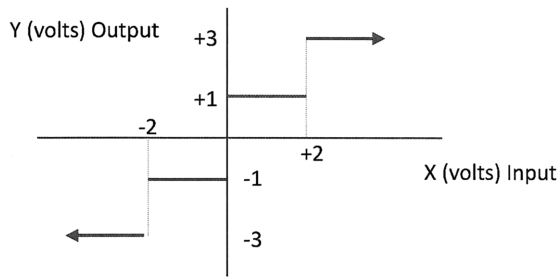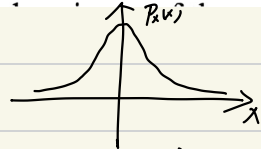1. (a) Consider a random variable X which is zero mean Gaussian distributed with unit variance $\sigma^2 = 1$. Suppose X is applied to the uniform quantization with the output Y shown below in Figure 1.

(i) Evaluate and sketch the probability density function (PDF) of the output Y.

Answer (i) $X \sim N(0,1)$

$P[Y=3] = P[X \geq 2] = Q(2) = 2.275 \times 10^{-2}$

$P[Y=-3] = 2.275 \times 10^{-2}$

$P[Y=1] = P[0 \leq X < 2] = \frac{1}{2} \cdot Q(2) = 4.7725 \times 10^{-1}$

$P[Y=-1] = P[Y=1] = 4.7725 \times 10^{-1}$

Y	-3	-1	1	3
P_i	2.275×10^{-2}	4.7725×10^{-1}	4.7725×10^{-1}	2.275×10^{-2}

(ii) Determine the mean and variance of the output Y.

$E[Y] = 0$

$D[Y] = E[Y^2] - E^2[Y] = E[Y^2] = 9 \times 2 \times 2.275 \times 10^{-2} + 1 \times 2 \times 4.7725 \times 10^{-1} = 1.364$

(iii) Design a suitable symbol mapping for the source encoder above. If a source message m(t) is applied to this input quantizer in Figure 1. If it is sent PCM transmission with available bit rate 50×10^6 bits per sec (bps), determine the maximum message bandwidth W Hz for the system to work satisfactory.

Y	-3	-1	1	3
symbol	00	01	10	11

$R_{max} = 5 \times 10^7 \text{ bps}$

$R_s = \frac{5 \times 10^7 \text{ bps}}{2 \text{ bps}/\text{symbols}} = 2.5 \times 10^7 \text{ symbols}/s$

$R_s/W = 2 \qquad W = \frac{R_s}{2} = 1.25 \times 10^7 \text{ Hz}$

(b) A binary channel with the bit rate $R_b = 36000$ bits per second (bps) is available for PCM audio transmission. Find the appropriable values of the sampling rate f_s, the uniform quantization level L, and the binary digits n, assuming the maximum frequency contents of the source message is $f_M = 3.2$ kHz. Calculate the **peak** Signal-to-quantization Noise Ratio (SNR$_p$) for uniform quantization in terms of dB units. Explain the modification required for SNR$_a$ when the average power of sinusoidal tone signal is used instead and evaluate this value.
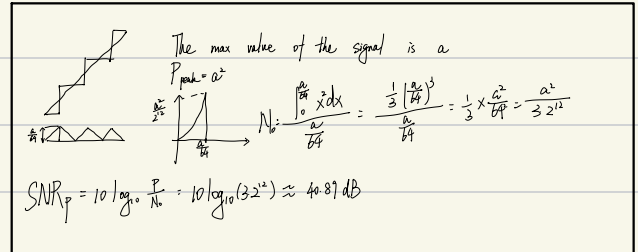
Answer:

$$f_s = 2f_M = 2 \times 3.2 \text{ kHz} = 6.4 \text{ kHz} = 6400 \text{ symbols/s}$$

$$R_b = 36000 \text{ bps}$$

$$\frac{R_b}{f_s} = 5.625$$

$$n = 5$$

$$L = 32$$

The max value of the signal is a

$$P_{peak} = a^2$$

$$N_0 = \frac{\int_0^{\frac{a}{64}} x^2 dx}{\frac{a}{64}} = \frac{\frac{1}{3}\left(\frac{a}{64}\right)^3}{\frac{a}{64}} = \frac{1}{3} \times \frac{a^2}{64^2} = \frac{a^2}{3 \cdot 2^{12}}$$

$$SNR_p = 10 \log_{10} \frac{P}{N_0} = 10 \log_{10}(3 \cdot 2^{12}) \approx 40.87 \text{ dB}$$

Assume X is a R.V. denotes the scope of amplitude of the audio signal.

$$X \sim U(-0.5, 31.5)$$
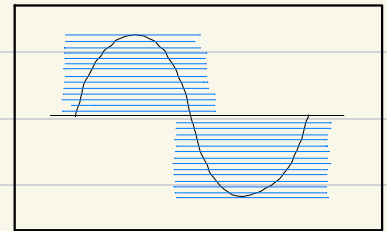
$$N_q = E\left[|X - [X]|^2\right]$$

$$Y = X - [X] \sim U(-0.5, 0.5)$$

$$E[Y] = 0 \qquad D[Y] = \frac{(b-a)^2}{12} = \frac{1}{12} = E\left[(Y-m_Y)^2\right] = E[Y^2] = \int_{-0.5}^{0.5} y^2 dy = \frac{1}{3} y^3\Big|_{-0.5}^{0.5} = \frac{1}{3}\left(\frac{1}{8} + \frac{1}{8}\right) = \frac{1}{12}$$

$$N_q = E[Y^2] = \frac{1}{12}$$

$$P_{peak} = 32^2$$

$$SNR_p = 10 \log_{10}\left(\frac{P_{peak}}{N_q}\right) = 40.87 \text{ dB}$$

Assume that U is an R.V. that denotes the phase.

$$U \sim U(0, 2\pi)$$

V is the R.V. that denotes the amplitude of sinewave.

$$V = \sin(U)$$

$$f_U(u) = \frac{1}{2\pi} \qquad 0 \leq u \leq 2\pi$$

$$u = \arcsin V$$

$$f_V(v) = f_U(u) \cdot \left|\frac{du}{dv}\right| = \frac{1}{2\pi} \frac{1}{\sqrt{1-v^2}} \qquad V \in [-1, 1]$$

$$W = 16V + 15.5 \qquad W \in [-0.5, 31.5] \qquad V = \frac{1}{16}W - 15.5$$

$$f_W(w) = f_V(v) \left|\frac{dv}{dw}\right| = \frac{1}{16} \cdot \frac{1}{2\pi} \frac{1}{\sqrt{1-\left(\frac{1}{16}w-15.5\right)^2}} = \frac{1}{32\pi} \frac{1}{\sqrt{1-(16w-15.5)^2}}$$
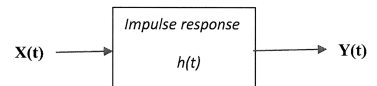
$$N = W - [W]$$

$$N_q = D[N]$$

$$SN_a = 10 \log_{10} \frac{P}{N_q} = 10 \log_{10} \frac{31.5^2}{N_q}$$

2. The input X(t) to a linear time invariant (LTI) system is a white noise with a two-sided power spectral density (PSD) $S_X(f) = N_0/2$ W/Hz. The impulse response $h(t)$ of the LTI system is

$$h(t) = \frac{1}{t_0} \exp\left(-\frac{\pi t^2}{t_0^2}\right),$$

Impulse response
X(t) → h(t) → Y(t)

with a parameter $t_0 > 0$.

(a) Derive the transfer function H(f) of the LTI system and autocorrelation function $R_Y(\tau)$ and the PSD $S_Y(f)$ of the random process Y(t) at the output of the LTI system.

Answer

$\exp(-\pi t^2) \longleftrightarrow \exp(-\pi f^2)$

$\exp\left(-\pi \frac{t^2}{t_0^2}\right) \longleftrightarrow t_0 \exp\left(-\pi t_0^2 f^2\right)$

$h(t) = \frac{1}{t_0} \exp\left(-\pi \frac{t^2}{t_0^2}\right) \longleftrightarrow H(f) = \exp\left(-\pi t_0^2 f^2\right)$

$S_Y(f) = |H(f)|^2 \cdot S_X(f) = \frac{N_0}{2} \exp\left(-2\pi t_0^2 f^2\right) = \frac{N_0}{2} \cdot H(\sqrt{2} f)$

$R_Y(\tau) = F^{-1}(S_Y(f)) = \frac{N_0}{2} \cdot \frac{1}{\sqrt{2}} \cdot h\left(\frac{\tau}{\sqrt{2}}\right) = \frac{N_0}{2\sqrt{2}} \cdot \frac{1}{t_0} \exp\left(-\pi \frac{\tau^2}{2 t_0^2}\right) = \frac{N_0}{2\sqrt{2} t_0} \cdot \exp\left(-\pi \frac{\tau^2}{2 t_0^2}\right)$

(b) Determine the noise equivalent bandwidth of the LTI system.

Answer.

$|H(f)|^2_{max} = 1$

$BW_{eq} = \frac{\int_{-\infty}^{+\infty} |H(f)|^2 df}{|H(f)|^2_{max}} = \int_{-\infty}^{+\infty} \exp\left(-2\pi t_0^2 f^2\right) df$

$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1 \qquad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-x^2/2\right) dx = 1$

$BW_{eq} = \int_{-\infty}^{+\infty} \exp\left(-(\sqrt{2\pi} t_0 f)^2\right) df = \int_{-\infty}^{+\infty} \exp\left[-\frac{1}{2}(2\sqrt{\pi} t_0 f)^2\right] df = \frac{1}{2\sqrt{\pi} t_0} \int_{-\infty}^{+\infty} \exp\left[-\frac{1}{2}(2\sqrt{\pi} t_0 f)^2\right] d(2\sqrt{\pi} t_0 f)$

$= \frac{1}{2\sqrt{\pi} t_0} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2} x^2\right) dx = \frac{1}{\sqrt{2} \cdot t_0}$

$BW_{eq} = \frac{1}{\sqrt{2} t_0}$

(c) Suppose there is a cascade of N identical LTI systems, so that the output Y(t) passed through each of these stages. Evaluate the noise equivalent bandwidth of this cascaded N stage overall system. Explain the significance of your result when the number of stages N increases.

Answer.

$\left[|H(f)|^2\right]^N_{max} = 1 \qquad \left[|H(f)|^2\right]^N = \exp\left(-2\pi N t_0^2 f^2\right)$

$BW_{eq} = \frac{\int_{-\infty}^{+\infty} \left[|H(f)|^2\right]^N df}{\left[|H(f)|^2\right]^N_{max}} = \int_{-\infty}^{+\infty} \exp\left(-2\pi N t_0^2 f^2\right) df = \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2} \cdot 4\pi N t_0^2 f^2\right) df = \int_{-\infty}^{+\infty} \exp\left[-\frac{1}{2}(2\sqrt{\pi N} t_0 f)^2\right] df$

$= \frac{1}{2\sqrt{\pi N} t_0 f} \int_{-\infty}^{+\infty} \exp\left[-\frac{1}{2}(2\sqrt{\pi N} t_0 f)^2\right] d(2\sqrt{\pi N} t_0 f)$

$= \frac{1}{\sqrt{2N} t_0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2} x^2\right) dx = \frac{1}{\sqrt{2N} t_0}$

4. (a) Given the following irreducible polynomial over the binary field,

$$g(X) = 1 + X + X^2 + X^4$$

   (i) Find the lowest-rate cyclic code whose generator polynomial is g(X). What is the rate of this code?

(i) Answer: $g(x) = 10111$

$$
\begin{array}{r}
1 \\
10111 \overline{\smash{)}10001} \\
\underline{10111} \\
110
\end{array}
\qquad
\begin{array}{r}
1 \\
10111 \overline{\smash{)}100001} \\
\underline{10111} \\
1\cdot11
\end{array}
\qquad
\begin{array}{r}
101 \\
10111 \overline{\smash{)}1000001} \\
\underline{10111} \\
11101 \\
\underline{10111} \\
1010
\end{array}
\qquad
\begin{array}{r}
1011 \\
10111 \overline{\smash{)}10000001} \\
\underline{10111} \\
11100 \\
\underline{10111} \\
10111 \\
\underline{10111} \\
0
\end{array}
$$

$$R = \frac{3}{7}$$

   (ii) Find the generator matrix, **G**, for this code in systematic form, and hence, determine the parity check matrix **H**.

(ii) (7,3)

$$
\begin{array}{r}
1 \\
10111 \overline{\smash{)}0010000} \\
\underline{10111} \\
0111
\end{array}
\qquad
\begin{array}{r}
10 \\
10111 \overline{\smash{)}0100000} \\
\underline{10\ 111} \\
11\ 10
\end{array}
\qquad
\begin{array}{r}
101 \\
10111 \overline{\smash{)}1000000} \\
\underline{10111} \\
11100 \\
\underline{10111} \\
1011
\end{array}
$$

$$G = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}
\qquad
H = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

   (iii) What is the error-correcting capability of this code?

Answer: $d_{min} = 4$

$$C = \left\lfloor \frac{d_{min}-1}{2} \right\rfloor = 1$$