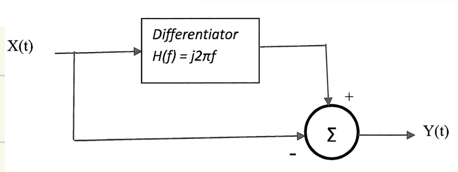
1. (a) X(t) is a stationary random process with autocorrelation function $R_X(\tau) = \exp(-\pi\tau^2)$. The input X(t) random process is passed through system shown in Figure 1 below. Determine the power spectral density of the output random process Y(t).



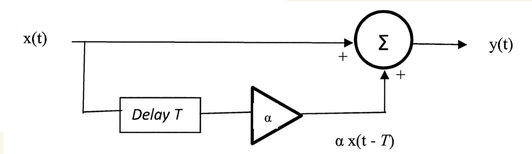
Answer: $R_X(\tau) = \exp(-\pi\tau^2) \longleftrightarrow X(f) = \exp(-\pi f^2)$

$|H(f)|^2 = 4\pi^2 f^2$

$Y(f) = [|H(f)|^2 - 1]\, X(f) = (4\pi^2 f^2 - 1)\exp(-\pi f^2)$

$\therefore$ PSD of $Y(t)$ is $Y(f) = (4\pi^2 f^2 - 1)\exp(-\pi f^2)$

(b) A multi-path transmission occurs when a transmitted energy signal x(t) arrives at the receiver, often by two or more paths of different delay T with a parameter α. A simple model for such a multipath communication channel is illustrated in Figure 2 below. The output of this channel is the received signal y(t).



(i) Find the impulse response h(t) and the transfer function $H(f)$ for this multi-path channel. Sketch the amplitude response $|H(f)|$ for α = 1 and α = 0.5.

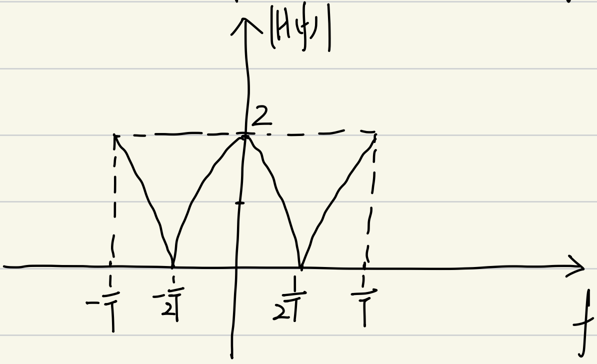
(i) Answer: $y(t) = x(t) + \alpha x(t-T) = h(t) \circledast x(t)$

$h(t) = \delta(t) + \alpha\,\delta(t-T)$

$H(f) = 1 + \alpha e^{-j2\pi fT}$

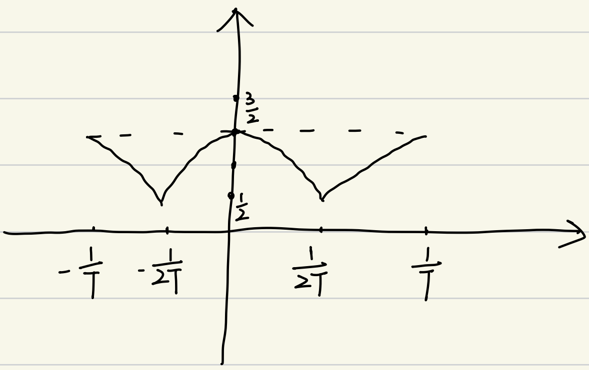
① $\alpha = 1$ $\quad H(f) = 1 + e^{-j2\pi fT} = (1+\cos 2\pi fT) - j\sin(2\pi fT)$

$|H(f)| = \sqrt{(1+\cos 2\pi fT)^2 + \sin^2 2\pi fT}$
$= \sqrt{2 + 2\cos 2\pi fT}$



② $\alpha = \frac{1}{2}$ $\quad H(f) = 1 + \frac{1}{2}e^{-j2\pi fT} = 1 + \frac{1}{2}\cos 2\pi fT - j\frac{1}{2}\sin 2\pi fT$

$|H(f)|^2 = \sqrt{(1+\frac{1}{2}\cos 2\pi fT)^2 + (\frac{1}{2}\sin 2\pi fT)^2} = \sqrt{1 + \frac{1}{4} + \cos 2\pi fT} = \sqrt{\frac{5}{4} + \cos 2\pi fT}$



(ii) Suppose $x(t) = \frac{1}{T}\,\mathrm{sinc}\left(\frac{t}{T}\right)$ is transmitted, derive the expression for the output energy spectral density $\psi_y(f)$ and the total energy **E** of the received signal y(t). When $\alpha = 0.5$, determine the value of T where the total energy at the output is unity.

Answer:

$$x(t) = \frac{1}{T}\,\mathrm{sinc}\left(\frac{t}{T}\right)$$

$$\mathrm{sinc}(2Wt) \longleftrightarrow \frac{1}{2W}\,\mathrm{rect}\left(\frac{f}{2W}\right)$$

$$x(t) \Longleftrightarrow X(f) = \mathrm{rect}(Tf)$$

$$\psi_x(f) = |X(f)|^2 = \mathrm{rect}(Tf)$$

$$H(f) = 1 + \alpha e^{-j2\pi fT}$$

$$|H(f)| = \sqrt{(1+\alpha\cos 2\pi fT)^2 + \alpha^2\sin^2 2\pi fT} = \sqrt{1+\alpha^2 + 2\alpha\cos 2\pi fT}$$

$$|H(f)|^2 = 1+\alpha^2 + 2\alpha\cos 2\pi fT$$

$$\psi_y(f) = \psi_x(f)\cdot|H(f)|^2 = \mathrm{rect}(Tf)\cdot\left[1+\alpha^2+2\alpha\cos 2\pi fT\right]$$

$$E = \int_{-\infty}^{+\infty}\psi_y(f)\,df = \int_{-\frac{1}{2T}}^{\frac{1}{2T}}\left(1+\alpha^2 + \underline{2\alpha\cos 2\pi fT}\right)df = \frac{1+\alpha^2}{T}$$

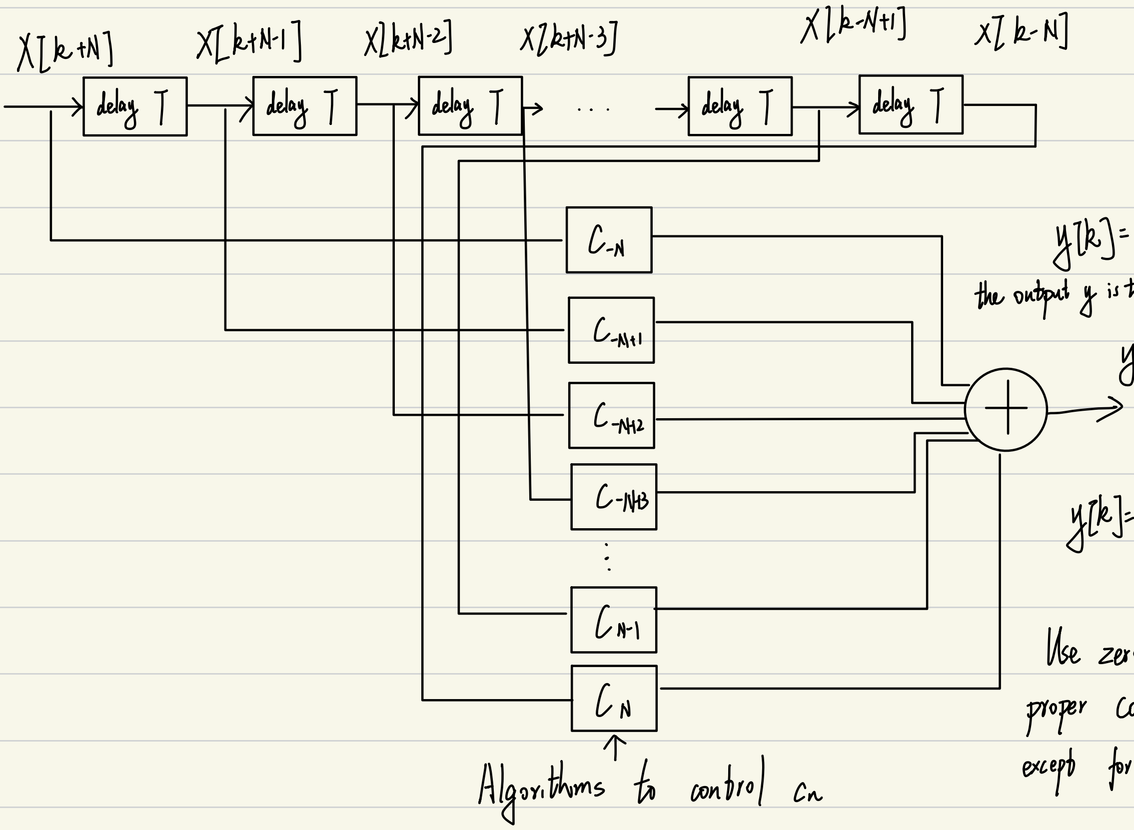
?

$$\alpha = \frac{1}{2} \qquad E = \frac{1+\frac{1}{4}}{T} = \frac{5}{4T}$$

$$\because E = 1 \qquad \therefore T = \frac{5}{4}$$

(iii) To compensate for the channel induced multi-path distortion, an equalization filter $H_{eq}(f) = 1/H(f)$ is often utilized. Sketch and briefly describe the architecture of the tapped-delay line or transversal filter having 2N+1 tap coefficient. State any necessary assumption needed.

(iii) Answer:



$$y[k] = \sum_{n=-N}^{N} C_n X_{k-n}$$

the output y is the convolution of input sequence and coefficient sequence

$$y[k] = \begin{cases} 1 & k=0 \\ 0 & k\neq 0 \end{cases}$$

Use zero-force algorithm to choose proper coefficient sequence to make y = 0 except for k=0

2. (a) Suppose $X(t)$ is stationary Gaussian random process with a constant mean value μ_X of 2 and the autocorrelation function $R_X(\tau) = E[X(t)\,X(t+\tau)] = 6\,[\,\exp(-0.2|\tau|) + 1]$. Let X_1 be the Gaussian random variable obtained by sampling the random process $X(t)$ at $t = t_1$. Determine the probability $P(\,X_1 \leqslant 1)$. **[Hint:** You may leave your answers in terms of the *Q*-function].

Answer: $E[X(t)] = 2 \qquad R_X(\tau) = E[X(t)\,X(t+\tau)] = 6\,\exp(-0.2|\tau|) + 1)$

$R_X(0) = E[X^2(t)] = 6e$

$Var[X(t)] = E[X^2(t)] - E^2[X(t)] = 6e - 4$

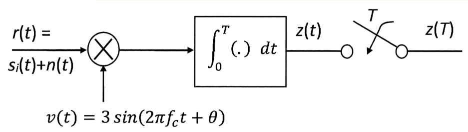
$X_1 \sim N(2, 6e-4)$

$\frac{X_1 - 2}{\sqrt{6e-4}} \sim N(0,1)$

$P\{X_1 \leq 1\} = P\left\{\frac{X_1-2}{\sqrt{6e-4}} \leq -\frac{1}{\sqrt{6e-4}}\right\} = 1 - Q\left(-\frac{1}{\sqrt{6e-4}}\right) = Q\left(\frac{1}{\sqrt{6e-4}}\right)$

(b) The binary PSK receiver is depicted in Figure 3 (on page 3). The received signal $r(t) = s_i(t) + n(t)$ consists of two components; The signal components are

$$s_i(t) = (-1)^i A\, sin(2\pi f_c t), \quad 0 \le t \le T, \quad i = 0{,}1.$$



Note that these two components are mutually independent. The probability of transmitting both signal components are equally likely, where A is the amplitude and T is the bit duration. The noise component $n(t)$ is additive white Gaussian noise (AWGN) with a zero mean and two-sided power spectral density $N_0/2$. The demodulating function is given as $v(t) = 3\, sin(2\pi f_c t + \theta)$, where the phase θ is a constant.

(i) Calculate the energy E_v of the demodulating function $v(t)$ at BPSK receiver over the bit duration.

(i) Answer : $E_v = \int_0^T v^2(t)\,dt = \int_0^T 9\sin^2(2\pi f_c t+\theta)\,dt = 9\int_0^T \sin^2(2\pi f_c t+\theta)\,dt$

$$= 9\int_0^T \frac{1-\cos(4\pi f_c t+2\theta)}{2}\,dt = 9\cdot\frac{T}{2} = \frac{9}{2}T$$

(ii) Determine signal components a_1 and a_2 at the output z(T) due to the two signals $s_i(t)$, $i = 0{,}1$, respectively. Find the optimum threshold γ_0 using the maximum-likelihood (ML) criterion.

(ii) Answer : $a_1 = \int_0^T A\sin 2\pi f_c t \cdot 3\sin(2\pi f_c t+\theta)\,dt$

$\sin A \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}$

$$= 3A\int_0^T \frac{\cos\theta + \cos(4\pi f_c t+\theta)}{2}\,dt$$

$$= \frac{3}{2}AT\cos\theta$$

$a_2 = -\frac{3}{2}AT\cos\theta$

$\because$ The probability of transmitting both signal components is equal likely.

$\therefore\ \gamma_0 = \frac{a_1+a_2}{2} = 0$

(iii) Find the mean and variance of the noise component in $z(T)$.

(iii) Answer : $n_0(T) = \int_0^T n(t)\cdot v(t)\,dt = 3\int_0^T \sin(2\pi f_c t+\theta)\,n(t)\,dt$

$E[n(t)] = 0$

$$E[n_0(T)] = E\left[3\int_0^T \sin(2\pi f_c t+\theta)\,n(t)\,dt\right] = 3\int_0^T \sin(2\pi f_c t+\theta)\,E[n(t)]\,dt = 0$$

$$Var[n_0(T)] = E[n_0^2(T)] - E^2[n_0(T)] = E[n_0^2(T)]$$

$$= E\left[\int_0^T n(t_1)\,v(t_1)\,dt_1 \int_0^T n(t_2)\,v(t_2)\,dt_2\right] = E\left[\int_0^T\int_0^T v(t_1)\,v(t_2)\,n(t_1)\,n(t_2)\,dt_1\,dt_2\right]$$

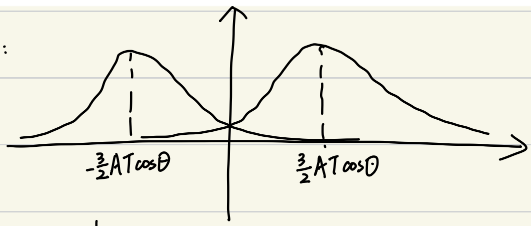
$$= \int_0^T\int_0^T v(t_1)\,v(t_2)\,E[n(t_1)\,n(t_2)]\,dt_1\,dt_2 = \int_0^T\int_0^T v(t_1)\,v(t_2)\,R_n(t_1-t_2)\,dt_1\,dt_2$$

$$= \int_0^T\int_0^T v(t_1)\,v(t_2)\,\frac{N_0}{2}\,\delta(t_1-t_2)\,dt_1\,dt_2 = \frac{N_0}{2}\int_0^T\int_0^T v(t_1)\,v(t_2)\,\delta(t_1-t_2)\,dt_1\,dt_2$$

$$= \frac{N_0}{2}\int_0^T v^2(t_1)\,dt_1 = \frac{N_0}{2}\cdot\frac{9}{2}T = \frac{9}{4}N_0 T$$

(iv) Compute the bit-error rate (BER) of the BPSK system in terms of Q-function, A, T, N_0, with θ.

(iv) Answer:



$$P(s_1) = P(s_0) = \frac{1}{2}$$

$$P(s_0|s_1) = P(s_1|s_0) = P\{z_1(T) < 0\} = P\left\{\frac{z_1(T) - \frac{3}{2}AT\cos\theta}{\frac{3}{2}\sqrt{N_0 T}} \leq -\frac{AT\cos\theta}{\sqrt{N_0 T}}\right\} = 1 - Q\left(-\frac{AT\cos\theta}{\sqrt{N_0 T}}\right) = Q\left(\frac{AT\cos\theta}{\sqrt{N_0 T}}\right)$$

$$= Q\left(\sqrt{\frac{A^2 T}{N_0}}\cos\theta\right)$$

$$P_E = P(s_0|s_1)\cdot P(s_1) + P(s_1|s_0)P(s_0) = Q\left(\sqrt{\frac{A^2T}{N_0}}\cos\theta\right)$$

(v) The noise $n(t)$ has a two-sided PSD $N_0/2 = 10^{-5}$ W/Hz . The demodulation function $v(t)$ is ahead with the phase $\theta = \pi/4$ and $A = 2$ V. Determine the maximum transmission bit rate R_b given the criteria for the BER $P_b \leq 10^{-4}$.

(v) Answer: $N_0/2 = 10^{-5}$ W/Hz $\quad N_0 = 2\times10^{-5}$ W/Hz $\quad \theta = \frac{\pi}{4} \quad A = 2$

$$P_b = Q\left(\sqrt{\frac{A^2T}{N_0}}\cos\theta\right) = Q\left(\frac{\sqrt{2}}{2}\sqrt{\frac{4T}{2\times10^{-5}}}\right) = Q\left(\sqrt{\frac{T}{1\times10^{-5}}}\right) \leq 10^{-4}$$

$$\therefore \sqrt{\frac{T}{1\times10^{-5}}} \geq 3.72$$

$$T \geq 1.38\times10^{-4}$$

$$R_b = \frac{1}{T} \leq 7226.27 \text{ bps}$$

$\therefore$ The maximum bit rate is $R_b = 7.22$ kbps.

3. (a) Design a (6, 2) single error correcting cyclic code by choosing the shortest possible generator polynomial, g(X).

Answer:

```
                 1 0 1
10101 ) 1 0 0 0 0 0 1
        1 0 1 0 1
        -------------
            1 0 1 0 1
            1 0 1 0 1
            ---------
                    0
```

The shortest $g(x) = 1 + x^2 + x^4$

(b) What is the probability of message error when using the code in part (a) if the binary symmetric channel symbol error probability is 10^{-3}? Consider only the first term.

Answer $p = 10^{-3}$

```
                1                            1
10101 ) 1 0 0 0 0 0          10101 ) 1 0 0 0 0
        1 0 1 0 1                    1 0 1 0 1
        ---------                    ---------
          1 0 1 0                    0 1 0 1
```

$$G = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$H = [I_{n-k} \; P^T] = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$d_{min} = 3$

Capatality of correct the error is $\frac{d_{min} - 1}{2} = 1$

X denotes the number of error symbols when we send a message $(6, 2)$

$X \sim b(6, 10^{-3})$

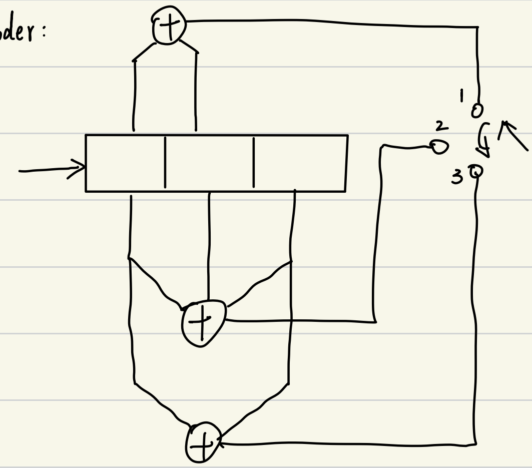
$$P_{message\_e} = P\{X \geq 2\} = 1 - P\{X = 0\} - P\{X = 1\} = 1 - (1 - 10^{-3})^6 - \binom{6}{1}(1 - 10^{-3})^5 \, 10^{-3} = 1.496 \times 10^{-5}$$

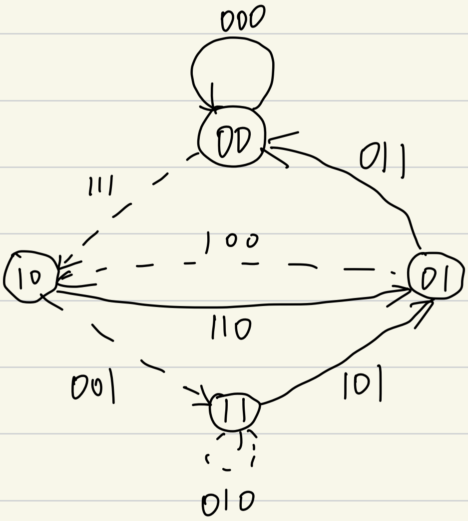
(c) A rate $\frac{1}{3}$ convolutional code is defined by the generator polynomials:

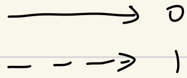
$g_1(X) = 1 + X$
$g_2(X) = 1 + X + X^2$
$g_3(X) = 1 + X^2$

(i) Draw the shift-register encoder and the state diagram of the code.

Encoder:

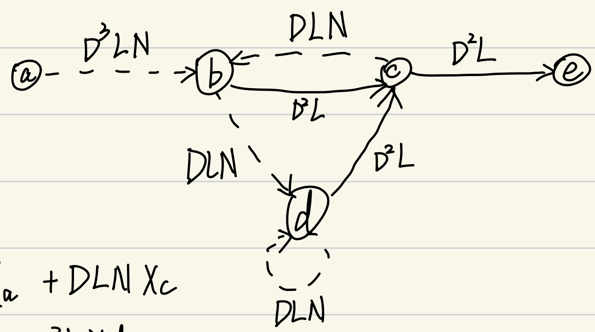






(ii) Determine the transfer function, T(D,L,N), of the encoder, where the exponents of D, L and N in the transfer function denote the Hamming distance, the number of branches, and branch transition caused by a bit one, respectively. What is the minimum free distance of the code? If the all-zero path is the correct path but the channel noise causes the receiver to choose the incorrect path with the minimum distance to the all-zero path, how many bit errors would have been made by the receiver?

Answer: $a = 00$ $b = 10$ $c = 01$ $d = 11$ $e = 00$



$X_b = D^3 L N X_a + DLN X_c$

$X_c = D^2 L X_b + D^2 L X_d$

$X_d = DLN X_b + DLN X_d$

$X_e = D^2 L X_c$

$$X_d = \frac{DLN}{1-DLN} X_b \qquad X_c = \left[D^2L + \frac{D^3L^2N}{1-DLN}\right] X_b$$

$$X_b = D^3LN X_a + DLN\left[D^2L + \frac{D^3L^2N}{1-DLN}\right] X_b$$

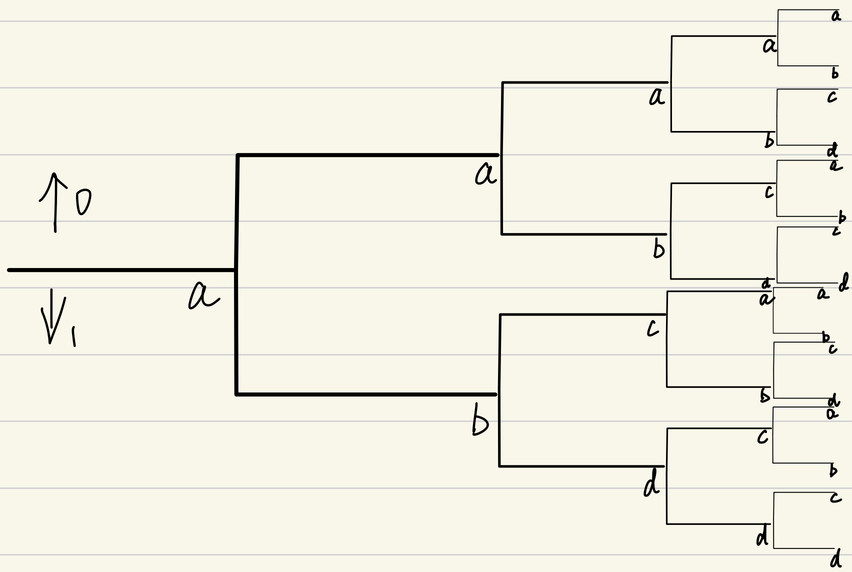
$$X_b = \frac{D^3LN}{1 - DLN\left[D^2L + \frac{D^3L^2N}{1-DLN}\right]} X_a$$

(iii) Decode the received sequence (starting from left to right), $\mathbf{r}$ = (111 000 001 101 110 011) using the stack algorithm. Illustrate by using the tree diagram and show the steps involved for the accumulated path metrics stack table. You may assume that the metric for the j^{th} branch of the r^{th} path is

$$\mu_j^{(r)} = 1 - 2d$$

where d is the Hamming distance between the received bits and the branch bits.

(14 Marks)



(b) The users in the cell of the CDMA system are distributed uniformly within it. The base-station (BS) is equipped with a 4-sector antenna with effective beamwidth of 100°. The voice data rate per user is 10 kbps and the system bandwidth is 2 MHz. Assume voice-activated discontinuous transmission is activated with half of the users in the system speaking 3/8 of the time and the rest speaking only 20% of the time. Assume also that perfect uplink power-control is implemented. The channel noise is an additive white Gaussian noise (AWGN). If only one terminal transmits the signal, the received energy per bit to noise spectral density ratio, $\frac{E_b}{N_0}$, at the BS is 25 dB. If the required $\frac{E_b}{N_0+I_0}$ (where I_0 is the total interference power spectral density) is 7 dB to operate satisfactorily, how many equal-power users in the cell can be supported?

Answer: $\frac{E_b}{N_0} = 25dB = 316.23$ $\quad \frac{N_0}{E_b} = 0.00316$ $\quad PG =$

$\frac{E_b}{N_0+I_0} = 7dB = 5.012$

$\frac{N_0+I_0}{E_b} = 0.19952$

$\frac{I_0}{E_b} = 0.19636$

$\frac{E_b}{I_0} = 5.0927$

$$\frac{S}{MAI} = \frac{E_b R_b}{I_0 R_c} = \frac{E_b}{I_0} \cdot \frac{1}{PG} = \frac{P}{(\frac{1}{2}\times\frac{3}{8}+\frac{1}{2}\times\frac{1}{5})PM} = \frac{1}{0.2875M}$$

$$\frac{E_b}{I_0} = \frac{PG}{0.2875M}$$