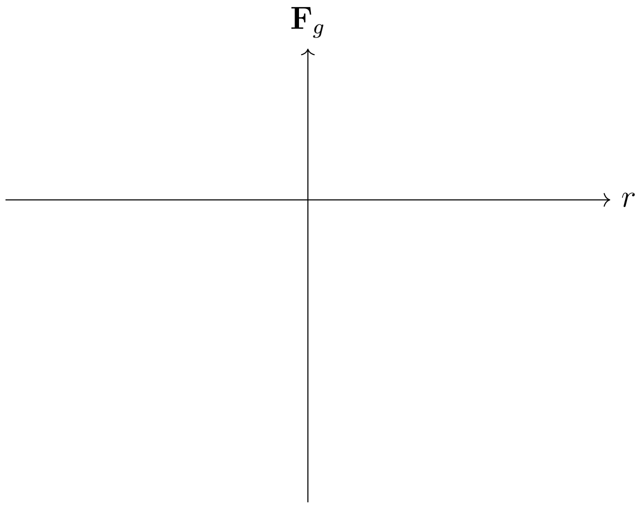
Recall that in any field of force, when an object in that field is moved through a small distance dr, its potential energy would be changed by dU. The force (due to the field) acting on the object placed at that region, $\vec{F}$, would be expressed as $\vec{F} = -\frac{dU}{dr}\hat{r}$.

Note:
Both sides of the equation are **vectors**, and the negative sign indicates that they are in opposite directions, i.e., the direction of gravitational force $\vec{F}$ acts in the direction of **decreasing** U.

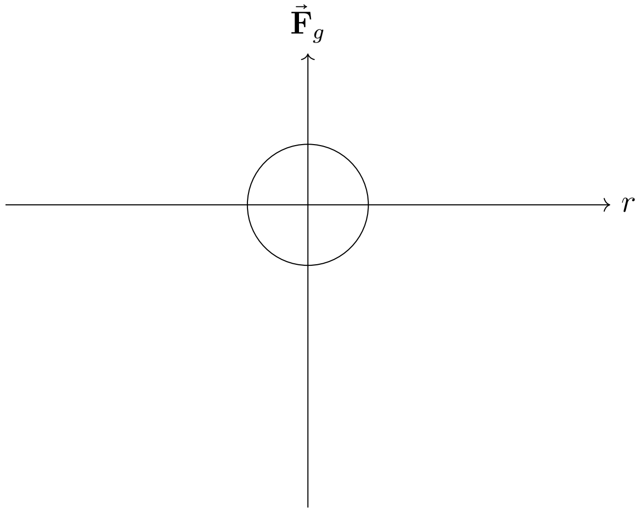
§7.5 Graphs

§7.5.1 Force-distance

Force-distance graph (point mass):

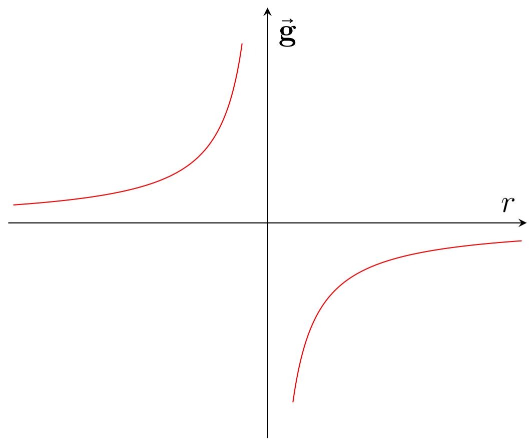


Force-distance graph (planet):

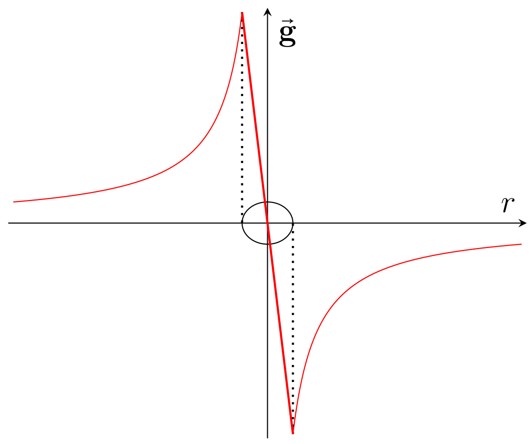


§7.5.2 Field-distance

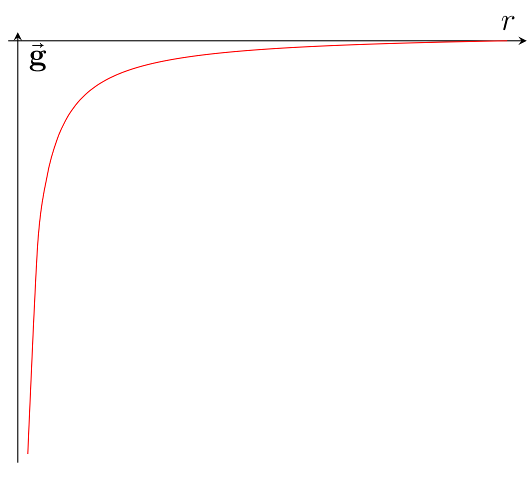
Field-distance graph (point mass)



Field-distance graph (planet)

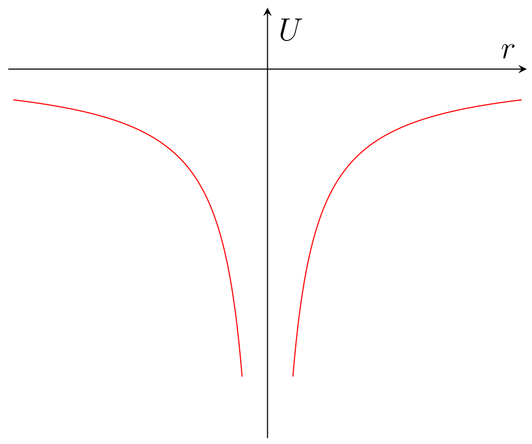


Field-distance graph between two masses

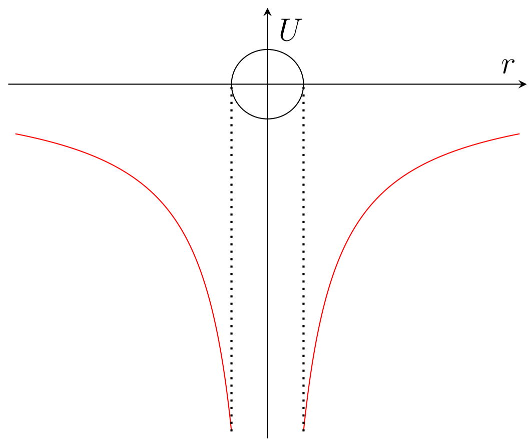


§7.5.3 Energy-distance

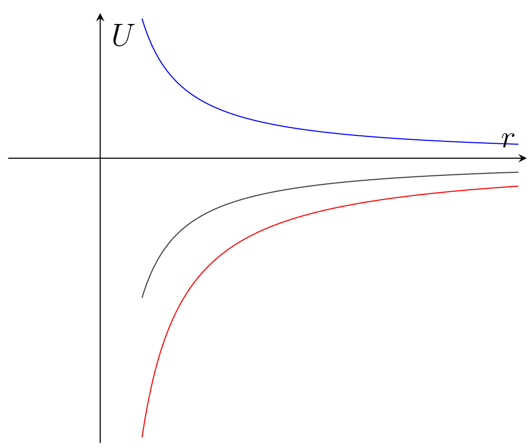
Energy-distance graph (point mass)



Energy-distance graph (planet)

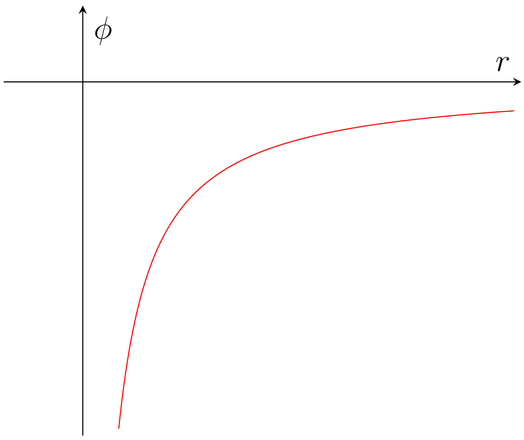


Energy-distance of satellite (GPE, KE, TE)

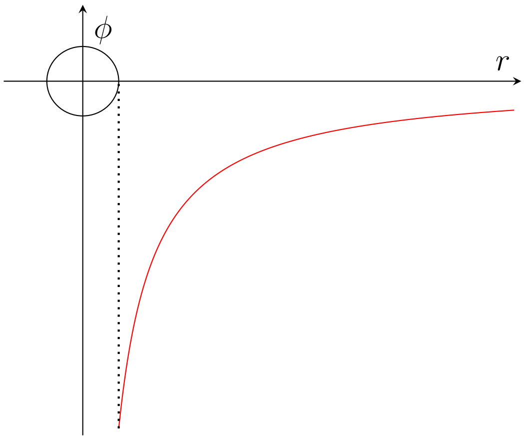


§7.5.4 Potential-distance

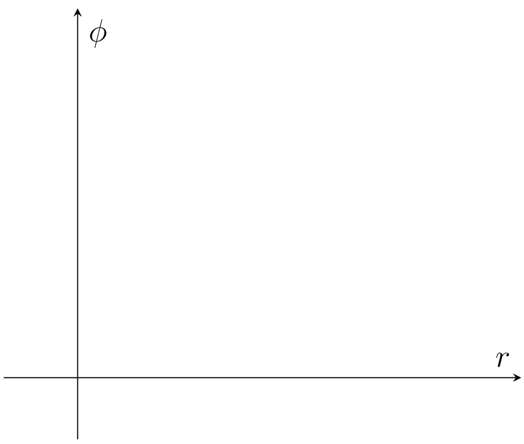
Potential-distance graph (point mass)



Potential-distance graph (planet)



Potential-distance graph between two masses



§7.6 Applications

§7.6.1 Orbit velocity

For a satellite of mass m orbiting a planet of mass M at a certain orbital velocity, **gravitational force provides centripetal force**.

$$F_g = F_c$$
$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

$$\boxed{v = \sqrt{\frac{GM}{r}}}$$

§7.6.2 Kinetic energy

For a satellite in orbit, **gravitational force provides centripetal force**.

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \implies mv^2 = \frac{GMm}{r} \implies \frac{1}{2}mv^2 = \frac{GMm}{2r}$$

$$\boxed{\mathrm{KE} = \frac{GMm}{2r}}$$

§7.6.3 Kepler's Third Law

Kepler's Third Law states that the ratio of the square of a body's orbital period to the cube of the axis of orbit is the same for all objects orbiting the same primary.

$$T^2 \propto r^3 \tag{50}$$

Derivation. **Gravitational force provides centripetal force.**

$$\frac{GMm}{r^2} = mr\omega^2 = mr\left(\frac{2\pi}{T}\right)^2$$

Making period T the subject,

$$T^2 = \frac{4\pi^2}{GM}r^3 \implies \boxed{T^2 \propto r^3}$$

□

§7.6.4 Escape speed

Escape speed: <u>minimum</u> speed required to escape the effect of a gravitational field.

By conservation of energy,

$$\text{KE}_i + U_i = \text{KE}_f + U_f$$

At infinity, $U_f = 0$ (by definition of gravitational potential energy) and $\text{KE} = 0$ (by definition of escape speed).

$$\frac{1}{2}mv^2 + \left(-\frac{GMm}{r}\right) = 0$$

Making v the subject,

$$\boxed{v = \sqrt{\frac{2GM}{r}}}$$

§7.6.5 Geostationary satellite

Definition 7.6: Geostationary satellite

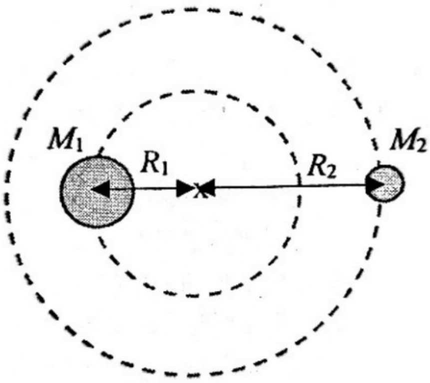
A satellite that appears stationary when observed from a fixed location from Earth.

Characteristics:

1. Orbital period is the same as the rotational period of Earth about its axis, i.e. $T = 24$ hr.
2. Moves in the same direction as the rotation of Earth about its own axis, i.e. from west to east.
3. Vertically above the equator, so that its axis of rotation is the same as the Earth.
    - Gravitational force by Earth is the resultant force that provides centripetal force for the satellite.
    - Gravitational force is directed towards centre of Earth, centripetal force is directed towards centre of orbit,
    - so centre of orbit must be centre of Earth.

§7.6.6 Binary star system

In a binary star system, two stars rotate about their *common* centre of mass.



For mass M_1, gravitational force provides centripetal force for orbit.

$$\begin{aligned} F_g &= F_c \\ \frac{GM_1M_2}{(R_1+R_2)^2} &= M_1R_1\omega^2 \\ \frac{GM_2}{(R_1+R_2)^2} &= R_1\omega^2 \end{aligned}$$

For mass M_2, gravitational force provides centripetal force for orbit.

$$\begin{aligned} F_g &= F_c \\ \frac{GM_1M_2}{(R_1+R_2)^2} &= M_2R_2\omega^2 \\ \frac{GM_1}{(R_1+R_2)^2} &= R_2\omega^2 \end{aligned}$$

Adding the two equations gives us the period of rotation:

$$\frac{G(M_1+M_2)}{(R_1+R_2)^2} = (R_1+R_2)\left(\frac{2\pi}{T}\right)^2$$

$$\boxed{T = \sqrt{\frac{4\pi^2(R_1+R_2)^3}{G(M_1+M_2)}}}$$

§7.7 Problems

Problem 7.1. Explain, in words, why there is a neutral point between two planets.

Answer. □

Problem 7.2. At a point on the surface of a uniform sphere of diameter d, the gravitational field due to the sphere is X. What would be the corresponding value on the surface of a uniform sphere of the same density but of diameter $2d$?

Solution.

$$g = \frac{GM}{r^2} = \frac{G\rho\left(\frac{4}{3}\pi r^3\right)}{r^2} = \frac{4}{3}G\rho\pi r = \frac{2}{3}G\rho\pi d \implies g \propto d$$

$$\frac{g_2}{g_1} = \frac{d_2}{d_1} \implies g_2 = \frac{2d}{d}X = 2X$$

□

Problem 7.3. Assume that the Earth is a point mass of 6.0×10^{24} kg and the Moon is a point mass of 7.4×10^{22} kg. The distance between them is 3.8×10^5 km.

Determine the position of a point from Earth where the gravitational field strength due to Earth and Moon is zero.

Solution. Let x be the distance from Earth to the point where the resultant gravitational field strength is zero.

At that point, gravitational field strength due to Earth g_E (directed towards Earth) is *balanced* by gravitational field strength due to Moon g_M (directed towards Moon).

$$\begin{aligned}
g_E &= g_M \\
\frac{GM_E}{x^2} &= \frac{GM_M}{\left(3.8 \times 10^8 - x\right)^2} \\
\frac{6.0 \times 10^{24}}{x^2} &= \frac{7.4 \times 10^{22}}{\left(3.8 \times 10^8 - x\right)^2} \\
\left(\frac{3.8 \times 10^8 - x}{x}\right)^2 &= \frac{7.4 \times 10^{22}}{6.0 \times 10^{24}} \\
\frac{3.8 \times 10^8}{x} - 1 &= \sqrt{\frac{7.4 \times 10^{22}}{6.0 \times 10^{24}}}
\end{aligned}$$

Solving this gives us $x = 3.4 \times 10^8$ m. □

Remark. Remember this way to solve similar questions (do not solve quadratically).

- Move x to one side.
- Take square root to reduce it to a linear equation.

Problem 7.4 (2014 P1 Q13)**.** X and Y are two stars of equal mass. The points P and Q are equidistant from X and Y.

Which graph best shows the variation in magnitude of the total gravitational field strength g due to the stars when moving from P to Q?

Answer. □

Problem 7.5 (2015 P1 Q12)**.** A meteorite of mass m initially has zero velocity relative to a planet. The meteorite falls from a large distance to the planet of mass M and radius R. The planet has no atmosphere.

The graph shows the potential ϕ of the meteorite in the gravitational field at a distance r from the centre of the planet.

Which expression is equal to the maximum kinetic energy of the meteorite as it hits the surface?

Answer. □

Problem 7.6 (2018 P1 Q11)**.**

Problem 7.7 (2015 P2 Q4)**.** The planet Jupiter has many moons. Explain why the gravitational field strength at the position of each moon has the same magnitude and direction as the centripetal acceleration of the moon.

Answer.
The attractive gravitational force exerted by Jupiter on each moon provides the centripetal force required to sustain the circular motion about the centre of Jupiter.

The gravitational field strength at each moon's respective position, which is its gravitational force per unit mass, acts towards the centre of Jupiter.

Therefore, each moon's centripetal acceleration, which is its centripetal force per unit mass, must have the same magnitude as its gravitational force per unit mass as well as direction also towards the centre of Jupiter. □

Problem 7.8 (2017 P2 Q2)**.** Charon is one of the moons of Pluto. When viewed from above, Pluto and Charon rotate in the same direction about their axes.

A space probe on the surface of Pluto is able to observe Charon over a time of several days. Suggest what the space probe observes as a result of

(i) the period of rotation of Pluto about its axis equalling the orbital period of Charon,

(ii) equal periods of rotation about their axes for both Pluto and Charon.

Answer.

(i) The space probe observes Charon continuously if Charon orbits in the same direction as Pluto, and periodically if they orbit in opposite directions.

(ii) The space probe observes the same view of Charon's surface continuously as Charon's synchronous orbit about Pluto causes its near side to face the space probe permanently.

$\square$

Part III

Thermal Physics

§8 Temperature and Ideal Gases

§8.1 Thermal Equilibrium

Temperature: a measure of degree of hotness of an object. Thermal energy moves from object at higher temperature to object at lower temperature.

Remark. Temperature does not measure amount of thermal energy; it only indicates direction of heat flow.

Heat: (thermal) energy that flows from region of higher to lower temperature[1].

Definition 8.1: Thermal equilibrium

When two objects in thermal contact are in thermal equilibrium, there is no *net* heat transfer between them. They are at the same temperature.

Definition 8.2: Zeroth law of Thermodynamics

If objects A and B are separately in thermal equilibrium with a third object C, then A and B are also in thermal equilibrium with each other.

Remark. Object C can function as a thermometer to determine if temperatures of two objects are the same. This allows us to determine whether temperatures of two objects are same, without both objects being in contact.

[1] through conduction, convection and radiation.

§8.2 Temperature Scales

§8.2.1 Empirical Scale

Empirical temperature scale: scale of temperature based on variation of some physical property with temperature.

1. Choose appropriate thermometric property

   Thermometric property varies **linearly** with temperature. Some examples include:

   - volume of fixed mass of liquid (liquid-in-glass thermometer)
   - resistance of metal (platinum resistance thermometer)
   - pressure of fixed mass of gas at constant volume (constant volume gas thermometer)
   - e.m.f. produced between junctions of dissimilar metals at different temperatures (thermocouple thermometer)

2. Select two fixed points

   Usually **ice point** of water (0 °C) and **steam point** of water (100 °C).

3. Calibrate thermometer

   Place it in systems of lower and upper fixed points. Record values of thermometric quantity. Assume linear relationship between these two points.

Remark. The thermometric quantity must have a **unique** value at every temperature, i.e. must show a one-one graph.

If the value of the thermometric property is X_θ at temperature θ, then

$$\frac{\theta}{100} = \frac{X_\theta - X_i}{X_s - X_i} \tag{51}$$

where X_i is the value of X at ice point, X_s is the value of X at steam point.

We can generalise this to give us the following ratio:

$$\frac{\theta_2 - \theta_1}{\theta_3 - \theta_1} = \frac{X_2 - X_1}{X_3 - X_1} \tag{52}$$

Remark. This equation should not be memorised, as problems usually do not simply provide temperatures at ice point and steam point; instead, use ratios of temperatures and the given thermometric property.

Remark. The assumption of linearity of the thermometric properties may be wrong or inaccurate; instead, the actual behaviour of the thermometric property is non-linear. Hence empirical scales are always slightly wrong, except at the fixed points.

§8.2.2 Thermodynamic scale

Absolute zero: temperature at which all substances have minimum internal energy.

Thermodynamic temperature scale: does not depend on thermometric property of any particular substance, has fixed points at absolute zero and triple point of water[2].

To convert temperatures measured in °C to K,

$$T\,(\mathrm{K}) = T\,(^\circ\mathrm{C}) + 273.15$$

[2] The particular temperature and pressure (273.16 K, 4.58 mmHg) at which the three states of water (solid, liquid, vapour) can co-exist at equilibrium, i.e. transition state curves meet.

§8.3 Equation of State

For a fixed amount of gas, the following relationships can be deduced experimentally.

- **Charles' Law**: $V \propto T$ at constant p
- **Boyle's Law**: $p \propto \frac{1}{V}$ for constant T
- **Gay–Lussac's Law**: $p \propto T$ for constant V

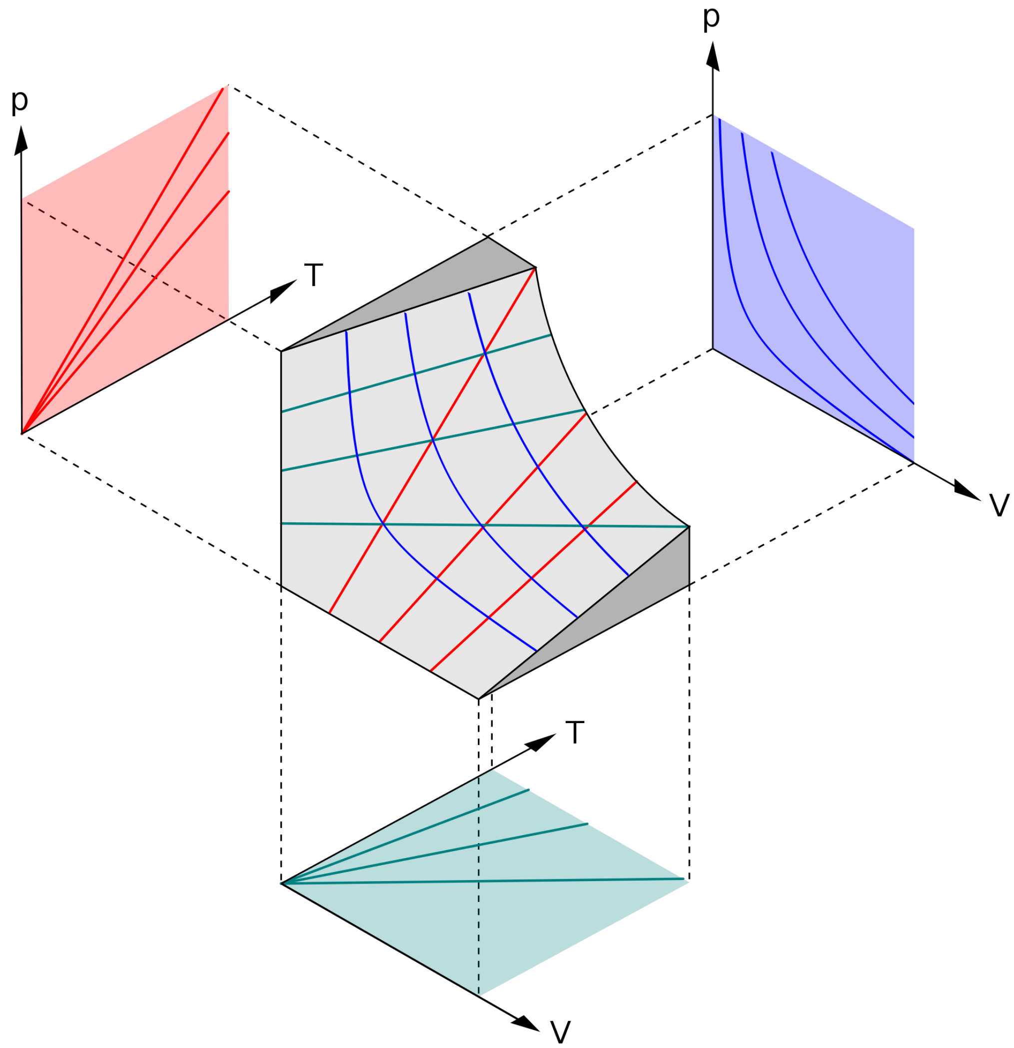
Combining the above give us the **equation of state** of an ideal gas, in *moles*:

$$pV = nRT \tag{53}$$

where **molar gass constant** $R = 8.31\ \mathrm{J\,mol^{-1}\,K^{-1}}$.

Definition 8.3: Ideal gas

A hypothetical gas that obeys the equation of state $pV = nRT$ perfectly for all pressure p, volume V, amount of substance n, and temperature T.



Mole n: amount of substance that contains the same number of particles as the number of atoms in 0.012 kg (or 12 g) of carbon-12.

Avogardo constant N_A: number of particles in one mole of substance.

$$N_A = \frac{N}{n} = 6.02 \times 10^{23} \text{ mol}^{-1}$$

Boltzmann's constant: gas constant per molecule.

$$k_B = \frac{R}{N_A} = 1.38 \times 10^{-23} \text{ J.K}^{-1}$$

Rewriting the equation of state in *number of molecules* gives us

$$pV = Nk_BT \tag{54}$$

Molar mass of a substance Mr: mass of one mole of substance.

$$Mr = \frac{m}{n}$$

Molar volume of a gas V_m: volume of one mole of gas.

$$V_m = \frac{V}{n}$$

§8.4 Kinetic theory of gases

Assumptions of the kinetic theory of gases:

1. Gas consists of a very large number of particles.
2. Gas particles are in constant random motion and obey Newton's laws of motion.
3. No forces of attraction or repulsion between gas particles except during collision.
4. Gas particles behave as perfectly elastic, identical, hard spheres,
5. Total volume of the gas particles is negligible compared to volume of container.
6. Time duration of collision is negligible compared to time interval between collisions.

Explain how molecular movement causes the pressure exerted by a gas.

- Gas molecules have rapid and random motion.
- When they hit the walls of the container, they exert a force.
- Pressure = Force/Area

We have the following relationship.

$$pV = \frac{1}{3}Nm\langle c^2\rangle \tag{55}$$

See Appendix for the derivation.

Related equations include

$$pV = \frac{1}{3}Nm\langle c^2\rangle \iff \frac{1}{3}M\langle c^2\rangle \iff \frac{1}{3}nM_r\langle c^2\rangle$$

Using this, we can deduce the mean translational kinetic energy of one molecule.

$$\langle \mathrm{KE}\rangle = \frac{1}{2}m\langle c^2\rangle = \frac{3}{2}k_BT \tag{56}$$

This means that mean KE of a molecule of an ideal gas is *proportional* to the thermodynamic temperature.

Thus total translational kinetic energy of N molecules is given by

$$N\langle \mathrm{KE}\rangle = \frac{3}{2}Nk_BT = \frac{3}{2}nRT \tag{57}$$

Problems

Problem 8.1. A resistance thermometer gives a resistance of 20 Ω when the temperature is known to be −10 °C. When the temperature is 110°C, the resistance thermometer has a resistance of 500 Ω. What is the temperature when the resistance is 360 Ω?

Solution.

$$\frac{\theta - (-10)}{110 - (-10)} = \frac{360 - 20}{500 - 20} \implies \boxed{\theta = 75°}$$

□

Problem 8.2. Two bulbs, X of volume 100 cm^3 and Y of volume 50 cm^3, are connected with a tube of negligible volume. A valve prevents gas to flow between the two bulbs. Initially bulb X is filled with an ideal gas at 10°C to a pressure of 3×10^5 Pa. Bulb Y is filled with the same ideal gas at 100°C to a pressure of 1×10^5 Pa. The valve is opened and the temperature of X and Y are maintained at their initial temperatures. Determine the new equilibrium pressure of the system.

Solution. Both bulbs exchange molecules until pressure is equal.

Total number of particles is conserved.

$$n_{X,i} + n_{Y,i} = n_{X,f} + n_{Y,f}$$
$$\frac{P_{X,i}V_x}{T_X} + \frac{P_{Y,i}V_Y}{T_Y} = \frac{P_f V_X}{T_X} + \frac{P_f V_Y}{T_Y}$$

$\therefore$ $\boxed{P_f = 2.45 \times 10^5 \text{ Pa}}$ □

§9 First Law of Thermodynamics

§9.1 Specific Heat Capacity, Latent Heat

Definition 9.1: Specific heat capacity c

Quantity of heat required to raise the temperature of a unit mass of a substance by 1 K.

$$Q = mc\Delta T \tag{58}$$

Definition 9.2: Specific latent heat L

$$Q = mL \tag{59}$$

Phase change: transition from one state of matter to another. During a phase change, latent heat is given off or absorbed, temperature of the object does not change.

§9.2 Internal Energy

The **state** of a system is defined by its pressure p, volume V, and thermodynamic temperature T.[3]

Definition 9.3: Internal energy U

Sum of *kinetic energy* due to random motion of molecules, and *potential energy* due to intermolecular forces of attraction.

$$U = \mathrm{KE}_{\mathrm{random}} + \mathrm{PE}_{\mathrm{random}} \tag{60}$$

For an ideal gas, no intermolecular attraction hence no random PE. Internal energy is only the sum of random distribution of KE of molecules.

$$U = \mathrm{KE}_{\mathrm{random}} = N\langle \mathrm{KE}\rangle = \frac{3}{2}Nk_BT$$

This means increase in temperature of an ideal gas indicates increase in mean random KE of molecules. This means an increase in sum of random KE, and thus increase in internal energy.

§9.3 First Law of Thermodynamics

Work done W is area under pressure-volume graph

$$W = \int p\,\mathrm{d}V$$

[3]not solid, liquid or gas (there are *phases*). The concept of state must be clear!

Definition 9.4: First Law of Thermodynamics

Internal energy of a system depends only on its state. Internal energy of system is the sum of the heat supplied to system and work done on system.

$$\Delta U = Q_{\text{to}} + W_{\text{on}} \tag{61}$$

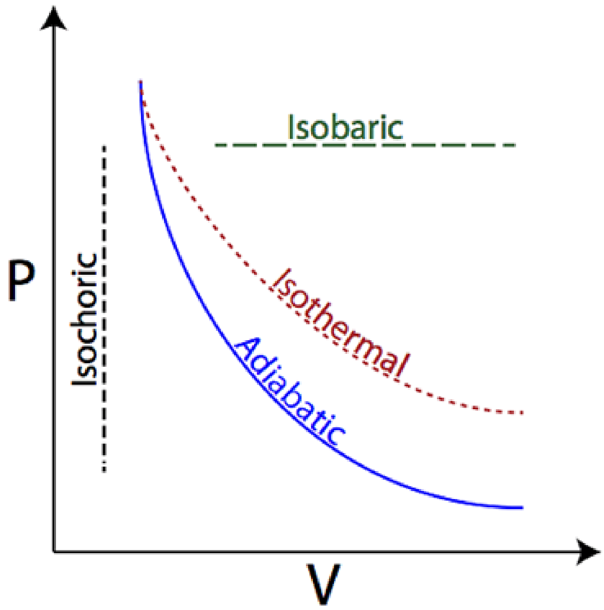
where ΔU is the change in internal energy, Q_{to} is heat supplied to system, W is work done on the system.

§9.3.1 Thermodynamic processes

Quasi-static process: idealised process where the change in state is made infinitesimally slowly so that at each instant, the system can be assumed to be at a thermodynamic equilibrium with itself and with the environment.

The **state** of a system can be represented by a point on $p - V$ graph.

Isotherm: hyperbolic $p - V$ graph (since pV is constant at constant temperature)



- Isothermal expansion/contraction: **constant temperature**

  Change in state occurs along an isotherm. Constant temperature implies constant mean random KE and hence constant total random KE of molecules. For an ideal gas, it means constant internal energy.

  Both p and V change, but pV is constant.

  $\Delta U = 0$

- Isobaric expansion/contraction: **constant pressure**

  Constant pressure means system's environment is not changing e.g. system experiences atmospheric pressure.

Final state is on a different isotherm, meaning temperature changes. Volume of changes.

Work done by system = $-W = p\Delta V$

- **Isochoric** heating/cooling: **constant volume**

  Constant volume means system is confined in a rigid container, implying zero work done by and on the system.

  Final state is on a different isotherm, meaning the temperature changes. Pressure also changes.

  $W = 0$

- **Adiabatic** expansion/contraction: **no heat exchange between system and environment**

  During adiabatic expansion, system does work by pushing back its environment. Since no heat is supplied to system for it to do the work, it uses its thermal energy, thus temperature decreases.

  Final state is at lower isotherm than initial state. Pressure also decreases.

  Paths for adiabatic changes are steeper than isotherms.

  $Q = 0$

Temperature change - ΔU change

Volume change - W done

Cyclic process: system starts and ends with same state, no change in internal energy

Part IV

Oscillation and Waves

§10 Oscillations

Free oscillation: object oscillates with constant amplitude, with no external force acting on it

Definition 10.1: Amplitude x_0

Magnitude of maximum displacement from equilibrium position.

Definition 10.2: Period T

Time taken for one complete oscillation.

Definition 10.3: Frequency f

Number of complete cycles per unit time.

Definition 10.4: Angular frequency ω

Measure of the rate of change of phase angle of the body's motion with time with respect to the centre of its oscillation.

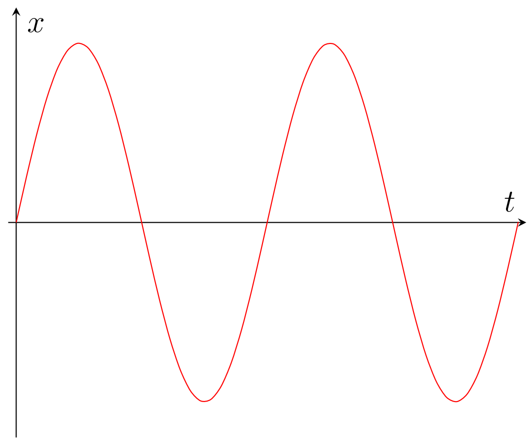
$$\omega = 2\pi f = \frac{2\pi}{T} \tag{62}$$

§10.1 Kinematics

Displacement-time:

When the body is at equilibrium position at $t = 0$,

$$x = x_0 \sin \omega t \tag{63}$$

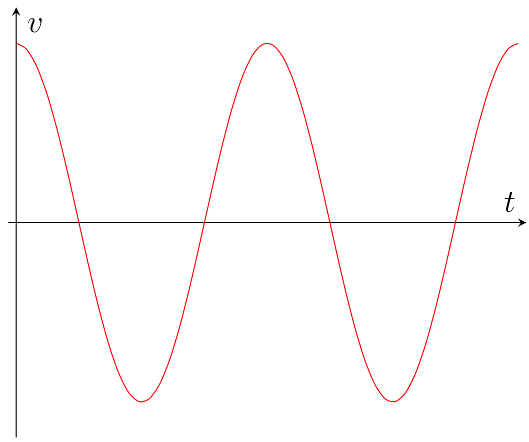


When the body is at extreme position at $t = 0$,

$$x = x_0 \cos \omega t$$

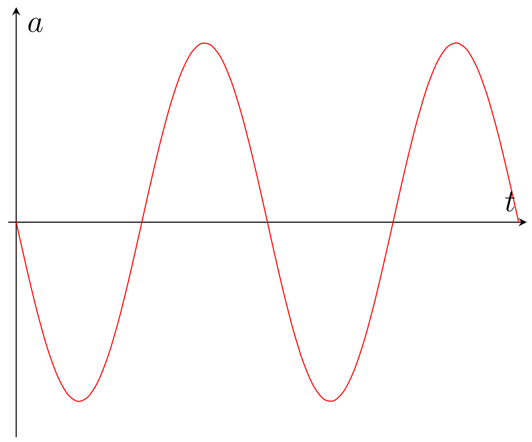
Velocity-time:

$$v = v_0 \cos \omega t \tag{64}$$



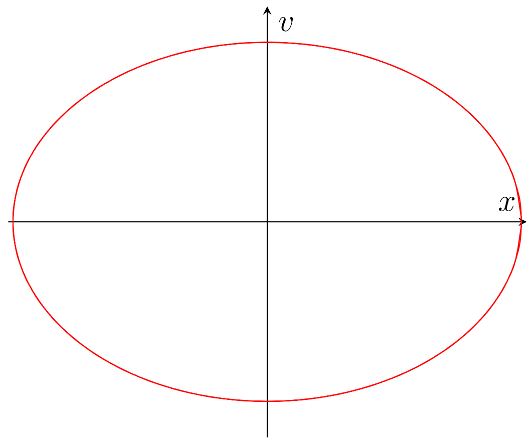
Acceleration-time:

$$a = -a_0 \sin \omega t \tag{65}$$



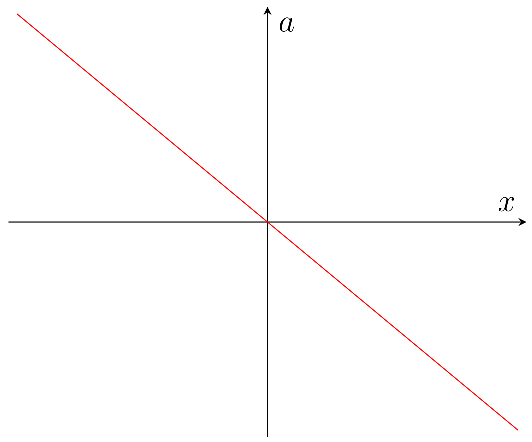
Velocity-displacement:

$$v = \pm\omega\sqrt{{x_0}^2 - x^2} \tag{66}$$



Acceleration-displacement:[4]

$$a = -\omega^2 x \tag{67}$$



[4]This is known as the defining equation of simple harmonic motion.

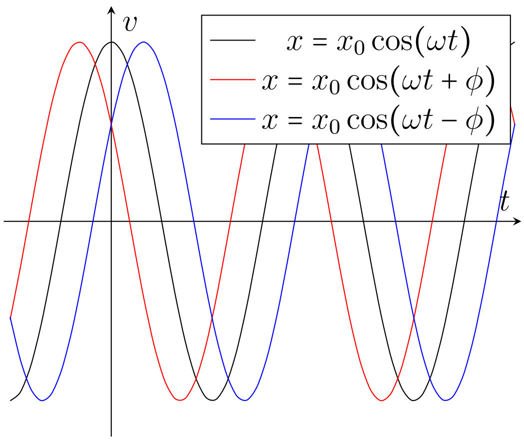
Definition 10.5: Phase

Angle which gives a measure of the fraction of a cycle that has been completed by the oscillating particle or wave.

Definition 10.6: Phase difference ϕ

Angle which gives a measure of how much one oscillation is out of step with another.

- Graph of $x = x_0 \cos(\omega t + \phi)$ is displaced to the left. Motion **leads** by time $\dfrac{\phi}{\omega}$.
- Graph of $x = x_0 \cos(\omega t - \phi)$ is displaced to the right. Motion **lags** by time $\dfrac{\phi}{\omega}$.



§10.2 Energy

For an oscillator in simple harmonic motion, total energy is sum of **kinetic energy** and **potential energy**.

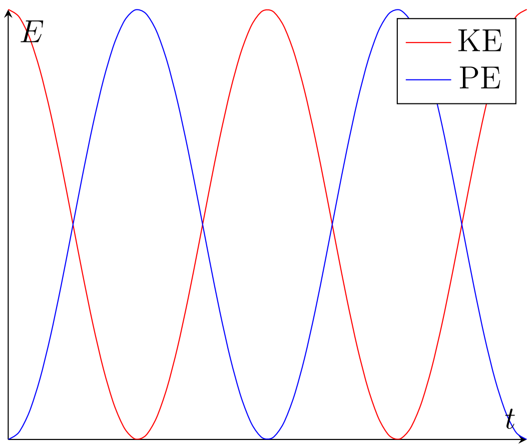
$$E = E_k + E_p$$

$$E_k = \frac{1}{2}m\omega^2{x_0}^2\cos^2\omega t = \frac{1}{2}m\omega^2\left({x_0}^2 - x^2\right) \tag{68}$$

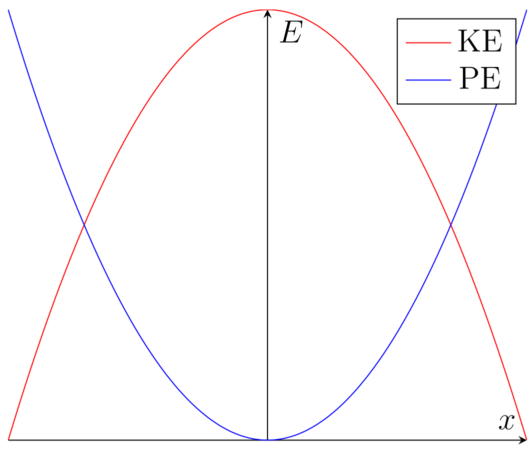
$$E_p = \frac{1}{2}m\omega^2{x_0}^2\sin^2\omega t = \frac{1}{2}m\omega^2x^2 \tag{69}$$

$$E = E_{k,max} = E_{p,max} = \frac{1}{2}m\omega^2{x_0}^2$$

Energy-time graph (for one period):



Energy-displacement graph (for one period):



§10.3 Simple harmonic motion

Definition 10.7: Simple harmonic motion

Oscillatory motion where acceleration is directly proportional to displacement from a fixed point, and this acceleration is always in the opposite direction to its displacement.

$$a \propto -x$$

§10.3.1 Examples

Spring-mass system

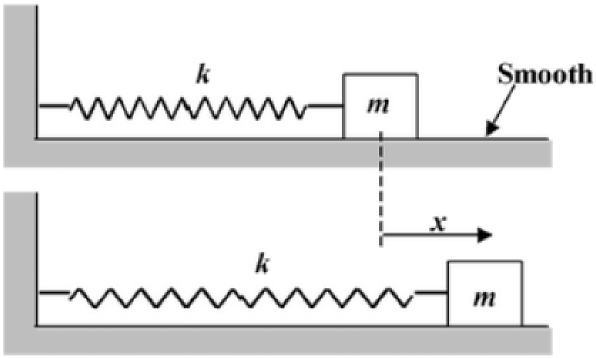


Figure 1: Spring-mass system

Restoring force is $F = -kx$. By Newton's 2nd law,

$$\sum F = ma = -kx \implies a = -\frac{k}{m}x$$

Comparing with $a = -\omega^2 x$,

$$\omega = \sqrt{\frac{k}{m}}$$

Hence frequency is

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Simple pendulum

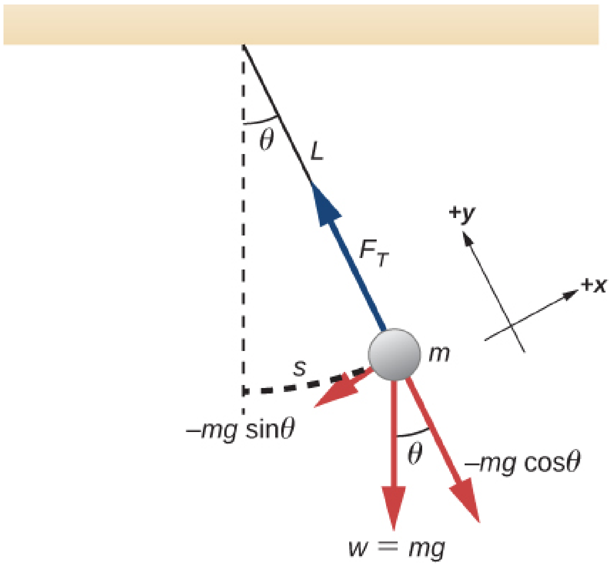


Figure 2: Simple pendulum

Restoring force is the component of the bob's weight, $mg\sin\theta$, that is tangential to the circumference of its swing.

For small θ, by small angle approximation, $mg\sin\theta \approx mg\theta$, where $\theta \approx \frac{x}{l}$.

By Newton's 2nd Law,

$$\sum F = ma = -mg\theta = -\frac{mgx}{l} \implies a = -\frac{g}{l}x$$

Comparing with $a = -\omega^2 x$,

$$\omega = \sqrt{\frac{g}{l}}$$

Hence frequency is

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$$

This indicates that the angular velocity or the period of a simple pendulum is independent of the mass of the weight and the amplitude of oscillation. This is called Galileo's **isochronism of pendulum**.

Vertical spring-mass system

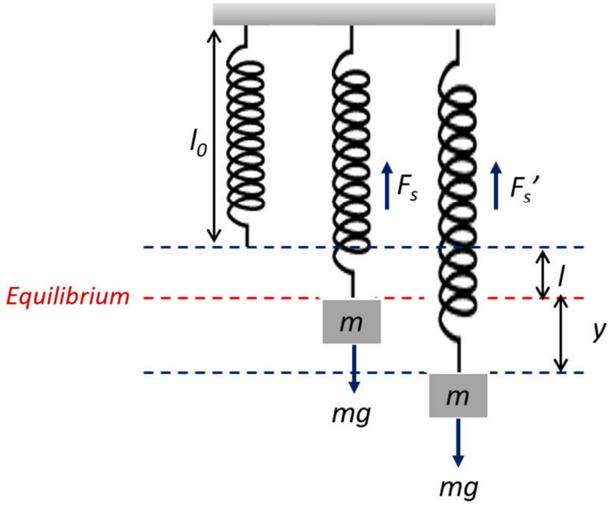


Figure 3: Vertical spring-mass system

At equilibrium, spring force balances weight: $ke = mg$.

At lowest point, by Newton's 2nd law,

$$\sum F = mg - k(e + x) = ma \implies a = -\frac{k}{m}x$$

Comparing with $a = -\omega^2 x$,

$$\omega^2 = \frac{k}{m} \implies f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Floating block

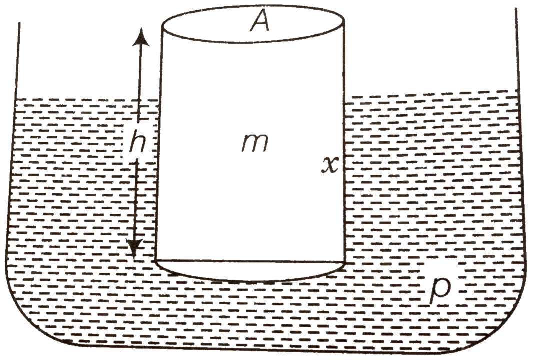


Figure 4: Floating block

Restoring force is the difference between upthrust exerted by water on block and the block's weight. By Newton's 2nd law,

$$\sum F = mg - \rho(A(h + x))g = -\rho(Ax)g = ma \implies a = -\frac{\rho A g}{m}x$$

Comparing with $a = -\omega^2 x$,

$$\omega = \sqrt{\frac{\rho A g}{m}}$$

Hence frequency is

$$f = \frac{1}{2\pi}\sqrt{\frac{\rho A g}{m}}$$

§10.4 Damped oscillation

Definition 10.8: Damping

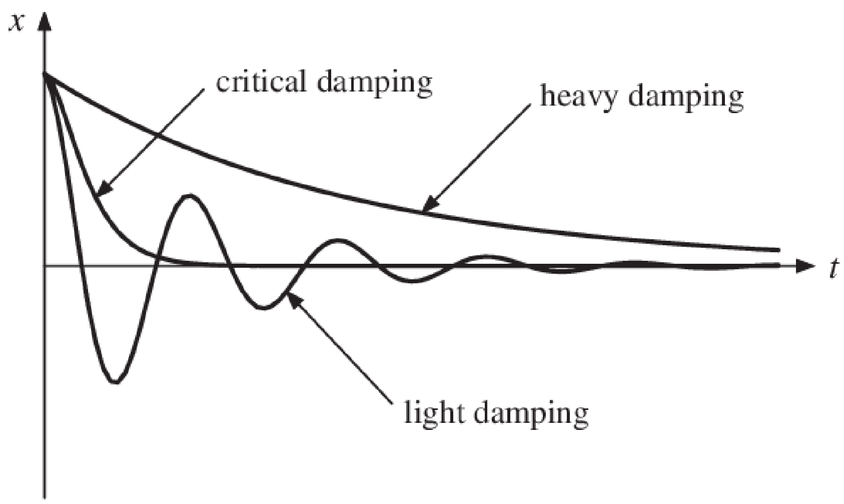
Energy is lost from system as a result of dissipative forces.

Definition 10.9: Damped oscillation

Amplitude decreases with time due to loss of energy to surroundings as a result of resistive forces acting on system.

Degrees of damping:

1. **Light damping**: continues to oscillate, amplitude decreases gradually with time but period remains almost the same
2. **Heavy damping**: does not oscillate, takes a long time to return to equilibrium e.g. door damper
3. **Critical damping**: does not oscillate, returns to equilibrium in the shortest possible time
   e.g. damping system of car



The equation for undamped free oscillation is

$$x = x_0 \cos \omega t$$

The equation for damped oscillation is

$$x = x_0 e^{-\frac{b}{2m}t} \cos \omega t$$

where b is the damping constant.

§10.5 Forced oscillation

Definition 10.10: Forced oscillation

Continual input of energy by an external applied force, to compensate the energy loss due to damping, in order to maintain amplitude of oscillation.
The system oscillates at the frequency of the external periodic force.

§10.6 Resonance

Definition 10.11: Natural frequency f_0

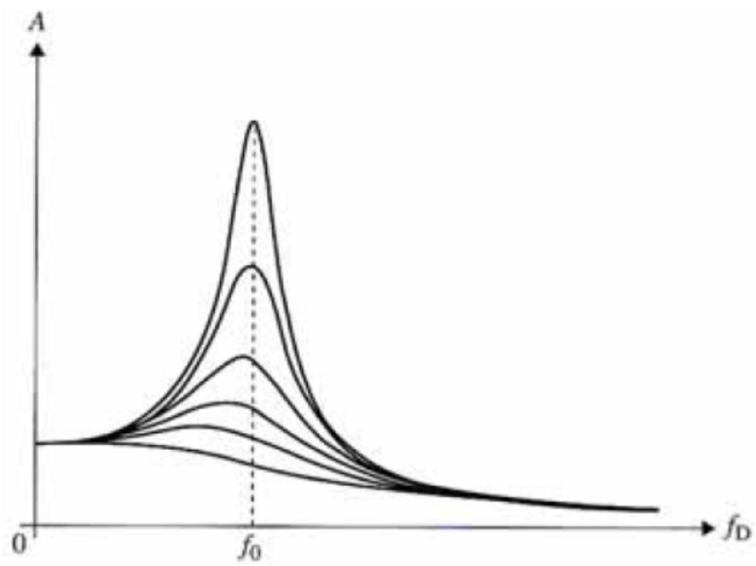
Frequency at which a body oscillates after an initial disturbance.

Definition 10.12: Resonance

Amplitude of the oscillator reaches a maximum when driving frequency equals natural frequency of the oscillator, resulting in maximum transference of energy to the oscillator.

$$f = f_0$$

Amplitude of forced oscillation changes with driving frequency. When $f = f_0$, resonance occurs, amplitude is maximum.



Effect of increased damping on resonance curve:

- Lower at all frequencies
- Flatter peak
- Peak shifts to left slightly

Applications of resonance: magnetic resonance imaging (MRI)

Drawbacks of resonance: bridge design to prevent collapse due to resonant oscillations

Problems

Problem 10.1. A cylinder of radius R, length h, density ρ_0 floats upright in a fluid of density ρ_1. It is given a small vertical displacement, and undergoes undamped harmonic motion with angular frequency ω.

Calculate ω^2.

Solution. Using Archimedes' principle: the force on the cylinder is equal to the weight of the water displaced, which is

$$F = mg = -\rho_1(\pi R^2 d)g$$

where d is the vertical displacement.

This acts as a spring force $F = -kd$. The spring constant k of a harmonic oscillator of mass m is related to the angular frequency ω by $k = m\omega^2$; in this case, mass $m = \rho_0 \cdot \pi R^2 h$.

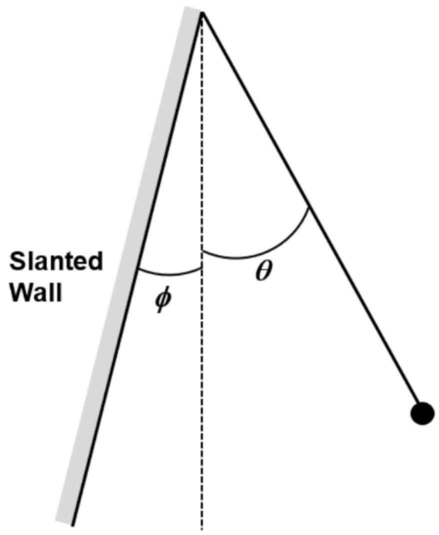
Putting everything together,

$$k = \rho_1 \pi R^2 g = \rho_0 \pi R^2 h \omega^2$$

$$\boxed{\omega^2 = \frac{\rho_1 g}{\rho_0 h}}$$

□

Problem 10.2. The figure below shows a simple pendulum consisting of a small mass at the end of a light, inextensible string. It swings from an initial position of $\theta = 10°$, for which it would have a period T_0. It hits a slanted wall elastically, which is at angle $\phi = 5°$ to the vertical.



When the pendulum hits the wall, what is the new period of oscillation, in terms of T_0?

Solution. Simple harmonic motion implies $\theta = \theta_0 \cos \omega t$, where $\theta_0 = 10°$ before the collision, and given the period we know $\omega = \dfrac{2\pi}{T_0}$.

Let T be time taken to swing from initial position to $-5°$.

$$-5° = 10° \cos \frac{2\pi T}{T_0} \implies \frac{2\pi T}{T_0} = \frac{2\pi}{3} \implies T = \frac{T_0}{3}$$

Hence new period is $\boxed{\dfrac{2T_0}{3}}$. □

§11 Wave Motion

§11.1 Progressive Waves

Definition 11.1: Progressive wave

Wave in which energy is carried from one point to another *by means of vibrations or oscillations* within the waves, without transporting matter.

§11.1.1 Key Terms

Definition 11.2: Displacement y

Distance in a specific direction of a point on the wave from its equilibrium position.

Definition 11.3: Amplitude A

Maximum displacement of any point on the wave from its equilibrium position.

Definition 11.4: Period T

Time taken for one complete oscillation of a point in the wave.
(Time taken for wave to travel a distance of one wavelength)

Definition 11.5: Frequency f

Number of oscillations per unit time of a point on the wave.

Definition 11.6: Wavelength λ

Minimum distance between any two points of the wave with the *same phase* at the *same instant*.

Definition 11.7: Wave speed v

Speed with which energy is transmitted by wave.

$$v = f\lambda \tag{70}$$

Remark. For EM waves, $v = 3 \times 10^8$ m.s^{-1} in vacuum; for sound waves, $v = 330$ m.s^{-1} in air.

Definition 11.8: Wavefront

An imaginary line or surface joining points that are in phase.

§11.1.2 Transverse and Longitudinal Waves

Definition 11.9: Transverse wave

Direction of vibration of the wave particles is perpendicular to *direction of transfer of energy* of the wave.

Definition 11.10: Longitudinal wave

Direction of vibration of the wave particles is parallel to the *direction of transfer of energy* of the wave.

Mechanical wave: wave that requires a medium for propagation. E.g.: sound waves, Water waves

Electromagnetic wave: wave consisting of oscillating electric and magnetic fields that are perpendicular to each other and to the direction of transfer of energy of wave. It does not require medium for transmission. It can travel through a vacuum at spedd of light.

§11.1.3 Graphical Representations

- **Displacement-distance graph**

  Displacement of particles at a particular instant in time.

  Determine wavelength, amplitude

- **Displacement-time graph**

  Displacement of a single particle varies with time.

  Determine period, amplitude

- **Pressure-distance graph** (longitudinal waves)

§11.1.4 Phase and Phase Difference

Phase: angle which indicates fraction of a cycle completed by particle / wave.

Phase difference $\Delta\phi$: angle which indicates how much one wave / particle is *out of step* with another.

- **In phase**: in step with one another ($\Delta\phi = 0$)
- **Out of phase**: not in step with one another ($\Delta\phi \neq 0$)
- **Anti-phase**: out of phase by half cycle ($\Delta\phi = \pi$)

For displacement-distance graph,

$$\Delta\phi = \frac{\Delta x}{\lambda} \times 2\pi \tag{71}$$

For displacement-time graph,

$$\Delta\phi = \frac{\Delta t}{T} \times 2\pi \tag{72}$$

§11.1.5 Energy, Intensity

From SHM, energy associated with oscillation is proportional to square of amplitude:

$$E = \frac{1}{2}m\omega^2 {x_0}^2 = \frac{1}{2}m(2\pi f)^2 {x_0}^2 \implies \boxed{E \propto f^2 A^2}$$

Definition 11.11: Intensity I

Rate of energy transmitted per unit area perpendicular to direction of wave velocity.

$$I = \frac{P}{\text{Area}} \tag{73}$$

As a wave spreads out, its amplitude decreases. This suggests that intensity I of a wave is related to amplitude A. At constant f,

$$I \propto E \text{ and } E \propto A^2 \implies \boxed{I \propto A^2}$$

Sources of transmission

- Three-dimensional transmission: point source emits energy radially outwards onto spherical surface (surface area of $4\pi r^2$)

$$I \propto \frac{1}{r^2}$$

- Two-dimensional transmission: point source emits energy radially outwards onto cylindrical surface (surface area of $2\pi rh$)

$$I \propto \frac{1}{r}$$

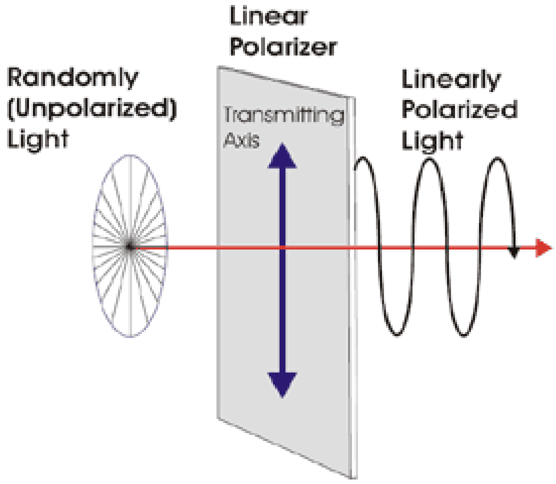
§11.2 Polarisation

Definition 11.12: Polarisation

Oscillations of wave are *confined to only one direction*, in the plane normal to the direction of transfer of energy of wave.

Remark. Polarisation is associated only with *transverse waves*. Longitudinal waves (e.g. sound waves) cannot be polarised, as they do not have oscillations in the plane normal to direction of transfer of energy of wave.

Polariser: optical filter that only allows waves whose direction of oscillation is parallel to transmission axis to pass through.



§11.2.1 Unpolarised wave

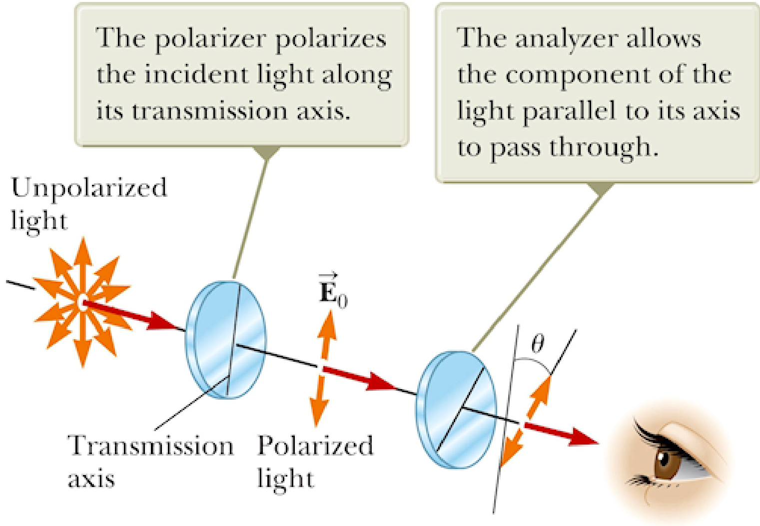
When unpolarised wave is incident on a polariser, intensity is halved[5], amplitude unchanged.

$$I = \frac{1}{2} I_0 \tag{74}$$

§11.2.2 Polarised wave

Polariser only allows *component* of oscillation *parallel* to transmission axis to pass through.

[5] because incident unpolarised wave is a random mixture of all states of polarisation, so vertical and horizontal components are, on average, equal.



Let I_0 denote intensity of linearly polarised light, I denote intensity after passing through analyser, θ denotes angle between direction of polarisation of incident wave and the polarising axis. Then

$$I_0 \propto {A_0}^2 \tag{1}$$

Component of A_0 parallel to transmission axis of analyser is $A = A_0 \cos\theta$. Hence $I \propto (A_0 \cos\theta)^2$, or

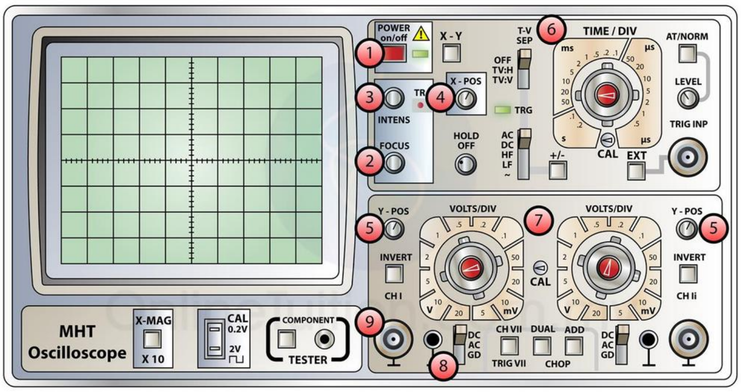
$$I \propto {A_0}^2 \cos^2\theta \tag{2}$$

Dividing (2) by (1) and rearranging gives **Malus' Law**:

$$I = I_0 \cos^2\theta \tag{75}$$

Remark. I is maximum when transmission axes of polariser and analyser are aligned, i.e. $\theta = 0°$; I is minimum when transmission axes are perpendicular to each other, i.e. $\theta = 90°$.

§11.3 Cathode Ray Oscilloscope (c.r.o.)



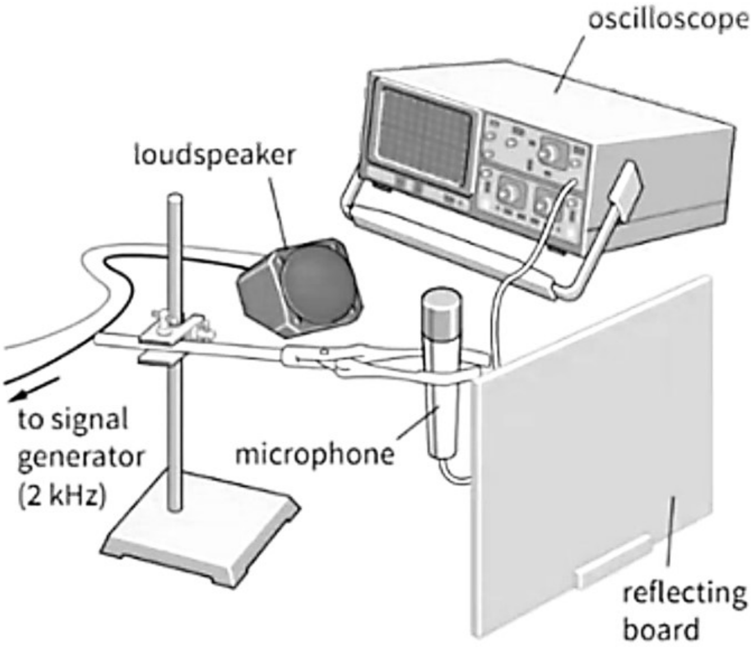
Determine frequency of sound wave

1. Signal fed through microphone into c.r.o.
2. Turn on time-base of c.r.o., trace on screen displays displacement against time.
3. Adjust time-base until stationary trace obtained.
4. Find period T, calculate frequency f using $f = \frac{1}{T}$.

Determine wavelength of sound wave (using stationary waves)

1. Loudspeaker delivers sound via signal generator. Incident wave is directed towards and is reflected at reflecting board.
2. Superposition of incident wave and reflected wave produces stationary wave in the space between loudspeaker and reflecting board.
3. Connect microphone to c.r.o. (time-base switched off[6]). Move it along the line between loudspeaker and reflecting board.
4. When microphone is at positions of minimum amplitude (nodes), signal displayed is minimum; at positions of maximum amplitude (antinodes), signal displayed is maximum.
5. Measure distance across several nodes / antinodes, calculate average distance d between two adjacent nodes / antinodes.
6. Separation of two adjacent nodes is equal to half a wavelength: $d = \frac{1}{2}\lambda$.

[6] so that the spot does not move across the screen. The spot moves up and down the screen, and the height of the vertical trace gives a measure of the amplitude of the sound



Exercise 11.1

Suggest why it is easier to determine accurately the position of a node rather than an antinode.

Answer. The amplitude of oscillation at a node is zero. Hence it is easier to see the spot on the screen lie right on the horizontal line. □

Exercise 11.2

Explain why it is better to measure the distance across several nodes.

Answer. Measuring a longer distance produces a smaller relative error of measurement. □

Problems

§12 Superposition

§12.1 Principle of Superposition

Definition 12.1: Principle of superposition

When two or more waves of the *same type* meet at a point at the same time, the *displacement* of the resultant wave is the vector sum of the displacements of the individual waves at that point at that time.

§12.2 Stationary Waves

Definition 12.2: Node

Region of destructive superposition where the two waves meet *antiphase*. Displacement is permanently zero (or minimum amplitude).

Definition 12.3: Antinode

Region of constructive superposition where the two waves meet *in phase*. Displacement is maximum amplitude.

Distance between two adjacent nodes / antinodes:

$$d = \frac{1}{2}\lambda$$

Definition 12.4: Stationary wave

A stationary wave is formed when two progressive waves of the *same type* of *equal amplitude*, *equal frequency*, *equal speed* travelling in opposite directions meet and undergo superposition with each other.

- Wave profile does not advance.
- Positions of wave elements oscillating with maximum amplitudes (antinodes) and minimum amplitudes (nodes) are fixed with time.
- Vibrational energy of the wave is not transmitted from one point to another.

Progressive vs stationary waves

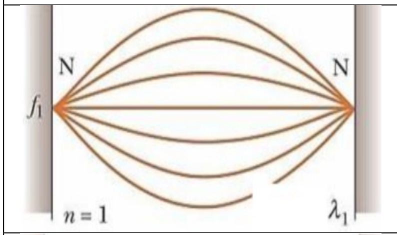
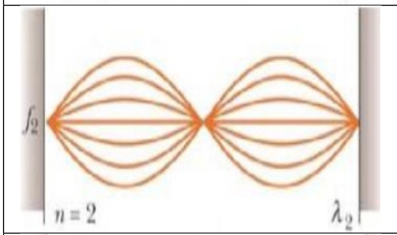
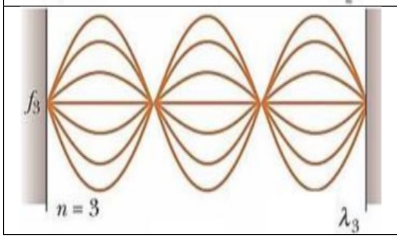
	Progressive wave	**Stationary wave**
Amplitude	Same amplitude for all particles in wave motion	Amplitude of oscillating particles varies from zero (at node) to a maximum (at antinode)
Frequency	Particles vibrate in SHM with frequency of progressive wave	Particles vibrate in SHM with frequency of stationary wave
Wavelength	Shortest distance between two points in phase	Twice the distance between a pair of adjacent nodes / antinodes
Phase	All particles within one wavelength have different phases	All particles within a loop vibrate in phase, paricles in adjacent loops are antiphase
Wave profile	Wave profile advances with the speed of wave	Wave profile does not advance
Energy	Energy is transported in the direction of travel of wave	Energy is stored within vibratory motion of stationary wave

Drawing of stationary waves: (draw the two extremes)

§12.2.1 Transverse stationary waves

Stretched string **Experiment set-up:** String of length L is stretched tightly between fixed supports. When plucked, progressive wave produces, reflected at fixed ends, travel backwards. Incident and reflected waves superpose to form a stationary wave.

Node: fixed ends

Mode of vibration	Wavelength	Frequency	Harmonic	Overtone
	$\lambda_1 = 2L$	$f_1 = \dfrac{v}{\lambda_1} = \dfrac{v}{2L}$	1st	-
	$\lambda_2 = L$	$f_2 = \dfrac{v}{\lambda_2} = \dfrac{v}{L} = 2f_1$	2nd	1st
	$\lambda_3 = \dfrac{2}{3}L$	$f_3 = \dfrac{v}{\lambda_3} = \dfrac{3v}{2L} = 3f_1$	3rd	2nd

Integer number of lops must fit exactly into the length of the string, that is

$$L = n\left(\frac{\lambda}{2}\right)$$

Generalising, the n-th harmonic is given by

$$f_n = n\left(\frac{v}{2L}\right)$$

§12.2.2 Longitudinal stationary waves

Experiment set-up: Sound wave enters pipe via open end, travels from open end towards closed end, reflected when it hits wall of closed end. Incident and reflected waves superpose to form a stationary wave.

Node: closed end[7]

Antinode: open end[8]

Closed pipe

Open pipe

[7] At the nodes, the air particles are not moving. Hence, they allow powder to collect at the nodes.

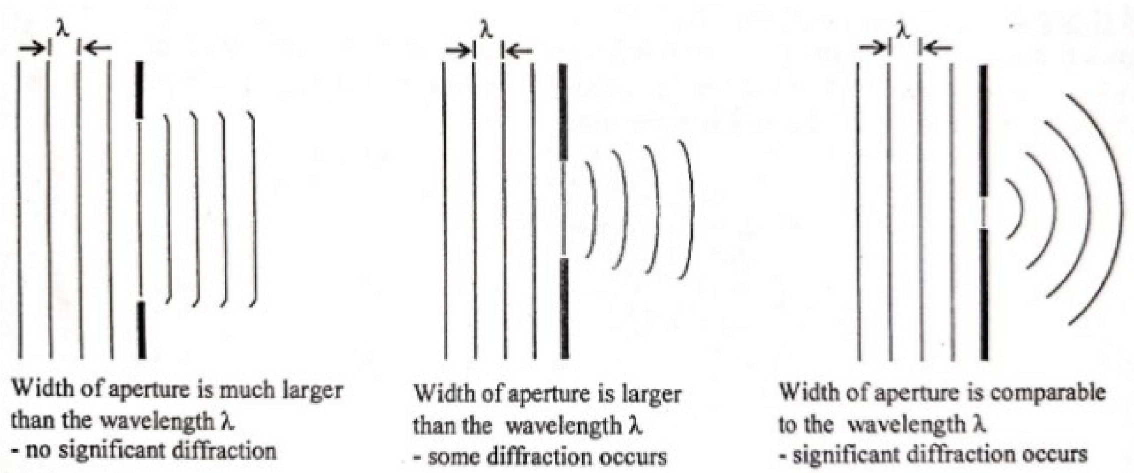
[8] At the antinodes, the air particles are moving with maximum amplitude. Hence, they "push" the powder away from the antinodes to the nodes.

End corrections In practice, antinode at open end occurs *slightly outside* the pipe.

§12.3 Diffraction

Definition 12.5: Diffraction

Spreading of waves at edge of obstacle, or through slit, so that waves do not travel in straight lines.



Condition: aperture size (size of slit) is comparable to wavelength (same order), i.e. $b \approx \lambda$.

§12.3.1 Single slit diffraction

Mimima Positions of minima (points of zero intensity) occur at angles

$$\sin\theta = m\frac{\lambda}{b}$$

where $m = \pm 1, \pm 2, \dots$

Hence position of first minima is at angle

$$\sin\theta = \frac{\lambda}{b} \tag{76}$$

Rayleigh criterion Resolution of two objects is the ability to see as distinct two objects that are distinct.

Rayleigh criterion states that for two patterns to be *just resolved*, the central maximum of one must lie on the first minimum of the other.

Minimum angle for two sources to be resolved:

$$\theta \approx \frac{\lambda}{b} \tag{77}$$

§12.4 Interference

Definition 12.6: Coherent waves

Two waves have a constant phase difference between them (with respect to time).

Definition 12.7: Interference

Superposition of *coherent* waves which results in change in overall intensity.

Definition 12.8: Constructive interference

When two waves meet *in phase* at a point, resultant displacement is the *sum* of magnitudes of individual displacements of the two waves.

Path difference is a whole number of wavelengths, i.e. $n\lambda$.

Definition 12.9: Destructive interference

When two waves meet *antiphase* at a point, resultant displacement is the *difference* of magnitude of individual displacements of the two waves.

Path difference is an odd number of half wavelengths, i.e. $(n + \frac{1}{2})\lambda$

Conditions for two-source interference fringes to be *observable*:

1. Waves must *meet*
2. Waves must be *coherent*
3. Waves have (approximately) equal amplitudes
4. Transverse waves must be either unpolarised or polarised in the same plane
5. Split separation is of same order as wavelength, i.e $b \approx \lambda$

§12.4.1 Young's double slit diffraction

$$\lambda = \frac{ax}{D} \tag{78}$$

§12.5 Diffraction Grating

Problems

Problem 12.1. A sound source of frequency 2500 Hz is placed several metres from a plane reflecting wall in a large chamber containing a gas. A microphone, connected to a c.r.o., is used to detect nodes and antinodes formed along the normal from the source to the wall. The microphone is moved from one node through 20 antinodes to another node, across a distance of 1.90 m.

Calculate the speed of sound in the gas.

Solution. The sequence of nodes and antinodes are given by

$$N \quad A \quad N \quad \cdots \quad N \quad A \quad N$$

where the distance between two nodes is $d = \dfrac{\lambda}{2}$.

There are 20 half-wavelengths, so

$$20\left(\frac{\lambda}{2}\right) = 1.90 \implies \lambda = 0.190 \text{ m}$$

The wavelength of the stationary wave is equal to that of the progressive wave that formed it.

Hence $v = f\lambda = (2500)(0.190) = \boxed{475 \text{ m s}^{-1}}$. □

Part V

Electricity and Magnetism

§13 Electric Fields

Types of particles

- proton: charge $+e$
- electron: charge $-e$
- α-particle: charge $+2e$

Definition 13.1: Coulomb's Law

Electric force between two point charges is proportional to the product of the charges and inversely proportional to the square of their separation.

$$\vec{\mathbf{F}} = k\frac{Qq}{r^2}\hat{\mathbf{r}} \tag{79}$$

where the constant of proportionality is

$$k = \frac{1}{4\pi\varepsilon_0}$$

where **permittivity of free space** $\varepsilon_0 = 8.85 \times 10^{-12}$ F.m^{-1} and can be taken to be equal to that of air unless specified otherwise.

Remark. Electric force is repulsive when $Qq > 0$ and attractive when $Qq < 0$.

Principle of Superposition: When more than two charges are present, net force on any one charge is the vector sum of the forces exerted on it by the other charges. For example, if three charges are present, the resultant force experienced by q_3 due to q_1 and q_2 is

$$\vec{\mathbf{F}}_3 = \vec{\mathbf{F}}_{13} + \vec{\mathbf{F}}_{23}$$

To generalise, for a system of n charges, the net force experienced by the j-th particle is

$$\vec{\mathbf{F}}_j = \sum_{i=1, i\neq j}^{n} \vec{\mathbf{F}}_{ij}$$

Definition 13.2: Electric field

Region of space where a charge experiences an electric force.

Representation of electric field using **field lines** (lines of force):

- An electric field line indicates the direction of the force a positive charge would experience if it is placed at that point in the field (at a normal to surface of charge).
- The number of field lines per unit cross-sectional area is proportional to the **electric field strength**.
- Electric field lines are directed away from positive to negative charges, never intersect each other, and are never created or annihilated in vacuum.

Definition 13.3: Electric field strength $\vec{\mathbf{E}}$

Electric force per unit positive charge on a *small test charge* placed at that point.

$$\vec{\mathbf{E}} = \frac{F}{q} = k\frac{Q}{r^2} \tag{80}$$

Remark. We take charge q to be infinitesimally small so that the field it generates does not disturb that of the "source charge", i.e. charge Q.

Comparison between electric field and gravitational field:

- Qualitative aspect: Gravitational force results from interaction between masses; electric force results from interaction between charges.
- Quantative aspect: Both fields are inverse square law fields.

Definition 13.4: Electric potential V

Work done per unit positive charge by an external force in bringing a *small test charge* from infinity to that point.

$$V = \frac{W}{q} = k\frac{Q}{r} \tag{81}$$

where W is the work done on the charge.

Positive charges move from places of high potential to lower potential, EPE increases.

Negative charges move from places of low potential to higher potential, EPE decreases.

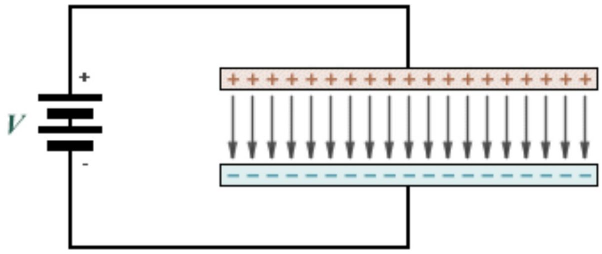
Definition 13.5: Electric potential energy U

When two point charges Q and q are at a distance r apart, electric potential energy U of the *system of two charges* is given by

$$U = qV = k\frac{Qq}{r} \tag{82}$$

Remark. EPE can be negative, if one charge is positive and the other is negative.

§13.1 Parallel plates

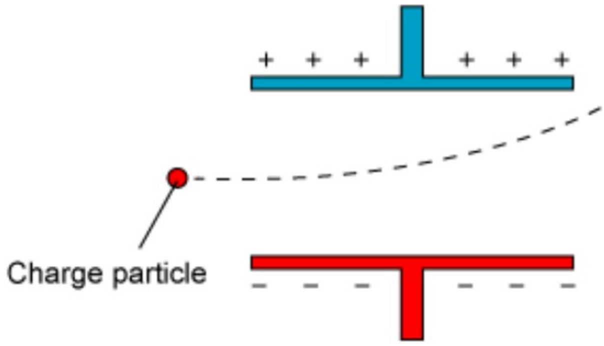


Electric field set up is uniform, hence electric field strength E is constant. Thus $F = qE$.

$$E = \frac{|\Delta V|}{d} \tag{83}$$

where $|\Delta V|$ is the potential difference across the plates, d is the separation of the plates.

§13.1.1 Charge Moving Perpendicularly to an Electric Field



Motion of charged particle in electric field is **parabolic** in nature.

Proof. By Newton's 2nd Law, acceleration is given by

$$a = \frac{F}{m} = \frac{qE}{m} = \frac{q|\Delta V|}{md}$$

which is constant.

Assuming particle is initially at rest, then velocity is given by

$$v = u + at = \frac{qE}{m}t$$

When the particle projection is perpendicular to the direction of the electric field, then motion is in the upward direction (along y-axis). Thus displacement in y-direction is

$$y = ut + \frac{1}{2}at^2 = \frac{qE}{2m}t^2$$

Since $v = xt$, eliminating time dependence gives us

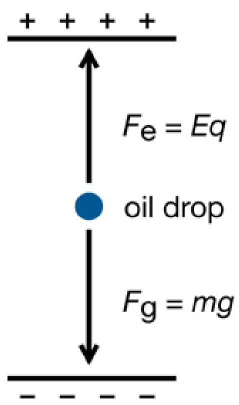
$$y = \frac{qE}{2mv^2}x^2$$

The y-x relation is a parabola. Hence the particle follows a parabolic trajectory. $\square$

§13.1.2 Millikan's oil drop

Physicist Robert Millikan's experiment involves spraying tiny oil droplets into a vertical chamber with two metal plates on either end. The oil droplets became charged. When they entered the chamber, they began to fall under the influence of gravity. He then stopped the free-falling droplets and reversed their direction of motion by applying a voltage across the two metal plates.

He measured the velocity of a single oil droplet in the electric field to determine the electrical force F acting on it. This allowed him to determine the charge on the oil droplet, since $q = \frac{F}{E}$.



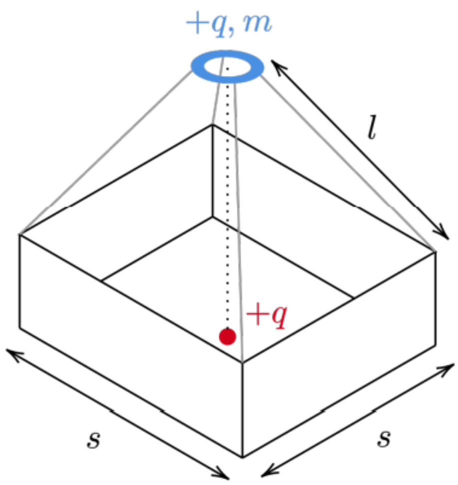
By measuring the charge of many droplets and comparing them, he reasoned that the smallest difference in charge among all the droplets would be due to the presence of one extra electron. That small difference in charge would then be equal to the charge of a single electron or the elementary charge. He then discovered that the charges of the oil droplets were always integer multiples of 1.60×10^{-19} C. He reasoned this must be the charge of a single electron, a value that is referred to as the **elementary unit of charge**.

Problems

Problem 13.1. Josiah bought a small engagement ring of mass $m = 1.00 \times 10^{-3}$ kg, which he wanted to present to his fiancée in a box with a square base of side length $s = 0.100$ m and negligible height. On opening the box, he wanted the ring to hover a short distance above its centre. To achieve this, he hid a positive point charge $+q$ under the centre of the box and applied the same positive charge $+q$ to the ring. To constrain the ring to hover directly above the centre of the box, he tied four thin inextensible strings of length $l = 0.120$ m to the ring and secured them to the four corners of the box.

Suppose the ring is small enough to be approximated by a point charge. What is the minimum charge q required to ensure the four strings remain taut while the ring hovers above the box?

Leave your answer to 3 significant figures in units of μ C.

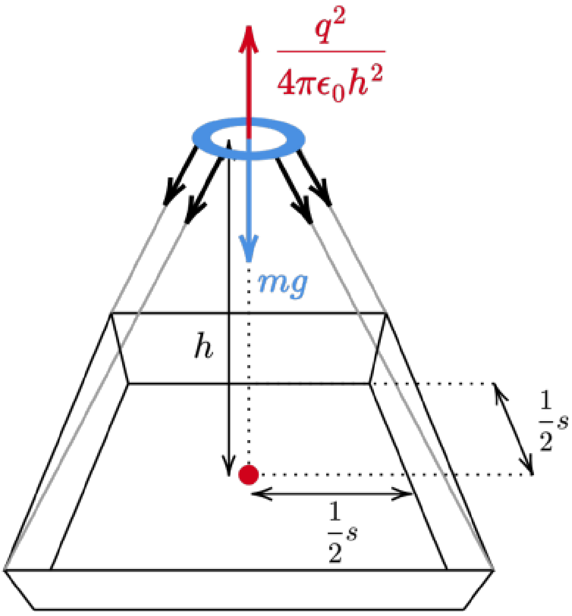


Solution. Three types of forces act on the hovering ring: the electrostatic repulsion from the hidden point charge, the weight of the ring, and the tension from the strings. These forces must cancel for the ring to hover in place.

Let h be the height above the box at which the ring hovers, and let us define the downwards direction to be positive. As shown in the diagram, the electrostatic repulsion is $-\dfrac{q^2}{4\pi\varepsilon_0 h^2}$ while the weight from the ring is $+mg$.

Since the net (downwards) force from the tension of the four strings, T, balances the gravitational and electrostatic forces on the ring, we have:

$$T = \frac{q^2}{4\pi\varepsilon_0 h^2} - mg$$



For the strings to remain taut, the tensions in the string must be non-negative, which implies that:

$$T = \frac{q^2}{4\pi\varepsilon_0 h^2} - mg \geq 0 \implies q^2 \geq 4\pi\varepsilon_0 mgh^2$$

From the same diagram above and Pythagoras's theorem, we can infer that:

$$l^2 = \left(\frac{s}{2}\right)^2 + \left(\frac{s}{2}\right)^2 + h^2 \implies h = \sqrt{l^2 - \frac{s^2}{2}}$$

Substituting this expression for h into the previous equation and isolating q implies that the charge must be at least:

$$q \geq \sqrt{4\pi\varepsilon_0 mg\left(l^2 - \frac{s^2}{2}\right)}$$

$$\boxed{q \approx 0.101\,\mu\mathrm{C}}$$

□

§14 Current of Electricity

§14.1 Electric current

Elementary charge:

$$e = 1.6 \times 10^{-19} \text{ C}$$

- Protons are positively charged, with a charge $+e$.
- Electrons are negatively charged, with a charge $-e$.
- Ions carry charges that are multiples of $+e$ and $-e$.

Definition 14.1: Electric current I

Rate of flow of charge.

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t}$$

For a steady current I, we have

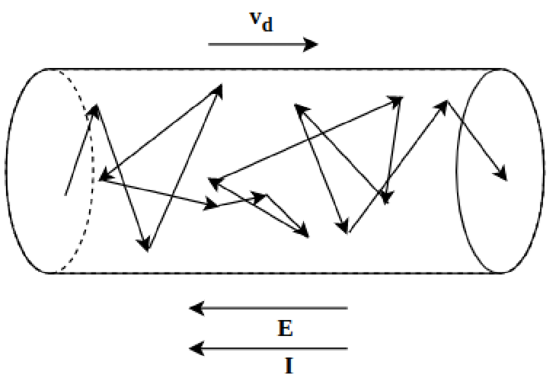
$$Q = It \tag{84}$$

Remark. By convention, the direction of current is the direction that positive charges move. However, remember that current is due to the flow of electrons; lattice atoms/ions do not move.

Transport equation:

$$I = nAvq \tag{85}$$

where v is the **drift velocity**.



Remark. Drift velocity of charge carriers is much lower than the maximum velocity that charge carriers could achieve from the potential difference applied to the wire.

This is because charge carriers experience electrical force in all directions because of collisions with lattice ions. This produces a range of velocities. The drift velocity is an average.

Remark. When a domestic lighting circuit is switched on, the lights come on almost immediately. This is because when the switch is on, all electrons in the wire and filament start to move together.

§14.2 Potential difference and electromotive force

Definition 14.2: Potential difference V

Work done per unit charge when electrical energy is converted into non-electrical energy when the charge passes from one point to the other.

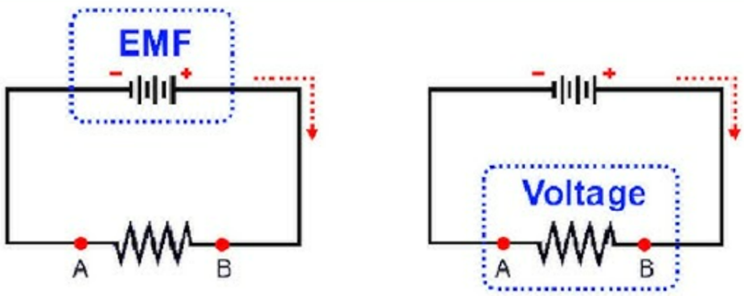
$$V = \frac{W}{Q} \tag{86}$$

Definition 14.3: Electromotive force ε

Work done per unit charge when non-electrical energy is converted into electrical energy when the charge is moved around a complete circuit.

$$\varepsilon = \frac{W}{Q} \tag{87}$$

Potential difference	**Electromotive force**
Refers only to source	Refers to any two points in the circuit
Amount of non-electrical energy converted into electrical energy	Amount of electrical energy into non-electrical energy
Always exists as it is a source of energy	Only exists if current is flowing



§14.3 Resistance

Definition 14.4: Resistance R

Ratio of potential difference across component to current flowing through it.

$$R = \frac{V}{I} \tag{88}$$

Resistivity is a property unique to the material.

$$R = \rho \frac{l}{A} \tag{89}$$

Definition 14.5: Ohm's Law

Current flowing through conductor is directly proportional to potential difference applied across it, provided that physical conditions remain constant.

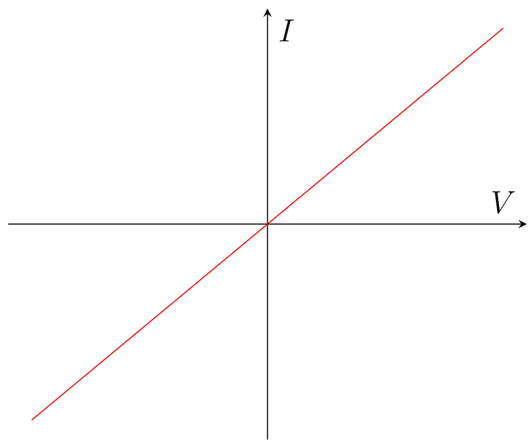
$$I \propto V \tag{90}$$

§14.4 I-V characteristics

An ohmic resistor obeys Ohm's Law. For non-ohmic resistors that do not obey Ohm's Law, factors that cause resistance to deviate are:

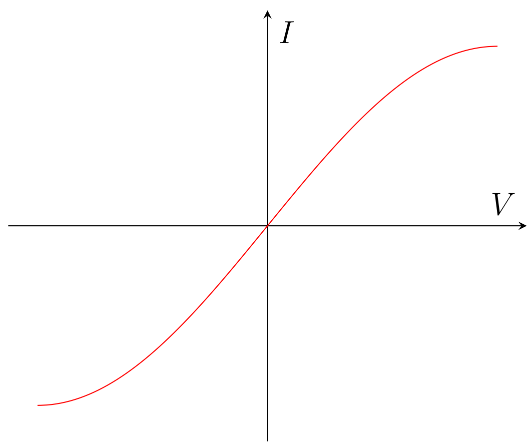
1. Number density of charge carriers n (decrease resistance)
2. Amplitude of atomic vibrations of lattice atoms (increase resistance)

§14.4.1 Ohmic resistor



Resistance is **constant**: as p.d. increases, current increases proportionately.

§14.4.2 Filament lamp



Resistance **increases**: as p.d. increases, current increases less than proportionately.

Alternative explanation:
The gradient of the line joining the origin to each point on the curve decreases as p.d. increases.
Since resistance is the reciprocal of the gradient, resistance increases as p.d. increases.

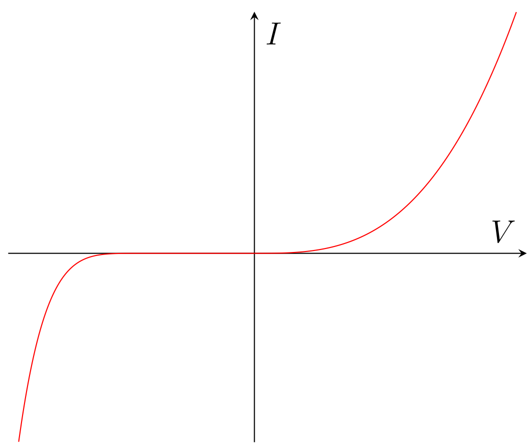
- As temperature increases, amplitude of atomic vibrations of lattice atoms increases.
- n does not increase significantly.
- Overall effect is resistance increases.

Useful problem solving technique:

To find a particular resistance value on an $I-V$ graph, draw the $I-V$ graph of an ohmic conductor with the particular resistance value.

§14.4.3 Semiconductor diode

What is a semiconductor diode?
A diode is a two-terminal electronic component that has a low resistance to the flow of current in one direction thus allowing the passage of current in one direction (**forward bias**) whereas there will be a high resistance in the other, thus restricting the flow of current in that direction (**reverse bias**).



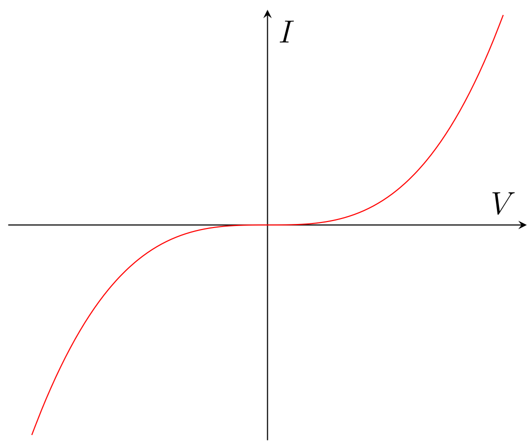
For forward-biased region, resistance **decreases**: as p.d. increases, current increases more than proportionately.

- As temperature increases, electrons in semiconductor are more likely to have sufficient energy to escape from atom, so n increases significantly.
- Increase in rate of interaction of electrons with vibrating atoms.

- Increase in n predominates over increase in rate of interactions of electrons with lattice. Overall effect is resistance decreases.

For reverse-biased region, resistance is **infinitely high**: no current flow through diode until breakdown voltage.

§14.4.4 Negative Temperature Coefficient (NTC) thermistor



Resistance **decreases**: as p.d. increases, current increases more than proportionately.

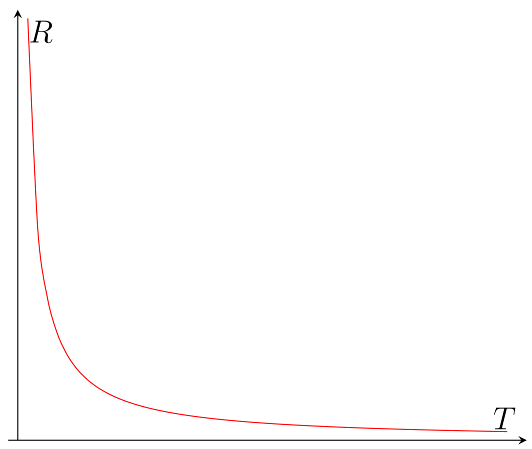
- As temperature increases, electrons are more likely to have sufficient energy to escape from atom, so n increases significantly.
- Increase in rate of interaction of electrons with vibrating atoms.
- Increase in n predominates over rate of interactions of electrons with lattice. Overall effect is resistance decreases.

Remark. As temperature of thermistor increases, its resistance decreases.

Since e.m.f. of battery remains unchanged, current increases when resistance decreases, causing greater power to be generated in the thermistor.

This will result in a further increase in temperature of thermistor, decrease in its resistance, leading to **thermal runaway** which could cause overheating.

Resistance-temperature characteristic: resistance decreases as temperature increases



§14.4.5 Light Dependent Resistor (LDR)

Similar to NTC thermistor.

§14.5 Power

Electrical power dissipated by the conductor:

$$\begin{aligned}P &= VI \\ P &= I^2R \\ P &= \frac{V^2}{R}\end{aligned} \tag{91}$$

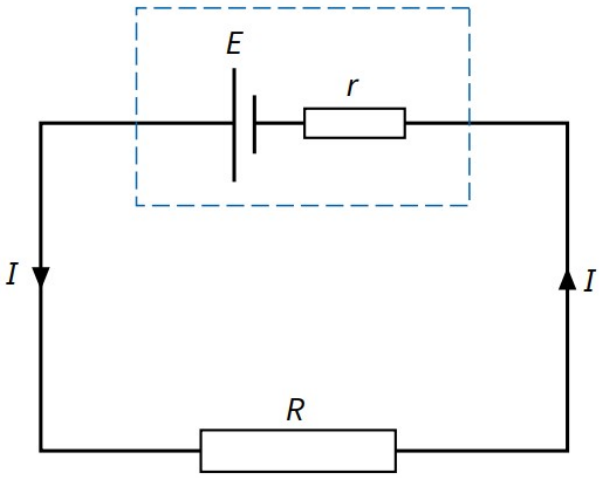
§14.6 Internal resistance

Let r denote internal resistance of battery, R denote resistance of load, V denote terminal p.d. of battery, ε denote e.m.f.

For an **ideal** battery, $r = 0$. This gives us

$$\varepsilon = V$$

For a **real** battery, $r \neq 0$. We imagine the internal resistance as another load of resistance r connected with R in series.



Since e.m.f. is sum of p.d.,

$$\begin{aligned}\varepsilon &= I(R+r) \\ \varepsilon &= V + Ir\end{aligned} \tag{92}$$

This gives us

$$\boxed{I = \frac{\varepsilon}{R+r}}$$

Problems

Problem 14.1. The terminals of a battery are connected to a load resistance. As the battery increases, its internal resistance increases. How does this affect the ability of the battery to deliver energy?

Answer. Terminal p.d. decreases. Power delivered to the load decreases (p.d. decrease, resistance constant) as more energy dissipated as heat by internal resistance of battery.

□

Problem 14.2. By considering a practical source with e.m.f. E and internal resistance r connected in series with an electrical device of resistance R, determine

(i) an expression for output efficiency of the source, $\eta = \dfrac{\text{useful power output}}{\text{total power generated}}$.

(ii) the value of R in terms of r such that maximum power is delivered to the device, and the output efficiency of the source when it is used to operate an electrical device at maximum power.

Answer.

(i) Total power generated:

$$P_{gen} = I^2(R + r)$$

Useful power output:

$$P_{out} = I^2 R$$

Efficiency:

$$\boxed{\eta = \frac{R}{R + r}}$$

(ii) Current in the circuit:

$$I = \frac{E}{R + r}$$

Power output at the load:

$$P_{out} = I^2 R = \left(\frac{E}{R + r}\right)^2 R$$

For maximum power output,

$$\frac{\mathrm{d}P_{out}}{\mathrm{d}R} = 0 \implies \boxed{R = r}$$

Output efficiency is $\boxed{0.5}$ (50%).

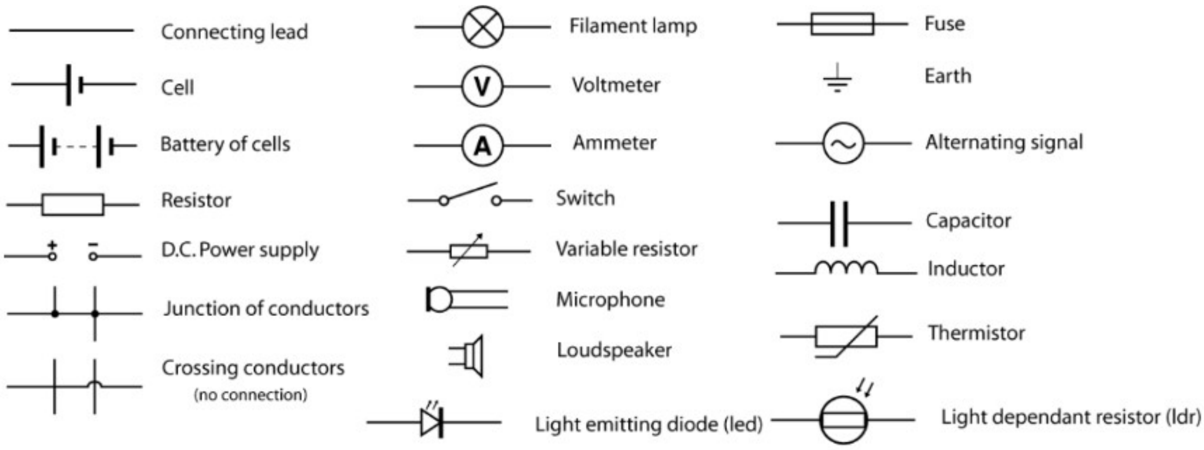
□

§15 D.C. Circuits

Direct current (d.c.): current flowing only in *one direction*

§15.1 Circuit symbols and diagrams

Circuit symbols:



§15.2 Series and parallel arrangements

§15.2.1 Current

Current divides up where a circuit splits into multiple branches.

Definition 15.1: Kirchhoff's Current Law

Algebraic sum of the currents at a junction of a circuit is zero.

$$\sum_{\text{junction}} I_i = 0 \tag{93}$$

Currents entering the junction are given a positive (+) sign, currents leaving the junction are given a negative (−) sign.

This means sum of currents entering a junction = sum of currents leaving the junction

$$\sum I_{\text{in}} = \sum I_{\text{out}}$$

Proof. By the Principle of Conservation of Charge, the total charge that enters a junction per unit time must be equal to the total charge that leaves the same junction per unit time. □

Series Currents at all points are same, equal to total current.

$$I_1 = \cdots = I_n = I$$

Parallel Total current is sum of individual currents.

$$I = I_1 + \cdots + I_n$$

§15.2.2 Voltage

Definition 15.2: Kirchhoff's Voltage Law

Algebraic sum of all electrical potential changes around any closed loop is zero.

$$\sum_{\text{junction}} \Delta V_i = 0 \tag{94}$$

This means the sum of the e.m.f.s around any loop in a circuit is equal to the sum of the p.d.s around the loop.

$$\sum \varepsilon = \sum V_{\text{drop}}$$

Proof. By the Law of Conservation of Energy, energy supplied by the source is equal to energy dissipated by resistors.

For constant current and time,

$$\left(I \sum \varepsilon\right) t = \left(I^2 \sum R\right) t$$
$$\sum \varepsilon = I \sum R$$
$$\sum \varepsilon = \sum V$$

□

Remark. Current always flows from higher potential to lower potential.

Series e.m.f. is sum of voltages.

$$V_1 + \cdots + V_n = \varepsilon$$

Parallel Voltages are same, equal to e.m.f.

$$\varepsilon = V_1 = \cdots = V_n$$

§15.2.3 Resistance

Series Effective resistance is sum of individual resistances

$$R_{\text{eff}} = R_1 + \cdots + R_n \tag{95}$$

Derivation. Consider two resistors of resistances R_1 and R_2 connected in series.

According to Kirchhoff's current law, the current in each resistor is the same. The p.d. V across the combination is equal to the sum of the p.d.s across the two resistors:

$$V = V_1 + V_2$$

Since $V = IR$, $V_1 = IR_1$ and $V_2 = IR_2$, we can write:

$$IR = IR_1 + IR_2$$

Cancelling the common factor of current I gives:

$$R = R_1 + R_2$$

For three or more resistors, the equation for total resistance R becomes:

$$R = R_1 + R_2 + R_3 + \cdots$$

□

Parallel Reciprocal of effective resistance is sum of reciprocals of individual resistances

$$\frac{1}{R_{\text{eff}}} = \frac{1}{R_1} + \cdots + \frac{1}{R_n} \tag{96}$$

Remark. Effective resistance of resistors in parallel is always lower than the lowest resistance in the network.

Derivation. Consider two resistors of resistances R_1 and R_2 connected in parallel. The total current is divided between them.

Using Kirchhoff's current law,

$$I = I_1 + I_2$$

Applying Kirchhoff's voltage law to the loop that contains the two resistors,

$$I_1R_1 - I_2R_2 = 0$$

(because there is no source of e.m.f. in the loop)

This suggests that the two resistors have the same p.d. across them. Hence we can write $I = \frac{V}{R}$, $I_1 = \frac{V_1}{R_1}$ and $I_2 = \frac{V_2}{R_2}$.

Substituting these into $I = I_1 + I_2$ and cancelling the common factor V,

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

For three or more resistors, the equation for total resistance R becomes

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots$$

□

§15.2.4 Measuring instruments

Instrument	Measured quantity	Assumption[9]	Connection
ammeter	current (in one direction)	zero (internal) resistance	in series
galvanometer	current (in both directions)	zero (internal) resistance	in series
voltmeter	potential difference	infinite resistance	in parallel

§15.3 Potential divider

Definition 15.3: Potential divider rule

If a voltage exists across several resistors connected in series, then the voltage across each resistor is proportional to the total resistance.

$$V_1 = \frac{R_1}{R_T} V \tag{97}$$

Derivation. Consider two resistors R_1 and R_2 are connected in *series*. The *same* current flows through both resistors.

Hence p.d. across R_1 is given by

$$V_1 = IR_1 = \left(\frac{E}{R_1 + R_2}\right) R_1 = \left(\frac{R_1}{R_1 + R_2}\right) E$$

or

$$\frac{V_1}{E} = \frac{R_1}{R_T}$$

This means that the two resistors divide the total p.d. into fractions according to their resistance. □

Using the potential-divider principle for a continuous resistor (e.g. a resistance wire) which has constant resistance per unit length, the voltage drop across a section of the wire AB is proportional to the length of AB as a fraction to the wire's total length.

$$V_{AB} = \frac{\ell_{AB}}{\ell_T} \times V$$

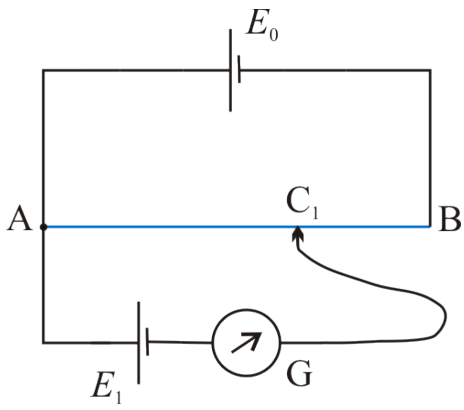
Consider a wire of non-uniform resistance per unit length, who resistance is described by the function $R(x)$. Integrating over the length of the wire L gives the total resistance of the wire:

$$R = \int_0^L R(x)\,\mathrm{d}x$$

where the length x is measured from one fixed end of the wire.

§15.3.1 Potentiometer

A **potentiometer** is a device used for comparing potential differences *without drawing a current*[10] from the circuit involved. It is used to measure the unknown e.m.f. of a source E_1, using another source of known e.m.f. E_0.



Proposition 15.1. Along the slide wire, $V \propto \ell$.

Proof. p.d. across AB is

$$V_{AB} = IR_{AB} = I\left(\frac{\rho \ell_{AB}}{A}\right)$$

Similarly, p.d. across AY is

$$V_{AC} = IR_{AC} = I\left(\frac{\rho \ell_{AC}}{A}\right)$$

Hence it is evident that

$$\frac{V_{AC}}{V_{AB}} = \frac{\ell_{AC}}{\ell_{AB}} \implies \boxed{V \propto \ell}$$

□

Remark. Recall that there needs to be a potential difference for current to flow.

Objective: Locate the **balance point** to determine **balance length**.

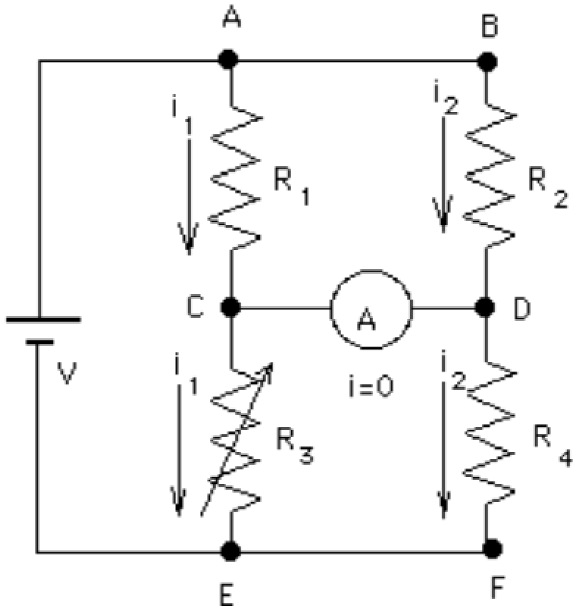
- At the balance point, No current flows through galvanometer. Galvanometer registers zero reading.
- This means no potential difference between point A and +ve terminal of E_1, and between point C and −ve terminal of E_1. Hence p.d. between A and C is equal to p.d. across E_1, thus $E_1 = V_{AC}$.
- Since p.d. is proportional to length across slide wire, we have

$$E_1 = V_{AC} = \frac{\ell_{AC}}{\ell_{AB}} V_{AB}$$

[10] This is in contrast to a voltmeter. A non–ideal voltmeter draws some current from the circuit as it does not have infinitely high resistance.

§15.3.2 Wheatstone bridge

The Wheatstone bridge is a way to measure the resistance of unknown resistors placed in the position of R_4, where the resistance of variable resistor R_3 is adjustable.



Since no current flows in the ammeter, no potential difference between points C and D. Hence $V_C = V_D$.

Since no current flows through the ammeter, $I_1 = I_3$ and $I_2 = I_4$. It also follows (from the fact that points C and D have the same potential) that the voltage drop across R_1 is the same as the voltage drop across resistor R_2. Hence

$$I_1 R_1 = I_2 R_2 \tag{1}$$

Similarly, voltage drop across R_3 is the same as voltage drop across R_4 so

$$I_1 R_3 = I_2 R_4 \tag{2}$$

Dividing (1) by (2) gives us

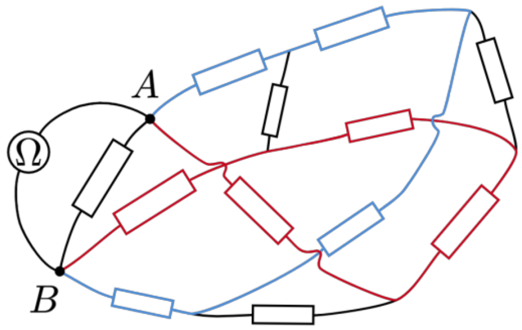
$$\frac{R_1}{R_3} = \frac{R_2}{R_4} \tag{3}$$

Solving (3) for the unknown resistor R_4,

$$R_4 = \frac{R_2}{R_1} R_3$$

Problems

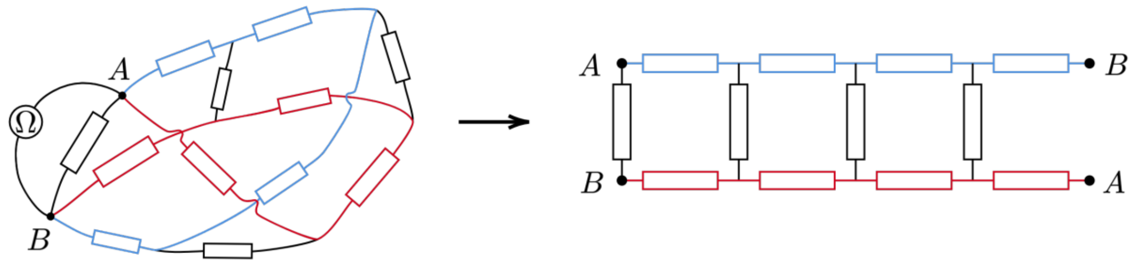
Problem 15.1 (Möbius Strip)**.** Two wires are each strung with 4 resistors, along with another 4 resistors that bridge pairs of resistors across both wires. The wires are then twisted together to form a Möbius strip, as shown below.



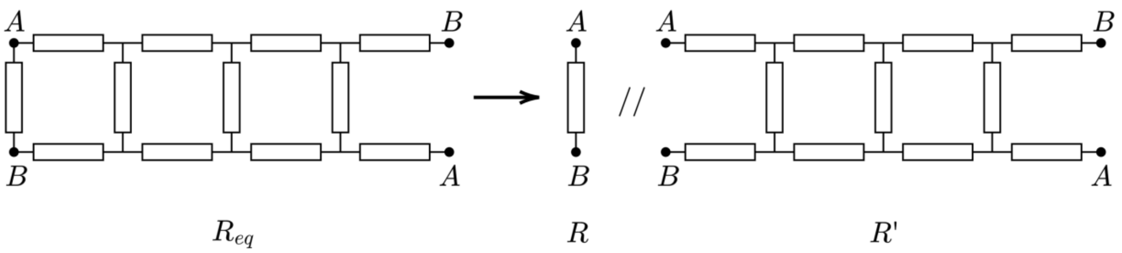
Every resistor has identical resistance $R = 1.0\,\Omega$. Determine the equivalent resistance R_{eq} between the points A and B in the Möbius strip.

Leave your answer to 2 significant figures in units of Ω*.*

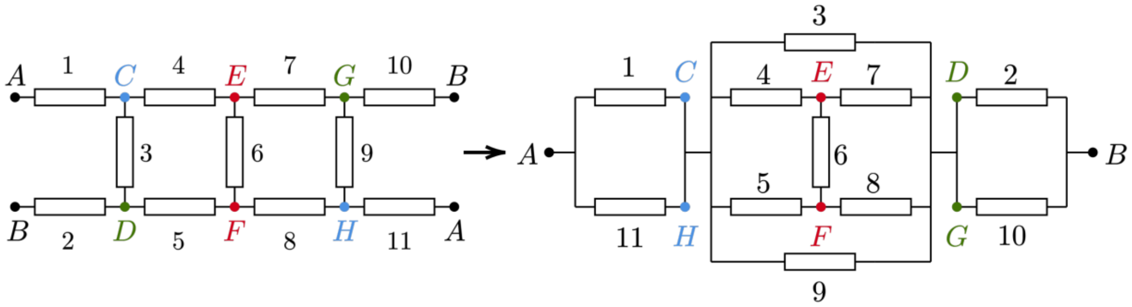
Solution. Imagine "slicing" the Möbius strip along line AB and then opening up and untwisting the circuit. The circuit can then be redrawn in its deconstructed form:



To determine the resistance R_{eq} across AB, we can treat the 1 bridging resistor, of resistance R, to be in parallel with the rest of the circuit, of unknown resistance R'.



Let us now focus on finding this unknown resistance R'. Notice that due to the symmetry of this circuit, there are 3 pairs of equipotential points as marked below (each pair takes a different colour). As such, we obtain the following equivalent circuit after combining points A and B on both ends:



From here, we can determine the value of R' (noting that resistor 6 can be disregarded since its two ends are equipotential):

$$R' = \frac{1}{\frac{1}{R} + \frac{1}{R}} + \frac{1}{\frac{1}{R} + \frac{1}{2R} + \frac{1}{2R} + \frac{1}{R}} + \frac{1}{\frac{1}{R} + \frac{1}{R}} = \frac{4}{3}R$$

We can hence calculate the resistance R_{eq} of the complete circuit:

$$R_{eq} = \frac{1}{\frac{1}{R} + \frac{1}{R'}} = \frac{4}{7}R \approx \boxed{0.57\,\Omega}$$

□

Problem 15.2 (Hexagonmania)**.** Roger is bored, so he decides to use his collection of uniform thin copper rods, each of resistance $R = 1.00\,\Omega$, to create a rigid compound shape shown below. The copper rods form seven regular hexagons. Calculate the effective resistance R_{AB} between points A and B.

Leave your answer to 3 significant figures in units of Ω.

Solution. □

§16 Electromagnetism

§16.1 Concept of a magnetic field

Magnets produce magnetic fields.

- Hard magnetic materials like cobalt and nickel are difficult to magnetise but tend to retain their magnetism.
- Soft magnetic materials like iron are easily magnetised but tend to lose their magnetism easily.

Definition 16.1: Magnetic field

Region of space where a magnetic pole, a current-carrying conductor or a moving charge particle will experience a force.

A magnetic field can be produced by:

1. permanent magnets
2. current-carrying conductors

§16.1.1 Magnetic field lines

- The field is represented by lines of force, starting from the North pole and ending at the South pole.
- The tangent to the magnetic field line at a point in the magnetic field gives the direction of the field at that point.
- The number of lines per unit cross section area is an indication of the strength of the field.

Magnetic field strength at a point is denoted by B.

Arrow: Magnetic field acting in direction of arrow

Cross: Magnetic field acting into plane of paper

Dot with a circle: Magnetic field acting out of the plane of paper

Use right hand rule to determine direction of magnetic field due to current.

§16.1.2 Magnetic Flux Patterns

Due to current in long straight wire magnetic flux density B:

$$B = \frac{\mu_0 I}{2\pi r} \tag{98}$$

where μ_0 is the vacuum magnetic permeability constant.

Due to current in flat circular coil magnetic flux density B:

$$B = \frac{\mu_0 N I}{2r} \tag{99}$$

Due to current in long solenoid magnetic flux density B:

$$B = \mu_0 n I \tag{100}$$

§16.2 Magnetic force

§16.2.1 Force on a current-carrying conductor

Direction of the force can be determined using Fleming's left hand rule.

Magnitude of the force:

$$F = BIL\sin\theta \tag{101}$$

where θ is the angle between the magnetic field vector and the direction of the current.

In vector form:

$$\mathbf{F} = \mathbf{IL} \times \mathbf{B}$$

Definition 16.2: Magnetic flux density B

Force acting per unit length of a conductor which carries unit current and is of right angles to the magnetic field.

$$B = \frac{F}{IL\sin\theta}$$

Tesla is the unit of magnetic flux density equivalent to a force of 1 N experienced by a straight conductor of length 1 m and carrying a current of 1 A when it is placed perpendicular to the magnetic field.

§16.2.2 Force between current-carrying conductors

Currents in same direction: attract each other

Currents in opposite direction: repel each other

Force of interaction:

$$F \propto \frac{I_1 I_2}{d}$$

$$F_1 = BI_1 L = \frac{\mu_0 I_2}{2\pi r} I_1 L \tag{102}$$

§16.2.3 Force on a moving charge

As current consists of moving charges, it can be deduced that a moving charged particle also experiences an electromagnetic force. Consider a charge q travelling at constant speed v at an angle θ to magnetic field of flux density B.

Assume the charge travels a distance in time t, so $v = \dfrac{L}{t}$ thus $L = vt$.

Equation for force on conductor $F = BIL\sin\theta$ can be rearranged as

$$F = B\left(\frac{q}{t}\right)(vt)\sin\theta$$
$$F = Bqv\sin\theta \tag{103}$$

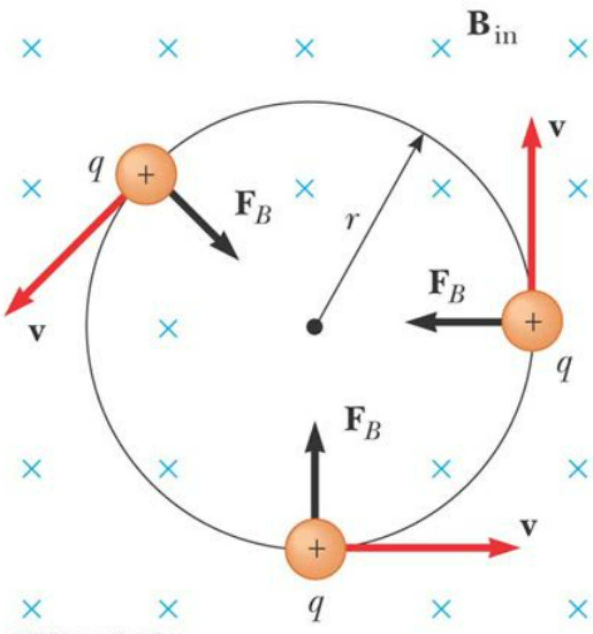
§16.2.4 Circulating Charge

If v and B are *perpendicular*, force will make the charge undergo uniform circular motion.

Magnetic force provides centripetal force:

$$F_B = F_c \implies Bqv = \frac{mv^2}{r} \implies \boxed{r = \frac{mv}{Bq}}$$

This means that for larger v, r is larger, and vice versa.



Period of revolution:

$$T = \frac{2\pi r}{v} = \frac{2\pi\left(\frac{mv}{Bq}\right)}{v} = \frac{2\pi m}{Bq}$$

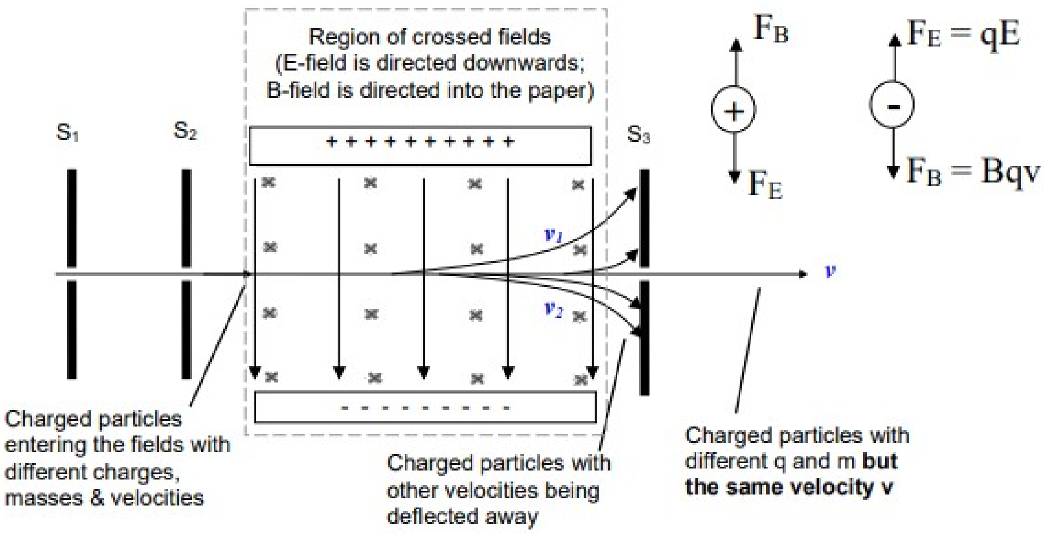
which is independent of v.

If v and B are not perpendicular, $0° < \theta < 90°$, charge moves in a helical path (i.e. it spirals forward).

§16.2.5 Use of Crossed Fields

Uniform E and B fields could be set up perpendicular to each other such that they exert equal forces of opposite directions on a moving charged particle. This setup may be referred to as crossed fields.

A **velocity selector** emits a stream of charged particles (e.g. electrons) of a specific velocity.



A beam of charged particles with a range of velocities $v_1, v_2, \ldots, v_n$ pass through a region where there is a crossed field.

If the charged particles were electrons, then each electron in the crossed field experiences an upward electric force, and a downward magnetic force. (For positively charged particles: a downward electric force and an upward magnetic force)

For the particles to pass through undeflected, electric force and magnetic force are equal in magnitude:

$$F_B = F_E \implies Bqv = qE \implies \boxed{v = \frac{E}{B}}$$

Problems

§17 Electromagnetic Induction

Electromagnetic induction is the phenomenon where an e.m.f. is induced due to a changing magnetic field.

§17.1 Magnetic flux

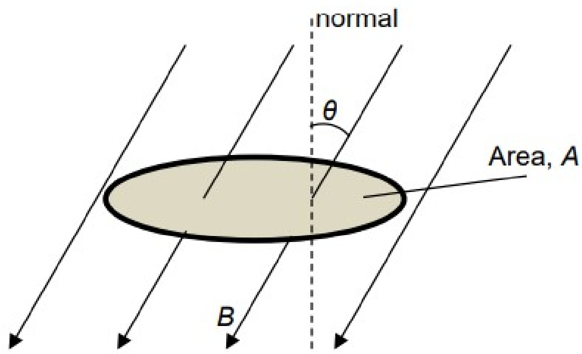
Definition 17.1: Magnetic flux Φ

Product of the component of the magnetic field density normal to the plane of the surface and the area of the surface.

$$\Phi = B_{\perp}A = BA\cos\theta \tag{104}$$

where θ is the angle between the normal of the plane and the magnetic field.

Unit of magnetic flux is the weber (Wb). **Weber** is the flux of a uniform magnetic field B of flux density 1 T, through a plane surface of area A of 1 m^2, placed normally to the B field.



If area A is bounded by a coil and the coil has N turns, then the total magnetic flux passing through the coil (**magnetic flux linkage** through the coil) is

$$N\Phi = NBA\cos\theta \tag{105}$$

From the equation, magnetic flux linkage through a coil depends on

- number of turns in coil N
- magnitude of magnetic flux density B
- surface area A
- orientation of the coil θ with respect to the direction of B (θ is the angle between the magnetic flux density and the normal of area of coil)

Definition 17.2: Magnetic flux linkage

Product of magnetic flux passing through the coil and number of turns on the coil.

§17.2 Laws of electromagnetic induction

It was discovered experimentally that a changing magnetic field could induce an electric current in a circuit.

Definition 17.3: Faraday's law

When the magnetic flux linkage with a circuit is changed, an induced e.m.f. is set up whose magnitude is directly proportional to rate of change of magnetic flux linkage.

$$\varepsilon \propto \frac{\mathrm{d}(N\Phi)}{\mathrm{d}t}$$

Definition 17.4: Lenz's law

Induced e.m.f. (and hence current flow in a closed circuit)[a] is in a direction so as to produce effects which oppose the change that produces it.

$$\varepsilon = -\frac{\mathrm{d}(N\Phi)}{\mathrm{d}t} = -\frac{\mathrm{d}(NBA\cos\theta)}{\mathrm{d}t} \tag{106}$$

[a] A current is induced only if there is a complete circuit. An e.m.f. is always induced when there is a change in flux linkage.

Lenz's Law is a consequence of the principle of conservation of energy.

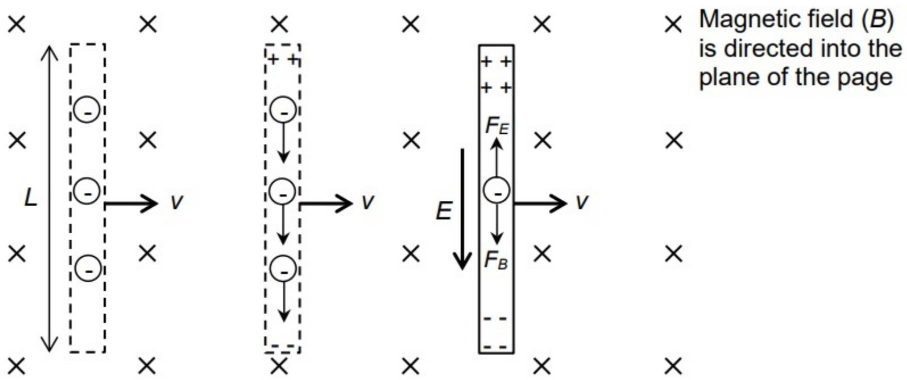
- As the external agent brings the magnet towards the coil, by Lenz's law, a current is induced in such a direction that the coil opposes, (i.e. repels) the approaching magnet.
- Consequently, work has to be done by the external agent to overcome this opposition (the repulsive force).
- It is this work done which is the source of the electrical energy.

Fleming's Right Hand rule: determine direction of induced e.m.f. (and hence current in closed circuit).

- seCond finger represents Current
- First finger represents magnetic Field
- thumb represents force (or direction of motion)

§17.2.1 Motional e.m.f. (Cutting of magnetic field)

Motional e.m.f. is the e.m.f. induced in a conductor moving through a uniform magnetic field.



- When a conductor (wire) moves at a constant velocity in a uniform magnetic field, the moving conductor cuts magnetic field (or flux). An induced e.m.f. is generated when there is a cutting of flux.
- When the conductor is pushed to move to the right, free electrons in the conductor move to the right, so current flows to the left. Based on Fleming's left-hand rule, each electron experiences downward magnetic force.
- Under the influence of this force, electrons move to the lower end of the conductor and accumulate there, leaving a net positive charge at the upper end. As a result of charge separation, electric field (directed downward) is produced inside the conductor.
- Charges accumulate at both ends until downward magnetic force is balanced by upward electric force.

$$F_E = F_B \implies qE = qvB \implies E = vB$$

Since the electric field in the conductor is uniform, potential difference across the ends of conductor of length L is given by $\Delta V = EL$ thus

$$\varepsilon = BLv \tag{107}$$

Therefore an e.m.f. ε is induced as long as the conductor continues to move through the uniform magnetic field. If the direction of the motion is reversed, polarity of induced e.m.f. is also reversed.

In summary, induced e.m.f. is also proportional to the rate of flux cutting.

For a moving conductor on a circuit with a resistance R connected in series,

$$\Delta V = BLv \implies \boxed{I = \frac{BLv}{R}}$$

§17.3 Applications

§17.3.1 Generator

Electric generators take in energy by work and transfer it out by electrical transmission.

§17.3.2 Eddy currents

When a conductor is subjected to a changing magnetic flux, induced e.m.f. causes currents to flow. If the conductor is in the shape of a loop, induced current flows around the loop. However if the conductor is a solid plate, induced currents, known as **Eddy currents**, flow simultaneously along many different paths in swirls.

This can be demonstrated by allowing a flat copper or aluminium plate attached to the end of a rigid bar to swing back and forth through a magnetic field. As the plate enters the field, the plate cuts the magnetic field/flux. The changing magnetic flux induces an e.m.f. in the plate. The field in the plate is not uniform and the rate of cutting is not the same over the whole plate, so different e.m.f. are induced in different parts of plate, leading to eddy currents in the plate.

[figure]

As the swinging plate enters the field at position 1, the flux due to the external magnetic field into the page through the plate is increasing, hence by Lenz's law the induced eddy current must provide its own magnetic field out of the page. Direction of the induced current is thus anticlockwise. The opposite is true as the plate leaves the field at position 2, where the current is clockwise.

Applications set up a braking system which can rapidly change kinetic energy to other forms of energy. This can be taken a step further if a circuit can be built to channel the electrical energy from the kinetic energy back into the battery. This is what most hybrid cars do.

Stop rollercoasters, galvanometers, voltmeters and ammeters.

Drawbacks Eddy currents are dissipated as heating in the conductor (i.e. the conductor gets heated up).

Loss of energy in applications such as a transformer, as induced currents do work and raise the temperature of the iron core and cause energy loss.

Eddy currents can be reduced by eliminating paths for the current flow, for example, by cutting slits in the plate. These slits will prevent large eddy currents from occurring.

Problems

§18 Alternating Current

Definition 18.1: Alternating current (a.c.)

Current that varies periodically with time in magnitude and direction.
(Polarity of the voltage source constantly changes)

§18.1 Characteristics

Period T: time taken for the current to undergo one complete cycle.

Frequency f: number of complete cycles undergone by the current per unit time.

Angular frequency ω: frequency in terms of radians per unit time rather than cycles per unit time.

$$\omega = \frac{2\pi}{T}$$

Peak value I_0: maximum magnitude of the current attained in each cycle.

Peak-to-peak value I_{pp}: difference between the maximum and minimum values of the current within one cycle.

For a sinusoidal wave,

$$I_{pp} = 2I_0$$

Mean Value $\langle I \rangle$: average value of a.c. over a given time interval.

Root mean square value $I_{\text{r.m.s.}}$: value of alternating current that is equal to the steady direct current which would dissipate heat at the same average rate in a given resistor.

The most commonly encountered form of a.c. is the **sinusoidal** form, that is, it varies with time according to a sine or cosine function.

$$I = I_0 \sin \omega t \tag{108}$$

Similarly, potential difference across the resistor is given by

$$V = V_0 \sin \omega t$$

§18.2 Transformer

§18.2.1 Functioning

§18.2.2 Turns ratio

§18.2.3 Power loss

Practical transformers lose power through

- **Joule heating**

  The wires used for the windings of the coils have resistance and so heating occurs, resulting in power loss $P = I^2R$. Thicker wires made of material with low resistivity (i.e. high purity copper) are used to reduce this power loss.

- **Eddy currents**

  The alternating magnetic flux induces eddy currents in the iron core and cause heating. This effect is reduced by laminating the iron core. Laminations reduce the area of circuits in the core, and thus reduce the e.m.f. induced and current flowing within the core, which leads to a reduction in the energy lost.

- **Hysteresis loss**

  Magnetisation of the core is repeatedly reversed by the alternating magnetic field. The energy required to magnetise the core (while the current is increasing) is not entirely recovered during demagnetisation. The difference in energy is lost as heat in the core. The energy loss is kept to a minimum by using a magnetic material with low hysteresis loss.

- **Flux leakage**

  The flux due to the primary may not all link to the secondary coil if the coil is badly designed or has air gaps in it. When flux is “leaked “to the surrounding, power is loss and thus not all the power from the primary coil can be transferred to the secondary coil.

§18.3 Rectification with a diode

Problems

Part VI

Modern Physics

§19 Quantum Physics

§19.1 Photoelectric effect

Definition 19.1: Photoelectric effect

Emission of electrons from metal surface when electromagnetic radiation of sufficiently high frequency is incident on it.

Experimental observations from the photoelectric effect experiment:

1. No electrons are emitted if the frequency of the EM radiation is below a minimum frequency (called the threshold frequency f_0), regardless of the intensity of the radiation.
2. Photoelectric current is proportional to the intensity of radiation, for a fixed frequency (because the rate of emission of electrons $\propto$ rate of incidence of photons)
3. Max KE of photo-electrons depends only on the frequency and the work function ϕ of the metal used, not the intensity. (Note: Emitted electrons have a range of kinetic energy, ranging from zero to a certain maximum value.)
4. Emission of electrons begins instantaneously (i.e. no (measurable) time lag between emission and illumination) even if the intensity is low.

(1), (2) and (3) cannot be explained by Classical Wave Theory of Light; they provide evidence for the particulate[11] nature of EM radiation.

Failure of the classical wave theory to explain the photoelectric effect

- According to the "Particle Theory of Light", EM radiation consists of a stream of particles/ photons/ discrete energy packets, each of energy hf.
- An electron is ejected when a single photon of sufficiently high frequency, transfers ALL its energy in a discrete packet to the electron.
- According to equation, $hf - \phi = \frac{1}{2}m_e v^2$, if the energy of the photon hf < the minimum energy required for emission (ϕ), no emission can take place, no matter how intense the light may be. Explains observation (1)
- This also explains why, (even at very low intensities), as long as $hf > \phi$, emission takes place without a time delay between illumination of the metal and ejection of electrons. Explains observation(4)

[11] particle-like

§19.2 Energy of a photon

Particulate nature of electromagnetic radiation:

- Electromagnetic radiation can be said to be particulate in nature in addition to being waved in nature.
- A beam of light consists of small discrete quanta of electromagnetic energy known as photons.
- Photons transfer either all or none of their energy to another particle instantaneously, contrary to the wave theory which states that the energy transfer is continuous.

Definition 19.2: Photon

A discrete packet (or quantum) of energy of an electromagnetic radiation with energy hf.

Energy of a photon is given by

$$E = \hbar f \tag{109}$$

where **Planck constant** $\hbar = 6.63 \times 10^{-34}$ J s.

§19.3 Wave-particle duality

Waves can exhibit particle-like characteristics and particles can exhibit wave-like characteristics.

de Broglie wavelength of a particle:

$$\lambda = \frac{h}{p} \tag{110}$$

Packets of EM radiation of wavelength λ would therefore possess a momentum $p = \frac{h}{\lambda}$. When photons are incident on a surface, they therefore exert a force on the surface, resulting in a pressure on the surface. This pressure is known as "radiation pressure".

Using $KE = \frac{p^2}{2m}$, wavelength of a particle can be related to its KE by

$$\lambda = \frac{h}{\sqrt{2m(KE)}} \tag{111}$$

§19.4 Energy levels in atoms

§19.5 Line spectra

§19.6 X-ray spectra

§19.7 Uncertainty principle

Definition 19.3: Heisenberg position-momentum uncertainty principle

It is impossible to measure the exact position and momentum of a body at the same time. It can be expressed as

$$\Delta p \Delta x \geq \hbar \tag{112}$$

where Δp and Δx denote the uncertainties in the momentum and position of the particle respectively.

Problems

§20 Nuclear Physics

§20.1 The nucleus

§20.2 Isotopes

§20.3 Nuclear processes

§20.4 Mass defect and nuclear binding energy

§20.5 Radioactive decay

§20.6 Biological effects of radiation

Problems

§A Derivations

§A.1 Kinematics

Equations of motion

$$v = u + at$$

Derivation. This is from the definition of acceleration. □

$$s = \frac{1}{2}(u + v)t$$

Derivation. Computing the area under a velocity-time graph (which has the shape of a trapezoid) gives the displacement. □

$$s = ut + \frac{1}{2}at^2$$

Derivation. Substitute eq. (6) into eq. (7) to remove v. □

$$v^2 = u^2 + 2as$$

Derivation. Rewrite eq. (6) to give $t = \dfrac{v - u}{a}$ which we can substitute into eq. (7) to remove t. □

§A.2 Forces

Spring constant

For springs in parallel,

$$k_{\text{eff}} = \sum_i k_i$$

Derivation. Extension of all springs is the same, total force is the sum of forces acting on all springs.

$$F = \sum_i F_i \implies k_{\text{eff}} x = \sum_i F_i x \implies k_{\text{eff}} = \sum_i k_i$$

□

For springs in series,

$$\frac{1}{k_{\text{eff}}} = \sum_i \frac{1}{k_i}$$

Derivation. Force acting on all springs is the same, total extension is the sum of extensions of all springs.

$$x = \sum_i x_i \implies \frac{F}{k_{\text{eff}}} = \sum_i \frac{F}{k_i} \implies \frac{1}{k_{\text{eff}}} = \sum_i \frac{1}{k_i}$$

□

Pressure

$$P = \rho g h$$

Derivation. Given a liquid column of height h and cross-sectional area A, of density rho.

$$m = \rho V = (Ah)\rho$$

Weight W of the liquid column above A is

$$W = mg = \rho V g = Ah\rho g$$

Hence pressure on area A is given by

$$\rho = \frac{F}{A} = \frac{Ah\rho g}{A} = \rho g h$$

□

Upthrust

$$U = W_{\text{displaced}}$$

Derivation. Consider a solid cylinder of height h and cross-sectional area A, submerged in a liquid of density ρ.

Pressure on the top surface is given by

$$p_1 = \rho g h_1 + p_0$$

Hence downward force on top surface is

$$F_1 = (\rho g h_1 + p_0)A$$

Similarly, pressure on the bottom surface is given by

$$p_2 = \rho g h_2 + p_0$$

Upward force on bottom surface is

$$F_2 = (\rho g h_2 + p_0)A$$

Hence, the resultant upward force (upthrust) on the cylinder is

$$\begin{aligned} U &= F_2 - F_1 \\ &= \rho g(h_2 - h_1)A \\ &= \rho g h A \\ &= \rho g V_{\text{displaced}} \\ &= m_{\text{displaced}} g \end{aligned}$$

which is equal to the weight of fluid displaced by the object. □

§A.3 Work, Energy, Power

Work done by gas

$$W = p\Delta V$$

Derivation. Consider gas at pressure p in a syringe which has a frictionless piston of cross-sectional area A, then the force exerted by gas on piston is $F = pA$. If the gas expands slowly (pressure of gas remains constant) against a constant external pressure moving outwards a displacement s, then force F is constant.

Work done by gas in expanding from V_1 to V_2 is given by

$$W = Fs = (pA)s = p(As) = p\Delta V = p(V_2 - V_1)$$

When the gas expands ($V_2 > V_1$), work done by gas is positive.

When the gas expands ($V_2 < V_1$), work done by gas is negative. □

Gravitational potential energy

$$\text{GPE} = mgh$$

Derivation. Consider an object being raised from height h_1 to height h_2 by a constant force F equal and opposite to the weight mg of the object (so that object does not gain KE).

$$F = mg$$

Work done by force F changes gravitational potential energy, is given by

$$\begin{aligned} W &= F\Delta h \\ &= F(h_2 - h_1) \\ &= mg(h_2 - h_1) \\ &= mgh_2 - mgh_1 \\ &= \text{GPE}_f - \text{GPE}_i \end{aligned}$$

Therefore, gravitational potential energy is $\text{GPE} = mgh$. □

Kinetic energy

$$\text{KE} = \frac{1}{2}mv^2$$

Derivation. Consider a stationary body of mass m which moves a horizontal displacement s under the action of a constant net force F. Since the force is constant, body moves with constant acceleration a.

By Newton's 2nd Law of Motion,

$$F = ma$$

The final velocity v of the body is given by

$$v^2 = u^2 + 2as \implies v^2 = 2as \implies s = \frac{v^2}{2a}$$

Hence, work done on the body is

$$W = Fs = ma\left(\frac{v^2}{2a}\right) = \frac{1}{2}mv^2$$

Work done by force F increases the kinetic energy of the body.

Therefore, the kinetic energy of a body at speed v is $\mathrm{KE} = \dfrac{1}{2}mv^2$. □

Instantaneous power

$$P = Fv$$

Derivation.

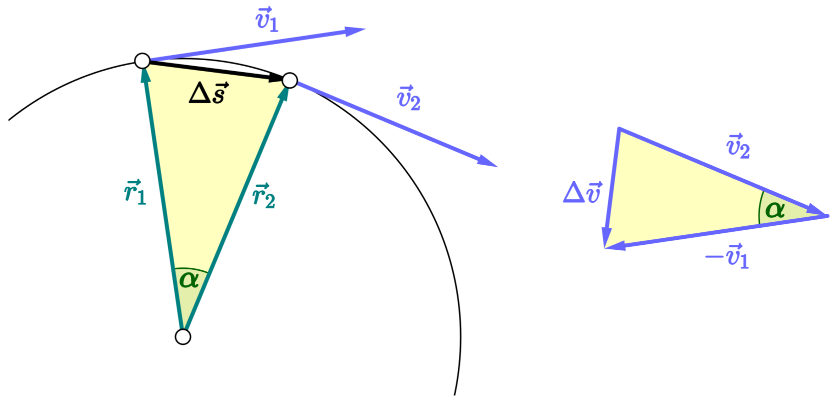
$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}(Fs)}{\mathrm{d}t} = F\frac{\mathrm{d}s}{\mathrm{d}t} = Fv$$

□

§A.4 Circular motion

$$a = \frac{v^2}{r} = r\omega^2 \tag{113}$$

Derivation. The two velocity vectors v_1 and v_2 can be rearranged to form a triangle which is similar to the triangle formed by the radii r_1, r_2 and displacement s.



Using similar triangles,

$$\frac{\Delta s}{r} = \frac{\Delta v}{v} \implies \frac{\Delta s}{\Delta t} \cdot \frac{1}{r} = \frac{\Delta v}{\Delta t} \cdot \frac{1}{v}$$

Taking limits where $\Delta t \to 0$,

$$\frac{1}{r}\left(\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}\right) = \frac{1}{v}\left(\lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}\right) \implies \frac{1}{r}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}t} \implies \frac{v}{r} = \frac{a}{v} \implies a = \frac{v^2}{r}$$

□

§A.5 Gravitational Field

Gravitational field strength

$$g = -\frac{GM}{r^2}$$

Derivation. Newton's Law of Gravitation states that

$$F = -\frac{GMm}{r^2}$$

By the definition of gravitational field strength,

$$g = \frac{F}{m} = \frac{-\frac{GMm}{r^2}}{m} = -\frac{GM}{r^2}$$

□

Gravitational field strength near the surface

Derivation. Consider gravitational field strength at height h above the surface of Earth. Near the surface of Earth, h is small compared to the radius R of Earth, i.e. $h \ll R$.

$$g = \frac{GM}{(h+R)^2} \approx \frac{GM}{R^2}$$

□

Gravitational potential energy

$$U = -\frac{GMm}{r}$$

Derivation. Consider moving a mass m from infinity to a point at distance r from a mass M at *constant speed* (so that kinetic energy of mass m does not change).

From the definition of gravitational potential energy,

$$U = \int_{\infty}^{r} \vec{\mathbf{F}}_{\text{ext}} \, \mathrm{d}r$$

Since the external force acts in the opposite direction of gravitational force, $\vec{\mathbf{F}}_{\text{ext}} = -\vec{\mathbf{F}}_g$.

$$U = \int_{\infty}^{r} -\vec{\mathbf{F}}_g \, \mathrm{d}r = \int_{\infty}^{r} \frac{GMm}{r^2} \, \mathrm{d}r$$

Computing the integral,

$$U = \left[-\frac{GMm}{r}\right]_{\infty}^{r} = -\frac{GMm}{r}$$

□

§A.6 Temperature and Ideal Gases

Kinetic theory of gas

$$pV = \frac{1}{3}Nm\langle c^2\rangle$$

Derivation. Consider an ideal gas consisting of N identical molecules, each of mass m, in a cubical container of side length L.

Since the molecules move randomly, they do not have any preferred direction of travel along the x-, y- and z-axes. We expect that one-third of the N molecules move along each axis.

Consider a one-dimensional case along the x-axis. One gas molecule of mass m approaches and collides elastically with the wall with velocity c_x, and leaves the wall with velocity $-c_x$. Change in momentum of gas molecule is given by

$$\Delta p = -2mc_x$$

By conservation of linear momentum, wall experiences change in momentum of

$$\Delta p_{\text{wall}} = 2mc_x$$

After the collision, assume this molecule continues its motion uninterrupted. It travels a total distance of $2L$ back and forth, and collides with the same wall. The time interval between successive collisions is given by

$$\Delta t = \frac{2L}{c_x}$$

Rate of change of momentum of wall due to one molecule is given by

$$\frac{\Delta p_{\text{wall}}}{\Delta t} = -\frac{m{c_x}^2}{L}$$

By Newton's 2nd law, net force on wall is rate of change of momentum of wall due to all N molecules, given by

$$\begin{aligned} F_{\text{wall}} &= \frac{m{c_{x1}}^2}{L} + \cdots + \frac{m{c_{xN}}^2}{L} \\ &= \frac{m}{L}\left({c_{x1}}^2 + \cdots + {c_{xN}}^2\right) \\ &= \frac{Nm}{L}\langle {c_x}^2\rangle \end{aligned}$$

where mean square speed $\langle {c_x}^2\rangle$ is defined as

$$\langle {c_x}^2\rangle = \frac{{c_{x1}}^2 + \cdots + {c_{xN}}^2}{N}$$

Since the area of the wall is L^2, pressure on wall is given by

$$p = \frac{F}{L^2} = \frac{Nm}{V}\langle {c_x}^2\rangle \tag{1}$$

since $V = L^3$.

Applying Pythagoras' Theorem to 3D velocity vector of molecule gives

$$c^2 = {c_x}^2 + {c_y}^2 + {c_z}^2 \implies \langle c^2 \rangle = \langle {c_x}^2 \rangle + \langle {c_y}^2 \rangle + \langle {c_z}^2 \rangle$$

Since there are $\dfrac{N}{3}$ molecules moving along each axis, $\langle {c_x}^2 \rangle = \langle {c_y}^2 \rangle = \langle {c_z}^2 \rangle$. Hence

$$\langle {c_x}^2 \rangle = \frac{1}{3}\langle c^2 \rangle$$

Substituting this into (1) gives us

$$pV = \frac{1}{3}Nm\langle c^2 \rangle$$

as desired. □

Remark. How to remember the derivation:

- Context, find pressure $P = \frac{F}{A}$
- Origin of force F: collision of one, N gas molecules with wall. Relate rate of change of δp to $F = \frac{\mathrm{d}p}{\mathrm{d}t}$
- Generalise for N molecules, apply mean square speed

Remark. The root mean square is an estimation of a statistical "average". Statistically, the **root mean squared speed** of gas molecules means taking the square root of the sum of the squares of the speeds of all gas molecules.

$$c_{\text{r.m.s.}} = \sqrt{\frac{{c_1}^2 + {c_2}^2 + \cdots + {c_n}^2}{n}}$$

§A.7 Oscillations

Velocity

$$v = \pm\omega\sqrt{{x_0}^2 - x^2}$$

Derivation. Displacement is given by

$$x = x_0 \sin \omega t$$

By differentiation, velocity is given by

$$v = \omega x_0 \cos \omega t$$

Using Pythagorean identity $\sin^2 x + \cos^2 x = 1$, the given equation can be easily derived. □

§A.8 Wave Motion

Wave speed

$$v = f\lambda$$

Derivation. One wavelength λ is distance travelled by wave during one complete oscillation of the source.

Distance travelled by wave during n complete oscillations is $n\lambda$.

Speed of wave is distance per unit time, $v = \dfrac{n\lambda}{t}$.

Frequency is number of oscillations per unit time, $f = \dfrac{n}{t}$.

Hence

$$f\lambda = \left(\frac{n}{t}\right)\lambda = \frac{n\lambda}{t} = v.$$

$\square$

§A.9 Current of Electricity

Transport equation

$$I = nAvq$$

Derivation. Consider a current I passing through a section of a wire of cross-sectional area A.

We define

- n as number density of charge carriers (number per unit volume)
- q as amount of charge of each charge carrier
- v as drift velocity of charge carriers

$$I = \frac{Q}{t} = \frac{Nq}{t} = \frac{nVq}{t} = \frac{nAxq}{t} = nAvq$$

□

§B Summary of Key Quantities, Symbols and Units

Quantity	Symbol	Unit
Displacement (or equivalent)	s, x	m
Velocity	v	m.s^{-1}
Acceleration	a	m.s^{-2}
Force	F	N
Weight	W	N
Normal force	N	N
Tension	T	N
Friction	f	N
Upthrust	U	N
Moment	M	N.m
Work done	W	J
Energy	E	J
Power	P	W
Efficiency	η	
Angular displacement	θ	rad
Angular velocity	ω	rad.s^{-1}
Centripetal acceleration	a	m.s^{-2}
Centripetal force	F_c	N
Period	T	s
Frequency	f	Hz
Gravitational force	F_g	N
Gravitational field strength	g	N.kg^{-1}
Gravitational potential energy	U	J
Gravitational potential	ϕ	J.kg^{-1}

Quantity	**Symbol**	**Unit**
Pressure	p	Pa
Volume	V	m^3
Moles (amount)	n	mol
Temperature	T	K
Number of gas particles	N	–
Mass of gas	M	kg
Molar mass	M_r	$g\,mol^{-1}$
Molar volume	V_m	m^3
Amplitude	x_0	m
Angular frequency	ω	$rad\,s^{-1}$
Phase	ϕ	rad
Phase difference	$\Delta\phi$	rad
Current	I	A
Potential difference	V	V
Electromotive force	ε	V
Resistance	R	Ω
Resistivity	ρ	$\Omega\,m$